New phenomenology in the first–order thermodynamics of scalar–tensor gravity for Bianchi universes

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The phase space of Bianchi I universes in vacuum Brans–Dicke gravity is analyzed in terms of physical variables. The behaviour of the solutions of the field equations near the fixed points (which are solutions of Einstein gravity) is compared with basic ideas of the recent first–order thermodynamics of scalar-tensor gravity, elucidating new phenomenology.

I. INTRODUCTION

Shortly after the introduction of general relativity (GR), researchers began looking for alternative theories of gravity, moved by pure curiosity about how things could be different in nature [1, 2]. More concrete motivation emerged with the birth of quantum field theory, when the question arose of how to reconcile the two biggest physics discoveries of the twentieth century, quantum mechanics and GR. The reason is that, as soon as one introduces the lowest–order quantum corrections to GR, one simultaneously causes deviations from it in the form of higher derivative terms in the field equations or extra degrees of freedom [3, 4]. This situation does not change in string theory: the simplest string theory, the bosonic string, has a low–energy limit that reproduces not GR, but an $\omega = -1$ Brans–Dicke gravity [5, 6]. The prototype of alternative gravity is scalar–tensor gravity, which contains only a scalar degree of freedom ϕ in addition to the two massless spin two modes familiar from GR. The original Jordan–Brans–Dicke theory [7] was later generalized to wider scalar–tensor theories [8–11] in which the "Brans–Dicke coupling" parameter became a function $\omega(\phi)$ of the scalar field ϕ, which was also endowed with a potential $V(\phi)$.

There is independent motivation for the study of alternative theories of gravity coming from cosmological observations. The 1998 discovery, made with type Ia supernovae, that the present expansion of the universe is accelerated led to introducing overnight a completely *ad hoc* dark energy with very exotic properties (equation of state parameter close to $w \simeq -1$), of completely unkown nature and comprising approximately 70% of the energy content of the universe (see [12] for a review). A wide range of dark energy models have been proposed in the literature, but none is compelling and this state of affairs is deeply unsatisfactory from the theoretical point of view. For this reason, many cosmologists have turned to questioning whether, instead, we do not understand gravity on the largest (cosmological) scales and dark energy simply does not exist [13, 14]. This idea had led to formulating and testing modified gravity models. Among a spectrum of possibilities, the most popular models belong to the so–called metric $f(R)$ gravity class (see [15–17] for reviews). Metric $f(R)$ gravity contains only one extra massive, propagating, scalar degree of freedom and, therefore, falls into the wider category of scalar–tensor gravity [15–17]. Even Starobinski inflation [18], the first scenario of inflation and the one currently favored by observations [19], is based on quadratic corrections to the Einstein–Hilbert action and is ultimately a scalar–tensor theory.

In the past decade, the problem of finding the most general scalar–tensor theory expressed by field equations that are at most of second order led to the rediscovery and intense study of the older Horndeski gravity [20]. This sought–for property was found to belong not to Horndeski gravity but to the more general Degenerate Higher Order Scalar–Tensor (DHOST) theories, a subclass of higher order gravities in which a degeneracy condition brings the order of the field equations back to two (*e.g.*, [21–33], see [34–36] for reviews).

Given the wide spectrum of scalar–tensor gravities (not to mention other alternatives richer in propagating degrees of freedom that are difficult to identify and count [37]), what is the role of GR in this landscape? A proposal is well–known in the context of emergent gravity, in which the field equations can be deduced as an emergent or collective property of underlying degrees of freedom and are not fundamental. The seminal paper by Jacobson [38] derived the Einstein equations of GR from purely thermodynamical considerations, an idea referred to as "thermodynamics of spacetime". This feat was repeated with quadratic $f(R)$ gravity producing a new picture: GR is somehow a state of "thermal equilibrium" of gravity, while alternative theories correspond to entropy generation and to excited thermal states [39]. This view has been very influential and has generated a large literature but, unfortunately, no substantial progress has been made since its early days. Specifically, the "temperature of gravity" (or other order parameter) has not been identified and no equation describing the relaxation of alternative gravity to GR has been proposed, although there are reasons to believe that such phenomena could have occurred in the early universe [40, 41].

Recently a new proposal has been advanced, known as the first–order thermodynamics of scalar–tensor gravity

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[42–46]: it has nothing to do with Jacobson's thermodynamics of spacetime, although some of its ideas share the same spirit. It is a far–reaching analogy (but, still, only an analogy) between the description of the effective stress–energy tensor of the scalar degree of freedom ϕ of scalar–tensor gravity and the stress–energy tensor of a dissipative fluid. By writing the field equations of scalar–tensor gravity as effective Einstein equations, the contributions of ϕ and of its first and second derivatives form an effective stress–energy tensor $T_{ab}^{(\phi)}$ which has the form of a dissipative fluid stress–energy tensor [42] (this fact has been known for a long time for special theories or special geometries [47, 48] and has been recently recognized also for "viable" Horndeski gravity [49, 50]). Specifically, if the scalar field gradient $\nabla^a\phi$ is timelike and future–oriented [45], its normalized version

$$u^a \equiv \frac{\nabla^a\phi}{\sqrt{-\nabla^c\phi\nabla_c\phi}} \tag{1.1}$$

can be seen as the four–velocity of an effective fluid with stress–energy tensor of the form

$$T_{ab}^{(\phi)} = \rho u_a u_b + P h_{ab} + \pi_{ab} + q_a u_b + q_b u_a \,, \tag{1.2}$$

where ρ is an effective energy density, P is an effective isotropic pressure, π_{ab} is an effective anisotropic, trace–free, stress tensor, and q^a is an effective heat flux density. Here $h_{ab} \equiv g_{ab} + u_a u_b$ is the Riemannian three-space metric seen by observers comoving with this fluid (with $h^a{}_b$ the projector onto this 3-space), while π_{ab} and q^a are purely spatial:

$$h_{ab}u^a = h_{ab}u^b = 0\,, \quad \pi_{ab}u^a = \pi_{ab}u^b = 0\,, \quad q_c u^c = 0\,. \tag{1.3}$$

The fact that this $T_{ab}^{(\phi)}$ has the dissipative fluid form contains no physics: the decomposition (1.2) holds true for any symmetric two–index tensor [46]. However, when one takes seriously this dissipative fluid structure and tries to apply to it Eckart's [51] theory of dissipative fluids [42–44] one discovers that, miraculously, Eckart's constitutive relation

$$q_a = -\mathcal{K} h_{ab} \left(\nabla^b \mathcal{T} + T\dot{u}^b\right) \tag{1.4}$$

holds. Here $\mathcal{T}$ is the temperature of the dissipative fluid, $\mathcal{K}$ is the thermal conductivity, and $\dot{u}^a$ is the fluid four–acceleration. Equation (1.4) is nothing but the relativistic generalization of Fourier's law with the addition of an inertial term proportional to the four–acceleration, which takes into account the fact that heat is a form of energy and its transport contributes to the energy flux in a way that was absent in pre–relativistic physics [51]. The unexpected fact that this relation holds for the effective ϕ–fluid makes it possible to define the product $\mathcal{KT}$ for scalar–tensor gravity.

Let us refer, for simplicity, to "first–generation" (*i.e.*, pre–Horndeski) scalar–tensor gravity: in the Jordan frame, the gravitational sector of the theory is described by the action[1]

$$S_{\mathrm{ST}} = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla^c\phi\nabla_c\phi - V(\phi)\right] \tag{1.7}$$

where $\phi > 0$ is the Brans-Dicke scalar (approximately equivalent to the inverse of the effective gravitational coupling strength G^{-1}), the function $\omega(\phi)$ (which was a strictly constant parameter in the original Brans-Dicke theory) is the "Brans-Dicke coupling", and $V(\phi)$ is a scalar field potential (absent in the original Brans-Dicke theory). When $\nabla^a\phi$ is timelike and future–oriented, the product of effective thermal conductivity and effective temperature is found to be [43, 44]

$$\mathcal{KT} = \frac{\sqrt{-\nabla^c\phi\nabla_c\phi}}{8\pi\phi}\,. \tag{1.8}$$

It is apparent that GR, reproduced for $\phi = \text{const.} > 0$, corresponds to $\mathcal{KT} = 0$. It is rather intuitive that, when extra degrees of freedom with respect to GR are excited, gravity is in some sense excited and "hotter" than the GR state in which the extra degrees of freedom are absent. The "temperature of gravity" $\mathcal{T}$ is, in this sense, a temperature *relative to the GR state of equilibrium*.

The first–order thermodynamics of scalar–tensor gravity has been extended [50] to "viable" Horndeski gravity, *i.e.*, to the subclass in which gravitational waves propagate at the speed of light, and applied to various situations in cosmology and other contexts [46, 53–60] (see [61] for a recent review). Two basic qualitative ideas emerge from these studies: near spacetime singularities gravity is "hot", *i.e.*, it diverges from GR; the expansion of the three–space perceived by comoving observers (with 3–metric h_{ab}) "cools" gravity, bringing it closer to GR. These ideas have been tested against various situations of physical interest or mathematical convenience (for which it is possible to draw analytically definite conclusions). Here we continue this program. While it was natural to apply the first–order thermodynamics of scalar–tensor gravity to Friedmann–Lemaître–Robertson–Walker (FLRW) cosmology [54, 62], here we extend the description to anisotropic Bianchi universes. For simplicity, we confine ourselves to Brans–Dicke gravity and to the simplest anisotropic cosmologies described by spatially flat Bianchi I geometries. To describe the dynamics we resort to a phase space view and, contrary

[1] We adopt the notation of Ref. [52]: the signature of the spacetime metric g_{ab} is $-+++$ and units are used in which the the speeed of light c and Newton's constant G are unity. The Ricci tensor and Ricci scalar are

$$R_{ab} \equiv R^c{}_{acb} = \partial_c\Gamma^c_{ba} - \partial_b\Gamma^c_{ca} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ca}\,, \tag{1.5}$$

$$R \equiv R^a{}_a = \partial_c\Gamma^c_{ba} - \partial_b\Gamma^c_{ca} + \Gamma^c_{cd}\Gamma^d_{ba} - \Gamma^c_{bd}\Gamma^d_{ca}\,. \tag{1.6}$$

to the literature that we are aware of, we choose as phase space variables the average Hubble function H, the scalar field ϕ, and its time derivative $\dot{\phi}$ that have a direct physical meaning instead of other variables obtained by various non–linear combinations of physical ones (for this reason we cannot avail ourselves of existing phase space analyses).

II. FIELD EQUATIONS AND BIANCHI I GEOMETRY

The (Jordan frame) vacuum field equations obtained by varying the action (1.7) with respect to the inverse metric g^{ab} and the scalar ϕ are

$$\begin{aligned} R_{ab} - \frac{1}{2}\, g_{ab} R &= \frac{\omega}{\phi^2}(\nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi) \\ &\quad + \frac{1}{\phi}\left(\nabla_a \nabla_b \phi - g_{ab} \Box \phi\right) - \frac{V}{2\phi} g_{ab}\,, \end{aligned} \tag{2.1}$$

$$\Box \phi = \frac{1}{2\omega+3}\left(\phi\, \frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi}\, \nabla^c \phi \nabla_c \phi\right) . \tag{2.2}$$

In the following we assume $V(\phi) \geq 0$, constant Brans–Dicke coupling ω, and $2\omega+3>0$ to avoid phantom scalar fields ϕ.

The line element of a Bianchi I universe is

$$ds^2 = -dt^2 + A^2(t) dx^2 + B^2(t) dy^2 + C^2(t) dz^2 \tag{2.3}$$

in Cartesian comoving coordinates (t, x, y, z), where $A(t), B(t)$, and $C(t)$ are the scale factors associated with the three orthogonal spatial directions. This anisotropic universe has average scale factor $a(t) \equiv (ABC)^{1/3}$ and average Hubble parameter

$$H \equiv \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) \equiv \frac{1}{3}\left(H_A + H_B + H_C\right) , \tag{2.4}$$

where an overdot denotes differentiation with respect to the cosmic time t, $H_i \equiv \dot{A}_i / A$ (where $A_i = A, B$, or C), and

$$\begin{aligned} H^2 &= \frac{1}{9}\left(H_A^2 + H_B^2 + H_C^2 + 2H_A H_B + 2H_A H_C \right. \\ &\quad \left. + 2H_B H_C\right) . \end{aligned} \tag{2.5}$$

The shear scalar Σ for this scalar–tensor gravity, computed in terms of ϕ and its derivatives, is given by [42]

$$\begin{aligned} \sigma &\equiv \sqrt{\frac{1}{2}\, \sigma_{ab} \sigma^{ab}} \\ &= (-\nabla^e \phi \nabla_e \phi)^{-3/2} \left\{ \frac{1}{3} (\nabla_a \nabla_b \phi \nabla^a \phi \nabla^b \phi)^2 \right. \\ &\quad + \frac{1}{2} (\nabla^e \phi \nabla_e \phi)^2 \left[\nabla^a \nabla^b \phi \nabla_a \nabla_b \phi - \frac{1}{3} (\Box \phi)^2 \right] - (\nabla^e \phi \nabla_e \phi) \\ &\quad \left. \times \left(\nabla_a \nabla_b \phi \nabla^b \nabla_c \phi - \frac{1}{3} \Box \phi \nabla_a \nabla_c \phi \right) \nabla^a \phi \nabla^c \phi \right\}^{1/2} , \end{aligned} \tag{2.6}$$

where σ_{ab} is the shear tensor [52]. For clarity, we use the shear variable $\Sigma \equiv \frac{1}{2}\sigma^2$ which, in the Bianchi I geometry (2.3), assumes the form

$$\begin{aligned} \Sigma &= \frac{1}{6A^2B^2C^2}\left(B^2C^2\dot{A}^2 + A^2C^2\dot{B}^2 + A^2B^2\dot{C}^2 \right. \\ &\quad \left. - ABC^2\dot{A}\dot{B} - AB^2C\dot{A}\dot{C} - A^2BC\dot{B}\dot{C} \right) \\ &= \frac{B^2C^2\dot{A}^2 + A^2C^2\dot{B}^2 + A^2B^2\dot{C}^2}{6A^2B^2C^2} \\ &\quad - \frac{C\dot{A}\dot{B} + B\dot{A}\dot{C} + A\dot{B}\dot{C}}{6ABC} \\ &= \frac{1}{6}\left(H_A^2 + H_B^2 + H_C^2 - H_A H_B - H_A H_C - H_B H_C\right) \\ &= \frac{1}{4}\left(H_A^2 + H_B^2 + H_C^2\right) - \frac{3}{4} H^2 \\ &= \frac{3H^2}{2} - \frac{1}{2}\left(H_A H_B + H_A H_C + H_B H_C\right) . \end{aligned} \tag{2.7}$$

Σ vanishes if and only if all the components of σ_{ab} vanish [63], in which case the Bianchi geometry reduces to a FLRW one.

The only non–vanishing Christoffel symbols of the geometry (2.3) are

$$\begin{aligned} \Gamma^t_{xx} &= A\dot{A}\,, \quad \Gamma^t_{yy} = B\dot{B}\,, \quad \Gamma^t_{zz} = C\dot{C}\,, \\ \Gamma^x_{tx} &= \Gamma^x_{xt} = \frac{\dot{A}}{A}\,, \quad \Gamma^y_{ty} = \Gamma^y_{yt} = \frac{\dot{B}}{B}\,, \\ \Gamma^z_{tz} &= \Gamma^z_{zt} = \frac{\dot{C}}{C}\,, \end{aligned} \tag{2.8}$$

while the non–vanishing components of the Ricci tensor are

$$R_{tt} = -\left(\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}\right), \tag{2.9}$$

$$R_{xx} = \frac{ABC\ddot{A}+\left(AC\dot{B}+AB\dot{C}\right)\dot{A}}{BC}, \tag{2.10}$$

$$R_{yy} = \frac{ABC\ddot{B}+\left(BC\dot{A}+AB\dot{C}\right)\dot{B}}{AC}, \tag{2.11}$$

$$R_{zz} = \frac{ABC\ddot{C}+\left(BC\dot{A}+AC\dot{B}\right)\dot{C}}{AB}, \tag{2.12}$$

and the Ricci scalar reads

$$R = 2\left(\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+H_AH_B+H_BH_C+H_AH_C\right) \tag{2.13}$$

$$= 6\left(\dot{H}+2H^2+\frac{2\Sigma}{3}\right). \tag{2.14}$$

The time–time component of the Brans–Dicke field equations (2.1) is

$$H^2=\frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2+\frac{V}{6\phi}+\frac{2\Sigma}{3}-H\,\frac{\dot{\phi}}{\phi}, \tag{2.15}$$

while the spatial components read

$$-\omega ABC\dot{\phi}^2+ABCV\phi-2\phi^2\left(A\ddot{B}C+A\dot{B}\dot{C}+AB\ddot{C}\right) -2ABC\phi\ddot{\phi}-2\phi\dot{\phi}\left(A\dot{B}C+AB\dot{C}\right)=0, \tag{2.16}$$

$$-\omega ABC\dot{\phi}^2+ABCV\phi-2\phi^2\left(\ddot{A}BC+\dot{A}B\dot{C}+AB\ddot{C}\right) -2ABC\phi\ddot{\phi}-2\phi\dot{\phi}\left(\dot{A}BC+AB\dot{C}\right)=0, \tag{2.17}$$

$$-\omega ABC\dot{\phi}^2+ABCV\phi-2\phi^2\left(\ddot{A}BC+\dot{A}\dot{B}C+A\ddot{B}C\right) -2ABC\phi\ddot{\phi}-2\phi\dot{\phi}\left(\dot{A}BC+A\dot{B}C\right)=0. \tag{2.18}$$

and the trace of the field equation (2.1) is

$$\dot{H}=-\frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2+\frac{V}{3\phi}-\frac{2\Sigma}{3}-2H^2-\frac{\left(\ddot{\phi}+3H\dot{\phi}\right)}{2\phi}. \tag{2.19}$$

Equation (2.2) for the scalar field is

$$\ddot{\phi}+3H\dot{\phi}+\frac{\phi V'-2V}{2\omega+3}=0, \tag{2.20}$$

where a prime denotes differentiation with respect to ϕ. Combining these three equations yields

$$\dot{H}=-\frac{\omega}{2}\left(\frac{\dot{\phi}}{\phi}\right)^2-2\Sigma+2H\,\frac{\dot{\phi}}{\phi}+\frac{(\phi V'-2V)}{2\phi\,(2\omega+3)}. \tag{2.21}$$

By inserting Eq. (2.20) for the scalar field ϕ into (2.19) and combining the result with Eq. (2.15), one obtains

$$\dot{H}=-3H^2-\frac{H\dot{\phi}}{\phi}+\frac{\phi V'(\phi)+(2\omega+1)V(\phi)}{2\phi(2\omega+3)}. \tag{2.22}$$

To summarize, the field equations to be solved for the scalar field $\phi(t)$ and the Hubble function $H(t)$ are Eq. (2.20) and Eq. (2.22), respectively. Once these quantities are known, Eq. (2.15) gives the shear Σ.

III. PHASE SPACE

Let us discuss the phase space of Bianchi I cosmologies in vacuum Brans–Dicke gravity, where the dynamics is due entirely to the Brans–Dicke scalar ϕ. We use the variables $\left(H,\phi,\dot{\phi}\right)$ which are physical: the Hubble parameter H is a cosmological observable (although its actual value is subject to a very significant tension [64, 65]), while ϕ is the extra scalar degree of freedom of scalar–tensor gravity in addition to the two spin zero massless modes of GR contained in the metric g_{ab}. The strength of the gravitational coupling $G\simeq\phi^{-1}$ is measured directly by Cavendish experiments and its time variation (*i.e.*, $\dot{G}$ and, consequently, $\dot{\phi}$) is subject to observational constraints [66, 67]. By contrast, much of the existing literature on Bianchi cosmologies uses variables which are complicated functions of H,ϕ, and $\dot{\phi}$ and do not have direct physical interpretation. Although they may make the study of the phase space dynamics more convenient from the formal point of view, they have no direct physical meaning. Here we want to interpret the dynamics and compare it with the first–order thermodynamics of spacetime, therefore we must use physical variables.

Equation (2.15) yields the shear

$$\Sigma\left(H,\phi,\dot{\phi}\right)=\frac{3}{2}\left[H^2+H\,\frac{\dot{\phi}}{\phi}-\frac{\omega}{6}\left(\frac{\dot{\phi}}{\phi}\right)^2-\frac{V}{6\phi}\right] \tag{3.1}$$

as a function of the three phase space variables, therefore Σ is not an independent variable, although it will be relevant in our analysis.

With our choice of variables the fixed points in the phase space, if they exist, have necessarily the form $\left(H,\phi,\dot{\phi}\right)=(H_0,\phi_0,0)$, with H_0 and $\phi_0>0$ constants. They are solutions of the Einstein equations located on the "GR plane" $\dot{\phi}=0$ of the phase space identified by constant scalar field, the condition that reproduces GR.

Using the notation $V(\phi_0) \equiv V_0$ and $V'(\phi_0) \equiv V'_0$, Eq. (3.1) that must be satisfied by the fixed points gives

$$\Sigma_0 = \frac{3}{2}\left(H_0^2 - \frac{V_0}{6\phi_0}\right), \tag{3.2}$$

while the mix of Eqs. (2.19) and (2.20) gives, using $\dot{H} = \omega' = 0$,

$$H_0^2 = \frac{V_0}{6\phi_0} - \frac{\Sigma_0}{3} + \frac{\phi_0 V'_0 - 2V_0}{4\phi_0\,(2\omega+3)} \tag{3.3}$$

and the scalar field equation degenerates into

$$V'_0 = \frac{2V_0}{\phi_0} \tag{3.4}$$

(which is non–negative since $V \geq 0$ and $\phi > 0$) reducing Eq. (3.3) to

$$H_0^2 = \frac{V_0}{6\phi_0} - \frac{\Sigma_0}{3}. \tag{3.5}$$

Comparing Eqs. (3.2) and (3.5), one obtains $\Sigma_0 = 0$: as expected, the shear vanishes at the fixed points, which have $A = B = C \equiv a(t)$ and the average Hubble function $H(t)$ coincides with the FLRW one. Equation (3.3) then becomes $H_0^2 = V_0/(6\phi_0)$, or

$$H_0 = \pm\sqrt{\frac{V_0}{6\phi_0}} = \pm\sqrt{\frac{V'_0}{12}}. \tag{3.6}$$

The degenerate fixed points corresponding to $V_0 = V'_0 = H_0 = 0$ are Minkowski spaces, while those corresponding to $V_0 > 0$, $V'_0 > 0$, and $H_0 = \pm\sqrt{\frac{V_0}{6\phi_0}}$ are de Sitter spaces.[2] When ϕ becomes a constant ϕ_0 and $V(\phi_0) \equiv V_0$ is positive, the theory of gravity reduces to GR with a positive cosmological constant $\Lambda = V_0$.

In the study of exact solutions of the field equations, one sometimes find solutions $\left(H(t), \phi(t), \dot{\phi}(t)\right)$ with $\phi(t) \to 0^+$ at late times $t \to +\infty$: these are pathological as the effective gravitational coupling $G_{\text{eff}} \to +\infty$. The line $\phi = 0$ in the "GR plane" $\dot{\phi} = 0$ corresponds to singularities at which G_{eff} changes sign, but it does so by going through $G_{\text{eff}} = \infty$ (a similar situation occurs with conformally coupled scalar fields [68, 69]). Exact solutions with these properties are unphysical and cannot be regarded as GR solutions, even though the first–order thermodynamics of scalar–tensor gravity does not, strictly speaking, indicate a pathology or a gross deviation from GR in this situation (see Appendix A).

A. Stability of the equilibrium points

Let us examine the stability of the fixed points with respect to homogeneous perturbations described by

$$\phi(t) \;=\; \phi_0 + \delta\phi(t), \tag{3.7}$$

$$H(t) \;=\; H_0 + \delta H(t). \tag{3.8}$$

Evolution equations for the perturbations $\delta\phi(t), \delta H(t)$ are obtained from Eq. (2.20) for the scalar field ϕ and Eq. (2.22) for H.

Let us begin with the stability of the scalar field. By expanding the scalar field potential, $V(\phi) \simeq V_0 + V'_0\,\delta\phi$, and using the zero–order field equation (2.20), one obtains the linearized equation for $\delta\phi$

$$\delta\ddot{\phi} + 3H_0\delta\dot{\phi} + \omega_0^2\,\delta\phi = 0, \tag{3.9}$$

where

$$\omega_0^2 \equiv \frac{\phi_0 V''_0 - V'_0}{2\omega+3} = \frac{\phi_0 V''_0 - 2V_0/\phi_0}{2\omega+3} = \frac{\phi_0 V''_0 - 12H_0^2}{2\omega+3}. \tag{3.10}$$

The *ansatz*

$$\delta\phi(t) = \delta_0\,\mathrm{e}^{\alpha t} \tag{3.11}$$

with δ_0 and α constants yields the algebraic equation

$$\alpha^2 + 3H_0\alpha + \omega_0^2 = 0 \tag{3.12}$$

with roots

$$\alpha_{(\pm)} = \frac{-3H_0 \pm \sqrt{9H_0^2 - 4\omega_0^2}}{2} \equiv \frac{1}{2}\left(-3H_0 \pm \sqrt{\Delta}\right) \tag{3.13}$$

and two modes $\delta\phi_{(\pm)}(t) = \delta_0\,\mathrm{e}^{\alpha_{(\pm)}t}$ (this applies if $\alpha_{(\pm)} \neq 0$; the case $\alpha_{(\pm)} = 0$ is discussed separately).

- If $\Delta < 0$, corresponding to $\omega_0^2 > 9H_0^2/4$ and ω_0 real, then

  $$\delta\phi_{(\pm)}(t) = \delta_0\,\mathrm{e}^{-\frac{3H_0 t}{2}}\,\mathrm{e}^{\pm\frac{i}{2}\sqrt{|\Delta|}\,t}: \tag{3.14}$$

  the second exponential in the right–hand side oscillates while, as $t \to +\infty$, the first exponential diverges if $H_0 < 0$ and decays if $H_0 > 0$ (it remains constant if $H_0 = 0$). In this case the fixed point is stable if $H_0 \geq 0$ and unstable if $H_0 < 0$.

- If $\Delta = 0$, corresponding to $\omega_0^2 = 9H_0^2/4$ (and ω_0 real), the scalar field perturbation is simply $\delta\phi(t) = \delta_0\,\mathrm{e}^{-\frac{3H_0 t}{2}}$ and is stable if $H_0 \geq 0$, unstable if $H_0 < 0$.

- If $\Delta > 0$, corresponding to $\omega_0^2 < 9H_0^2/4$ then it is convenient to write

  $$\alpha_{(\pm)} = \frac{3H_0}{2}\left(-1 \pm \sqrt{1 - \left(\frac{2\omega_0}{3H_0}\right)^2}\right). \tag{3.15}$$

[2] With an abuse of nomenclature, we refer to the fixed points with $H_0 > 0$ as "expanding de Sitter spaces" and to those with $H_0 < 0$ as "contracting de Sitter spaces".

If ω_0 is real, corresponding to $\phi_0 V_0'' \geq 12H_0^2$, then $\sqrt{1-\left(\frac{2\omega_0}{3H_0}\right)^2} < 1$ and $-1 \pm \sqrt{1-\left(\frac{2\omega_0}{3H_0}\right)^2} < 0$, hence $\alpha_{(\pm)} < 0$ if $H_0 > 0$, or $\alpha_{(\pm)} = 0$ if $H_0 = 0$ (corresponding to a Minkowski fixed point), or $\alpha_{(\pm)} > 0$ if $H_0 < 0$. Fixed points with $H_0 \geq 0$ are stable while those with $H_0 < 0$ are unstable.

If instead ω_0 is imaginary, $\omega_0 = i|\omega_0|$, corresponding to $\phi_0 V_0'' < 12H_0^2$, then we have $\sqrt{-1+\left(\frac{2\omega_0}{3H_0}\right)^2} > 0$ and $-1 - \sqrt{1-\left(\frac{2\omega_0}{3H_0}\right)^2} < 0$. The mode $\delta\phi_{(+)}$ is unstable (*i.e.*, $\alpha_{(+)} > 0$) if $H_0 > 0$, while the other mode $\delta\phi_{(-)}$ is unstable (*i.e.*, $\alpha_{(-)} > 0$) if $H_0 < 0$. In short, when ω_0 is imaginary and $H_0 \neq 0$ there is always a unstable mode and the fixed point is unstable.

If instead ω_0 is imaginary and $H_0 = 0$ (Minkowski fixed point), the equation for the scalar field perturbations reduces to

$$\delta\ddot{\phi} - |\omega_0^2|\delta\phi = 0\,, \tag{3.16}$$

which describes an unstable inverted harmonic oscillator.

An exception not included in the previous discussion is the situation in which $V = 0$ and $H_0 = 0$, $\omega_0 = 0$ corresponding to a Minkowski space. In this case, Eq. (3.9) reduces to $\ddot{\phi} = 0$, which has a linear solution and this Minkowski space is unstable.

Let us consider now the perturbation $\delta H(t)$. Using the zero–order equations, Eq. (2.22) gives the linearized equation of motion for δH

$$\delta\dot{H} + 6H_0\delta H = -\frac{H_0}{\phi_0}\,\delta\dot{\phi} + \frac{\left[V_0''\phi_0 + (2\omega+1)\,\frac{V_0}{\phi_0}\right]}{2\phi_0\,(2\omega+3)}\,\delta\phi \tag{3.17}$$

and Eq. (3.10) yields

$$\delta\dot{H} + 6H_0\delta H = -\frac{H_0}{\phi_0}\,\delta\dot{\phi} + \frac{\left(\omega_0^2 + 6H_0^2\right)}{2\phi_0}\,\delta\phi\,. \tag{3.18}$$

Using the explicit form $\delta\phi_{(\pm)}(t)$ of the scalar field perturbation gives

$$\delta\dot{H} + 6H_0\delta H = \beta_{(\pm)}\delta\phi\,, \tag{3.19}$$

where

$$\beta_{(\pm)} = \frac{\omega_0^2 + 6H_0^2 - 2\alpha_{(\pm)}H_0}{2\phi_0}\,. \tag{3.20}$$

The solution of this inhomogeneous ordinary differential equation is

$$\begin{aligned}\delta H_{(\pm)}(t) &= \frac{\beta_{(\pm)}}{\alpha_{(\pm)} + 6H_0}\,\delta\phi + C\,\mathrm{e}^{-6H_0 t} \\ &= \frac{\left(H_0 - \alpha_{(\pm)}\right)}{2\phi_0}\,\delta\phi + C\,\mathrm{e}^{-6H_0 t}\,,\end{aligned} \tag{3.21}$$

where C is an integration constant. The perturbation $\delta H(t)$ diverges for $H_0 < 0$ regardless of the behaviour of the scalar field perturbation $\delta\phi$.

To summarize:

- Contracting de Sitter spaces are always unstable fixed points, which can be understood as the effect of anti–friction in the (anti–)damped harmonic oscillator equation (3.9).
- Expanding de Sitter fixed points are stable if $\phi_0 V_0'' > 12H_0^2$ and unstable if $\phi_0 V_0'' < 12H_0^2$.
- Minkowski fixed points are (marginally) stable if ω_0 is real (corresponding to $\phi_0 V_0'' \geq 12H_0^2$) and unstable otherwise. The exception is the Minkowski space obtained for $V \equiv 0, H_0 = 0$, which is unstable.

Let us consider now the ratio of the shear variable to the expansion variable Σ/H_0^2, which quantifies the amount of anisotropy and the departure from GR (remember that the fixed points, when they exist, all lie in the GR plane $\dot{\phi} = 0$ and $\Sigma = 0$). Since the equilibrium points are isotropic de Sitter spaces, the shear (3.1) is purely perturbative and given by

$$\begin{aligned}\Sigma \;=\; \delta\Sigma = \frac{3}{2}\Bigg[&(H_0+\delta H)^2 + \frac{\delta\dot{\phi}}{\phi_0+\delta\phi}\,(H_0+\delta H) \\ &-\frac{\omega}{6}\left(\frac{\delta\dot{\phi}}{\phi_0+\delta\phi}\right)^2 - \frac{(V_0+V_0'\delta\phi)}{6\,(\phi_0+\delta\phi)}\Bigg]\end{aligned} \tag{3.22}$$

which, to first order, reduces to

$$\frac{\delta\Sigma}{H_0^2} = \frac{3}{2H_0}\left[2\delta H + \frac{\delta\dot{\phi}}{\phi_0} - \frac{H_0}{\phi_0}\,\delta\phi\right]. \tag{3.23}$$

Inserting the solution for the perturbations $\delta\phi, \delta H$ into this equation and using Eq. (3.12) to express ω_0^2 yields

$$\frac{\delta\Sigma}{H_0^2} = \frac{3C}{H_0}\,\mathrm{e}^{-6H_0 t}\,, \tag{3.24}$$

(Since $\Sigma \simeq \delta\Sigma \geq 0$, one deduces that $\mathrm{sign}\,(C) = \mathrm{sign}\,(H_0)$.)

The ratio Σ/H_0^2 vanishes as $t \to +\infty$ for all solutions that are perturbations of expanding de Sitter fixed points, and diverges in the same late–time limit near contracting de Sitter fixed points.

Let us now examine the significance of these results with respect to the first–order thermodynamics of spacetime of Refs. [42–46]. We emphasize that the following discussion is meaningful only because physical phase space variables $\left(H, \phi, \dot{\phi}\right)$ have been chosen at the outset.

de Sitter solutions with non–constant scalar field, which are possible in scalar–tensor gravity but not in GR, have been discussed in Ref. [57].

B. Comparison with the first-order thermodynamics of scalar–tensor gravity

In Brans–Dicke theory, the temperature of gravity relative to the GR state of equilibrium [42–46] is given by Eq. (1.8). For linear homogeneous perturbations of the fixed points, it reads

$$\mathcal{K}\mathcal{T} = \frac{|\alpha_{(\pm)}||\delta\phi_{(\pm)}|}{8\pi\phi_0} \geq 0\,. \tag{3.25}$$

The fixed points, which lie in the GR plane with $\dot{\phi} = 0$ clearly correspond to $\mathcal{K}\mathcal{T} = 0$. $\mathcal{K}\mathcal{T}$ assumes the same form as in a FLRW universe, but the solution $\phi(t)$ is, in general, different in Bianchi I and in FLRW universes.

If the orbit of a solution in the $\left(H, \phi, \dot{\phi}\right)$ phase space lies near an expanding de Sitter fixed point and is attracted to it, the anisotropic three-space expands, and the solution converges to the zero–temperature state of equilibrium, while $\Sigma/H_0^2 \to 0$ and this three-space isotropizes. The cooling of gravity ($\mathcal{K}\mathcal{T} \to 0$) is indeed another way of saying that GR is a late–time attractor of the dynamics. The situation is not so trivial, however, because there are exceptions for $\phi_0 V_0'' < 12H_0^2$. In this case three–space still expands exponentially but the de Sitter fixed point nearby is a repellor. It is still the case that $H_0 > 0$ and $\Sigma/H_0^2 \to 0$ as $t \to +\infty$. How do we understand this situation in the light of scalar–tensor thermodynamics? The answer comes from examining the equation ruling the approach to/departure from the GR equilibrium state derived in Refs. [42–46]

$$\frac{d\left(\mathcal{K}\mathcal{T}\right)}{d\tau} = 8\pi\left(\mathcal{K}\mathcal{T}\right)^2 - \Theta\mathcal{K}\mathcal{T} + \frac{\Box\phi}{8\pi\phi}\,, \tag{3.26}$$

where τ is the comoving time of the effective ϕ–fluid (*i.e.*, the proper time of observers comoving with this fluid and with four–velocity (1.1)) and Θ is the expansion scalar of the fluid [52]. In a Bianchi I universe $\Theta = 3H$ and $\tau = t$. Near a fixed point (which lies in the GR plane) $\mathcal{K}\mathcal{T}$ is a first–order quantity and Eq. (3.26) reduces, to linear order, to

$$\frac{d\left(\mathcal{K}\mathcal{T}\right)}{dt} = -3H_0\mathcal{K}\mathcal{T} - \frac{\left(\delta\ddot{\phi} + 3H_0\delta\dot{\phi}\right)}{8\pi\phi_0} \tag{3.27}$$

or, in the light of the previous discussion,

$$\frac{d\left(\mathcal{K}\mathcal{T}\right)}{dt} = -\frac{3H_0|\delta\dot{\phi}|}{8\pi\phi_0} + \frac{\omega_0^2\delta\phi}{8\pi\phi_0}\,. \tag{3.28}$$

For expanding de Sitter spaces with purely imaginary ω_0 it is $\omega_0^2 < 0$. In order for the scalar field gradient $\nabla^a\phi = -\dot{\phi}\,\delta^a{}_0$ to be future–oriented it must be $\dot{\phi} < 0$, which implies that $\delta\phi = \delta_0\,\mathrm{e}^{\alpha t} < 0$, or $\delta_0 < 0$. Then, for the exceptional expanding de Sitter fixed points with imaginary ω_0, we have

$$\begin{aligned} \frac{d\left(\mathcal{K}\mathcal{T}\right)}{dt} &= -\frac{3H_0\alpha|\delta\phi|}{8\pi\phi_0} + \frac{\omega_0^2\delta\phi}{8\pi\phi_0} \\ &= \frac{|\delta\phi|}{8\pi\phi_0}\left(-3H_0\alpha - \omega_0^2\right) \\ &= \frac{\alpha^2}{8\pi\phi_0}\,|\delta\phi| > 0 \end{aligned} \tag{3.29}$$

using Eq. (3.12): $\mathcal{K}\mathcal{T}$ always grows near these repellors, describing the departure from the GR equilibrium state. The reason for this behaviour is clearly due to the third term $\Box\phi/(8\pi\phi)$ in the right–hand side of Eq. (3.26).

IV. CONCLUSIONS

Two basic insights have been obtained thus far in the first–order thermodynamics of scalar–tensor (including viable Horndeski) gravity [42–46]. The first one is that the expansion of the three–space seen by observers comoving with the effective ϕ–fluid "cools" gravity. The second is that gravity is "hot" (*i.e.*, $\mathcal{K}\mathcal{T} \to +\infty$) near spacetime singularities.

The idea that "expansion cools gravity" (*i.e.*, $\mathcal{K}\mathcal{T} \to 0$) was deduced in Refs. [42–46] using situations in which $\Box\phi = 0$. The lesson from the present study is that this statement is not always true when $\Box\phi \neq 0$. The term $\Box\phi/(8\pi\phi)$ in Eq. (3.26) cannot be expressed unambigously in terms of $\mathcal{K}\mathcal{T}$ or its powers or derivatives. This third term in the right–hand side of (3.26) is reminiscent of entropy generation terms in non–equilibrium thermodynamics and it is fair to say that it is the dynamics of the scalar field itself, embodied in $\Box\phi/\phi$, that drives gravity away from the GR equilibrium state.

When $\omega = $ const., $V(\phi) \equiv 0$, and in the presence of conformally invariant matter (for example, in the radiation era), it is $\Box\phi = 0$ because Eq. (2.2) in the presence of matter and with a quadratic potential $V(\phi) = m^2\phi^2/2$ becomes

$$\Box\phi = \frac{8\pi T^{(\mathrm{m})}}{2\omega + 3} \tag{4.1}$$

where $T^{(\mathrm{m})}$ is the trace of the matter energy–momentum tensor. $\Box\phi$ vanishes in the presence of conformally invariant matter with zero trace, such as a radiation fluid in the radiation era, during which the expansion of space causes gravity to approach GR. This phenomenon was indeed reported in FLRW scalar–tensor cosmology [40, 41]. However, was not expected in other cosmological eras. The convergence of scalar–tensor to GR cosmology has been debated at length and we hope that our approach can shed some light on this issue, which we will discuss in a future publication.

Another lesson garnered from the discussion of the previous section is that the degree of anisotropy Σ/H_0^2 commonly used in the literature on Bianchi universes does not tell the full story about the approach to, or departure from the GR state because it tends to zero for the exceptional expanding de Sitter fixed points with imaginary ω_0 that are phase space repellors.

To conclude, more research is needed to understand scalar–tensor gravity (and even more for Horndeski gravity) from the point of view of first–order thermodynamics. We remind the reader that this formalism is, ultimately, only an analogy; nevertheless, it is proving useful from the theoretical point of view and it is building up to a consistent framework to understand at least scalar–tensor gravity in the increasingly wider spectrum of alternatives to GR.

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Appendix A THE PATHOLOGICAL LINE $\phi = 0$ IN THE $\dot{\phi} = 0$ PLANE OF THE PHASE SPACE

Let us consider an exact Bianchi I solution of Brans–Dicke gravity that asymptotes to a $\phi = 0$ solution. Assuming $V(\phi) \equiv 0$, which yields $\Box\phi = 0$, consider the power–law *ansatz* for the scalar field

$$\phi(t) = \phi_0 t^\alpha \tag{A.1}$$

where ϕ_0 is a positive constant, $t > 0$, and α is assumed to be negative to guarantee that the gradient $\nabla^a\phi$ is future–oriented. The corresponding Hubble function

$$H(t) = \frac{1-\alpha}{3t} \tag{A.2}$$

is always positive, describing an expanding universe, and $H(t) \to 0^+$ as $t \to +\infty$. The shear

$$\Sigma(t) = -\frac{\left(3\alpha^2\omega + 4\alpha^2 - 2\alpha - 2\right)}{12t^2} \tag{A.3}$$

is positive if

$$\omega < -\frac{2\,(2\alpha+1)\,(\alpha-1)}{3\alpha^2}\,. \tag{A.4}$$

It is interesting that the quantity Σ/H^2, which measures the ratio of anisotropy to expansion, remains exactly constant during the evolution of this universe, signalling that GR (which corresponds to exactly vanishing Σ) is not approached. Formally, for this solution it is

$$\mathcal{K}\mathcal{T} = \frac{|\dot{\phi}|}{8\pi\phi} = \frac{|\alpha|}{8\pi t} \to 0^+ \quad \text{as } t \to +\infty\,. \tag{A.5}$$

Although the expansion of 3–space "cools" this Brans–Dicke gravity, the zero temperature limit is not GR and is indeed a physical pathology corresponding to infinite $G_{\rm eff}$, which should be excluded from the range of physical possibilities. This means that a grain of salt is needed in the physical interpretation of the first–order thermodynamics of scalar–tensor gravity (which is not defined for $\phi = 0$). In any case, the Minkowski space obtained for $V \equiv 0$, $H_0 = 0$, $\omega_0 = 0$ is unstable, as seen in Sec. III A.

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