THE ARENS–MICHAEL ENVELOPE OF A SOLVABLE LIE ALGEBRA IS A HOMOLOGICAL EPIMORPHISM

O. YU. ARISTOV

Abstract. The Arens–Michael envelope of the universal enveloping algebra of a finite-dimensional complex Lie algebra is a homological epimorphism if and only if the Lie algebra is solvable. The necessity was proved by Pirkovskii in [Proc. Amer. Math. Soc. 134, 2621–2631, 2006]. We prove the sufficiency.

Introduction

We are interested in the question under what conditions, for a given finite-dimensional complex Lie algebra $\mathfrak{g}$, the universal completion (called the Arens–Michael envelope) of $U(\mathfrak{g})$ (its universal enveloping algebra) is a homological epimorphism. This question is implicitly contained in the works of J. Taylor published in early 70s [32, 33, 34, 35] and explicitly formulated by Pirkovskii in [27, § 9, Problem 1].

Homological epimorphisms. Flat homomorphisms play an important role in algebra and analysis but in some respects the requirement of flatness is too strong and must be replaced by a weaker condition. The concept of homological epimorphism was introduced J. Taylor [34] as a weakened form of flatness under the name of “absolute localization” and has been rediscovered several times in different contexts and under different names — “lifting”, “stably flat homomorphism”, “isocohomological morphism”, “homotopy epimorphism”; see references in [10, Remark 3.16]. An important property for applications is that homological epimorphisms preserve homologies and cohomologies [34, Propositions 1.4 and 1.7]. We are interested in the functional analytic version of this concept that comes back to Taylor and consider homological epimorphisms of complete locally convex algebras with jointly continuous multiplication (we call them $\widehat{\otimes}$-algebras); see Definition 4.2.

An example of the use of homological epimorphisms is contained in Meyer’s preprint [24], where Connes’s calculation of cyclic cohomology of a smooth non-commutative torus in [15, § 6] is simplified. In addition, it was noted by Pirkovskii in [25] that flat homomorphisms are not sufficient for characterization of open embeddings of Stein varieties but a weak version of the notion of homological epimorphism is adequate for this task; for an extension of results to general Stein spaces see [10, 13]. Another motivation for studying of homological epimorphisms in the context of functional analysis comes from non-commutative spectral theory; see the introduction to [27] for a detailed discussion and interpretation of Taylor’s ideas.

Arens–Michael envelopes. The idea to study Arens–Michael envelopes and, in particular, to look for conditions under which they are homological homomorphisms also belongs to Taylor. Recall that the Arens–Michael envelope of a topological algebra A is the universal completion with respect to the class of Banach algebras (equivalently, the completion relative to the locally convex topology determined by all continuous submultiplicative seminorms on A). Roughly speaking, this completion is responsible for continuous representations of A on Banach spaces; see details in § 2. A prototype example of an Arens–Michael envelope is the embedding of the polynomial algebra $\mathbb{C}[z_1,\dots,z_n]$ to the algebra of entire functions $\mathcal{O}(\mathbb{C}^n)$.

Taylor found explicit descriptions of the Arens–Michael envelopes of certain finitely-generated associative algebras and proved that in some cases the envelope is a homological epimorphism; also, he provided first counterexamples [34].

History and the main result. Let $U(\mathfrak{g})$ be the universal enveloping algebra of a finite-dimensional complex Lie algebra $\mathfrak{g}$. The question of whether the Arens–Michael envelope homomorphism $U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ is a homological epimorphism has been studied since 70s. The affirmarive answer in the case when $\mathfrak{g}$ is abelian, and the negative answer in the case when it is semisimple, was given by Taylor in [34]. Three decades later, in [26], Pirkovskii showed that the *solvability a necessary condition*. At the same time, the study of the nilpotent case was started by Dosiev [16, 17]. It was continued by Pirkovskii in [27] and recently completed by the author in [2], where it was shown that the nilpotency is a sufficient condition. Thus, only the case when $\mathfrak{g}$ is solvable but not nilpotent remained uninvestigated (with one exception — the two-dimensional non-abelian algebra; see [27]). Here we prove that the *solvability is also a sufficient condition*; see Theorem 4.4. Thus, the final answer is that $U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ is a homological epimorphism if and only if $\mathfrak{g}$ is solvable.

The following papers of the author precede this work.

- In [2], the nilpotent case is considered. The structure of $\widehat{U}(\mathfrak{g})$ is described and proved that Arens–Michael envelope is a homological epimorphism (using a method different from that applied in this article).
- In [3], the structure of the algebra of analytic functionals on a connected complex Lie group is discussed. In particular, the results there can be applied to $\widehat{U}(\mathfrak{g})$ for a general finite-dimensional Lie algebra. Some improvements is obtained in [8].
- A decomposition into iterated analytic smash products (which is essential here) is constructed in [9].
- The decomposition mentioned above exists not only for the Arens–Michael envelope but for other completions. Some preliminary work for these completions is done in [6, 7]. Formally, [9] is based on those preliminary results but, in fact, the case of the Arens–Michael envelope can be studied without reference to [6, 7].

Our main result is of independent interest but, on the other hand, this text is a part of a big project on completions of universal enveloping algebras and algebras of analytic functionals. Homological properties related with the first topic (universal enveloping algebras) is a subject of this paper. The author plans to study homological properties of algebras of analytic functionals and their completions in a separate article.

The result was announced at the seminar "Algebras in Analysis" in Moscow State University in 2018. The author later discovered a gap in the proof proposed at the time. This text contains an argument in which the gap is filled.

Contents

1. Main ideas of the proof

The proof is based on two circles of ideas: author's results on decomposition of $\widehat{U}(\mathfrak{g})$ into an iterated analytic smash products obtained in [9] and a powerful technique of relative homological epimorphisms developed by Pirkovskii in [28]. In fact, the analytic part of preliminary work is contained in [9] and preceding articles. Pirkovskii's technique, which is applied below, is more algebraic in nature.

When studying homological epimorphisms, the question of finding a convenient projective resolution is important. To construct such a resolution one can use a decomposition of $U(\mathfrak{g})$ into an iterated Ore extension. In group homology, a construction of a resolution of the trivial module of a semi-direct product is well known (see, e.g., [21, Chapter 5, § 2, pp. 250–251]). It is not hard to generalize this construction for cocommutative Hopf algebras and transfer it into the context of functional analysis. (This topic will be discussed elsewhere.) Unfortunately, it is not clear whether or not the tensor product functor (4.1) sends such projective resolution to a similar resolution. In [28], to overcome this kind of difficulty it was proposed to use relative homological homomorphisms instead of usual. We show below that, with an appropriate modification, this idea works well in our case.

Outline of Pirkovskii's results. In [28, Theorem 9.12], Pirkovskii proved that the Arens–Michael envelope is a homological epimorphism for a number of finitely-generated non-commutative algebras. In this paper, we are interested only in generalizing of Parts (vi) and (vii) of the above theorem, the parts that concern the two-dimensional non-abelian and three-dimensional Heisenberg Lie algebra. This section contains an extract of argument in these cases. We postpone the necessary definitions until § 4.

In both the two-dimensional and three-dimensional cases considered in [28], a decomposition of $\mathfrak{g}$ into a semidirect sum induces a decomposition of $U(\mathfrak{g})$ into an Ore extension of a commutative algebra. Namely, $U(\mathfrak{g})$ is isomorphic to $\mathbb{C}[x][t;\delta]$, where $\delta = y\frac{d}{dy}$, in the first case and $\mathbb{C}[x,y][t;\delta]$, where $\delta = y\frac{\partial}{\partial x}$ in the second. As a corollary, $\widehat{U}(\mathfrak{g})$ can be represented as an analytic Ore extension $\mathcal{O}(\mathbb{C}, R_\delta; \widehat{\delta})$, where R_δ is a certain completion of R and $\widehat{\delta}$ is the extension δ to R_δ [28, Theorem 5.1]. (Here and below $\mathcal{O}$ stands for holomorphic functions.)

Pirkovskii used a two-step argument. First he showed that $R \to R_\delta$ is a homological homomorphism and next that ι is a one-sided relative homological homomorphism with respect to R_δ. Both conditions are included into the hypotheses of the following result.

Theorem 1.1. [28, Theorem 9.1(ii)] *Let $(f,g)\colon (A,R) \to (B,S)$ be an R-S-homomorphism from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B. Suppose that*

(1) *g is a homological epimorphism;*
(2) *f is a left or right relative homological epimorphism;*
(3) *A is projective in R-mod and B is projective in S-mod;*

(4) *A is an R-$\widehat{\otimes}$-algebra of (f_2)-finite type.*
Then f is a homological epimorphism.

In the case of low-dimensional Lie algebras, consider R-R_δ-homomorphism

$$(\iota, g)\colon (U(\mathfrak{g}), R) \to (\widehat{U}(\mathfrak{g}), R_\delta),$$

where g denotes the $R \to R_\delta$. To prove that ι is a homological epimorphism it suffices to check Conditions (1)–(4) in Theorem 1.1 for (ι, g).

For Ore extensions and analytical Ore extensions, Condition (3) easily follows from definitions. An Ore extension of R is relatively (f_2)-quasi-free R-algebra [28, Example 8.1], which in particular implies that it is of (f_2)-finite type [28, Definition 8.5 and Proposition 7.3.]. Thus Condition (4) holds. (Note that the use of relatively quasi-free algebras was an important innovation in this topic.) Checking of Conditions (1) and (2) is more challenging.

To prove Condition (2) Pirkovskii applied the following result.

Theorem 1.2. [28, Theorem 7.6(ii)] *Let A, R, S be $\widehat{\otimes}$-algebras and $g : R \to S$ an epimorphism of $\widehat{\otimes}$-algebras. Suppose that A is an R-$\widehat{\otimes}$-algebra and its Arens–Michael envelope $\widehat{A}$ is an S-$\widehat{\otimes}$-algebra such that the pair (ι_A, g) is an R-S-homomorphism from A to $\widehat{A}$. Assume also that A is relatively quasi-free over R. Then $\iota_A\colon A \to \widehat{A}$ is a two-sided relative homological epimorphism.*

It is essential in the proof that $\iota_A : A \to \widehat{A}$ satisfies to the unique extension property for derivations (Property (UDE)); see the corresponding definition in § 3.

Verifying Condition (1) is arduous since $R \to R_\delta$ is not an Arens–Michael envelope in general. To surmount this challenge Pirkovskii applied another technique, which we do not use in this paper; see details in the proof of Parts (vi) and (vii) of Theorem 9.12 in [28].

In the high dimensional case, it is natural to use induction. Specifically, if an algebra A has a chain of subalgebras $\mathbb{C} = R_0 \subset R_1 \subset \cdots \subset R_n = A$, where each of R_i is relatively quasi-free over R_{i-1}, then an iterative application of Theorem 1.1 enable us with a tool for proving that $A \to \widehat{A}$ is a homological epimorphism; see the introduction to [28]. This plan works well in some situations but unfortunately cannot be applied to all high-dimensional solvable Lie algebras. Indeed, a finite-dimensional solvable complex Lie algebra $\mathfrak{g}$ admits an iterated semidirect sum decomposition,

$$\mathfrak{g} = ((\cdots(\mathfrak{f}_1 \rtimes \mathfrak{f}_2) \rtimes \cdots) \rtimes \mathfrak{f}_n,$$

where $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ are 1-dimensional. Moreover, this decomposition actually induces an decomposition of the universal enveloping algebra into an iterated Ore extension; see, e.g., [23, pp. 33–34, 1.7.11(iv)]. Nevertheless, in this case, $R_{n-1} \to (R_{n-1})_\delta$ is not usually an Arens–Michael envelope and, moreover, $(R_{n-1})_\delta$ cannot be represented as an analytic Ore extension.

Plan of the proof. Our strategy is to apply Theorem 1.1 using iterations. But the scheme proposed in [28] needs to be modified, as is obvious from previous considerations.

1. We use iterated analytic smash products instead of analytic Ore extensions; see Theorem 2.5. An analytic Ore extension with trivial twisting is an analytic smash product but when the twisting is non-trivial this is not the case. (Thus most of parts of Theorem

9.12 in [28] are not covered by our approach.) Some preliminaries on Hopf $\widehat{\otimes}$-algebras, analytic smash products and Arens–Michael envelopes is contained in § 2.

2. We use completions of $\mathbb{C}[z]$ other than $\mathcal{O}(\mathbb{C})$, the algebras $\mathfrak{A}_s$; see definition in (2.7). The essential point is that $\mathcal{O}(\mathbb{C}) \to \mathfrak{A}_s$ and some induced homomorphisms of analytic Ore extensions satisfy Property (UDE). To prove theses facts we apply an auxiliary result on pushouts; see Theorem 3.4 and preliminary results in § 2.

3. We get an Arens–Michael envelope only on the final step of iteration. So we have to replace Theorem 1.2 by a more general result on homomorphisms satisfying Property (UDE), Theorem 4.7. Combining it with a decomposition result essentially proved in [9] (Theorem 2.5), we finally deduce Theorem 4.4.

2. Hopf $\widehat{\otimes}$-algebras, analytic smash products and Arens–Michael envelopes

Consider the bifunctor $(-)\,\widehat{\otimes}\,(-)$ of complete projective tensor product on the category of complete locally convex spaces and Hopf algebras in the corresponding symmetric monoidal category. We call them Hopf $\widehat{\otimes}$-*algebras* (read 'topological Hopf algebras'); see [22] or [27]. Also, $\widehat{\otimes}$-algebras and $\widehat{\otimes}$-(bi)modules, i.e., complete locally convex algebras and (bi)modules with jointly continuous multiplication, are considered. We assume that each algebra contains an identity and each module is unital. A $\widehat{\otimes}$-(bi)module over a $\widehat{\otimes}$-algebra A is referred as an A-$\widehat{\otimes}$-(bi)module. We also assume that $\widehat{\otimes}$-algebra homomorphisms preserve identity and are continuous (as well as $\widehat{\otimes}$-module morphisms).

Generalized Sweedler notation. The Sweedler notation is widely used in the Hopf algebra theory instead of the tensor notation because the latter is not always convenient. It was noted in [1, § 2.4] that this notation can be generalized to topological Hopf algebras. A version of the generalized Sweedler notation sufficient for our purposes is described in [9] and we briefly recall it here.

In the classical Sweedler notation, the comultiplication Δ on a Hopf algebra has the form

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)}. \tag{2.1}$$

Here an arbitrary representation of $\Delta(h)$ is taken.

Before using this type of notation for Hopf $\widehat{\otimes}$-algebras note that, in the Fréchet space context, we can treat (2.1) not as a finite sum but as a convergent series. However, in general this is not always possible and we write $\Delta(h)$ as the limit of a net of finite sums of elementary tensors:

$$\Delta(h) = \lim_{\nu} \sum_{i=1}^{n_\nu} h^{\nu,i}_{(1)} \otimes h^{\nu,i}_{(2)}.$$

In the case of a Hopf $\widehat{\otimes}$-algebra, (2.1) is a short notation for such a representation.

For example, in the full form, the coassociativity axiom, $(1\otimes\Delta)\Delta(h)=(\Delta\otimes 1)\Delta(h)$, can be written as

$$\lim_\nu\left(\sum_{i=1}^{n_\nu} h^{\nu,i}_{(1)}\otimes\left(\lim_\mu\sum_{j=1}^{m_\mu}(h^{\nu,i}_{(2)})^{\mu,j}_{(1)}\otimes(h^{\nu,i}_{(2)})^{\mu,j}_{(2)}\right)\right)=$$
$$=\lim_\nu\left(\left(\lim_\lambda\sum_{j=1}^{l_\lambda}(h^{\nu,i}_{(2)})^{\lambda,j}_{(1)}\otimes(h^{\nu,i}_{(2)})^{\lambda,j}_{(2)}\right)\otimes\sum_{i=1}^{n_\nu}h^{\nu,i}_{(2)}\right), \quad (2.2)$$

where

$$\Delta(h^{\nu,i}_{(1)})=\lim_\lambda\sum_{j=1}^{l_\lambda}(h^{\nu,i}_{(2)})^{\lambda,j}_{(1)}\otimes(h^{\nu,i}_{(2)})^{\lambda,j}_{(2)}\quad\text{and}\quad\Delta(h^{\nu,i}_{(2)})=\lim_\mu\sum_{j=1}^{m_\mu}(h^{\nu,i}_{(2)})^{\mu,j}_{(1)}\otimes(h^{\nu,i}_{(2)})^{\mu,j}_{(2)}.$$

In the generalized Sweedler notation, it takes the same form as in the classical Sweedler notation:

$$\sum h_{(1)}\otimes\left(\sum(h_{(2)})_{(1)}\otimes(h_{(2)})_{(2)}\right)=\left(\sum(h_{(1)})_{(1)}\otimes(h_{(1)})_{(2)}\right)\otimes h_{(2)}.$$

By the iterated limit theorem [20, Chapter 2, p. 69, Theorem 4], both iterated limits in (2.2) can be replaced by a simple limit of a net. In particular, this means that the iteration of the comultipication can be written as

$$(1\otimes\Delta)\Delta(h)=\sum h_{(1)}\otimes h_{(2)}\otimes h_{(3)},$$

where a simple limit is also implied.

Also, in this notation, the antipode axiom for a Hopf $\widehat{\otimes}$-algebra takes the form

$$\sum S(h_{(1)})h_{(2)}=\varepsilon(h)1=\sum h_{(1)}S(h_{(2)}).$$

This formula stands for

$$\lim_\nu\sum_{i=1}^{n_\nu}S(h^{\nu,i}_{(1)})h^{\nu,i}_{(2)}=\varepsilon(h)1=\lim_\nu\sum_{i=1}^{n_\nu}h^{\nu,i}_{(1)}S(h^{\nu,i}_{(2)}).$$

Analytic smash products. In this section, we recall the necessary information on analytic smash products, which were introduced in [26]. In the definitions we follow [9], where some historical remarks can also be found.

Let H be a Hopf $\widehat{\otimes}$-algebra and A a $\widehat{\otimes}$-algebra that is a left H-$\widehat{\otimes}$-module. Then $A\mathbin{\widehat{\otimes}}A$ and $\mathbb{C}$ are left H-$\widehat{\otimes}$-modules with respect to

$$h\cdot(a\otimes b):=\sum(h_{(1)}\cdot a)\otimes(h_{(2)}\cdot b)\qquad(h\in H,\ a,b\in A);$$
$$h\cdot\lambda:=\varepsilon(h)\lambda,\qquad(h\in H,\ \lambda\in\mathbb{C}).$$

Recall that A is called a (left) *H-$\widehat{\otimes}$-module algebra* if the linearized multiplication $\mu\colon A\mathbin{\widehat{\otimes}}A\to A$ and the unit map $\mathbb{C}\to A$ are left H-module morphisms, i.e.,

$$h\cdot(ab)=\sum(h_{(1)}\cdot a)(h_{(2)}\cdot b)\quad\text{and}\quad h\cdot 1=\varepsilon(h)1\qquad(h\in H,\ a,b\in A). \qquad (2.3)$$

For a $\widehat{\otimes}$-algebra homomorphism $\psi\colon H\to B$ consider the *adjoint action* H on B:

$$h\cdot b:=\sum\psi(h_{(1)})\,b\,\psi(S(h_{(2)}))\qquad(h\in H,\ b\in B). \qquad (2.4)$$

It is easy to see that B is an H-$\widehat{\otimes}$-module algebra with respect to this action.

Definition 2.1. Let H be a Hopf $\widehat{\otimes}$-algebra and A an H-$\widehat{\otimes}$-module algebra. The *analytic smash product* $A\mathbin{\widehat{\#}}H$ is defined as a $\widehat{\otimes}$-algebra endowed with $\widehat{\otimes}$-algebra homomorphisms $i\colon A\to A\mathbin{\widehat{\#}}H$ and $j\colon H\to A\mathbin{\widehat{\#}}H$ such that the following conditions holds.

(A) i is a $\widehat{\otimes}$-module algebra homomorphisms with respect to the adjoint action associated with j.

(B) For every $\widehat{\otimes}$-algebra B, $\widehat{\otimes}$-algebra homomorphism $\psi\colon H\to B$ and H-$\widehat{\otimes}$-module algebra homomorphism $\varphi\colon A\to B$ (with respect to the adjoint action associated with ψ) there is a unique $\widehat{\otimes}$-algebra homomorphism θ such that the diagram

$$\begin{tikzcd} & A\mathbin{\widehat{\#}}H \arrow[dd, dashed, "\theta"] & \\ A \arrow[ur, "i"] \arrow[dr, "\varphi"'] & & H \arrow[ul, "j"'] \arrow[dl, "\psi"] \\ & B & \end{tikzcd}$$

is commutative.

The explicit construction is as follows. The formula

$$(a\otimes h)(a'\otimes h')=\sum a(h_{(1)}\cdot a')\otimes h_{(2)}h' \qquad (h,h'\in H;\ a,a'\in A). \tag{2.5}$$

determines a multiplication on $A\mathbin{\widehat{\otimes}}H$. Endowed with the maps $i: a\mapsto a\otimes 1$ and $j\colon h\mapsto 1\otimes h$ and this multiplication, $A\mathbin{\widehat{\otimes}}H$ is an analytic smash product.

Remark 2.2. In the case when $H=\mathcal{O}(\mathbb{C})$ and act on A by a derivation, the analytic smash product coincides with the corresponding analytic Ore extension [28, Remarks 4.4 and 4.2]. The same holds in the pure algebraic case, i.e., when $\mathbb{C}[z]$ acts by a derivation.

The following lemma follows directly from the definitions.

Lemma 2.3. [9, Lemma 3.10] *Let H and K be Hopf $\widehat{\otimes}$-algebras, R a H-$\widehat{\otimes}$-module algebra and S a K-$\widehat{\otimes}$-module algebra. If $\beta: H\to K$ is a Hopf $\widehat{\otimes}$-algebra homomorphism and $\alpha\colon R\to S$ is a $\widehat{\otimes}$-algebra homomorphism that is an H-$\widehat{\otimes}$-module morphism (i.e., $\alpha(h\cdot r)=\beta(h)\cdot\alpha(r)$ when $h\in H$, $r\in R$), then the formula*

$$\alpha\mathbin{\widehat{\#}}\beta\colon R\mathbin{\widehat{\#}}H\to S\mathbin{\widehat{\#}}K\colon r\otimes h\mapsto \alpha(r)\otimes\beta(h) \tag{2.6}$$

determines a $\widehat{\otimes}$-algebra homomorphism.

Smash product decomposition of Arens–Michael envelopes. In what follows we need a family of Hopf $\widehat{\otimes}$-algebras importance of which for finding Arens–Michael envelopes was discovered in [4] and [8].

For $s\in[0,+\infty)$ put

$$\mathfrak{A}_s:=\Big\{a=\sum_{n=0}^{\infty}a_nz^n\colon \|a\|_{r,s}:=\sum_{n=0}^{\infty}|a_n|\,\frac{r^n}{n!^s}<\infty\ \forall r>0\Big\}, \tag{2.7}$$

where z is treated as a formal variable, and endow $\mathfrak{A}_s$ with the topology determined by the seminorm family $(\|\cdot\|_{r,s};\ r>0)$. Denote also $\mathbb{C}[[z]]$, the algebra of all power series in z, by $\mathfrak{A}_\infty$.

Lemma 2.4. *Let $s \in [0, +\infty]$. Then $\mathfrak{A}_s$ is a Hopf $\widehat{\otimes}$-algebra with respect to the operations continuously extended from $\mathbb{C}[z]$ and so the natural embedding $\beta\colon \mathbb{C}[z] \to \mathfrak{A}_s$ is a Hopf $\widehat{\otimes}$-algebra homomorphism.*

The assertion of the lemma is proved in [8, Example 2.4] when $s \neq \infty$ and can be directly checked when $s = \infty$.

Recall that an *Arens–Michael algebra* is a topological algebra whose topology can be determined by a family of submultiplicative seminorms. The *Arens–Michael envelope* of a topological $\mathbb{C}$-algebra A is a pair $(\widehat{A}, \iota_A)$, where $\widehat{A}$ is an Arens–Michael algebra and ι_A is a continuous homomorphism $A \to \widehat{A}$ such that for every Arens–Michael algebra B and every continuous homomorphism $\varphi\colon A \to B$ there is a unique continuous homomorphism $\widehat{\varphi}\colon \widehat{A} \to B$ making the diagram

$$\begin{tikzcd} A \arrow[r, "\iota_A"] \arrow[dr, "\varphi"'] & \widehat{A} \arrow[d, dashed, "\widehat{\varphi}"] \\ & B \end{tikzcd}$$

commutative. More concrete, we can take as ι_A the completion homomorphism with respect to the topology determined by all possible (continuous) submultiplicative seminorms on A.

In the case when A is an arbitrary associative $\mathbb{C}$-algebra, we can endow it with the strongest locally convex topology or, equivalently, take arbitrary homomorphism φ in the definition. In the Lie algebra case, we can replace $U(\mathfrak{g})$ by $\mathfrak{g}$ in the diagram (assuming that φ is a Lie algebra homomorphism) and say that $\widehat{U}(\mathfrak{g})$ is the Arens–Michael envelope of $\mathfrak{g}$.

Our proof of the main result, Theorem 4.4, is based on the following structural theorem, which is easily implied by results in [9].

Theorem 2.5. *Let $\mathfrak{g}$ be a finite-dimensional solvable complex Lie algebra and fix an iterated semidirect sum decomposition $\mathfrak{g} = ((\cdots(\mathfrak{f}_1 \rtimes \mathfrak{f}_2) \rtimes \cdots) \rtimes \mathfrak{f}_n$, where $\mathfrak{f}_1, \dots, \mathfrak{f}_n$ are 1-dimensional. Then there is a non-increasing sequence $i_1, \dots, i_n$ in $[0, \infty]$ such that*

$$\widehat{U}(\mathfrak{g}) \cong (\cdots(\mathfrak{A}_{i_1} \mathbin{\widehat{\#}} \mathfrak{A}_{i_2}) \mathbin{\widehat{\#}} \cdots) \mathbin{\widehat{\#}} \mathfrak{A}_{i_n} \tag{2.8}$$

Moreover, the homomorphism $U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ is compatible with the decomposition

$$U(\mathfrak{g}) \cong (\cdots(U(\mathfrak{f}_1) \# U(\mathfrak{f}_2)) \# \cdots) \# U(\mathfrak{f}_n)$$

induced by the decomposition of $\mathfrak{g}$.

Here the compatibility of decompositions means that at each step we have a smash product with a homomorphism of the form β considered in Lemma 2.4 as described in Lemma 2.3. (Note that $U(\mathfrak{f}_i) \cong \mathbb{C}[z]$ for every i.)

Proof. We use reduction to algebras of analytic functionals. Let G be a simply connected complex Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. Also, let $\mathscr{A}(G)$ denote the corresponding algebra of analytic functionals and $\widehat{\mathscr{A}(G)}$ the Arens–Michael algebra of $\mathscr{A}(G)$. Then the natural embedding $U(\mathfrak{g}) \to \mathscr{A}(G)$ induces a topological isomorphism $\widehat{U}(\mathfrak{g}) \to \widehat{\mathscr{A}(G)}$ [2, Proposition 2.1]. So we can apply results on $\widehat{\mathscr{A}(G)}$, Theorems 6.5 and 6.4 in [9], which give the existence of the decomposition in (2.8) and the compatibility with the decomposition of $U(\mathfrak{g})$, respectively. $\square$

Smash product and pushouts. We use the results in this section when proving Theorem 3.4.

Proposition 2.6. *Let H be a Hopf $\widehat{\otimes}$-algebra and $\alpha : R \to S$ an H-$\widehat{\otimes}$-module algebra homomorphism. If α has dense range, then*

$$\begin{tikzcd} R \arrow[r, "i_R"'] \arrow[d, "\alpha"'] & R \mathbin{\widehat{\#}} H \arrow[d, "\alpha \mathbin{\widehat{\#}} 1"] \\ S \arrow[r, "i_S"'] & S \mathbin{\widehat{\#}} H, \end{tikzcd}$$

where i_R and i_S are the canonical homomorphisms, is a pushout diagram.

Proof. Let $\varphi : S \to B$ and $\chi : R \mathbin{\widehat{\#}} H \to B$ be $\widehat{\otimes}$-algebra homomorphisms such that $\varphi\alpha = \chi i_R$. We want to use the universal property in Definition 2.1. Put $\psi := \chi j$, where $j : H \to R \mathbin{\widehat{\#}} H$ is the canonical homomorphism. It suffices to find a $\widehat{\otimes}$-algebra homomorphism θ making the diagram

$$\begin{tikzcd} R \arrow[r, "i_R"] \arrow[d, "\alpha"'] & R \mathbin{\widehat{\#}} H \arrow[d, "\alpha \mathbin{\widehat{\#}} 1"'] \arrow[ddr, "\chi"] & H \arrow[l, "j"] \arrow[dd, "\psi"] \\ S \arrow[r, "i_S"] \arrow[drr, "\varphi"'] & S \mathbin{\widehat{\#}} H \arrow[dr, dashed, "\theta"] & \\ & & B \end{tikzcd}$$

commutative and prove that it is unique.

If $h \in H$ and $r \in R$, then by the definition of the adjoint action in (2.4), we have that

$$h \cdot \varphi\alpha(r) = \sum \psi(h_{(1)})\, \varphi\alpha(r)\, \psi(S(h_{(2)})) = \chi\left(\sum j(h_{(1)})\, i_R(r)\, j(S(h_{(2)}))\right) = \chi i_R(h \cdot r).$$

(The last equality follows from the fact that i_R is an H-$\widehat{\otimes}$-module morphism with respect to the adjoint action.) Thus $h \cdot \varphi\alpha(r) = \varphi\alpha(h \cdot r)$. Hence $\varphi\alpha$ is an H-$\widehat{\otimes}$-module morphism.

Further, by the hypothesis, α is also an H-$\widehat{\otimes}$-module morphism and so is φ since the range of α is dense. Hence, by the universal property of smash product written for $S \mathbin{\widehat{\#}} H$, there is a unique $\widehat{\otimes}$-algebra homomorphism θ such that $\theta i_S = \varphi$ and $\theta(\alpha \mathbin{\widehat{\#}} 1)j = \psi$.

Since $\varphi\alpha$ is an H-$\widehat{\otimes}$-module algebra homomorphism, it follows from the universal property of smash product written for $R \mathbin{\widehat{\#}} H$ that a homomorphism χ such that $\chi i_R = \varphi\alpha$ and $\chi j = \psi$ is unique. Therefore $\theta(\alpha \mathbin{\widehat{\#}} 1) = \chi$.

To complete the proof we need to show that θ such that $\theta i_S = \varphi$ and $\theta(\alpha \mathbin{\widehat{\#}} 1) = \chi$ is unique. If θ' is another $\widehat{\otimes}$-algebra homomorphism with these properties, then $\theta'(\alpha \mathbin{\widehat{\#}} 1)i_R = \varphi\alpha$ and $\theta'(\alpha \mathbin{\widehat{\#}} 1)j = \psi$. The uniqueness statement above implies that $\theta' = \theta$. □

Proposition 2.7. *Let $\beta\colon H\to K$ be a Hopf $\widehat{\otimes}$-algebra homomorphism and S a K-$\widehat{\otimes}$-module algebra. If β has dense range, then*

$$\begin{tikzcd} S\mathbin{\widehat{\#}}H \arrow[d, "1\mathbin{\widehat{\#}}\beta"'] & H \arrow[l, "j_H"'] \arrow[d, "\beta"] \\ S\mathbin{\widehat{\#}}K & K, \arrow[l, "j_K"'] \end{tikzcd}$$

where j_H and j_K are the canonical homomorphisms, is a pushout diagram.

Proof. Let $\psi\colon K\to B$ and $\chi\colon S\mathbin{\widehat{\#}}H\to B$ be $\widehat{\otimes}$-algebra homomorphisms such that $\psi\beta=\chi j_H$. We use the universal property in Definition 2.1 as in the proof of Proposition 2.6. Put $\varphi:=\chi i$, where $i\colon S\to S\mathbin{\widehat{\#}}H$ is the canonical homomorphism. It suffices to find a $\widehat{\otimes}$-algebra homomorphism θ making the diagram

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dd, "\varphi"'] & S\mathbin{\widehat{\#}}H \arrow[d, "1\mathbin{\widehat{\#}}\beta"] \arrow[ddl, "\chi"'] & H \arrow[l, "j_H"'] \arrow[d, "\beta"] \\ & S\mathbin{\widehat{\#}}K \arrow[dl, dashed, "\theta"] & K, \arrow[l, "j_K"] \arrow[dll, "\psi"] \\ B & & \end{tikzcd}$$

commutative and prove that it is unique.

Recall that S is endowed with the action $h\cdot s=\beta(h)\cdot s$. If $h\in H$ and $s\in S$, then by the definition of the adjoint action, we have

$$\beta(h)\cdot\varphi(s)=\sum\psi(\beta(h_{(1)}))\,\varphi(s)\,\psi(\beta(S(h_{(2)})))=\\ \chi\Bigl(\sum j_H(h_{(1)})\,i(s)\,j_H(S(h_{(2)}))\Bigr)=\chi(i(\beta(h)\cdot s)).$$

(The last equality follows from the fact that i is an H-$\widehat{\otimes}$-module morphism with respect to the adjoint action associated with j_H.) Hence $k\cdot\varphi(s)=\varphi(k\cdot s)$ when $k=\beta(h)$. Therefore φ is a K-$\widehat{\otimes}$-module morphism since the range of β is dense.

Similarly, using density again, we have that $(1\mathbin{\widehat{\#}}\beta)j_H$ is a K-$\widehat{\otimes}$-module morphism with respect to the adjoint action associated with j_K. Hence, by the universal property of smash product written for $S\mathbin{\widehat{\#}}K$, there is a unique $\widehat{\otimes}$-algebra homomorphism θ such that $\theta j_K=\psi$ and $\theta(1\mathbin{\widehat{\#}}\beta)i=\varphi$. The first equality implies that $\theta(1\mathbin{\widehat{\#}}\beta)j_H=\psi\beta$.

Being a K-$\widehat{\otimes}$-module morphism, φ is also an H-$\widehat{\otimes}$-module morphism. Then the universal property of smash product written for $S\mathbin{\widehat{\#}}H$ implies that a homomorphism χ such that $\chi i=\varphi$ and $\chi j_H=\psi\beta$ is unique. Therefore $\theta(1\mathbin{\widehat{\#}}\beta)=\chi$.

To complete the proof we need to show that θ such that $\theta j_K=\psi$ and $\theta(1\mathbin{\widehat{\#}}\beta)=\chi$ is unique. If θ' is another $\widehat{\otimes}$-algebra homomorphism with these properties, then $\theta'(1\mathbin{\widehat{\#}}\beta)i=\varphi$ and $\theta'(1\mathbin{\widehat{\#}}\beta)j_H=\psi\beta$. The uniqueness statement above implies that $\theta'=\theta$. ☐

3. Unique extension property for derivations

When A is a $\widehat{\otimes}$-algebra and X is an A-$\widehat{\otimes}$-bimodule, we denote the vector space of continuous derivations from A to X by $\operatorname{Der}(A,X)$. It is clear that a $\widehat{\otimes}$-algebra homomorphism

$\varphi\colon A \to B$ induces the linear map

$$\widetilde{\varphi}_X\colon \mathrm{Der}(B,X) \to \mathrm{Der}(A,X)\colon D \mapsto Df.$$

Definition 3.1. We say that a homomorphism $\varphi\colon A \to B$ of $\widehat{\otimes}$-algebras satisfies *Property (UDE)* (the unique extension property for derivations) if $\widetilde{\varphi}_X\colon\ \mathrm{Der}(B,X) \to \mathrm{Der}(A,X)$ is bijective for each B-$\widehat{\otimes}$-bimodule X.

It is proved in [28] that the Arens–Michael envelope always satisfies to Property (UDE).

In what follows we denote the pushout of $\widehat{\otimes}$-algebra homomorphisms $A \to B_1$ and $A \to B_2$ using relative free algebra notation, i.e., as $B_1 *_A B_2$. Note that the category of $\widehat{\otimes}$-algebras is cocomplete and, in particular, pushout exists. But we do not need this fact because we use only Propositions 2.6 and 2.7, which in particular imply the existence of pushouts.

The following result is an analytic version of Proposition 5.2 in [14].

Proposition 3.2. *In the category of unital $\widehat{\otimes}$-algebras, Property (UDE) is preserved by pushouts.*

For the proof we need an auxiliary proposition. Note that every $(B_1 *_A B_2)$-$\widehat{\otimes}$-bimodule is both a B_1-$\widehat{\otimes}$- and B_2-$\widehat{\otimes}$-bimodule with the multiplications given by lifting along homomorphisms.

Proposition 3.3. *Let $\varphi_1 : A \to B_1$ and $\varphi_2 : A \to B_2$ be homomorphisms of $\widehat{\otimes}$-algebras and X be a $(B_1 *_A B_2)$-$\widehat{\otimes}$-bimodule. Suppose that $D_1 \in \mathrm{Der}(B_1,X)$, $D_2 \in \mathrm{Der}(B_2,X)$ and $D_1\varphi_1 = D_2\varphi_2$. Then there exists a unique continuous derivation D making the diagram*

$$\begin{tikzcd}
A \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & B_2 \arrow[d, "\varkappa_2"'] \arrow[ddr, "D_2"] & \\
B_1 \arrow[r, "\varkappa_1"] \arrow[drr, "D_1"'] & B_1 *_A B_2 \arrow[dr, dashed, "D"] & \\
 & & X
\end{tikzcd}$$

commutative.

Proof. Consider the locally convex space $(B_1 *_A B_2) \oplus X$ as a $\widehat{\otimes}$-algebra by letting

$$(c,x)(c',y) = (cc', cy + xc') \qquad (c, c' \in B_1 *_A B_2,\ x, y \in X).$$

For $j = 1,2$, define a $\widehat{\otimes}$-algebra homomorphism by

$$\psi_j\colon B_j \to (B_1 *_A B_2) \oplus X\colon b \mapsto (\varkappa_j(b), D_j(b)).$$

By the pushout property, there is a unique homomorphism

$$\psi\colon B_1 *_A B_2 \to (B_1 *_A B_2) \oplus X$$

such that $\psi\varkappa_j = \psi_j$ $(j = 1,2)$. Write ψ as $c \mapsto (\alpha(c), D(c))$, where α and D are continuous linear maps. It is easy to see that α is an endomorphism of $B_1 *_A B_2$ and D is an α-derivation, i.e., $D(cc') = \alpha(c)D(c') + D(c)\alpha(c')$ $(c, c' \in B_1 *_A B_2)$. Since $\alpha\varkappa_j = \varkappa_j$ $(j = 1,2)$, the uniqueness of ψ implies that $\alpha = 1$. Therefore D is a derivation such that $D\varkappa_j = D_j$ $(j = 1,2)$. Finally, D is unique since so is ψ. $\square$

Proof of Proposition 3.2. Let

$$\begin{tikzcd} A \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & B_2 \arrow[d, "\varkappa_2"] \\ B_1 \arrow[r, "\varkappa_1"'] & B_1 *_A B_2 \end{tikzcd}$$

be a pushout diagram such that φ_1 satisfies Property (UDE).

Take a $(B_1 *_A B_2)$-$\widehat{\otimes}$-bimodule X and $D_2 \in \mathrm{Der}(B_2, X)$. Since $D_2\varphi_2$ is a derivation of A and φ_1 satisfies Property (UDE), there exists $D_1 \in \mathrm{Der}(B_1, X)$ such that $D_1\varphi_1 = D_2\varphi_2$. By the universal property for derivations in Proposition 3.3, there is $D \in \mathrm{Der}(B_1 *_A B_2, X)$ such that $D_1 = D\varkappa_1$ and $D_2 = D\varkappa_2$. Hence, $\mathrm{Der}(B_1 *_A B_2, X) \to \mathrm{Der}(B_2, X)$ is surjective.

Note that $\widetilde{\varphi}_X$ is injective for each X if and only if φ is an epimorphism; cf. the Fréchet algebra case in [10, Theorem 3.20]. Since epimorphisms are preserved by pushouts in an arbitrary category, $\mathrm{Der}(B_1 *_A B_2, X) \to \mathrm{Der}(B_2, X)$ is injective for each X. Thus, $\varkappa_2$ satisfies Property (UDE). $\square$

Now we apply Proposition 3.2 to analytic smash products.

Theorem 3.4. *Let $\beta\colon H \to K$ be a Hopf $\widehat{\otimes}$-algebra homomorphism, R and S be an H- and K-$\widehat{\otimes}$-module algebras, resp., and $\alpha\colon R \to S$ be an H-$\widehat{\otimes}$-module algebra homomorphism. If each of α and β has dense range and satisfies Property (UDE), then so is the $\widehat{\otimes}$-algebra homomorphism $\alpha \mathbin{\widehat{\#}} \beta\colon R \mathbin{\widehat{\#}} H \to S \mathbin{\widehat{\#}} K$ defined in Lemma 2.3.*

Proof. The density obviously inherits. Further, write $\alpha \mathbin{\widehat{\#}} \beta$ as the following composition:

$$R \mathbin{\widehat{\#}} H \xrightarrow{\alpha \mathbin{\widehat{\#}} 1} S \mathbin{\widehat{\#}} H \xrightarrow{1 \mathbin{\widehat{\#}} \beta} S \mathbin{\widehat{\#}} K.$$

It is easy to see that Property (UDE) is stable under composition. So it suffices to show that $\alpha \mathbin{\widehat{\#}} 1$ and $1 \mathbin{\widehat{\#}} \beta$ have this property. Since it is preserved by pushouts according to Proposition 3.2, the result follows from Propositions 2.6 and 2.7, which assert that $\alpha \mathbin{\widehat{\#}} 1$ and $1 \mathbin{\widehat{\#}} \beta$ are obtained by pushouts with α and β, respectively. $\square$

4. Homological epimorphisms and relatively quasi-free algebras

Definitions and statement of main result. Our main reference on the homological theory of topological algebras is Helemskii's book [18] (the Russian edition is known as 'first black book'). Additional facts regarding a more general relative theory can be found in [28].

Suppose that R is a $\widehat{\otimes}$-algebra. Recall that an *R-$\widehat{\otimes}$-algebra* is a pair (A, η_A), where A is a $\widehat{\otimes}$-algebra and $\eta_A : R \to A$ is a $\widehat{\otimes}$-algebra homomorphism. Note that each A-$\widehat{\otimes}$-module is automatically an R-$\widehat{\otimes}$-module via the restriction-of-scalars functor along η_A. We denote by (A, R)-mod the category of left A-$\widehat{\otimes}$-modules endowed with the split exact structure. This means that the admissible sequences in (A, R)-mod are those that split by R-$\widehat{\otimes}$-module morphisms; cf. [28, Appendix, Example 10.1 and 10.3]. In particular, when $R = \mathbb{C}$, we recover the standard definition of an admissible (or $\mathbb{C}$-split) sequence of A-$\widehat{\otimes}$-modules used in [18]. When considering $\widehat{\otimes}$-bimodules over an R-$\widehat{\otimes}$-algebra A (from the left) and an S-$\widehat{\otimes}$-algebra B (from the right), we denote the corresponding category by (A, R)-mod-(B, S). In the case when $R = S = \mathbb{C}$, we write simply A-mod-B. When we talk on a projective resolution in (A,R)-mod-(B,S), we mean a complex consisting of objects projective in the corresponding exact category and splitting in S-mod-R.

Let A be a $\widehat{\otimes}$-algebra and let X and Y be a right and left A-$\widehat{\otimes}$-modules, respectively. Then the *projective A-module tensor product* $X\mathbin{\widehat{\otimes}_A} Y$ is defined as the completion of the quotient space of $X\mathbin{\widehat{\otimes}} Y$ by the closure of the linear hull of all elements of the form $x\cdot a\otimes y - x\otimes a\cdot y$ ($x\in X$, $y\in Y$, $a\in A$); see [18].

Recall that an *R-S-homomorphism* from an R-$\widehat{\otimes}$-algebra to an S-$\widehat{\otimes}$-algebra is defined as a pair (f,g), where $f\colon A\to B$ and $g\colon R\to S$ are $\widehat{\otimes}$-algebra homomorphisms, such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \\ R \arrow[u, "\eta_A"] \arrow[r, "g"] & S \arrow[u, "\eta_B"'] \end{tikzcd}$$

is commutative. In the case when $S=R$, we obtain an *R-homomorphism.*

Definition 4.1. [28, Definition 6.2] An R-S-homomorphism $f\colon A\to B$ from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B is called a *two-sided relative homological epimorphism* if f is an epimorphism of $\widehat{\otimes}$-algebras and A is acyclic relative to the functor

$$B\mathbin{\widehat{\underset{A}{\otimes}}}(-)\mathbin{\widehat{\underset{A}{\otimes}}} B\colon (A,R)\text{-}\mathsf{mod}\text{-}(A,R)\to (B,S)\text{-}\mathsf{mod}\text{-}(B,S),$$

i.e., it sends some (equivalently, each) projective resolution of A in (A,R)-mod-(A,R) to a sequence that splits in S-mod-S.

We omit the definitions of *left and right relative homological epimorphisms*; for details see [28, Definition 6.1] with corrections in [29, 30]. The only fact that we need here is that a two-sided relative homological epimorphism is also left and right [28, Proposition 6.5].

In the case when $R=S=\mathbb{C}$, Definition 4.1 takes more simple form:

Definition 4.2. An epimorphism $A\to B$ of $\widehat{\otimes}$-algebras is said to be *homological* if A is acyclic relative to the functor

$$B\mathbin{\widehat{\underset{A}{\otimes}}}(-)\mathbin{\widehat{\underset{A}{\otimes}}} B\colon A\text{-}\mathsf{mod}\text{-}A\to B\text{-}\mathsf{mod}\text{-}B; \tag{4.1}$$

see [34, Definition 1.3], [27, Definition 3.2] or [28, Definition 6.3].

Lemma 4.3. *Every homological epimorphism from a finitely generated algebra to a Fréchet algebra satisfies Property (UDE).*

Proof. It follows from the definitions that every homological epimorphism is 1-pseudoflat in the sense of [10]. The argument for the claim that a 1-pseudoflat epimorphism from a finitely generated to a Fréchet algebra satisfies Property (UDE) is exactly the same as for [10, Theorem 3.24] (in the corrected version of [11]), where the case of a 1-pseudoflat epimorphism between Fréchet algebras is considered; see also details in [12]. $\square$

The following theorem is our main result.

Theorem 4.4. *Let $\mathfrak{g}$ be a finite-dimensional complex Lie algebra. Then the Arens-Michael envelope $U(\mathfrak{g})\to\widehat{U}(\mathfrak{g})$ is a homological epimorphism if and only if $\mathfrak{g}$ is solvable.*

For the proof we need a result on homological epimorphisms for analytic smash products with the Hopf $\widehat{\otimes}$-algebra $\mathfrak{A}_s$ defined in (2.7); see Theorem 4.6 below.

Smash products with $\mathfrak{A}_s$. The following proposition is the first step.

Proposition 4.5. *Let $s \in [0,+\infty]$. Then $\beta\colon \mathbb{C}[z] \to \mathfrak{A}_s$ in Lemma 2.4 is a homological epimorphism.*

Proof. Since $\mathfrak{A}_s$ is a Hopf $\widehat{\otimes}$-algebra by Lemma 2.4, it follows from [27, Proposition 3.7] that it suffices to check that $\mathbb{C}[z] \to \mathfrak{A}_s$ is a weak localization, i.e., the following conditions hold:

(1) for some projective resolution $0 \leftarrow \mathbb{C} \leftarrow P_\bullet$ in $\mathbb{C}[z]$-mod the complex $0 \leftarrow \mathfrak{A}_s \widehat{\otimes}_{\mathbb{C}[z]} \mathbb{C} \leftarrow \mathfrak{A}_s \widehat{\otimes}_{\mathbb{C}[z]} P_\bullet$ is admissible;

(2) the natural map $\mathfrak{A}_s \widehat{\otimes}_{\mathbb{C}[z]} \mathbb{C} \to \mathbb{C}$ is a topological isomorphism.

Condition (2) is clearly satisfied. To check Condition (1) note that

$$0 \leftarrow \mathbb{C} \xleftarrow{\varepsilon} \mathbb{C}[z] \xleftarrow{d_0} \mathbb{C}[z] \leftarrow 0,$$

where $\varepsilon : z \mapsto 0$ and $[\delta_0(a)](z) := az$, is a projective resolution in $\mathbb{C}[z]$-mod. Applying $\mathfrak{A}_s \widehat{\otimes}_{\mathbb{C}[z]}(-)$ to this resolution we get

$$0 \leftarrow \mathbb{C} \longleftarrow \mathfrak{A}_s \longleftarrow \mathfrak{A}_s \leftarrow 0, \tag{4.2}$$

where the differentials are defined by the same formulas.

Consider the map

$$\sigma\colon \mathfrak{A}_s \to \mathfrak{A}_s : a \mapsto \sum_{n=0}^{\infty} a_{n+1} z^n.$$

Suppose that $s < \infty$ and let C be a positive constant such that $n^s \leqslant C\, 2^n$ for every $n \in \mathbb{Z}_+$. Then for each $r > 0$ we have

$$\|\sigma(a)\|_{r,s} = \sum_{n=0}^{\infty} |a_{n+1}| \frac{r^n}{n!^s} \leqslant \sum_{n=0}^{\infty} \frac{n^s}{r} |a_n| \frac{r^n}{n!^s} \leqslant \frac{C}{r} \|a\|_{2r,s}.$$

Thus σ is well defined and continuous. It is easy to see that the sequence in (4.2) splits by σ and hence it is admissible. The case when $s = \infty$ is obvious.

Thus Condition (1) is also satisfied and therefore $\mathbb{C}[z] \to \mathfrak{A}_s$ is a weak localization for every s. $\square$

Recall that when an algebra R is endowed with a derivation, the action induced by this derivation turns R into a $\mathbb{C}[z]$-module algebra and so one can consider the smash product $R \mathbin{\#} \mathbb{C}[z]$.

Theorem 4.6. *Let $s \in [0,+\infty]$, R be an associative algebra of countable dimension, S be a $\widehat{\otimes}$-algebra, and $\alpha\colon R \to S$ be a homological epimorphism with dense range. Suppose that S is an $\mathfrak{A}_s$-$\widehat{\otimes}$-module algebra, respectively and R is endowed with a derivation is a such way that α becomes a $\mathbb{C}[z]$-module morphism. Then*

$$\alpha \mathbin{\widehat{\#}} \beta\colon R \mathbin{\#} \mathbb{C}[z] \to S \mathbin{\widehat{\#}} \mathfrak{A}_s$$

is also a homological epimorphism.

To prove Theorem 4.6 we establish a generalization of Theorem 1.2 and next apply Theorem 1.1.

Recall that an R-$\widehat{\otimes}$-algebra A is said to be *relatively quasi-free* over R if any admissible singular R-extension of A is split; see [28, Definition 7.1] and the discussion therein.

Theorem 4.7. *Let (f,g) be an R-S-homomorphism from an R-$\widehat{\otimes}$-algebra A to a S-$\widehat{\otimes}$-algebra B. Suppose that*
(1) *$g\colon R\to S$ has dense range;*
(2) *$f\colon A\to B$ satisfies Property (UDE) and has dense range;*
(3) *A is relatively quasi-free over R.*
Then f is a two-sided relative homological epimorphism.

For the proof we need a relative version of Property (UDE). If A is an R-$\widehat{\otimes}$-algebra and X is a B-$\widehat{\otimes}$-bimodule, then a derivation $D: A\to X$ is called an *R-derivation* if it is an R-bimodule morphism (or, equivalently, of left or right R-module morphism). The subspace of $\operatorname{Der}(A,X)$ consisting of R-derivations is denoted by $\operatorname{Der}_R(A,X)$. An R-S-homomorphism $(f,g)\colon (A,R)\to(B,S)$ induces the linear map

$$\operatorname{Der}_S(B,X)\to\operatorname{Der}_R(A,X)\colon D\mapsto Df;$$

see details in [28].

Proposition 4.8. *Let $(f,g): (A,R)\to(B,S)$ be an R-S-homomorphism from an R-$\widehat{\otimes}$-algebra A to an S-$\widehat{\otimes}$-algebra B. Suppose that each of f and g has dense range and f satisfies Property (UDE). Then the canonical map $\operatorname{Der}_S(B,X)\to\operatorname{Der}_R(A,X)$ is a bijection for every B-$\widehat{\otimes}$-bimodule X.*

Proof. Since $\operatorname{Der}(B,X)\to\operatorname{Der}(A,X)$ is injective, so is $\operatorname{Der}_S(B,X)\to\operatorname{Der}_R(A,X)$.

To prove the surjectivity take $D\in\operatorname{Der}_R(A,X)$. Since f satisfies Property (UDE), there is $D'\in\operatorname{Der}(B,X)$ such that $D=D'f$. To complete the proof we need to show that D' is an S-derivation, that is,

$$D'(\eta_B(s)b)=\eta_B(s)\cdot D'(b)\qquad\text{for every } s\in S \text{ and } b\in B. \tag{4.3}$$

Since each of f and g has dense range, it suffices to verify (4.3) in the case when $s=g(r)$ and $b=f(a)$ for some $r\in R$ and $a\in A$. Then $\eta_B(s)b=\eta_B(g(r))f(a)=f(\eta_A(r))f(a)=f(\eta_A(r)a)$ and hence $D'(\eta_B(s)b)=D(\eta_A(r)a)$. Since D is an R-derivation, we have $D(\eta_A(r)a)=\eta_A(r)\cdot D(a)$. On the other hand,

$$\eta_A(r)\cdot D(a)=f(\eta_A(r))\cdot D(a)=\eta_B(g(r))\cdot D'f(a)=\eta_B(s)\cdot D'(b).$$

Thus (4.3) holds. □

Remark 4.9. The conditions of Proposition 4.8 can be weakened by assuming that g is an epimorphism of $\widehat{\otimes}$-algebras; cf. [28, Proposition 3.8]. But we do nit need this variant.

If A is an R-$\widehat{\otimes}$-algebra, then a representing object of the functor $\operatorname{Der}_R(A,-)$ from A-mod-A to the category of sets exists and is denoted by $\Omega^1_R A$ (it is called the *bimodule of relative differential* 1*-forms* of A); for details see [28, p. 82].

Fix an R-S-homomorphism and a B-$\widehat{\otimes}$-bimodule X. From the universal property of Ω^1 we have the commutative diagram

$$\begin{array}{ccc} \operatorname{Der}_S(B,X) & \longrightarrow & \operatorname{Der}_R(A,X) \\ \| & & \| \\ {}_B\mathbf{h}_B(\Omega^1_S B,X) & \longrightarrow & {}_A\mathbf{h}_A(\Omega^1_R A,X), \end{array} \tag{4.4}$$

where $\mathbf{h}_B(-,-)$ denotes the vector space of B-$\widehat{\otimes}$-bimodule morphisms (and similarly for A); see [28, eq. (7.3) and p. 83]. Also, consider the natural map

$$ {}_A\mathbf{h}_A(\Omega^1_R A, X) \to {}_B\mathbf{h}_B(B \mathop{\widehat{\otimes}}_A \Omega^1_R A \mathop{\widehat{\otimes}}_A B, X) \tag{4.5}$$

and the composition

$$ {}_B\mathbf{h}_B(\Omega^1_S B, X) \to {}_B\mathbf{h}_B(B \mathop{\widehat{\otimes}}_A \Omega^1_R A \mathop{\widehat{\otimes}}_A B, X). \tag{4.6}$$

Substituting $\Omega^1_S B$ for X, we obtain the morphism

$$B \mathop{\widehat{\otimes}}_A \Omega^1_R A \mathop{\widehat{\otimes}}_A B \to \Omega^1_S B$$

corresponding to the identity morphism of $\Omega^1_S B$.

Proposition 4.10. (cf. [28, Proposition 7.4]) *Under the hypotheses of Proposition 4.8, the morphism $B \widehat{\otimes}_A \Omega^1_R A \widehat{\otimes}_A B \to \Omega^1_S B$ defined above is a B-$\widehat{\otimes}$-bimodule isomorphism.*

Proof. Let X be an arbitrary B-$\widehat{\otimes}$-bimodule. Note that the map in (4.5) is always bijective. Also, under the hypotheses of Proposition 4.8, the top arrow in (4.4) is a bijection and so is the bottom arrow.

Thus we have that the map in (4.6) is bijective for every X, which implies that $B \widehat{\otimes}_A \Omega^1_R A \widehat{\otimes}_A B \to \Omega^1_S B$ is an isomorphism. □

Proof of Theorem 4.7. By [28, Proposition 7.3], A is relatively quasi-free over R if and only if $\Omega^1_R A$ is projective in (A, R)-mod-(A, R). Hence the sequence

$$0 \leftarrow A \xleftarrow{m} A \mathop{\widehat{\otimes}}_R A \xleftarrow{j} \Omega^1_R A \leftarrow 0 \tag{4.7}$$

is a projective resolution in (A, R)-mod-(A, R). (Here m is the multiplication on A and j is the bimodule morphism corresponding to the inner derivation $-\operatorname{ad}_{1\otimes 1}\colon A \to A \widehat{\otimes}_R A\colon a \mapsto [1 \otimes 1, a]$.)

Since f satisfies Property (UDE), it is an epimorphism. Then the canonical map $B \widehat{\otimes}_A B \to B$ is a topological isomorphism. Also, since g is an epimorphism, the canonical map $B \widehat{\otimes}_R B \to B \widehat{\otimes}_S B$ is a topological isomorphism; for both properties of epimorphisms see [28, Proposition 6.1]. Applying the functor $B \widehat{\otimes}_A (-) \widehat{\otimes}_A B$ to the sequence in (4.7) and using Proposition 4.10, we get the sequence

$$0 \leftarrow B \leftarrow B \mathop{\widehat{\otimes}}_S B \leftarrow \Omega^1_S B \leftarrow 0,$$

which splits in B-mod-S by [28, Proposition 7.2]. Hence it also splits in S-mod-S. This implies that f is a two-sided relative homological epimorphism; see Definition 4.1. □

A projective (A, R)-$\widehat{\otimes}$-bimodule P satisfies the *finiteness condition* (f_2) if it is a retract of a bimodule $A \widehat{\otimes}_R M \widehat{\otimes}_R A$, where $M \in R$-mod-R is isomorphic to the R-$\widehat{\otimes}$-bimodule R^n for some n [28, Definition 8.3]. (We do not need the alternative condition (f_1) from [28].) Also, an R-$\widehat{\otimes}$-algebra A is called an algebra of (f_2)-*finite type* if in (A, R)-mod-(A, R) it has a finite projective resolution all of whose bimodules satisfy the finiteness condition (f_2) [28, Definition 8.4].

Now we can prove the main result of the section.

Proof of Theorem 4.6. Recall that we denote by β the Hopf $\widehat{\otimes}$-algebra homomorphism $\mathbb{C}[z] \to \mathfrak{A}_s$. Put $A = R \mathbin{\#} \mathbb{C}[z]$, $B = S \mathbin{\widehat{\#}} \mathfrak{A}_s$, $g = \alpha$ and $f = \alpha \mathbin{\widehat{\#}} \beta$ (the latter is well defined by Lemma 2.3 since α is a $\mathbb{C}[z]$-module morphism). It suffices to check the conditions in Theorem 1.1, i.e.,

(1) g is a homological epimorphism;
(2) f is a left or right relative homological epimorphism;
(3) A is projective in R-mod and B is projective in S-mod;
(4) A is an R-$\widehat{\otimes}$-algebra of (f_2)-finite type.

Condition (1) holds by the hypothesis.

Condition (2) follows from Theorem 4.7. Indeed, β is a homological epimorphisms by Proposition 4.5. Hence α and β satisfy Property (UDE) (the first by the hypothesis and the second by Lemma 4.3). Since the ranges are dense, Theorem 3.4 implies that f also satisfies Property (UDE). Since R is a $\mathbb{C}[z]$-module algebra, we have the action by a derivation. Therefore $R \mathbin{\#} \mathbb{C}[z]$ is an Ore extension (see Remark 2.2) and hence it is relatively quasi-free by [28, Proposition 7.9]. Thus all the conditions in Theorem 4.7 hold and therefore f is a two-sided relative homological epimorphism. Finally, note that every two-sided relative homological epimorphism is a one-sided relative homological epimorphism [28, Proposition 6.5].

To verify Condition (3) note that $S \mathbin{\widehat{\#}} \mathfrak{A}_s$ is isomorphic to $S \mathbin{\widehat{\otimes}} \mathfrak{A}_s$ as an S-$\widehat{\otimes}$-module and so it is projective being a free module (see, e.g., [18, Chapter III, Proposition 1.25]). Similarly, $R \mathbin{\#} \mathbb{C}[z]$ is a projective R-module.

Since R has countable dimension, each Ore extension of R (without twisting) is a relatively (f_2)-quasi-free R-algebra [28, Proposition 7.9 and Example 8.1], which in particular implies that it is of (f_2)-finite type [28, Definition 8.5 and Proposition 7.3.]. Thus Condition (4) holds. □

The proof of the main theorem. Now we are in a position to prove the central result of the paper.

Proof of Theorem 4.4. For the necessity see [26, Theorem 3.6].

To show the sufficiency consider the decomposition in (2.8). We claim that the truncated homomorphism

$$(\cdots(U(\mathfrak{f}_1) \mathbin{\#} U(\mathfrak{f}_2)) \mathbin{\#} \cdots) \mathbin{\#} U(\mathfrak{f}_k) \longrightarrow (\cdots(\mathfrak{A}_{i_1} \mathbin{\widehat{\#}} \mathfrak{A}_{i_2}) \mathbin{\widehat{\#}} \cdots) \mathbin{\widehat{\#}} \mathfrak{A}_{i_k} \tag{4.8}$$

(which exists since the decompositions of $U(\mathfrak{g})$ and $\widehat{U}(\mathfrak{g})$ are compatible) is a homological epimorphism for every $k = 1, \dots, n$.

Proceeding by induction, note first that $U(\mathfrak{f}_1) \cong \mathbb{C}[z]$ for every k. If $k = 1$, then the assertion is given by Lemma 4.3. Assume now that the claim holds for $k - 1$. Then the homomorphism in (4.8) can be written as $R \mathbin{\#} \mathbb{C}[z] \to S \mathbin{\widehat{\#}} \mathfrak{A}_{i_k}$, where R and S denote the algebras obtained on the $(k-1)$th step.

Note that R has countable dimension and $g\colon R \to S$ has dense range. Since, in addition, g is a homological epimorphism by the induction hypothesis, Theorem 4.6 implies that $R \mathbin{\#} \mathbb{C}[z] \to S \mathbin{\widehat{\#}} \mathfrak{A}_{i_k}$ is also a homological epimorphism. The claim is proved.

Putting $k = n$ we conclude that $U(\mathfrak{g}) \to \widehat{U}(\mathfrak{g})$ is a homological epimorphism. □

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Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, China

Email address: aristovoyu@inbox.ru