SPHERICALLY SYMMETRIC EINSTEIN-SCALAR-FIELD EQUATIONS FOR SLOWLY PARTICLE-LIKE DECAYING NULL INFINITY

CHUXIAO LIU[1,4] AND XIAO ZHANG[2,3,4]

Abstract. We show that the spherically symmetric Einstein-scalar-field equations for small slowly particle-like decaying initial data at null infinity have unique global solutions.

1. Introduction

Spherically symmetric spacetime metrics can be written as

$$ds^2 = -gqdu^2 - 2gdudr + r^2(d\theta^2 + s\sin^2\theta d\psi^2) \tag{1.1}$$

in Bondi coordinates, see, e.g. [3, 8, 9], where $g(u,r)$ and $q(u,r)$ are C^2 nonnegative functions over $(0,\infty)$. The null frames are

$$\vec{n} = \frac{1}{\sqrt{gq}}D, \quad \vec{l} = \frac{1}{\sqrt{gq^{-1}}}\frac{\partial}{\partial r}, \quad e_1 = \frac{1}{r}\frac{\partial}{\partial\theta}, \quad e_2 = \frac{1}{r\sin\theta}\frac{\partial}{\partial\psi},$$

where

$$D = \frac{\partial}{\partial u} - \frac{q}{2}\frac{\partial}{\partial r}$$

is the derivative along the incoming light rays.

Throughout the paper, we denote $\bar{f}(r)$ the integral average of integrable function $f(r)$ over $[0,r]$

$$\bar{f}(r) = \frac{1}{r}\int_0^r f(r')dr'.$$

For a real spherically symmetric C^2 scalar field $\phi(u,r)$ on $(0,\infty)\times(0,\infty)$, the Einstein-scalar field equations are

$$R_{\mu\nu} = 8\pi\partial_\mu\phi\partial_\nu\phi, \quad \Box\phi = 0. \tag{1.2}$$

Under the following regularity conditions at $r=0$ and boundary condition at $r=\infty$,

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Regularity Condition I: For each u,

$$\lim_{r\to 0}\left(-r\frac{\partial g}{\partial u}+r\frac{\partial q}{\partial u}+\frac{r^2}{2g}\frac{\partial g}{\partial r}-8\pi r^2\Big(\frac{\partial \phi}{\partial u}-\frac{q}{2}\frac{\partial \phi}{\partial r}\Big)^2\right)=0. \tag{1.3}$$

Regularity Condition II: For each u,

$$\lim_{r\to 0}\big(r\phi\big)=\lim_{r\to 0}\big(rq\big)=0. \tag{1.4}$$

Boundary Condition: For each u,

$$\lim_{r\to\infty} g=\lim_{r\to\infty} q=1, \tag{1.5}$$

the Einstein-scalar field equations (1.2) are equivalent to the following systems, see, e.g. [9]

$$\begin{cases} g=\exp\left\{-4\pi\displaystyle\int_r^\infty \frac{(h-\bar{h})^2}{r'}dr'\right\},\\ q=\bar{g}=\dfrac{1}{r}\displaystyle\int_0^r g dr',\\ Dh=\dfrac{g-\bar{g}}{2r}(h-\bar{h}). \end{cases} \tag{1.6}$$

The Bondi mass $M_B(u)$, the final Bondi mass M_{B1}, the Bondi-Christodoulou mass $M(u)$ and the final Bondi-Christodoulou mass M_1 are given as follows [1, 3, 8, 9]

$$M_B(u)=\lim_{r\to\infty}\frac{r}{2}\Big(1-\bar{g}\Big),\quad M_{B1}=\lim_{u\to\infty}M_B(u),$$
$$M(u)=\lim_{r\to\infty}\frac{r}{2}\Big(1-\frac{\bar{g}}{g}\Big),\quad M_1=\lim_{u\to\infty}M(u).$$

In [3, 4, 5], Christodoulou proved the global existence and uniqueness of classical solutions for spherically symmetric Einstein-scalar-field equations with small initial data and the generalized solutions in the large for particle-like decaying null infinity. He also studied the asymptotic behaviour of the generalized solutions and proved the formation of black holes of mass M_1 surrounded by vacuum when the final Bondi-Christodoulou mass $M_1\neq 0$ [6].

Under the double null coordinates

$$ds^2=-\Omega^2 dudv+r^2\left(d\theta^2+\sin^2\theta d\psi^2\right),$$

Christodoulou solved the characteristic initial value problem for small bounded variation norm and showed the global existence and uniqueness of classical solution in spherically symmetric case for particle-like decaying null infinity[7]. These results was extended to more general case by Luk-Oh-Yang [10, 11], which are summarized as follows.

Theorem 1.1. *Let C_{u_0} be the initial curve with $v \geq u_0$ which satisfies*

$$\partial_v r\Big|_{C_{u_0}} = \frac{1}{2}, \quad r(u_0, u_0) = m(u_0, u_0) = 0.$$

Suppose the data on the initial curve is given by

$$2\partial_v(r\phi)(u_0, v) = \Phi(v),$$

where $\Phi : [u_0, \infty) \to \mathbb{R}$ is a smooth function which satisfies

$$\int_u^v \big|\Phi(u_0, v')\big| dv' \leq \epsilon\big(v - u\big)^{1-\gamma}, \quad |\Phi(u_0, v)| + \Big|\frac{\partial \Phi}{\partial v}(u_0, v)\Big| \leq \epsilon$$

for any $v \geq u \geq u_0$, where $\gamma > 0$ is certain positive constant. Then there exists a unique global solution to the spherically symmetric Einstein-scalar field equations. And the resulting spacetime is future causally geodesically complete. Moreover, the solution satisfies the following uniform priori estimates,

$$\partial_v r > \frac{1}{3}, \quad -\frac{1}{6} > \partial_u r > -\frac{2}{3}, \quad \frac{2m}{r} < \frac{1}{2}$$

and

$$\begin{aligned} |\phi| &\leq C\epsilon \min\{1, r^{-\gamma}\}, \\ |\partial_v(r\phi)| &\leq C\left(|\Phi(v)| + \epsilon \min\{1, r^{-\gamma}\}\right), \\ |\partial_u(r\phi)| &\leq C\epsilon, \\ |\partial_v^2(r\phi)| + |\partial_v^2 r| + |\partial_u^2(r\phi)| + |\partial_u^2 r| &\leq C\epsilon, \end{aligned}$$

where $C > 0$ is a constant depending only on γ. Furthermore, if Φ satisfies the strong asymptotic flatness condition

$$\sup_{v \in [u_0, \infty)} \big\{(1 + v)^{\epsilon}|\Phi(v)| + (1 + v)^{\epsilon+1}|\partial_v \Phi(v)|\big\} \leq A_0 < \infty$$

for some $A_0 > 0$ and $\epsilon > 1$. Then the following estimates

$$\begin{aligned} |\phi| &\leq A_1 \min\Big\{u^{-w}, r^{-1}u^{-(w-1)}\Big\}, \\ |\partial_v(r\phi)| &\leq A_1 \min\big\{u^{-w}, r^{-w}\big\}, \\ |\partial_u(r\phi)| &\leq A_1 u^{-w}, \\ |\partial_v^2(r\phi)| &\leq A_1 \min\Big\{u^{-(w+1)}, r^{-(w+1)}\Big\}, \\ |\partial_u^2(r\phi)| &\leq A_1 u^{-(w+1)}, \\ |\partial_v^2 r| &\leq A_1 \min\big\{u^{-3}, r^{-3}\big\}, \\ |\partial_u^2 r| &\leq A_1 u^{-3}, \end{aligned}$$

hold for some $A_1 > 0$ and $w = \min\{\epsilon, 3\}$.

In [8], Liu and Zhang proved the global existence and uniqueness for classical solutions with small initial data, and for generalized solutions with large initial data for wave-like decaying null infinity.

Theorem 1.2. *Let $\epsilon \in (0,2]$. Given initial data $\breve{h}(r) \in C^1[0,\infty)$. Denote*

$$d_0 = \inf_{b>0} \sup_{r\geq 0} \left\{ \left(1+\frac{r}{b}\right)^{1+\epsilon} \left|\breve{h}(r)\right| + \left(1+\frac{r}{b}\right)^{1+\epsilon} \left| b\frac{\partial \breve{h}}{\partial r}(r)\right| \right\}.$$

Then there exists $\delta > 0$ such that if $d_0 < \delta$, there exists a unique global classical solution

$$h(u,r) \in C^1\Big([0,\infty)\times[0,\infty)\Big)$$

of (1.6) which satisfies the initial condition $h(0,r) = \breve{h}(r)$ and the decay property

$$|h(u,r)| \leq \frac{C}{\left(1+\frac{u}{2}+r\right)^{1+\epsilon}}, \quad \left|\frac{\partial h}{\partial r}(u,r)\right| \leq \frac{C}{\left(1+\frac{u}{2}+r\right)^{1+\epsilon}}$$

for some constant C depending on ϵ only. Moreover, the corresponding spacetime is future causally geodesically complete with vanishing final Bondi mass.

Theorem 1.3. *For any initial data $\breve{h}(r) \in C^1[0,\infty)$ which satisfies*

$$\breve{h}(r) = O\Big(\frac{1}{r^{1+\epsilon}}\Big), \quad \frac{\partial \breve{h}}{\partial r}(r) = O\Big(\frac{1}{r^{1+\epsilon}}\Big)$$

as $r \to \infty$ for some $\epsilon \in (0,2]$, there exists at least one global generalized solution which has the same data as a classical solution coincides with it in the domain of existence of the latter.

We refer to [2, 12] for the spherically symmetric Einstein-scalar field equations with nontrivial potential for particle-like decaying null infinity, and to [9] for wave-like decaying null infinity.

In this paper, we prove the global existence and uniqueness for classical solutions for small slowly particle-like decaying null infinity.

Theorem 1.4. *Let $\epsilon \in (0,1)$. Given initial data $\breve{h}(r) \in C^1[0,\infty)$. Denote*

$$d_0 = \inf_{b>0} \sup_{r\geq 0} \left\{ \left(1+\frac{r}{b}\right)^{\epsilon} \left|\breve{h}(r)\right| + \left(1+\frac{r}{b}\right)^{1+\epsilon} \left| b\frac{\partial \breve{h}}{\partial r}(r)\right| \right\}.$$

Then there exists $\delta > 0$ such that if $d_0 < \delta$, there exists a unique global classical solution

$$h(u,r) \in C^1\Big([0,\infty)\times[0,\infty)\Big)$$

of (1.6) which satisfies the initial condition $h(0,r) = \breve{h}(r)$ and the decay property

$$|h(u,r)| \leq \frac{C}{(1+u+r)^{\epsilon}}, \quad \left|\frac{\partial h}{\partial r}(u,r)\right| \leq \frac{C}{(1+u+r)^{1+\epsilon}}$$

for some constant C depending on ϵ only. Moreover, the corresponding spacetime is future causally geodesically complete. If we further assume $\epsilon \in (\frac{1}{2}, 1)$, then the final Bondi mass vanishes.

The paper is organized as follow: In Section 2, we derive the main estimates. In Section 3, we prove the main theorem.

2. Main lemma

In this section, we derive the estimates on the iteration solution as well as its partial derivative with respect to r. Denote X the space of C^1 functions with the finite norm

$$\|h\|_X = \sup_{u\geq 0, r\geq 0}\left\{(1+u+r)^{\epsilon}|h(u,r)| + (1+u+r)^{1+\epsilon}\Big|\frac{\partial h}{\partial r}(u,r)\Big|\right\} < \infty.$$

As in [3, 8], for $h_n \in X$, let h_{n+1} be the solution of the equation

$$D_n h_{n+1} - \frac{g_n - \bar{g}_n}{2r} h_{n+1} = -\frac{g_n - \bar{g}_n}{2r}\bar{h}_n \tag{2.1}$$

with the initial data

$$h_{n+1}(0,r) = h(0,r),$$

where g_n and D_n are the metric given by (1.6) and the derivative along the incoming light rays corresponding to h_n.

Lemma 2.1. *Given the initial data $h(0,r)$ and the nth-iteration solution $h_n(u,r)$ which are C^1 and satisfy*

$$\|h(0,r)\|_X = d, \quad \|h_n(u,r)\|_X = x.$$

Then the solution of (2.1) satisfies

$$\|h_{n+1}\|_X \leq C\exp{(Cx^2)}(d+x^3)(2+x^2) \tag{2.2}$$

for some constant $C > 0$.

Proof: By the assumption, we have

$$\begin{aligned}
|\bar{h}_n(u,r)| &\leq \frac{1}{r}\int_0^r |h_n(u,s)|ds \\
&\leq \frac{x}{r}\int_0^r \frac{ds}{(1+u+s)^{\epsilon}} \\
&= \frac{x}{(1-\epsilon)r}\left[(1+u+r)^{1-\epsilon} - (1+u)^{1-\epsilon}\right] \qquad (2.3)\\
&\leq \frac{x}{(1-\epsilon)r}\frac{(1+u+r) - (1+u)^{1-\epsilon}(1+u+r)^{\epsilon}}{(1+u+r)^{\epsilon}} \\
&\leq \frac{x}{(1-\epsilon)(1+u+r)^{\epsilon}}.
\end{aligned}$$

Using it, we estimate $|(h_n - \bar{h}_n)(u,r)|$ in the following.

(1) For $0 \le r \le 1+u$, using (2.11) in [8], we obtain

$$|(h_n - \bar{h}_n)(u,r)| \le \frac{5xr(1+u)^{1-\epsilon}}{\epsilon(1-\epsilon)(1+u+r)^2}.$$

(2) For $r \ge 1+u$, we have

$$\begin{aligned}|(h_n - \bar{h}_n)(u,r)| &\le |h_n(u,r)| + |\bar{h}_n(u,r)| \\ &\le \frac{x}{(1+u+r)^\epsilon} + \frac{x}{1-\epsilon}\frac{1}{(1+u+r)^\epsilon} \\ &\le \frac{2x}{1-\epsilon}\frac{(1+u+r)}{(1+u+r)^{1+\epsilon}} \\ &\le \frac{4xr}{(1-\epsilon)\,(1+u+r)^{1+\epsilon}}.\end{aligned}$$

Let $c = \frac{5}{\epsilon(1-\epsilon)}$. Then, for $r \ge 0$,

$$|(h_n - \bar{h}_n)(u,r)| \le \begin{cases} \dfrac{cxr\,(1+u)^{1-\epsilon}}{(1+u+r)^2}, & 0 \le r \le 1+u, \\ \dfrac{cxr}{(1+u+r)^{1+\epsilon}}, & r \ge 1+u. \end{cases} \tag{2.4}$$

Thus

$$|(h_n - \bar{h}_n)(u,r)| \le \frac{cxr}{(1+u+r)^{1+\epsilon}}. \tag{2.5}$$

Let $k = \exp\left(-\frac{2\pi c^2 x^2}{\epsilon}\right)$, $0 < k < 1$. Since $g(u,r)$ is monotonically increasing with respect to r, we obtain

$$\begin{aligned}\bar{g}_n(u,r) &\ge g_n(u,0) \\ &\ge \exp\left[-4\pi\int_0^\infty \frac{(h_n - \bar{h}_n)^2}{s}ds\right] \\ &\ge \exp\left[-4\pi c^2 x^2 \int_0^\infty \frac{ds}{(1+u+s)^{1+2\epsilon}}\right] \\ &\ge \exp\left[-\frac{2\pi c^2 x^2}{\epsilon(1+u)^{2\epsilon}}\right] \ge k.\end{aligned} \tag{2.6}$$

Claim: Let $c_1 = \frac{4\pi c^2}{\epsilon}$. We have

$$(g_n - \bar{g}_n)(u,r) \le \begin{cases} \dfrac{c_1 x^2 r^2}{(1+u)^{2\epsilon-1}\,(1+u+r)^3}, & 0 \le r \le 1+u, \\ \dfrac{c_1 x^2 r}{(1+u)^{2\epsilon}\,(1+u+r)}, & r \ge 1+u. \end{cases} \tag{2.7}$$

Indeed, for $0 \le r \le 1+u$, (2.7) is a direct consequence of (2.5) and (2.14) in [8]. For $r \ge 1+u$, by using (2.5) and

$$\frac{\partial g_n}{\partial r} = \frac{4\pi(h_n - \bar{h}_n)^2 g_n}{r},$$

we obtain

$$\begin{aligned}
(g_n - \bar{g}_n)(u,r) &= \frac{1}{r}\int_0^r \int_{r'}^r \frac{\partial g_n}{\partial s} ds dr' \\
&\le \frac{2\pi c^2 x^2}{\epsilon r} \int_0^r \left[\frac{1}{(1+u+r')^{2\epsilon}} - \frac{1}{(1+u+r)^{2\epsilon}}\right] dr' \\
&\le \frac{2\pi c^2 x^2}{\epsilon r} \left[\frac{r}{(1+u)^{2\epsilon}} - \frac{r}{(1+u+r)^{2\epsilon}}\right] \\
&= \frac{2\pi c^2 x^2}{\epsilon} \frac{(1+u+r)^2 - (1+u)^{2\epsilon}(1+u+r)^{2-2\epsilon}}{(1+u)^{2\epsilon}(1+u+r)^2} \\
&\le \frac{c_1 x^2 r}{(1+u)^{2\epsilon}(1+u+r)}.
\end{aligned}$$

Thus, the claim follows and it also implies that

$$(g_n - \bar{g}_n)(u,r) \le \frac{c_1 x^2 r}{(1+u)^{2\epsilon+1}}. \tag{2.8}$$

Therefore

$$\begin{aligned}
\left| -\frac{g_n - \bar{g}_n}{2r} \bar{h}_n \right| &\le \frac{1}{2r} \frac{c_1 x^2 r}{(1+u)^{2\epsilon+1}} \frac{x}{1-\epsilon} \frac{1}{(1+u+r)^{\epsilon}} \\
&\le \frac{c_1 x^3}{2(1-\epsilon)(1+u)^{2\epsilon+1}(1+u+r)^{\epsilon}}.
\end{aligned}$$

For the characteristic

$$r(u) = \chi_n(u; r_0),$$

we use (2.6) and, e.g. (3.26) in [12], to obtain

$$\begin{aligned}
1 + u + r(u) &\ge \frac{k}{2}(1 + u_1 + r_1), \\
1 + r_0 &\ge \frac{k}{2}(1 + r_1 + u_1).
\end{aligned}$$

They yield

$$|h(0, r_0)| \le \frac{d}{(1+r)^{\epsilon}} \le \frac{2^{\epsilon} d}{k^{\epsilon}(1+r_1)^{\epsilon}}, \tag{2.9}$$

$$\int_0^{u_1} \left[\frac{g_n - \bar{g}_n}{2r}\right]_{\chi_n} du \le \frac{c_1 x^2}{2} \int_0^{\infty} \frac{du}{(1+u)^{2\epsilon+1}} \le \frac{c_1 x^2}{4\epsilon}. \tag{2.10}$$

Let $c_2 = \frac{c_1}{2^{2-\epsilon}k^{\epsilon}\epsilon(1-\epsilon)}$. Integrating (2.1) along the characteristic, we obtain

$$
\begin{aligned}
&(1+u_1+r_1)^{\epsilon}\left|h_{n+1}(u_1,r_1)\right| \\
&\le |h(0,r_0)| \exp\left\{\int_0^{u_1}\left[\frac{g_n-\bar{g}_n}{2r}\right]_{\chi_n} du\right\} \\
&\quad + \int_0^{u_1} \exp\left\{\int_u^{u_1}\left[\frac{g_n-\bar{g}_n}{2r}\right]_{\chi_n} du'\right\}\left|-\frac{g_n-\bar{g}_n}{2r}\bar{h}_n\right|_{\chi_n} du \\
&\le c_2 \exp\left(c_2 x^2\right)\left(d+c_2 x^3\right).
\end{aligned}
\tag{2.11}
$$

Next we estimate $\frac{\partial h_{n+1}}{\partial r}$. Using (9.16) in [3] and (2.23) in [8], we obtain

$$
D_n \frac{\partial h_{n+1}}{\partial r} - \frac{g_n-\bar{g}_n}{r}\frac{\partial h_{n+1}}{\partial r} = f_1,
\tag{2.12}
$$

where

$$
\begin{aligned}
f_1 =& \frac{1}{2}\frac{\partial^2 \bar{g}_n}{\partial r^2}(h_{n+1}-\bar{h}_n) - \frac{g_n-\bar{g}_n}{2r}\frac{\partial \bar{h}_n}{\partial r}, \\
\frac{\partial^2 \bar{g}_n}{\partial r^2} =& -\frac{2(g_n-\bar{g}_n)}{r^2} + \frac{4\pi(h_n-\bar{h}_n)^2}{r^2} g_n.
\end{aligned}
$$

Using (2.4) and (2.7), for $0 \le r \le 1+u$ and $r \ge 1+u$, we obtain

$$
\left|\frac{\partial^2 \bar{g}_n}{\partial r^2}\right| \le
\begin{cases}
\dfrac{2}{r^2}\dfrac{c_1 x^2 r^2}{(1+u)^{2\epsilon-1}(1+u+r)^3} + \dfrac{4\pi}{r^2}\dfrac{c^2 x^2 r^2 (1+u)^{2-2\epsilon}}{(1+u+r)^4} \\
\dfrac{2}{r^2}\dfrac{c_1 x^2 r}{(1+u)^{2\epsilon}(1+u+r)} + \dfrac{4\pi}{r^2}\dfrac{c^2 x^2 r^2}{(1+u+r)^{2+2\epsilon}}
\end{cases}
$$

respectively. Therefore

$$
\left|\frac{\partial^2 \bar{g}_n}{\partial r^2}\right| \le \frac{(2c_1+4\pi c^2)x^2}{(1+u)^{2\epsilon+1}(1+u+r)}.
\tag{2.13}
$$

Let $c_3 = \frac{c_2(2\pi c^2+c_1)}{1-\epsilon}$. Using (2.3), (2.11), (2.5) and (2.8), we obtain

$$
\begin{aligned}
\frac{1}{2}\left|\frac{\partial^2 \bar{g}_n}{\partial r^2}\right| \cdot \left|h_{n+1}-\bar{h}_n\right| &\le \frac{c_3 x^2 (d+x+x^3)\exp\left(c_2 x^2\right)}{(1+u)^{2\epsilon+1}(1+u+r)^{1+\epsilon}}, \\
\left|\frac{g_n-\bar{g}_n}{2r}\frac{\partial \bar{h}_n}{\partial r}\right| &= \left|\frac{(g_n-\bar{g}_n)(h_n-\bar{h}_n)}{2r^2}\right| \\
&\le \frac{1}{2r^2}\frac{c_1 x^2 r}{(1+u)^{2\epsilon+1}}\frac{c x r}{(1+u+r)^{1+\epsilon}} \\
&\le \frac{c c_1 x^3}{2(1+u)^{2\epsilon+1}(1+u+r)^{1+\epsilon}}.
\end{aligned}
$$

Let $c_4 = c_3 + \frac{cc_1}{2}$. We obtain

$$|f_1| \le \frac{c_4 x^2 (d + x + x^3) \exp\left(c_2 x^2\right)}{(1+u)^{2\epsilon+1}\,(1+u+r)^{1+\epsilon}}. \tag{2.14}$$

Similar to (2.9), we have

$$\left|\frac{\partial h}{\partial r}(0, r_0)\right| \le \frac{2^{1+\epsilon} d}{k^{1+\epsilon}\,(1+u+r)^{1+\epsilon}}. \tag{2.15}$$

Let $c_5 = \max\left\{2c_2, \frac{2^\epsilon c_4}{\epsilon k^{1+\epsilon}}\right\}$. Using (2.10), (2.14) and (2.15), we obtain

$$\begin{aligned} \left|\frac{\partial h_{n+1}}{\partial r}(u_1, r_1)\right| \le & \left|\frac{\partial h}{\partial r}(0, r_0)\right| \exp\left\{\int_0^{u_1} \left[\frac{g_n - \bar{g}_n}{r}\right]_{\chi_n} du\right\} \\ & + \int_0^{u_1} \exp\left\{\int_u^{u_1} \left[\frac{g_n - \bar{g}_n}{r}\right]_{\chi_n} du'\right\} [|f_1|]_{\chi_n} du \\ \le & \frac{c_5 (d + x^3)(1 + x^2) \exp\left(c_5 x^2\right)}{(1 + u_1 + r_1)^{1+\epsilon}}. \end{aligned} \tag{2.16}$$

Let $C = 2\max\{c_5, c_2^2\}$. Using (2.11), we have

$$\|h_{n+1}\|_X \le C \exp(Cx^2)(d + x^3)(2 + x^2).$$

Thus proof of the lemma is complete. Q.E.D.

3. Proof of the main theorem

In this section we prove Theorem 1.4. Denote $\{h_n\}$ the sequence of the iteration solutions of (2.1). We first show that $\{h_n\}$ converges in function space

$$Y = \left\{ h \in C^0[0,\infty) \times [0,\infty) \middle| \|h\|_Y < \infty \right\},$$

where

$$\|h\|_Y = \sup_{u \ge 0, r \ge 0} \left\{ (1+u+r)^\epsilon |h(u,r)| \right\}.$$

Lemma 3.1. *Assume for some $n > 0$ such that*

$$\|h_{n-1}\|_X \le x, \quad \|h_n\|_X \le x$$

for some constant $x > 0$. Then there exists $F(x) \in (0, \frac{1}{2})$ such that

$$\|h_{n+1} - h_n\|_Y \le F(x) \|h_n - h_{n-1}\|_Y.$$

Proof: From (9.27) in [3], the following equation holds

$$D_n(h_{n+1} - h_n) - \frac{g_n - \bar{g}_n}{2r}(h_{n+1} - h_n) = f_2, \tag{3.1}$$

where

$$f_2 = \frac{\bar{g}_n - \bar{g}_{n-1}}{2}\frac{\partial h_n}{\partial r} - \frac{g_n - \bar{g}_n}{2r}(\bar{h}_n - \bar{h}_{n-1}) + \frac{g_n - \bar{g}_n - g_{n-1} + \bar{g}_{n-1}}{2r}(h_n - \bar{h}_{n-1}).$$

Similar to (2.3), we have

$$\left|\bar{h}_n - \bar{h}_{n-1}\right| \leq \frac{\|h_n - h_{n-1}\|_Y}{(1-\epsilon)\,(1+u+r)^{\epsilon}},$$

then

$$\left|h_n - h_{n-1} - \bar{h}_n + \bar{h}_{n-1}\right| \leq \frac{2\|h_n - h_{n-1}\|_Y}{(1-\epsilon)\,(1+u+r)^{\epsilon}}. \tag{3.2}$$

Similar to (2.4), we have

$$\left|h_n + h_{n-1} - \bar{h}_n - \bar{h}_{n-1}\right| \leq \begin{cases} \dfrac{2cxr\,(1+u)^{1-\epsilon}}{(1+u+r)^2}, & 0 \leq r \leq 1+u \\ \dfrac{2cxr}{(1+u+r)^{1+\epsilon}}, & r \geq 1+u. \end{cases} \tag{3.3}$$

Then (3.2) and (3.3) imply that

$$\left|(h_n - \bar{h}_n)^2 - (h_{n-1} - \bar{h}_{n-1})^2\right| \leq \begin{cases} \dfrac{4cxr\,(1+u)^{1-\epsilon}\,\|h_n - h_{n-1}\|_Y}{(1-\epsilon)\,(1+u+r)^{2+\epsilon}}, & 0 \leq r \leq 1+u \\ \dfrac{4cxr\|h_n - h_{n-1}\|_Y}{(1-\epsilon)\,(1+u+r)^{1+2\epsilon}}, & r \geq 1+u. \end{cases} \tag{3.4}$$

Thus,

$$|g_n - g_{n-1}| \leq 4\pi \int_r^{\infty} \left|(h_n - \bar{h}_n)^2 - (h_{n-1} - \bar{h}_{n-1})^2\right| \frac{ds}{s} \leq \frac{16\pi cx\|h_n - h_{n-1}\|_Y}{\epsilon(1-\epsilon)\,(1+u+r)^{2\epsilon}}. \tag{3.5}$$

Therefore

$$|\bar{g}_n - \bar{g}_{n-1}| \leq \frac{1}{r}\int_0^r |g_n - g_{n-1}|dr \leq \frac{16\pi cx\|h_n - h_{n-1}\|_Y}{\epsilon(1-\epsilon)(1+u)^{2\epsilon}}. \tag{3.6}$$

Let $c_6 = \frac{8\pi c c_5}{\epsilon(1-\epsilon)}$. Using (3.6) and (2.16), we have

$$\begin{aligned}\left|\frac{\bar{g}_n - \bar{g}_{n-1}}{2}\frac{\partial h_n}{\partial r}\right| \leq & \frac{8\pi cx\|h_n - h_{n-1}\|_Y}{\epsilon(1-\epsilon)(1+u)^{2\epsilon}} \\ & \times \frac{c_5 \exp{(c_5 x^2)}(d+x^3)(1+x^2)}{(1+u_1+r_1)^{1+\epsilon}} \\ \leq & \frac{c_6 \exp{(c_5 x^2)}(d+x^3)(x+x^3)\|h_n - h_{n-1}\|_Y}{(1+u)^{2\epsilon+1}\,(1+u+r)^{\epsilon}}.\end{aligned} \tag{3.7}$$

Using (2.3) and (2.8), we have

$$\begin{aligned} \left|\frac{g_n-\bar{g}_n}{2r}(\bar{h}_n-\bar{h}_{n-1})\right| \leq & \frac{1}{2r}\frac{c_1x^2r}{(1+u)^{2\epsilon+1}}\frac{\|h_n-h_{n-1}\|_Y}{(1-\epsilon)\,(1+u+r)^{\epsilon}} \\ \leq & \frac{c_1x^2\|h_n-h_{n-1}\|_Y}{(1-\epsilon)\,(1+u)^{2\epsilon+1}\,(1+u+r)^{\epsilon}}. \end{aligned} \tag{3.8}$$

Let $c_7 = \frac{16\pi^2c^3}{\epsilon^2(1-\epsilon)}$. Using (2.5), (3.4) and (3.5), we have

$$\begin{aligned} &\frac{1}{2r}\left|g_n-g_{n-1}-(\bar{g}_n-\bar{g}_{n-1})\right| \\ &\quad\leq \frac{1}{2r^2}\int_0^r\int_{r'}^r\left|\frac{\partial(g_n-g_{n-1})}{\partial s}\right|dsdr' \\ &\quad\leq \frac{2\pi}{r^2}\int_0^r\int_{r'}^r|g_n|\left|(h_n-\bar{h}_n)^2-(h_{n-1}-\bar{h}_{n-1})^2\right|\frac{ds}{s}dr' \\ &\quad\quad+\frac{2\pi}{r^2}\int_0^r\int_{r'}^r|g_n-g_{n-1}||h_{n-1}-\bar{h}_{n-1}|^2\frac{ds}{s}dr' \\ &\quad\leq \frac{32\pi^2c^3(x+x^3)\|h_n-h_{n-1}\|_Y}{\epsilon(1-\epsilon)r^2}\int_0^r\int_{r'}^r\frac{dsdr'}{(1+u+s)^{1+2\epsilon}} \\ &\quad\leq \frac{c_7(x+x^3)\|h_n-h_{n-1}\|_Y}{r^2}\int_0^r\left[\frac{1}{(1+u)^{2\epsilon}}-\frac{1}{(1+u+r)^{2\epsilon}}\right]dr' \\ &\quad\leq \frac{2c_7(x+x^3)\|h_n-h_{n-1}\|_Y}{(1+u)^{2\epsilon+1}}. \end{aligned}$$

Thus, using (2.3) and (2.11), we obtain

$$\begin{aligned} &\frac{1}{2r}\left|g_n-g_{n-1}-(\bar{g}_n-\bar{g}_{n-1})\right|\left|h_n-\bar{h}_{n-1}\right| \\ &\quad\leq \frac{2\exp{(c_2x^2)}c_2^2c_7(d+x^3)(x+x^3)}{(1+u)^{1+2\epsilon}\,(1+u+r)^{\epsilon}}. \end{aligned} \tag{3.9}$$

Let $c_8 = c_6 + \frac{c_1}{1-\epsilon} + 2c_2^2c_7$. Using (3.7), (3.8) and (3.9), we obtain

$$|f_2| \leq \frac{c_8\exp{(c_8x^2)}(d+x+x^3)(x+x^3)\|h_n-h_{n-1}\|_Y}{(1+u)^{2\epsilon+1}\,(1+u+r)^{\epsilon}}. \tag{3.10}$$

Integrating (3.1) along the characteristic, and using (2.10), (3.10), we have

$$(1+u_1+r_1)^{\epsilon}\left|(h_{n+1}-h_n)(u_1,r_1)\right| \leq F(x)\|h_n-h_{n-1}\|_Y,$$

where

$$F(x) = \frac{2^{\epsilon}c_8\exp{(2c_8x^2)}(d+x+x^3)(x+x^3)}{2\epsilon k^{\epsilon}}.$$

Obviously,

$$F(0)=0, \quad F'(x)\geq 0.$$

Thus $F(x)$ is monotonically increasing. Therefore there exists $x_1 > 0$ such that, for any $x \in (0, x_1)$,

$$0 < F(x) < \frac{1}{2}.$$

This gives proof of the lemma. Q.E.D.

Next, we show that the sequences $\{h_n\}$ and $\{\frac{\partial h_n}{\partial r}\}$ are uniformly bounded and equicontinuous.

Lemma 3.2. *There exists $\tilde{x}$ such that for any $x \in (0, \tilde{x})$, the sequence $\{h_n\}$ are uniformly bounded by x and converges in the space Y if*

$$\Phi(x) = \frac{x \exp\left(-Cx^2\right)}{C(2+x^2)} - x^3 \geq d.$$

Moreover, $\{h_n\}$ and $\{\frac{\partial h_n}{\partial r}\}$ are uniformly bounded and equicontinous.

Proof: By Lemma 2.1, we know that

$$\|h_{n+1}\|_X \leq C \exp(Cx^2)(d + x^3)(2 + x^2).$$

Denote

$$\Phi(x) = \frac{x \exp\left(-Cx^2\right)}{C(2+x^2)} - x^3.$$

Then we have

$$\Phi(0) = 0, \quad \Phi'(0) > 0.$$

Thus, there exists x_0 such that $\Phi(x)$ is monotonically increasing on $[0, x_0]$ and attains its maximum at the point x_0. Let

$$\tilde{x} = \min\{x_0, x_1, 1\},$$

where x_1 is given in Lemma 3.1. Then for any $x \in (0, \tilde{x})$,

$$\Phi(x) \geq d, \quad \|h_n\|_X \leq x \Longrightarrow \|h_{n+1}\|_X \leq x.$$

By induction, $\|h_n\| \leq x$ for all $n \in \mathbb{N}$, i.e., $\{h_n\}$ is uniformly bounded by x. By the proof of Lemma 2.3 in [8], $\{\frac{\partial h_n}{\partial r}\}$ and $\{\frac{\partial h_n}{\partial u}\}$ are uniformly bounded. Thus, $\{h_n\}$ is equicontinuous.

By Lemma 3.1, we obtain that for any $x \in (0, \tilde{x})$,

$$\|h_{n+1} - h_n\|_Y < \frac{1}{2}\|h_n - h_{n-1}\|_Y.$$

This implies that $\{h_n\}$ converges in space Y.

For any $u \geq 0$, $0 \leq r_1 < r_2$, let $\chi_n(u; r_1)$ and $\chi_n(u; r_2)$ be two characteristics through u-slice at $r = r_1$ and $r = r_2$ respectively. Let $k' = \exp\left(\frac{c_1 x^2}{4\epsilon}\right)$.

By (4.29) of [4] and (2.8), we obtain

$$\begin{aligned}\left|\frac{\chi_n(u;r_2)-\chi_n(u;r_1)}{r_2-r_1}\right| &\le \sup_{s\in[r_1,r_2]} \exp\left\{\frac{1}{2}\int_u^{u_1}\left[\frac{\partial\bar g}{\partial r}\right]_{\chi_n(u';s)} du'\right\} \\ &\le k'.\end{aligned} \tag{3.11}$$

For any differentiable function f, denote

$$B(f)(u) = f(u,\chi_n(u;r_1)) - f(u,\chi_n(u;r_2)).$$

We have

$$|B(f)(u)| \le \sup\left|\frac{\partial f}{\partial r}\right| k'(r_2-r_1).$$

Now we use the arguments for proving Lemma 2.3 in [8] to prove that $\{\frac{\partial h_n}{\partial r}\}$ is equicontinuous. Let

$$\psi(u) = \frac{\partial h_{n+1}}{\partial r}(u,\chi_n(u;r_1)) - \frac{\partial h_{n+1}}{\partial r}(u,\chi_n(u;r_2)). \tag{3.12}$$

Differentiate (3.12), we have

$$\psi'(u) - \frac{(g_n-\bar g_n)(u,\chi_n(u;r_1))}{\chi_n(u;r_1)}\psi(u) = \sum_{i=1}^4 A_i, \tag{3.13}$$

where

$$\begin{aligned}
A_1 &= \frac{\partial h_{n+1}}{\partial r}(u,\chi_n(u;r_2)) B\Big(\frac{g_n-\bar g_n}{r}\Big)(u),\\
A_2 &= \frac{1}{2}B\Big(\frac{\partial^2 \bar g_n}{\partial r^2}(h_{n+1}-\bar h_{n+1})\Big)(u),\\
A_3 &= \frac{1}{2}B\Big(\frac{\partial^2 \bar g_n}{\partial r^2}(\bar h_{n+1}-\bar h_n)\Big)(u),\\
A_4 &= -\frac{1}{2}B\Big(\frac{\partial \bar g_n}{\partial r}\frac{h_n-\bar h_n}{r}\Big)(u).
\end{aligned}$$

From (2.13), we have

$$\begin{aligned}
|A_1| &\le \left|\frac{\partial h_{n+1}}{\partial r}\right|\cdot\left|\frac{\partial^2 \bar g_n}{\partial r^2}\right| k'(r_2-r_1)\\
&\le k'(r_2-r_1)\frac{x}{(1+u)^{1+\epsilon}}\frac{(2c_1+4\pi c^2)x^2}{(1+u)^{2\epsilon+1}(1+u+r)}\\
&\le \frac{k'(2c_1+4\pi c^2)x^3}{(1+u)^{3+3\epsilon}}(r_2-r_1).
\end{aligned} \tag{3.14}$$

From (2.42) in [8], we know

$$\begin{aligned}
\frac{\partial^3 \bar g_n}{\partial r^3} =& \frac{6(g_n-\bar g_n)}{r^3} - \frac{16\pi(h_n-\bar h_n)^2 g_n}{r^3}\\
&+ \frac{8\pi(h_n-\bar h_n)g_n}{r^2}\frac{\partial(h_n-\bar h_n)}{\partial r} + \frac{16\pi^2(h_n-\bar h_n)^4 g_n}{r^3}.
\end{aligned}$$

Using (2.4) and (2.8), we obtain

$$\left|\frac{\partial^3 \bar{g}_n}{\partial r^3}\right| \le \begin{cases} \dfrac{(6c_1 + 32\pi c^2)x^2 + 16\pi^2 c^4 x^4}{r\,(1+u)^{2\epsilon+1}}, & 0 \le r \le 1+u, \\ \dfrac{(6c_1 + 32\pi c^2)x^2 + 16\pi^2 c^4 x^4}{(1+u)^{2\epsilon+2}}, & r \ge 1+u, \end{cases} \tag{3.15}$$

Let $c_9 = 10cc_1 + 40\pi c^3 + 16\pi^2 c^5$. Then (2.4), (3.15) imply that

$$|A_2| \le \frac{k' c_9 (x^3 + x^5)}{(1+u)^{3\epsilon+2}}(r_2 - r_1).$$

By Lemma 3.1, we know

$$h_{n+1} - h_n \to 0$$

uniformly. Then the argument for proving Lemma 2.3 in [8] gives

$$\bar{h}_{n+1} - \bar{h}_n \to 0.$$

By (2.13), we have

$$\left|(1+u)^{2\epsilon+1}\frac{\partial^2 \bar{g}_n}{\partial r^2}\right| \le (2c_1 + 4\pi c^2)x^2.$$

Thus,

$$(1+u)^{2\epsilon+1}\frac{\partial^2 \bar{g}_n}{\partial r^2}(\bar{h}_{n+1} - \bar{h}_n) \to 0$$

uniformly. Therefore,

$$(1+u)^{2\epsilon+1}\frac{\partial^2 \bar{g}_n}{\partial r^2}(\bar{h}_{n+1} - \bar{h}_n)$$

is equicontinuous. Hence for $\eta > 0$, there exists $t > 0$ such that

$$|\chi_n(u; r_2) - \chi_n(u; r_1)| \le t \Longrightarrow |A_3| \le \frac{2\epsilon\eta}{3k'^2\,(1+u)^{2\epsilon+1}}.$$

Taking $s_1 = \frac{t}{k'}$, we have

$$\begin{aligned} r_2 - r_1 \le s_1 &\Longrightarrow |\chi_n(u; r_2) - \chi_n(u; r_1)| \le t \\ &\Longrightarrow |A_3| \le \frac{2\epsilon\eta}{3k'^2\,(1+u)^{2\epsilon+1}}. \end{aligned}$$

Similarly,

$$|A_4| \le \frac{k'(3cc_1 + 2\pi c^4)x^3}{(1+u)^{3+3\epsilon}}(r_2 - r_1).$$

Taking

$$s_2 = \frac{2\epsilon\eta}{3k'^3[(2c_1 + 4\pi c^2 + c_9 + 3cc_1 + 2\pi c^4)x^3 + c_9 x^5]},$$

we have

$$r_2 - r_1 \le s_2 \Longrightarrow |A_1| + |A_2| + |A_4| \le \frac{2\epsilon\eta}{3k'^2\,(1+u)^{2\epsilon+1}}.$$

By (2.8), we have

$$\exp\Big(\int_0^{u_1} \frac{(g_n - \bar{g}_n)(u, \chi_n(u; r_1))}{\chi_n(u; r_1)} du\Big) \le k'^2.$$

By Lemma 2.3 in [8], there exists $s_3 > 0$ such that

$$r_2 - r_1 \le s_3 \Longrightarrow \left|\frac{\partial h}{\partial r}(0, \chi_n(0; r_1)) - \frac{\partial h}{\partial r}(0, \chi_n(0; r_2))\right| \le \frac{\eta}{3k'^2}.$$

Therefore, integrating (3.13), we obtain

$$\begin{aligned}\psi(u_1) =& \psi(0) \exp\Big(\int_0^{u_1} \frac{(g_n - \bar{g}_n)(u, \chi_n(u; r_1))}{\chi_n(u; r_1)} du\Big) \\ &+ \int_0^{u_1} \Big[\exp\Big(\int_u^{u_1} \frac{(g_n - \bar{g}_n)(u, \chi_n(u; r_1))}{\chi_n(u; r_1)} du\Big)\Big] \sum_{i=1}^{4} A_i du.\end{aligned}$$

Let $s = \min\{s_1, s_2, s_3\}$. Then

$$r_2 - r_1 \le s \Longrightarrow |\psi(u_1)| \le \eta \Longleftrightarrow \left|\frac{\partial h_{n+1}}{\partial r}(u_1, r_1) - \frac{\partial h_{n+1}}{\partial r}(u_1, r_2)\right| \le \eta.$$

Thus, $\{\frac{\partial h_{n+1}}{\partial r}\}$ is equicontinuous with respect to r.

The equicontinuous of $\{\frac{\partial h_{n+1}}{\partial r}\}$ with respect to u can be proved by the equiboundedness of $D_n \frac{\partial h_{n+1}}{\partial r}$. Thus proof of the lemma is complete. Q.E.D.

Proof of Theorem 1.4. Denote

$$\delta = \max_{[0, \tilde{x}]} \{\Phi(x)\}.$$

For $d_0 \le \delta$, we can find x such that $d_0 \le \Phi(x)$. This implies that Lemma 2.1, Lemma 3.1 and Lemma 3.2 hold. Given initial data $\breve{h}(r)$ with $d_0 \le \delta$, then there exists $a > 0$ such that

$$\hat{d}_0 = \sup_{r \ge 0} \left\{\Big(1 + \frac{r}{a}\Big)^{\epsilon} \left|\breve{h}(r)\right| + \Big(1 + \frac{r}{a}\Big)^{1+\epsilon} \left|a\frac{\partial \breve{h}}{\partial r}(r)\right|\right\} < \delta.$$

Consider the new initial data

$$\hat{h}(0, r) = \breve{h}(ar).$$

By using the same argument as these in [3, 8], there exists a unique global classical solution $\hat{h}(u, r)$ with slowly particle-like decaying null infinity satisfying the initial data $\hat{h}(0, r)$. By scaling group invariance (2.1) [3], we find that

$$h(u, r) = \hat{h}\left(\frac{u}{a}, \frac{r}{a}\right)$$

is a unique global classical solution of (2.1) satisfying the initial data $\breve{h}(r)$. Moreover h satisfies

$$|h(u,r)| \leq \frac{C}{(1+u+r)^{\epsilon}}, \qquad \left|\frac{\partial h}{\partial r}(u,r)\right| \leq \frac{C}{(1+u+r)^{1+\epsilon}}.$$

Now (2.4) and (2.6) imply that

$$k \leq g,\, \bar{g} \leq 1$$

By Lemma 2.1, h, $\bar{h}$, g, $\bar{g}$ and their partial derivatives are all uniformly bounded. Thus, using the same argument as [8], we can show that the corresponding spacetime is future casually geodecsically complete.

Finally, using the same argument as [8], we have

$$\begin{aligned} 1-g(u,r) &\leq 4\pi \int_r^{\infty} \frac{(h-\bar{h})^2}{r'} dr' \\ &\leq 4\pi c^2 x^2 \int_r^{\infty} \frac{r'}{(1+u+r')^{2+2\epsilon}} dr' \\ &\leq \frac{2\pi c^2 x^2}{\epsilon(1+u+r)^{2\epsilon}}. \end{aligned}$$

Therefore

$$\frac{r}{2}\Big(1-g\Big) \leq \frac{\pi c^2 x^2}{\epsilon} \frac{r}{(1+u+r)^{2\epsilon}}.$$

We obtain

$$\frac{1}{2} < \epsilon < 1 \Longrightarrow \lim_{r\to\infty} \frac{r}{2}\Big(1-g\Big) = 0.$$

Hence the Bondi-Christodoulou mass is equivalent to Bondi mass

$$M(u) = M_B(u).$$

From [3], we have

$$\begin{aligned} M(u) &= 2\pi \int_0^{\infty} \frac{\bar{g}}{g}(h-\bar{h})^2 dr \\ &\leq \frac{2\pi c^2 x^2}{2\epsilon - 1} \frac{1}{(1+u)^{2\epsilon-1}}. \end{aligned}$$

Therefore,

$$\frac{1}{2} < \epsilon < 1 \Longrightarrow M_1 = \lim_{u\to\infty} M(u) = 0.$$

Thus proof of the theorem is complete. Q.E.D.

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[1]School of Mathematics and Information Science, Guangxi University, Nanning, Guangxi 530004, PR China

[2]Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

[3]School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, PR China

[4]Guangxi Center for Mathematical Research, Guangxi University, Nanning, Guangxi 530004, PR China

Email address: `cxliu@gxu.edu.cn`[1,4]

Email address: `xzhang@amss.ac.cn`[2,3,4]