

Quantum Relaxation for Solving Multiple Knapsack Problems

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Combinatorial problems are a common challenge in business, requiring finding optimal solutions under specified constraints. While significant progress has been made with variational approaches such as QAOA, most problems addressed are unconstrained (such as Max-Cut). In this study, we investigate a hybrid quantum-classical method for constrained optimization problems, particularly those with knapsack constraints that occur frequently in financial and supply chain applications. Our proposed method relies firstly on relaxations to local quantum Hamiltonians, defined through commutative maps. Drawing inspiration from quantum random access code (QRAC) concepts, particularly Quantum Random Access Optimizer (QRAO), we explore QRAO's potential in solving large constrained optimization problems. We employ classical techniques like Linear Relaxation as a presolve mechanism to handle constraints and cope further with scalability. We compare our approach with QAOA and present the final results for a real-world procurement optimization problem: a significant sized multi-knapsack-constrained problem.

I. INTRODUCTION

The concept of leveraging quantum computers to generate approximate solutions for NP-hard combinatorial problems dates back more than two decades, initially introduced through quantum adiabatic eigenstate evolution [1]. This foundational idea evolved into the Quantum Approximate Optimization Algorithm (QAOA) [2] and Variational Quantum Eigensolver [3], which entails variational optimization of quantum parameters [4]. With considerable interest in harnessing quantum advantage for classical combinatorial problems, extensive research has scrutinized the performance of QAOA [5–7]. These quantum optimization techniques rely on establishing a bijective mapping between the space of classical binary variables and logical basis states of a collection of qubits, as originally outlined [2]. Within the QAOA framework, the cost function optimized on a quantum computer typically features a classical maximal eigenstate, which doesn't strictly necessitate superposition and entanglement for preparation. This contrasts with quantum many-body Hamiltonians or quantum chemistry, where extremal eigenstates often exhibit significant entanglement. In such cases, quantum computers benefit from a natural memory advantage in storing the ground state.

While QAOA and VQE represent classical-quantum hybrid algorithms tailored for near-term quantum devices capable of executing only shallow circuits, several critical issues persist including the concerns of scalability. Given that QAOA and VQE encode one classical bit into one qubit, the maximum number of qubits for near-term

quantum devices are limited. Consequently, the sizes of problem instances that can be addressed are severely restricted. This limitation poses a significant obstacle to the widespread applicability of these algorithms in tackling larger-scale computational challenges.

QRAO [8] offers a solution to this challenge by encoding multiple classical bits (typically three or fewer) into one qubit. By doing so, it facilitates the generation of approximate solutions for combinatorial problems that seek extremal eigenstates of local quantum Hamiltonians. These local quantum Hamiltonians represent relaxations of the original combinatorial problems. For each element in the image of a combinatorial cost function, it becomes feasible to construct a quantum state with the same Hamiltonian expectation value. This integration between classical combinatorial optimization and quantum relaxation is a key feature of QRAO, making it a promising approach for addressing complex optimization challenges using quantum-inspired techniques [9]. Recent results of QRAO give hints of its robustness to quantum noise [10], and its power to leverage quantum entanglement to obtain optimal and better solutions [11]. However, the problems addressed by QRAO were all unconstrained optimization.

In this study, our objective is to address a complex constrained supply chain problem using QRAO. We will explore the effectiveness of QRAO in addressing the inherent complexities of constrained supply chain problems, by solving a Multiple Knapsack Problem (MKP) and comparing it with the well-studied QAOA approach which is effective in solving unconstrained problems such as Max-Cut [13]. We will also be scaling up a real-world multiple knapsack-based Risk-Aware Procurement Optimization problem involving ≥ 100 variables and demonstrating the prospect of combining QRAO with a classical method in operations research, namely Linear Relaxation (LR)[14]. In this work, we aim to merge quantum and classical com-

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putation to expand the utility of quantum algorithms, enabling the solution of larger instances. Through this endeavor, we demonstrate the scalability of QRAO and its efficacy in handling larger problem sizes, where traditional methods like QAOA encounter limitations due to the requirement of more qubits and memory.

The subsequent sections of this paper are structured as follows: In Section. II we described the formulation for Multiple Knapsack Problem (MKP) and the Risk-Aware Procurement Optimization problem, followed by Section. III where we introduced the details of Quantum Random Access Encoding. In Sec. IV we explain our approach on using Linear Relaxation and in Section. V we present the results of the comparison of QAOA and QRAO on MKP problem and the performance of QRAO with Linear Relaxation on Risk-Aware Procurement Optimization problem. Finally in Section. VI we provide the concluding remarks and the future outlook of the work.

This research will contribute valuable insights into the practical applicability of quantum and classical hybrid optimization methods for complex optimization challenges in finance and supply chain management.

II. PROBLEM SETTINGS

We consider two problem settings. First, the Multiple Knapsack Problem, which is a strongly NP-hard problem with no fully polynomial-time approximation scheme (FPTAS) (unlike the standard Knapsack Problem). On this problem, we compare the QAOA and QRAO performance on multiple randomly generated test instances. Next, we consider a real-world supply chain problem akin to an MKP but with an added risk dimension. This is a significantly larger problem, which cannot be solved via QAOA due to the current limit of hardware and memory. The two problems are described as follows:

A. Multiple Knapsack Problem (MKP)

The Multiple Knapsack Problem (MKP) [15] is a generalization of the standard knapsack problem (KP) from a single knapsack to m knapsacks with (possibly) different capacities. MKP involves allocating a subset of n items to m different knapsacks, aiming to maximize the total profit of the selected items while ensuring that the capacity of each knapsack is not exceeded. This problem finds applications in various fields such as naval and financial management, where resources need to be efficiently allocated while considering capacity constraints.

We are given n items that need to be distributed among m knapsacks, each with a distinct capacity c_i for $i = 1, \dots, m$. Each item j has an associated profit p_j and weight w_j . The objective is to select m disjoint subsets of items, ensuring that the items in subset i fit within the capacity c_i of knapsack i while maximizing the total profit of the selected items. Formally, the MKP can

be formulated as an Integer Linear Programming (ILP) problem as follows:

$$\begin{aligned} \text{Maximize: } & \sum_{i=1}^m \sum_{j=1}^n p_j x_{ij} \\ \text{Subject to: } & \sum_{j=1}^n w_j x_{ij} \leq c_i, \quad i = 1, \dots, m \\ & \sum_{i=1}^m x_{ij} \leq 1 \quad j = 1, \dots, n, \\ & x_{ij} \in \{0, 1\}, \quad i = 1, \dots, m, \quad j = 1, \dots, n \end{aligned} \quad (1)$$

where $x_{ij} = 1$ if item j is assigned to knapsack i , and $x_{ij} = 0$ otherwise. It is usual to assume that the coefficients p_j , w_j , and c_i are positive integers, and to avoid trivial cases we demand that:

$$\begin{aligned} \max_{j=1, \dots, n} w_j &\leq \max_{i=1, \dots, m} c_i \\ \max_{i=1, \dots, m} c_i &\leq \max_{j=1, \dots, n} w_j \\ \sum_{j=1}^n w_j &\leq \max_{i=1, \dots, m} c_i \end{aligned} \quad (2)$$

Note that we assume every item j can fit into at least one knapsack; otherwise, the item should be disregarded. Furthermore, if the second inequality is breached, we can disregard the smallest knapsack, as it will not accommodate any item. Lastly, the last inequality prevents a trivial solution where all items can fit into the largest knapsack.

B. Risk-Aware Procurement Optimization

Optimization techniques have found wide application in addressing various challenges within supply chain and logistics management, particularly in mitigating risks and managing uncertainties. For instance, the classic Newsvendor Problem [16] tackles the dilemma of making purchasing decisions amidst uncertain demand, while disruptions in the supply chain can be managed through portfolio approaches that assess the impact of delays across multiple periods [17]. In scenarios where the ramifications of disruptions can be precisely quantified, stochastic optimization techniques are typically employed, solving deterministically across numerous generated scenarios [18]. However, when data limitations hinder the accurate realization of scenarios for optimization, methods that directly incorporate risk into the optimization process become essential. This can involve integrating risk minimization as a joint objective alongside cost minimization or constraining cost minimization problems with risk considerations [19].

Supply chain disruptions, whether stemming from unforeseen events like the COVID-19 pandemic or more

routine challenges such as labor disputes and adverse weather conditions, pose significant risks to global companies. These disruptions can lead to delays in deliveries, missed orders, and financial losses. Addressing these challenges requires a comprehensive approach that combines accurate risk quantification with cost-effective decision-making. Following the methodology outlined in [20], we first establish a supplier risk score metric by analyzing various data sources and identifying key risk factors through factor analysis. With these risk scores in hand, we develop a risk-constrained optimization model to formulate strategic procurement plans for multinational computer manufacturing firms. We construct the following simplified model for a typical procurement setting, with multiple parts to procure from multiple suppliers.

We assign a risk score $r_i \in [0, 9]$, for each supplier, i , derived from Supplier Risk Analysis [20]. The per part cost for procuring part j from supplier i is known in advance and denoted as $c_{i,j}$. For simplicity we assume every supplier can produce all parts, in the case where a supplier doesn't produce certain parts, the respective costs will be set to a large value. For each part j , there is a demand d_j to be fulfilled by sourcing from multiple suppliers and a risk tolerance level ψ_j . The objective is to choose suppliers to fulfill the demands for each part at minimized costs while managing the risk of suppliers within a given tolerance. (4) represents the demand constraint and (5) ensures the weighted average risks for each part are within the given thresholds.

TABLE I: Notations used in the model

Symbol	Description
i	Supplier index from set \mathcal{I}
j	Part index from set \mathcal{J}
$y_{i,j}$	Number of part j obtained from supplier i
r_i	Risk score for supplier i
ψ_j	Risk tolerance level for part j
$c_{i,j}$	Per part cost of procuring part j from supplier i
d_j	Demand for part j

Following the notations given in Table I, the following is the MILP formulation:

$$\text{Minimize: } \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} c_{i,j} y_{i,j} \quad (3)$$

$$\text{Subject to: } \sum_{i \in \mathcal{I}} y_{i,j} \geq d_j \quad \forall j \in \mathcal{J} \quad (4)$$

$$\sum_{i \in \mathcal{I}} r_i y_{i,j} \leq \psi_j d_j \quad \forall j \in \mathcal{J} \quad (5)$$

III. QUANTUM RANDOM ACCESS CODES

The representation of n qubits as a vector in \mathbb{C}^{2^n} may initially suggest a higher information capacity compared

to classical n bits. However, it is worth noting that according to the Holevo bound [21], transferring n -bit classical information without error requires n qubits. Nevertheless, if some errors are permissible, it becomes possible to encode multiple classical bits into a single qubit using $(n, 1, p)$ -QRAC codes [22]. In this context, (n, m, p) -QRAC refers to quantum random access codes that encode n classical bits into m qubits.

Definition 1: An $(n, 1, p)$ -QRA coding is a function that maps n -bit strings $x \in \{0, 1\}^n$ to 1-qubit states ρ_x satisfying the following condition that for every $i \in \{1, 2, \dots, n\}$ there exists a POVM

$$E^i = \{E_0^i, E_1^i\}$$

such that

$$\text{Tr}(E_{x_i}^i \rho_x) \geq p$$

The POVM E^i is integral to the decoding process, enabling the extraction of the i -th encoded bit x_i from the measured encoded state ρ_x with probability p . It's important to note that $(n, 1, p)$ -QRA codes lack significance when $p \leq \frac{1}{2}$, as $p = \frac{1}{2}$ implies random selection of binary bits. Additionally, (n, m, p) -QRA coding, where $m \geq 2$, follows a similar definition. Notably, specific instances include $(2, 1, 0.85)$ - and $(3, 1, 0.78)$ -QRA coding employed in QRAO [8]. However, further extension to $(n, 1, p)$ coding with $n \geq 4$ and $p > \frac{1}{2}$ is impractical. This limitation arises from the geometric constraint where a three-dimensional ball cannot be partitioned into sixteen non-empty regions using only four planes [23].

Example 1: $(2, 1, 0.85)$ -QRA Encoding

This encoding maps $x_1, x_2 \in \{0, 1\}^2$ to a pure state $\rho_{x_1 x_2} = |\psi(x_1 x_2)\rangle\langle\psi(x_1 x_2)|$, considering the map:

$$(x_1, x_2) \mapsto \rho_{x_1, x_2} := \frac{1}{2} \left(I + \frac{1}{\sqrt{2}} \left((-1)^{x_1} X + (-1)^{x_2} Z \right) \right) \quad (6)$$

where

$$\begin{aligned} |\psi(0, 0)\rangle &= \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} |1\rangle \\ |\psi(0, 1)\rangle &= \cos \frac{3\pi}{8} |0\rangle + \sin \frac{3\pi}{8} |1\rangle \\ |\psi(1, 0)\rangle &= \cos \frac{5\pi}{8} |0\rangle + \sin \frac{5\pi}{8} |1\rangle \\ |\psi(1, 1)\rangle &= \cos \frac{7\pi}{8} |0\rangle + \sin \frac{7\pi}{8} |1\rangle \end{aligned}$$

Then, this map is a $(2, 1, 0.85)$ -QRA coding with the POVMs

$$E^1 = \{|+\rangle\langle+|, |-\rangle\langle-|\}, \quad E^2 = \{|0\rangle\langle0|, |1\rangle\langle1|\},$$

The measurements described above are conducted in both the X and computational bases. The X basis measurement is specifically aimed at decoding the first classical bit, whereas the computational basis measurement is employed to decode the second classical bit. This dual measurement approach facilitates the extraction of both classical bits encoded within the quantum state.

Example 2: (3, 1, 0.78)-QRA Encoding Considering the map:

$$(x_1, x_2, x_3) \mapsto \rho_{x_1, x_2, x_3} := \frac{1}{2} \left(I + \frac{1}{\sqrt{3}} \left((-1)^{x_1} X + (-1)^{x_2} Y + (-1)^{x_3} Z \right) \right) \quad (7)$$

For every pair of $(x_1, x_2, x_3), \rho_{x_1, x_2, x_3}$ is a pure state and can be written in the form $\rho_{x_1, x_2, x_3} = |\psi(x_1, x_2, x_3)\rangle\langle\psi(x_1, x_2, x_3)|$, where:

$$\begin{aligned} |\psi(0, 0, 0)\rangle &= \cos \tilde{\theta}|0\rangle + e^{\frac{\pi i}{4}} \sin \tilde{\theta}|1\rangle \\ |\psi(0, 0, 1)\rangle &= \sin \tilde{\theta}|0\rangle + e^{\frac{\pi i}{4}} \cos \tilde{\theta}|1\rangle \\ |\psi(0, 1, 0)\rangle &= \cos \tilde{\theta}|0\rangle + e^{\frac{-\pi i}{4}} \sin \tilde{\theta}|1\rangle \\ |\psi(0, 1, 1)\rangle &= \sin \tilde{\theta}|0\rangle + e^{\frac{-\pi i}{4}} \cos \tilde{\theta}|1\rangle \\ |\psi(1, 0, 0)\rangle &= \cos \tilde{\theta}|0\rangle + e^{\frac{3\pi i}{4}} \sin \tilde{\theta}|1\rangle \\ |\psi(1, 0, 1)\rangle &= \sin \tilde{\theta}|0\rangle + e^{\frac{3\pi i}{4}} \cos \tilde{\theta}|1\rangle \\ |\psi(1, 1, 0)\rangle &= \cos \tilde{\theta}|0\rangle + e^{\frac{-3\pi i}{4}} \sin \tilde{\theta}|1\rangle \\ |\psi(1, 1, 1)\rangle &= \sin \tilde{\theta}|0\rangle + e^{\frac{-3\pi i}{4}} \cos \tilde{\theta}|1\rangle \end{aligned} \quad (8)$$

where $\tilde{\theta}$ satisfies the condition $(\cos(\tilde{\theta}))^2 = \frac{1}{2} + \frac{1}{2\sqrt{3}} > 0.79$. Then this map is a (3, 1, 0.79)-QRA codings with the POVMs

$$\begin{aligned} E^1 &= \{|+\rangle\langle+|, |-\rangle\langle-|\}, E^2 = \{|+\iota\rangle\langle+\iota|, |-\iota\rangle\langle-\iota|\} \\ E^3 &= \{|0\rangle\langle 0|, |1\rangle\langle 1|\} \end{aligned} \quad (9)$$

The measurements described above are conducted in the X , Y , and computational bases. Each measurement is specifically performed to decode the corresponding classical bit encoded within the quantum state. By employing these multiple bases, the decoding process enables the extraction of classical information encoded within the quantum system.

Utilizing QRAO [8], multiple classical bits are compactly encoded into a reduced number of qubits using Quantum Random Access Codes (QRACs), as explained earlier. For instance, employing a (3, 1)-QRAC, three classical binary variables x_1, x_2 , and x_3 are mapped to a single qubit through the application of the Pauli X , Y , and Z operators, respectively. In comparison to methods like QAOA or VQE, QRAO boasts a constant-factor space complexity advantage. Consequently, our focus in this study lies on leveraging QRAO with a (3, 1)-QRAC approach.

The objective is to simplify the optimization problem by directing it towards the exploration of the maximum eigenstate of the relaxed Hamiltonian H_{relax} . To achieve this, we first map classical binary variables into qubits through the construction of a relaxed Hamiltonian. This process involves performing a graph coloring of the instance graph G , (made from the objective problem) using the Large Degree First (LDF) method [24], with time complexity of $O(|V(G)| \log |V(G)| + \deg(G)|V(G)|)$, where $\deg(G)$ represents the maximum degree of the graph G . Upon completion of the LDF algorithm, the vertices are partitioned into the set V_c associated with the color $c \in C$. Here, the color of the i -th vertex v_i is denoted as $\text{color}(i)$. This partitioning satisfies the following condition:

$$e_{i,j} \in E(G) \Rightarrow \text{color}(i) \neq \text{color}(j)$$

Next, we allocate $\left\lceil \frac{|V_c|}{3} \right\rceil$ qubits for each color $c \in C$, enabling up to three vertices to be assigned to a single qubit. These vertices are ordered, and the Pauli operators X , Y , and Z are assigned accordingly. Subsequently, we employ variational methods like VQE to explore the maximum eigenstate of H_{relax} . Unlike the diagonal structure of the original Hamiltonian, H_{relax} comprises non-classical states, characterized by superposition and entanglement, as its maximal eigenstates. Consequently, the eigenstate obtained for H_{relax} cannot be directly linked to the classical solution due to its quantum nature. Instead, it represents a quantum state corresponding to the relaxed solution of the optimization problem, where the constraint that the solution must be a binary vector is lifted. To recover the classical solution, we employ quantum state rounding algorithms, as proposed in [8].

The first rounding algorithm, Pauli rounding, deciphers the encoded three classical bits in each qubit using the POVM outlined in Eq. 9. Essentially, this procedure involves measuring the j -th qubit with sufficient repetitions, determining the majority measurement outcome, and assigning it as the rounded value of the corresponding classical bit. However, Pauli rounding may encounter limitations when the relaxed state exhibits significant entanglement, preventing its representation in the form of $\rho_1 \otimes \rho_2 \otimes \dots \otimes \rho_n$. In such cases, the algorithm's effectiveness may be compromised due to the oversight of correlations among the qubits.

In contrast to Pauli rounding, the second rounding algorithm, known as Magic Rounding, mitigates the aforementioned issue. This method aims to decode three classical variables simultaneously from a single qubit. For a comprehensive understanding of Magic Rounding, further details can be found in [8] and [12].

IV. PROPOSED APPROACH

An integer linear program (ILP) consists of a linear program requiring variables to be integers. ILPs serve as expressive tools for formulating combinatorial optimization problems. Nonetheless, solving ILPs optimally is NP-hard.

One standard method to find an approximate solution for a combinatorial optimization problem is via Linear Relaxation:

- Formulate the optimization problem as an ILP.
- Derive a linear program (LP) from the ILP by relaxing the integrality constraints on variables. This resulting LP termed a relaxation of the original problem, retains the same objective function but operates over a broader solution set, leading to $\text{opt}(LP) \leq \text{opt}(ILP)$ for minimization problem.
- Solve the LP optimally using an efficient linear programming algorithm and rounding variables to integers by some rounding techniques.

In the current NISQ era, quantum algorithms still suffer from scaling to large-size problems. In our approach, we apply linear relaxation as a method to reduce the problem size. Given the LP relaxed solution, the binary decision variables that are solved to extreme values, i.e. very close to 0 or 1, will be rounded accordingly and fixed, leaving a reduced-size problem for QRAO to tackle.

V. RESULTS

We conducted initial investigations to ascertain the optimal ansatz and associated parameters for leveraging $(3, 1, p)$ -QRAO by solving a simpler version of our problem. All experiments are conducted with Qiskit's [25] AerSimulator.

Notably, our findings consistently favored the *EfficientSU2* ansatz employing full entanglement over alternatives such as *PauliTwoDesign* and *RealAmplitudes*. This preference stems from the intrinsic characteristics described in Eq. 8, where we see that $(3, 1, p)$ -QRAC has complex amplitudes. In contrast, the *RealAmplitudes* ansatz exclusively addresses real-valued parameters. The superiority of *EfficientSU2*, denoted by its designation, lies in its design tailored for hardware efficiency within $SU(2)$ 2-local circuits. These circuits comprise layers of single-qubit operations interconnected by $SU(2)$ and CX entanglements, where $SU(2)$ denotes the special unitary group of degree 2. Such circuits are characterized by 2×2 unitary matrices with determinant 1, exemplified by the Pauli rotation gates.

As observed in Table II, for the $(3, 1, p)$ -QRAO scenario, the *EfficientSU2* ansatz demonstrates superior performance with five repetitions and full entanglement. Conversely, as previously discussed, the *RealAmplitudes* ansatz exhibits notably poor performance in this context.

Reps.	EfficientSU2			PauliTwoDesign			RealAmplitudes		
	Circ.	Full	Linear	Circ.	Full	Linear	Circ.	Full	Linear
0	27	30	33	20	19	18	0	0	0
1	18	5	2	4	9	9	0	0	0
2	21	27	12	26	24	14	0	0	0
3	32	20	10	3	4	2	0	0	0
4	23	31	12	1	7	7	0	0	0
5	30	38	24	12	5	18	0	0	0

TABLE II: Number of solved instances for various ansätze, employing different types of entanglement, with increasing repetitions.

A. QAOA vs QRAO on Multiple Knapsack Problem

Subsequently, we conducted a comparative analysis between QAOA and $(3, 1, p)$ -QRAO, evaluating their performances on 20 randomly generated, non-trivial (guaranteed by Eq. (2)) instances of the Multiple Knapsack Problem (MKP). The problem sizes comprised a maximum of 20 binary variables. In QAOA, each variable necessitated one qubit, while QRAO utilized fewer qubits due to compression as shown in Tab. (III) and Tab. (IV) respectively.

Instance	# vars	# qubits	Optimal	QAOA Obj.	Opt. gap
1	13	13	20	20	0%
2	18	18	25	25	0%
3	17	17	15	15	0%
4	20	20	15	12	20%
5	15	15	20	20	0%
6	15	15	19	19	0%
7	16	16	17	17	0%
8	20	20	19	16	15.79%
9	14	14	19	19	0%
10	18	18	15	13	13.33%
11	16	16	9	8	11.11%
12	18	18	32	32	0%
13	19	19	14	infeasible	N.A.
14	14	14	15	15	0%
15	17	17	15	15	0%
16	13	13	18	18	0%
17	16	16	14	11	21.43%
18	15	15	12	12	0%
19	14	14	12	12	0%
20	15	15	19	19	0%

TABLE III: Multiple Knapsack Problem QAOA Results

In QAOA, we employed the default *QAOAAnsatz*, while in $(3, 1, p)$ -QRAO, we used the *EfficientSU2* ansatz with full entanglement, based on the findings presented in Tab. II. In both the cases, we used COBYLA as the classical optimizer. The rounding scheme employed for QRAO adhered to the Magic Rounding method.

The optimal solutions' objective values, obtained by CPLEX, is used to compare with the solutions obtained

Instance	# vars	# qubits	Optimal	QRAO Obj.	Opt. gap
1	13	5	20	20	0%
2	18	6	25	25	0%
3	17	6	15	15	0%
4	20	7	15	13	13.33%
5	15	5	20	20	0%
6	15	5	19	19	0%
7	16	6	17	17	0%
8	20	7	19	19	0%
9	14	5	19	16	15.79%
10	18	6	15	infeasible	N.A.
11	16	6	9	5	44.44%
12	18	6	32	32	0%
13	19	7	14	14	0%
14	14	5	15	15	0%
15	17	6	15	15	0%
16	13	5	18	18	0%
17	16	6	14	14	0%
18	15	5	12	12	0%
19	14	5	12	12	0%
20	15	5	19	9	52.63%

TABLE IV: Multiple Knapsack Problem QRAO Results

from QAOA/QRAO. The optimality gap is the absolute difference between the solved objective value and the optimal objective value, over the optimal objective value in percentage.

The results in Tab. V is an overall summary for Tab. III and Tab. IV, which showed a slightly better performance for QRAO.

Method	Feasible	Optimal
QAOA	95%	70%
QRAO	95%	75%

TABLE V: Multiple Knapsack Problem performance comparison of QAOA vs QRAO results

B. QRAO with Linear Relaxation on Risk-Aware Procurement Optimization Problem

We generated multiple random instances of the Risk-Aware Procurement Optimization problem, each featuring approximately 100 binary variables. The sheer scale of these problems surpasses the current hardware limitations of QAOA, rendering them challenging to tackle. Moreover, as the size of the problems expands, the efficiency of QRAO is similarly impacted. To mitigate these hurdles, we employed the Linear Relaxation technique as a strategic response. We first tried with a random number of variables being fixed, and compared it to 90% of the variables being fixed. This approach aids in navigat-

ing the complexities posed by the increasing dimensions of the problems at hand, offering a viable pathway forward.

Method	Feasible	Optimal
QRAO-LR	100%	80%
QRAO-LR 90%	70%	50%

TABLE VI: QRAOs' (with Linear Relaxation) performance comparison

VI. CONCLUSION & FUTURE OUTLOOK

This paper introduces a hybrid methodology aimed at addressing constraint optimization problems prevalent in business contexts. It focuses on two knapsack problem variants: the Multiple Knapsack Problem (MKP) and MKP incorporating an additional risk dimension. Our analysis reveals that, despite utilizing fewer qubits, QRAO achieves performance comparable to QAOA. Notably, QRAO demonstrates an added advantage by enabling problem-solving for problems ranging ≥ 100 binary variables. This scalability advantage is particularly significant as QAOA encounters challenges with errors stemming from its high qubit and memory demands at such scales.

Moreover, our study demonstrates that by integrating the classical technique of Linear Relaxation, we can achieve a further reduction in qubit demand. This integration not only minimizes the resources needed but also enhances the quality of the obtained results. By leveraging Linear Relaxation alongside quantum approaches, our method offers a comprehensive solution framework that optimizes both resource utilization and solution efficacy. This combination underscores the versatility and effectiveness of our hybrid approach in tackling complex optimization challenges across diverse business domains.

Expanding on this study, we aim to enhance our approach by integrating stochastic elements into the problem formulation. This entails addressing uncertainty surrounding both demand and supply variables [28]. Additionally, we intend to explore alternative encoding schemes such as multi-basis encoding (MBE) [26] and leverage polynomial space compression techniques [27]. Furthermore, we will investigate the potential of employing various classical methods to streamline the problem-solving process. These adaptations are anticipated to broaden the scope of our methodology and enhance its effectiveness across diverse problem landscapes.

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