# THE DENSITY OF GABOR SYSTEMS IN EXPANSIBLE LOCALLY COMPACT ABELIAN GROUPS

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Abstract. We investigate the reproducing properties of Gabor systems within the context of expansible groups. These properties are established in terms of density conditions. The concept of density that we employ mirrors the well-known Beurling density defined in Euclidean space, which is made possible due to the expansive structure. Along the way, for groups with an open and compact subgroup, we demonstrate that modulation spaces are continuously embedded in Wiener spaces. Utilizing this result, we derive the Bessel condition of Gabor systems. Additionally, we construct Gabor orthonormal bases with arbitrarily small or large densities, enabling us to conclude that a Comparison Theorem, such as the one proven to be valid in the Euclidean case, cannot hold in this context. Finally, we establish that Gabor frames possess the Homogeneous Approximation Property.

## 1. Introduction

A Gabor system is a set of functions in $L^2(\mathbb{R}^d)$ that are obtained by translating a single window in time and frequency along a set $\Lambda \subseteq \mathbb{R}^{2d}$. Their structure makes them of particular importance in many applications such as wireless communication, analysis and description of speech signals or music signals and more (see for instance, [10, 11, 22]). Because of this, it is important to study their reproducing properties, that is, to understand which properties of the generating window and the set $\Lambda$ of time-frequency shifts guarantee the Gabor system to be an orthonormal basis, a Bessel sequence, a frame, or a Riesz basis. When the set $\Lambda$ does not have a particular structure, these reproducing properties are stated in terms of its Beurling density.

Consider a subset $\Lambda$ in $\mathbb{R}^d$. The upper and lower Beurling density of $\Lambda$, denoted by $D^+(\Lambda)$ and $D^-(\Lambda)$ respectively, captures its asymptotic behavior within balls of varying radius. In mathematical terms, they are given by:

$$D^+(\Lambda) = \limsup_{r\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap B_r(x))}{r^d}, \qquad \text{and} \qquad D^-(\Lambda) = \liminf_{r\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap B_r(x))}{r^d}$$

where $B_r(x)$ denotes the ball centered in $x$ with radius $r$ and $\#$ indicates the cardinality of a set. A very complete survey on the results about density of Gabor systems is [14].

Beurling density is a concept that is also related to conditions for sampling and interpolation. See, for example, Landau's result in [18]. In order to explore the validity of Landau's result in the context of locally compact abelian groups that are compactly generated, Gröchenig, Kutyniok, and Seip in [12], provide a definition of Beurling density by means of a comparison with a canonical lattice of reference. We refer to [1] for related results on the existence of sampling and interpolation sets near to the critical density in LCA groups.

In this paper we focus on studying Gabor systems in terms of density within the framework of expansible locally compact abelian groups, that is, locally compact abelian (LCA) groups which have an open and compact subgroup and an expansive automorphism. (See Definition 2.2.1 for

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more details). One example of a such an expansible LCA group is the space of $p$-adic numbers $\mathbb{Q}_p$.

This type of LCA group does not possess lattices, and thus, the notion of density given in [12] needs to be reformulated. However, the presence of the expansive automorphism allows us to elaborate the concept of density in the spirit of Beurling density in $\mathbb{R}^d$. A similar approach was given in [21].

Then, given an expansible LCA group $G$, a window $\varphi \in L^2(G)$ and a set $\Lambda \subseteq G \times \widehat{G}$, where $\widehat{G}$ denotes the Pontryagin dual group of $G$, we explore whether or not the Gabor system generated by $\varphi$ by time-frequency shifts along $\Lambda$, $S(\varphi, \Lambda)$, is a Bessel sequence or a frame of $L^2(G)$ depending on the density of $\Lambda$. We first show that if $S(\varphi, \Lambda)$ is a Bessel sequence in $L^2(G)$, then $\Lambda$ must be of finite density. Conversely, when $\Lambda$ is of finite density and $\varphi$ has some decay, $S(\varphi, \Lambda)$ is a Bessel sequence. Our results are thus analogues for expansible LCA groups of [5, Theorem 3.1] and [14, Theorem 12], which are in the Euclidean setting.

Surprisingly, in the framework in which we work, we can construct Gabor systems that are orthonormal bases, with the set of time-frequency shifts having arbitrarily small density in some cases and arbitrarily big density in others. This contrasts with what happens in the Euclidean case, where if a Gabor system $S(\varphi, \Lambda)$ is a frame of $L^2(\mathbb{R}^d)$, the lower density of $\Lambda$ must be bigger than 1 (see [5, Theorem 1.1]). Another consequence of our construction is that it is not possible to obtain a Comparison Theorem in this context such as [5, Theorem 3.6]. However, we were able to prove that, in the setting of expansible LCA groups, a Gabor system that is a frame has the well-known Homogeneous Approximation Property (see Theorem 5.3.3).

On the other hand, we move to a more abstract setting where we deal with unitary and projective representations of a locally compact group having an open and compact subgroup. We based our analysis in the notion of modulation spaces introduced in [8, 9]. We prove that the generalized wavelet transform (voice transform) induced by the (unitary or projective) representation, continuously maps the modulation space of order $p$ into the Wiener space $W(\mathcal{C}, \ell^p)$. We then apply these resuls to the case were the projective representation is the one given by time-frequency translations, resulting in Gabor systems.

The article is structured as follows: In Section 2, we recall certain definitions about LCA groups and expansive automorphisms as well as what we will need about frame theory. We investigate in Section 3 the concept of density. We establish its (essentially) independence with respect to the chosen automorphism used to defining the density and we show an equivalent condition for a set to have finite upper density. Moving to Section 4, we use the theory of modulation spaces developed by Feichtinger and Gröchenig in [8, 9]. When the underlying locally compact group has an open and compact subgroup, we establish one of the pivotal results of this work, namely, the continuous inclusion of modulation spaces into Wiener spaces through the generalized wavelet transform. Finally, in Section 5 we prove that some classical properties of Gabor systems hold true in our context and provide examples which show that there are others that do not.

## 2. Preliminaries

In this section we fix the context where we will work in. We will also recall some aspects of frame theory that we will need.

2.1. **LCA groups.** A *locally compact abelian group $G$ – LCA group* for short – is an abelian group which is also a locally compact topological space such that both multiplication and inversion are homeomorphism of the space. We will always assume that the topology is Hausdorff. See, e.g., [16] for a general reference

Given an LCA group $G$ written additively, we denote by $\widehat{G}$ its Pontryagin dual group. By $m_G$ we denote a Haar measure associated to $G$ (with the desired normalization defined below). Since the dual of the dual group is topologically isomorphic to the original group, for $\xi \in \widehat{G}$ and $x \in G$ we write $\langle x, \xi \rangle$ to indicate the character $\xi$ applied to $x$ (i.e. $\xi(x)$) or the character $x$ applied to $\xi$. For a subgroup $H \subseteq G$, its annihilator is denoted by $H^\perp$ and is defined as $H^\perp = \{\xi \in \widehat{G} : \langle h, \xi \rangle = 1 \,\forall\, h \in H\}$. It is well known that if $H$ is closed, $H^\perp$ is a closed subgroup of $\widehat{G}$.

For a closed subgroup $H$ of $G$, the dual of the quotient group $G/H$, $\widehat{G/H}$ is algebraically and topologically isomorphic to $H^\perp$ and $\widehat{H}$ is algebraically and topologically isomorphic to $\widehat{G}/H^\perp$.

When $H \subseteq G$ is an open and compact subgroup, then so is $H^\perp \subseteq \widehat{G}$. As a consequence, the quotients $G/H$ and $\widehat{G}/H^\perp$ are discrete abelian groups.

For an LCA group $G$ with an open and compact subgroup $H$, we consider the following normalization of the Haar measures involved: we fix $m_G$ such that $m_G(H) = 1$ and $m_{\widehat{G}}(H^\perp) = 1$. As explained in [16, Comment (31.1)], this choice guarantees that the Fourier transform between $L^2(G)$ and $L^2(\widehat{G})$ is an isometry. We take $m_H = m_G\,|_H$, $m_{\widehat{H}} = m_{\widehat{G}}\,|_{H^\perp}$ and $m_{G/H}$ and $m_{\widehat{G}/H^\perp}$ to be the counting measures. Then, the Fourier transforms between $L^2(H)$ and $L^2(\widehat{G}/H^\perp)$ and between $L^2(G/H)$ and $L^2(H^\perp)$ are isometries.

2.2. **Expansive automorphisms.** Let $G$ be an LCA group. The group of homeomorphic automorphisms of $G$ into itself is denoted by $\mathrm{Aut}(G)$. For a given $A \in \mathrm{Aut}(G)$, the measure $\mu_A$ defined by $\mu_A(U) = m_G(AU)$ where $U$ is a Borel set of $G$ is a non-zero Haar measure on $G$. Therefore, there is a unique positive number $|A|$, the so-called *modulus of* $A$, such that $\mu_A = |A| m_G$.

For $A \in \mathrm{Aut}(G)$, there is an adjoint $A^* \in \mathrm{Aut}(\widehat{G})$ defined as $\langle Ax, \gamma \rangle = \langle x, A^*\gamma \rangle$ for all $x \in G$ and $\gamma \in \widehat{G}$. It holds that $|A^*| = |A|$.

We next present the definition of expansive automorphisms as given in [2], where they were used to define a wavelet theory over local fields. See also [21].

**Definition 2.2.1.** [2, Definition 2.5] *Let $G$ be an LCA group and $H \subseteq G$ an open and compact subgroup, and let $A \in \mathrm{Aut}(G)$. We say that $A$ is expansive with respect to $H$ if the next two conditions hold true:*

*(1) $H \subsetneq AH$;*
*(2) $\bigcap_{n \leq 0} A^n H = \{0\}$.*

*When $H$ is fixed or clear from the context, we will simply say that $A$ is expansive.*

There exist many groups $G$ with expansive automorphism $A$. We now give several examples.

**Example 2.2.2.** [2, Example 2.10] *Let $p$ be a prime number. Define the $p$-adic valuation over $\mathbb{Q}$ as $|p^r x|_p = p^{-r}$ for all $r \in \mathbb{Z}$ and all $x \in \mathbb{Q}$ such that the numerator and denominator of $x$ are both relatively prime to $p$. Then $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the $p$-adic valuation. Each element of $\mathbb{Q}_p$ may be represented as a Laurent series*

$$\mathbb{Q}_p = \left\{ \sum_{n \geq n_0} a_n p^n : n_0 \in \mathbb{Z} \text{ and } a_n \in \{0, 1, \ldots, p-1\} \right\},$$

*where addition "carries" rather than is modular. That is,*

$$(p-1)p + 1p = p^2 \neq 0p.$$

*Then $\mathbb{Q}_p$ is a locally compact abelian group with group operation addition and topology defined via the p-adic valuation. The Laurent series are not formal as they converge in the topology.*

*Then the p-adic integers $\mathbb{Z}_p$, defined as the set of power series*

$$\mathbb{Z}_p = \left\{ \sum_{n\geq 0} a_n p^n : a_n \in \{0, 1, \ldots, p-1\} \right\},$$

*is an open and compact subset of $\mathbb{Q}_p$. Further characterizations of $\mathbb{Z}_p$ are that $\mathbb{Z}_p$ is the unit ball (with respect to p-adic valuation) of $\mathbb{Q}_p$ and the closure of $\mathbb{Z}$ in $\mathbb{Q}_p$.*

*The p-adic numbers are self-dual. Let $\{\cdot\} : \mathbb{Q}_p \to \mathbb{Q}_p$ be defined as*

$$\left\{ \sum_{n\geq n_0} a_n p^n \right\} = \sum_{n=n_0}^{\max\{0,n_0\}} a_n p^n.$$

*Then each $y \in \mathbb{Q}_p$ defines an element of $\widehat{\mathbb{Q}}_p$ as $\langle \cdot, y\rangle = \exp(2\pi i \{\cdot y\})$.*

*If we consider $A : \mathbb{Q}_p \to \mathbb{Q}_p$ to be the morphism given by $Ax = p^{-1}x$, then $A$ is an automorphism of $\mathbb{Q}_p$ and it is easy to see that is expansive with respect to $\mathbb{Z}_p$.*

**Example 2.2.3.** [2, Example 2.11] *Let $p$ be a prime number, where $\mathbb{F}_p$ is the field of order $p$. The additive group of the field $\mathbb{F}_p((t))$ of formal Laurent series in variable $t$:*

$$\mathbb{F}_p((t)) = \left\{ \sum_{n\geq n_0} a_n t^n : n_0 \in \mathbb{Z} \text{ and } a_n \in \mathbb{F}_p \right\},$$

*where addition is modular rather than "carries":*

$$(p-1)t + 1t = 0t = 0$$

*is an LCA group with respect to the topology defined from an analog of the p-adic valuation. The set of formal power series*

$$\mathbb{F}_p[[t]] = \left\{ \sum_{n\geq 0} a_n t^n : a_n \in \mathbb{F}_p \right\}$$

*is an open and compact subgroup. One possible expansive automorphism is multiplying by $t^{-1}$. However, the structure of $\mathbb{F}_p((t))$ yields a richer collection of automorphisms.*

Both classes of the LCA groups above are also fields. Further examples may be formed by considering the additive groups of finite field extensions or vector spaces over the above examples.

**Example 2.2.4.** [2, Example 2.14] *Let $G_1$ be an LCA group with an open and compact subgroup $H_1$ and $G_2$ be a nontrivial discrete abelian group. If we consider $G = G_1 \times G_2$, then $G$ is an LCA group with aopen and compact subgroup $H = H_1 \times \{0\}$. Further, the annihilator $H^\perp$ is $H_1 \times \widehat{G_2}$ and if $A_1$ is an automorphism of $G_1$ which is expansive with respect to $H_1$, then $A = A_1 \times id_{G_2}$ is an automorphism of $G$ which is expansive with respect to $H$. However, in this case, the union of all positive iterates $A^n H$ can not cover $G$ since $AH = A_1 H_1 \times \{0\}$.*

Following [21], we shall call an LCA group which admits an expansive automorphism an *expansible* group. Expansiveness can also be characterized by the action of the adjoint automorphism as the next lemma shows, whose proof can be found in [2, Lemma 2.6].

**Lemma 2.2.5.** *Let $G$ be an LCA with an open and compact subgroup $H \subseteq G$, and let $A \in$* $\mathrm{Aut}(G)$.

*(i)* $H \subseteq AH$ *if and only if* $H^\perp \subseteq A^* H^\perp$.
*(ii)* $H \subsetneq AH$ *if and only if* $H^\perp \subsetneq A^* H^\perp$.
*(iii)* *If* $H \subseteq AH$*, then*

$$\bigcap_{n\le 0} A^n H = \{0\} \iff \bigcup_{n\ge 0} A^{*n} H^\perp = \widehat{G}. \tag{1}$$

Given an LCA group $G$ with an open and compact subgroup $H$ and expansive automorphism $A : G \to G$, we define for $n \in \mathbb{Z}$ and $x \in G$,

$$Q_n(x) = x + A^n H.$$

One may think of this as the "ball" with "center" $x$ and "radius" $|A|^n$, keeping in mind that if $y \in Q_n(x)$ then $Q_n(x) = Q_n(y)$, so the choice of "center" is not unique. In the case that $G = \mathbb{Q}_p$, $H = \mathbb{Z}_p$ , and $A$ is multiplication by $1/p$, the $Q_n(x)$ are precisely the balls in the metric induced by the $p$-adic valuation. Note that each $Q_n(x)$ is an open and compact subset of $G$ and also a coset of $A^n H$ in $G$. Moreover, by [15, Theorem 4.5], $\{Q_n(x)\}_{n\in\mathbb{Z},x\in G}$ is a basis for the topology of $G$.

In this paper, we will be dealing with the specific scenario of forming "balls" in $G \times \widehat{G}$, where $G$ satisfies the hypothesis above. In that case, we write for $n \in \mathbb{Z}$ and $(x, \gamma) \in G \times \widehat{G}$

$$\begin{aligned} Q_n(x,\gamma) &= (x,\gamma) + (A^n \otimes (A^*)^n)\,(H \times H^\perp) \\ &= (x + A^n H) \times \left(\gamma + (A^*)^n H^\perp\right) = Q_n(x) \times Q_n(\gamma). \end{aligned} \tag{2}$$

Note that in this case, the "radius" of $Q_n(x, \gamma)$ is $(|A||A^*|)^n = |A|^{2n}$.

When $A$ is expansive with respect to $H$ and $A^*$ is expansive with respect to $H^\perp$, we have that $\{Q_n(x,\gamma)\}_{n\in\mathbb{Z},x\in G,\gamma\in\widehat{G}}$ is a basis for the topology of $G \times \widehat{G}$. This is because for $n \le m, x \in G$ and $\gamma \in \widehat{G}$, $Q_n(x) \times Q_m(\gamma) \subseteq Q_m(x, \gamma)$.

2.3. **Frames and Riesz bases.** Let $\{\varphi_i\}_{i\in I}$ be a family of elements in a separable Hilbert space $\mathcal{H}$. It is said that $\{\varphi_i\}_{i\in I}$ is a *frame for* $\mathcal{H}$ if there exist constants $A$,$B > 0$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, \varphi_i\rangle|^2 \le B\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{3}$$

The constants $A, B$ are called *frame bounds*. The *frame operator* defined as $Sf = \sum_{i\in I}\langle f, \varphi_i\rangle\varphi_i$ for $f \in \mathcal{H}$, is a bounded, invertible, and positive operator from $\mathcal{H}$ onto itself. This provides the well known frame decomposition

$$f = S^{-1}Sf = \sum_{i\in I}\langle f, \varphi_i\rangle\phi_i \quad \forall f \in \mathcal{H},$$

where $\phi_i = S^{-1}\varphi_i$. The family $\{\phi_i\}_{i\in I}$ is also a frame for $\mathcal{H}$, which is called the *canonical dual frame*, and has frame bounds $B^{-1}$, $A^{-1}$. Any other frame $\{\widetilde{\phi}_i\}_{i\in I}$ satisfying $f = \sum_{i\in I}\langle f, \varphi_i\rangle\widetilde{\phi}_i \quad \forall f \in \mathcal{H}$, is called a *dual frame*, and it is well known that a frame can have dual frames besides the canonical one (typically infinitely many). For a general reference on frame theory, see, e.g., [4].

Riesz bases are special cases of frames and can be characterized as those frames which are biorthogonal to their canonical dual frame, i.e., such that $\langle \varphi_i, \phi_j\rangle = \delta_{ij}$.

A family $\{\varphi_i\}_{i\in I}$ which satisfies the right inequality in (3) (but possibly not the left) is called a *Bessel sequence* for $\mathcal{H}$.

## 3. Density with respect to expansive automorphisms

In this section, using the balls defined above, we will consider the concept of density, which will extend that of the well known Beurling density for Euclidean spaces $\mathbb{R}^d$. This concept has been considered before in LCA groups that are compactly generated in [1, 12], however, with another approach. In [21] the authors work with the same density as here, but they consider a slightly different notion of expansive automorphism. In fact, their automorphism only satisfies condition (2) of Definition 2.2.1.

**Definition 3.0.1.** *Let $G$ be an LCA group, $H \subseteq G$ an open and compact subgroup and $A \in \operatorname{Aut}(G)$ expansive with respect to $H$. For $n \in \mathbb{Z}$, a sequence (countable or uncountable) $\Lambda \subseteq G$ is said to be $(A,n)$-uniformly separated if $\#\{\Lambda \cap Q_n(x)\} \leq 1$ for all $x \in G$. We say that $\Lambda$ is simply uniformly separated if it is $(A,n)$-uniformly separated for some $n \in \mathbb{Z}$. Additionally, $\Lambda$ is said to be $A$-separated if it is a finite union of uniformly separated sequences.*

Recall that, if $G$ is an LCA group and $H \subseteq G$ is a subgroup, a *section* of a quotient group $G/H$ is a measurable set of representatives, and it contains exactly one element of each coset. As we already said, if $H \subseteq G$ is an open and compact subgroup, since $H^\perp$ is also compact, then $G/H$ is a discrete group. Thus, every section $C \subseteq G$ for the quotient $G/H$ must be discrete as well. This is because for $x \in C$, $x = C \cap (x+H)$ and then, since $x+H$ is an open set in $G$, $C$ is discrete with respect to the topology of $G$.

With this in mind, we can say that $\Lambda$ is $(A,n)$-uniformly separated if and only if $\#\{\Lambda \cap Q_n(A^n c)\} \leq 1$ for all $c \in C$. This is a direct consequence of the fact that for every $n \in \mathbb{Z}$, $\{Q_n(A^n c)\}_{c \in C}$ is a partition of $G$, that is, $G = \bigcup_{c \in C} Q_n(A^n c)$ where the union is disjoint, and that $Q_n(A^n c) = Q_n(x)$ for every $x \in Q_n(A^n c)$.

**Definition 3.0.2.** *Let $G$ be an LCA group, $H \subseteq G$ an open and compact subgroup and $A \in \operatorname{Aut}(G)$ expansive with respect to $H$. For a sequence $\Lambda \subseteq G$, the upper and lower Beurling density of $\Lambda$ are defined by*

$$D_A^+(\Lambda) := \limsup_{n \to +\infty} \frac{1}{|A|^n} \max_{x \in G} \#\{\Lambda \cap Q_n(x)\},$$

*and*

$$D_A^-(\Lambda) := \liminf_{n \to +\infty} \frac{1}{|A|^n} \min_{x \in G} \#\{\Lambda \cap Q_n(x)\},$$

*respectively. If $D_A^+(\Lambda) = D_A^-(\Lambda)$ we say that $\Lambda$ has uniform density $D_A(\Lambda) = D_A^+(\Lambda) = D_A^-(\Lambda)$.*

The analogy with the Beurling density defined in $\mathbb{R}^d$ [5, 14, 21] is clear from the definition noting that, since $m_G$ is invariant under translations and $m_G(H) = 1$, we have that $|A|^n = m_G(A^n H) = m_G(Q_n(x))$ for any $x \in G$.

At first glance, the Beurling density seems to depend on the automorphism. However, if both the automorphism and their adjoint are expansive, the density becomes independent of the automorphism choice, as we show next. First, we observe some properties about sections that will be useful.

**Remark 3.0.3.** *Given $A \in \operatorname{Aut}(G)$ expansive with respect to an open and compact subgroup $H$ of $G$, the quotient $AH/H$ must be finite. This is because $AH$ is compact and then, it may be covered by finite (disjoints) cosets $x+H$. Let $C_0 \subseteq AH$ be a finite section for $AH/H$. Therefore, $AH = \bigcup_{c_0 \in C_0} H + c_0$, and then we have that $m_G(AH) = \#C_0$ which implies $|A| = \#C_0$. Moreover, since $AH \subseteq G$ is also an open and compact subgroup, we can take $C_1$ a discrete section for $G/AH$. An easy computation shows that the set $C := C_0 + C_1$ must be a (discrete) section for $G/H$. From now on, we will consider sections for $G/H$ of this form.*

**Lemma 3.0.4.** *Let $G$ be an LCA group, $H \subseteq G$ an open and compact subgroup and $A, B \in \operatorname{Aut}(G)$ be expansive with respect to $H$ such that $A^*$ and $B^*$ are expansive with respect to $H^\perp$ as well. Then, for every sequence $\Lambda \subseteq G$,*

$$D_A^+(\Lambda) = D_B^+(\Lambda) \quad \text{and} \quad D_A^-(\Lambda) = D_B^-(\Lambda).$$

*Proof.* Because of the expansiveness of $A^*$ we know by (1) that $G = \bigcup_{n \geq 0} A^n H$. Let $n \in \mathbb{N}$ be fixed. Then there exists $k_0 = k_0(n) \in \mathbb{N}$ such that $B^n H \subseteq A^{k_0} H$ because $B^n H$ is compact and $\{A^k H\}_{k \in \mathbb{N}}$ is an open cover of nested sets.

Let $X = \{x_i\}_{i \in I} \subseteq A^{k_0} H$ be a section of $A^{k_0} H / B^n H$. Then

$$A^{k_0} H = \bigcup_{i \in I} (B^n H + x_i),$$

and for $x \in G$,

$$Q_{k_0}^A(x) = A^{k_0} H + x = \bigcup_{i \in I} (B^n H + x_i) + x = \bigcup_{i \in I} Q_n^B(x_i + x).$$

By intersecting with $\Lambda$ and taking the maximum cardinal over $G$ we obtain the following inequality,

$$\begin{aligned} \max_{x \in G} \#\{Q_{k_0}^A(x) \cap \Lambda\} &= \max_{x \in G} \#\{\bigcup_{i \in I} Q_n^B(x_i + x) \cap \Lambda\} \\ &\leq \sum_{i \in I} \max_{x \in G} \#\{Q_n^B(x_i + x) \cap \Lambda\} \\ &= \sum_{i \in I} \max_{x \in G} \#\{Q_n^B(x) \cap \Lambda\} = (\#I) \max_{x \in G} \#\{Q_n^B(x) \cap \Lambda\}. \end{aligned}$$

Furthermore, $\#(A^{k_0} H / H) = \#(A^{k_0} H / B^n H)\, \#(B^n H / H)$ and then

$$\#I = \#(A^{k_0} H / B^n H) = \frac{|A|^{k_0}}{|B|^n}.$$

Finally, we obtain

$$\frac{\max_{x \in G} \#\{Q_{k_0}^A(x) \cap \Lambda\}}{|A|^{k_0}} \leq \frac{|A|^{k_0}}{|B|^n} \frac{\max_{x \in G} \#\{Q_n^B(x) \cap \Lambda\}}{|A|^{k_0}} = \frac{\max_{x \in G} \#\{Q_n^B(x) \cap \Lambda\}}{|B|^n}.$$

If we take $\limsup_{n \to \infty}$ in the last inequality we have that $D_A^+(\Lambda) \leq D_B^+(\Lambda)$. By using the expansiveness of $B^*$ and doing the same reasoning we obtain that $D_A^+(\Lambda) = D_B^+(\Lambda)$. A similar procedure proves that $D_A^-(\Lambda) = D_B^-(\Lambda)$.

□

As a consequence of the above result and in order to keep as clear as posible the exposition, we choose to omit the subscript $A$ in the density and simply right $D^+(\Lambda)$, $D^-(\Lambda)$ and $D(\Lambda)$.

The next lemma shows a characterization of the sequences $\Lambda$ whose upper density is finite and provides a valid version of [5, Lemma 2.3] in this context. Our proof is based on the group structure. In [21, Theorem 3.7], the authors proved the same result based on [21, Lemma 2.3], which is not satisfied in our case. (See Example 2.2.4 and [21, Lemma 2.3, item (ii)]).

**Lemma 3.0.5.** *Let $G$ be an LCA group, $H \subseteq G$ an open and compact subgroup and $A \in \operatorname{Aut}(G)$ expansive with respect to $H$. If $\Lambda \subset G$ is a sequence, then the following conditions are equivalent:*

(i) $D^+(\Lambda) < \infty$;
(ii) *For some* $n \in \mathbb{Z}$ *there exists* $N_n > 0$ *such that* $\#\{\Lambda \cap Q_n(A^n c)\} \leq N_n$ *for all* $c \in C$*, where* $C$ *is a section for* $G/H$*.*

*Additionaly, if* (ii) *holds for some* $n \in \mathbb{Z}$*, it holds for every* $n \in \mathbb{Z}$*.*

*Proof.* (i) $\Rightarrow$ (ii). It is obvious from the definition of $D^+(\Lambda)$.
(ii) $\Rightarrow$ (i). Let $n \in \mathbb{Z}$ and $N_n > 0$ be such that $\#\{\Lambda \cap Q_n(A^n c)\} \leq N_n$ for all $c \in C$ and call for every $c \in C$, $\Lambda \cap Q_n(A^n c) = \{\lambda_{1,c}, \cdots, \lambda_{r,c}\}$ with $r = r(c) \leq N_n$. Now, for $j \in \{1, \cdots, N_n\}$ and $c_0 \in C_0$ set $\Lambda_{j,c_0} := \{\lambda_{j,c_0+c_1} : c_1 \in C_1\}$. By construction, each element of $\Lambda_{j,c_0}$ lies in a different coset of $G/A^n H$. Then, since there are at most $|A| N_n$ many $\Lambda_{j,c_0}$ sets, we have that any coset of $G/A^n H$ contains at most $|A| N_n$ elements of $\Lambda$.

Now, since

$$A^{n+1}H = A^n(AH) = \bigcup_{c_0 \in C_0} (A^n H + A^n c_0)$$

and

$$G = A^n G = \bigcup_{c_0 \in C_0, c_1 \in C_1} (A^n H + A^n c_0 + A^n c_1) = \bigcup_{c_1 \in C_1} \left( \bigcup_{c_0 \in C_0} (A^n H + A^n c_0) + A^n c_1 \right),$$

each coset of $G/(A^{n+1}H)$ has at most $|A|^2 N_n$ elements of $\Lambda$. Continuing by induction, we see that if $m > n$, then each coset of $G/(A^m H)$ has at most $|A|^{m-n+1} N_n$ elements of $\Lambda$.

As a consequence,

$$\begin{aligned} D^+(\Lambda) &= \limsup_{m \to +\infty} \frac{1}{|A|^m} \max_{x \in G} \#\{\Lambda \cap Q_m(x)\} \\ &\leq \limsup_{m \to +\infty} \frac{1}{|A|^m} |A|^{m-n+1} N_n \\ &= \frac{N_n}{|A|^{n-1}} < \infty. \end{aligned}$$

Suppose now that (ii) fails for some $n \in \mathbb{Z}$. Then, $\max_{x \in G} \#\{\Lambda \cap Q_m(x)\} = +\infty$ for every $m \geq n$. As a consequence, $D^+(\Lambda) = +\infty$. This completes the proof. $\square$

**Remark 3.0.6.** *Note that if* $\Lambda \subset G$ *is a sequence and* $D^+(\Lambda) < \infty$*, we can deduce from the proof of Lemma 3.0.5 that for every fixed* $n \in \mathbb{Z}$*,* $\Lambda = \bigcup_{j \in \{1, \cdots, N_n\}, c_0 \in C_0} \Lambda_{j,c_o}$ *where the union is disjoint. Moreover, every set* $\Lambda_{j,c_o}$ *is a uniformly separated sequence because we saw that each element of* $\Lambda_{j,c_o}$ *lies in a different coset of* $G/A^n H$ *and then* $\#\{\Lambda_{j,c_o} \cap [A^n H + A^n c]\} \leq 1$ *for every* $c \in C$*. Therefore, when* $D^+(\Lambda) < \infty$ *we have that* $\Lambda$ *must be a finite union of uniformly separated sequences; that is,* $\Lambda$ *must be* $A$*-separated. Since uniformly separated sequences must be countable,* $\Lambda$ *must be countable as well.*

**Example 3.0.7.** *Let* $G$ *be an LCA group,* $H \subseteq G$ *an open and compact subgroup and* $C$ *section of the quotient* $G/H$*. Consider* $\Lambda := C$*. Then, we have that for all* $c \in C$*,* $\#\{C \cap Q_0(c)\} = 1$ *and proceeding as in the proof of Lemma 3.0.5 we get* $\#\{C \cap Q_n(A^n c)\} = |A|^n$ *for all* $n \in \mathbb{Z}$*. Therefore,* $D^+(C) = D^-(C) = 1 = D(C)$*.*

## 4. Modulation spaces on locally compact groups

In this section, we consider $G$ a locally compact group (non necessarily abelian), with an open and compact subgroup $H$. Using definitions and lemmas from the seminal papers [8, 9], we will prove that the generalized wavelet transform continuously maps modulation spaces into Wiener spaces.

Let $G$ be a locally compact group with right Haar measure $m_G$. To emphasize that $G$ needs not be abelian, along this section we use multiplicative notation. Given $\mathcal{H}$ a Hilbert space, a unitary representation of $G$ on $\mathcal{H}$ is a continuous homomorphism $\pi : G \to \mathcal{U}(\mathcal{H})$. For $f, g \in \mathcal{H}$ we consider $\mathcal{V}_g f : G \to \mathbb{C}$, the *generalized wavelet transform* (also called *voice transform* or *representation coefficients*) of $f$ with respect to the window $g$ defined by

$$\mathcal{V}_g f(x) := \langle f, \pi(x)g \rangle, \quad x \in G.$$

The set of *analyzing vectors* on $G$ is given by

$$\mathcal{A} := \{g \in \mathcal{H} : \mathcal{V}_g g \in L^1(G)\}. \tag{4}$$

Note that for $g \in \mathcal{H}$ and $x \in G$, since $\pi$ is a representation, $\mathcal{V}_{\pi(x)g}(\pi(x)g)(y) = \mathcal{V}_g g(x^{-1}yx)$ for all $y \in G$. Thus, $\mathcal{A}$ is invariant under $\pi$. Then, when $\pi$ is an irreducible unitary representation, that is, without proper invariant subspaces, $\mathcal{A}$ must be a dense linear subspace of $\mathcal{H}$.

In the remainder of this section, we assume that $H \subseteq G$ is an open and compact subgroup. Examples of non-abelian groups with that property follows.

**Example 4.0.1.** *Fix $p$ prime and $n \geq 2$. Then the general linear group $\mathrm{GL}_n(\mathbb{Q}_p)$ is a non-abelian locally compact group with well-understood representation theory, and $\mathrm{GL}_n(\mathbb{Z}_p)$ is an open and compact subgroup* [3].

We denote the space of continuous functions on $G$ as $\mathcal{C} := \mathcal{C}(G)$. For $\varphi \in \mathcal{C}$ and every $x \in G$ we see that $\|\chi_{Hx} \cdot \varphi\|_\infty = \sup_{y \in Hx} |\varphi(y)| < \infty$, where $\chi_A$ is the characteristic function of $A$ that takes the value 1 for $x \in A$ and 0 otherwise.

For $1 \leq p < \infty$, the Wiener space $W(\mathcal{C}, L^p)$ is defined as

$$W(\mathcal{C}, L^p) := \{\varphi \in \mathcal{C} : \int_G \|\chi_{Hx} \cdot \varphi\|_\infty^p dm_G(x) < \infty\}.$$

It turns out that $W(\mathcal{C}, L^p)$ equipped with the norm $\|\varphi\|_{W(\mathcal{C},L^p)} := \left(\int_G \|\chi_{Hx} \cdot \varphi\|_\infty^p dm_G(x)\right)^{1/p}$ is a Banach space.

Additionally, if $C$ is a section of $G/H$ and $1 \leq p < \infty$, let us denote by $W(\mathcal{C}, \ell^p)$ the space

$$W(\mathcal{C}, \ell^p) := \{\varphi \in \mathcal{C} : \sum_{x \in C} \|\chi_{Hx} \cdot \varphi\|_\infty^p < \infty\}.$$

We note that $W(\mathcal{C}, \ell^p)$ does not depend on the choice of the section $C$. Furthermore, it holds that $W(\mathcal{C}, \ell^p)$ endowed with the norm given by $\|\varphi\|_{W(\mathcal{C},\ell^p)} := \left(\sum_{x \in C} \|\chi_{Hx} \cdot \varphi\|_\infty^p\right)^{1/p}$ is a Banach space.

Moreover, it can be seen that actually $W(\mathcal{C}, \ell^p) = W(\mathcal{C}, L^p)$ and the norms $\|\cdot\|_{W(\mathcal{C},L^p)}$ and $\|\cdot\|_{W(\mathcal{C},\ell^p)}$ coincide as we show in the next lemma.

**Lemma 4.0.2.** *Let $G$ be a locally compact group with an open and compact subgroup. Then, $W(\mathcal{C}, \ell^p) = W(\mathcal{C}, L^p)$. Moreover, for $\varphi \in \mathcal{C}$,*

$$\|\varphi\|_{W(\mathcal{C},\ell^p)} = \|\varphi\|_{W(\mathcal{C},L^p)}.$$

*Proof.* Let $\varphi \in \mathcal{C}$. Since $G = \bigcup_{c\in C} Hc$ where the union is disjoint, $C$ is a section of $G/H$ and $m_G(H) = 1$, we can write

$$\begin{aligned}\|\varphi\|^p_{W(\mathcal{C},L^p)} &= \int_G \|\chi_{Hx}\cdot\varphi\|^p_\infty dm_G(x) = \sum_{c\in C}\int_{Hc}\|\chi_{Hx}\cdot\varphi\|^p_\infty dm_G(x)\\ &= \sum_{c\in C}\int_{Hc}\|\chi_{Hc}\cdot\varphi\|^p_\infty dm_G(x) = \sum_{c\in C}\|\chi_{Hc}\cdot\varphi\|^p_\infty\\ &= \|\varphi\|^p_{W(\mathcal{C},\ell^p)}.\end{aligned}$$

Then $\varphi \in W(\mathcal{C},\ell^p)$ if and only if $\varphi \in W(\mathcal{C},L^p)$. □

We now consider the following subset of $\mathcal{A}$,

$$\mathcal{B} = \{g \in \mathcal{H} : \mathcal{V}_g g \in W(\mathcal{C}, L^1)\}, \tag{5}$$

which turns out to be also invariant under $\pi$. As before, when $\pi$ is irreducible, $\mathcal{B}$ is dense in $\mathcal{H}$. Moreover, when additionally $G$ is abelian, as a consequence of [9, Lemma 7.2], $\mathcal{A} = \mathcal{B}$.

Fixing an arbitrary non-zero element $g \in \mathcal{A}$, the space $\mathscr{M}^1(G)$ is given by

$$\mathscr{M}^1(G) := \{f \in \mathcal{H} : \mathcal{V}_g f \in L^1(G)\},$$

and it is called a *modulation space*. It is a Banach space with the norm $\|f\|_{\mathscr{M}^1} := \|\mathcal{V}_g f\|_{L^1}$. The set $\mathscr{M}^1$ is independent of the choice of $g$; i.e., different vectors in $\mathcal{A}$ give the same space with equivalent norms (see, for instance, [8, Theorem 4.2]). Considering the topological dual of $\mathscr{M}^1$, denoted as $(\mathscr{M}^1(G))'$, and $p \in [1,\infty]$, it is said that $f \in \mathscr{M}^p(G)$, the *modulation space of order* $p$, if $f \in (\mathscr{M}^1(G))'$ and $\|\mathcal{V}_g f\|_{L^p(G)} < +\infty$. These spaces are Banach spaces with the norm $\|f\|_{\mathscr{M}^p} = \|\mathcal{V}_g f\|_{L^p}$, and they are also independent of the choice of the window $g$. For more details on these spaces we refer to [8, Section 4].

It is known [9, Theorem 8.1] that for general locally compact groups, when $g \in \mathcal{B}$, $\mathcal{V}_g$ maps the space $\mathscr{M}^p$ into $W(\mathcal{C},\ell^p)$. We will prove now that, when $G$ has an open and compact subgroup, $\mathcal{V}_g$ maps $\mathscr{M}^p$ into $W(\mathcal{C},\ell^p)$ continuously.

**Proposition 4.0.3.** *Let $G$ be a locally compact group with an open and compact subgroup $H$, $\pi : G \to \mathcal{U}(\mathcal{H})$ an irreducible unitary representation of $G$, $g \in \mathcal{B}$, and $1 \le p \le +\infty$. If $f \in \mathscr{M}^p(G)$, then $\mathcal{V}_g f \in W(\mathcal{C},\ell^p)$. Furthermore, there exists a constant $K > 0$ such that*

$$\|\mathcal{V}_g f\|_{W(\mathcal{C},\ell^p)} \le K\|f\|_{\mathscr{M}^p} \tag{6}$$

*for every $f \in \mathscr{M}^p(G)$.*

*Proof.* By [9, Theorem 8.1] we know that for $f \in \mathscr{M}^p(G)$ we have $\mathcal{V}_g f \in W(\mathcal{C},L^p)$, and that for each set $X = \{x_i\}_{i\in I}$ which is a section of $G/H$, the linear operator given by

$$R_X : f \mapsto (\mathcal{V}_g f(x_i))_{i\in I}$$

maps $\mathscr{M}^p(G)$ in $\ell^p(I)$ continuously; i.e., there exists $K_X > 0$ such that

$$\|(\mathcal{V}_g f(x_i))_{i\in I}\|_{\ell^p} \le K_X\|f\|_{\mathscr{M}^p} \quad \forall f \in \mathscr{M}^p(G).$$

Consider $\mathcal{F} = \{R_X : \text{X is a section of } G/H\}$ and fix $f \in \mathscr{M}^p(G)$. Then, since $\mathcal{V}_g f$ is continuous, we have that

$$\sup_{R_X\in\mathcal{F}}\|R_X(f)\|_{\ell^p} = \sup_{X=\{x_i\}_{i\in I}}\|(\mathcal{V}_g f(x_i))_{i\in I}\|_{\ell^p} = \|(\mathcal{V}_g f(\tilde{x_i}))_{i\in I}\|_{\ell^p}$$

where $X$ is a section of $G/H$ and, for each $i \in I$, $\tilde{x_i} \in H + x_i$ is a point that maximizes $|\mathcal{V}_g f|$ on $H + x_i$. Then, by the uniform boundedness principle there exists $K > 0$ such that

$$\sup_{R_X \in \mathcal{F}} \|R_X\| = \sup_{R_X \in \mathcal{F}} K_X \leq K.$$

Consequently, we have for all sections $\{x_i\}_{i \in I}$ of $G/H$ that the following inequality holds

$$\|(\mathcal{V}_g f(x_i))_{i \in I}\|_{\ell^p} \leq K \|f\|_{\mathscr{M}^p},$$

for every $f \in \mathscr{M}^p(G)$. Now, note that, for each $f \in \mathscr{M}^p(G)$, $\|\mathcal{V}_g f\|_{W(\mathcal{C}, \ell^p)} = \|(\mathcal{V}_g f(\tilde{x_i}))_{i \in I}\|_{\ell^p}$ for a proper section $\tilde{X} = \{\tilde{x_i}\}_{i \in I}$. Thus,

$$\|\mathcal{V}_g f\|_{W(\mathcal{C}, \ell^p)} \leq K \|f\|_{\mathscr{M}^p} \quad \forall f \in \mathscr{M}^p(G).$$

□

We shall see now that there is a valid version of Proposition 4.0.3 for projective representations.

For this, let $G$ be a locally compact group and recall that a *projective representation* is a continuous mapping $\Pi : G \to \mathcal{U}(\mathcal{H})$ for which there exists a continuous function $\alpha : G \times G \to \mathbb{T}$, called a *2-cocycle*, such that $\Pi(x)\Pi(y) = \alpha(x, y)\Pi(xy)$. It is usual to call $\Pi$ an *$\alpha$-projective representation* to emphasize the dependence of $\Pi$ on $\alpha$. As we did for unitary representations, we define the *generalized wavelet transform* corresponding to a projective representation as

$$\mathcal{V}_g^{\Pi} f(x) := \langle f, \Pi(x) g \rangle,$$

for $f, g \in \mathcal{H}$.

Every projective representation of $G$ induces a unitary representation on the *Mackey group* associated to $G$. The last is defined as follows: if $G$ is a locally compact group and $\alpha$ is a 2-cocycle, as a topological space, it is just $G \times \mathbb{T}$ and the product is given by

$$(x_1, \tau_1)(x_2, \tau_2) = (x_1 x_2, \tau_1 \tau_2 \alpha(x_1, x_2)),$$

for $x_1, x_2 \in G$ and $\tau_1, \tau_2 \in \mathbb{T}$. The Mackey group associated to $G$ is a locally compact group and its Haar measure is given by the product of the Haar measures on $G$ and $\mathbb{T}$. Then, for a $\alpha$-projective representation $\Pi : G \to \mathcal{U}(\mathcal{H})$, define $\pi : G \times \mathbb{T} \to \mathcal{U}(\mathcal{H})$ as

$$\pi(x, \tau) := \tau \Pi(x), \tag{7}$$

for $\tau \in \mathbb{T}, x \in G$. This mapping $\pi$ turns out to be a unitary representation, and it is irreducible when $\Pi$ is. Note that for every $f, g \in \mathcal{H}$, $x \in G$ and $\tau \in \mathbb{T}$ we have

$$\mathcal{V}_g^{\Pi} f(x) = \tau \langle f, \tau \Pi(x) g \rangle = \tau \langle f, \pi(x, \tau) g \rangle = \tau \mathcal{V}_g f(x, \tau),$$

with $\mathcal{V}_g f$ being the generalized wavelet transform associated to $\pi$. As a consequence, the sets $\mathcal{A}$ and $\mathcal{B}$ given in (4) and (5) respectively, remain equal if we use the generalized wavelet transform induced by $\Pi$ instead of the one induced by the unitary representation given by (7). Then, the same holds for modulation spaces.

Therefore, we obtain the corresponding version of Proposition 4.0.3 for projective representations. This result is a particular case of [13, Theorem 2.5], with a significant distinction being our successful demonstration that the inclusion given by the Wavelet transform is continuous.

**Theorem 4.0.4.** *Let $G$ be a locally compact group with an open and compact subgroup, $\Pi : G \to \mathcal{U}(\mathcal{H})$ a irreducible projective representation of $G$, $g \in \mathcal{B}$ and $1 \leq p \leq +\infty$. If $f \in \mathscr{M}^p(G)$ then $\mathcal{V}_g^{\Pi} f \in W(\mathcal{C}, \ell^p)$. Furthermore, there exists a constant $K > 0$ such that*

$$\|\mathcal{V}_g^{\Pi} f\|_{W(\mathcal{C}, \ell^p)} \leq K \|f\|_{\mathscr{M}^p} \tag{8}$$

*for every $f \in \mathscr{M}^p(G)$.*

*Proof.* Note that if $H \subseteq G$ is the open and compact subgroup of $G$, then $H \times \mathbb{T}$ is an open and compact subgroup of the Mackey group of $G$. Also note that if $C$ is a section of $G/H$, then $C \times \{1\}$ is a section of $(G \times \mathbb{T})/(H \times \mathbb{T})$. Therefore, for each $c \in C$ we have

$$\|\chi_{Hc} \cdot \mathcal{V}_g^{\Pi} f\|_\infty = \|\chi_{Hc \times \mathbb{T}} \cdot \mathcal{V}_g f\|_\infty,$$

and then

$$\|\mathcal{V}_g^{\Pi} f\|_{W(\mathcal{C},\ell^p)} = \|\mathcal{V}_g f\|_{W(\mathcal{C}(G \times \mathbb{T}),\ell^p)}.$$

The result follows by applying Proposition 4.0.3. □

## 5. Reproducing properties of Gabor systems

In this section we study Bessel and frame conditions on Gabor systems in terms of density. To be precise, let us fix $G$, an LCA group. For every $x \in G$, the *translation operator* by $x$ of a function $f \in L^2(G)$ is given by

$$T_x f(y) = f(y - x), \quad \text{for } m_G\text{-a.e. } y \in G.$$

For $\xi \in \widehat{G}$, the *modulation operator* by $\xi$ of a function $f \in L^2(G)$ is defined by

$$M_\xi f(y) = \langle y, \xi \rangle f(y), \quad \text{for } m_G\text{-a.e. } y \in G..$$

Now, given a funcion $\varphi \in L^2(G)$ and a sequence $\Lambda \subseteq G \times \widehat{G}$, we define the *Gabor system generated by* $\varphi$ *and* $\Lambda$ as

$$S(\varphi, \Lambda) = \{M_\xi T_x \varphi\}_{(x,\xi)\in\Lambda}.$$

In order to establish frame conditions on $S(\varphi, \Lambda)$ in terms of density of $\Lambda$, we assume that $G$ has an open and compact subgroup $H$, and we fix $A \in \mathrm{Aut}(G)$ expansive with respect to $H$ such that $A^*$ is expansive with respect to $H^\perp$. Then, the densities of $\Lambda \subseteq G \times \widehat{G}$ are

$$D^+(\Lambda) := \limsup_{n\to+\infty} \frac{1}{|A|^{2n}} \max_{(x,\xi)\in G\times\widehat{G}} \#\{\Lambda \cap Q_n(x, \xi)\},$$

and

$$D^-(\Lambda) := \liminf_{n\to+\infty} \frac{1}{|A|^{2n}} \min_{(x,\xi)\in G\times\widehat{G}} \#\{\Lambda \cap Q_n(x, \xi)\},$$

where $Q_n(x, \xi)$ is defined as in (2).

On the other hand, note that translation and modulation operators are unitary in $L^2(G)$, and they satisfy the intertwining relationship $M_\xi T_x f = \langle x, \xi\rangle T_x M_\xi f$ for all $x \in G$, $\xi \in \widehat{G}$ and for all $f \in L^2(G)$. Thus, the Gabor representation $\Pi : G \times \widehat{G} \to \mathcal{U}(L^2(G))$ given by

$$\Pi(x, \xi) := M_\xi T_x \tag{9}$$

is an irreducible projective representation with 2-cocycle given by $\alpha((x_1, \xi_1), (x_2, \xi_2)) = \langle x_1, \xi_2\rangle$. Then, we can make use of the tools described in the previous section. In particular, we have defined the well-know *short-time Fourier transform*

$$V_g f(x, \xi) := \langle f, M_\xi T_x g\rangle = \mathcal{V}_g^{\Pi} f(x, \xi),$$

where, $f, g \in L^2(G)$, and $(x, \xi) \in G \times \widehat{G}$. For fixed $f, g \in L^2(G)$, $V_g f$ is well defined and continuous on $G \times \widehat{G}$.

When we consider translations along a section of $G/H$ and modulations along a section of $\widehat{G}/H^\perp$ of the function $\chi_H$, it turns out that the obtained Gabor system is an ortonormal basis for $L^2(G)$. See [13, Theorem 2.7, Case II] for a proof of this fact.

**Lemma 5.0.1.** *Let $C$ and $D$ be sets of coset representatives of $G/H$ and $\widehat{G}/H^{\perp}$ respectively, and consider $\Lambda = C \times D$. Then, $S(\chi_H, \Lambda)$ is an orthonormal basis for $L^2(G)$.*

5.1. **Bessel sequences.** In this section we show one necessary and one sufficient condition for a Gabor system to be a Bessel sequence of $L^2(G)$.

We begin by proving that if a Gabor system is a Bessel sequence, then $\Lambda$ must have finite upper density. This was proved before for $\mathbb{R}^d$ in [5, Theorem 3.1].

**Theorem 5.1.1.** *Let $\varphi \in L^2(G)$ and let $\Lambda \subseteq G \times \widehat{G}$. If $S(\varphi, \Lambda)$ is a Bessel sequence in $L^2(G)$, then $D^+(\Lambda) < \infty$.*

*Proof.* Let $f \in L^2(G)$ with $\|f\|_2 = 1$ such that $\langle \varphi, f\rangle \neq 0$ and define $A_\varphi f : G \times \widehat{G} \to \mathbb{R}_{\geq 0}$

$$A_\varphi f(x, \xi) := |V_\varphi f(x, \xi)|.$$

Then, $A_\varphi f$ is continuous on $G \times \widehat{G}$.

As $A_\varphi f \neq 0$, there exists $(x_0, \xi_0) \in G \times \widehat{G}$ and $n_0 \in \mathbb{Z}$ such that $\eta := \inf\{A_\varphi f(x, \xi) : (x, \xi) \in Q_{n_0}(x_0, \xi_0)\} > 0$.

If we had $D^+(\Lambda) = \infty$, by Lemma 3.0.5 for each $N > 0$ there should exist some $(x_N, \xi_N)$ such that $\#\{\Lambda \cap Q_{n_0}(x_N, \xi_N)\} \geq N$ .

Now, note that if $(x, \xi) \in Q_{n_0}(x_N, \xi_N)$, then $(x, \xi) - (x_N, \xi_N) + (x_0, \xi_0) \in Q_{n_0}(x_0, \xi_0)$ and thus

$$|\langle f, M_{\xi - \xi_N + \xi_0} T_{x - x_N + x_0} \varphi\rangle| \geq \eta.$$

Since $|\langle f, M_{\xi - \xi_N + \xi_0} T_{x - x_N + x_0} \varphi\rangle| = |\langle M_{\xi_0 - \xi_N} T_{x_0 - x_N} f, M_\xi T_x \varphi\rangle|$ we have that

$$\sum_{(x,\xi) \in \Lambda \cap Q_{n_0}(x_N, \xi_N)} |\langle M_{\xi_0 - \xi_N} T_{x_0 - x_N} f, M_\xi T_x \varphi\rangle|^2 \geq \eta^2 N,$$

for all $N \in \mathbb{N}$. Hence, $S(\varphi, \Lambda)$ can not be a Bessel sequence because $\|M_{\xi_0 - \xi_N} T_{x_0 - x_N} f\|_2 = \|f\|_2 = 1$. □

The above theorem extends to a finite union of Gabor systems. More precisely, let $\Lambda_1, \ldots, \Lambda_r \subseteq G \times \widehat{G}$ be sequences each indexed in $I_1, \ldots, I_r$, respectively. That is, $\Lambda_k = \{(x_{i,k}, \xi_{i,k})\}_{i \in I_k}$, for $1 \leq k \leq r$. Define $I = \{(i, k) : i \in I_k, 1 \leq k \leq r\}$ and $\Lambda$ as the sequence $\{(x_{i,k}, \xi_{i,k}) : (i, k) \in I\}$. By abuse of notation, we will simply write $\Lambda = \bigcup_{k=1}^r \Lambda_k$ and say that $\Lambda$ is the *disjoint union of* $\Lambda_1, \ldots, \Lambda_r$. Once this is clear, we can state the result that we announced.

**Theorem 5.1.2.** *For $1 \leq k \leq r$, let $\varphi_k \in L^2(G)$ and $\Lambda_k \subseteq G \times \widehat{G}$ a sequence. Consider $\Lambda$ the disjoint union of $\Lambda_1, \ldots, \Lambda_r$. If $\bigcup_{k=1}^r S(\varphi_k, \Lambda_k)$ is a Bessel sequence, then $D^+(\Lambda) < \infty$.*

*Proof.* Note that since $\bigcup_{k=1}^r S(\varphi_k, \Lambda_k)$ is a Bessel sequence, so is $S(\varphi_k, \Lambda_k)$ for every $1 \leq k \leq r$. Then, by Theorem 5.1.1 we have that $D^+(\Lambda_k) < \infty$ for every $1 \leq k \leq r$.

Now, since $\Lambda$ is the disjoint union of $\Lambda_1, \ldots, \Lambda_r$, for each $n \in \mathbb{Z}$ and $(x, \xi) \in G \times \widehat{G}$,

$$\#\{\Lambda \cap Q_n(x, \xi)\} = \sum_{k=1}^r \#\{\Lambda_k \cap Q_n(x, \xi)\},$$

and as a consequence

$$\sum_{k=1}^r D^-(\Lambda_k) \leq D^-(\Lambda) \leq D^+(\Lambda) \leq \sum_{k=1}^r D^+(\Lambda_k).$$

From here the conclusion follows. □

In what follows we shall prove a weaker converse of Theorem 5.1.1. For this to be true, we have to assume that the generating function of the Gabor system must have a particular decay; that is, it must be a function of $\mathscr{M}^1(G)$.

As a consequence of [17, Theorem 4.7] we have that, when the representation involved is the Gabor representation (9), $\mathcal{A} = \mathcal{B} = \mathscr{M}^1(G)$, and then, Theorem 4.0.4 holds for $g \in \mathscr{M}^1(G)$.

Then, we have the following result, which is a generalization of [14, Theorem 12] to our setting.

**Theorem 5.1.3.** *Let $\varphi \in \mathscr{M}^1(G)$, $\varphi \neq 0$ and $\Lambda \subseteq G \times \widehat{G}$ any sequence with $D^+(\Lambda) < +\infty$. Then $S(\varphi, \Lambda)$ is a Bessel sequence.*

*Proof.* By Lemma 3.0.5, since $D^+(\Lambda) < +\infty$ there exists $N_0 > 0$ such that $\#\{\Lambda \cap Q_0(c,d)\} \leq N_0$ for all $(c,d) \in C \times D$. Then for $f \in L^2(G)$,

$$\begin{aligned}
\sum_{(x,\xi)\in\Lambda} |\langle f, M_\xi T_x \varphi\rangle|^2 &= \sum_{(x,\xi)\in\Lambda} |V_\varphi f(x,\xi)|^2 \\
&= \sum_{(c,d)\in C\times D} \sum_{(x,\xi)\in\Lambda\cap Q_0(c,d)} |V_\varphi f(x,\xi)|^2 \\
&\leq \sum_{(c,d)\in C\times D} N_0 \sup_{(x,\xi)\in Q_0(c,d)} |V_\varphi f(x,\xi)|^2 \\
&= N_0 \|V_\varphi f\|^2_{W(\mathcal{C},\ell^2)} \leq N_0 K \|f\|^2_{\mathscr{M}^2},
\end{aligned}$$

where the last inequality is the result of Theorem 4.0.4. Finally, since $\|f\|_{\mathscr{M}^2} = \|V_\varphi f\|_2$ and by the well-known orthogonality relationship of the short-time Fourier transform (see [8, Section 2.2], $\|V_\varphi f\|_2 = \|f\|_2 \|\varphi\|_2$, we conclude the result. □

It is known that every locally compact abelian group $G$ is algebraically and topologically isomorphic to $\mathbb{R}^d \times G_0$, where $G_0$ is an LCA group with an open and compact subgroup (see, e.g., [6]). For the case where $G_0$ is an expansible group we can combine Theorem 5.1.3 with [14, Theorem 12] to prove a similar statement for the product group $\mathbb{R}^d \times G_0$.

**Proposition 5.1.4.** *Let $G_0$ be an expansible LCA group. Take $g_1 \in M^1(\mathbb{R}^d)$, $g_2 \in M^1(G_0)$ and $\Lambda = \Lambda_1 \times \Lambda_2$, where $\Lambda_1 \subseteq \mathbb{R}^{2d}$ and $\Lambda_2 \subseteq G_0 \times \widehat{G_0}$. Suppose that $\Lambda_1$ and $\Lambda_2$ have finite upper density, where the density of $\Lambda_1$ is the usual Beurling density defined in the introduction, and the density of $\Lambda_2$ is as in Definition 3.0.2. If we consider $g \in L^2(\mathbb{R}^d \times G_0)$ given by $g = g_1 \otimes g_2$, then $S(g, \Lambda)$ is a Bessel sequence for $L^2(\mathbb{R}^d \times G_0)$*

*Proof.* First observe that for $(x_1, x_2, \xi_1, \xi_2) \in (\mathbb{R}^d \times G_0) \times (\mathbb{R}^d \times \widehat{G_0})$ and $f \in L^2(\mathbb{R}^d \times G_0)$, we have

$$\begin{aligned}
V_g f(x_1,x_2,\xi_1,\xi_2) &= \int_{\mathbb{R}^d}\int_{G_0} f(y_1,y_2)\overline{g(y_1-x_1,y_2-x_2)\langle y_1,\xi_1\rangle\langle y_2,\xi_2\rangle}\, dm_{G_0}(y_2)dy_1 \\
&= \int_{\mathbb{R}^d}\int_{G_0} f(y_1,y_2)\overline{g_1(y_1-x_1)g_2(y_2-x_2)\langle y_1,\xi_1\rangle\langle y_2,\xi_2\rangle}\, dm_{G_0}(y_2)dy_1 \\
&\int_{\mathbb{R}^d} \overline{g_1(y_1-x_1)\langle y_1,\xi_1\rangle} \left(\int_{G_0} f(y_1,y_2)\overline{g_2(y_2)\langle y_2,\xi_2\rangle}\, dm_{G_0}(y_2)\right) dy_1 \\
&= V_{g_1}\left(V_{g_2} f_\bullet(x_2,\xi_2)\right)(x_1,\xi_1),
\end{aligned}$$

where by $f_\bullet$ we denote the function the is obtained when the first variable of $f$ is fixed.

On the other hand, since both upper densities are finite, we know from Lemma 3.0.5 and [5, Lemma 2.3] that there exist $N_0$ and $N_1 \in \mathbb{N}$ such that

$$\begin{aligned}&\#(\Lambda_2 \cap Q_0(x,\xi)) \leq N_0 \text{ for all } (x,\xi) \in G_0 \times \widehat{G_0} \text{ and}\\ &\#(\Lambda_1 \cap B_1(t)) \leq N_1 \text{ for all } t \in \mathbb{R}^{2d},\end{aligned}$$

where $B_1(0) = [0,1]^{2d}$ and $B_1(t) = B_1(0) + t$.

Putting this together we obtain

$$\begin{aligned}\sum_{(x_1,x_2,\xi_1,\xi_2)\in\Lambda} |V_g f(x_1,x_2,\xi_1,\xi_2)|^2 &= \sum_{(x_2,\xi_2)\in\Lambda_2}\sum_{(x_1,\xi_1)\in\Lambda_1} |V_{g_1}\left(V_{g_2} f_\bullet(x_2,\xi_2)\right)(x_1,\xi_1)|^2\\ &= \sum_{(x_2,\xi_2)\in\Lambda_2}\sum_{j\in\mathbb{Z}^{2d}}\sum_{(x_1,\xi_1)\in\Lambda_1\cap B_1(j)} |V_{g_1}\left(V_{g_2} f_\bullet(x_2,\xi_2)\right)(x_1,\xi_1)|^2\\ &\leq \sum_{(x_2,\xi_2)\in\Lambda_2}\sum_{j\in\mathbb{Z}^{2d}} N_1 \sup_{(x_1,\xi_1)\in\Lambda_1\cap B_1(j)} |V_{g_1}\left(V_{g_2} f_\bullet(x_2,\xi_2)\right)(x_1,\xi_1)|^2\\ &= N_1 \sum_{(x_2,\xi_2)\in\Lambda_2} \|V_{g_1}(V_{g_2} f_\bullet(x_2,\xi_2))\|^2_{W(\mathbb{R}^d)}\\ &\leq N_1 \sum_{(x_2,\xi_2)\in\Lambda_2} \|V_{g_2} f_\bullet(x_2,\xi_2)\|^2_{L^2(\mathbb{R}^d)} K_1 \|g_1\|^2_{L^2(\mathbb{R}^d)},\end{aligned}$$

where in the last inequality we used [14, Theorem 12] and the fact that $V_{g_2} f_\bullet(x_2,\xi_2) \in L^2(\mathbb{R}^d)$ for a.e. $(x_2,\xi_2) \in G_0 \times \widehat{G_0}$.

Now, using Fubini and Theorem 4.0.4 we have

$$\begin{aligned}N_1 K_1 \|g_1\|^2_{L^2(\mathbb{R}^d)} \sum_{(x_2,\xi_2)\in\Lambda_2} \|V_{g_2} f_\bullet(x_2,\xi_2)\|^2_{L^2(\mathbb{R}^d)} &= K_0 \sum_{(x_2,\xi_2)\in\Lambda_2} \int_{\mathbb{R}^d} |V_{g_2} f_{y_1}(x_2,\xi_2)|^2 dy_1\\ &= K_0 \int_{\mathbb{R}^d} \sum_{(c,d)\in C\times D}\sum_{(x_2,\xi_2)\in\Lambda_2\cap Q_0(c,d)} |V_{g_2} f_{y_1}(x_2,\xi_2)|^2 dy_1\\ &\leq K_0 \int_{\mathbb{R}^d} \sum_{(c,d)\in C\times D} N_0 \sup_{(x_2,\xi_2)\in\Lambda_2\cap Q_0(c,d)} |V_{g_2} f_{y_1}(x_2,\xi_2)|^2 dy_1\\ &= K_0 N_0 \int_{\mathbb{R}^d} \|V_{g_2} f_{y_1}\|^2_{W(G_0)} dy_1\\ &\leq K \int_{\mathbb{R}^d} \|f_{y_1}\|^2_{L^2(G_0)} dy_1\\ &= K \int_{\mathbb{R}^d}\int_{G_0} |f(y_1,y_2)|^2 dy_2 dy_1 = K\|f\|^2_{L^2(\mathbb{R}^d\times G_0)}.\end{aligned}$$

Since $\sum_{(x_1,x_2,\xi_1,\xi_2)\in\Lambda} |\langle f, M_{(\xi_1,\xi_2)} T_{(x_1,x_2)} g\rangle|^2 = \sum_{(x_1,x_2,\xi_1,\xi_2)\in\Lambda} |V_g f(x_1,x_2,\xi_1,\xi_2)|^2$, the above computation shows that $S(g,\Lambda)$ is a Bessel sequence.

□

5.2. **Riesz bases and frames.** In the Euclidean case, it is well known that if $\Lambda$ and $\Gamma$ are sequences of $\mathbb{R}^{2d}$ and for $\psi, \phi \in L^2(\mathbb{R}^d)$, $S(\psi,\Lambda)$ is a frame and $S(\phi,\Gamma)$ is a Riesz basis of $L^2(\mathbb{R}^d)$, then the upper density (resp., lower) of $\Gamma$ will be lower or equal than the upper density (resp., lower) of $\Lambda$ (see [5, Theorem 3.6]). The following corollaries can be derived from this result.

**Corollary 5.2.1.** *Let $g \in L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$.*

*(i) If $S(g,\Lambda)$ is a Riesz basis of $L^2(\mathbb{R}^d)$ then $D(\Lambda) = 1$.*
*(ii) If $S(g,\Lambda)$ is a frame of $L^2(\mathbb{R}^d)$ then $D^-(\Lambda) \geq 1$.*

In our context, none of these statements are true, as we will prove with the following example. This also indicates that it is not possible to have a density comparison theorem like the one valid in the Euclidean case.

**Example 5.2.2.** *Let $G$ be expansive with $A$ and $H$ given. $\varphi := \chi_{A^{-1}H}$ and $\Lambda = \Lambda_1 + \Lambda_2$ with $\Lambda_1$ and $\Lambda_2$ sections of $G/H \times \widehat{G}/H^\perp$ and $(H/A^{-1}H) \times (H^\perp/A^{*-1}H^\perp)$ respectively. We have that $A^{-1}H$ is an open and compact subgroup, $A$ is expansive with respect to $A^{-1}H$ and $\Lambda$ is a section of $(G/A^{-1}H) \times (\widehat{G}/A^{*-1}H^\perp)$. Then $S(\chi_{A^{-1}H}, \Lambda)$ is an orthonormal basis for $L^2(G)$ by Lemma 5.0.1.*

*We want to calculate $D(\Lambda)$. For $n \in \mathbb{N}$ we have*

$$\begin{aligned}\max_{(x,\xi)\in G\times\widehat{G}} \#(Q_n(x,\xi) \cap \Lambda) &\leq \sum_{\lambda_2\in\Lambda_2} \max_{(x,\xi)\in G\times\widehat{G}} \#(Q_n(x,\xi) \cap (\Lambda_1 + \lambda_2))\\ &= \sum_{\lambda_2\in\Lambda_2} \max_{(x,\xi)\in G\times\widehat{G}} \#(Q_n(x,\xi) \cap \Lambda_1)\\ &= \#\Lambda_2 \max_{(x,\xi)\in G\times\widehat{G}} \#(Q_n(x,\xi) \cap \Lambda_1)\\ &= |A|^2 \max_{(x,\xi)\in G\times\widehat{G}} \#(Q_n(x,\xi) \cap \Lambda_1).\end{aligned}$$

*Dividing by $|A|^{2n}$ and taking* lim sup *we obtain that*

$$D^+(\Lambda) \leq |A|^2 D^+(\Lambda_1) = |A|^2.$$

*By doing analogous calculation with the minimum, we obtain that $D^-(\Lambda) \geq |A|^2 D^-(\Lambda_1) = |A|^2$ and therefore $D(\Lambda) = |A|^2$.*

Using the same procedure as in Example 5.2.2, we can construct orthonormal Gabor bases with arbitrarily large density or arbitrarily close to 0 density. For this, it suffices to consider, for each $k \in \mathbb{Z}$, the function $\varphi_k = \chi_{A^kH}$ and $\Lambda_k$ a section of $(G/A^kH) \times (\widehat{G}/A^{*k}H^\perp)$. We then have that $D(\Lambda_k) = |A|^{-2k}$ and $S(\varphi_k, \Lambda_k)$ is an orthonormal basis for $L^2(G)$.

5.3. **Homogeneous Approximation Property.** An important tool that also holds in this context is the *Homogeneous Approximation Property* (HAP) for Gabor frames. The HAP in $L^2(\mathbb{R}^d)$ was introduced by Ramanathan and Steger in [19] in order to prove the Comparison Theorem, but is a fundamental result of independent interest. It essentially states that for any irregular Gabor frame $S(\varphi,\Lambda)$ in $L^2(\mathbb{R}^d)$, the rate of approximation of a Gabor frame expansion of a function $f$ is invariant under time-frequency shifts of $f$. What Ramanathan and Steger established corresponds to what is subsequently identified in the literature as *weak HAP* . In the Euclidean case, Christensen, Deng and Heil demonstrate in [5] that every Gabor frame possesses this particular property. We will prove that the same holds true in our context.

**Definition 5.3.1.** *Assume that $S(\varphi,\Lambda) = \{M_\xi T_x \varphi\}_{(x,\xi)\in\Lambda}$ is a frame for $L^2(G)$, and let $\Phi = \{\phi_{x,\xi}\}_{(x,\xi)\in\Lambda}$ denote its canonical dual frame. For each $n \in \mathbb{Z}$ and $(y,\gamma) \in G \times \widehat{G}$, set*

$$W(n,y,\gamma) = \operatorname{span}\{\phi_{x,\xi} : (x,\xi) \in \Lambda \cap Q_n(y,\gamma)\}.$$

*We say that $S(\varphi,\Lambda)$ has the Homogeneous Approximation Property (HAP) if*

$$\forall f\in L^2(G) \text{ and } \forall \epsilon>0,\ \exists N\in\mathbb{N} \text{ such that}$$
$$\operatorname{dist}(M_\gamma T_y f, W(N,y,\gamma))<\epsilon,\ \ \forall (y,\gamma)\in G\times\widehat{G}.$$

Our goal is to prove that every frame of the form $\bigcup_{k=1}^r S(\varphi_k,\Lambda_k)$ has the HAP. We need the next lemma.

**Lemma 5.3.2.** *Set $\varphi:=\chi_H$ and let $n\in\mathbb{Z}$ be fixed. Then, there exists $C>0$ such that for every $f\in L^2(G)$ and every $(x,\xi)\in G\times\widehat{G}$, we have*

$$|\langle\varphi, M_\xi T_x f\rangle|^2\le C\iint_{Q_n(x,\xi)}|\langle\varphi, M_\gamma T_y f\rangle|^2\,dm_G(y)\,dm_{\widehat{G}}(\gamma). \tag{10}$$

*Further, when $n\le 0$, equality holds in (10) when $C=|A|^{-2n}$. For arbitrary $n\in\mathbb{Z}$, $C=1$ may be chosen to make the inequality hold.*

*Proof.* In order to simplify the notation we write $dy:=dm_G(y)$ and $d\gamma:=dm_{\widehat{G}}(\gamma)$. We first show that the statement holds for $(x,\xi)=(0,0)$ and $n\le 0$, noting that we may apply Fubini-Tonelli's theorem to manipulate the integral since $H\times H^\perp$ is compact and $f\in L^2(G)$:

$$\begin{aligned}
\iint_{Q_n(0,0)}|\langle\varphi, M_\gamma T_y f\rangle|^2\,dy\,d\gamma &= \iint_{Q_n(0,0)}|\langle T_{-y}M_{-\gamma}\varphi, f\rangle|^2\,dy\,d\gamma = \iint_{Q_n(0,0)}|\langle M_{-\gamma}T_{-y}\varphi, f\rangle|^2\,dy\,d\gamma\\
&=\int_{A^nH}\int_{(A^*)^nH^\perp}\left(\int_G\langle z,-\gamma\rangle\chi_H(z+y)\overline{f(z)}\,dz\right)\left(\int_G\langle w,\gamma\rangle\chi_H(w+y)f(w)\,dw\right)dy\,d\gamma\\
&=\int_G\overline{f(z)}\int_G f(w)\int_{(A^*)^nH^\perp}\langle z,-\gamma\rangle\langle w,\gamma\rangle\int_{A^nH}\chi_H(z+y)\chi_H(w+y)\,dy\,d\gamma\,dw\,dz\\
&=\int_G\overline{f(z)}\int_G f(w)\int_{(A^*)^nH^\perp}\langle w-z,\gamma\rangle\left(\int_G\chi_{(A^nH)\cap(-z+H)\cap(-w+H)}(y)\,dy\right)d\gamma\,dw\,dz.
\end{aligned}$$

Since $A$ is expansive and $n$ is nonpositive,

$$\chi_{(A^nH)\cap(-z+H)\cap(-w+H)}(y)=\begin{cases}\chi_{A^nH}(y); & z,w\in H\\ 0; & \text{otherwise}\end{cases},$$

hence

$$\int_G\chi_{(A^nH)\cap(-z+H)\cap(-w+H)}(y)\,dy=\begin{cases}m_G(A^nH); & z,w\in H\\ 0; & \text{otherwise}\end{cases}=|A|^n\chi_H(z)\chi_H(w).$$

Therefore,

$$\begin{aligned}
\iint_{Q_n(0,0)}|\langle\varphi, M_\gamma T_y f\rangle|^2\,dy\,d\gamma &= \int_G\overline{f(z)}\int_G f(w)\int_{(A^*)^nH^\perp}\langle w-z,\gamma\rangle|A|^n\chi_H(z)\chi_H(w)\,d\gamma\,dw\,dz\\
&=|A|^n\int_G\overline{f(z)}\int_G f(w)m_{\widehat{G}}((A^*)^nH^\perp)\chi_H(z)\chi_H(w)\,dw\,dz\\
&=|A|^{2n}\int_G\overline{f(z)}\int_G f(w)\chi_H(z)\chi_H(w)\,dw\,dz=|A|^{2n}\int_G\overline{f(z)}\chi_H(z)\,dz\int_G f(w)\chi_H(w)\,dw\\
&=|A|^{2n}|\langle\varphi,f\rangle|^2,
\end{aligned}$$

where we have used that $(A^*)^nH^\perp\subseteq H^\perp$ since $A$ is expansive and $n$ is nonpositive.

Now, for arbitrary $(x,\xi)\in G\times\widehat{G}$ and $n\le 0$, replace $f$ by $M_\xi T_x f$ in the calculations above to obtain

$$\begin{aligned}
|\langle\varphi, M_\xi T_x f\rangle|^2 &= |A|^{-2n}\iint_{Q_n(0,0)} |\langle\varphi, M_\gamma T_y M_\xi T_x f\rangle|^2\, dy\, d\gamma \\
&= |A|^{-2n}\iint_{Q_n(0,0)} |\langle\varphi, \overline{(y,\xi)} M_\gamma M_\xi T_y T_x f\rangle|^2\, dy\, d\gamma \\
&= |A|^{-2n}\iint_{Q_n(0,0)} |\langle\varphi, M_{\gamma+\xi} T_{y+x} f\rangle|^2\, dy\, d\gamma \\
&= |A|^{-2n}\iint_{Q_n(x,\xi)} |\langle\varphi, M_\gamma T_y f\rangle|^2\, dy\, d\gamma.
\end{aligned}$$

Finally, for $n \geq 1$, we note that

$$|\langle\varphi, M_\xi T_x f\rangle|^2 = \iint_{Q_0(x,\xi)} |\langle\varphi, M_\gamma T_y f\rangle|^2\, dy\, d\gamma \leq \iint_{Q_n(x,\xi)} |\langle\varphi, M_\gamma T_y f\rangle|^2\, dy\, d\gamma.$$

□

If $\varphi_1, \ldots, \varphi_r \in L^2(G)$, $\Lambda_1, \ldots, \Lambda_k \subseteq G \times \widehat{G}$ is such that $\bigcup_{k=1}^r S(\varphi_k, \Lambda_k)$ is a frame for $L^2(G)$ let us denote its canonical dual frame as

$$\{\phi_{k,x,\xi} : (x,\xi) \in \Lambda_k, 1 \leq k \leq r\}.$$

Given $n \in \mathbb{Z}$ and $(y,\gamma) \in G \times \widehat{G}$ we recall that

$$W(n,y,\gamma) := \operatorname{span}\{\phi_{k,x,\xi} : (x,\xi) \in Q_n(y,\gamma) \cap \Lambda_k, 1 \leq k \leq r\}.$$

Note that since $\bigcup_{k=1}^r S(\varphi_k, \Lambda_k)$ is a frame for $L^2(G)$, $S(\varphi_k, \Lambda_k)$ is a Bessel sequence for every $1 \leq k \leq r$. Then, by Theorem 5.1.1, $D^+(\Lambda_k) < \infty$, and by Lemma 3.0.5, $\#(Q_n(y,\gamma) \cap \Lambda_k)$ is finite for every $1 \leq k \leq r$. Hence, $W(n,y,\gamma)$ is a finite dimensional subspace of $L^2(G)$ and thus closed. The following result states that for this type of frame the HAP always holds.

**Theorem 5.3.3.** *Let $\varphi_1, \ldots, \varphi_r \in L^2(G)$, $\Lambda_1, \ldots, \Lambda_k \in G \times \widehat{G}$ be such that $\bigcup_{k=1}^r S(\varphi_k, \Lambda_k)$ is a frame for $L^2(G)$. Then, for every $f \in L^2(G)$ the following condition holds:*

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, (y,\gamma) \in G \times \widehat{G},\ d(M_\gamma T_y f, W(N,y,\gamma)) < \varepsilon.$$

*Proof.* Let

$$K = \{f \in L^2(G) : \forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, (y,\gamma) \in G \times \widehat{G},\ d(M_\gamma T_y f, W(N,y,\gamma)) < \varepsilon\}.$$

Then, it is easily seen that $K$ is a subspace that is closed.

We want to show that $M\gamma T_y\varphi \in K$ for all $(y,\gamma) \in C \times D$ where $\varphi := \chi_H$ and $C$ and $D$ are like in Lemma 5.0.1. By this lemma, we know that $\{M\gamma T_y\varphi\}_{(y,\gamma)\in C\times D}$ is an orthonormal basis for $L^2(G)$ and then we will conclude that $K = L^2(G)$.

Consider $\Lambda$ to be the disjoint union of $\Lambda_1, \ldots, \Lambda_r$. Then, by Theorem 5.1.2 we know that $D^+(\Lambda) < \infty$. Therefore, by Remark 3.0.6 for each fixed $\tilde{n} \in \mathbb{Z}$ there exists a partition of each $\Lambda_k$ into disjoint sets $\Lambda_1^{(k)}, \ldots, \Lambda_{n_k}^{(k)}$ such that $\#\{\Lambda_j^{(k)} \cap Q_{\tilde{n}}(x,\xi)\} \leq 1$ for every $j = 1, \ldots, n_k$, every $k = 1, \ldots, r$ and every $(x,\xi) \in G \times \widehat{G}$. Then, in order to simplify the notation, we can fix $\tilde{n} \in \mathbb{Z}$ and assume that each $\Lambda_k$ is uniformly separated for that $\tilde{n}$.

From here, we deduce that for $(x_1,\xi_1), (x_2,\xi_2) \in \Lambda_k$, $Q_{\tilde{n}}(x_1,\xi_1) \cap Q_{\tilde{n}}(x_2,\xi_2) = \emptyset$.

Now, fix $(x_0,\xi_0) \in C \times D$ and take any $(y,\gamma) \in G \times \widehat{G}$. Since $\bigcup_{k=1}^{r} S(\varphi_k,\Lambda_k)$ is a frame, we can write

$$M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi) = \sum_{k=1}^{r} \sum_{(x,\xi)\in\Lambda_k} \langle M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi), M_\xi T_x \varphi_k\rangle \phi_{k,x,\xi}.$$

Now, since for every $n \in \mathbb{Z}$, $\sum_{k=1}^{r}\sum_{(x,\xi)\in Q_n(y,\gamma)\cap\Lambda_k} \langle M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi), M_\xi T_x\varphi_k\rangle\phi_{k,x,\xi}$ is an element of $W(n,y,\gamma)$, we have that

$$\begin{aligned}
&d(M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi),W(n,y,\gamma))^2\\
&\qquad\le \|M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi) - \sum_{k=1}^{r}\sum_{(x,\xi)\in Q_n(y,\gamma)\cap\Lambda_k}\langle M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi), M_\xi T_x\varphi_k\rangle\phi_{k,x,\xi}\|^2\\
&\qquad= \|\sum_{k=1}^{r}\sum_{(x,\xi)\in\Lambda_k\setminus Q_n(y,\gamma)}\langle M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi), M_\xi T_x\varphi_k\rangle\phi_{k,x,\xi}\|^2\\
&\qquad\le \tilde{C}\sum_{k=1}^{r}\sum_{(x,\xi)\in\Lambda_k\setminus Q_n(y,\gamma)}|\langle M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi), M_\xi T_x\varphi_k\rangle|^2\\
&\qquad= \tilde{C}\sum_{k=1}^{r}\sum_{(x,\xi)\in\Lambda_k\setminus Q_n(y,\gamma)}|\langle \varphi, M_{\xi-\gamma-\xi_0}T_{x-y-x_0}\varphi_k\rangle|^2\\
&\qquad\le \tilde{C}C\sum_{k=1}^{r}\sum_{(x,\xi)\in\Lambda_k\setminus Q_n(y,\gamma)}\iint_{Q_{\tilde{n}}(x-y-x_0,\xi-\gamma-\xi_0)}|\langle\varphi, M_\delta T_z\varphi_k\rangle|^2\,d\delta\,dz, \qquad (11)
\end{aligned}$$

where in the last inequality we have used Lemma 5.3.2. Note that by a simple change of variables

$$\iint_{Q_{\tilde{n}}(y+x_0-x,\gamma+\xi_0-\xi)}|\langle\varphi, M_\delta T_z\varphi_k\rangle|^2\,dz\,d\delta = \iint_{Q_{\tilde{n}}(x-y-x_0,\xi-\gamma-\xi_0)}|\langle\varphi, M_\delta T_z\varphi_k\rangle|^2\,dz\,d\delta.$$

Since balls with center in $\Lambda_k$ an radius $\tilde{n}$ are disjoint, we have that for a fixed $k \in \{1,\dots,r\}$

$$\bigcup_{(x,\xi)\in\Lambda_k\setminus Q_n(y,\gamma)} Q_{\tilde{n}}(x-y-x_0,\xi-\gamma-\xi_0)$$

is a disjoint union. Moreover, for every $n > \tilde{n}$

$$\bigcup_{(x,\xi)\in\Lambda_k\setminus Q_n(y,\gamma)} Q_{\tilde{n}}(x-y-x_0,\xi-\gamma-\xi_0) \subseteq \left(G\times\widehat{G}\right)\setminus Q_{n-\tilde{n}}(x_0,\xi_0),$$

and then, combining this with (11) we obtain that

$$d(M_\gamma T_y(M_{\xi_0}T_{x_0}\varphi),W(n,y,\gamma))^2 \le \tilde{C}C\sum_{k=1}^{r}\iint_{(G\times\widehat{G})\setminus Q_{n-\tilde{n}}(x_0,\xi_0)}|\langle\varphi, M_\delta T_z\varphi_k\rangle|^2\,dz\,d\delta.$$

The last integral can be made as small as we want by taking $n$ large enough, independently of $(y,\gamma)$. □

**Corollary 5.3.4.** *Let $\varphi_1,\dots,\varphi_r \in L^2(G)$, $\Lambda_1,\dots,\Lambda_k \in G\times\widehat{G}$ be such that $\bigcup_{k=1}^{r} S(\varphi_k,\Lambda_k)$ is a frame for $L^2(G)$. Then for each $f \in L^2(G)$ and each $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that*

$$\forall (y,\gamma)\in G\times\widehat{G}, \quad \forall k>0, \quad \forall (x,\xi)\in Q_k(y,\gamma) \qquad d(M_\xi T_x f, W(N+k,y,\gamma)) < \epsilon.$$

*Proof.* Simply note that if $(x,\xi)\in Q_k(y,\gamma)$, then $W(N,x,\xi)\subset W(N+k,y,\gamma)$ and therefore $d(M_\xi T_x f, W(N+k,y,\gamma)) \le d(M_\xi T_x f, W(N,x,\xi))$. □

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