A Multi-Period Black-Litterman Model*

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The Black-Litterman model is a framework for incorporating forward-looking expert views in a portfolio optimization problem. Existing work focuses almost exclusively on single-period problems and assumes that the horizon of expert forecasts matches that of the investor. We consider a multi-period generalization where the horizon of expert views may differ from that of a dynamically-trading investor. By exploiting an underlying graphical structure relating the asset prices and views, we derive the conditional distribution of asset returns when the price process is geometric Brownian motion. We also show that it can be written in terms of a multi-dimensional Brownian bridge. The new price process is an affine factor model with the conditional log-price process playing the role of a vector of factors. We derive an explicit expression for the optimal dynamic investment policy and analyze the hedging demand associated with the new covariate. More generally, the paper shows that Bayesian graphical models are a natural framework for incorporating complex information structures in the Black-Litterman model.

Key words: Black-Litterman Model, Forward-looking Views, Kalman Smoothing Equations, Brownian Bridge, Portfolio Allocation, Hedging Strategies

1. Introduction

The Black-Litterman model (Black and Litterman (1991, 1992)) is a framework for incorporating forward-looking expert views in a one-step portfolio optimization problem. The model uses a backward-looking equilibrium model like the Capital Asset Pricing Model (CAPM) as a preliminary (prior) forecast and Bayes' rule to combine this with forward-looking expert views. The updated return distribution is used in a single-period mean-variance optimization to obtain an asset allocation. It has been observed that portfolios constructed using this approach tend to be less concentrated in a small number of assets and less sensitive to model inputs than those obtained without expert views.

The classical Black-Litterman model has two limitations which to our knowledge have not been explored in the literature: Forecasts are assumed to match the investment horizon of the investor,

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and investors are restricted to making single-period decisions. When views coincide with an external event like the outcome of a firm's quarterly earnings report, they are unlikely to match the investment horizon of the investor. The classical Black-Litterman model does not accommodate such an information structure as it does not specify how the predictions over a different horizon are related to the return distributions of interest to the investor.

Summary of contributions

1. We formulate a Bayesian graphical model that accommodates a stochastic model for asset prices and views about returns over multiple horizons. This allows us to transform expert forecasts over multiple horizons into recursive equations for the conditional price dynamics for the investor. More generally, Bayesian graphical models are a convenient framework to model more complex information structures and the update equations derived for our model can be used by a single-period Black-Litterman investor with views over multiple horizons.

2. We show that the log-returns process, which was Brownian motion with drift under the prior model, is a mean-reverting process after conditioning on views. It also defines a vector of covariates in the drift of the conditional price process. The consitional log-returns process is related to Kalman smoothing because views are forward looking and hence provide information about future states. More generally, one can imagine using forward looking views as covariates in a linear regression model of returns. This is indeed the case, though covariates (conditional log-returns) and their coefficients need to be updated over time.

3. We generalize the classical Black-Litterman model to the dynamic setting by formulating a dynamic portfolio choice problem in terms of the conditional price process. Although this is a factor model, we are able to derive an explicit expression for the optimal dynamic portfolio. This is surprising as the optimal portfolio for a dynamic factor model is usually expressed in terms of the solution of a system of ordinary differential equations. We show that the optimal portfolio consists of a mean-variance term and a hedging demand for changes in the conditional log-returns ("views covariate"), which we fully characterize.

4. We show that the hedging demand for the views covariate is large when views are informative, and that the dynamic Black-Litterman investor holds a smaller portfolio of risky assets than a single-period Black-Litterman investor with the same information set. This difference increases as views become more informative.

5. While the recursive update equations for the conditional log-returns process is all we need to formulate and solve the resulting dynamic portfolio choice problem, we also show that the conditional log-returns can be expressed in terms of a multi-dimensional Brownian bridge. The hitting time of each component of the Brownian bridge is endogenously determined by the correlation

structure of the assets and the statistical properties of the views, and generally extends beyond the horizon of the investor. (This connection between Kalman smoothing and the multidimensional Brownian bridge appears novel).

Literature review

The Black-Litterman model was introduced by Black and Litterman (1991, 1992), then expanded and discussed with greater detail in Bevan (1998) and He and Litterman (2002). The Bayesian interpretation of the Black-Litterman model was introduced by Qian and Gorman (2001) and expanded by Cheung (2009); for a survey we refer to Walters (2011). Much of the literature on the Black-Litterman model stays close to the classical setting where investments occur over a single-period and the horizon of the views matches that of the investor. Chen and Lim (2020) show how complex information structures and uncertainty about the equilibrium model can be modeled using a Bayesian graphical model for a single-period problem. We formulate a graphical model that accommodates continuous time asset price dynamics and expert views over multiple horizons. Analogous to the blending of the prior model and views in the classical Black-Litterman model, we derive recursive update equations for the price process conditional on views

Multi-period versions of the Black-Litterman model are proposed in Davis and Lleo (2013) and van der Schans and Steehouwer (2017). Both model asset prices as a Hidden Markov Model, Davis and Lleo (2013) in continuous time and van der Schans and Steehouwer (2017) in discrete time, with experts providing noisy views about the current value of the hidden market state. While such views can occur, it differs from the typical Black-Litterman model where experts are forward-looking. Davis and Lleo (2013) and van der Schans and Steehouwer (2017) derive filtered estimates of the latent (current) state of the Markov chain. In contrast, due to the forward-looking views, the conditional price process in our paper is more closely related to a smoothed estimate of a future value of the state. We also show that the smoothed estimate can be written in terms of a multi-dimensional Brownian bridge.

A contribution we wish to highlight is to the literature on multi-dimensional Brownian bridge. The classical Brownian bridge (Siegrist (2022) and Pinsky and Karlin (2011)) is the stochastic process $B(t) = \{W(t)|W(T) = y\}$ obtained after conditioning on the terminal value of a one-dimensional Brownian motion $W(t)$. The classical literature characterizes its distributional properties and shows that it is the solution of a linear stochastic differential equation. One-dimensional Brownian bridge has been used in the finance literature to model price dynamics of an insider trader (Aksamit and Jeanblanc (2017) and Peralta Hernández (2018)). The notion of multi-dimensional Brownian bridge is more complex because the terminal value and time of *each component*, as well as the correlation between them, needs to be defined and there does not appear to be a canonical definition of this process. Applications of multi-dimensional versions of Brownian bridge in finance are quite scarce; one example we know of is Angoshtari and Leung (2020) who use it to model futures prices. We also mention Atkinson and Singham (2015) who consider a multi-dimensional Brownian bridge where the starting and ending values of each element of the Brownian motion is known in advance and the hitting times are derived from the correlation structure.

One contribution of this paper is to generalize the classical definition of Brownian bridge to the case where observations of the terminal value of a multi-dimensional Brownian motion is observed with noise. We derive a stochastic differential equation for generalized Brownian bridge and fully characterize its joint distribution. We show that its components are correlated one-dimensional Brownian bridges with different hitting times that are determined endogenously by the correlation structure of the original Brownian motion and the noisy terminal observation equation. Finally, we show that the conditional log-returns process from the Black-Litterman model can be written in terms of this generalized Brownian bridge. More generally, the connection between noisy views of the future value of a multi-dimensional Brownian motion and the stochastic process it defines seems to be of independent interest.

Outline

We briefly review the classical Black-Litterman model in Section 2 and introduce a graphical model for the multi-period generalization in Section 3. We also derive the price dynamics conditioned on the views and show that it is an affine model with a new covariate $X^y(t)$ that dynamically blends the views and real-time price information. We define the notion of generalized multi-dimensional Brownian bridge in 4 and provide an interpretation of this covariate in terms of this process. We formulate and solve the associated dynamic portfolio choice problem in 5. In section 6, we provide numerical results and simulations, and show how our analysis can cover multiple extensions of the Black-Litterman model in section 7. Finally, section 8 concludes and discusses the implication of this work beyond financial applications.

2. The Black-Litterman Model

To keep the paper self-contained, we now provide a brief review of the classical Black-Litterman model. Building on the single period model in Chen and Lim (2020) we adopt a graphical Bayesian representation of the joint distribution of views and asset prices that allows us to capture more complex view structure.

2.1. Equilibrium Model

Consider a financial market of N risky assets with rate of return r and one risk-free asset with rate r_f. The Black-Litterman model assumes that r is normally distributed

$$r \sim \mathcal{N}\big(\mu, \Sigma\big) \tag{1}$$

and selects the mean return μ using a backward-looking equilibrium model such as the Capital Asset Pricing Model (CAPM) (see Sharpe (1964)), or an inverse optimization method (see Bertsimas et al. (2012), Sharpe (1974)). In a Bayesian setting, equation (1) can be considered a prior distribution on the unrealized returns, with views being noisy observations which are used to updated the prior distribution (1). For a comparison between the two approaches refer to Subekti et al. (2021).

2.2. Expert views

The investor receives forward-looking views about risky asset returns. Conditional on the realized returns r, views are assumed to be normally distributed

$$Y|r \sim \mathcal{N}\big(Pr, \Omega\big), \tag{2}$$

where P is a linear mapping from the set of returns to the set of views and captures the relationship between each view and the vector of returns r. The covariance matrix Ω models the accuracy of the views. Methods for determining Ω from empirical data are discussed in Granger and Newbold (1974) and Winkler and Makridakis (1983).

To illustrate the idea, consider a market with assets, A, B, and C, with $r = [r_A, r_B, r_C]^\top$ being the vector of returns. Before the realization of the returns, the investor receives expert views concerning the three assets. We distinguish here between two types of views:

1. Absolute views: The expert gives a direct forecast about the return of one of the assets; for example, 'The return of asset A will be 5%',
2. Relative views: The expert compares the returns of two or more assets; For example, 'Company C will outperform company B by 10%'.

In this case, the expert is giving a noisy forecast of the realized return

$$Pr = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} r_A \\ r_B \\ r_C \end{bmatrix} = \begin{bmatrix} r_A \\ r_C - r_B \end{bmatrix},$$

which by (2) is a sample of a two-dimensional normal random vector with mean Pr and covariance matrix Ω. In this example, the realization y of $Y(0,T)$ is

$$y = \begin{pmatrix} y_A \\ y_{C-B} \end{pmatrix} = \begin{pmatrix} 5\% \\ 10\% \end{pmatrix}.$$

2.3. Graphical Representation

Bayesian graphical models provide a clear and intuitive framework for capturing uncertainty, dependencies, and causal relationships among variables (see for example Chen and Lim (2020)). We represent random variables as nodes with unobserved random variables (the vector of unrealized returns) as circles, and observed random variables (experts views) as squares. Edges represent conditional dependencies.

Expert views in the classical single-period Black-Litterman model can be interpreted as noisy observations of unrealizaed returns, which can be represented as shown in Figure 1.

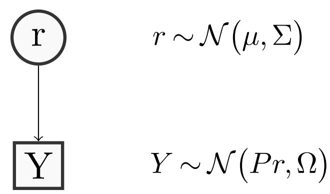

Figure 1 **Bayesian network of the classical Black-Litterman model**

2.4. Posterior Distribution of the Returns

We can use the view (2) to update the equilibrium returns (1) using Bayes' rule. Specifically, the vector of returns r given the view $Y(0,T) = y$ is still normal

$$r|Y = y \sim \mathcal{N}(\mu_{BL}, \Sigma_{BL}), \tag{3}$$

with mean and covariance

$$\begin{aligned} \mu_{BL} &= \mathbb{E}\Big[r|Y = y\Big] = \big(\Sigma^{-1} + P^\top \Omega^{-1} P\big)^{-1} \big(\Sigma^{-1}\mu + P^\top \Omega^{-1} y\big), \\ \Sigma_{BL} &= \mathbb{V}\Big[r|Y = y\Big] = \big(\Sigma^{-1} + P^\top \Omega^{-1} P\big)^{-1}. \end{aligned} \tag{4}$$

Note that the posterior mean μ_{BL} is a combination of the prior mean μ and the view y weighted by their respective precision matrices Σ^{-1} and Ω^{-1}. The precision of the posterior return increases from Σ^{-1} to $\Sigma_{BL}^{-1} = \Sigma^{-1} + P^\top \Omega^{-1} P$ after the update.

2.5. Optimal Portfolio

Using the updated return distribution in (4), the optimal portfolio is obtained by solving a mean-variance optimization problem

$$\max_{\pi} \pi^\top \mathbb{E}\Big[r|Y = y\Big] + (1 - \pi^\top \mathbf{1}_N) r_f - \frac{\gamma}{2} \pi^\top \mathbb{V}\Big[r|Y = y\Big] \pi \tag{5}$$

where $\gamma \in (0, \infty)$ is the risk-aversion parameter. The optimal portfolio is

$$\begin{aligned} \pi^* &= \frac{1}{\gamma} Var^{-1}\Big[r|Y = y\Big] (\mathbb{E}\Big[r|Y = y\Big] - r_f \mathbf{1}_N) \\ &= \frac{1}{\gamma} (\Sigma_{BL})^{-1} (\mu_{BL} - r_f \mathbf{1}_N), \end{aligned} \tag{6}$$

where $\mathbf{1}_N = (1, \ldots, 1)^\top \in \mathbb{R}^N$ vector of ones.

3. The Multi-Period Black-Litterman Model

We formulate a continuous time version of the Black-Litterman model with forward-looking expert views. We derive the price dynamics conditional on these views and show that it is closely related to the Kalman smoothing equations.

Throughout the paper, we assume that all random variables and stochastic processes are defined on the common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All vectors are column vectors. For a vector $S \in \mathbb{R}^N$, we denote its i^{th} element by S_i, $i \in [N]$, and the diagonal matrix of stock prices by $D(S) = \text{diag}(S_1, \dots, S_N) \in \mathbb{R}^{N \times N}$. For a matrix L, we use L_i to denote its i^{th} column, and I_N to denote the N by N identity matrix. $\top$ is used for the transpose operator.

3.1. Financial Market

Consider a financial market of N risky assets and one risk-free asset. The interest rate r_f for the risk-free asset is assumed to be constant and its price $S_0(t)$ satisfies

$$\frac{dS_0(t)}{S_0(t)} = r_f dt. \tag{7}$$

For $i \in [N]$, the price $S_i(t)$ of the risky asset i evolves as a geometric Brownian motion

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + dW_i(t) \tag{8}$$

where $W(t)$ is a vector of N-correlated Brownian motions with

$$W(t) \sim \mathcal{N}\big(0, t\Sigma\big).$$

As in the classical Black-Litterman model, the drift μ_i of stock i is set to equal the expected return obtained from a backward-looking equilibrium model such as CAPM, and σ_i is the associated volatility. If ρ_{ij} is the correlation between $W_i(t)$ and $W_j(t)$, we can write the covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1N}\sigma_1\sigma_N \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2N}\sigma_2\sigma_N \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N}\sigma_1\sigma_N & \rho_{2N}\sigma_2\sigma_N & \cdots & \sigma_N^2 \end{pmatrix}.$$

There are various methods for specifying the covariance matrix Σ (see, e.g., He and Litterman (2002), Walters (2013)). Let $X(t) = (X_1(t), \dots, X_N(t))^\top \in \mathbb{R}^N$ denote the vector of the log-returns

$$X_i(t) = \log\left(\frac{S_i(t)}{S_i(0)}\right).$$

Then

$$X(t) = t\mu^x + W(t) = t\mu^x + LV(t) \sim \mathcal{N}\big(t\mu^x, t\Sigma\big) \tag{9}$$

is a multivariate Brownian motion with drift where $\mu^x = (\mu_1^x, \dots, \mu_N^x)^\top \in \mathbb{R}^N$ is the vector of drifts $\mu_i^x = \mu_i - \sigma_i^2/2$ for the $i \in [N]$ log returns, $L \in \mathbb{R}^{N\times N}$ is a lower triangular matrix such that $LL^\top = \Sigma$, the so-called Cholesky decomposition of Σ (for details about the proof of existence and uniqueness of L we refer to Moler and Stewart (1978) and Higham (2009)), and $V(t) = L^{-1}W(t)$ is a N- dimensional standard Brownian motion. We adopt the parameterization (9) of the log return throughout the paper. We use $\mathcal{F}_t := \sigma(W_s; s \le t)$ to denote the natural filtration generated by the Brownian motion $\{W(t), t \in [0,T]\}$. It is easy to see that $V(t)$ generates the same filtration as $W(t)$.

Equation (9) is analogous to the equilibrium model in the classical single period problem. We now introduce the model for forward-looking views. Analogous to the classical model (3)–(4), we then show show how they can be used to update the stochastic model of returns.

3.2. Expert Views

A key difference between our model and Davis and Lleo (2013) and van der Schans and Steehouwer (2017) is that our expert gives forward-looking views about future returns, for instance, a view at time t_1 about the return between t_1 and t_2. As in the classical Black-Litterman model, we model these views as noisy observations of a linear mapping of the vector of log-returns $X(t_2) - X(t_1)$. Note that views about the returns can be transformed to views about the log-returns, and vice versa.

An important component of the views model is the specification of the noise in the prediction. We assume that the noise is increasing in the return horizon $t_2 - t_1$. Let $Y(t_1, t_2) \in \mathbb{R}^K$ be the vector of K views given at time t_1, about the value of the log-returns vector at time t_2. Conditioned on the true log-returns being $X(t_2) - X(t_1)$, we assume that $Y(t_1, t_2)$ is normally distributed

$$Y(t_1,t_2)|X(t_2) - X(t_1) = P(X(t_2) - X(t_1)) + f(t_2 - t_1)\epsilon \sim \mathcal{N}\big(P(X(t_2) - X(t_1)), f^2(t_2 - t_1)\Omega\big)$$

where $\epsilon \sim \mathcal{N}\big(0, \Omega\big)$ captures the dependence between the K views, $P \in \mathbb{R}^{K\times N}$ is a linear mapping from returns to views, and the non-negative and increasing scalar function $f(t)$ describes how the horizon affects the uncertainty in the views. We assume for simplicity that all views are given at the start of the investment horizon and the horizon of the views match the investment problem, so $t_1 = 0$ and $X(t_1) = 0$ and $t_2 = T$, and that the variance of the view is linear in the horizon of the forecast $f(t) = \sqrt{t}$. That is

$$Y(0,T)|X(T) = PX(T) + \sqrt{T}\epsilon \sim \mathcal{N}\big(PX(T), T\Omega\big). \tag{10}$$

We show in Section 6 how the model can be extended to accommodate views at multiple future points. We assume in the rest of the paper that the log-returns and the views covariance matrices,

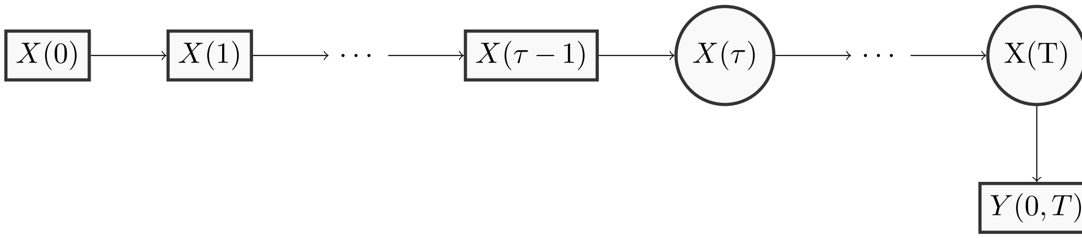

Figure 2 **Bayesian network of the Multi-Period Black-Litterman model.** The figure shows a discrete time version of the problem where $t = 0, \dots, T$, and the noisy view $Y(0,T)$ of the log-return $X(T)$ is revealed at $t = 0$. An investor at time $0 \leq \tau \leq T$ knows $\{X(0), \dots, X(\tau-1)\}$ and the forward-looking view $Y(0,T)$.

respectively Σ and Ω, are invertible, which implies that the market is arbitrage free[1] (see Dhaene et al. (2020)) and that expert views are not redundant[2].

Figure 2 shows a Bayesian network representation of the discrete time version of the problem where $t = 0, \dots, T$, and the noisy view $Y(0,T)$ of the log-returns $X(T)$ is revealed at $t = 0$. An investor at the beginning of the investment period $\tau \in \{0, 1, \dots, T\}$ knows the past realizations of the log-returns $\{X(0), \dots, X(\tau-1)\}$ and the forward-looking view $Y(0,T)$ that was given at $t = 0$ and uses this to update her beliefs about the distribution of future log-returns $\{X(\tau), \dots, X(T)\}$.

3.3. Conditional Dynamics of the Asset Price

We now derive the dynamics of $X(t)$ conditioned on the forward-looking view $Y(0,T) = y$. Let $X^y(t)$ denote the conditional process $X(t)|Y(0,T) = y$ and $\mathcal{F}_t^Y := \sigma(\mathcal{F}_t \vee \sigma(Y(0,T)))$ the filtration containing the total information available to the investor at time t. The proof of the following result can be found in the Appendix.

Proposition 1. *Suppose that the price process satisfies* (8) *and expert views* $Y(0,T)$ *satisfy* (10). *Assume that* $PL_j \neq 0$ *for* $j \in [N]$. *Conditional on* $Y(0,T) = y$, *the log-returns* $X(t)$ *satisfy*

$$dX^y(t) = \left(\mu^x + \frac{1}{T}\beta_1\big(y - TP\mu^x\big) - \frac{1}{T-t}\beta_2(t)\big(X^y(t) - \mathbb{E}[X^y(t)]\big)\right)dt + dW^y(t), \tag{11}$$

where

$$\beta_1 = \Sigma P^\top (P\Sigma P^\top + \Omega)^{-1} \in \mathbb{R}^{N\times K},$$

$$\beta_2(t) = \big[I_N - \big(I_N + (1 - \frac{t}{T})\Sigma P^\top \Omega^{-1} P\big)^{-1}\big] \in \mathbb{R}^{N\times N},$$

and

$$\mathbb{E}[X^y(t)] = t\mu^x + \frac{t}{T}\beta_1\big(y - TP\mu^x\big)$$

[1] It is a sufficient (but not necessary) condition for the market to be arbitrage free.

[2] In practice, even if an expert is redundant (his view can be written as a linear combination of other views), we can add a small noise to the view to make Ω positive-definite.

is the expected log-return over the horizon $[0,t]$ *given* $Y(0,T)=y$. $W^y(t)\sim\mathcal{N}\big(0,t\Sigma\big)$ *is a* $N-$*dimensional Brownian motion adapted to the filtration* $\mathcal{F}_t^Y$. *Conditional on the views, the stock price* $S^y(t)=S(t)|(Y(0,T)=y)$ *has dynamics*

$$dS^y(t)=D(S^y(t))\big(\tilde{\mu}(t,X^y(t))dt+dW^y(t)\big) \tag{12}$$

where $D(S^y(t))$ *a diagonal matrix with elements* $\{S_i^y(t), i\in[N]\}$ *and*

$$\tilde{\mu}(t,x)=\mu+\frac{1}{T}\beta_1\big(y-TP\mu^x\big)-\frac{1}{T-t}\beta_2(t)\big(x-\mathbb{E}[X^y(t)]\big)$$

is the new drift of the stock price.

Remark 1. The condition $PL_j\neq 0$, for $j\in[N]$ ensures that forward-looking views give information about each element of the Brownian motion $V(t)=L^{-1}W(t)$. In the next section, we show how this condition can be dropped without affecting the results.

If we interpret $Y(0,T)$ as a noisy observation of $X(T)$, the distribution of $X(t)$ given $Y(0,T)=y$ (and hence the dynamics of $X^y(t)$) is similar to the Kalman smoother. The view $Y(0,T)=y$ changes the probability measure for the investor from $\mathbb{P}$ to $\mathbb{Q}=\mathbb{P}(\,\cdot\,|Y(0,T)=y)$.

For the information structure in Figure 2, the investor at time t has the forward-looking views $Y(0,T)=y$ provided at the beginning of the investment period and the history of log returns on $[0,t]$ including $X^y(t)=x$. Using this information

$$X^y(t+dt)-x=\left(\mu^x+\frac{1}{T}\beta_1\big(y-TP\mu^x\big)-\frac{1}{T-t}\beta_2(t)\big(x-\mathbb{E}[X^y(t)]\big)\right)dt+dW^y(t)$$

predicts log-returns over the interval $[t,t+dt]$ where β_1 and $\beta_2(t)$ are coefficients of a linear regression model with covariates $y-TP\mu^x$ and $x-\mathbb{E}[X^y(t)]$ and $dW^y(t)$ is the uncertainty in the prediction. The conditional price process (12) is no longer Geometric Brownian motion but has a drift which is a function of time, the vector of views y, and the conditional log-return $X^y(t)=x$ as a predictor.

To provide an understanding of how these coefficients depend on the problem structure, consider the case of a single risky asset and one forward-looking view.

Example 1. Suppose we have a single asset with price $S(t)\in\mathbb{R}$ that is geometric Brownian motion with drift $\mu\in\mathbb{R}$ and volatility $\sigma\in\mathbb{R}$. Let $X(t)\in\mathbb{R}$ be its log-return. Given a noisy forward-looking view $y\sim X(T)+\epsilon$ where $\epsilon\sim\mathcal{N}\big(0,T\omega^2\big)$ is independent of $W(t)$, the conditional log returns satisfies

$$dX^y(t)=\left(\mu^x+\frac{1}{\tilde{T}}(y-\mu^xT)-\frac{1}{\tilde{T}-t}(X^y(t)-\mathbb{E}[X^y(t)])\right)dt+\sigma dW^y(t) \tag{13}$$

where $\tilde{T} = T(1 + \frac{\omega^2}{\sigma^2})$ and $\mu^x = \mu - \frac{\sigma^2}{2}$. The stochastic differential equation (SDE) (13) has an explicit solution

$$X^y(t) = \mu^x t + \frac{t}{\tilde{T}}(y - \mu^x T) + \sigma(\tilde{T} - t)\int_0^t \frac{1}{\tilde{T} - s} dW^y(s), \text{ for } t \in [0, T]. \tag{14}$$

Observe from (14) that the view induces an adjustment in the posterior mean that is proportional to the difference between the view y and $\mu^x T$ so prices drift upwards if the forecast y exceeds prior expectations $\mu^x T = \mathbb{E}[Y(0,T)]$. For an alternative interpretation note that we can write

$$X^y(t) = \mu^x t + \sigma B(t), \text{ for } t \in [0, T]$$

where

$$B(t) = \frac{t}{\tilde{T}}\frac{1}{\sigma}(y - \mu^x T) + (\tilde{T} - t)\int_0^t \frac{1}{\tilde{T} - s} dW^y(s), \text{ for } t \in [0, T].$$

Specifically, $B(t)$ is nothing but the restriction to $[0, T]$ of a Brownian bridge from 0 to $\frac{1}{\sigma}(y - \mu^x T)$ with a hitting time $\tilde{T} \geq T$ that is increasing in the uncertainty in the view. It now follows that

$$\tilde{X}(t) := \mu^x t + \sigma B(t), \text{ for } t \in [0, \tilde{T}]$$

is a Brownian bridge on $[0, \tilde{T}]$ from 0 to $\tilde{X}(\tilde{T}) = y$ with drift μ^x and the original conditional log returns process (14) is the restriction of this Brownian bridge to $[0, T]$.

4. Forward-looking Views and Brownian Bridge

Example 1 shows that the conditional log-returns process for a single asset problem can be written in terms of a Brownian bridge. We now explore this connection in the multi-asset case.

4.1. One-dimensional case

The classical Brownian bridge (Siegrist (2022) and Pinsky and Karlin (2011)) is the stochastic process that is obtained after conditioning on the terminal value of a one-dimensional Brownian motion $W(t)$ on the closed interval $[0, T]$.

Definition 1. Let $W(t) \in \mathbb{R}$ be a Brownian motion with initial value $W(0) = a$. Then the process $\{B(t) = (W(t)|W(T) = y), t \in [0, T]\}$ is called a Brownian bridge (Bb) from a to y with hitting time T.

The following result characterizes properties of one-dimensional Brownian bridge (Aksamit and Jeanblanc (2017) and Gasbarra et al. (2007)).

Proposition 2. *A stochastic process $B(t) \in \mathbb{R}$ is a Brownian bridge (Bb) from a to y with hitting time T if*

1. $B(0)=a$ *and* $B(T)=y$ *(with probability* 1*),*
2. $\{B(t), t\in[0,T]\}$ *is a Gaussian process,*
3. $\mathbb{E}[B(t)]=a+\frac{t}{T}(y-a)$ *for* $t\in[0,T]$*,*
4. $cov\big(B(t),B(s)\big)=\min\{s,t\}-\dfrac{st}{T}$*, for* $s,t\in[0,T]$*,*
5. *With probability* 1*,* $t\to B(t)$ *is continuous in* $[0,T]$*.*

The Brownian bridge $B(t)$ *is the solution to the SDE*

$$\begin{cases} dB(t) &= \dfrac{y-B(t)}{T-t}dt+dW^y(t) \\ B(0) &= a, \end{cases} \tag{15}$$

where $W^y(t)$ *is a Brownian motion. The explicit solution of this equation is*

$$B(t)=a+\frac{t}{T}(y-a)+(T-t)\int_0^t \frac{1}{T-s}dW^y(s). \tag{16}$$

We now generalize Definition 1 to the case when we have a noisy observation of the terminal value of a Brownian motion of the form $Y(0,T)=W(T)+\epsilon$. Conditional on $Y(0,T)$, we show that the Brownian motion is the restriction to $[0,T]$ of a Brownian bridge with a hitting time $\tilde{T}>T$ (Figure 3). This enables us to connect the conditional log returns process from the Black-Litterman model with Bb.

Proposition 3. *Let* $W(t)\in\mathbb{R}$ *be a standard Brownian motion such that* $W(0)=a$*. Let* $T>0$ *and suppose we observe a sample from* $Y(0,T)=W(T)+\epsilon$ *at* $t=0$ *where* $\epsilon\sim\mathcal{N}\big(0,T\omega^2\big)$ *is independent of* $W(T)$*. Then the stochastic process* $\{B(t)=(W(t)|Y(0,T)=y), t\in[0,T]\}$ *is a restriction of a Brownian bridge (Bb) from* a *to* y *with hitting time* $\tilde{T}=T(1+\omega^2)$ *to the interval* $[0,T]$*. Additionally,* $B(t)$ *is the solution the SDE*

$$\begin{cases} dB(t) &= \dfrac{y-B(t)}{\tilde{T}-t}dt+dW^y(t),\ t\in[0,T] \\ B(0) &= a, \end{cases}$$

where the solution is

$$B(t)=a+\frac{t}{\tilde{T}}(y-a)+(\tilde{T}-t)\int_0^t \frac{1}{\tilde{T}-s}dW^y(s),\ \ for\ t\in[0,\tilde{T}].$$

Returning to Example 1, observe that

$$\begin{aligned} X^y(t) &= (X(t)|Y(0,T)=y) \\ &= \mu^x t+\sigma\big(W(t)|Y(0,T)=y\big). \end{aligned}$$

It now follows from Proposition 3 that $B(t)=(W(t)|Y(0,T)=y)$ is the restriction to the interval $[0,T]$ of a Bb from 0 to $\dfrac{1}{\sigma}(y-\mu^x T)$ with hitting time $\tilde{T}=T(1+\dfrac{\omega^2}{\sigma^2})$, and hence

$$X^y(t)=\mu^x t+\sigma B(t).$$

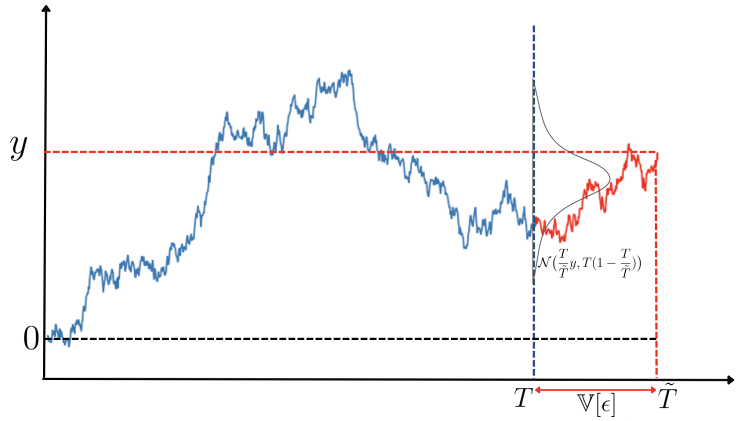

Figure 3 **Relationship between the forward-looking view and the Bb.** The view $Y(0,T) = y$ changes the distribution of the Brownian motion $W(t)$, for $t \in [0,T]$, to a Brownian bridge hitting y at a time $\tilde{T} = T + \mathbb{V}[\epsilon]$.

4.2. Multidimensional Case

Let $W(t)$ is an N-dimensional correlated Brownian motion. We now consider a generalization of Brownian bridge

$$\Big\{B(t) = \big(W(t)|Y(0,T) = y\big), t \in [0,T]\Big\} \tag{17}$$

where

$$Y(0,T) = PW(T) + \epsilon$$

is a vector of noisy observations of its terminal value $W(T)$ with view matrix P and ϵ is a normal random vector. The classical Brownian bridge (Definition 1) assumes $W(t)$ is one-dimensional and $W(T)$ is observed without noise while Proposition 3 allows noisy observations of $W(T)$. The multi-dimensional version is more nuanced than the single-dimensional case because information about each component of $W(t)$ is obtained not only through $Y(0,T)$ but through noisy observations of other components of $W(t)$ which are correlated. We illustrate this in the following example.

Example 2. Let $W_1(t)$ and $W_2(t)$ be standard Brownian motions with correlation $\rho \in (0,1]$

$$dW_1(t)dW_2(t) = \rho dt.$$

Let $\epsilon \sim \mathcal{N}\big(0,\omega^2\big)$ and suppose we have a noisy observation $Y(0,T) = W_2(T) + \epsilon$ of $W_2(T)$. Let $\big\{B(t) \equiv \big(B_1(t), B_2(t)\big)^\top = (W(t)|Y(0,T) = y), t \in [0,T]\big\}$ be the distribution of $W(t)$ condition on the vector of observations $Y(0,T) = y$. It follows from Proposition 3 that $B_2(t)$ is the restriction to $[0,T]$ of a Bb $(W_2(t)|W_2(\tilde{T}_2) = y)$ with terminal value $W_2(\tilde{T}_2) = y$ at $\tilde{T}_2 = T + \omega^2$. Additionally, the correlation between the two Brownian motions transforms the view about $W_2(T)$ to a noisy observation of $W_1(T)$ and it can also be shown (see Appendix) that $B_1(t)$ is the restriction of a Bb

$(W_1(t)|W_1(\tilde{T}_1) = \frac{y}{\rho})$ with terminal value $W_1(\tilde{T}_1) = \frac{y}{\rho}$ at $\tilde{T}_1 = \frac{T+\omega^2}{\rho^2}$ to $[0,T]$. Note that $\tilde{T}_1 \geq \tilde{T}_2$ with equality if and only if $\rho = 1$; there is less information about W_1 than W_2. If $\rho = 0$, then $B_1(t)$ is unaffected by the observation of $W_2(T)$ and remains a standard Brownian motion.

4.2.1. Main Results. The following result generalizes Proposition 3 for classical Browning bridge to the multi-dimensional case with noisy observations (17).

THEOREM 1. *Let $W(t)$ be an N- dimensional Brownian motion such that $W(t) \sim \mathcal{N}\big(a, t\Sigma\big)$, where $\Sigma \in \mathbb{R}^{N\times N}$ is symmetric positive definite with Cholesky decomposition L ($\Sigma = LL^\top$). Let $Y(0,T) = PW(T) + \epsilon$ where $P \in \mathbb{R}^{K\times N}$ is such that $PL_j \neq 0$ for $j \in [N]$ and $\epsilon \sim \mathcal{N}\big(0, T\Omega\big)$ for some symmetric positive definite covariance matrix $\Omega \in \mathbb{R}^{K\times K}$. Given $Y(0,T) = y$, the stochastic process $\{B(t) = (W(t)|Y(0,T) = y), t \in [0,T]\}$ satisfies*

1. *$B(0) = a$ (with probability 1).*
2. *B is a Gaussian process.*
3. *For $t \in [0,T]$,*

$$\mathbb{E}[B(t)] = a + \frac{t}{T}\beta_1\big(y - Pa\big)$$

where

$$\beta_1 = \Sigma P^\top (P\Sigma P^\top + \Omega)^{-1}.$$

4. Cov $\big(B(t), B(s)\big) = L\big(\min\{s,t\}I_N - stH\big)L^\top$ *for $s,t \in [0,T]$ where*

$$H = \frac{1}{T}(PL)^\top (P\Sigma P^\top + \Omega)^{-1} PL \in \mathbb{R}^{N\times N}$$

is a symmetric, positive-definite $N \times N$ matrix.

5. *With probability 1, $t \to B_i(t)$ is continuous in $[0,T]$ for $i \in [N]$.*

The properties of $\{B(t), t \in [0,T]\}$ are similar to those of the one-dimensional Bb in Proposition 2. In particular, for every $i \in [N]$, $(L^{-1}B(t))_i$ is a one-dimensional Brownian bridge with hitting time

$$\tilde{T}_i = \frac{1}{H_{i,i}} = T\big((PL_i)^\top (P\Sigma P^\top + \Omega)^{-1} PL_i\big)^{-1}. \tag{18}$$

For $i \neq j$, $(L^{-1}B(t))_i$ and $(L^{-1}B(t))_j$ are generally dependent. We define the vector of hitting times $\tilde{T} = [\tilde{T}_1, \cdots, \tilde{T}_N]^\top \in \mathbb{R}^N$ where each element is given by (18).

REMARK 2. It is worth noting that the diagonal elements of the matrix H are strictly positive. This follows from the fact that the covariance matrices Σ and Ω are positive definite, along with the assumption $PL_i \neq 0$, for $i \in [N]$. Thus, $(PL_i)^\top (P\Sigma P^\top + \Omega)^{-1} PL_i$ are strictly positive, and the vector $\tilde{T}$ is always well-defined.

The following result characterizes the dynamics of the process $\{B(t) = (W(t)|y), t \in [0,T]\}$ defined in Theorem 1. It generalizes the properties of one-dimensional Brownian bridge given in Proposition 2 to the multi-dimensional noisy-observation setting.

THEOREM 2. *Consider the stochastic process $\{B(t), t \in [0,T]\}$ defined in Theorem 1. Then*

$$B(t) = a + \frac{t}{T}\beta_1\big(y - Pa\big) + L\bar{B}(t) \tag{19}$$

where

$$\beta_1 = \Sigma P^\top (P\Sigma P^\top + \Omega)^{-1}$$

and $\bar{B}(t)$ is a solution of the SDE

$$d\bar{B}(t) = -\frac{dt}{T-t}\bar{\beta}_2(t)\bar{B}(t) + dV^y(t). \tag{20}$$

Here, $V^y(t)$ is a standard $N-$dimensional Brownian motion adapted to the enlarged filtration $\mathcal{F}_t^Y$ and

$$\bar{\beta}_2(t) = I_N - L^{-1}\big(\Sigma^{-1} + (1 - \frac{t}{T})P^\top \Omega^{-1} P\big)^{-1}(L^{-1})^\top \in \mathbb{R}^{N\times N}.$$

$\{\bar{B}(t) \in \mathbb{R}^N, t \in [0,T]\}$ is a 0 mean stochastic process and each element $\{\bar{B}_i(t), t \in [0,T]\}$ $(i \in [N])$ is a restriction to the interval $[0,T]$ of a Brownian bridge from 0 to 0 with hitting time $\tilde{T}_i$ defined in (18). The covariance matrix of $\bar{B}(t)$ is

$$\mathrm{Cov}(\bar{B}_i(t), \bar{B}_j(s)) = \begin{cases} \min\{s,t\} - \dfrac{st}{\tilde{T}_i}, & \text{for } i = j, \\ -\dfrac{st}{H_{i,j}}, & \text{for } i \neq j. \end{cases}$$

REMARK 3. To understand the condition $PL_j \neq 0$, notice that when $PL_j = 0$, the view $Y(0,T) = y$ gives no additional information about the Brownian motion $\{V_j(t), t \in [0,T]\}$, where $V(t) = L^{-1}W(t)$ is the vector of $N-$independent Brownian motions derived from $W(t)$. Therefore, $Y(0,T)$ and $V_j(t)$ are independent and the Brownian bridge $\{V_j(t)|y, t \in [0,T]\}$ remains a Brownian motion (in this case, its hitting time $\tilde{T}_j$ is infinite). The condition can be dropped by separating the elements of $B(t)$ into a set $\mathcal{I}$ where $PL_i \neq 0$ for $i \in \mathcal{B}$, and a set $\mathcal{J}$ where $PL_j = 0$ for $j \in \mathcal{J}$, with $\mathcal{I} \cup \mathcal{J} = [N]$, and $\mathcal{I} \cap \mathcal{J} = \emptyset$. The process $B_{\mathcal{I}}(t) = \{B_i(t), i \in \mathcal{I}\}$ is then a vector of Brownian bridges and Theorem 2 applies, whereas $B_{\mathcal{J}}(t) = \{B_j(t), j \in \mathcal{J}\}$ is a vector of Brownian motions where $B_j(t) = V_j(t)$ for $j \in \mathcal{J}$. Without loss of generality, we assume in the rest of the paper that $PL_j \neq 0$ for all $j \in [N]$.

4.2.2. Hitting Times of the Brownian Bridges. We characterize the relationship between the hitting times $\tilde{T}_i$ of the Brownian bridges $\{\bar{B}_i(t), t \in [0,T]\}$ $(i \in [N])$ and the uncertainty in the views $Y(0,T)$ captured by the covariance matrix $\Omega \in \mathbb{R}^{K\times K}$. This extends the observation made in Example 1 from the one-dimensional case where noise in the view translates to a larger hitting time $\tilde{T}$ for the Bb.

For the multidimensional case, given two covariance matrices Ω^1 and Ω^2, we say that Ω^1 is greater than or equal Ω^2 $(\Omega^1 \succeq \Omega^2)$ if their difference is positive semi-definite $(\Omega^1 - \Omega^2 \succeq 0)$.

PROPOSITION 4. *Consider the process $\{\bar{B}(t), t \in [0,T]\}$ satisfying (20). Then for each $i \in [N]$, the hitting time $\tilde{T}_i$ satisfying (18) of the Brownian bridge $\{\bar{B}_i(t), t \in [0,T]\}$, is strictly larger than the views horizon T and is increasing in the views covariance matrix Ω.*

The hitting times $\tilde{T}_i$, $i \in [N]$, given by (18) are strictly larger than the views horizon T as there is always noise in the views (Ω is non-singular). Furthermore, for two covariance matrices Ω^1 and Ω^2 such that $\Omega^1 \succeq \Omega^2$, their respective hitting times satisfy $\tilde{T}_i^1 \geq \tilde{T}_i^2$, for $i \in [N]$. This shows that views with covariance Ω^1 contain less information about the risky assets compared to those with covariance Ω^2.

4.2.3. Summary We have defined a generalization (17) of the classical one-dimensional Brownian bridge to allow the Brownian motion $W(t)$ to be multi-dimensional and correlated and the observation of its terminal value $Y(0,T) = PW(T) + \epsilon$ to be vector valued and noisy. Theorems 1 and 2 extend the properties of classical Brownian motion to this generalized setting. Each component of generalized Brownian bridge is a classical one-dimensional Brownian bridge with a hitting time (18) that is determined endogenously by the statistical properties of the Brownian motion and the observations. Proposition 4 shows that the vector of hitting times is increasing in the uncertainty of the observations. To our knowledge, the relationship between the conditional distribution of Brownian motion with noisy views and multi-dimensional Brownian bridge through the SDE (20) is novel and could be of independent interest.

4.3. Application: Black-Litterman model

We now show how the conditional process $\{B(t), t \in [0,T]\}$, as derived in Theorems 1 – 2, can be used to obtain the posterior dynamics of the log-returns (11).

Recall the price process (8), log-returns process $X(t)$ given by (9) and the vector of views $Y(0,T)$ in (10). Given $Y(0,T) = y$, the conditional log-returns

$$X^y(t) = t\mu^x + B(t)$$

where

$$B(t) = \big(W(t)|Y(0,T) = y\big) = \big(W(t)|PW(T) + \epsilon = y - TP\mu^x\big)$$

is a generalized Brownian bridge. By Theorem 2,

$$B(t) = \frac{t}{T}\beta_1(y - TP\mu^x) + L\bar{B}(t)$$

where

$$\beta_1 = \Sigma P^\top (P\Sigma P^\top + \Omega)^{-1} \in \mathbb{R}^{N\times K}$$

and $\bar{B}(t)$ is a vector of dependent Brownian bridges that solve the SDE (20) with

$$\bar{\beta}_2(t) = I_N - L^{-1}\Big(\Sigma^{-1} + \Big(1 - \frac{t}{T}\Big)P^\top\Omega^{-1}P\Big)^{-1}(L^{-1})^\top.$$

To recover the SDE (11) for $X^y(t)$, define

$$\begin{aligned}\beta_2(t) &= L\bar{\beta}_2(t)L^{-1}\\ &= I_N - \big(I_N + (1 - \frac{t}{T})\Sigma P^\top\Omega^{-1}P\big)^{-1} \in \mathbb{R}^{N\times N}.\end{aligned}$$

Observing that $L\bar{B}(t) = X^y(t) - \mathbb{E}[X^y(t)]$, it follows from (20) that

$$L\bar{B}(t) = -\int_0^t \frac{1}{T-s}\beta_2(s)(X^y(t) - \mathbb{E}[X^y(t)])ds + LV^y(t) \tag{21}$$

where $V^y(t)$ is an N-dimensional standard Brownian motion and

$$B(t) = \frac{t}{T}\beta_1(y - TP\mu^x) - \int_0^t \frac{1}{T-s}\beta_2(s)(X^y(t) - \mathbb{E}[X^y(t)])ds + W^y(t)$$

where $W^y(t) = LV^y(t)$ is an N-dimensional Brownian motion with $W^y(t) \sim N(0, t\Sigma)$. It follows that $X^y(t) = \mu^x t + B(t)$ solves (11). This derivation also shows that $\mu^x + \frac{1}{T}\beta_1(y - TP\mu^x)$ is the drift of the conditional log-return (11) and (21) is a linear combination of N dependent Brownian bridges $\bar{B}(t)$, which generalizes the observation from Example 1 to the multi-asset case.

5. Optimal Portfolio Problem

The investor is seeking to maximize the expected utility of wealth at the end of the time horizon T. At the beginning of the investment period, she has access to expert views $Y(0,T) = y$ sampled from (10). By Proposition 1, risky asset prices are given by

$$dS^y(t) = D(S^y(t))\big(\tilde{\mu}(t, X^y(t))dt + dW^y(t)\big)$$

where the drift

$$\tilde{\mu}(t,x) = \mu + \frac{1}{T}\beta_1\big(y - TP\mu^x\big) - \frac{1}{T-t}\beta_2(t)\big(x - \mathbb{E}[X^y(t)]\big)\big)$$

and factor $X^y(t)$ is the conditional log-returns process (11) with mean

$$\mathbb{E}[X^y(t)] = t\mu^x + \frac{t}{T}\beta_1\big(y - TP\mu^x\big).$$

The investor dynamically chooses the proportion $\pi(t)$ of her wealth to be invested in the risky assets. We assume that $\pi(t) \in \mathcal{A}$ where the class of admissible policies

$$\mathcal{A} = \left\{\pi : [0,T] \to \mathbb{R}^N, \pi \text{ is adapted to } \{\mathcal{F}_t^Y\}_{t\in[0,T]}, \int_0^T |\pi(t)|^2 dt \leq \infty\right\}.$$

Under the assumption that the portfolio is self-financing, the investor's wealth satisfies

$$dZ(t) = Z(t)\bigg(r_f dt + \pi(t)^\top\big(\tilde{\mu}(t, X^y(t)) - r_f \mathbf{1}_N\big)dt + \pi(t)^\top dW^y(t)\bigg) \tag{22}$$

where r_f is the risk-free rate. We assume that the investor has an isoelastic utility function

$$U(Z) = \frac{Z^{1-\gamma}}{1-\gamma}$$

with relative risk aversion γ and maximizes the expected utility of her terminal wealth at the end of the horizon T. Her value function is

$$V(t,z,x) = \max_{\pi\in\mathcal{A}} \mathbb{E}\big[U(Z(T))|X^y(t) = x, Z(t) = z\big]$$

where $X^y(t)$ and $Z(t)$ satisfy (11) and (22).

5.1. Value Function

The Hamilton-Jacobi-Bellman (HJB) partial differential equation is

$$\max_\pi \bigg\{\frac{\partial V}{\partial t} + z\big(r_f + \pi(t)^\top(\tilde{\mu}(t,x) - r_f\mathbf{1}_N\big)\nabla_z V + \big(\tilde{\mu}(t,x) - \frac{1}{2}\operatorname{diag}(\Sigma)\big)^\top \nabla_x V + \frac{1}{2}z^2\pi(t)^\top\Sigma\pi(t)\nabla_z^2 V \\ + \frac{1}{2}\operatorname{Tr}(\Sigma\nabla_x^2 V) + z\pi^\top(t)\Sigma\nabla_{x,z}^2 V\bigg\} = 0$$

with terminal condition

$$V(T,z,x) = \frac{1}{1-\gamma}z^{1-\gamma}.$$

The optimal investment policy is

$$\pi^*(t) = \underbrace{-\frac{\nabla_z V}{z\nabla_z^2 V}\Sigma^{-1}\big(\tilde{\mu}(t,x) - r_f\mathbf{1}_N\big)}_{\text{Mean-Variance Holding}}\underbrace{-\frac{\nabla_{x,z}^2 V}{z\nabla_z^2 V}}_{\text{Hedging}}.$$

The first component of the optimal portfolio is the mean-variance holding while the second hedges changes in the value function that are driven by changes in the factor $X^y(t)$. The following Proposition gives an explicit expression of the value function and the optimal investment policy in terms of a system of ordinary differential equations.

PROPOSITION 5. *Suppose $\gamma > 1$. The solution to the HJB equation is*

$$V(t,z,x) = \frac{z^{1-\gamma}}{1-\gamma}\exp(g(t,x))$$

where

$$g(t,x) = \frac{1}{2}x^\top A(t)x + x^\top b(t) + c(t).$$

The matrix $A(t)$ is symmetric negative semi-definite for $t \in [0,T)$ and satisfies a Ricatti equation

$$\begin{cases} A'(t) + \dfrac{1-\gamma}{\gamma}\eta_t\Sigma\eta_t + \dfrac{1}{\gamma}(A(t)\Sigma\eta_t + \eta_t\Sigma A(t)) + \dfrac{1}{\gamma}A(t)\Sigma A(t) = 0, \\ A(T) = 0, \end{cases} \tag{23}$$

where

$$\eta_t = -P^\top((T-t)P\Sigma P^\top + T\Omega)^{-1}P, \tag{24}$$

$b(t)$ solves a system of linear ODEs

$$\begin{cases} b'(t) + \dfrac{1}{\gamma}\big(\eta_t + A(t)\big)\Sigma b(t) + \dfrac{1-\gamma}{\gamma}\big(\eta_t + A(t)\big)(\alpha_t - r_f\mathbf{1}_N) + A(t)\big(\alpha_t - \dfrac{1}{2}\operatorname{diag}(\Sigma)\big) = 0, \\ b(T) = 0, \end{cases} \tag{25}$$

where

$$\alpha_t = \mu + \frac{1}{T}\beta_1(y - TP\mu^x) - \Sigma\eta_t\mathbb{E}[X^y(t)],$$

and $c(t)$ is the solution of

$$\begin{cases} c'(t) + (1-\gamma)r_f + \dfrac{1}{2}\operatorname{Tr}\big(A(t)\Sigma\big) + \dfrac{1-\gamma}{2\gamma}\big(\alpha_t - r_f\mathbf{1}_N\big)^\top\Sigma^{-1}\big(\alpha_t - r_f\mathbf{1}_N\big) + \big(\alpha_t - \dfrac{1}{2}\operatorname{diag}(\Sigma)\big)^\top b(t) \\ \qquad + \dfrac{1-\gamma}{\gamma}\big(\alpha_t - r_f\mathbf{1}_N\big)^\top b(t) + \dfrac{1}{2\gamma}b^\top(t)\Sigma b(t) = 0, \\ c(T) = 0. \end{cases} \tag{21b}$$

There exists a unique optimal allocation policy

$$\pi^*(t) = \frac{1}{\gamma}\Sigma^{-1}\big(\tilde{\mu}(t,x) - r_f\mathbf{1}_N\big) + \frac{1}{\gamma}\frac{\partial g}{\partial x}(t,x) \tag{26}$$

where

$$\frac{\partial g}{\partial x}(t,x) = A(t)x + b(t).$$

5.2. Optimal Policy

The optimal policy (26) consists of a mean-variance term and a hedging demand

$$\frac{1}{\gamma}\frac{\partial g}{\partial x}(t,x) = \frac{1}{\gamma}(A(t)x + b(t)). \tag{27}$$

The hedging demand hedges changes in the value function that occur from the changes in the predictor $X^y(t)$.

To get an intuition about the hedging demand, suppose there is a single asset and that $X^y(t)=x$ is the current value of its log-return. Since in the one asset case $A(t)$ is strictly negative for $t<T$, $g(t,x)$ is strictly concave in x with a global maximum at $x_0(t)=-A(t)^{-1}b(t)$. Suppose that

$$X^y(t)=x<x_0(t)=-A(t)^{-1}b(t).$$

Since $V(t,z,x)$ is negative and $\dfrac{\partial g}{\partial x}(t,x)$ is positive

$$\frac{\partial V}{\partial x}(t,z,x)=V(t,z,x)\frac{\partial g}{\partial x}(t,x)<0$$

so the value function is decreasing in x. It follows that an increase (decrease) in the log-return results in a decrease (increase) in the value function. An investor wanting to hedge this risk will execute a trade which generates a return that is negatively correlated with the change in the value function. Since $dX^y(t)$ and the investor's return are positively correlated, such a hedging strategy is long the risky asset, which is exactly what we see in (27). A similar argument explains why the hedging demand is negative when $x>x_0(t)$.

The following result shows that the Ricatti equation (23) and the system of ODEs (25), and hence the hedging demand, have explicit expressions. This is typically not the case when returns are given by affine factor models and is a consequence of the structure of this application.

Proposition 6. *Suppose $\gamma>1$. Let*

$$M(t)=(\gamma-1)(1-\frac{t}{T})P^\top\Omega^{-1}P\Big(\gamma\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\Big)^{-1}\in\mathbb{R}^{N\times N} \tag{28}$$

and η_t be given by (24). Then

$$A(t)=M(t)\eta_t,\ \ \textit{for}\ t\in[0,T]$$

is the solutions of the Ricatti equation (23) and

$$b(t)=M(t)\Sigma^{-1}(\alpha_t-r_f\mathbf{1}_N),\ \ \textit{for}\ t\in[0,T]$$

is the solution of the system of ODEs (25). The hedging demand (27) is

$$\frac{1}{\gamma}\frac{\partial g}{\partial x}(t,x)=\frac{1}{\gamma}M(t)\Sigma^{-1}\big(\tilde\mu(t,x)-r_f\mathbf{1}_N\big)=M(t)\pi^*_{MV}(t)$$

where

$$\pi^*_{MV}(t)=\frac{1}{\gamma}\Sigma^{-1}\big(\tilde\mu(t,x)-r_f\mathbf{1}_N\big)$$

is the mean-variance term in (26).

The matrix $M(t)$ is essentially the ratio of the precision of the views $(1-\frac{t}{T})P^\top\Omega^{-1}P$ to the precision of the return

$$\gamma\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P$$

(with adjustments for risk-aversion) and can be interpreted as a measure of the information content of the views. It is increasing in the precision of the views. The hedging demand is obtained by scaling the mean-variance holding $\pi^*_{\mathrm{MV}}(t)$ by $M(t)$.

Intuitively, views become more important as their precision increases, so one expects that the value function is more sensitive to changes in the predictor $X^y(t)$ as views become more accurate, requiring a larger hedge. This is exactly what we see with the hedge increasing in $M(t)$. Observe too that $M(t)$ and hence the hedging demand vanish as we approach the maturity date T. Intuitively, the cost-to-go function becomes less sensitive to the predictor $X^y(t)$ as the time remaining in the market diminishes, which reduces the hedging demand. Finally, $M(t)=0$ when $\gamma=1$ so there is no hedging demand for an investor with log-utility; this is not too surprising.

Proposition 6 allows us to write the optimal dynamic portfolio in an even simpler form that facilitates comparison to a single-period Black-Litterman investor.

THEOREM 3. *Let $\gamma>1$. The optimal holding for the dynamic Black-Litterman investor is*

$$\pi^*(t)=\frac{1}{\gamma}\big(\Sigma^{-1}_{MPBL}\big)\big(\tilde{\mu}(t,x)-r_f\mathbf{1}_N\big) \tag{29}$$

where

$$\Sigma_{MPBL}=\Big(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\Big)^{-1}+\frac{1}{\gamma}\Big\{\Sigma-\big(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\big)^{-1}\Big\}.$$

Observing that

$$\pi^*_{BL|t}=\frac{1}{\gamma}\Sigma^{-1}_{BL|t}\big(\tilde{\mu}(t,x)-r_f\mathbf{1}_N\big)$$

where

$$\Sigma_{BL|t}=\Big(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\Big)^{-1}$$

is holding of a one-step Black-Litterman investor with the same information set at time t who optimizes mean-variance utility over the remaining time $T-t$, it follows from the observation $\Sigma_{MPBL}=\Sigma_{BL|t}+\frac{1}{\gamma}\big(\Sigma-\Sigma_{BL|t}\big)\succeq\Sigma_{BL|t}$ that the dynamic investor has a smaller portfolio holding in the sense that $\|\pi^*(t)\|_2\leq\|\pi_{BL|t}\|_2$. In particular, the difference in the variances

$$\Sigma-\Sigma_{BL|t}=\Sigma-\Big(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\Big)^{-1}$$

is the reduction in uncertainty when the view $Y(0,T)$ is available. It is large when the view is informative, being largest when the view is fresh ($t=0$) and monotonically decreasing in t as more returns data is used to update the posterior. The difference in holdings between the dynamic and single-period investor increases as views become more informative ($\Sigma - \Sigma_{BL|t}$ increases).

6. Experiments

The multi-period Black-Litterman model is more flexible than the classical model because the investor can trade dynamically and express views over horizons that differ from the investor's. The objective of the first experiment is to illustrate the benefits of dynamic trading.

We compare three approaches:

- Multi-period BL (MPBL): The investor has access to the prior market model and forward-looking views, makes an allocation decision at the beginning of the investment period, and keeps adjusting her position as she observes new realizations of the assets;
- Classical BL (CBL): The investor has access to the prior market model and forward-looking views, makes an allocation decision at the beginning of the investment period, and keeps her position fixed;
- Standard Markowitz mean-variance (MV): A myopic investor who solves a single-period mean-variance problem over the horizon $[0,T]$ using the expected returns and variances implied by the prior model.

We consider a market of $N=5$ risky assets (A, B, C, D, and F), one-risk free asset with fixed return $r_f = 3\%$ per year, and $K=3$ forward-looking views about returns over a $T=1$ year horizon. The views matrix $P=(p_1,p_2,p_3)^\top \in \mathbb{R}^{K\times N}$ with $p_1=(1,-1,0,0,0)$, $p_2=(1,0,0,0,-1)$, and $p_3=(0,0,1,0,0)$. Forward-looking views are given about the difference in returns between asset A and asset B, the difference in returns between asset A and asset F, and the return of asset C. No view is expressed about the return of asset D.

6.1. Sensitivity to the Noise in the Expert Views

We evaluate the effect of the noise in the views on the performance of the portfolio. We consider a covariance matrix for the expert views $\mathbb{V}\big[Y(0,T)|X(T)\big]=\alpha T\Omega$, where α varies in the interval $[0,10]$ corresponding to absolute certainty about views ($\alpha=0$) to significant uncertainty ($\alpha=10$).

We construct the Certainty Equivalent (CE) for investors with risk aversions $\gamma=2$, and $\gamma=10$ for different levels of view uncertainty ($\alpha\in[0,10]$). The CE is defined as the fixed return rate r_c that satisfies $U\big(Z(0)e^{Tr_c}\big)=\mathbb{E}\big[U\big(Z(T)\big)\big]$ with $U(Z)=\dfrac{1}{1-\gamma}Z^{1-\gamma}$. In other words, the investor is

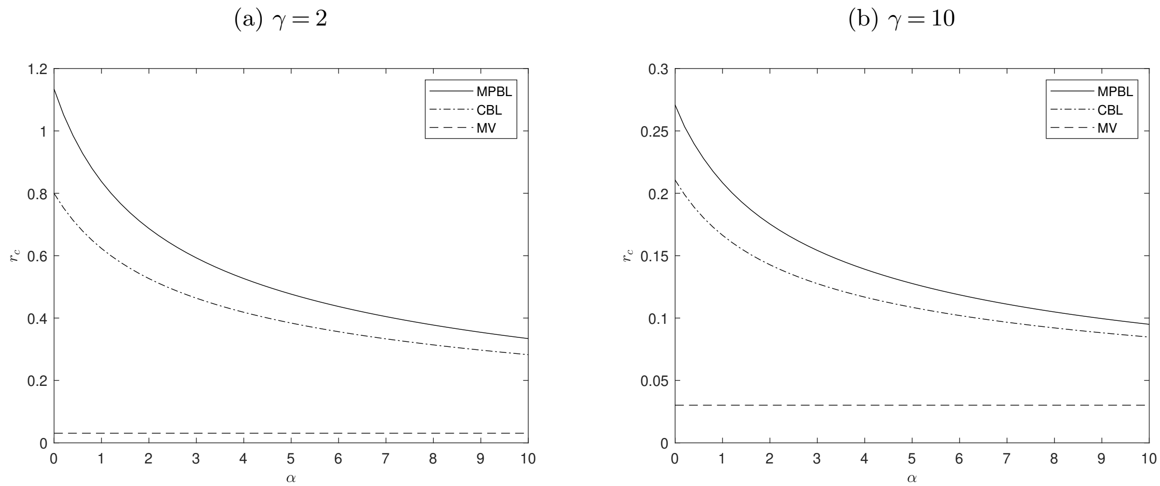

Figure 4 Portfolio performances as a function of noise in the views. The figure compares the Certainty Equivalent for three investors, adopting the MPBL, CBL, and MV strategies, respectively, for different levels of noise in the forward-looking views under low risk-aversion ($\gamma = 2$) and high risk-aversion ($\gamma = 10$).

indifferent between investing in the market and putting her money in a risk-free asset with a fixed return r_c.

Figure 4 compares the Certainty Equivalent (CE) for portfolios formed using the MPBL, CBL, and MV methods for an investor with low ($\gamma = 2$) and high ($\gamma = 10$) risk-aversion. MPBL and CBL consistently outperform MV, even when views are uncertain. Observe too that the CEs for MPBL and CBL differ significantly (figure 4a), with the difference ranging from 50% ($\alpha = 0$) to 16% ($\alpha = 10$). This is also the case for the investor with high risk-aversion ($\gamma = 10$), where the CE for MPBL exceeds CBL by at least 16% (figure 4b).

6.2. Optimal Allocation and Hedging Strategies

We consider two assets C and D. The investor holds an absolute view about the log-return of C but has no direct view of asset D. In the case of Asset D, our model uses the correlation structure of the assets to derive a prediction about its future return. Figure 5 shows the proportion of the investor's wealth invested in asset C when views are absolutely certain ($\alpha = 0$) and uncertain ($\alpha = 2$). In plots 5b and 5d we show realized log-returns and expected log-return for investors with and without views.

When $\alpha = 0$, the view about asset C is absolute and the investor knows the exact value of log-return $X_C(T)$, which explains the difference between prior expectation of the log-return $\mathbb{E}[X_C(t)]$ and the expectation $\mathbb{E}[X_C(t)|y]$ conditional on views. Consequently, the buy-and-hold under MV and CBL are very different.

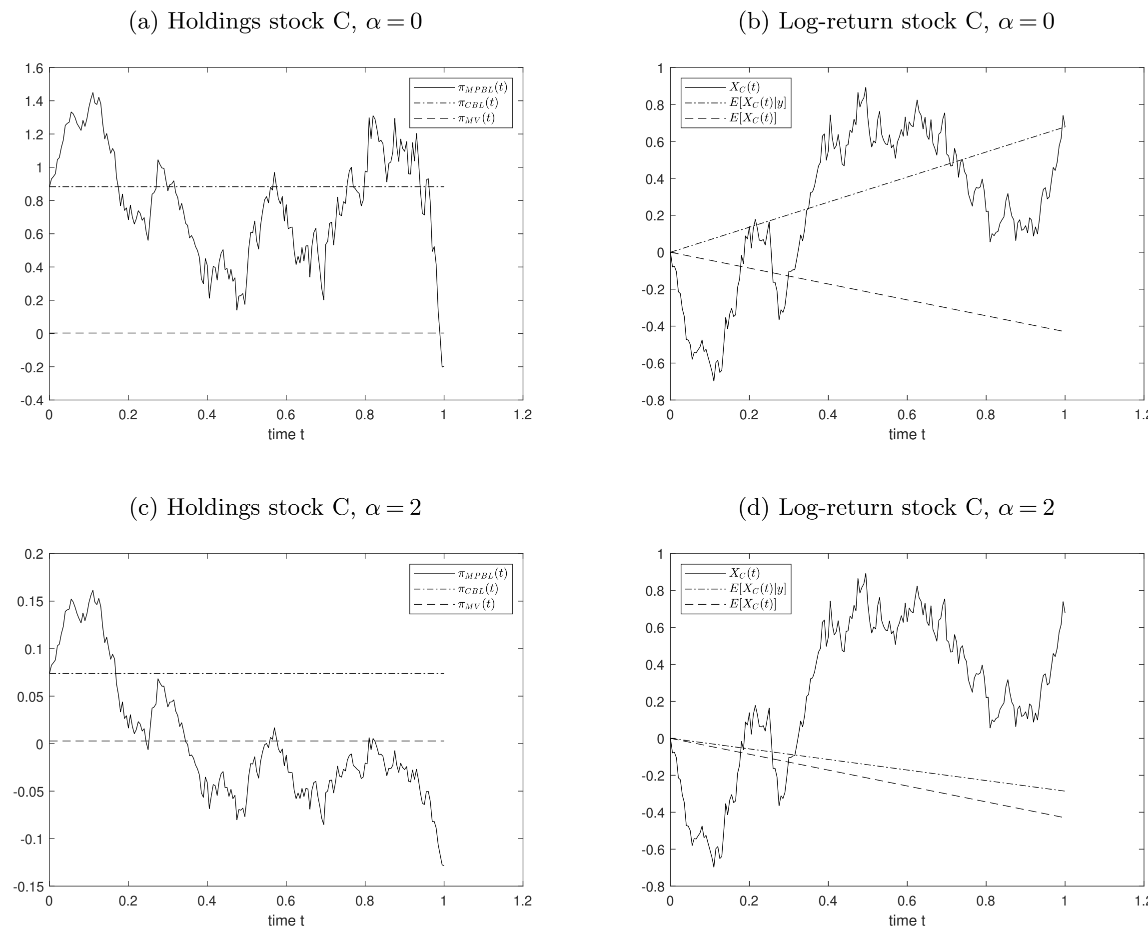

Figure 5 Optimal allocation policies under MPBL, CBL, and MV strategies. Optimal allocation strategies for asset C under the MPBL, CBL, and MV approaches. Specifically, the MV strategy is compared to the unconditional log-returns $\mathbb{E}[X_C(T)]$, the CBL strategy is compared to the conditional log-returns $\mathbb{E}[X_C^y(T)]$, and the MPBL strategy is compared to the actual realizations of the log-returns $X_C^y(t)$.

The holding holding for MPBL at $t = 0$ is similar to that of CBL since both investors have access to the identical information. However, as the investor begins observing returns, she updates her covariate $X^y(t)$ and adjusts her holding. Whenever the log-return falls below a certain threshold, the holding in asset C increases in anticipation of a future price increase.

We make similar observations when views are uncertain ($\alpha = 2$) though the difference between conditional and unconditional means is now smaller (Figure 5d). Consequently, holdings under CBL and MV are more similar (figure 5c). However, the MPBL investor continues to rebalance, resulting in a greater deviation from the CBL strategy.

Figure 6 illustrates the optimal allocation strategy for asset D under MPBL, CBL, and MV for different levels of expert uncertainty ($\alpha = 0$, and $\alpha = 2$). Despite the absence of direct views of the

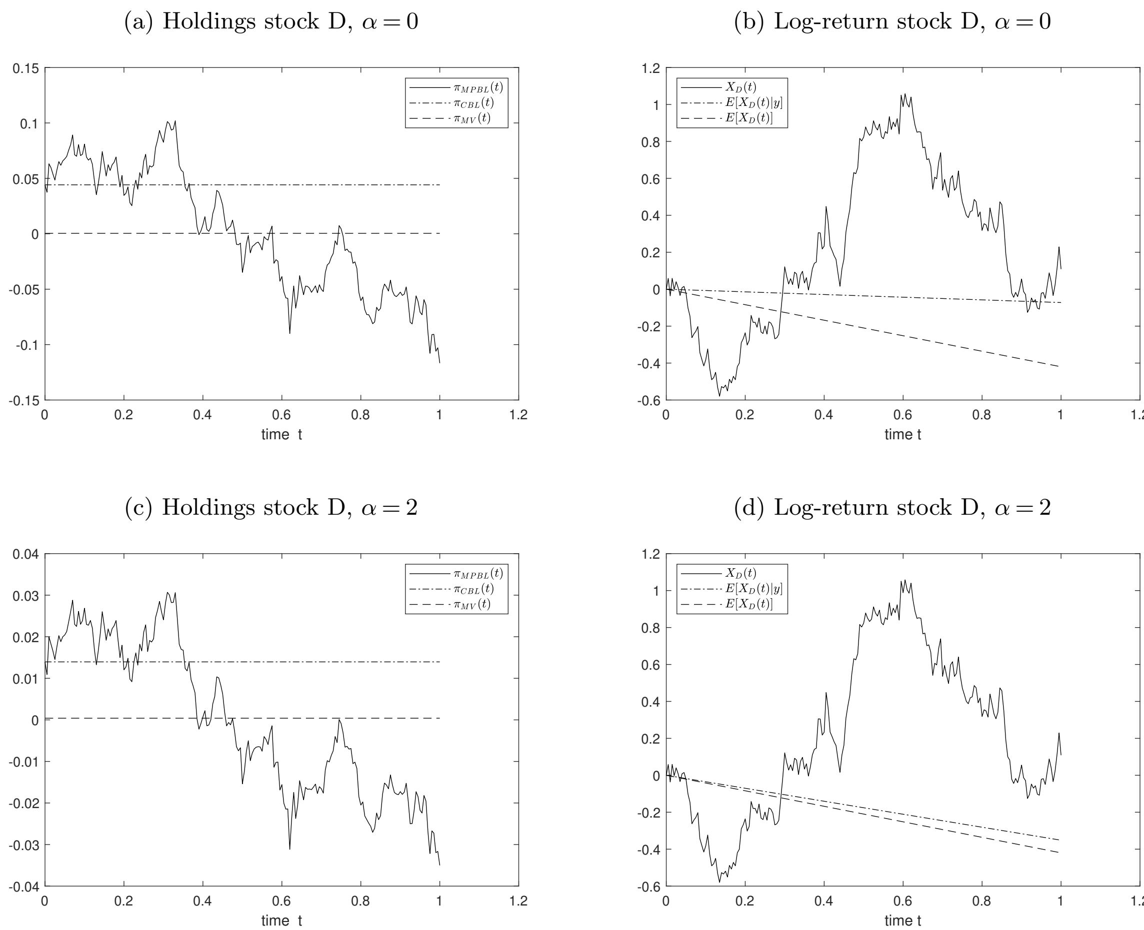

Figure 6 Optimal allocation policies under MPBL, CBL, and MV strategies. Optimal allocation strategies for asset D under the MPBL, CBL, and MV approaches. Specifically, the MV strategy is compared to the unconditional log-returns $\mathbb{E}[X_D(T)]$, the CBL strategy is compared to the conditional log-returns $\mathbb{E}[X_D^y(T)]$, and the MPBL strategy is compared to the actual realizations of the log-returns $X_D^y(t)$.

asset, predictions about its future value can still be made using the correlation structure of the assets. When $\alpha = 0$, the prediction about the return of asset D remains uncertain ($6b$) and the holdings under CBL and MV are close to each other ($6a$) compared to the case of asset C (5a). The difference becomes smaller as the noise in the views increases.

6.3. Uncertainty, Correlation, and Brownian Bridges

We now examine the relationship between view uncertainty, the correlation of the assets, and the hitting time of the associated Brownian bridges. We consider a simple setting with two correlated Brownian motions $W_1(t)$ and $W_2(t)$ where

$$dW_1(t)dW_2(t) = \rho dt.$$

At time $t = 0$, the investor gets a view about the value of $W_2(T)$ at time $T = 10$

$$Y(0,T)|W_2(T) = W_2(T) + \epsilon \sim \mathcal{N}\big(W_2(T), \omega^2\big).$$

We showed in Example 2 that both $\{W_1(t)|Y(0,T) = y, t \in [0,T]\}$ and $\{W_2(t)|Y(0,T) = y, t \in [0,T]\}$ are a restriction to $[0,T]$ of Brownian bridges with hitting times $\tilde{T}_1$ and $\tilde{T}_2$, respectively.

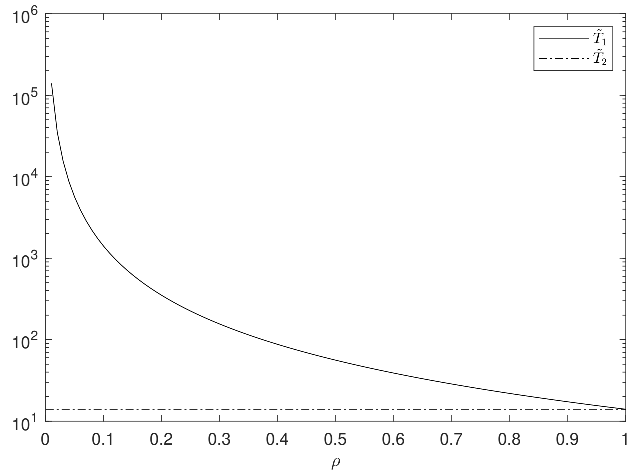

Figure 7 Impact of the correlation on the hitting time of the Bb. Comparison of the hitting times of the two Brownian bridges for a fixed level of uncertainty in the view ($\omega^2 = 4$) as a function of the correlation between the Brownian motions.

Figure 7 compares the hitting times of both Brownian bridges for a fixed level of view uncertainty ($\omega^2 = 4$). The hitting time of the Brownian bridge for asset 2 is $\tilde{T}_2 = T + \omega^2 = 14$, and that of asset 1 depends on the correlation between the two assets. When they are independent ($\rho = 0$), the hitting time $\tilde{T}_1$ is infinite because the view on asset 2 provides no information about asset 1. In this case, the Bb for asset 1 remains a Brownian motion. As the correlation between the two processes increases, the difference between the hitting times $\tilde{T}_1$ and $\tilde{T}_2$ diminishes because the view of asset 2 provides information about asset 1. When the two assets are perfectly correlated ($\rho = 1$) the hitting times are equal.

In figure 8, we fix the correlation coefficient ρ to be 0.5, and compare the hitting times of the Brownian bridges for varying degrees if uncertainty in the view ($\omega^2 \in [0, 10^2]$). When the view is certain ($\omega = 0$), $\tilde{T}_2 = T$ as we know for sure the terminal value of asset 2. $\tilde{T}_1$ is always greater than T due to the imperfect correlation between the two assets.

As we increase the view uncertainty, $\tilde{T}_1$ increases at a faster rate than $\tilde{T}_2$; The information about asset 1 disappears faster than the information about asset 2 due to the correlation between the two

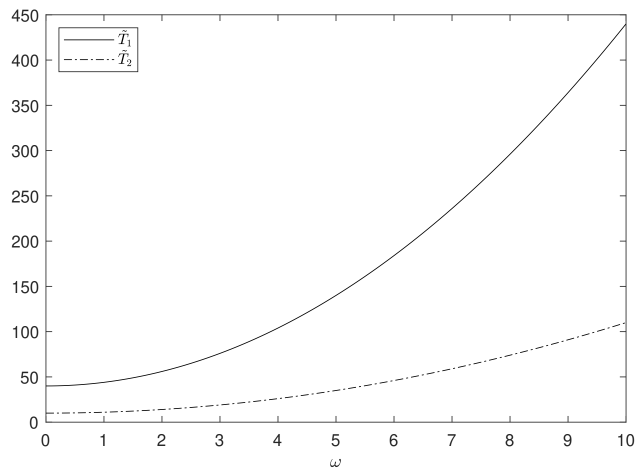

Figure 8 Impact of the noise on the hitting time of the Bb. Comparison of the hitting times of the two Brownian bridges for a fixed level of correlation between the Brownian motions ($\rho = 0.5$) as a function of the noise in the view.

assets being less than 1. This is consistent with observations from Section 6.2 where uncertainty in the view affected the prediction regarding stock D more significantly than stock C.

7. Extensions

In this part, we show that our approach can be used to solve a large range of settings within the Black-Litterman framework. We explore two distinct cases to illustrate this:

1. The classical Black Litterman model discussed in section 2,
2. A generalization of the multi-period Black-Litterman where the expert has varying views horizons.

7.1. The Classical Black Litterman model

We consider the classical setting where the investor has access to expert views at the beginning of the investment period ($t=0$) about the realization of the returns at the end of the investment horizon ($t=T$). The investor derives the conditional distribution of the log-return process $X^y(T) = (X(T)|Y(0,T)=y)$ at time to maturity T. She then decides her optimal investment strategy using a one step mean-variance optimization approach. Figure 9 shows the Bayesian network.

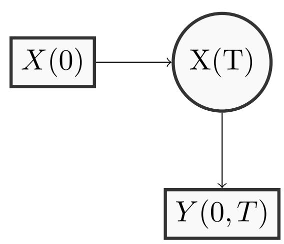

Figure 9 **Bayesian network of the classical Black-Litterman model**

The results of Theorems 1 – 2 are used to derive the posterior distribution and the optimal investment strategy for this setting.

COROLLARY 1. *Recall the price process* (8)*, log-returns process* $X(t)$ *given by* (9) *and the vector of views* $Y(0,T)$ *in* (10)*. Given* $Y(0,T)=y$*, the conditional log-returns is derived in Section 4.3*

$$X^y(t) = t\mu^x + \frac{t}{T}\beta_1(y - TP\mu^x) + L\bar{B}(t)$$

where $\bar{B}(t)$ *is a* 0 *mean* $N-$*dimensional Brownian bridge with*

$$\mathbb{V}[\bar{B}(t)] = t(I_N - \frac{t}{T}(PL)^\top(P\Sigma P^\top + \Omega)^{-1}PL).$$

The conditional log-returns process is then a multivariate normal distribution with

$$\mu_{BL} = \mathbb{E}[X^y(T)] = \left(\Sigma^{-1} + P^\top\Omega^{-1}P\right)^{-1}\left(\Sigma^{-1}T\mu^x + P^\top\Omega^{-1}y\right),$$
$$\Sigma_{BL} = \mathbb{V}[X^y(T)] = T\left(\Sigma^{-1} + P^\top\Omega^{-1}P\right)^{-1}.$$

The optimal portfolio is a mean-variance holding

$$\pi^* = \frac{1}{\gamma}(\Sigma_{BL})^{-1}(\mu_{BL} - r_f\mathbf{1}_N),$$

with γ *the risk-aversion parameter.*

The results of Corollary 1 are similar to those in Section 2 where the mean of returns μ, the assets' covariance Σ, and the views covariance Ω are scaled by T. This demonstrates that our approach can replicate the classical Black-Litterman equations (Black and Litterman (1991, 1992)).

7.2. The Multi-Period Black-Litterman Model With Different Experts Horizons

In this part, we explore a scenario where an expert gives forward-looking views with varying horizons. The view $j \in [K]$ is given at time $t=0$ regarding the realization of the log-returns vector at time $t = T_j$. Conditional on the true return being $X(T_j)$, we assume the view is normal with

$$Y_j(0,T_j)|X(T_j) = p_j^\top X(T_j) + \sqrt{T_j}\epsilon_j \sim \mathcal{N}\big(p_j^\top X(T_j), T_j\omega_j^2\big), \text{ for } j \in [K], \tag{30}$$

where $p_j \in \mathbb{R}^N$ is a linear mapping from the set of returns to the view j,

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_K \end{pmatrix} \sim \mathcal{N}\big(0,\Omega\big)$$

is the uncertainty of the views, and $\Omega \in \mathbb{R}^{K\times K}$ is a positive definite covariance matrix. $Y(0,T) = \big(Y_1(0,T_1),\dots,Y_K(0,T_K)\big)^\top$ is the vector of views and $T = (T_1,\dots,T_k)^\top \in \mathbb{R}^K$ the vector of horizons.

Without loss of generality we order the views by their respective time horizons ($T_i \le T_j$ for $i \le j$).

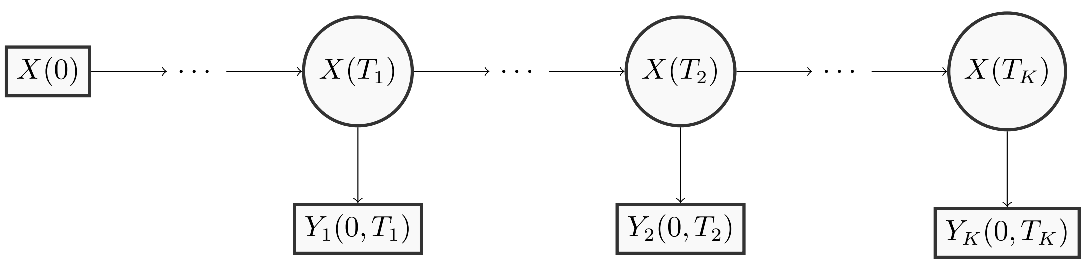

Figure 10 Bayesian network of the Multi-Period Black-Litterman model with Different Views Horizons. The figure shows a discrete time version of the problem where $t = 0,\dots,T_K$, and the noisy views $Y(0,T_j)$ of the log-return $X(T_j)$ for $j \in [K]$ are revealed at $t = 0$

Figure 10 shows a Bayesian network of the discrete time version of the problem where the views $\{Y(0,T_j)\}_{j=1\dots K}$ are revealed at the same time $t = 0$. Note that the view $Y_2(0,T_2)$ is not only an observation of the log-return vector $X(T_2)$ but does also contain information about the log-return $X(T_1)$. This is because $X(T_2)$ is by itself a noisy observation of $X(T_1)$. Extending this, all views $Y_j(0,T_j)_{j=1\dots K}$ can be considered as noisy observations of the log-return $X(T_1)$. Consequently, if we represent the vector $Y(0,T_j)_{j=1\dots K}$ as observations related to the log-return $X(T_1)$, we can use Proposition 1 to derive the posterior dynamics of the asset returns. The subsequent proposition shows how views from different time horizons can be effectively transformed into observations concerning the same time horizon.

Proposition 7. *Consider a stock price following equation (8), and expert views following (30). Define the transformation*

$$\bar{Y}(0,T) = Y(0,T) - \bar{\mu}(T_1,T)$$

where

$$\bar{\mu}_j(T_1,T) = (T_j - T_1)p_j^\top \mu^x, \ \textit{for } j \in [K].$$

Then, conditional on the true realization of $X(T_1)$, the views $\bar{Y}(0,T)$ are Gaussian with

$$\bar{Y}(0,T)|X(T_1) = PX(T_1) + \bar{\epsilon} \sim \mathcal{N}\big(PX(T_1), T\bar{\Omega}\big)$$

where $P = (p_1,\ldots,p_K)^\top \in \mathbb{R}^{K\times N}$, and $\bar{\Omega}$ is positive definite with

$$\bar{\Omega} = P\bar{\Omega}^W P' + \bar{\Omega}^V$$

where

$$\begin{cases} \bar{\Omega}_{ij}^V = \dfrac{\sqrt{T_iT_j}}{T}\Omega_{ij} \\ \bar{\Omega}_{ij}^W = \dfrac{1}{T}\min\{T_i - T_1, T_j - T_1\}\Sigma_{ij}. \end{cases}$$

Furthermore, both $\bar{Y}$ and Y are adapted to the same filtration, and

$$X(t)|Y(0,T) = X(t)|\bar{Y}(0,T), \ \textit{for } t \in [0,T_1].$$

Proposition 7 shows how a vector of views, each with distinct time horizons, can be transformed into a vector of views with a single time horizon T_1 by leveraging the graphical structure of the model. Once transformed, Proposition 1 can be applied to derive the conditional dynamics.

Corollary 2. *Suppose that the price process satisfies (8) and expert views $Y(0,T)$ satisfy (30). Assume that $PL_j \neq 0$ for $j \in [N]$. Conditional on $Y(0,T) = y$, the log-returns $X(t)$ satisfy*

$$dX^y(t) = \left(\mu^x + \frac{1}{T}\beta_1\big(y - TP\mu^x\big) - \frac{1}{T-t}\beta_2(t)\big(X^y(t) - \mathbb{E}[X^y(t)]\big)\right)dt + dW^y(t)$$

where

$$\beta_1 = \Sigma P^\top (P\Sigma P^\top + \bar{\Omega})^{-1} \in \mathbb{R}^{N\times K},$$

$$\beta_2(t) = \left[I_N - \big(I_N + (1 - \frac{t}{T})\Sigma P^\top \bar{\Omega}^{-1} P\big)^{-1}\right] \in \mathbb{R}^{N\times N},$$

and

$$\mathbb{E}[X^y(t)] = t\mu^x + \frac{t}{T}\beta_1\big(y - TP\mu^x\big)$$

is the expected log-return over the horizon $[0,t]$ given $Y(0,T) = y$. $W^y(t) \sim \mathcal{N}\big(0, t\Sigma\big)$ is a $N-$dimensional Brownian motion adapted to the filtration $\mathcal{F}_t^Y$.

Corollary 2 shows that having access to views with varying time horizons affect the conditional dynamics solely through the covariance matrix Ω. This further supports our results in Section 4, where we show that the hitting times of the Brownian bridge associated with the log-returns are increasing in the covariance of the views.

REMARK 4. We note that the results in Corollary 2 are applicable for $t \in [0, T_1]$. More generally, for $j \in [K+1]$ and $t \in [T_{j-1}, T_j]$ – where $T_0 = 0$ and $T_{K+1} = \infty$, the conditional dynamics of the log-returns can be obtained by observing that at $t = T_{j-1}$, the investor has access to the $K - j + 1$ views $\{Y_s(0, T_s)\}_{s=j\dots K}$. Additionally, by adopting a similar analysis, this result can be extended to the case where views are given at different time points[3] with no added difficulty.

8. Conclusion and Further Research

In this paper, we formulate a dynamic version of the Black-Litterman model with forward-looking expert views. We derive the dynamics of the conditional price process when asset prices are log-normal using techniques from Kalman smoothing, and show additionally that the conditional log-returns process can be written in terms of a multi-dimensional Brownian bridge with drift. In the process, we define a generalized notion of Brownian bridge, where noisy estimates of the terminal value (in place of exact values) of a multi-dimensional Brownian motion are given, provide a characterization of its distributional properties and derive the stochastic differential equation for its evolution. We also show that the components of the generalized bridge are correlated one-dimensional Brownian bridges with hitting times that are endogenously determined by the correlation structure between the elements of the original Brownian motion and the noisy observations. The conditional price process is now an affine model with the conditional log-returns playing the role of a predictor. We formulate a dynamic portfolio choice problem in terms of the conditional price process for an investor with the expert views. Although it is a factor model, we are able to derive (quite surprisingly) very explicit expressions for the optimal dynamic portfolio, which consists of a mean-variance holding and a hedging demand.

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[3] The views can be expressed as $Y_j(t_j, T_j)$, for $j \in [K]$, where the view j is given at time $t = t_j$ about the value of the log-returns at time $t = T_j$.

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Prerequisites

We recall some properties of conditioning over Gaussian vectors. Consider a Gaussian vector $(X,Y) \in \mathbb{R}^{N+K}$, where $X \in \mathbb{R}^N$ and $Y \in \mathbb{R}^K$, with respective means μ_X and μ_Y, and covariance matrices Σ_{XX} and Σ_{YY}. Denote by Σ_{XY} the cross covariance between X and Y. The conditional distribution of X given $Y=y$ remains Gaussian, with conditional mean given by

$$\mathbb{E}[X|Y=y] = \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(y-\mu_Y),$$

and conditional variance

$$\mathbb{V}[X|Y=y] = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{XY}^{\top}.$$

We furher recall Woodburry's identity matrix (see Hager (1989)). Let $\Sigma \in \mathbb{R}^{N\times N}$ and $\Omega \in \mathbb{R}^{K\times K}$ be two invertible square matrices, and let $U \in \mathbb{R}^{N\times K}$ and $V \in \mathbb{R}^{K\times N}$. We have

$$(\Sigma + U\Omega V)^{-1} = \Sigma^{-1} - \Sigma^{-1}U(\Omega^{-1} + V\Sigma^{-1}U)V\Sigma^{-1}.$$

EC.1. Proof of Proposition 1

The proof of the proposition can be split into two parts: We first derive the drift and volatility of the conditional log-returns process using Kalman smoothing equations, then use their expressions to prove that the process can be written as a solution of a Stochastic Differential Equation (SDE). Finally, the SDE of the conditional price process follows directly by Itô's Lemma.

We first start by deriving the conditional mean and covariance of the log-returns process.

EC.1.1. Conditional Mean and Covariance

The asset price $S(t)$ follows a geometric Brownian motion with drift $\mu \in \mathbb{R}^N$ and a positive definite covariance matrix $\Sigma \in \mathbb{R}^{N\times N}$

$$dS(t) = D(S(t))\big(dt\cdot\mu + dW(t)\big).$$

It follows that the log-returns vector is Gaussian with

$$X(t) = t\mu^x + W(t) \sim \mathcal{N}\big(t\mu^x, t\Sigma\big),$$

with $\mu^x = \mu - \dfrac{1}{2}\operatorname{diag}(\Sigma) \in \mathbb{R}^N$.

Conditioned on the true realization of the log-returns at time T, the expert views are Gaussian and defined as

$$Y(0,T)|X(T) = PX(T) + \sqrt{T}\epsilon \sim \mathcal{N}\big(PX(T), T\Omega\big) \in \mathbb{R}^K,$$

where $\Omega \in \mathbb{R}^{K\times K}$ is a positive definite covariance matrix. The vector $(X(t), Y(0,T))$ is then Gaussian with

$$\begin{pmatrix} X(t) \\ Y(0,T) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} t\mu^x \\ TP\mu^x \end{pmatrix}, M \right),$$

where the covariance matrix $M \in \mathbb{R}^{(N+K)\times(N+K)}$ is positive definite with block representation

$$M = \begin{pmatrix} t\Sigma & t\Sigma P^\top \\ tP\Sigma & T(P\Sigma P^\top + \Omega) \end{pmatrix}.$$

As the log-returns vector and the expert views are jointly Gaussian, the distribution of any subset of the log-return condition on any subset of the views will also be Gaussian. It follows that the conditional distribution of the log-returns given expert views is fully specified by its mean and covariance. We start by deriving the consitional mean

$$\begin{aligned}
\mathbb{E}[X(t)|Y(0,T)=y] &= \mathbb{E}[X(t)] + \mathrm{Cov}(X(t),Y(0,T))\mathbb{V}[Y(0,T)]^{-1}(y - \mathbb{E}[Y(0,T)]) \\
&= t\mu^x + \frac{t}{T}\Sigma P^\top (P\Sigma P^\top + \Omega)^{-1}(y - TP\mu^x) \\
&= t\mu^x + \frac{t}{T}\beta_1(y - TP\mu^x),
\end{aligned}$$

where

$$\beta_1 = \Sigma P^\top (P\Sigma P^\top + \Omega)^{-1}.$$

The conditional covariance is

$$\begin{aligned}
\mathbb{V}[X(t)|Y(0,T)=y] &= \mathbb{V}[X(t)] - \mathrm{Cov}(X(t),Y(0,T))\mathbb{V}[Y(0,T)]^{-1}\,\mathrm{Cov}(Y(0,T),X(t)) \\
&= t\Sigma - \frac{t^2}{T}\Sigma P^\top (P\Sigma P^\top + \Omega)^{-1} P\Sigma.
\end{aligned}$$

By using a similar argument, we can prove that the covariance between the log-return at times $t \le T$ and $\tau \le T$ conditioned on the views $Y(0,T) = y$ is

$$\begin{aligned}
\mathrm{Cov}(X(t),X(\tau)|Y(0,T)=y) &= \mathbb{E}\big[(X(t) - \mathbb{E}[X(t)|Y(0,T)=y])(X(\tau) - \mathbb{E}[X(s)|y])|Y(0,T)=yy\big] \\
&= \mathbb{E}[X(t)X(\tau)|Y(0,T)=y] - \mathbb{E}[X(t)|y]\mathbb{E}[X(\tau)|Y(0,T)=y] \\
&= \min\{t,\tau\}\Sigma - \frac{\tau t}{T}\Sigma P^\top (P\Sigma P^\top + \Omega)^{-1} P\Sigma.
\end{aligned}$$

Now we use these results to derive the distribution of the conditional dynamics $dX(t)|Y(0,T) = y$.

EC.1.2. Conditional Dynamics

Now, we derive the distribution of the dynamics of the conditional process $X(t)|(Y(0,T) = y)$. We first consider the process

$$dX(t)|(Y(0,T)=y, \{X(\tau)_{\tau\le t}\}) = \lim_{dt\to 0} \big(X(t+dt) - X(t)\big)\big|\big(Y(0,T)=y, \{X(\tau)_{\tau\le t}\}\big),$$

that is the limiting distribution of the log-returns over the interval $[t, t+dt]$, conditioned on the past observed log-returns $\{X(\tau), \tau \le t\}$, and the forward-looking views vector y. Since the log-returns vector is driven by a Brownian motion $W(t)$, it is Markovian and the information contained in the historical data $\{X(\tau), \tau \leq t\}$ is all stored in the last state $X(t)$, thus

$$\big(X(t+dt) - X(t)\big)|\big(Y(0,T) = y, \{X(\tau)_{\tau\leq t}\}\big) = \big(X(t+dt) - X(t)\big)|\big(Y(0,T) = y, X(t)\big).$$

Furthermore, as $X(t)$ and $Y(0,T)$ are Gaussian, it follows that $\big(X(t+dt)|Y(0,T) = y, X(t)\big)$ is also Gaussian. Thus, it is fully identified by its mean and covariance matrix that we derive next.

Mean of the conditional dynamics. Consider the random variable $Z(t) = (X(t), Y(0,T))$, and its realization $z = (x, y)$. The vector $(X(t+dt), Z(t))$ is jointly Gaussian, and we can express the conditional expectation of the log-returns at time $t + dt$ given $Z(t) = z$ as

$$\begin{aligned}\mathbb{E}\big[X(t+dt)|X(t) = x, Y(0,T) = y\big] &= \mathbb{E}\big[X(t+dt)|Z(t) = z\big] \\ &= \mathbb{E}\big[X(t+dt)\big] + \mathrm{Cov}(X(t+dt), Z(t))\mathbb{V}^{-1}[Z(t)](z - \mathbb{E}[Z(t)]),\end{aligned}$$

where the covariance variance of $Z(t)$ is

$$\mathbb{V}[Z(t)] = \begin{pmatrix} t\Sigma & t\Sigma P^\top \\ tP\Sigma & T(P\Sigma P^\top + \Omega) \end{pmatrix}, \tag{EC.1}$$

and the covariance between $X(t+dt)$ and $Z(t)$ is given by

$$\mathrm{Cov}(X(t+dt), Z(t)) = \begin{pmatrix} t\Sigma \\ (t+dt)\Sigma P^\top \end{pmatrix}^\top.$$

Now we derive the explicit expression of the inverse of the covariance matrix (EC.1). By Lu and Shiou (2002), the inverse of a 2×2 block matrix R is

$$R^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{pmatrix},$$

where R has a size $N + K \times 2N$, A and B have sizes $N \times N$, and C and D have sizes $K \times N$, and D is a non-singular matrix. Furthermore, by using Woodburry's identity matrix we get

$$(A - BD^{-1}C)^{-1} = A^{-1} - A^{-1}B(D + CA^{-1}B)CA^{-1},$$

Now, by letting $A = B = t\Sigma$, $C = tP\Sigma$, and $D = T(P\Sigma P^\top + \Omega)$ (and some mathematical expansion), we have

$$\mathrm{Cov}(X(t+dt), Z(t))\mathbb{V}^{-1}[Z(t)] = \begin{pmatrix} I_N - dt\cdot\Sigma P^\top((T-t)P\Sigma P^\top + T\Omega)^{-1}P \\ dt\cdot\Sigma P^\top((T-t)P\Sigma P^\top + T\Omega)^{-1} \end{pmatrix},$$

thus, the conditional mean is

$$\begin{aligned}\mathbb{E}\big[X(t+dt)|X(t) = x, Y(0,T) = y\big] = (t+dt)\mu^x + X(t) - t\mu^x - dt\cdot\Sigma P^\top((T-t)P\Sigma P^\top + T\Omega)^{-1}P(X(t) - t\mu^x) \\ + dt\cdot\Sigma P^\top((T-t)P\Sigma P^\top + T\Omega)^{-1}(y - TP\mu^x),\end{aligned}$$

which can be expressed as

$$\mathbb{E}\big[X(t+dt)-X(t)|X(t)=x,Y(0,T)=y\big]=dt\cdot\big(\mu^x-\Sigma P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}P(X(t)-t\mu^x)+\Sigma P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}(y-TP\mu^x)\big), \tag{EC.2}$$

and by using Woodburry's matrix identity, we can prove that

$$\Sigma P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}P=\frac{1}{T-t}\beta_2(t),$$

and

$$\Sigma P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}=\frac{1}{T}\big(I_N+\frac{t}{T-t}\beta_2(t)\big)\beta_1,$$

where the coefficients β_1 and $\beta_2(t)$ are given by

$$\beta_1=\Sigma P^\top(P\Sigma P^\top+\Omega)^{-1},$$

and

$$\beta_2(t)=I_N-(I_N+(1-\frac{t}{T})\Sigma P^\top\Omega^{-1}P)^{-1}.$$

Equation (EC.2) can then be expressed as

$$\mathbb{E}\big[X(t+dt)-X(t)|X(t)=x,Y(0,T)=y\big]=dt\cdot(\mu^x-\frac{1}{T-t}\beta_2(t)(X(t)-t\mu^x)+\frac{1}{T}(I_N+\frac{t}{T-t}\beta_2(t))\beta_1(y-TP\mu^x)),$$

and recall from EC.1.1 that the conditional mean of the log-returns is

$$\mathbb{E}[X(t)|Y(0,T)=y]=t\mu^x+\frac{t}{T}\beta_1(y-TP\mu^x),$$

thus, the drift of the conditional process $dX(t)|Y(0,T)=y$ is

$$\mathbb{E}\big[X(t+dt)-X(t)|X(t)=x,Y(0,T)=y\big]=dt\cdot\big(\mu^x+\frac{1}{T}\beta_1(y-TP\mu^x)-\frac{1}{T-t}\beta_2(t)(X(t)-\mathbb{E}[X(t)|y])\big), \tag{EC.3}$$

Remark EC.1. Notice that the term $\frac{1}{T-t}\beta_2(t)$ is well defined for $t=T$, where it is equal to $\frac{1}{T}\Sigma P^\top\Omega^{-1}P$. Additionaly, when $\Omega\to 0$ (the views are certain), the term $\frac{1}{T-t}\beta_2(t)$ goes to infinity and the log-returns process $X(t)|y$ converges to its mean $\mathbb{E}[X(t)|y]$ at time T.

Covariance of the conditional dynamics. A similar argument can be conducted to prove that the covariance of the conditional dynamics is

$$\begin{aligned}\mathbb{V}\big[X(t+dt)|X(t)=x,Y(0,T)=y\big] &= \mathbb{V}\big[X(t+dt)|Z=z\big]\\ &= \mathbb{V}\big[X(t+dt)\big]-\mathrm{Cov}(X(t+dt),Z)\mathbb{V}^{-1}[Z]\,\mathrm{Cov}(X(t+dt),Z)^\top\\ &= dt\cdot\Sigma-(dt)^2\cdot\Sigma P^\top((T-t)P\Sigma P^\top+T\Sigma)^{-1}P\Sigma\\ &= dt\cdot\Sigma+o(dt),\end{aligned} \tag{EC.4}$$

and for $ds\le dt$, it can prove in the same way that the covariance between the two processes $X(t+ds)$ and $X(t+dt)$ conditioned on the log-returns $X(t)=x$ and the views $Y(0,T)=y$ is

$$\mathrm{Cov}\left[X(t+dt),X(t+ds)|X(t)=x,Y(0,T)=y\right]=ds\cdot\Sigma-(ds)^2\cdot\Sigma P^\top((T-t)P\Sigma P^\top+T\Sigma)^{-1}P\Sigma.$$

(EC.3) and (EC.4) give the drift and volatility of the conditional process $dX(t)|Y(0,T)=y$, next we prove that we can write the process $X(t)|Y(0,T)=y$ as a solution to a SDE.

EC.1.3. SDE of the conditional log-returns process

We now prove that the conditional process $X(t)|Y(0,T)=y$ is a solution to an SDE with drift and volatility given by (EC.3) and (EC.4), respectively. We first define

$$\beta_3(t)=t\mu^x+\frac{t}{T}\beta_1(Y(0,T)-TP\mu^x)-\int_0^t\frac{1}{T-s}\beta_2(s)(X(s)-\mathbb{E}[X(s)|Y(0,T)]ds,\ \text{ for } t\in[0,T].$$

If the process

$$W^y(t)=X(t)-\beta_3(t)$$

is a Brownian motion in the enlarged filtration $\mathcal{F}_t^Y:=\sigma(\mathcal{F}_t\vee\sigma(Y(0,T)))$, then we can write

$$dW^y(t)=dX(t)|y-d\beta_3(t),\ \text{ for } t\in[0,T],$$

and from that we get the SDE representation of $X(t)|Y(0,T)=y$.

To prove that $W^y(t)$ is a Brownian motion in the filtration $\mathcal{F}_t^y$, we refer to Levy's Characterization of a Brownian motion (see for example Durrett (1996;2018;)).

Theorem EC.1 **(Levy's characterization of a Brownian motion)**. *Let the stochastic process $W^y=(W_1^y,\dots,W_N^y)$ be a N- dimensional local martingale with $W^y(0)=0$. Then, the following is equivalent:*

1. *W^y is a Brownian motion on the underlying filtered probability space with $W^y(t)\sim\mathcal{N}\big(0,t\Sigma\big)$.*
2. *W^y has quadratic covariations $[W_i^y(t),W_j^y(t)]=\Sigma_{ij}t$ for $1\le i,j\le N$.*

As $X(t)$ and $\beta_3(t)$ are both continuous processes, it follows that W^y is also continuous. Now, we would like to prove that is it a local martingale in the filtration $\mathcal{F}_t^Y$, that is for $s \le t$, we need to show that

$$\mathbb{E}[W^y(t) - W^y(s)|\mathcal{F}_s^Y] = 0.$$

We first have

$$\begin{aligned}\mathbb{E}[X(t) - X(s)|\mathcal{F}_s^Y] &\overset{(a)}{=} \mathbb{E}[X(t) - X(s)|X(s), Y(0,T)] \\ &\overset{(b)}{=} (t-s)\big(\mu^x + \frac{1}{T}\beta_1(y - TP\mu^x) - \frac{1}{T-s}\beta_2(s)(X(s) - \mathbb{E}[X(s)|Y(0,T)])\big),\end{aligned}$$

where (a) follows from the Markov property of $X(t)$, and (b) is derived from (EC.2). Therefore, we can write

$$\begin{aligned}\mathbb{E}[W^y(t) - W^y(s)|\mathcal{F}_s^Y] &= \mathbb{E}[W^y(t) - W^y(s)|X(s), Y(0,T)] \\ &= \mathbb{E}[X(t) - X(s)|X(s), Y(0,T)] - \mathbb{E}[\beta_3(t) - \beta_3(s)|X(s), Y(0,T)] \\ &= \underbrace{-\frac{t-s}{T-s}\beta_2(s)(X(s) - \mathbb{E}[X(s)|Y(0,T)])}_{RHS_1} \\ &\quad + \underbrace{\mathbb{E}\big[\int_s^t \frac{1}{T-u}\beta_2(u)(X(u) - \mathbb{E}[X(u)|Y(0,T)]du|X(s), Y(0,T)\big]}_{RHS_2},\end{aligned}$$

by Fubini's theorem, the second term of the right-hand side can be expressed as

$$\begin{aligned}RHS_2 &= \int_s^t \frac{1}{T-u}\beta_2(u)\mathbb{E}\big[(X(u) - \mathbb{E}[X(u)|Y(0,T)]|X(s), Y(0,T)\big]du \\ &= \int_s^t \frac{1}{T-u}\beta_2(u)\big(\mathbb{E}[X(u)|X(s), Y(0,T)] - \mathbb{E}[X(u)|Y(0,T)]\big)du \\ &\overset{(c)}{=} \int_s^t \frac{1}{T-u}\beta_2(u)(I_N + \frac{u-s}{T-s}\beta_2(s))(X(s) - \mathbb{E}[X(s)|Y(0,T)])du \\ &= \left\{\int_s^t \frac{1}{T-u}\beta_2(u)(I_N + \frac{u-s}{T-s}\beta_2(s))du\right\}(X(s) - \mathbb{E}[X(s)|Y(0,T)]),\end{aligned}$$

where (c) follows from EC.1.1. Now, if we prove that

$$\frac{1}{T-u}\beta_2(u)(I_N + \frac{u-s}{T-s}\beta_2(s)) = \frac{1}{T-s}\beta_2(s), \ \forall u \ge s,$$

we will have $RHS_1 + RHS_2 = 0$, and therefore

$$\mathbb{E}[W^y(t) - W^y(s)|\mathcal{F}_s^Y] = 0$$

and $W^y(t)$ is a local martingale in $\mathcal{F}_t^Y$.

Recall from EC.1.2 that

$$\frac{1}{T-u}\beta_2(u) = \Sigma P^\top((T-u)P\Sigma P^\top + T\Omega)^{-1}P, \ \forall u \ge 0,$$

thus, for $0 \le s \le u \le T$ we can write

$$\frac{1}{T-u}\beta_2(u) = \Sigma P^\top((T-s)P\Sigma P^\top + T\Omega - (u-s)P\Sigma P^\top)^{-1}P,$$

by applying Woodbury's matrix identity, we find that

$$\begin{aligned}\frac{1}{T-u}\beta_2(u) &= \frac{1}{T-s}\beta_2(s) + \frac{u-s}{(T-s)^2}\beta_2(s)(I_N - \frac{u-s}{T-s}\beta_2(s))^{-1}\beta_2(s) \\ &= \frac{1}{T-s}\beta_2(s)(I_N - \frac{u-s}{T-s}\beta_2(s))^{-1},\end{aligned}$$

therefore, we get

$$\frac{1}{T-u}\beta_2(u)(I_N + \frac{u-s}{T-s}\beta_2(s)) = \frac{1}{T-s}\beta_2(s),\ \forall u \ge s,$$

and thus

$$\mathbb{E}[W^y(t) - W^y(s)|\mathcal{F}_s^Y] = 0,\ \forall 0 \le s \le t \le T,$$

and W^y is a local martingale in the filtration $\mathcal{F}^Y$.

The quadratic variation of W follows directly from (EC.4), where $[W_i^y(t), W_j^y(t)] = \Sigma_{ij}t$, for $1 \le i,j \le N$. Thus, by Levy's theorem EC.1, W^y is a Brownian motion adapted to the filtration $\mathcal{F}_t^Y$. We can now write $X(t)$ in the filtration $\mathcal{F}_t^y$ as

$$X(t) = W^y(t) + \beta_3(t),$$

by conditioning on the event $\{Y(0,T) = y\}$ on both sides of the equation (notice that $W^y(t)$ is independent of $Y(0,T)$, and $\beta_3(t)$ is not random), we get

$$X(t)|y = W^y(t) + \beta_3(t),$$

where the SDE is

$$\begin{aligned}dX(t)|y &= dW^y(t) + d\beta_3(t)|y \\ &= dt\big(\mu^x + \frac{1}{T}\beta_1(y - TP\mu^x) - \frac{1}{T-t}\beta_2(t)(X(t)|y - \mathbb{E}[X(t)|y]\big) + dW^y(t).\end{aligned}$$

EC.1.4. Conditional Price Process

The stock price process can be obtained directly from the log-returns by noting that

$$S(t)|(Y(0,T) = y) = S(0)\exp\big(X(t)|(Y(0,T) = y)\big). \tag{EC.5}$$

We apply Itô's lemma to (EC.5), for $i \in [N]$, we have

$$\begin{aligned}dS_i(t)|y &= S_i(0)e^{X_i(t)|y}dX_i(t)|y + \frac{1}{2}S_i(0)e^{X_i(t)|y}(dX_i(t)|y)^2 \\ &= S_i(t)|y(dX_i(t)|y + (dX_i(t)|y)^2),\end{aligned}$$

with

$$(dX_i(t)|y)^2 = (dW_i^y(t))^2 = \sigma_i^2 dt,$$

therefore, the conditional dynamics of asset prices are

$$\begin{aligned} dS(t)|y &= D(S(t)|y)\big(dX(t)|y + \frac{1}{2}\operatorname{diag}(\Sigma)dt\big) \\ &= D(S(t)|y)\big(\tilde{\mu}(t, X(t)|y)dt + dW^y(t)\big), \end{aligned}$$

with drift

$$\begin{aligned} \tilde{\mu}(t,x) &= \mu - \frac{1}{2}\operatorname{diag}(\Sigma) + \frac{1}{T}\beta_1\big(y - TP\mu^x\big) - \frac{1}{T-t}\beta_2(t)\big(x - \mathbb{E}[X(t)|y]\big) + \frac{1}{2}\operatorname{diag}(\Sigma) \\ &= \mu + \frac{1}{T}\beta_1\big(y - TP\mu^x\big) - \frac{1}{T-t}\beta_2(t)\big(x - \mathbb{E}[X(t)|y]\big), \end{aligned}$$

which completes the proof. □

EC.2. Proofs of the Results in Section 4

EC.2.1. Proof of Proposition 3

Let $W(t) \in \mathbb{R}$ be a standard Brownian motion with variance $\mathbb{V}[W(t)] = t$ and initial value $W(0) = a$. At time $t = 0$, we observe a sample y of the random variable $Y(0,T) = W(T) + \epsilon$ where $\epsilon \sim \mathcal{N}\big(0, T\omega^2\big)$ is independent of W. We show here that the conditional process $\{B(t) = (W(t)|Y(0,T) = y), t \in [0,T]\}$ is a restriction of a Brownian bridge from a to y with hitting time $\tilde{T} = T(1+\omega^2)$ to the interval $[0,T]$.

From Definition 1, a process $B(t)$ is defined as a Brownian bridge from a to y with hitting time $\tilde{T}$ if it satisfies

1. $B(0) = a$, and $B(\tilde{T}) = y$ (with probability 1),
2. $\{B(t), t \in [0,\tilde{T}]\}$ is a Gaussian process,
3. $\mathbb{E}[B(t)] = a + \frac{t}{\tilde{T}}(y-a)$ for $t \in [0,\tilde{T}]$,
4. $\operatorname{Cov}\big(B(t), B(s)\big) = \min\{s,t\} - \dfrac{st}{\tilde{T}}$, for $s,t \in [0,\tilde{T}]$,
5. With probability 1, $t \to B(t)$ is continuous in $[0,\tilde{T}]$.

We show in this proof that $B(t) = (W(t)|Y(0,T) = y)$ satisfies the above properties for $t \in [0,T]$. We first have $B(0) = W(0) = a$, and since the vector $(W(t), Y(0,T))$ is jointly Gaussian, the conditional process $B(t) = (W(t)|Y(0,T))$ is normally distributed and satisfies 2. It is therefore fully identified by its mean and variance, and we have

$$\begin{aligned} \mathbb{E}[B(t)] &= \mathbb{E}[W(t)|Y(0,T) = y] \\ &= \mathbb{E}[W(t)] + \operatorname{Cov}(W(t), Y(0,t))\mathbb{V}^{-1}[Y(0,T)](y - \mathbb{E}[Y(0,T)]) \\ &= a + \frac{t}{T+\omega^2}(y-a) \\ &= a + \frac{t}{\tilde{T}}(y-a), \text{ for } t \in [0,T]. \end{aligned}$$

Thus, $B(t)$ satisfies 3. Now let $s,t\in\mathbb{R}$ with $s\le t$, we have

$$\begin{aligned}\mathrm{Cov}(B(t),B(s)) &= \mathrm{Cov}(W(t),W(s)|Y(0,T)=y)\\ &= \mathbb{E}\big[\big(W(t)-\mathbb{E}[W(t)|Y(0,T)=y]\big)\big(W(s)-\mathbb{E}[W(s)|Y(0,T)=y]\big)|Y(0,T)=y\big]\\ &= \mathbb{E}\big[W(t)W(s)|Y(0,T)=y\big]-\mathbb{E}\big[W(s)|Y(0,T)=y\big]\mathbb{E}\big[W(t)|Y(0,T)=y\big],\end{aligned} \tag{EC.6}$$

by the law of total expectation we get

$$\mathbb{E}\big[W(t)W(s)|Y(0,T)=y\big]=\mathbb{E}\big[W(s)\mathbb{E}[W(t)|W(s),Y(0,T)=y]|Y(0,T)=y\big],$$

where it can easily be seen that

$$\mathbb{E}[W(t)|W(s),Y(0,T)=y]=\frac{\tilde{T}-t}{\tilde{T}-s}W(s)+\frac{t-s}{\tilde{T}-s}y,$$

therefore, the conditional expectation of the product $W(t)W(s)$ is

$$\mathbb{E}\big[W(t)W(s)|Y(0,T)=y\big]=\frac{\tilde{T}-t}{\tilde{T}-s}\mathbb{E}[W^2(s)|Y(0,T)=y]+\frac{t-s}{\tilde{T}-s}y\cdot\mathbb{E}[W(s)|Y(0,T)=y]. \tag{EC.7}$$

It follows (EC.6) and (EC.7) that

$$\mathrm{Cov}(B(t),B(s))=\min\{s,t\}-\frac{st}{\tilde{T}},\ \text{ for } t\in[0,T]$$

and $B(t)$ satisfies 4. For the last point, as $t\to W(t)$ is continuous in $\mathbb{R}^+$, and the information y is given at time 0 and expires at time T, the process $t\to W(t)|Y(0,T)=y$ is also continuous for $t\in[0,T]$ and $B(t)$ satisfies 5. Therefore, $\{B(t)=(W(t)|Y(0,T)=y),t\in[0,T]\}$ is a restriction of a Brownian bridge from a to y with hitting time $\tilde{T}=T+\omega^2$ to the interval $[0,T]$. This concludes the proof. □

EC.2.2. Proof of Example 2

Let $W_1(t)$ and $W_2(t)$ be two standard Brownian motions with correlation $\rho\in(0,1]$. It can easily be shown that there exist a standard Brownian motion $W_3(t)$ such that:

1. $W_3(t)$ is independent of $W_1(t)$,
2. $W_2(t)=\rho W_1(t)+\sqrt{1-\rho^2}W_3(t)$.

Given a view y_2 sampled from $Y_2(0,T)=W_2(T)+\epsilon$ where ϵ is a centered Gaussian noise independent of $W_2(t)$ with variance ω^2, we show that the two processes $B_1(t)=\big(W_1(t)|Y_2(0,T)=y_2\big)$ and $B_2(t)=\big(W_2(t)|Y_2(0,T)=y_2\big)$ are Brownian bridges restricted to the interval $[0,T]$. The latter is directly deduced from Proposition 3, where $B_2(t)=(W_2(t)|Y_2(0,T)=y_2)$ is a restriction of a Bb from 0 to y_2 with hitting time $\tilde{T}_2=T+\omega^2$ to $[0,T]$. Here, we prove the same for $\{B_1(t),t\in[0,T]\}$.

We start by showing how the view y_2 about the the Brownian motion $W_2(T)$ can be transformed to a view y_1 about the Brownian motion $W_1(T)$. Consider the random variable $Y_1(0,T)$ such that

$$Y_1(0,T) = \frac{1}{\rho} Y_2(0,T),$$

we have

$$\begin{aligned}
Y_1(0,T) &= \frac{1}{\rho} Y_2(0,T) \\
&= \frac{1}{\rho}(W_2(T) + \epsilon) \\
&\overset{(a)}{=} W_1(T) + \frac{\sqrt{1-\rho^2}}{\rho} W_3(T) + \frac{1}{\rho}\epsilon \\
&= W_1(T) + \bar{\epsilon},
\end{aligned}$$

where (a) follows from the decomposition of the Brownian motion $W_2(t)$ into $W_1(t)$ and $W_3(t)$, and $\bar{\epsilon}$ is the noise term in the view $Y_1(0,T)$ with distribution

$$\bar{\epsilon} \sim \mathcal{N}\big(0, \frac{\omega^2 + (1-\rho^2)T}{\rho^2}\big).$$

Therefore, we have

$$\begin{aligned}
B_1(t) &= W_1(t) | (Y_2(0,T) = y_2) \\
&= W_1(t) | (Y_1(0,T) = y_1),
\end{aligned}$$

from Proposition 3, the conditional process $B_1(t) = (W_1(t) | Y_1(0,T) = y_1)$ is a Bb from 0 to y_1 with hitting time $\tilde{T}_1$ restricted to the interval $[0,T]$, with

$$\begin{aligned}
\tilde{T}_1 &= T + \mathbb{V}[\bar{\epsilon}] \\
&= \frac{\omega^2 + T}{\rho^2}.
\end{aligned}$$

When $\rho = 0$, notice that $W_1(t)$ and $Y_2(0,T)$ are independent, therefore

$$\begin{aligned}
B_1(t) &= W_1(t) | (Y_2(0,T) = y_2) \\
&= W_1(t),
\end{aligned}$$

this is also equivalent to having $\tilde{T}_1 = \infty$ (notice that a Brownian bridge with infinite hitting time is a Brownian motion). This concludes the proof. □

EC.2.3. Proof of Proposition 1

Let $W(t)$ be an $N-$dimensional Brownian motion starting at $a \in \mathbb{R}^N$, with

$$W(t) \sim \mathcal{N}\big(a, t\Sigma\big).$$

At $t = 0$, we have access to an information y sampled from the random variable

$$Y(0,T) = PW(T) + \epsilon,$$

where $P \in \mathbb{R}^{K\times N}$ is a linear mapping such that $PL_j \neq 0$ for $j \in [N]$, and ϵ is normally distributed with $\epsilon \sim \mathcal{N}\big(0, T\Omega\big)$. We define the conditional process $B(t) \in \mathbb{R}^N$ as the Brownian motion $W(t)$ conditioned on the forward-looking views $Y(0,T) = y$

$$B(t) = W(t)|(Y(0,T) = y), \text{ for } t \in [0,T].$$

Since the vector $(W(t), Y(0,T))$ is Gaussian, the conditional process $B(t)$ is also Gaussian, and with probability 1, we have

$$B(0) = W(0)|(Y(0,T) = y) = a.$$

Additionally, the conditional expectation is

$$\begin{aligned}
\mathbb{E}[B(t)] &= \mathbb{E}[W(t)|Y(0,T) = y] \\
&= \mathbb{E}[W(t)] + \mathrm{Cov}(W(t), Y(0,T))\mathbb{V}^{-1}[Y(0,T)]\big(y - \mathbb{E}[Y(0,T)]\big) \\
&= a + t\Sigma P^\top (TP\Sigma P^\top + \Omega)^{-1}(y - Pa) \\
&= a + \frac{t}{T}\Sigma P^\top (P\Sigma P^\top + \frac{1}{T}\Omega)^{-1}(y - Pa).
\end{aligned}$$

For $s, t \in \mathbb{R}$ with $s \leq t$, the covariance between $B(t)$ and $B(s)$ is

$$\begin{aligned}
\mathrm{Cov}(B(t), B(s)) &= \mathrm{Cov}(W(t), W(s)|Y(0,T) = y) \\
&= \mathbb{E}\big[\big(W(t) - \mathbb{E}[W(t)|Y(0,T) = y]\big)\big(W(s) - \mathbb{E}[W(s)|Y(0,T) = y]\big)|Y(0,T) = y\big] \\
&= \mathbb{E}\big[W(t)W(s)|Y(0,T) = y\big] - \mathbb{E}\big[W(s)|Y(0,T) = y\big]\mathbb{E}\big[W(t)|Y(0,T) = y\big],
\end{aligned}$$

by using the law of total expectation (and some further mathematical expansion), we can prove that

$$\begin{aligned}
\mathrm{Cov}(B(t), B(s)) &= s\Sigma - \frac{st}{T}\Sigma P^\top (P\Sigma P^\top + \frac{1}{T}\Omega)^{-1} P\Sigma \\
&= L\big(sI_N - \frac{st}{T}(PL)^\top (P\Sigma P^\top + \frac{1}{T}\Omega)^{-1} PL\big)L^\top \\
&= L\big(sI_N - stH\big)L^\top,
\end{aligned}$$

where

$$H = \frac{1}{T}(PL)^\top (P\Sigma P^\top + \Omega)^{-1} PL \in \mathbb{R}^{N\times N}.$$

As the covariance matrices Σ and Ω are positive definite, and the linear mapping matrix P satisfies the condition $PL_i \neq 0$, for $i \in [N]$, it is easy to see that for a non-zero vector $z \in \mathbb{R}^N$, we have

$$z^\top H z = \frac{1}{T}(PLz)^\top (P\Sigma P^\top + \Omega)^{-1} PLz \geq 0,$$

therefore, H is positive semi-definite[4]. Finally, as the Brownian motion $t \to W_i(t)$ is continuous for every $i \in [N]$, and the view y is given at time 0 and expires at time T (the view causes no jumps in the interval $(0,T)$), the process $t \to B_i(t) = (W_i(t)|Y(0,T) = y)$ is also continuous for $t \in [0,T]$ and $i \in [N]$. This concludes the proof. □

[4] It is not positive definite as there can exist a vector $z \in \mathbb{R}^N$ such that $z \neq 0$ and $PLz = 0$.

EC.2.4. Proof of Theorem 2

We define the stochastic process $\{\bar{B}(t), t \in [0,T]\}$ as

$$\bar{B}(t) = L^{-1}(B(t) - \mathbb{E}[B(t)]),$$

where $\{B(t), t \in [0,T]\}$ satisfies the properties in Proposition 5. It is easy to see that $\bar{B}(t)$ satisfies the following

1. $\bar{B}(0) = 0$ (with probability 1),
2. $\bar{B}$ is a Gaussian process,
3. $\mathbb{E}[\bar{B}(t)] = 0$, for $t \in [0,T]$,
4. $cov\big(\bar{B}_i(t), \bar{B}_j(s)\big) = \begin{cases} \min\{s,t\} - \dfrac{st}{\tilde{T}_i}, & \text{if } i = j, \\ -\dfrac{st}{H_{i,j}}, & \text{if } i \neq j, \end{cases}$
5. With probability 1, $t \to \bar{B}_i(t)$ is continuous in $[0,T]$ for $i \in [N]$.

From the properties above, it is evident that each $\bar{B}_i(t)$, where $i \in [N]$, satisfies Definition 1 and is therefore a Brownian bridge from 0 to 0 with hitting time $\tilde{T}_i$ restricted to the interval $[0,T]$. Thus, there marginals satisfy

$$d\bar{B}_i(t) = \frac{y - \bar{B}_i(t)}{\tilde{T}_i - t} dt + dW_i^y(t), \text{ for } i \in [N].$$

However, these Brownian bridges are correlated with

$$\text{Cov}(\bar{B}_i(t), \bar{B}_j(t)) = -\frac{t^2}{\tilde{T}_{ij}} \text{ for } i \neq j, \ t \in [0,T],$$

as such, the SDE representation of the multidimensional process $\bar{B}(t)$ can not be deduced from the marginals.

We now derive the SDE representation of $\bar{B}(t)$. Similarly to the proof of Proposition 1, we first derive the drift and volatility of the dynamics $d\bar{B}(t)$, then show that it admits a SDE representation. Let $\{V_j(t), j \in [N]\}$ be a vector of N-independent Brownian motions with

$$V(t) \sim \mathcal{N}\big(0, tI_N\big),$$

and consider the forward-looking views

$$\bar{Y}(0,T) = (PL)V(T) + \epsilon,$$

where $\epsilon \sim \mathcal{N}\big(0, T\Omega\big)$. It can be easily proven that the process $\bar{B}(t)$ has the same distribution as the Brownian motion $V(t)$ conditioned on the views $\bar{Y}(0,T) = 0$

$$\bar{B}(t) \overset{d}{=} V(t) | (\bar{Y}(0,T) = 0).$$

Conditioned on its realization at time t, we can write the expectation of the process at time $t+dt$ as

$$\begin{aligned}\mathbb{E}[\bar{B}(t+dt)|\bar{B}(t)] &= \mathbb{E}[V(t+dt)|V(t), Y(0,T)=0] \\ &= \left((t+dt)I_N \ t(PL)^\top\right)\begin{pmatrix} tI_N & t(PL)^\top \\ (t+dt)PL & T((PL)(PL)^\top+\Omega)\end{pmatrix}^{-1}\begin{pmatrix}V(t)\\0\end{pmatrix},\end{aligned}$$

through further mathematical expansion, we prove that

$$\mathbb{E}[\bar{B}(t+dt)|\bar{B}(t)] = \bar{B}(t) - \frac{dt}{T}(PL)^\top\left((1-\frac{t}{T})P\Sigma P^\top+\Omega\right)^{-1}PL\bar{B}(t),$$

and by using Woordburry's matrix identity we have

$$\begin{aligned}\frac{1}{T}(PL)^\top\left((1-\frac{t}{T})P\Sigma P^\top+\Omega\right)^{-1}PL &= \frac{1}{T-t}\left(I_N + L^{-1}(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P)^{-1}(L^\top)^{-1}\right. \\ &= \frac{1}{T-t}\bar{\beta}_2(t),\end{aligned}$$

with

$$\bar{\beta}_2(t) = I_N - L^{-1}\left(\Sigma^{-1}+(T-t)P^\top\Omega^{-1}P\right)^{-1}(L^{-1})^\top \in \mathbb{R}^{N\times N}.$$

Thus, we can write the change of expectation as

$$\mathbb{E}[\bar{B}(t+dt)-\bar{B}(t)|\bar{B}(t)] = -\frac{dt}{T-t}\bar{\beta}_2(t)\bar{B}(t). \tag{EC.8}$$

By following a similar argument, we prove that the covariance of $\bar{B}(t)$ is

$$\begin{aligned}\mathbb{V}[\bar{B}(t+dt)-\bar{B}(t)|\bar{B}(t)] &= \mathbb{V}[V(t+dt)|V(t), Y(0,T)=0] \\ &= \left(tI_N \ t(PL)^\top\right)\begin{pmatrix} tI_N & t(PL)^\top \\ (t+dt)PL & T((PL)(PL)^\top+\Omega)\end{pmatrix}^{-1}\begin{pmatrix}tI_N\\tPL\end{pmatrix} \\ &= dtI_N - (dt)^2\bar{\beta}_2(t) \\ &= dtI_N + o(dt).\end{aligned} \tag{EC.9}$$

Now we would like to prove that process $\bar{B}(t)$ is a solution to a Stochastic Differential Equation where the drift and volatility are given by (EC.8) and (EC.9), respectively. We can prove this in the same way as in Proposition 1 by using Levy's characterization of a Brownian motion. To see this, recall that

$$\bar{B}(t) = L^{-1}(B(t)-\mathbb{E}[B(t)]),$$

additionally, we can write

$$\begin{aligned}X^y(t) &= \mu^x t + W(t)|(Y(0,T)=y) \\ &= \mu^x t + B(t),\end{aligned}$$

thus

$$\bar{B}(t) = L^{-1}(X^y(t)-\mathbb{E}[X^y(t)]).,$$

From Proposition 1, it follows that $\bar{B}(t)$ admits a SDE representation. From (EC.8) and (EC.9), we get

$$d\bar{B}(t) = -\frac{dt}{T-t}\bar{\beta}_2(t)\bar{B}(t)dt + dV^y(t),$$

where $V^y(t) = L^{-1}W^y(t)$ is a vector of N independent Brownian motions. This concludes the proof.

□

EC.2.5. Proof of Proposition 4

Consider the conditional process $\{\bar{B}(t), t \in [0,T]\}$ satisfying (20), we showed in the proof of Theorem 2 that each element $\bar{B}_i(t)$, for $i \in [N]$, is a Brownian bridge from 0 to 0 with hitting time $\tilde{T}_i$ restricted to the interval $[0,T]$, with hitting time

$$\tilde{T}_i = T\big((PL_i)^\top (P\Sigma P^\top + \Omega)^{-1} PL_i\big)^{-1} > 0,$$

As the covariance matrices Σ and Ω are positive definite, then so is the matrix $(P\Sigma P^\top + \Omega)^{-1}$. Furthermore, as we assume that $PL_i \neq 0$, for $i \in [N]$, it follows that $\tilde{T}_i < \infty$. Now we prove that the hitting times are strictly larger than the views horizon T. For $i \in [N]$, we have

$$\tilde{T}_i = \frac{1}{H_{ii}},$$

where

$$\begin{aligned}
H &= \frac{1}{T}(PL)^\top (P\Sigma P^\top + \Omega)^{-1} PL \\
&\overset{(a)}{=} \frac{1}{T} L^\top P^\top \big(\Omega^{-1} - \Omega^{-1} P(\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1} P^\top \Omega^{-1}\big) PL \\
&= \frac{1}{T} L^\top \big(P^\top \Omega^{-1} P(I_N - (\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1} P^\top \Omega^{-1} P)\big) L \\
&= \frac{1}{T} L^\top \big(I_N - \Sigma^{-1}(\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1}\big) L \\
&= \frac{1}{T}\big(I_N - L^{-1}(\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1}(L^{-1})^\top\big),
\end{aligned}$$

where (a) follows from Woodburry's matrix identity. Thus, the hitting time $\tilde{T}_i$, for $i \in [N]$, can be written as

$$\frac{1}{\tilde{T}_i} = \frac{1}{T}\big(1 - \ell_i^{-1}(\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1}(\ell_i^{-1})^\top\big)^{-1},$$

with ℓ_i^{-1}, the i^{th} line of L^{-1}, the inverse of the Cholesky decomposition matrix. As the matrix $(\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1}$ is positive definite and $\ell_i^{-1} \neq 0$ (because L^{-1} is invertible), we have

$$\ell_i^{-1}(\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1}(\ell_i^{-1})^\top > 0,$$

therefore

$$\frac{1}{\tilde{T}_i} = \frac{1}{T}\big(1 - \ell_i^{-1}(\Sigma^{-1} + P^\top \Omega^{-1} P)^{-1}(\ell_i^{-1})^\top\big)^{-1} < \frac{1}{T}, \tag{EC.10}$$

and $\tilde{T}_i > 0$, it follows that

$$\tilde{T}_i > T, \quad \text{for } i \in [N].$$

Now we prove that the hitting times are increasing in the covariance matrix Ω. Consider two positive definite matrices Ω^1 and Ω^2 such that $\Omega^1 \succeq \Omega^2$ (the matrix $\Omega^1 - \Omega^2$ is positive semi-definite), and for $i \in [N]$, let $\tilde{T}_i^1$ and $\tilde{T}_i^2$ be their respective hitting times, we first have

$$(\Omega^2)^{-1} \succeq (\Omega^1)^{-1},$$

and since Σ is positive definite, it follows that

$$(\Sigma^{-1} + P\top(\Omega^1)^{-1}P)^{-1} \succeq (\Sigma^{-1} + P\top(\Omega^2)^{-1}P)^{-1}, \text{ for } i \in [N],$$

thus, we have

$$\ell_i^{-1}(\Sigma^{-1} + P^\top(\Omega^1)^{-1}P)^{-1}(\ell_i^{-1})^\top \geq \ell_i^{-1}(\Sigma^{-1} + P^\top(\Omega^2)^{-1}P)^{-1}(\ell_i^{-1})^\top, \text{ for } i \in [N],$$

and from (EC.10), it follows that

$$\tilde{T}_i^1 \geq \tilde{T}_i^2, \text{ for } i \in [N].$$

Additionally, if $\Omega^1 \succ \Omega^2$ (the matrix $\Omega^1 - \Omega^2$ is positive definite), we get

$$\tilde{T}_i^1 > \tilde{T}_i^2, \text{ for } i \in [N].$$

This concludes the proof. □

EC.2.6. Proof of Application 4.3

Consider the log-returns process satisfying (8)

$$X(t) = t\mu^x + W(t),$$

where $W(t) \sim \mathcal{N}(0, t\Sigma)$ a $N-$dimensional Brownian motion. Let y be the expert views vector sampled from (10). Conditional on $Y(0,T) = y$, we have

$$\begin{aligned} X^y(t) &= X(t)|(Y(0,T) = y) \\ &= t\mu^x + W(t)|(Y(0,T) = y) \\ &= t\mu^x + B(t), \end{aligned}$$

where $B(t)$ satisfies (19). From Theorem 2, we can write

$$B(t) = \mathbb{E}[B(t)] + L\bar{B}(t),$$

where $\bar{B}(t)$ is a zero mean stochastic process satisfying the SDE (20). Furthermore, we have

$$\begin{aligned} dX^y(t) &= dt\mu^x + dB(t) \\ &\overset{(a)}{=} dt\mu^x + \frac{dt}{T}\beta_1(y - TP\mu^x) + Ld\bar{B}(t) \\ &= dt\big(\mu^x + \frac{dt}{T}\beta_1(y - TP\mu^x)\big) - \frac{1}{T-t}L\bar{\beta}_2(t)\bar{B}(t) + LdV^y(t) \\ &\overset{(b)}{=} dt\big(\mu^x + \frac{dt}{T}\beta_1(y - TP\mu^x)\big) - \frac{1}{T-t}L\bar{\beta}_2(t)L^{-1}(B(t) - \mathbb{E}[B(t)] + LdV^y(t), \end{aligned}$$

where (a) follows from (20) and (b) from (19). Additionally, notice that we have

$$X^y(t) - \mathbb{E}[X^y(t)] = B(t) - \mathbb{E}[B(t)],$$

and that

$$L\bar{\beta}_2(t)L^{-1} = \beta_2(t),$$

thus, the conditional log-returns is a solution to the following SDE

$$dX^y(t) = dt\big(\mu^x + \frac{dt}{T}\beta_1(y - TP\mu^x)\big) - \frac{1}{T-t}\beta_2(t)(X^y(t) - \mathbb{E}[X^y(t)] + LdV^y(t).$$

and by definition, we have that $LV^y(t) = W^y(t)$, therefore

$$dX^y(t) = dt\big(\mu^x + \frac{dt}{T}\beta_1(y - TP\mu^x)\big) - \frac{1}{T-t}\beta_2(t)(X^y(t) - \mathbb{E}[X^y(t)]\big) + dW^y(t),$$

which complete the proof. □

EC.3. Proofs of the Results in Section 5

EC.3.1. Proof of Proposition 5

Given an observed log-returns x and a level of wealth z at time t, the investor's value function is

$$V(t,z,x) = \max_{\pi\in\mathcal{A}} \mathbb{E}\big[U(Z(T))|X^y(t) = x, Z(t) = z\big],$$

where

$$U(Z(T)) = \frac{Z(T)^{1-\gamma}}{1-\gamma}$$

is her utility at the end of the investment horizon T, and γ her risk aversion ($\gamma \geq 0$ and $\gamma \neq 1$). By Itô's Lemma, the dynamics of the value function are

$$\begin{aligned} dV(t,Z(t),X^y(t)) =& \frac{\partial V}{\partial t}dt + \nabla_z V dZ(t) + \frac{1}{2}\nabla_z^2 V(dZ(t))^2 + (\nabla_x V)^\top dX^y(t) + \frac{1}{2}(dX^y(t))^\top(\nabla_x^2 V)dX^y(t) \\ &+ (dX^y(t))^\top \nabla_{z,x}^2 V dZ(t). \end{aligned} \tag{EC.11}$$

Under the assumption that the market is self-financing, the wealth process satisfies

$$dZ(t) = Z(t)\bigg(r_f dt + \pi(t)^\top\big(\tilde{\mu}(t,X^y(t)) - r_f\mathbf{1}_N\big)dt + \pi(t)^\top dW^y(t)\bigg),$$

it follows that

$$(dZ(t))^2 = \pi^\top(t)\Sigma\pi(t)Z^2(t)dt.$$

The conditional log-returns is given by SDE (11) and can be expressed as

$$\begin{aligned}
dX^y(t) =& \left(\mu^x + \frac{1}{T}\beta_1\big(y - TP\mu^x\big) - \frac{1}{T-t}\beta_2(t)\big(X^y(t) - \mathbb{E}[X^y(t)]\big)\right)dt + dW^y(t)\\
=& (\tilde{\mu}(t, X^y(t)) - \frac{1}{2}\operatorname{diag}(\Sigma))dt + dW^y(t).
\end{aligned}$$

Therefore, by substituting the above dynamics in (EC.11), we express the HJB as

$$\begin{aligned}
\max_{\pi\in\mathcal{A}}\{&\frac{\partial V}{\partial t} + z\big(r_f + \pi(t)^\top(\tilde{\mu}(t,x) - r_f\mathbf{1}_N)\big)\nabla_z V \ + \big(\tilde{\mu}(t,x) - \frac{1}{2}\operatorname{diag}(\Sigma)\big)^\top\nabla_x V + \frac{1}{2}z^2\pi(t)^\top\Sigma\pi(t)\nabla_z^2 V\\
&+ \frac{1}{2}\operatorname{Tr}(\Sigma\nabla_x^2 V) + z\pi^\top(t)\Sigma\nabla_{x,z}^2 V\} = 0.
\end{aligned} \tag{EC.12}$$

The HJB is concave in $\pi(t)$ for each $t \in [0,T]$, therefore, the optimal investment strategy $\pi^*(t)$ can be directly derived by taking the first order derivative with respect to π

$$(\tilde{\mu}(t,x) - r_f\mathbf{1}_N)\nabla_z V + z^2\pi^*(t)\Sigma\nabla_z^2 V + z\Sigma\nabla_{x,z}^2 V = 0,$$

the optimal solution is then

$$\begin{aligned}
\pi^*(t) &= -\frac{1}{z^2\nabla_Z^2 V}\Sigma^{-1}\big(\nabla_z V(\tilde{\mu}(t, x - r_f\mathbf{1}_N) + z\Sigma\nabla_{x,z}^2 V\big)\\
&= -\frac{\nabla_z V}{z\nabla_z^2 V}\Sigma^{-1}\big(\tilde{\mu}(t,x) - r_f\mathbf{1}_N\big) - \frac{\nabla_{x,z}^2 V}{z\nabla_z^2 V}.
\end{aligned} \tag{EC.13}$$

Now, we prove that a function of the following form is a solution to the HJB (EC.12)

$$V(t,z,x) = \frac{z^{1-\gamma}}{1-\gamma}\exp(\frac{1}{2}x^\top A(t)x + x^\top b(t) + c(t)), \tag{EC.14}$$

where $A(t) \in \mathbb{R}^{N\times N}$, $b(t) \in \mathbb{R}^N$, and $c(t) \in \mathbb{R}$ are to be determined. By exploiting the structure of (EC.14), we get the following

$$\begin{cases}
\frac{\partial V}{\partial t} & = (\frac{1}{2}x^\top A'(t)x + x^\top b'(t) + c'(t))V,\\
\nabla_z V & = \frac{1-\gamma}{z}V,\\
\nabla_x V & = (A(t)x + b(t))V,\\
\nabla_z^2 V & = \frac{-\gamma(1-\gamma)}{z^2}V,\\
\nabla_x^2 V & = (A(t) + (A(t)x + b(t))(A(t)x + b(t))^\top)V,\\
\nabla_{x,z}^2 V & = \frac{1-\gamma}{z}(A(t)x + b(t))V,
\end{cases}$$

by substituting (EC.14) in (EC.12), we can write the HJB as

$$\begin{aligned}&\frac{1}{2}x^\top A'(t)x+x^\top b'(t)+c'(t)+\frac{1-\gamma}{\gamma}\big(\Sigma^{-1}(\tilde{\mu}(t,x)-r_f\mathbf{1}_N)+A(t)x+b(t)\big)^\top(\tilde{\mu}(t,x)-r_f\mathbf{1}_N)\\&-\frac{1-\gamma}{2\gamma}\big(\Sigma^{-1}(\tilde{\mu}(t,x)-r_f\mathbf{1}_N)+A(t)x+b(t)\big)^\top\Sigma\big(\Sigma^{-1}(\tilde{\mu}(t,x)-r_f\mathbf{1}_N)+A(t)x+b(t)\big)\\&+\frac{1}{2}\operatorname{Tr}\big(A(t)\Sigma+(A(t)x+b(t))(A(t)x+b(t))^\top\Sigma\big)+\frac{1-\gamma}{\gamma}\big(\Sigma^{-1}(\tilde{\mu}(t,x)-r_f\mathbf{1}_N)+A(t)x+b(t)\big)^\top\Sigma(A(t)x+b(t))\\&+(\tilde{\mu}(t,x)-\frac{1}{2}\operatorname{diag}(\Sigma))^\top(A(t)x+b(t))+(1-\gamma)r_f=0.\end{aligned} \tag{EC.15}$$

Now we derive the system of ODEs by separation of variables. We first define $\eta_t\in\mathbb{R}^{N\times N}$ and $\alpha_t\in\mathbb{R}^N$ such as

$$\begin{aligned}\tilde{\mu}(t,x)&=\mu+\frac{1}{T}\beta_1\big(y-TP\mu^x\big)-\frac{1}{T-t}\beta_2(t)\big(x-\mathbb{E}[X(t)|y]\big)\big)\\&=\alpha_t+\Sigma\eta_t x,\end{aligned}$$

it is easy to see that η_t is symmetric with

$$\begin{aligned}\eta_t&=-\frac{1}{T-t}\Sigma^{-1}\beta_2(t)\\&=-P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}P,\end{aligned} \tag{EC.16}$$

and

$$\alpha_t=\mu+\frac{1}{T}\beta_1(y-TP\mu^x)-\Sigma\eta_t\mathbb{E}[X^y(t)]. \tag{EC.17}$$

By separation of variables, the HJB (EC.15) can be split into the following system of ODEs:

$$A'(t)+\frac{1-\gamma}{\gamma}\eta_t\Sigma\eta_t+\frac{1}{\gamma}(A(t)\Sigma\eta_t+\eta_t\Sigma A(t))+\frac{1}{\gamma}A(t)\Sigma A(t)=0, \tag{EC.18}$$

$b(t)$ solves a system of linear ODEs

$$b'(t)+\frac{1}{\gamma}\big(\eta_t+A(t)\big)\Sigma b(t)+\frac{1-\gamma}{\gamma}\big(\eta_t+A(t)\big)(\alpha_t-r_f\mathbf{1}_N)+A(t)\big(\alpha_t-\frac{\operatorname{diag}(\Sigma)}{2}\big)=0,$$

and $c(t)$ is obtained by direct integration

$$\begin{aligned}&c'(t)+(1-\gamma)r_f+\frac{1}{2}\operatorname{Tr}\big(A(t)\Sigma\big)+\frac{1-\gamma}{2\gamma}\big(\alpha_t-r_f\mathbf{1}_N\big)^\top\Sigma^{-1}\big(\alpha_t-r_f\mathbf{1}_N\big)+\big(\alpha_t-\frac{\operatorname{diag}(\Sigma)}{2}\big)^\top b(t)\\&\quad+\frac{1-\gamma}{\gamma}\big(\alpha_t-r_f\mathbf{1}_N\big)^\top b(t)+\frac{1}{2\gamma}b^\top(t)\Sigma b(t)=0.\end{aligned}$$

Thus, (EC.14) solves the HJB (EC.12) where the coefficients $A(t)$, $b(t)$, and $c(t)$ are given by the above differential equations. Additionaly, as at time T

$$V(T,z,x)=U(z),\ \forall x$$

we get the terminal conditional $A(T)=0$, $b(T)=0$, and $c(T)=0$.

Now we prove that $A(t)$, the solution to (EC.18) is symmetric and negative-semi definite. Since $A(T)=0$ is symmetric, and the matrices Σ and $\eta_t\Sigma\eta_t$ are also symmetric, it follows that $A(t)$ is symmetric for $t\in[0,T]$. Further, Σ is positive definite and $\eta_t\Sigma\eta_t$ is positive semi-definite so

$$\frac{1-\gamma}{\gamma}\eta_t\Sigma\eta_t$$

is negative semi-definite when $\gamma>1$. Additionally, we have as terminal condition $A(T)=0$ which is negative semi-definite. It follows from Abou-Kandil et al. (2003), that $A(t)$ is negative semi-definite when $\gamma>1$ (and positive semi-definite when $\gamma<1$). To see this more clearly, notice that we can write (EC.18) as

$$\lim_{dt\to 0} A(T)-A(T-dt) = -\frac{1-\gamma}{\gamma}\eta_t\Sigma\eta_t \succeq 0,$$

and as $A(T)=0$, we have that $A(T-dt)$ is negative semi-definite. We can show in the same way that $A(t)$ is negative semi-definite for all $t\in[0,T]$.

Finally, by taking into account the structure of the value function $V(t,z,x)$, the optimal investment policy (EC.13) can be simplified with

$$\begin{aligned}\pi^*(t) &= -\frac{\nabla_z V}{z\nabla_z^2 V}\Sigma^{-1}\big(\tilde{\mu}(t,x)-r_f\mathbf{1}_N\big)-\frac{\nabla_{x,z}^2 V}{z\nabla_z^2 V}\\ &= \frac{1}{\gamma}\big(\Sigma^{-1}(\tilde{\mu}(t,x)-r_f\mathbf{1}_N)+A(t)x+b(t)\big),\end{aligned}$$

which completes the proof. □

EC.3.2. Proof of Proposition 6

The key to this proof hinges on deriving a solution for the Ricatti equation (23). Once an explicit solution is obtained, it enables us to solve the system of ordinary differential equations (ODEs) (25), and derive an explicit characterization for the hedging demand (27).

However, it is important to note that deriving solutions to a differential Ricatti equation is not inherently straightforward as it is non-linear. Nonetheless, if we find a solution $A_0(t)\in\mathbb{R}^{N\times N}$ that satisfies (23), then the solution set can be expressed as

$$A(t) = Q(t) + A_0(t),$$

where $Q(t)$ is a solution to a different Riccati differential equation, which in practice is easier to solve. We start by introducing the following Lemma

Lemma EC.1. *η_t satisfying (EC.16) is symmetric negative semi-definite for $t\in[0,T]$ with*

$$\frac{\partial\eta_t}{\partial t} = -\eta_t\Sigma\eta_t.$$

Given EC.1, we can directly deduce that $A_0(t) = -\eta_t$ solves (23). Specifically

$$A_0'(t) = \eta_t \Sigma \eta_t = -\frac{1-\gamma}{\gamma}\eta_t \Sigma \eta_t - \frac{1}{\gamma}(A_0(t)\Sigma \eta_t + \eta_t \Sigma A_0(t)) - \frac{1}{\gamma} A_0(t) \Sigma A_0(t).$$

It follows that the solutions to (23) satisfy

$$\begin{cases} A(t) = Q(t) - \eta_t \\ A(T) = 0, \end{cases} \tag{EC.19}$$

where $Q(t)$ satisfies the following Ricatti differential equation

$$\begin{cases} Q'(t) + \frac{1}{\gamma} Q(t) \Sigma Q(t) = 0 \\ Q(T) = \eta_T. \end{cases} \tag{EC.20}$$

From Lemma EC.1, we saw that for $\eta_t = -P^\top((T-t)P\Sigma P^\top + T\Omega)^{-1}P$, we have

$$\frac{d\eta_t}{dt} + \eta_t \Sigma \eta_t = 0.$$

Thus, it is straightforward that by defining ζ_t similarly to η_t, but replacing Σ with $\frac{1}{\gamma}\Sigma$ such that

$$\zeta_t = -P^\top(\frac{T-t}{\gamma} P\Sigma P^\top + T\Omega)^{-1}P, \tag{EC.21}$$

we get

$$\frac{d\zeta_t}{dt} + \frac{1}{\gamma}\zeta_t \Sigma \zeta_t = 0.$$

Additionally, we have

$$\zeta_T = -P^\top(T\Omega)^{-1}P = \eta_T,$$

so ζ_t is a solution to (EC.20). Therefore, the solution to the Ricatti differential equation (23) is unique with

$$A(t) = \zeta_t - \eta_t, \text{ for } t \in [0, T], \tag{EC.22}$$

where ζ_t and η_t are given by (EC.21) and (EC.16), respectively.

Now we prove that

$$A(t) = \zeta_t - \eta_t = M(t)\eta_t,$$

where $M(t)$ satisfies

$$M(t) = (\gamma - 1)(1 - \frac{t}{T})P^\top \Omega^{-1} P \Big(\gamma \Sigma^{-1} + (1 - \frac{t}{T})P^\top \Omega^{-1} P\Big)^{-1} \in \mathbb{R}^{N \times N}. \tag{EC.23}$$

From (EC.22), notice that we can write $A(t)$ as

$$A(t) = -P^\top(C(t)^{-1} - F(t)^{-1})P,$$

where

$$C(t) = \frac{T-t}{\gamma} P\Sigma P^\top + T\Omega,$$

and

$$F(t) = (T-t)P\Sigma P^\top + T\Omega.$$

Furthermore, we have

$$\begin{aligned} C(t) &= \frac{1-\gamma}{\gamma}(T-t)P\Sigma P^\top + F(t) \\ &\overset{(a)}{=} F(t)^{-1} - F(t)^{-1}P\big(\frac{\gamma}{(1-\gamma)(T-t)}\Sigma^{-1} + P^\top F(t)^{-1}P)^{-1}P^\top F(t)^{-1}. \end{aligned}$$

where (a) follows from Woodburry's matrix identity. We can then write $A(t)$ as

$$\begin{aligned} A(t) &= -P^\top(C(t)^{-1} - F(t)^{-1})P \\ &= P^\top F(t)^{-1}P\big(\frac{\gamma}{(1-\gamma)(T-t)}\Sigma^{-1} - P^\top F(t)^{-1}P\big)^{-1}P^\top F(t)^{-1}P \\ &= -\eta_t\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1} + \eta_t\big)^{-1}\eta_t. \end{aligned}$$

Now proving that

$$\begin{aligned} -\eta_t\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1} + \eta_t\big)^{-1} &= (\gamma-1)(1-\frac{t}{T})P^\top\Omega^{-1}P\Big(\gamma\Sigma^{-1} + (1-\frac{t}{T})P^\top\Omega^{-1}P\Big) \\ &= M(t), \end{aligned}$$

is straightforward and follows from Woodburry's matrix identity. The solution to the ricatti (23) can then be expressed as

$$A(t) = M(t)\eta_t,$$

where $M(t)$ satisfies (EC.23).

Now we give the solution to the system of ODEs (25) in the following Lemma

LEMMA EC.2. *The solution to the system of ODEs is*

$$b(t) = M(t)\Sigma^{-1}(\alpha_t - r_f\mathbf{1}_N),$$

where $M(t)$ and α_t satisfy (EC.23) *and* (EC.17) *, respectively.*

From Lemma EC.2, we write the hedging demand as

$$\begin{aligned}
\frac{1}{\gamma}\frac{\partial g}{\partial x}(t,x) &= \frac{1}{\gamma}(A(t)x + b(t)) \\
&= \frac{1}{\gamma}(M(t)\eta_t x + M(t)\Sigma^{-1}(\alpha_t - r_f)) \\
&= \frac{1}{\gamma}(M(t)(\eta_t x + \Sigma^{-1}(\alpha_t - r_f))) \\
&= \frac{1}{\gamma}(M(t)\Sigma^{-1}(\alpha_t + \Sigma\eta_t x - r_f)) \\
&\overset{(b)}{=} \frac{1}{\gamma}(M(t)\Sigma^{-1}(\tilde{\mu}(t,x) - r_f)) \\
&\overset{(c)}{=} M(t)\Sigma^{-1}\pi^*_{MV}(t),
\end{aligned}$$

where (b) follows from the definition of α_t and η_t, and (c) from the definition of the mean-variance term where

$$\pi^*_{MV}(t) = \frac{1}{\gamma}\Sigma^{-1}(\tilde{\mu}(t,x) - r_f).$$

This conclude the proof. □

EC.3.3. Proof of Theorem 3

From Proposition 5, we have

$$\pi^*(t) = \frac{1}{\gamma}\left(\Sigma^{-1}\big(\tilde{\mu}(t,x) - r_f\mathbf{1}_N\big) + \frac{\delta g}{\delta x}(t,x)\right),$$

and from Proposition 6, we get

$$\frac{\partial g}{\partial x} = M(t)\Sigma^{-1}(\tilde{\mu}(t,x) - r_f),$$

where

$$M(t) = (\gamma - 1)(1 - \frac{t}{T})P^\top\Omega^{-1}P\Big(\gamma\Sigma^{-1} + (1 - \frac{t}{T})P^\top\Omega^{-1}P\Big)^{-1}.$$

Therefore, we can write the optimal investment strategy as

$$\pi^*(t) = \frac{1}{\gamma}\big(I_N + M(t)\big)\Sigma^{-1}(\tilde{\mu}(t,x) - r_f).$$

Now we prove that

$$\Big(\big(I_N + M(t)\big)\Sigma^{-1}\Big)^{-1} = \Sigma_{\mathrm{MPBL}},$$

with

$$\Sigma_{\mathrm{MPBL}} = \big(\Sigma^{-1} + (1 - \frac{t}{T})P^\top\Omega^{-1}P\big)^{-1} + \frac{1}{\gamma}\big(\Sigma - \big(\Sigma^{-1} + (1 - \frac{t}{T})P^\top\Omega^{-1}P\big)^{-1}\big).$$

From the proof of Proposition 6, we have

$$
\begin{aligned}
M(t) &= -\eta_t\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1}+\eta_t\big)^{-1} \\
&= -I_N + \frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1}\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1}+\eta_t\big)^{-1}.
\end{aligned}
$$

Additionally, from (EC.23), $M(t)$ is positive semi-definite for $\gamma > 1$. It follows that $I_N + M(t)$ is positive definite, and thus, invertible. We can then write

$$
\begin{aligned}
\Big(\big(I_N+M(t)\big)\Sigma^{-1}\Big)^{-1} &= \Sigma\big(I_N+M(t)\big)^{-1} \\
&= \frac{(\gamma-1)(T-t)}{\gamma}\Sigma\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1}+\eta_t\big)\Sigma \\
&= \frac{(\gamma-1)(T-t)}{\gamma}\Sigma\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1}-P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}P\big)\Sigma.
\end{aligned}
$$

By using Woodburry's matrix identity we prove that

$$
P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}P = \frac{1}{T-t}\Sigma^{-1} - \frac{1}{T-t}\Sigma^{-1}\big(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\big)^{-1}\Sigma^{-1},
$$

thus, we get

$$
\begin{aligned}
\Big(\big(I_N+M(t)\big)\Sigma^{-1}\Big)^{-1} &= \frac{(\gamma-1)(T-t)}{\gamma}\Sigma\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1}-P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}P\big)\Sigma \\
&= \Sigma - \frac{\gamma-1}{\gamma}\Big(\Sigma-\big(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\big)^{-1}\Big) \\
&= \frac{1}{\gamma}\Sigma + (1-\frac{1}{\gamma})\big(\Sigma^{-1}+(1-\frac{t}{T})P^\top\Omega^{-1}P\big)^{-1} \\
&= \Sigma_{\text{MPBL}}.
\end{aligned}
$$

Therefore, we have

$$
\pi^*(t) = \frac{1}{\gamma}\Sigma_{\text{MPBL}}^{-1}(\tilde{\mu}(t,x)-r_f),
$$

which completes the proof. □

EC.3.4. Proof of Lemma EC.1

From Proposition 5, we have

$$
\eta_t = -P^\top((T-t)P\Sigma P^\top+T\Omega)^{-1}P,
$$

it is clear that η_t is symmetric and negative semi-definite as Σ and Ω are both symmetric positive definite. Additionally, for an invertible matrix $R(t)\in\mathbb{R}^{K\times K}$, we know that its derivative can be expressed as

$$
\frac{\partial}{\partial t}(R^{-1}(t)) = -R^{-1}(t)\frac{\partial}{\partial t}(R(t))R^{-1}(t). \tag{EC.24}
$$

Now, we define

$$
R(t) = (T-t)P\Sigma P^\top + T\Omega \in \mathbb{R}^{K\times K},
$$

since Σ and Ω are positive definite, it follows that $R(t)$ is invertible. We have

$$\begin{aligned}\frac{\partial}{\partial t}\eta_t &= -P^\top \frac{\partial}{\partial t}(R^{-1}(t))P \\ &= P^\top R^{-1}(t)\frac{\partial}{\partial t}(R(t))R^{-1}(t)P \\ &= -P^\top\big((T-t)P\Sigma P^\top + T\Omega\big)^{-1}P\Sigma P^\top\big((T-t)P\Sigma P^\top + T\Omega\big)^{-1} \\ &= -\eta_t\Sigma\eta_t,\end{aligned}$$

which completes the proof. □

EC.3.5. Proof of Lemma EC.2

Here, we want to solve the following system

$$\begin{cases} b'(t) + \frac{1}{\gamma}\big(\eta_t + A(t)\big)\Sigma b(t) + \frac{1-\gamma}{\gamma}\big(\eta_t + A(t)\big)(\alpha_t - r_f\mathbf{1}_N) + A(t)\big(\alpha_t - \frac{1}{2}\operatorname{diag}(\Sigma)\big) = 0 \\ b(T) = 0, \end{cases} \tag{EC.25}$$

where $A(t) = M(t)\eta_t$.

Let

$$q(t) = M(t)\Sigma^{-1}(\alpha_t - r_f\mathbf{1}_N),$$

we now show that $q(t)$ is the unique solution of (EC.25).

As $M(T) = 0$, we have that $q(T) = 0$, thus, it satisfies the terminal condition. Furthermore, its derivative is

$$q'(t) = M'(t)\Sigma^{-1}(\alpha_t - r_f\mathbf{1}_N) + M(t)\Sigma^{-1}\frac{\partial\alpha_t}{\partial t}.$$

From (EC.17) and Lemma EC.1, we have

$$\frac{\partial\alpha_t}{\partial t} = -\eta_t(\alpha_t - \frac{1}{2}\operatorname{diag}(\Sigma)),$$

we can then write

$$q'(t) = M'(t)\Sigma^{-1}(\alpha_t - r_f\mathbf{1}_N) - A(t)(\alpha_t - \frac{1}{2}\operatorname{diag}(\Sigma)).$$

Therefore, if we prove that

$$M'(t)\Sigma^{-1}(\alpha_t - r_f\mathbf{1}_N) = -\frac{1}{\gamma}\big(\eta_t + A(t)\big)\Sigma q(t) - \frac{1-\gamma}{\gamma}\big(\eta_t + A(t)\big)(\alpha_t - r_f\mathbf{1}_N), \tag{EC.26}$$

then $q(t)$ would satisfy (EC.25).

We first start by deriving the explicit expression of $M'(t)$. From the proof of Proposition 6, we have

$$M(t) = -\eta_t\big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1} + \eta_t\big)^{-1}.$$

We define the invertible matrix

$$s(t) = \big(\frac{\gamma}{(\gamma-1)(T-t)}\Sigma^{-1} + \eta_t\big)^{-1} \in \mathbb{R}^{N\times N},$$

we then have

$$\begin{aligned} M'(t) &= -(\eta_t s(t))' \\ &\stackrel{(a)}{=} \eta_t \Sigma \eta_t - \eta_t s'(t) \\ &\stackrel{(b)}{=} \eta_t \Sigma \eta_t + \eta_t s(t)(s^{-1}(t))' s(t) \\ &= \eta_t \Sigma \eta_t - \frac{1-\gamma}{\gamma}\eta_t\big(I_N - s(t)\eta_t\big)\Sigma\big(I_N - \eta_t s(t)\big), \end{aligned} \tag{EC.27}$$

where (a) follow from Lemma EC.1 and (b) from the expression of the derivative of the inverse. Now we introduce the following Lemma

Lemma EC.3. *The matrices $\eta_t\Sigma \in \mathbb{R}^{N\times N}$ and $\eta_t s(t) \in \mathbb{R}^{N\times N}$ commute for all $t \in [0,T)$, i.e.,*

$$(\eta_t\Sigma)(\eta_t s(t)) = (\eta_t s(t))(\eta_t \Sigma), \text{ for } t \in [0,T).$$

Thus, (EC.27) becomes

$$\begin{aligned} M'(t) &= \eta_t \Sigma \eta_t - \frac{1-\gamma}{\gamma}\eta_t\big(I_N - s(t)\eta_t\big)\Sigma\big(I_N - \eta_t s(t)\big) \\ &\stackrel{(a)}{=} \frac{1}{\gamma}(\eta_t - \eta_t s(t)\eta_t)\Sigma\eta_t s(t) - \frac{1-\gamma}{\gamma}(\eta_t - \eta_t s(t)\eta_t)\Sigma \\ &\stackrel{(b)}{=} -\frac{1}{\gamma}(\eta_t + M(t)\eta_t)\Sigma M(t) - \frac{1-\gamma}{\gamma}(\eta_t + M(t)\eta_t)\Sigma \\ &\stackrel{(c)}{=} -\frac{1}{\gamma}(\eta_t + A(t))\Sigma M(t) - \frac{1-\gamma}{\gamma}(\eta_t + A(t))\Sigma, \end{aligned}$$

where (a) follows from Lemma EC.3, (b) from the definition of $s(t)$, and (c) from the expression of $A(t)$. It follows that

$$\begin{aligned} M'(t)\Sigma^{-1}(\alpha_t - r_f\mathbf{1}_N) &= -\frac{1}{\gamma}(\eta_t + A(t))\Sigma M(t)\Sigma^{-1}(\alpha_t - r_f\mathbf{1}_N) - \frac{1-\gamma}{\gamma}(\eta_t + A(t))(\alpha_t - r_f\mathbf{1}_N) \\ &= -\frac{1}{\gamma}(\eta_t + A(t))\Sigma q(t) - \frac{1-\gamma}{\gamma}(\eta_t + A(t))(\alpha_t - r_f\mathbf{1}_N). \end{aligned}$$

thus, we have proved that (EC.26) is true, and $q(t)$ satisfies (EC.25). Furthermore, we can write the ODE in (EC.25) as

$$b'(t) = f(t, b(t)),$$

with terminal value $b(T) = 0$, where

$$f(t,b) = -\frac{1}{\gamma}\big(\eta_t + A(t)\big)\Sigma b - \frac{1-\gamma}{\gamma}\big(\eta_t + A(t)\big)(\alpha_t - r_f\mathbf{1}_N) - A(t)\big(\alpha_t - \frac{1}{2}\operatorname{diag}(\Sigma)\big).$$

It is clear that f is continuous in b. Furthermore, since η_t, α_t, and $A(t)$ are continuous in t, it follows that f is also continuous in t. Thus, from Cauchy–Lipschitz theorem (see for example Arnold (2006)) the solution to (EC.25) is unique. Therefore

$$b(t) = M(t)\Sigma^{-1}(\alpha_t - r_f \mathbf{1}_N)$$

is the unique solution to the system, which completes the proof. □

EC.3.6. Proof of Lemma EC.3

We now show that the matrices $\eta_t\Sigma \in \mathbb{R}^{N\times N}$ and $\eta_t s(t) \in \mathbb{R}^{N\times N}$ commute for all $t\in[0,T)$. We have

$$s(t) = \left(\lambda_t \Sigma^{-1} + \eta_t\right)^{-1} \in \mathbb{R}^{N\times N},$$

with

$$\lambda_t = \frac{\gamma}{(\gamma-1)(T-t)}.$$

We can then write

$$\begin{aligned}
\eta_t s(t)\eta_t\Sigma &= \eta_t s(t)\left(\eta_t + \lambda_t\Sigma^{-1} - \lambda_t\Sigma^{-1}\right)\Sigma \\
&= \eta_t (I_N - \lambda_t s(t)\Sigma^{-1})\Sigma \\
&= \eta_t\Sigma - \lambda_t\eta_t s(t) \\
&= \eta_t\Sigma(I_N - \lambda_t\Sigma^{-1}s(t)) \\
&= \eta_t\Sigma\left(\eta_t + \lambda_t\Sigma^{-1} - \lambda_t\Sigma^{-1}\right)s(t) \\
&= \eta_t\Sigma\eta_t s(t),
\end{aligned}$$

which complete the proof. □

EC.4. Proofs of the Results in Section 7

EC.4.1. Proof of Proposition 7

As the views are ordered according to their horizon, we have that $T_1 = \min\{T_j, j\in[K]\}$. From (9), we can write for $j\in[K]$

$$\begin{aligned}
X(T_j) &= T_j\mu^x + W(T_j) \\
&= T_1\mu^x + W(T_1) + (T_j - T_1)\mu^x + W(T_j) - W(T_1) \\
&= X(T_1) + (T_j - T_1)\mu^x + W(T_j) - W(T_1),
\end{aligned}$$

thus, the log-returns at time T_j can be expressed in terms of the log-returns at time T_1 plus some extra terms. It follows that for $j\in[K]$, the view $Y_j(0,T_j)$ can be transformed to a view about the log-return realization at time T_1

$$\begin{aligned}
Y_j(0,T_j) &= p_j^\top X(T_j) + \sqrt{T_j}\epsilon_j \\
&= p_j^\top (X(T_1) + (T_j - T_1)\mu^x + W(T_j) - W(T_1)) + \epsilon_j \\
&= p_j^\top X(T_1) + \bar{\mu}_j(T1,T) + \bar{\epsilon}_j,
\end{aligned}$$

with

$$\bar{\mu}_j(T_1,T) = (T_j - T_1) p_j^\top \mu^x,$$

is the additional bias in the view and is equal to the difference between the mean of the view at time T_j and at time T_1 unconditioned on the log-returns. And

$$\bar{\epsilon}_j = p_j^\top (W(T_j) - W(T_1)) + \sqrt{T_j}\epsilon_j.$$

is the uncertainty of the updated view. Notice that the noise term can be split into two part: The first one captures the uncertainty related to the structure of the log-returns, and the second one captures the uncertainty of the experts.

We can now write the vector of forward-looking views $Y(0,T) \in \mathbb{R}^K$ as

$$Y(0,T) = X(T_1) + \mu(T_1,T) + \bar{\epsilon},$$

where $\bar{\epsilon}$ is Gaussian with

$$\bar{\epsilon} \sim \mathcal{N}\big(0, T\bar{\Omega}\big),$$

and the covariance matrix $\bar{\Omega}$ takes the form

$$\bar{\Omega} = \bar{\Omega}^V + P\bar{\Omega}^W P^\top,$$

with

$$\bar{\Omega}^V_{ij} = \frac{\sqrt{T_i T_j}}{T}\Omega_{ij}, \text{ for } i,j \in [K],$$

is the covariance related to pushing the view horizon from time T_j to T_1, and

$$\bar{\Omega}^W_{ij} = \frac{1}{T}\min\{T_i - T_1, T_j - T_1\}\Sigma_{ij}, \text{ for } i,j \in [K].$$

is the covariance related to using the structure of the log-returns to predict the realization of $X(T_1)$ from that of $X(T_j)$. It follows that for

$$\bar{Y}(0,T) = Y(0,T) - \bar{\mu}(T_1,T),$$

conditioned on the true log-returns being $X(T_1)$, the view $\bar{Y}(0,T)$ is Gaussian with

$$\bar{Y}(0,T)|X(T_1) = PX(T_1) + \bar{\epsilon} \sim \mathcal{N}\big(PX(T_1), T\bar{\Omega}\big).$$

Furthermore, both $\bar{Y}$ and Y are adapted to the same filtration and contain the same information. Thus

$$X(t)|Y(0,T) = X(t)|\bar{Y}(0,T), \text{ for } t \in [0,T_1],$$

which complete the proof. $\square$

EC.4.2. Proof of Corollary 2

Since we showed that the conditional log-returns process can be written as

$$X^y(t) = X(t)|(\bar{Y}(0,T) = \bar{y}),$$

where

$$\bar{Y}(0,T)|X(T_1) \sim \mathcal{N}\big(PX(T_1), T\bar{\Omega}\big).$$

It follows from Corollary 2 that

$$X^y(t) = t\mu^x + \frac{t}{T}\beta_1(y - TP\mu^x) + L\bar{B}(t),$$

where

$$\beta_1(t) = \Sigma P^\top (P\Sigma P^\top + \bar{\Omega})^{-1},$$

and $\bar{B}(t)$ is a vector of $N-$Brownian bridges from 0 to 0 with correlation

$$\mathrm{Cov}(\bar{B}_i(t), \bar{B}_j(s)) = \begin{cases} \min\{s,t\} - \dfrac{st}{H_{i,i}}, & \text{for } i = j, \\ -\dfrac{st}{H_{i,j}}, & \text{for } i \neq j, \end{cases}$$

where

$$H = \frac{1}{T}(PL)^\top (P\Sigma P^\top + \bar{\Omega})^{-1} PL.$$

Finally, by using Itô's Lemma, we can directly prove that

$$\begin{aligned} dS^y(t) &= S(t)|(\bar{Y}(0,T) = \bar{y}) \\ &= d(S(0)e^{X^y(t)}) \\ &= D(S^y(t))(\tilde{\mu}(t, X^y(t))dt + dW^y(t)), \end{aligned}$$

where the drift is

$$\tilde{\mu}(t,X) = \mu + \frac{1}{T}\beta_1\big(y - TP\mu^x\big) - \frac{1}{T-t}\beta_2(t)\big(X(t)|y - \mathbb{E}[X^y(t)]\big)\big),$$

with

$$\beta_2(t) = \big[I_N - \big(I_N + (1 - \frac{t}{T})\Sigma P^\top \bar{\Omega}^{-1} P\big)^{-1}\big] \in \mathbb{R}^{N\times N}.$$

This concludes the proof. $\square$