CONTINUITY OF THE CONTINUED FRACTION MAPPING REVISITED

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Abstract. The continued fraction mapping maps a number in the interval $[0,1)$ to the sequence of its partial quotients. When restricted to the set of irrationals, which is a subspace of the Euclidean space $\mathbb{R}$, the continued fraction mapping is a homeomorphism onto the product space $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}$ is a discrete space. In this short note, we examine the continuity of the continued fraction mapping, addressing both irrational and rational points of the unit interval.

Contents

1. Introduction

Given $x \in [0,1)$, we define

$$d_1(x) \coloneqq \begin{cases} \lfloor 1/x \rfloor, & \text{if } x \neq 0; \\ \infty, & \text{if } x = 0, \end{cases} \quad \text{and} \quad T(x) \coloneqq \begin{cases} 1/x - \lfloor 1/x \rfloor, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0, \end{cases} \tag{1.1}$$

where $\lfloor \xi \rfloor$ denotes the largest integer not exceeding $\xi \in \mathbb{R}$, and define

$$d_n(x) \coloneqq d_1(T^{n-1}(x)) \tag{1.2}$$

for each $n \in \mathbb{N}$. The $d_n(x)$ are called the *partial quotients* of the continued fraction expansions. It is well-known that the algorithms (1.1) and (1.2) yield a unique

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continued fraction expansion

$$x = [d_1(x), d_2(x), \dots] \coloneqq \cfrac{1}{d_1(x) + \cfrac{1}{d_2(x) + \ddots}},$$

with the conventions $1/\infty = 0$ and $\infty + 0 = \infty$. For irrational $x \in [0,1)$, we have $(d_k(x))_{k\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, while for rational $x \in (0,1)$, we have $n \coloneqq \inf\{k \in \mathbb{N} : d_k(x) = \infty\} < \infty$ so that $x = [\underbrace{d_1(x), \dots, d_{n-1}(x)}_{n-1 \text{ terms}}, \infty, \infty, \dots]$ with $(d_k(x))_{k=1}^{n-1} \in \mathbb{N}^{n-1}$ (see [5, p. 4]).

The mapping from $[0,1)$ to $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ defined by

$$x \mapsto (d_1(x), d_2(x), d_3(x), \dots)$$

for each $x \in [0,1)$ is referred to as the *continued fraction mapping*. A classical result in the study of continued fractions is that the set of irrationals in $[0,1)$, equipped with the subspace topology inherited from the Euclidean space $\mathbb{R}$, and the product space $\mathbb{N}^{\mathbb{N}}$, where $\mathbb{N}$ is a discrete space, are homeomorphic via the continued fraction mapping.

In this paper, building upon our previous work [1, Section 3.2], where we studied the continuity of the Pierce expansion mapping, we address the continuity problem of the continued fraction mapping on $[0,1)$, encompassing rational numbers, and adjust the codomain and its topology accordingly. To achieve this, in Section 2, we provide some basic results on continued fractions. In Section 3, as the main results of this paper, we establish the continuity result for an extension of the continued fraction mapping to $[0,1]$ (see Theorem 3.12) and the continuity result for the original continued fraction mapping on $[0,1)$ (see Corollary 3.13).

Throughout the paper, we denote by $\mathbb{N}$ the set of positive integers, $\mathbb{N}_0$ the set of non-negative integers, and $\mathbb{N}_\infty \coloneqq \mathbb{N} \cup \{\infty\}$ the set of extended positive integers. Let $\mathbb{I} \coloneqq [0,1) \setminus \mathbb{Q}$, i.e., the set $\mathbb{I}$ is the set of irrationals in $[0,1)$. The Fibonacci sequence is denoted by $(F_n)_{n\in\mathbb{N}_0}$, with the recursive definition $F_{n+2} \coloneqq F_{n+1} + F_n$ for each $n \in \mathbb{N}_0$, where $F_0 \coloneqq 0$ and $F_1 \coloneqq 1$. Specifically, Binet's formula tells us that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right] \quad \text{for each } n \in \mathbb{N}_0. \tag{1.3}$$

2. Preliminaries of continued fractions

In this section, we list some basic properties of continued fractions. Most of the results in this section can be found in the classical textbook on number theory by Hardy and Wright [4], as well as in books more focused on the theory of continued fractions by Iosifescu and Kraaikamp [5] and Khinchin [**?**].

We slightly extend the domain of the continued fraction mapping to the closed unit interval $[0,1]$. For each $x \in [0,1]$, define

$$\overline{d}_1(x) \coloneqq \begin{cases} \lfloor 1/x \rfloor, & \text{if } x \neq 0; \\ \infty, & \text{if } x = 0, \end{cases} \quad \text{and} \quad \overline{T}(x) \coloneqq \begin{cases} 1/x - \lfloor 1/x \rfloor, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Furthermore, for each $x \in [0,1]$ and $n \in \mathbb{N}$, let

$$\overline{d}_n(x) \coloneqq \overline{d}_1(\overline{T}^{n-1}(x)).$$

By definitions, $\overline{T} = T$ and $\overline{d}_n = d_n$ for every $n \in \mathbb{N}$, on the original domain $[0,1)$. Define $f\colon [0,1] \to \mathbb{N}_\infty^{\mathbb{N}}$ by

$$f(x) \coloneqq (\overline{d}_k(x))_{k\in\mathbb{N}} = (\overline{d}_1(x), \overline{d}_2(x), \overline{d}_3(x), \dots)$$

for each $x \in [0,1]$. Clearly, f is well-defined by definition, and f is an extension of the continued fraction mapping in the sense that $f(x) = (d_k(x))_{k\in\mathbb{N}}$ for any $x \in [0,1)$. We call f the *extended continued fraction mapping*.

We shall consider a set closely related to the sequence of partial quotients of continued fractions. Put $\Sigma \coloneqq \Sigma_0 \cup \bigcup_{n\in\mathbb{N}} \Sigma_n \cup \Sigma_\infty$, where

$$\begin{aligned} \Sigma_0 &\coloneqq \{\infty\}^{\mathbb{N}}, \\ \Sigma_n &\coloneqq \mathbb{N}^{\{1,\dots,n\}} \times \{\infty\}^{\mathbb{N}\setminus\{1,\dots,n\}}, \quad \forall n \in \mathbb{N}, \\ \Sigma_\infty &\coloneqq \mathbb{N}^{\mathbb{N}}. \end{aligned}$$

For each $\sigma \coloneqq (\sigma_k)_{k\in\mathbb{N}} \in \mathbb{N}_\infty^{\mathbb{N}}$, put

$$\widetilde{\varphi}_k(\sigma) \coloneqq [\underbrace{\sigma_1, \sigma_2, \dots, \sigma_k}_{k \text{ terms}}, \infty, \infty, \dots]$$

for each $k \in \mathbb{N}_0$. Now, define $\widetilde{\varphi}\colon \mathbb{N}_\infty^{\mathbb{N}} \to [0,1]$ by

$$\widetilde{\varphi}(\sigma) \coloneqq \lim_{k\to\infty} \widetilde{\varphi}_k(\sigma)$$

for each $\sigma \in \mathbb{N}_\infty^{\mathbb{N}}$.

Lemma 2.1. *For any $\sigma \in \mathbb{N}_\infty^{\mathbb{N}} \setminus \Sigma_\infty$, there exists an $n \in \mathbb{N}$ such that $\widetilde{\varphi}(\sigma) = \widetilde{\varphi}_n(\sigma)$.*

Proof. Let $\sigma \coloneqq (\sigma_k)_{k\in\mathbb{N}} \in \mathbb{N}_\infty^{\mathbb{N}} \setminus \Sigma_\infty$. By definition, we can find an $n \in \mathbb{N}$ such that $\sigma_n = \infty$. Then, $1/\sigma_n = 0 = 1/(\sigma_n + r)$ for any $r \in \mathbb{R}$ due to convention. This implies that

$$\begin{aligned} \widetilde{\varphi}_n(\sigma) &= [\sigma_1, \dots, \sigma_n, \infty, \infty, \dots] \\ &= [\sigma_1, \dots, \sigma_n, \sigma_{n+1}, \dots, \sigma_{n+j}, \infty, \infty, \dots] = \widetilde{\varphi}_{n+j}(\sigma) \end{aligned}$$

for any $j \in \mathbb{N}$. Hence the lemma. $\square$

Upon combining Lemma 2.1 and the following proposition, we deduce that the mapping $\widetilde{\varphi}\colon \mathbb{N}_\infty^{\mathbb{N}} \to [0,1]$ is well-defined.

Proposition 2.2 (See [5, Proposition 1.1.2])**.** *For each $\sigma \in \Sigma_\infty$, the limit $\lim_{k\to\infty} \widetilde{\varphi}_k(\sigma)$ exists and belongs to $\mathbb{I}$.*

We define $\varphi\colon \Sigma \to [0,1]$ by

$$\varphi(\sigma) \coloneqq \widetilde{\varphi}(\sigma) \quad \text{for each } \sigma \in \Sigma. \tag{2.4}$$

Let $\sigma \in \Sigma \setminus \Sigma_0$ be arbitrary. Since $\varphi_n(\sigma) > 0$ for all $n \in \mathbb{N}$ by definition, we write

$$\varphi_n(\sigma) \coloneqq \frac{p_n^*(\sigma)}{q_n^*(\sigma)},$$

where $p_n^*(\sigma)$ and $q_n^*(\sigma)$ are coprime positive integers. Furthermore, define $p_{-1}^*(\sigma) \coloneqq 1$, $q_{-1}^*(\sigma) \coloneqq 0$, $p_0^*(\sigma) \coloneqq 0$, and $q_0^*(\sigma) \coloneqq 1$. (Notice that this resembles the definition of the *convergents* $p_n(x)/q_n(x)$, $n \in \mathbb{N}_0$, which are defined in terms of numbers in $[0,1)$.)

Proposition 2.3 (See [4**?** , 5])**.** *Let $\sigma \coloneqq (\sigma_k)_{k\in\mathbb{N}} \in \Sigma \setminus \Sigma_0$. If $\sigma \in \Sigma_n$ for some $n \in \mathbb{N}$, then the following hold:*

(i) $q_k^*(\sigma) = \sigma_k q_{k-1}^*(\sigma) + q_{k-2}^*(\sigma)$ *for each $k \in \{1,\dots,n\}$.*
(ii) $q_k^*(\sigma) \geq F_{k+1}$ *for each $k \in \{1,\dots,n\}$.*
(iii) $|\varphi(\sigma) - \varphi_k(\sigma)| \leq 1/[q_k^*(\sigma) q_{k+1}^*(\sigma)]$ *for each $k \in \mathbb{N}$.*

If $\sigma \in \Sigma_\infty$, then the equality and inequalities in parts (i)–(iii) hold for all $k \in \mathbb{N}$.

We remark that if the algorithms (1.1) and (1.2) were absent, then every rational number in $(0,1)$ would admit two distinct continued fraction expansions (see [4, Theorem 162] or [5, p. 3]). This is because, for any integer $k \geq 2$, we have $1/k = 1/[(k-1) + 1/1]$ with $k-1 \in \mathbb{N}$. The following proposition rephrases this fact in terms of two maps $f\colon [0,1] \to \Sigma$ and $\varphi\colon \Sigma \to [0,1]$.

Proposition 2.4. *For each $x \in [0,1]$, the following hold:*

(i) *If $x \in \mathbb{I} \cup \{0,1\}$, then we have $\varphi^{-1}(\{x\}) = \{\sigma\}$, where $\sigma \coloneqq f(x)$. Moreover, $\sigma \in \Sigma_\infty$ if $x \in \mathbb{I}$; $\sigma = (\infty,\infty,\dots) \in \Sigma_0$ if $x = 0$; $\sigma = (1,\infty,\infty,\dots) \in \Sigma_1$ if $x = 1$.*
(ii) *If $x \in (0,1) \cap \mathbb{Q}$, then we have $\varphi^{-1}(\{x\}) = \{\sigma, \sigma'\}$, where*

$$\sigma \coloneqq (\underbrace{\overline{d}_1(x), \overline{d}_2(x), \dots, \overline{d}_{n-1}(x)}_{n-1 \text{ terms}}, \overline{d}_n(x)) = f(x) \in \Sigma_n \cap f([0,1]),$$

$$\sigma' \coloneqq (\underbrace{\overline{d}_1(x), \overline{d}_2(x), \dots, \overline{d}_{n-1}(x)}_{n-1 \text{ terms}}, \overline{d}_n(x) - 1, 1) \in \Sigma_{n+1} \setminus f([0,1]),$$

for some $n \in \mathbb{N}$.

For a given $\sigma \coloneqq (\sigma_k)_{k\in\mathbb{N}} \in \Sigma$, we define $\sigma^{(n)} \coloneqq (\tau_k)_{k\in\mathbb{N}} \in \Sigma$, for each $n \in \mathbb{N}_0$, by

$$\tau_k \coloneqq \begin{cases} \sigma_k, & \text{if } k \in \{1,\dots,n\}; \\ \infty, & \text{otherwise,} \end{cases} \qquad \text{i.e.,} \quad \sigma^{(n)} \coloneqq (\underbrace{\sigma_1, \dots, \sigma_n}_{n \text{ terms}}, \infty, \infty, \dots).$$

Note that $\sigma^{(n)}$ is not in Σ_n in general. For instance, if $\sigma := (3, 5, \infty, \infty, \dots) \in \Sigma_2 \subseteq \Sigma$, we have $\sigma^{(3)} = \sigma \notin \Sigma_3$.

Fix $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$. Define the *cylinder set* associated with σ by

$$\Upsilon_\sigma := \{\upsilon \in \Sigma \colon \upsilon^{(n)} = \sigma\}.$$

We further define the *fundamental interval* associated with σ by

$$I_\sigma := f^{-1}(\Upsilon_\sigma) = \{x \in [0,1] : \overline{d}_k(x) = \sigma_k \text{ for all } k \in \{1, \dots, n\}\}.$$

It is legitimate to call I_σ as an interval due to the following proposition.

Proposition 2.5 (See [5, Theorem 1.2.2])**.** *For each $n \in \mathbb{N}$ and $\sigma := (\sigma_k)_{k\in\mathbb{N}} \in \Sigma_n$, the set I_σ is an interval with endpoints $\varphi(\sigma)$ and $\varphi(\widehat{\sigma})$, where $\widehat{\sigma} \in \Sigma_n$ is given by*

$$\widehat{\sigma} := (\underbrace{\sigma_1, \dots, \sigma_{n-1}}_{n-1 \text{ terms}}, \sigma_n + 1, \infty, \infty, \dots).$$

3. Main results and proofs

In this section, we establish and prove the main results of this paper regarding the continuity of the continued fraction mapping.

Equip $\mathbb{N}$ with the discrete topology, and denote its one-point compactification by $\mathbb{N}_\infty$. Define $\rho \colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{R}$ by

$$\rho(m,n) := \begin{cases} 1/m + 1/n, & \text{if } m \neq n; \\ 0, & \text{if } m = n, \end{cases}$$

for each $m, n \in \mathbb{N}_\infty$.

Proposition 3.1 ([1, Lemma 3.1])**.** *The mapping $\rho \colon \mathbb{N}_\infty \times \mathbb{N}_\infty \to \mathbb{R}$ is a metric on $\mathbb{N}_\infty$ and induces the topology of the one-point compactification on $\mathbb{N}_\infty$.*

Define $\rho^{\mathbb{N}} \colon \mathbb{N}_\infty^{\mathbb{N}} \times \mathbb{N}_\infty^{\mathbb{N}} \to \mathbb{R}$ by

$$\rho^{\mathbb{N}}(\sigma, \tau) := \sum_{k=1}^{\infty} \frac{1}{F_k^2} \rho(\sigma_k, \tau_k)$$

for each $\sigma := (\sigma_k)_{k\in\mathbb{N}}$ and $\tau := (\tau_k)_{k\in\mathbb{N}}$ in $\mathbb{N}_\infty^{\mathbb{N}}$. The following lemma is analogous to [1, Lemma 3.3].

Lemma 3.2. *The mapping $\rho^{\mathbb{N}} \colon \mathbb{N}_\infty^{\mathbb{N}} \times \mathbb{N}_\infty^{\mathbb{N}} \to \mathbb{R}$ is a metric on $\mathbb{N}_\infty^{\mathbb{N}}$ and induces the product topology on $\mathbb{N}_\infty^{\mathbb{N}}$. Consequently, the metric space $(\mathbb{N}_\infty^{\mathbb{N}}, \rho^{\mathbb{N}})$ is compact.*

Proof. For the first assertion, we just note that $\sum_{k=1}^{\infty} 1/F_k^2 < \infty$, which is clear from Binet's formula (1.3). We omit the remaining details, as the proof follows the same line as the standard proof that any countable product of metric spaces is metrizable. The second statement follows from Tychonoff's theorem, which tells us that the product space $\mathbb{N}_\infty^{\mathbb{N}}$ is compact. $\square$

Due to the preceding lemma, from now on, we may use $\mathbb{N}_\infty^{\mathbb{N}}$ to refer to both the product space and the metric space $(\mathbb{N}_\infty^{\mathbb{N}}, \rho^{\mathbb{N}})$.

Define $g\colon \mathbb{N}_\infty^{\mathbb{N}} \to \Sigma$ by

$$g(\sigma) \coloneqq \begin{cases} \sigma^{(n)}, & \text{if } \sigma^{(n)} \in \Sigma_n \text{ and } \sigma^{(n+1)} \notin \Sigma_{n+1} \text{ for some } n \in \mathbb{N}_0; \\ \sigma, & \text{if } \sigma \in \Sigma_\infty, \end{cases}$$

for each $\sigma \coloneqq (\sigma_k)_{k\in\mathbb{N}} \in \mathbb{N}_\infty^{\mathbb{N}}$. It is not hard to verify that g is well-defined and that g is surjective. Under the mapping g, each sequence forgets all terms following the first ∞, if any, and replaces the forgotten terms with ∞'s.

Lemma 3.3. *We have $\widetilde{\varphi} = \varphi \circ g$ on $\mathbb{N}_\infty^{\mathbb{N}}$.*

Proof. Note first that, for any $\sigma \in \mathbb{N}_\infty^{\mathbb{N}}$, we have $g(\sigma) \in \Sigma$, and so $\varphi(g(\sigma)) = \widetilde{\varphi}(g(\sigma))$ by definition of φ in (2.4). If $\sigma \in \Sigma_\infty$, then $g(\sigma) = \sigma$ by definition, and hence, $\widetilde{\varphi}(g(\sigma)) = \widetilde{\varphi}(\sigma)$. Now, assume $\sigma \in \mathbb{N}_\infty^{\mathbb{N}} \setminus \Sigma_\infty$, and put $\sigma \coloneqq (\sigma_k)_{k\in\mathbb{N}}$. Then, we can find the unique $n \in \mathbb{N}_0$ such that $\sigma^{(n)} \in \Sigma_n$ and $\sigma_{n+1} = \infty$, so that $g(\sigma) = \sigma^{(n)} \in \Sigma_n$. It follows by definition that

$$\widetilde{\varphi}(g(\sigma)) = [\sigma_1, \dots, \sigma_n, \infty, \infty, \dots] = [\sigma_1, \dots, \sigma_n, \infty, \sigma_{n+2}, \sigma_{n+3}, \dots] = \widetilde{\varphi}(\sigma),$$

and this completes the proof. □

The following lemma shows that the mapping $\widetilde{\varphi}\colon \mathbb{N}_\infty^{\mathbb{N}} \to [0,1]$ is Lipschitz, and therefore continuous.

Lemma 3.4. *For any $\sigma, \tau \in \mathbb{N}_\infty^{\mathbb{N}}$, we have $|\widetilde{\varphi}(\sigma) - \widetilde{\varphi}(\tau)| \le \rho^{\mathbb{N}}(\sigma, \tau)$.*

Proof. Let $\sigma \coloneqq (\sigma_k)_{k\in\mathbb{N}}, \tau \coloneqq (\tau_k)_{k\in\mathbb{N}} \in \mathbb{N}_\infty^{\mathbb{N}}$. If $\sigma = \tau$, the inequality holds trivially; hence, we suppose that σ and τ are distinct. If $\sigma_1 \neq \tau_1$, then

$$|\widetilde{\varphi}(\sigma) - \widetilde{\varphi}(\tau)| \le |\widetilde{\varphi}(\sigma)| + |\widetilde{\varphi}(\tau)| \le \frac{1}{\sigma_1} + \frac{1}{\tau_1} = \frac{1}{F_1^2}\rho(\sigma_1, \tau_1) \le \rho^{\mathbb{N}}(\sigma, \tau).$$

Assume that σ and τ share the initial block of length $n \in \mathbb{N}$, i.e., $\sigma^{(n)} = \tau^{(n)}$ and $\sigma_{n+1} \neq \tau_{n+1}$. If $\sigma^{(n)} = \tau^{(n)} \notin \Sigma_n$, then $\sigma_j = \tau_j = \infty$ for some $j \in \{1, \dots, n\}$, and it follows that $\widetilde{\varphi}(\sigma) = \widetilde{\varphi}(\tau)$; thus, the inequality holds. Now, assume $\sigma^{(n)} = \tau^{(n)} \in \Sigma_n$. Since $\sigma_{n+1} \neq \tau_{n+1}$, at least one of σ_{n+1} and τ_{n+1} is finite. Without loss of generality, we may further assume $\sigma_{n+1} \neq \infty$, so that

$$g(\sigma) \in \Sigma_l \quad \text{for some } l \in \mathbb{N}_\infty \setminus \{1, \dots, n, n+1\}. \tag{3.5}$$

Write $q_k^* = q_k^*(\sigma^{(n)}) = q_k^*(\tau^{(n)})$ for each $k \in \{0, 1, \dots, n\}$. We consider two cases separately according as $\tau_{n+1} = \infty$ or $\tau_{n+1} \neq \infty$.

Case I. Assume $\tau_{n+1} = \infty$. Then, by definitions, we have

$$\begin{aligned} \widetilde{\varphi}(\sigma^{(n)}) &= [\sigma_1, \dots, \sigma_n, \infty, \infty, \dots] = \varphi_n(g(\sigma)), \\ \widetilde{\varphi}(\tau) &= [\tau_1, \dots, \tau_n, \infty, \tau_{n+2}, \dots] = [\tau_1, \dots, \tau_n, \infty, \infty, \dots] = \widetilde{\varphi}(\tau^{(n)}). \end{aligned} \tag{3.6}$$

Hence, we find that

$$
\begin{aligned}
&|\widetilde{\varphi}(\sigma) - \widetilde{\varphi}(\tau)| \\
&= |(\widetilde{\varphi}(\sigma) - \widetilde{\varphi}(\sigma^{(n)})) - (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\tau^{(n)}))| && \text{since } \sigma^{(n)} = \tau^{(n)} \\
&= |\varphi(g(\sigma)) - \varphi_n(g(\sigma))| && \text{by Lemma 3.3 and (3.6)} \\
&\leq \frac{1}{q_n^*(g(\sigma)) q_{n+1}^*(g(\sigma))} && \text{by (3.5) and Proposition 2.3(iii)} \\
&= \frac{1}{q_n^*(g(\sigma))} \cdot \frac{1}{\sigma_{n+1} q_n^*(g(\sigma)) + q_{n-1}^*(g(\sigma))} && \text{by Proposition 2.3(i)} \\
&= \frac{1}{q_n^*} \cdot \frac{1}{\sigma_{n+1} q_n^* + q_{n-1}^*} && \text{since } \sigma^{(n)} = (g(\sigma))^{(n)} \in \Sigma_n \\
&< \frac{1}{(q_n^*)^2} \frac{1}{\sigma_{n+1}} = \frac{1}{(q_n^*)^2} \left(\frac{1}{\sigma_{n+1}} + \frac{1}{\tau_{n+1}} \right) && \text{since } 1/\tau_{n+1} = 0 \text{ by assumption} \\
&\leq \frac{1}{F_{n+1}^2} \left(\frac{1}{\sigma_{n+1}} + \frac{1}{\tau_{n+1}} \right) && \text{by Proposition 2.3(ii)} \\
&= \frac{1}{F_{n+1}^2} \rho(\sigma_{n+1}, \tau_{n+1}) \leq \rho^{\mathbb{N}}(\sigma, \tau),
\end{aligned}
$$

as desired.

Case II. Assume $\tau_{n+1} \neq \infty$. Then, $g(\tau) \in \Sigma_{l'}$ for some $l' \in \mathbb{N}_\infty \setminus \{1, \ldots, n, n+1\}$. An argument similar to the one in the preceding case results in

$$
\begin{aligned}
|\widetilde{\varphi}(\sigma) - \widetilde{\varphi}(\tau)| &= |(\widetilde{\varphi}(\sigma) - \widetilde{\varphi}(\sigma^{(n)})) - (\widetilde{\varphi}(\tau) - \widetilde{\varphi}(\tau^{(n)}))| \\
&\leq |\varphi(g(\sigma)) - \varphi_n(g(\sigma))| + |\varphi(g(\tau)) - \varphi_n(g(\tau))| \\
&\leq \frac{1}{q_n^*(g(\sigma)) q_{n+1}^*(g(\sigma))} + \frac{1}{q_n^*(g(\tau)) q_{n+1}^*(g(\tau))} \\
&= \frac{1}{q_n^*} \left(\frac{1}{\sigma_{n+1} q_n^* + q_{n-1}^*} + \frac{1}{\tau_{n+1} q_n^* + q_{n-1}^*} \right) \\
&< \frac{1}{(q_n^*)^2} \left(\frac{1}{\sigma_{n+1}} + \frac{1}{\tau_{n+1}} \right) \\
&\leq \frac{1}{F_{n+1}^2} \left(\frac{1}{\sigma_{n+1}} + \frac{1}{\tau_{n+1}} \right) = \frac{1}{F_{n+1}^2} \rho(\sigma_{n+1}, \tau_{n+1}) \leq \rho^{\mathbb{N}}(\sigma, \tau).
\end{aligned}
$$

This completes the proof of the lemma. □

Let $\sim_g$ be a binary relation on $\mathbb{N}_\infty^{\mathbb{N}}$ given by $\sigma \sim_g \tau$ if and only if $g(\sigma) = g(\tau)$ for $\sigma, \tau \in \mathbb{N}_\infty^{\mathbb{N}}$. Evidently, $\sim_g$ is an equivalence relation on $\mathbb{N}_\infty^{\mathbb{N}}$. Note that the map $\widetilde{g} \colon \mathbb{N}_\infty^{\mathbb{N}} / \sim_g \to \Sigma$ given by $\widetilde{g}([\sigma]) = g(\sigma)$, for each $[\sigma] \in \mathbb{N}_\infty^{\mathbb{N}} / \sim_g$, is a bijection. We equip $\mathbb{N}_\infty^{\mathbb{N}} / \sim_g$ with the quotient topology, and define

$$
\mathcal{T}_\Sigma \coloneqq \{\widetilde{g}(U) : U \text{ is open in } \mathbb{N}_\infty^{\mathbb{N}} / \sim_g\}.
$$

Lemma 3.5. *The set $\mathcal{T}_\Sigma$ defines a topology on Σ, and $\widetilde{g}\colon \mathbb{N}_\infty^\mathbb{N}/\sim_g \to (\Sigma, \mathcal{T}_\Sigma)$ is a homeomorphism.*

Proof. The lemma is clear from the definitions. $\square$

Lemma 3.6. *Let $\pi_g\colon \mathbb{N}_\infty^\mathbb{N} \to \mathbb{N}_\infty^\mathbb{N}/\sim_g$ denote the canonical projection. Then, the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{N}_\infty^\mathbb{N} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi_g"'] \arrow[dr, "g"] & {[0,1]} \\
\mathbb{N}_\infty^\mathbb{N}/\sim_g \arrow[r, "\widetilde{g}"'] & (\Sigma, \mathcal{T}_\Sigma) \arrow[u, "\varphi"']
\end{tikzcd}$$

In particular, $g\colon \mathbb{N}_\infty^\mathbb{N} \to (\Sigma, \mathcal{T}_\Sigma)$ is a continuous surjection.

Proof. The result follows from Lemma 3.3 and the paragraph preceding Lemma 3.5. $\square$

Lemma 3.7. *The graph of the equivalence relation $\sim_g$, i.e., the set $R_g \coloneqq \{(\sigma, \tau) \in \mathbb{N}_\infty^\mathbb{N} \times \mathbb{N}_\infty^\mathbb{N} : \sigma \sim_g \tau\}$, is closed in the product space $\mathbb{N}_\infty^\mathbb{N} \times \mathbb{N}_\infty^\mathbb{N}$.*

Proof. Since the product space $\mathbb{N}_\infty^\mathbb{N} \times \mathbb{N}_\infty^\mathbb{N}$ is metrizable as a finite product of the metric space $\mathbb{N}_\infty^\mathbb{N}$ (Lemma 3.2), it suffices to show that any convergent sequence in R_g has its limit in R_g.

Let $((\boldsymbol{\sigma}_k, \boldsymbol{\tau}_k))_{k\in\mathbb{N}}$ be a convergent sequence in R_g with the limit $(\sigma, \tau) \in \mathbb{N}_\infty^\mathbb{N} \times \mathbb{N}_\infty^\mathbb{N}$. Suppose $(\sigma, \tau) \in \mathbb{N}_\infty^\mathbb{N} \times \mathbb{N}_\infty^\mathbb{N} \setminus R_g$ for the sake of contradiction. Then, $\sigma \not\sim_g \tau$, or, equivalently, $g(\sigma) \neq g(\tau)$. Since $g(\sigma)$ and $g(\tau)$ are elements of Σ, we may write $g(\sigma) = ((g(\sigma))_j)_{j\in\mathbb{N}}$ and $g(\tau) = ((g(\tau))_j)_{j\in\mathbb{N}}$. Put $M \coloneqq \min\{j \in \mathbb{N} : (g(\sigma))_j \neq (g(\tau))_j\} < \infty$. Without loss of generality, we may assume that $(g(\sigma))_M \neq \infty$. Then, $(g(\sigma))_j \in \mathbb{N}$ for all $j \in \{1, \ldots, M-1\}$ and $g(\sigma) \in \Sigma_i$ for some $i \in \mathbb{N}_\infty \setminus \{1, \ldots, M\}$. There are two cases to consider according as $\tau_M \neq \infty$ or $\tau_M = \infty$.

Case I. Assume $\tau_M \neq \infty$. Since the first M terms of $g(\sigma)$ are all finite, we have $\sigma^{(M)} = ((g(\sigma))^{(M)} \in \Sigma_M$ by definition of g; similarly, $\tau^{(M)} = ((g(\tau))^{(M)} \in \Sigma_M$. Note that $\sigma_M \neq \tau_M$ by definition of M. Since $\boldsymbol{\sigma}_k \to \sigma$ and $\boldsymbol{\tau}_k \to \tau$ as $k \to \infty$ in the product space $\mathbb{N}_\infty^\mathbb{N}$, we know that $(\boldsymbol{\sigma}_k)^{(M)} = \sigma^{(M)} \in \Sigma_M$ and $(\boldsymbol{\tau}_k)^{(M)} = \tau^{(M)} \in \Sigma_M$ for all large enough k. Then, for such k, it must be that $(g(\boldsymbol{\sigma}_k))_M = \sigma_M$ and $(g(\boldsymbol{\tau}_k))_M = \tau_M$. But $\sigma_M \neq \tau_M$, and it follows that $g(\boldsymbol{\sigma}_k) \neq g(\boldsymbol{\tau}_k)$, i.e., $\boldsymbol{\sigma}_k \not\sim_g \boldsymbol{\tau}_k$, for all large enough k, a contradiction.

Case II. Assume $\tau_M = \infty$. Then, $(g(\boldsymbol{\sigma}_k))_M = \sigma_M$ for all large enough k as in the previous case, while $(g(\boldsymbol{\tau}_k))_M \to \infty$ as $k \to \infty$. In particular, $(g(\boldsymbol{\tau}_k))_M > \sigma_M$ for all sufficiently large k, and therefore, $g(\boldsymbol{\sigma}_k) \neq g(\boldsymbol{\tau}_k)$ for all such k. Thus, $\boldsymbol{\sigma}_k \not\sim_g \boldsymbol{\tau}_k$ for all but finitely many k's, contradicting that $(\boldsymbol{\sigma}_k, \boldsymbol{\tau}_k) \in R_g$ for all $k \in \mathbb{N}$.

This proves that the limit (σ, τ) must be in R_g in either case. Hence the lemma. $\square$

We state some standard facts in general topology in the following two propositions.

Proposition 3.8 (See [2, p. 105, Proposition 8]). *Let X be a compact space and $\sim$ an equivalence relation on X. If $X/\sim$ denotes the quotient space, then the following are equivalent:*

(i) *The graph of $\sim$, i.e., the set $\{(x, y) \in X \times X : x \sim y\}$, is closed in the product space $X \times X$.*
(ii) *The canonical projection $\pi \colon X \to X/\sim$ is a closed mapping.*
(iii) *$X/\sim$ is Hausdorff.*

Proposition 3.9 (See [3, p. 159, Proposition 17]). *Let X be a compact metrizable space and $\sim$ an equivalence relation on X. If the quotient space $X/\sim$ is Hausdorff, then $X/\sim$ is compact metrizable.*

Lemma 3.10. *The topological space $(\Sigma, \mathcal{T}_\Sigma)$ is compact metrizable.*

Proof. Since the graph of $\sim_g$ is closed in the product space $\mathbb{N}_\infty^{\mathbb{N}} \times \mathbb{N}_\infty^{\mathbb{N}}$ (Lemma 3.7), we infer, in view of Proposition 3.8, that the quotient space $\mathbb{N}_\infty^{\mathbb{N}}/\sim_g$ is Hausdorff. Now, since $\mathbb{N}_\infty^{\mathbb{N}}$ is compact metrizable (Lemma 3.2) and since the quotient space $\mathbb{N}_\infty^{\mathbb{N}}/\sim_g$ is Hausdorff, Proposition 3.9 tells us that $\mathbb{N}_\infty^{\mathbb{N}}/\sim_g$ is compact metrizable. Therefore, we conclude that $(\Sigma, \mathcal{T}_\Sigma)$ is compact metrizable, as a homeomorphic space to $\mathbb{N}_\infty^{\mathbb{N}}/\sim_g$ via the mapping $\widetilde{g} \colon \mathbb{N}_\infty^{\mathbb{N}}/\sim_g \to (\Sigma, \mathcal{T}_\Sigma)$ (Lemma 3.5). □

Lemma 3.11. *For each $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, the subspaces Υ_σ and $\Sigma \setminus \Upsilon_\sigma$ of $(\Sigma, \mathcal{T}_\Sigma)$ are compact metrizable.*

Proof. We first note that by Lemma 3.7 and Proposition 3.8, the canonical projection $\pi_g \colon \mathbb{N}_\infty^{\mathbb{N}} \to \mathbb{N}_\infty^{\mathbb{N}}/\sim_g$ is a closed mapping. But $g = \widetilde{g} \circ \pi_g$ by Lemma 3.6, where $\widetilde{g}$ is a homeomorphism (Lemma 3.5), and this implies that g is also a closed mapping.

Now, fix $n \in \mathbb{N}$ and $\sigma := (\sigma_k)_{k\in\mathbb{N}} \in \Sigma_n$. Consider the set

$$g^{-1}(\Upsilon_\sigma) = \{\sigma_1\} \times \cdots \times \{\sigma_n\} \times \mathbb{N}_\infty^{\mathbb{N}\setminus\{1,\dots,n\}},$$

which is clopen in $\mathbb{N}_\infty^{\mathbb{N}}$. Then, $g^{-1}(\Upsilon_\sigma)$ and $\mathbb{N}_\infty^{\mathbb{N}} \setminus g^{-1}(\Upsilon_\sigma)$ are closed in $\mathbb{N}_\infty^{\mathbb{N}}$, and hence, $\Upsilon_\sigma = g(g^{-1}(\Upsilon_\sigma))$ and $\Sigma \setminus \Upsilon_\sigma = g(\mathbb{N}_\infty^{\mathbb{N}} \setminus g^{-1}(\Upsilon_\sigma))$ are closed in $(\Sigma, \mathcal{T}_\Sigma)$ as g is a closed mapping by the preceding paragraph. Since $(\Sigma, \mathcal{T}_\Sigma)$ is compact metrizable (Lemma 3.10), its closed subspaces are compact metrizable. This completes the proof of the lemma. □

We are now ready to present the continuity theorem for the extended continued fraction mapping, which is an analogue of [1, Lemmas 3.9 and 3.10].

Theorem 3.12. *For the continuity of the mapping $f \colon [0, 1] \to (\Sigma, \mathcal{T}_\Sigma)$, the following hold:*

(i) *f is continuous at every irrational point and at two rational points 0 and 1.*

(ii) *f is one-sided continuous only at every rational point in $(0,1)$; precisely, if we let $x \in (0,1) \cap \mathbb{Q}$ and put $\varphi^{-1}(\{x\}) = \{\sigma, \tau\}$, then*

$$\lim_{\substack{t \to x \\ t \in I_\sigma}} f(t) = \sigma \quad \text{and} \quad \lim_{\substack{t \to x \\ t \notin I_\sigma}} f(t) = \tau.$$

Proof. (i) See the proof of [1, Lemma 3.9]. The proof follows the same line, using the compact metrizability of $(\Sigma, \mathcal{T}_\Sigma)$ (Lemma 3.10) and the fact that $\varphi^{-1}(\{x\})$ is a singleton for each $x \in \mathbb{I} \cup \{0,1\}$ (Proposition 2.4(i)).

(ii) See the proof of [1, Lemma 3.10]. The proof follows the same line, using the compact metrizability of the subspaces Υ_σ and $\Sigma \setminus \Upsilon_\sigma$ of $(\Sigma, \mathcal{T}_\Sigma)$ (Lemma 3.11) and the fact that $\varphi^{-1}(\{x\})$ is a doubleton for each $x \in (0,1) \cap \mathbb{Q}$ (Proposition 2.4(ii)). $\square$

Corollary 3.13. *The continued fraction mapping on $[0,1)$ to $(\Sigma, \mathcal{T}_\Sigma)$ defined by $x \mapsto (d_1(x), d_2(x), \dots)$ for each $x \in [0,1)$ is:*

(i) *Continuous at every irrational point and at 0.*
(ii) *One-sided continuous only at every rational point in $(0,1)$; precisely, if we let $x \in (0,1) \cap \mathbb{Q}$ and put $\varphi^{-1}(\{x\}) = \{\sigma, \tau\}$, then*

$$\lim_{\substack{t \to x \\ t \in I_\sigma}} f(t) = \sigma \quad \text{and} \quad \lim_{\substack{t \to x \\ t \notin I_\sigma}} f(t) = \tau.$$

Proof. The corollary is an immediate consequence of Theorem 3.12. $\square$

Theorem 3.14. *The mapping $\varphi\colon (\Sigma, \mathcal{T}_\Sigma) \to [0,1]$ is continuous.*

Proof. It is clear that $\widetilde{\varphi}\colon \mathbb{N}_\infty^\mathbb{N} \to [0,1]$ is continuous if and only if $\varphi\colon (\Sigma, \mathcal{T}_\Sigma) \to [0,1]$ is continuous. Indeed, for any open subset U of $[0,1]$, we have $\widetilde{\varphi}^{-1}(U) = \pi^{-1} \circ \widetilde{g}^{-1} \circ \varphi^{-1}(U)$ by Lemma 3.6, so that

$$\begin{aligned} \widetilde{\varphi}^{-1}(U) \text{ is open in } \mathbb{N}_\infty^\mathbb{N} &\iff \widetilde{g}^{-1} \circ \varphi^{-1}(U) \text{ is open in } \mathbb{N}_\infty^\mathbb{N} / \sim_g \\ &\iff \varphi^{-1}(U) \text{ is open in } (\Sigma, \mathcal{T}_\Sigma), \end{aligned}$$

where the first equivalence follows from the definition of the quotient topology and the second from the fact that $\widetilde{g}$ is a homeomorphism (Lemma 3.5). But then, the continuity of $\widetilde{\varphi}$ (Lemma 3.4) implies the continuity of φ. $\square$

Corollary 3.15. *The subspace $\mathbb{I}$ of $[0,1]$ and the subspace Σ_∞ of $(\Sigma, \mathcal{T}_\Sigma)$ are homeomorphic via the restriction of $f\colon [0,1] \to (\Sigma, \mathcal{T}_\Sigma)$ to $\mathbb{I}$ and the restriction of $\varphi\colon (\Sigma, \mathcal{T}_\Sigma) \to [0,1]$ to Σ_∞, serving as each other's continuous inverses.*

Proof. It is a standard fact that the sets $\mathbb{I}$ and Σ_∞ are in a bijective relation via the continued fraction mapping. Therefore, to obtain the corollary, we only need to combine Theorems 3.12(i) and 3.14. $\square$

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