Twist analysis of the spin-orbit correlation in QCD

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We present a QCD analysis of the twist-three parton distribution functions associated with the spin-orbit correlation of quarks and gluons in the nucleon. We derive a novel longitudinal momentum sum rule which may be regarded as the momentum version of the Jaffe-Manohar spin sum rule. The result is also applicable to spinless hadrons and nuclei with trivial modifications.

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I. INTRODUCTION

Quarks and gluons, fundamental constituents of the nucleon, carry angular momentum in the forms of spin and orbital angular momentum (OAM). These contributions must sum up to the spin of the nucleon, which is $\frac{1}{2}$, through the process of angular momentum coupling. This simple argument can be precisely formulated in QCD in terms of spin sum rules

$$\begin{aligned} \frac{1}{2} &= \frac{1}{2}\Delta\Sigma + \Delta G + L_q + L_g, &(1)\\ &= \frac{1}{2}\Delta\Sigma + L_q^{\rm kin} + J_g. &(2)\end{aligned}$$

(1) is referred to as the Jaffe-Manohar (JM) sum rule [1] where $\frac{1}{2}\Delta\Sigma = \frac{1}{2}\sum_q(\Delta q + \Delta\bar{q})$ and ΔG represent the quark and gluon helicity contributions, and $L_{q,g}$ are the canonical OAMs of quarks and gluons. On the other hand, (2) is called the Ji sum rule [2] where $L_q^{\rm kin}$ is the quark kinetic OAM and J_g is the total gluon angular momentum. These sum rules shed light on the origin of the nucleon spin, which is one of the main scientific goals of the future Electron-Ion Collider [3]. Ultimately, one would like to determine the precise numerical values of all the components in (1) and (2) and understand the intricate QCD dynamics behind them.

Needless to say, the sum rules (1), (2) are relevant only to hadrons with nonzero spin. For a spinless hadron, such as the pion, all the terms in (1) and (2) are zero. Of course, individual quarks and gluons do carry spin and OAM, but their averages are zero in a spinless or unpolarized hadron. However, their *correlations* can be nonzero in general, and may provide novel insights into the structure of hadrons with or without spin. One such example is a recent observation [4] that the helicity and OAM of quarks and gluons at small-x are maximally entangled in a quantum mechanical sense. This has been deduced from an analysis of the quark [5] and gluon [4, 6] spin-orbit correlations $C_{q,g}$ which measure how likely the helicity and OAM of individual partons are aligned or anti-aligned. Thus the study of $C_{q,g}$ offers a novel platform to discuss quantum entanglement effects inside hadrons and nuclei and their manifestation in experimental observables.

Motivated by this development, in this paper, we perform a QCD analysis of the quark and gluon spin-orbit correlations where 'spin' and 'orbit' refer to those in the JM decomposition (1). (Note that the gluon helicity and OAM can be defined only in the JM decomposition scheme (1).) We present the rigorous definition of the collinear parton distribution functions (PDFs) associated with spin-orbit correlations $C_{q,g}(x)$ and derive their exact relations to twist-two and twist-three generalized parton distributions (GPDs). Our approach closely parallels that in [7] which discussed the PDFs for parton OAMs $L_{q,g}(x)$. In a sense, $C_{q,g}(x)$ are the parity transforms of $L_{q,g}(x)$, so their operator structures are similar. In the quark sector, our results partly overlap with those in [8], but we provide more explicit expressions for the genuine twist-three terms. The results in the gluon sector are entirely new. In the end, we will establish a new longitudinal momentum sum rule

$$1 = \Delta\Sigma^{(3)} + \frac{1}{2}\Delta G^{(3)} - 3C_q^{(2)} - \frac{3}{2}C_g^{(2)} + \cdots , \tag{3}$$

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where the superscript (j) denotes the j-th moment of the corresponding PDF. (In this notation, $\Delta\Sigma = \Delta\Sigma^{(1)}$, etc. in (1).) (3) shows only the 'Wandzura-Wilczek' part [9]. The exact formula is given in (66) and includes off-forward matrix elements of genuine twist-three, quark-gluon-quark and three-gluon light ray operators, as well as a quark mass term related to certain quark transversity GPDs. This may be considered as the momentum version of the JM spin sum rule (1), and as such, it is equally fundamental to nucleon structure science.

II. SPIN-ORBIT CORRELATION AND GTMD

In this section, we introduce the spin-orbit correlations for quarks and gluons as moments of certain generalized TMDs (GTMD). Let us first consider the polarized quark GTMD [10][1]

$$\begin{aligned}\tilde{f}_q(x,\xi,k_\perp,\Delta_\perp) &= \int \frac{d^3z}{2(2\pi)^3} e^{ixP^+z^- - ik_\perp\cdot z_\perp} \langle p's'|\bar{q}(-z/2)W_\pm\gamma^+\gamma_5 q(z/2)|ps\rangle \\ &= \frac{-i}{2M}\bar{u}(p's')\left[\frac{\epsilon_{ij}k_\perp^i\Delta_\perp^j}{M^2}G^q_{1,1} + \frac{\sigma^{i+}\gamma_5}{P^+}(k^i_\perp G^q_{1,2} + \Delta^i_\perp G^q_{1,3}) + \sigma^{+-}\gamma_5 G^q_{1,4}\right]u(ps), \end{aligned} \tag{4}$$

where M is the nucleon mass, $P^\mu = \frac{p^\mu + p'^\mu}{2} = \left(P^+, 0_\perp, \frac{M^2+\Delta_\perp^2/4}{2P^+(1-\xi^2)}\right)$ and $\Delta^\mu = p'^\mu - p^\mu = \left(-2\xi P^+, \Delta_\perp, \frac{\xi(M^2+\Delta_\perp^2/4)}{P^+(1-\xi^2)}\right)$. We shall set $\xi \propto \Delta^+ = 0$ (hence also $\Delta^- = 0$) throughout this paper and keep only linear terms in $\Delta_\perp$ in the limit $p' \to p$. The spin vectors s, s' play no role in this paper and will be suppressed below unless otherwise stated. It is understood that we simultaneously take the limit $s' \to s$ and average over s. W_+ and W_- are staple-shaped Wilson lines in the fundamental representation

$$W_\pm = W_{-\frac{z^-}{2},\pm\infty} W_{-\frac{z_\perp}{2},\frac{z_\perp}{2}} W_{\pm\infty,\frac{z^-}{2}}, \qquad W_{\pm\infty,\frac{z^-}{2}} = \mathrm{P}\exp\left(-ig\int_{z^-/2}^{\pm\infty} dx^- A_a^+(x^-, z_\perp/2)t^a\right) \tag{5}$$

which connect the two points $\pm z/2$ via light-cone infinity $z^- = \pm\infty$. The quark spin-orbit correlation is defined as [5]

$$\begin{aligned} C_q &= -i\int_{-1}^1 dx \int dk_\perp \epsilon^{ij}\frac{\partial}{\partial\Delta^i}k^j\tilde{f}_q(x,k_\perp,\Delta_\perp)\Big|_{\Delta_\perp=0} \\ &= \int_{-1}^1 dx\int d^2k_\perp \frac{k_\perp^2}{M^2}G^q_{1,1}(x,k_\perp,0). \end{aligned} \tag{6}$$

As we shall see, the direction of the Wilson line W_+ or W_- does not matter due to PT-symmetry [11]. In terms of the polarized Wigner distribution [12] $\tilde{f}_q(x,k_\perp,b_\perp) = \int\frac{d^2\Delta_\perp}{(2\pi)^2}e^{-ib_\perp\cdot\Delta_\perp}\tilde{f}_q(x,k_\perp,\Delta_\perp)$

$$C_q = \int dx\int d^2k_\perp d^2b_\perp \epsilon^{ij}b^ik^j\tilde{f}_q(x,k_\perp,b_\perp). \tag{7}$$

This is a more intuitive formula which features the classical expression of OAM $\epsilon^{ij}b^ik^j$.

Let us also define the associated parton distribution function (PDF) by undoing the x-integral

$$C_q(x) = \int d^2k_\perp\frac{k_\perp^2}{M^2}G^q_{1,1}(x,k_\perp,0). \qquad (0<x<1) \tag{8}$$

For antiquarks, $C_{\bar{q}}(x) = -C_q(-x)$ such that $C_q = \int_0^1 dx(C_q(x) - C_{\bar{q}}(x))$. To find the operator structure of $C_q(x)$, we convert k^j in (6) into a $z_\perp$-derivative acting on the quark bilinear.

$$\begin{aligned}\int d^2k_\perp k^j\tilde{f}_q &= \frac{i}{2}\int\frac{dz^-}{4\pi}e^{ixP^+z^-}\langle p'|\bar{q}(-z/2)\gamma^+\gamma_5(W_{-\frac{z^-}{2},\frac{z^-}{2}}D^j_{\rm pure} - \overleftarrow{D}^j_{\rm pure}W_{-\frac{z^-}{2},\frac{z^-}{2}})q(z/2)|p\rangle\Big|_{z_\perp=0} \\ &\approx -\frac{i}{2}\epsilon^{ji}\Delta^iC_q(x). \end{aligned} \tag{9}$$

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[1] Our conventions are $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\epsilon^{0ij3} = \epsilon^{-+ij} = \epsilon^{ij}$ such that $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\lambda\gamma_5] = -4i\epsilon^{\mu\nu\rho\lambda}$. The sign of γ_5 is opposite to that in [7]. The covariant derivative is defined as $D^\mu = \partial^\mu + igA^\mu$. We also use the notations $\overleftarrow{D}^\mu = \overleftarrow{\partial}^\mu - igA^\mu$ and $\overleftrightarrow{D}^\mu = \frac{D^\mu - \overleftarrow{D}^\mu}{2}$. In this convention, the Yang-Mills equation reads $D_\mu F^{\mu\nu}_a = \sum_q g\bar{q}\gamma^\nu t^a q$. Transverse vectors such as $k^i = -k_i$ $(i=1,2)$ are contracted with the Euclidean signature $k_\perp\cdot z_\perp = k^iz^i$.

Here and below, the symbol $\approx$ means that only the linear term in $\Delta_\perp$ is kept. $D^\mu_{\rm pure}$ is the gauge covariant generalization of the partial (canonical) derivative ∂^μ

$$D^\mu_{\rm pure}(x) \equiv D^\mu(x) + i\int dw^- \mathcal{K}(w^- - x^-) W_{x^-,w^-} gF^{+\mu}(w^-, x_\perp) W_{w^-,x^-}, \tag{10}$$

with $\mathcal{K}(w^-) = \pm\theta(\pm w^-)$ corresponding to the two choices $W_\pm$. The additional term in (10) comes from the derivative of the Wilson line along the light-cone. We recall that $D^\mu_{\rm pure}$ defines the canonical OAM $L_{q,g} \sim \epsilon^{ij} b^i i D^j_{\rm pure}$ necessary for the gauge invariant completion of the Jaffe-Manohar sum rule (1) [11, 13]. The same operator naturally appears here since we use the staple-shaped Wilson line throughout.

The gluon spin-orbit correlation can be introduced in the same way. Starting from the polarized gluon GTMD

$$\begin{aligned} x\tilde{f}_g(x,\xi,k_\perp,\Delta_\perp) &= i\int \frac{d^3z}{(2\pi)^3P^+} e^{ixP^+z^- - ik_\perp\cdot z_\perp} \langle p'|\tilde{F}^{+\mu}(-z/2)\mathcal{W}_\pm F^+_{\ \mu}(z/2)|p\rangle \qquad (11)\\ &= \frac{-i}{2M}\bar{u}(p')\left[\frac{\epsilon_{ij}k^i\Delta^j}{M^2}G^g_{1,1} + \frac{\sigma^{i+}\gamma_5}{P^+}(k^iG^g_{1,2} + \Delta^iG^g_{1,3}) + \sigma^{+-}\gamma_5 G^g_{1,4}\right]u(p), \end{aligned}$$

where $\mathcal{W}_\pm$ is the staple-shaped Wilson line in the adjoint representation, we define the PDF of the gluon spin-orbit correlation as [4, 6]

$$\begin{aligned} xC_g(x) &\equiv -i\int d^2k_\perp \epsilon^{ij}\frac{\partial}{\partial\Delta^i}k^j x\tilde{f}_g(x,k_\perp,\Delta_\perp)\Big|_{\Delta_\perp=0} \\ &= \int d^2k_\perp \frac{k_\perp^2}{M^2}G^g_{1,1}(x,k_\perp,0). \qquad (12) \end{aligned}$$

Again the direction $\pm$ does not matter. Note that $C_g(x)$ is odd under $x \to -x$ so the first moment vanishes $\int_{-1}^1 dx C_g(x) = 0$ in contrast to the quark case. (Incidentally, the integral $\int_0^1 dx C_g(x)$ is divergent.) In terms of operators we find

$$\begin{aligned} \int d^2k_\perp k^j x\tilde{f}_g &= -\frac{1}{2}\int \frac{dz^-}{2\pi P^+} e^{ixP^+z^-}\langle p'|\tilde{F}^{+\mu}(-z/2)(\mathcal{W}_{-\frac{z^-}{2},\frac{z^-}{2}} D^j_{\rm pure} - \overleftarrow{D}^j_{\rm pure}\mathcal{W}_{-\frac{z^-}{2},\frac{z^-}{2}})F^+_{\ \mu}(z/2)|p\rangle\Big|_{z_\perp=0} \\ &\approx -\frac{i}{2}\epsilon^{ji}\Delta^i xC_g(x), \qquad (13) \end{aligned}$$

where $D^j_{\rm pure}$ is the same as before except that it is now in the adjoint representation. For simplicity, in the following we will omit subscripts on Wilson lines (e.g., $W_{-\frac{z^-}{2},\frac{z^-}{2}} \to W$). The reader can easily reinstate them considering gauge invariance.

III. QUARK SPIN-ORBIT CORRELATION

The definition of the spin-orbit correlations in terms of the Wigner/GTMD distributions is intuitive but somewhat heuristic, as it potentially overlooks the subtleties in properly defining transverse momentum dependent distributions beyond leading order. In order to avoid this issue, it is more advantageous to work in a purely collinear framework. In this section we follow the strategy developed in [7] (see also [14]) for the parton OAM PDFs $L_{q,g}(x)$ and adapt it to define and analyze the quark spin-orbit correlation $C_q(x)$.

A. Twist-3 qgq correlation functions

As a preliminary, we introduce the 'genuine twist-three' quark-gluon distributions. First, the 'F-type' correlators are parametrized as[2]

$$
\begin{aligned}
&\frac{1}{(P^+)^2}\int\frac{d\lambda d\tau}{(2\pi)^2}e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)}\langle p'|\bar{q}(-\lambda n/2)\gamma^+\gamma_5 gF^{+i}(\tau n)Wq(\lambda n/2)|p\rangle \\
&\approx \frac{-i}{2}\bar{u}(p')\gamma^i\gamma_5 u(p)\tilde{G}_{Fq}(x_1,x_2)+\frac{\epsilon^{ij}\Delta_j}{2P^+}\bar{u}(p')\gamma^+u(p)\tilde{A}_q(x_1,x_2)-\frac{i\Delta^i}{2P^+}\bar{u}(p')\gamma^+\gamma_5 u(p)\tilde{\Phi}_{Fq}(x_1,x_2) \\
&\approx \epsilon^{ij}\Delta_j\tilde{\Psi}_{Fq}(x_1,x_2)+\cdots, \qquad (16)
\end{aligned}
$$

where $n^\mu=\delta^\mu_-/P^+$ is a lightlike vector to make λ,τ dimensionless. In the last line we have kept only the linear terms in $\Delta_\perp$ in the limit $\Delta_\perp\to 0$ and redefined $\tilde{\Psi}_F(x,x')=\frac{1}{2}\tilde{G}_F(x,x')+\tilde{A}(x,x')$. $\tilde{\Phi}_F$ was the focus of [7] which concerned the parton OAM in a longitudinally polarized nucleon $\bar{u}'\gamma^+\gamma_5 u\approx 2s^+$. In the present work, this term drops out because we average over nucleon spins as mentioned before. Note that $\tilde{G}_F$ is one of the Efremov-Teryaev-Qiu-Sterman (ETQS) functions relevant to transverse single spin asymmetry (SSA) [15–17]. It is also familiar in the context of higher twist effects in polarized Deep Inelastic scattering [18]. This is the only term that survives in the forward limit and for a transversely polarized proton $\bar{u}'\gamma^i\gamma_5 u\approx 2s^i$. In the present kinematics the same spinor product gives rise to a linear term $\bar{u}'\gamma^i\gamma_5 u\approx i\epsilon^{ij}\Delta_j$. It is tempting to assume that $\tilde{A}=0$, in which case no new distribution is introduced and one can make a rather unexpected connection to the physics of SSA. However, we do not see valid reasons to neglect $\tilde{A}$ in general.

Another correlator without a γ_5 is

$$
\begin{aligned}
&\frac{1}{(P^+)^2}\int\frac{d\lambda d\tau}{(2\pi)^2}e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)}\langle p'|\bar{q}(-\lambda n/2)\gamma^+WgF^{+i}(\tau n)Wq(\lambda n/2)|p\rangle \\
&\approx \frac{\epsilon^{ij}}{2}\bar{u}(p')\gamma_j\gamma_5 u(p)G_{Fq}(x_1,x_2)-\frac{i\Delta^i}{2P^+}\bar{u}(p')\gamma^+u(p)A_q(x_1,x_2)+\frac{\epsilon^{ij}\Delta_j}{2P^+}\bar{u}(p')\gamma^+\gamma_5 u(p)\Phi_{Fq}(x_1,x_2) \\
&\approx -i\Delta^i\Psi_{Fq}(x_1,x_2)+\cdots, \qquad (17)
\end{aligned}
$$

with $\Psi_F(x,x')=\frac{1}{2}G_F(x,x')+A(x,x')$. Again Φ_F is the same as in [7] and G_F is the other ETQS function. Note that the local matrix element

$$
\frac{1}{(P^+)^2}\langle p'|\bar{q}\gamma^+gF^{+i}q|p\rangle\approx -i\Delta^i\int dx_1dx_2\Psi_{Fq}(x_1,x_2), \qquad (18)
$$

looks familiar in the context of the twist-three correction to the g_2 structure function for a transversely polarized nucleon [19]

$$
\frac{1}{(P^+)^2}\langle ps_\perp|\bar{q}\gamma^+gF^{+i}q|ps_\perp\rangle=\epsilon^{ij}s_j\int dx_1dx_2G_F(x_1,x_2)=4d_2\epsilon^{ij}s_j. \qquad (19)
$$

However, in general $d_2=\frac{1}{4}\int dx_1dx_2G_F$ is different from $\frac{1}{2}\int dx_1dx_2\Psi_F$ because of the A-term.

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[2] When $\Delta^+=0$ and hence also $\Delta^-=0$, one has the relations

$$
\begin{aligned}
2P^+\epsilon^{ij}\bar{u}(p')\gamma_j u(p) &= -i\Delta^i\bar{u}(p')\gamma^+\gamma_5 u(p), \\
\epsilon^{ij}\Delta_j\bar{u}(p')\gamma^+u(p) &= -2iP^+\bar{u}(p')\gamma^i\gamma_5 u(p)+2im\epsilon^{ij}\bar{u}(p')\sigma^+_{\ j}u(p), \\
\bar{u}(p')\gamma^+u(p) &= \frac{P^+}{m}\bar{u}(p')u(p)+\frac{i}{2m}\bar{u}(p')\sigma^{+i}\Delta_i u(p), \\
\bar{u}(p')\gamma^i u(p) &= \bar{u}(p')\frac{i\sigma^{ij}\Delta_j}{2m}u(p), \qquad (14)
\end{aligned}
$$

which follow from the Dirac equation and the Gordon identities. There are thus four independent spinor structures with one transverse index

$$
i\Delta^i\bar{u}(p')\gamma^+\gamma_5 u(p), \quad \epsilon^{ij}\Delta_j\bar{u}(p')\gamma^+u(p), \quad i\bar{u}(p')\gamma^i\gamma_5 u(p), \quad \epsilon^{ij}\Delta_j\bar{u}(p')u(p). \qquad (15)
$$

If we restrict to linear order in $\Delta_\perp$, the last one is redundant.

Next, we introduce the 'D-type' correlators

$$\frac{1}{P^+}\int \frac{d\lambda d\tau}{(2\pi)^2} e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)} \langle p'|\bar{q}(-\lambda n/2)\gamma^+\gamma_5 W \overleftrightarrow{D}^i(\tau n) W q(\lambda n/2)|p\rangle \approx \epsilon^{ij}\Delta_j \tilde{\Psi}_{Dq}(x_1,x_2), \tag{20}$$

$$\frac{1}{P^+}\int \frac{d\lambda d\tau}{(2\pi)^2} e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)} \langle p'|\bar{q}(-\lambda n/2)\gamma^+ W \overleftrightarrow{D}^i(\tau n) W q(\lambda n/2)|p\rangle \approx -i\Delta^i \Psi_{Dq}(x_1,x_2), \tag{21}$$

where we have already taken the $\Delta_\perp \to 0$ limit. From PT symmetry it follows that

$$\begin{aligned}
\tilde{\Psi}_F(x_1,x_2) &= -\tilde{\Psi}_F(x_2,x_1), \qquad \Psi_F(x_1,x_2) = \Psi_F(x_2,x_1), \\
\tilde{\Psi}_D(x_1,x_2) &= \tilde{\Psi}_D(x_2,x_1), \qquad \Psi_D(x_1,x_2) = -\Psi_D(x_2,x_1).
\end{aligned} \tag{22}$$

Moreover, by commuting $\overleftrightarrow{D}^i$ with W's, one can obtain the relation between the F, D-type correlators

$$\begin{aligned}
\tilde{\Psi}_D(x_1,x_2) &= P\frac{1}{x_1-x_2}\tilde{\Psi}_F(x_1,x_2) + \delta(x_1-x_2)C_q(x), \\
\Psi_D(x_1,x_2) &= P\frac{1}{x_1-x_2}\Psi_F(x_1,x_2),
\end{aligned} \tag{23}$$

where P denotes the principal value prescription. Note that we have identified the coefficient of the delta function with the quark spin-orbit correlation defined in (8). Indeed, a straightforward calculation gives, in the limit $\Delta_\perp \to 0$,

$$\begin{aligned}
P^+\epsilon^{ij}\Delta_j C_q(x) \;\approx\; \int \frac{d\lambda}{4\pi} e^{i\lambda x}\Bigg\{ &\langle p'|\bar{q}(-\lambda n/2)\gamma^+\gamma_5 (WD^i - \overleftarrow{D}^i W) q(\lambda n/2)|p\rangle \\
&+ \frac{i}{2P^+}\int d\tau \big(\epsilon(\tau-\lambda/2)+\epsilon(\tau+\lambda/2)\big)\langle p'|\bar{q}(-\lambda n/2)\gamma^+\gamma_5 W g F^{+i}(\tau n) W q(\lambda n/2)|p\rangle \Bigg\},
\end{aligned} \tag{24}$$

where $\epsilon(x) = x/|x|$ is the sign function. Writing $\epsilon(\tau \pm \lambda/2) = 2\theta(\tau\pm\lambda/2) - 1 = -2\theta(-(\tau\pm\lambda/2)) + 1$ and removing the ± 1 terms which do not contribute because $\tilde{\Psi}_F(x,x) = 0$, we see that (24) agrees with (9). Integrating (24) over x gives

$$P^+\epsilon^{ij}\Delta_j \int dx C_q(x) \;\approx\; \langle p'|\bar{q}\gamma^+\gamma_5 \overleftrightarrow{D}^i q|p\rangle + \frac{i}{2P^+}\int d\tau \epsilon(\tau)\langle p'|\bar{q}(0)\gamma^+\gamma_5 W g F^{+i}(\tau n) W q(0)|p\rangle. \tag{25}$$

In the second term, one can freely replace $\epsilon(\tau) \to \pm 2\theta(\pm\tau)$ without changing the value. The result then agrees with [8]. This term is due to the difference between the staple-shaped and straight Wilson lines, see [20] for a recent calculation in lattice QCD. It is an analog of the 'potential angular momentum' [11, 21] which accounts for the difference between the canonical OAM L_q in (1) and the kinetic OAM $L_q^{\rm kin}$ in (2).

We mention the well-known properties [22, 23] that for operators on the light-cone, as used in the operator definition of the Ψ's, the time-ordering is immaterial and hence the ordering of the fields can be changed (respecting Fermi-statistics). Moreover, their support is always limited to $-1 \le x_1, x_2 \le 1$. This applies equally to the pure gluon correlators defined later.

B. Equation of motion

We now derive an exact formula which relates $C_q(x)$ to generalized parton distributions (GPDs). We start with an identity that can be derived from the equation of motion (Dirac equation) $(i\not{D} - m_q)q = \bar{q}(i\overleftarrow{\not{D}} + m_q) = 0$

$$\begin{aligned}
\bar{q}(-z^-/2, 0_\perp)\gamma^i\gamma_5 (WD_- - \overleftarrow{D}_- W) q(z^-/2, 0_\perp) \;=\; &\bar{q}\gamma^+\gamma^5 (WD^i - \overleftarrow{D}^i W) q + i\epsilon^{ij}\bar{q}\gamma_j (WD_- + \overleftarrow{D}_- W) q \\
&- i\epsilon^{ij}\bar{q}\gamma^+ (WD_j + \overleftarrow{D}_j W) q - 2im_q\epsilon^{ij}\bar{q}\sigma^+{}_j W q,
\end{aligned} \tag{26}$$

where m_q is the quark mass. Taking the off-forward matrix element $\langle p'|...|p\rangle$ of this, we find

$$\begin{aligned}
2\frac{\partial}{\partial z^-}\langle p'|\bar{q}(-z^-/2)\gamma^i\gamma_5 W q(z^-/2)|p\rangle &= \langle p'|\bar{q}\gamma^+\gamma^5 (W\overleftrightarrow{D}^i + \overleftrightarrow{D}^i W) q|p\rangle + i\epsilon^{ij}\mathcal{D}_-\langle p'|\bar{q}\gamma_j W q|p\rangle \\
&- i\epsilon^{ij}\langle p'|\bar{q}\gamma^+ (W\overleftrightarrow{D}_j - \overleftrightarrow{D}_j W) q|p\rangle - i\epsilon^{ij}\langle p'|\partial_j(\bar{q}\gamma^+ W q)|p'\rangle - 2im_q\epsilon^{ij}\langle p'|\bar{q}\sigma^+{}_j W q|p\rangle,
\end{aligned} \tag{27}$$

where $\mathcal{D}_\mu f(-x,x) = \lim_{a^\mu\to 0}\frac{1}{a^\mu}\left(f(-x+a,x+a)-f(-x,x)\right)$ denotes the translation operator. Since we assume $\xi=0$, $\mathcal{D}_- \sim \Delta^+ = 0$ can be neglected. Let us now introduce the polarized quark GPDs

$$\langle p'|\bar{q}(-z^-/2)\gamma^\mu\gamma_5 W q(z^-/2)|p\rangle = \int dx e^{-ixP^+z^-}\bar{u}(p')\left[\gamma^\mu\gamma_5\tilde{H}_q + \frac{\gamma_5\Delta^\mu}{2M}\tilde{E}_q + \gamma_\perp^\mu\gamma_5\tilde{G}_q + \cdots\right]u(p), \tag{28}$$

where $\gamma_\perp^\mu = \delta_i^\mu\gamma^i$. This may not be a standard parametrization and requires an explanation. $\tilde{H}_q$ and $\tilde{E}_q$ are the usual twist-two GPDs relevant when $\mu=+$. When $\mu=\perp$, there will be additional terms, and we have collected in $\tilde{G}_q$ all terms which lead to the spinor product $\bar{u}'\gamma^i\gamma_5 u \approx i\epsilon^{ij}\Delta_j$ by using identities like (14).[3] In other words, (28) is arranged in such a way that the neglected terms do not give rise to the structure $i\epsilon^{ij}\Delta_j$ in the $\Delta_\perp\to 0$ limit. Lorentz covariance dictates that $\int dx\tilde{G}_q(x)=0$. The quark mass term can be parametrized by a 'transversity' GPD

$$\langle p'|\bar{q}(-z^-/2)\sigma^+_{\ j} W q(z^-/2)|p\rangle \approx -i\frac{P^+}{M}\Delta_j\int dx e^{-ixP^+z^-}H_{1q}(x), \tag{29}$$

where again we expanded to linear order in $\Delta_\perp$ and redefined the coefficient as H_1.[4] Inserting (28), (29) into (27) and comparing the coefficients of $\epsilon^{ij}\Delta_j$, we find

$$\begin{aligned} x(\Delta q(x)+\tilde{G}_q(x)) &= q(x) + \frac{1}{2}\int dx'\left(\tilde{\Psi}_{Dq}(x,x')+\tilde{\Psi}_{Dq}(x',x)-\Psi_{Dq}(x',x)+\Psi_{Dq}(x,x')\right) - \frac{m_q}{M}H_{1q}(x) \\ &= q(x) + C_q(x) + \int dx' P\frac{1}{x-x'}\left(\tilde{\Psi}_{Fq}(x,x')+\Psi_{Fq}(x,x')\right) - \frac{m_q}{M}H_{1q}(x), \end{aligned} \tag{30}$$

where $\Delta q(x) = \tilde{H}_q(x,0,0)$ is the polarized quark PDF. On the other hand, the GPD $\tilde{E}_q$ is absent due to the spin average (otherwise it would also not contribute since its contribution would be $O(\Delta^2)$). A result similar to (30) was previously derived in [8] although an exact comparison is difficult due to rather different notations for the genuine twist-three part.

C. Lorentz invariant relation

We can eliminate $\tilde{G}_q$ from (30) by utilizing a Lorentz invariant relation which we now derive. For this purpose, we extend (28) covariantly to off the light cone by replacing $\gamma_\perp^\mu \to \gamma^\mu - \frac{\gamma\cdot z}{P\cdot z}P^\mu - \frac{\gamma\cdot P}{P\cdot z}z^\mu + \cdots$,

$$\langle p'|\bar{q}(-z/2)\gamma^\mu\gamma_5 W q(z/2)|p\rangle = \int dx e^{-ixP\cdot z}\bar{u}(p')\left[\gamma^\mu(\tilde{H}_q+\tilde{G}_q) - \frac{\gamma\cdot z}{P\cdot z}P^\mu\tilde{G}_q + \cdots\right]\gamma_5 u(p), \tag{31}$$

where the neglected terms do not contribute in the following. Then, recalling that $P_\perp=0$ in our frame, we immediately find

$$\epsilon^{ij}\Delta_j\tilde{G}_q(x) = \int\frac{dz^-}{2\pi}(iP^+z^-)e^{ixP^+z^-}\frac{\partial}{\partial z_i}\langle p'|\bar{q}(-z/2)\gamma^+\gamma_5 W q(z/2)|p\rangle\Big|_{z^\mu=\delta^\mu_- z^-}. \tag{32}$$

A straightforward calculation gives

$$\begin{aligned} \tilde{G}_q(x) &= \int dx'\frac{d}{dx}\tilde{\Psi}_{Dq}(x,x') - \int dx' P\frac{1}{x-x'}\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\Big)\tilde{\Psi}_{Fq}(x,x') \\ &= \frac{d}{dx}C_q(x) - 2\int dx' P\frac{1}{(x-x')^2}\tilde{\Psi}_{Fq}(x,x'), \end{aligned} \tag{33}$$

where we have used (23) in the second equality. Inserting (33) into the equations of motion relation (30), we obtain the differential equation

$$x\frac{d}{dx}C_q(x) - C_q(x) = -x\Delta q(x) + q(x) + \int dx' P\frac{1}{x-x'}\Psi_{Fq}(x,x') + \int dx' P\frac{3x-x'}{x-x'}\tilde{\Psi}_{Fq}(x,x') - \frac{m_q}{M}H_{1q}(x), \tag{34}$$

---

[3] In the notation of [24], $\tilde{G}_q = \tilde{G}_2 + 2\tilde{G}_4$.
[4] In the notation of [25] $H_{1q} = 2\tilde{H}_T^q + E_T^q$.

With the boundary condition $C_q(\pm 1)=0$, this can be solved as[5]

$$
\begin{aligned}
C_q(x) &= x\int_x^{\epsilon(x)}\frac{dx'}{x'^2}x'\Delta q(x') - x\int_x^{\epsilon(x)}\frac{dx'}{x'^2}q(x') \\
&\quad -x\int_x^{\epsilon(x)}dx_1\int_{-1}^1 dx_2\frac{\tilde{\Psi}_{Fq}(x_1,x_2)}{x_1-x_2}P\frac{3x_1-x_2}{x_1^2(x_1-x_2)} \\
&\quad -x\int_x^{\epsilon(x)}dx_1\int_{-1}^1 dx_2\Psi_{Fq}(x_1,x_2)P\frac{1}{x_1^2(x_1-x_2)}+\frac{xm_q}{M}\int_x^{\epsilon(x)}\frac{dx'}{x'^2}H_{1q}(x').
\end{aligned}
\tag{37}
$$

Note that $\Delta q(x)$ survives in this relation even though we have systematically averaged over nucleon spins $s \approx s'$ as stated before. For antiquarks, $C_{\bar{q}}(x)=-C_q(-x)$. The first moment reads

$$
C_q \equiv \int_{-1}^1 dx C_q(x) = \frac{1}{2}\int_0^1 dxx(\Delta q(x)-\Delta\bar{q}(x)) - \frac{N_q}{2} - \int\frac{dxdx'}{x-x'}\tilde{\Psi}_{Fq}(x,x') + \frac{m_q}{2M}H_{1q}^{(1)}, \tag{38}
$$

where $N_q=\int_0^1 dx(q(x)-\bar{q}(x))$ is the number of valence quarks with flavor q and $H_{1q}^{(n)}\equiv\int_{-1}^1 dxx^{n-1}H_{1q}(x)$. Using (23) one can also write

$$
C'_q \equiv \int dxdx'\tilde{\Psi}_{Dq}(x,x') = \frac{1}{2}\int_0^1 dxx(\Delta q(x)-\Delta\bar{q}(x)) - \frac{N_q}{2} + \frac{m_q}{2M}H_{1q}^{(1)}, \tag{39}
$$

in agreement with [26]. C_q and C'_q differ by a genuine twist-three term featuring $\tilde{\Psi}_F$, see (25).

Our primary interest in this work is the second moment which reads

$$
\begin{aligned}
C_q^{(2)} \equiv \sum_q\int_{-1}^1 dxxC_q(x) &= \frac{1}{3}\int_0^1 dxx^2\Delta\Sigma(x) - \frac{1}{3}\sum_q(A_q+A_{\bar{q}}) \\
&\quad -\int dxdx'\sum_q\left[\frac{x}{x-x'}\tilde{\Psi}_{Fq}(x,x')+\frac{1}{6}\Psi_{Fq}(x,x')\right]+\sum_q\frac{m_q}{3M}H_{1q}^{(2)},
\end{aligned}
\tag{40}
$$

where $\Delta\Sigma(x)\equiv\sum_q(\Delta q(x)+\Delta\bar{q}(x))$ and we used the (anti-)symmetry of $\Psi_F,\tilde{\Psi}_F$. $A_q=\int_0^1 dxxq(x)$ is the fraction of the nucleon momentum carried by quarks with flavor q.

Finally, from (30), (38) and (40), it follows that

$$
\int_{-1}^1 dxx\tilde{G}_q(x) = -C'_q, \tag{41}
$$

$$
\int_{-1}^1 dxx^2\tilde{G}_q(x) = -\frac{2}{3}\int_0^1 dxx^2(\Delta q(x)+\Delta\bar{q}(x)) + \frac{2}{3}(A_q+A_{\bar{q}}) + \frac{1}{3}\int dxdx'\Psi_{Fq}(x,x') - \frac{2m_q}{3M}H_{1q}^{(2)}, \tag{42}
$$

and $\int_{-1}^1 dx\tilde{G}_q=0$ as mentioned before. (41) agrees with [24, 26]. (42) is consistent with [8] but disagrees with [24] where essentially the authors assumed $\int dx_1dx_2\Psi_F=0$, see a comment after (19).

---

[5] Alternatively, following [7], one may evaluate the linear combination

$$
z^\mu\left(\frac{\partial}{\partial z^\mu}\langle p'|\bar{q}(-z/2)\gamma^i\gamma_5 Wq(z/2)|p\rangle - \frac{\partial}{\partial z_i}\langle p'|\bar{q}(-z/2)\gamma_\mu\gamma_5 Wq(z/2)|p\rangle\right)\Big|_{z^\mu=\delta^\mu_- z^-}. \tag{35}
$$

in two ways, first inserting (31) and second using the equation of motion. This leads to

$$
\begin{aligned}
\frac{d}{dx}(x\Delta q)+x\frac{d}{dx}\tilde{G}_q(x) &= \frac{d}{dx}q(x)+\int dx'P\frac{1}{x-x'}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)\tilde{\Psi}_{Fq}(x,x') \\
&\quad +\int dx'P\frac{1}{x-x'}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)\Psi_{Fq}(x,x')-\frac{m_q}{M}\frac{d}{dx}H_{1q}(x).
\end{aligned}
\tag{36}
$$

Solving this equation for $\tilde{G}_q$ with the boundary condition $\tilde{G}_q(\pm 1)=0$ and inserting the solution into (30), one finds the same result (37).

IV. GLUON SPIN-ORBIT CORRELATION

A. Twist-3 three-gluon correlation functions

We now repeat essentially the same procedure for the gluon spin-orbit correlation $C_g(x)$. We first introduce the 'F-type' genuine twist-three, three-gluon correlator

$$\begin{aligned}&\frac{-1}{(P^+)^3}\int\frac{d\lambda d\tau}{(2\pi)^2}e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)}\langle p'|F^{+i}(-\lambda n/2)\mathcal{W}gF^{+j}(\tau n)\mathcal{W}F^{+k}(\lambda n/2)|p\rangle\\&\approx i\left(N(x_1,x_2)\epsilon^{ik}\epsilon^{jl}+N(x_2,x_2-x_1)\epsilon^{ij}\epsilon^{kl}-N(x_1,x_1-x_2)\epsilon^{jk}\epsilon^{il}\right)\Delta_l,\end{aligned}\tag{43}$$

where the field strength tensor in the middle should be understood as a matrix in the adjoint representation $F^{+j}_{ab}=F^{+j}_c(T^c)_{ab}=-if_{abc}F^{+j}_c$. (The rest of the color indices are understood to be contracted in the adjoint representation.) We can eliminate the antisymmetric tensor by using $\epsilon^{ik}\epsilon^{jl}=\delta^{ij}\delta^{kl}-\delta^{il}\delta^{jk}$. However, the parametrization (43) is particularly convenient when making a comparison with [7].[6] PT and permutation symmetries dictate that

$$N(x_1,x_2)=-N(x_2,x_1),\qquad N(-x_1,-x_2)=-N(x_1,x_2).\tag{45}$$

If we contract ik indices antisymmetrically,

$$\begin{aligned}&\frac{-\epsilon^{ik}}{(P^+)^3}\int\frac{d\lambda d\tau}{(2\pi)^2}e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)}\langle p'|F^{+i}(-\lambda n/2)\mathcal{W}gF^{+j}(\tau n)\mathcal{W}F^{+k}(\lambda n/2)|p\rangle\\&=\frac{1}{(P^+)^3}\int\frac{d\lambda d\tau}{(2\pi)^2}e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)}\langle p'|\tilde{F}^{+\rho}(-\lambda n/2)\mathcal{W}gF^{+j}(\tau n)\mathcal{W}F^{+}{}_{\rho}(\lambda n/2)|p\rangle\\&\approx i\left(2N(x_1,x_2)+N(x_2,x_2-x_1)-N(x_1,x_1-x_2)\right)\epsilon^{jl}\Delta_l\\&\equiv i\tilde{N}_F(x_1,x_2)\epsilon^{jl}\Delta_l,\end{aligned}\tag{46}$$

with $\tilde{N}_F(x_1,x_2)=-\tilde{N}_F(x_2,x_1)$ and $\tilde{N}_F(-x_1,-x_2)=-\tilde{N}_F(x_1,x_2)$. Let us also define

$$N_F(x_1,x_2)=N(x_2,x_2-x_1)+N(x_1,x_1-x_2),\tag{47}$$

with the properties $N_F(x_1,x_2)=N_F(x_2,x_1)$, $N_F(-x_1,-x_2)=-N_F(x_1,x_2)$.

Next introduce the 'D-type' correlator

$$\begin{aligned}&\frac{1}{(P^+)^3}\int\frac{d\lambda d\tau}{(2\pi)^2}e^{i\frac{\lambda}{2}(x_1+x_2)+i\tau(x_2-x_1)}\langle p'|\tilde{F}^{+\rho}(-\lambda n/2)\mathcal{W}\overleftrightarrow{D}^j(\tau n)\mathcal{W}F^{+}{}_{\rho}(\lambda n/2)|p\rangle\\&\approx i\tilde{N}_D(x_1,x_2)\epsilon^{jl}\Delta_l,\end{aligned}\tag{48}$$

which is symmetric $\tilde{N}_D(x_1,x_2)=\tilde{N}_D(x_2,x_1)=\tilde{N}_D(-x_1,-x_2)$. This is related to the F-type correlator as

$$\tilde{N}_D(x_1,x_2)=\frac{\tilde{N}_F(x_1,x_2)}{x_1-x_2}-\frac{1}{2}\delta(x_1-x_2)x_1C_g(x_1).\tag{49}$$

As in the quark case, we identified the coefficient of the delta function with the gluon spin-orbit correlation. One can readily check that this agrees with the previous definition (13).

---

[6] Having said this, we can still rewrite (43) as

$$\begin{aligned}&-i\left(\delta^{ik}\Delta^jN_F(x_1,x_2)-\delta^{ij}\Delta^kN_F(x_2,x_2-x_1)-\delta^{jk}\Delta^iN_F(x_1,x_1-x_2)\right)\\&\approx-\left(\delta^{ik}\epsilon^{jl}N_F(x_1,x_2)-\delta^{ij}\epsilon^{kl}N_F(x_2,x_2-x_1)-\delta^{jk}\epsilon^{il}N_F(x_1,x_1-x_2)\right)\bar{u}(p')\gamma^l\gamma_5u(p),\end{aligned}\tag{44}$$

where N_F is as defined in (47), in order to emphasize the connection to the three-gluon correlators relevant to transverse single spin asymmetry [27]. Indeed, for a transversely polarized nucleon in the forward limit, $\bar{u}\gamma^l\gamma_5u=2s^l$ and (44) agrees with the parametrization in [28] under the identification $N_F(x_1,x_2)\leftrightarrow N_{[28]}(x_1,x_2)$. However, again such a connection is obscured by a possible new distribution at finite $\Delta_\perp$ (analogs of $\tilde{A},A$ in (16), (17)).

B. Equation of motion

The twist structure of $C_g(x)$ can be completely determined by the equation of motion (Yang-Mills equation) and the Lorentz invariance relation. We first consider the equation of motion and define

$$J \equiv \langle p'|\frac{\partial}{\partial z^-}\left(\tilde{F}^{+\rho}(0)\mathcal{W}F^i{}_\rho(z^-)+\tilde{F}^{i\rho}(-z^-)\mathcal{W}F^+{}_\rho(0)\right)|p\rangle. \quad (50)$$

On one hand, this can be expressed in terms of the polarized gluon GPDs generalized to twist-three

$$\begin{aligned}&\langle p'|\tilde{F}^{\alpha\rho}(-z^-/2)\mathcal{W}F^\beta_\rho(z^-/2)+\tilde{F}^{\beta\rho}(-z^-/2)\mathcal{W}F^\alpha_\rho(z^-/2)|p\rangle \\ &= -i\int dx e^{-ixP^+z^-}\bar{u}(p')\left[\tilde{H}_g P^{(\alpha}\gamma^{\beta)}\gamma_5+\frac{P^{(\alpha}\Delta^{\beta)}}{2M}\gamma_5\tilde{E}_g+x\tilde{G}_g P^{(\alpha}\gamma_\perp^{\beta)}\gamma_5+\cdots\right]u(p), \quad (51)\end{aligned}$$

where $A^{(\alpha}B^{\beta)}=\frac{A^\alpha B^\beta+A^\beta B^\alpha}{2}$. $\tilde{H}_g,\tilde{E}_g$ are the standard twist-two helicity GPDs normalized as $\tilde{H}_g(x)=x\Delta G(x)$ in the forward limit. The twist-three $x\tilde{G}_g$ is defined in the same vein as $\tilde{G}_q$ in (28) and satisfies $\int dx x\tilde{G}_g(x)=0$. We find

$$J \approx -\frac{(P^+)^2}{2}\int dx e^{-ixP^+z^-}x^2(\Delta G(x)+\tilde{G}_g(x))\bar{u}'\gamma^i\gamma_5 u. \quad (52)$$

On the other hand, by explicitly carrying out the derivative, we obtain

$$\begin{aligned}J &= \tilde{F}^{+\rho}(0)\mathcal{W}D^+F^i{}_\rho(z^-)-\tilde{F}^{i\rho}(-z^-)\overleftarrow{D}^+\mathcal{W}F^+_\rho(0) \\ &= \tilde{F}^{+\rho}(0)\mathcal{W}(D^iF^+_\rho(z^-)-D_\rho F^{+i}(z^-))+(-\tilde{F}^+_\rho(-z^-)\overleftarrow{D}^i+\tilde{F}^{+i}(-z^-)\overleftarrow{D}_\rho)\mathcal{W}F^{+\rho}(0) \\ &\quad +\epsilon^{ij}D_\alpha F^{\alpha+}(-z^-)\mathcal{W}F^+_j(0) \\ &= \tilde{F}^{+\rho}(0)(\mathcal{W}\overleftrightarrow{D}^i+\overleftrightarrow{D}^i\mathcal{W})F^+_\rho(z^-)+\mathcal{D}_\rho(-\tilde{F}^{+\rho}\mathcal{W}F^{+i}+\tilde{F}^{+i}\mathcal{W}F^{+\rho}) \\ &\quad -i\int_0^{z^-}dw^-\tilde{F}^{+\rho}(0)\mathcal{W}gF^+{}_\rho(w^-)\mathcal{W}F^{+i}(z^-)+i\int_0^{z^-}dw^-\tilde{F}^{+i}(0)\mathcal{W}gF^+{}_\rho(w^-)\mathcal{W}F^{+\rho}(z^-) \\ &\quad +\sum_q\epsilon^{ij}\left(\bar{q}(z^-)\gamma^+WgF^+{}_j(0)Wq(z^-)+\bar{q}(-z^-)\gamma^+WgF^+{}_j(0)Wq(-z^-)\right), \quad (53)\end{aligned}$$

where for simplicity we omitted the symbol $\langle p'|...|p\rangle$. (Inside this matrix element one can freely translate the z^- coordinate.) In deriving this, we used the Yang-Mills equation $D_\mu F^{\mu\nu}_a=\sum_q g\bar{q}\gamma^\nu t^a q$, the Bianchi identity $D_\mu\tilde{F}^{\mu\nu}=0$ and its variant

$$D_\mu\tilde{F}_{\nu\lambda}=D_\nu\tilde{F}_{\mu\lambda}+D_\lambda\tilde{F}_{\nu\mu}+\epsilon_{\mu\nu\lambda\beta}D_\alpha F^{\alpha\beta}. \quad (54)$$

The second term of the last expression connects to the unpolarized gluon PDF

$$\begin{aligned}\langle p'|\mathcal{D}_j(-\tilde{F}^{+j}\mathcal{W}F^{+i}+\tilde{F}^{+i}\mathcal{W}F^{+j})|p\rangle &= i\epsilon^{ij}\Delta_j\langle p'|F^{+\mu}\mathcal{W}F^+_\mu|p\rangle \\ &= -i\epsilon^{ij}\Delta_j(P^+)^2\int dx e^{-ixP^+z^-}xG(x), \quad (55)\end{aligned}$$

where we used an identity $\epsilon^{ik}a^j-\epsilon^{jk}a^i=\epsilon^{ij}a^k$ for any two-dimensional vector $a_\perp$. Equating the two results, we obtain

$$\begin{aligned}\frac{1}{2}x^2(\Delta G+\tilde{G}_g) &= xC_g(x)+xG(x)-2\int dx'\frac{\tilde{N}_F(x,x')}{x-x'} \\ &\quad -2\int dx' P\frac{N_F(x,x')}{x-x'}+2\sum_q\int dX\,\Psi_{Fq}(X,x). \quad (56)\end{aligned}$$

where we abbreviated $\Psi_F\left(X+\frac{x}{2},X-\frac{x}{2}\right)\to\Psi_F(X,x)$.

C. Lorentz invariant relation

We now eliminate $\tilde{G}_g$ from (56) using a Lorentz invariant relation. As in the quark case, we generalize (51) to off the light-cone in the following way

$$\begin{aligned}&\langle p'|\tilde{F}^{\alpha\rho}(-z/2)\mathcal{W}F^{\beta}_{\ \rho}(z/2)+\tilde{F}^{\beta\rho}\mathcal{W}F^{\alpha}_{\ \rho}|p\rangle\\&=-\frac{i}{2}\int dx e^{-ixP\cdot z}\bar{u}(p')\left((P^\alpha\gamma^\beta+P^\beta\gamma^\alpha)(\tilde{H}_g+x\tilde{G}_g)-2\frac{\gamma\cdot z}{P\cdot z}P^\alpha P^\beta x\tilde{G}_g+\cdots\right)\gamma_5 u(p).\end{aligned} \tag{57}$$

This immediately gives

$$\epsilon^{ij}\Delta_j x\tilde{G}_g(x)=-2\int\frac{dz^-}{2\pi}e^{ixz^-P^+}z^-\frac{\partial}{\partial z_i}\langle p'|\tilde{F}^{+\rho}(-z/2)\mathcal{W}F^{+}_{\ \rho}(z/2)|p\rangle\Big|_{z^\mu=\delta^\mu_- z^-}. \tag{58}$$

After a straightforward calculation, (58) takes the form

$$\begin{aligned}x\tilde{G}_g(x)&=-2\int dx'\,\frac{\partial}{\partial x}\tilde{N}_D(x,x')+2\int dx'\,P\frac{1}{x-x'}\Big(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\Big)\tilde{N}_F(x,x')\\&=\frac{d}{dx}(xC_g(x))+4\int dx'\,P\frac{1}{(x-x')^2}\tilde{N}_F(x,x'),\end{aligned} \tag{59}$$

where we have used (51) in the second equality. Inserting (59) into the equations of motion relation (56) we get the differential equation

$$\begin{aligned}x\frac{d}{dx}C_g(x)-C_g(x)\equiv&-x\Delta G(x)+2G(x)-4\int dx'\,P\frac{2x-x'}{x(x-x')^2}\tilde{N}_F(x,x')\\&-\frac{4}{x}\int dx'\,P\frac{1}{x-x'}N_F(x,x')+\frac{4}{x}\sum_q\int dX\,\Psi_{Fq}(X,x).\end{aligned} \tag{60}$$

Solving (60) with the boundary conditions $C_g(\pm 1)=0$ we obtain[7]

$$\begin{aligned}C_g(x)\ =\ &x\int_x^{\epsilon(x)}\frac{dx'}{x'^2}x'\Delta G(x')-2x\int_x^{\epsilon(x)}\frac{dx'}{x'^2}G(x')-4x\sum_q\int_x^{\epsilon(x)}\frac{dx'}{x'^3}\int dX\Psi_{Fq}(X,x')\\&+4x\int_x^{\epsilon(x)}dx_1\int dx_2 P\frac{N_F(x_1,x_2)}{x_1^3(x_1-x_2)}+4x\int_x^{\epsilon(x)}dx_1\int dx_2\frac{\tilde{N}_F(x_1,x_2)}{x_1^3(x_1-x_2)}P\frac{2x_1-x_2}{x_1-x_2}.\end{aligned} \tag{63}$$

The Wandzura-Wilczek (WW) part agrees with the one reported in [6]. As pointed out already, $C_g(x)$ is odd in x, so the first moment vanishes $\int_{-1}^1 dxC_g(x)=0$. We define the second moment as

$$\begin{aligned}C_g^{(2)}\ &\equiv\ \int_0^1 dxxC_g(x)\\&=\ \frac{1}{3}\int_0^1 dxx^2\Delta G(x)-\frac{2}{3}A_g-\frac{2}{3}\sum_q\int dxdx'\Psi_{Fq}(x,x')+\int dxdx'\frac{\tilde{N}_F(x,x')}{x-x'}.\end{aligned} \tag{64}$$

---

[7] One can derive the same result by evaluating

$$z^\mu\left(\frac{\partial}{\partial z^\mu}z_\beta\langle p'|\tilde{F}^{i\rho}(-z/2)\mathcal{W}F^{\beta}_{\ \rho}(z/2)+\tilde{F}^{\beta\rho}\mathcal{W}F^{i}_{\ \rho}|p\rangle-\frac{\partial}{\partial z_i}z_\beta\langle p'|\tilde{F}_\mu^{\ \rho}\mathcal{W}F^{\beta}_{\ \rho}+\tilde{F}^{\beta\rho}\mathcal{W}F_{\mu\rho}|p\rangle\right)\Big|_{z^\mu\to\delta^\mu_- z^-}, \tag{61}$$

in two ways, first substituting (57) and second using the equation of motion. The result is

$$\begin{aligned}\frac{1}{2}\frac{d}{dx}(x\tilde{H}_g(x))+\frac{x^2}{2}\frac{d}{dx}\frac{\tilde{G}_g}{x}\ =\ &\frac{d}{dx}\left(xG(x)+2\sum_q\int dX\Psi_{qF}(X,x)\right)\\&-2\int P\frac{dx'}{x-x'}\left(\frac{d}{dx}+\frac{d}{dx'}\right)N_F(x,x')-2\int\frac{dx'}{x-x'}\left(\frac{d}{dx}-\frac{d}{dx'}\right)\tilde{N}_F(x,x').\end{aligned} \tag{62}$$

Solving this for $\tilde{G}_g$ and substituting it into (56), one recovers (63). This is similar to the method used in [7].

where $A_g = \int_0^1 dx x G(x)$ is the momentum fraction of the nucleon carried by gluons. Combining (64) with (40), we finally arrive at a new momentum sum rule

$$1 = \sum_q A_{q+\bar{q}} + A_g \tag{65}$$

$$= \Delta\Sigma^{(3)} + \frac{1}{2}\Delta G^{(3)} - 3C_q^{(2)} - \frac{3}{2}C_g^{(2)}$$
$$-\frac{3}{2}\int dx dx' \left[\sum_q \left(\frac{2x}{x-x'}\tilde{\Psi}_{Fq}(x,x') + \Psi_{Fq}(x,x')\right) - \frac{\tilde{N}_F(x,x')}{x-x'}\right] + \sum_q \frac{m_q}{M} H_{1q}^{(2)}. \tag{66}$$

where $\Delta\Sigma^{(3)} \equiv \int_0^1 dx x^2 \Delta\Sigma(x)$ and $\Delta G^{(3)} \equiv \int_0^1 dx x^2 \Delta G(x)$. This is the main result of this paper. Let us make a comparison with the spin sum rules (1), (2) which can be rewritten in the form

$$\frac{1}{2} = \frac{1}{2}\sum_q (A_{q+\bar{q}} + B_{q+\bar{q}}) + \frac{1}{2}(A_g + B_g) \tag{67}$$

$$= \frac{1}{2}\Delta\Sigma^{(1)} + \Delta G^{(1)} + L_q^{(1)} + L_g^{(1)}, \tag{68}$$

where $B_{q,g}$ are the second moments of the GPD $E_{q,g}$ and $J_{q,g} = \frac{1}{2}(A_{q,g} + B_{q,g})$ are the quark and gluon angular momenta in the Ji sum rule (2). The correspondence between (65), (66) and (67), (68) is obvious. In accordance with the existence of two well-known spin sum rules, we have now established the second momentum sum rule (66). Remarkably, this has been made possible only by introducing the spin-orbit correlations. We however note that, in contrast to the JM sum rule, genuine twist-three distributions and a mass term explicitly remain in (66). Also, the physical meaning of individual terms is less straightforward to interpret. Nevertheless, it is a novel and highly nontrivial way to characterize the partonic momentum structure in the nucleon.

Incidentally, the integral of (56) leads to another formula

$$\int_{-1}^1 dx x^2 \tilde{G}_g(x) = \frac{4}{3}A_g - \frac{2}{3}\int_0^1 dx x^2 \Delta G(x) + \frac{4}{3}\sum_q \int dx dx' \Psi_{Fq}(x,x'), \tag{69}$$

while $\int dx x \tilde{G}_g(x) = 0$ since $x\tilde{G}_g(x)$ is odd in x. Compare with (42). We can eliminate Ψ_F from these equations and obtain

$$\frac{3}{2}\int_{-1}^1 dx x^2 \left(4\sum_q \tilde{G}_q(x) - \tilde{G}_g(x)\right) = 4\sum_q A_{q+\bar{q}} - 2A_g - 4\Delta\Sigma^{(3)} + \Delta G^{(3)} - \sum_q \frac{4m_q}{M} H_{1q}^{(2)}. \tag{70}$$

The right hand side is expressed purely in terms of twist-two quantities.

V. DISCUSSIONS AND CONCLUSIONS

Spinless hadron—Unlike the Jaffe-Manohar sum rule (1), the new sum rule (66) is meaningful also for spinless hadrons and nuclei such as the pions and the helium-4 nucleus. The derivation in the previous sections is essentially unchanged except that the helicity PDFs $\Delta q, \Delta\bar{q}, \Delta G$ are set to zero everywhere. The resulting formulas are more concise

$$C_q = -\frac{N_q}{2} - \int \frac{dx dx'}{x-x'}\tilde{\Psi}_{Fq}(x,x') + \frac{m_q}{2M} H_{1q}^{(1)}, \tag{71}$$

$$1 = -3C_q^{(2)} - \frac{3}{2}C_g^{(2)} - \frac{3}{2}\int dx dx' \left[\sum_q \left(\frac{2x}{x-x'}\tilde{\Psi}_{Fq}(x,x') + \Psi_{Fq}(x,x')\right) - \frac{\tilde{N}_F(x,x')}{x-x'}\right] + \sum_q \frac{m_q}{M} H_{1q}^{(2)}. \tag{72}$$

The numerical impact of the quark mass term has been studied in [29] for the pion.

Small-x—From (37) and (63), one can immediately deduce that the small-x limit of $C_{q,g}(x)$ is dominated by the unpolarized PDFs

$$\begin{aligned} C_q(x) &\approx C_{\bar{q}}(x) \approx -x\int_x^1 dx' \frac{dx'}{x'^2} q(x') \approx -\frac{1}{2+c} q(x), \\ C_g(x) &\approx -2x\int_x^1 dx \frac{dx'}{x'^2} G(x') \approx -\frac{2}{2+c} G(x), \end{aligned} \tag{73}$$

where in the last expression we assumed a Regge behavior $q(x) \sim G(x) \sim 1/x^{1+c}$ $(0 < c < 1)$. If $c \ll 1$ as suggested by perturbation theory $c \propto \alpha_s$, this gives $C_q(x) \approx -\frac{1}{2} q(x)$ and $C_g(x) \approx -G(x)$, in agreement with independent calculations [4, 30]. Since $q(x), G(x) > 0$ at small-x, $C_{q,g}(x)$ are negative, meaning that the helicity and OAM of individual quarks and gluons are anti-aligned [4]. Related analyses of the OAM distributions $L_{q,g}(x)$ [31–33] suggest that the genuine twist-three terms do not change this conclusion, but it is worthwhile to explicitly confirm this.

In conclusion, we have derived exact formulas for the spin-orbit correlations for quarks $C_q(x)$ (37) and gluons $C_g(x)$ (63) which completely reveal their twist structure and are thus amenable to further first-principle analyses such as the effect of QCD evolution (cf. [31]). The study of these distributions, especially at small-x, is interesting in its own right as they can probe quantum entanglement between the helicity and OAM [4]. We have also pointed out that the new twist-three functions $\Psi_F, \tilde{\Psi}_F, N_F, \tilde{N}_F$ are at least partially related to the ETQS functions and the three-gluon correlators relevant to transverse single spin asymmetry. This should be a subject of more careful scrutiny.

Based on these results, we have derived a novel sum rule (66) which may be viewed as the momentum version of the Jaffe-Manohar spin sum rule (1). We emphasize that (66) is no less fundamental to the nucleon momentum structure than (1) is to the nucleon spin structure. We hope our finding further motivates experimental studies on spin-orbit correlations in the future [6, 34–36].

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