HOMOLOGICAL DIMENSIONS, THE GORENSTEIN PROPERTY, AND SPECIAL CASES OF SOME CONJECTURES

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ABSTRACT. Our purpose in this work is multifold. First, we provide general criteria for the finiteness of the projective and injective dimensions of a finite module M over a (commutative) Noetherian ring R. Second, in the other direction, we investigate the impact of the finiteness of certain homological dimensions of M if R is local, mainly when R is Cohen-Macaulay and with a partial focus on duals. Along the way, we produce various freeness criteria for modules. Finally, we give applications, including characterizations of when R is Gorenstein (and other ring-theoretic properties as well, sometimes in the prime characteristic setting), particularly by means of its anticanonical module, and in addition we address special cases of some long-standing conjectures; for instance, we confirm the 1985 conjecture of Vasconcelos on normal modules in case the module of differentials is almost Cohen-Macaulay.

1. Introduction

Part of modern homological commutative algebra is concerned with the problem of finding new characterizations, via module theory, of fundamental ring-theoretic properties of commutative rings such as the Gorenstein, the complete intersection, and the regular properties, among others (it is worth recalling that, in an appropriate setting, such properties also feature a strong geometric counterpart). One of our goals in this paper, for a given Noetherian local ring R, is to address this issue by studying several homological dimensions of a finitely generated R-module M, namely, the injective, projective, Gorenstein, and complete intersection dimensions of M, and of its algebraic dual – sometimes iterated with the canonical dual, once R possesses a canonical module – and then focus on the Gorenstein property of R, although we shall be able as well to provide characterizations of all the above-mentioned properties. For the case targeting Gorensteiness, if R is Cohen-Macaulay with canonical module ω_R then we investigate the case $M = \omega_R$, particularly via the R-dual Hom $_R(\omega_R, R)$, the so-called anticanonical module of R, which plays an important role on the properties of R itself; we refer to [20, Introduction] for a nice description of this interplay and its connections to other topics.

Furthermore, besides providing criteria for the finiteness of the injective and projective dimensions of M in Section 2 (for this task, we do not always assume that R is local or Noetherian; the main results in this section are Theorem 5 and Theorem 7), and as a fundamental step to achieve the goals described above, we address in Section 3 the problem of when the finiteness of suitable homological dimensions of the R-module $Hom_R(M,R)$ forces M to be free, or totally reflexive, or of complete intersection dimension zero, if R is a Cohen-Macaulay local Noetherian ring; one of the main results in this section is Proposition 19 and some of its byproducts, e.g., Corollary 21 and Proposition 23. Needless to say, freeness is a classical matter of interest, being in particular the protagonist of the celebrated Auslander-Reiten conjecture, to which we also contribute in one of our results (Corollary 26). See also Corollary 28 and Corollary 33. Criteria for the Gorensteiness of R (mentioned in the previous paragraph) and more on freeness of modules are presented in Section 4, where the main results are Proposition 34, Corollary

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42, Corollary 45 (this one treats the prime characteristic case) and Corollary 47 (which makes use of syzygies of strongly rigid modules).

Finally, in Section 5, we relate our investigation to issues involving celebrated modules such as modules of differentials and derivations as well as (co)normal modules; more precisely, we address long-standing problems such as the strong form of the Zariski-Lipman conjecture [25] about derivation modules, Berger's conjecture [8] on differential modules in dimension one, and Vasconcelos' conjecture [49, p. 373] about normal modules. More precisely, in the Cohen-Macaulay case, we settle the first one when the differential module is maximal Cohen-Macaulay (Corollary 50, which in fact proves a more general statement), and the third one when either the conormal module is maximal Cohen-Macaulay (Corollary 56, where, once again, a more general result is given) or the differential module is almost Cohen-Macaulay (Corollary 61). As to the second conjecture, we confirm its validity in case the canonical dual of the derivation module has finite projective dimension (Corollary 54).

A few conventions and notations. Throughout this note, unless explicitly stated differently, by a ring we mean a commutative Noetherian ring with non-zero identity, and by a finite module over a ring R we mean a finitely generated R-module. We denote projective dimension and injective dimension over R by pd_R and id_R , respectively.

2. FINITENESS CRITERIA FOR THE PROJECTIVE AND INJECTIVE DIMENSIONS VIA RING MAPS

The purpose of this first section is to derive general criteria for the finiteness of the projective and injective dimensions of finite modules (not necessarily over local rings) via a suitable ring homomorphism. Our first result is as follows.

Proposition 1. Let $R \to S$ be a ring homomorphism, with S not necessarily Noetherian, such that $\mathfrak{m}S \neq S$ for each $\mathfrak{m} \in \operatorname{Max}(R)$. Let M be a finite R-module. The following assertions hold true:

- (1) If, for each $\mathfrak{n} \in \operatorname{Max}(S)$, there exists an integer $h \geq 0$ (possibly depending on \mathfrak{n}) such that $\operatorname{Tor}_h^R(M, S/\mathfrak{n}) = 0$, then $\operatorname{pd}_R M < \infty$.
- (2) Assume S is module-finite over R. If, for each $n \in \text{Max}(S)$, there exists an integer $h \ge 0$ (possibly depending on n) such that $\text{Ext}_R^h(M, S/n) = 0$, then $\text{pd}_R M < \infty$.
- (3) Suppose dim $M < \infty$, and let $i > \dim M$ be an integer. If, for each $\mathfrak{n} \in \operatorname{Max}(S)$, we have $\operatorname{Ext}^i_R(S/\mathfrak{n},M) = 0$, then $\operatorname{id}_R M < i$ and R is locally Cohen-Macaulay on $\operatorname{Supp}_R(M)$.
- (4) Suppose dim $R < \infty$. If, for each $\mathfrak{n} \in \operatorname{Max}(S)$, we have $\operatorname{Ext}_R^J(S/\mathfrak{n}, M) = 0$ for all $j \gg 0$, then $\operatorname{id}_R M < \infty$ and R is locally Cohen-Macaulay on $\operatorname{Supp}_R(M)$.

Proof. (1) For a given $\mathfrak{m} \in \operatorname{Max}(R)$, we have $\mathfrak{m}S \neq S$ by hypothesis and so there exists $\mathfrak{n} \in \operatorname{Max}(S)$ such that $\mathfrak{m}S \subseteq \mathfrak{n}$. Also, by assumption, there is an integer $h \geq 0$ such that $\operatorname{Tor}_h^R(M,S/\mathfrak{n}) = 0$. As S/\mathfrak{n} is an R/\mathfrak{m} -vector space (possibly of infinite dimension), we get that S/\mathfrak{n} is a direct sum of (possibly infinitely many) copies of R/\mathfrak{m} , and hence $\operatorname{Tor}_h^R(M,R/\mathfrak{m}) = 0$ by [41, Proposition 7.6]. So, localizing at each $\mathfrak{m} \in \operatorname{Max}(R)$, we obtain $\operatorname{pd}_{R_\mathfrak{m}} M_\mathfrak{m} < \infty$. Now it is convenient to consider the so-called large restricted flat dimension $\operatorname{Rfd}_R M$ of M over R, which can be expressed as

$$\operatorname{Rfd}_R M = \sup \{\operatorname{depth} R_{\mathfrak p} - \operatorname{depth}_{R_{\mathfrak p}} M_{\mathfrak p} \mid \mathfrak p \in \operatorname{Spec}(R)\};$$

we refer to [6, Notes 1.6 and (1.0.1)]. It follows that, for each $\mathfrak{m} \in \operatorname{Max}(R)$,

$$\operatorname{pd}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} = \operatorname{depth} R_{\mathfrak{m}} - \operatorname{depth}_{R_{\mathfrak{m}}} M_{\mathfrak{m}} \leq \operatorname{Rfd}_R M < \infty,$$

where the finiteness of Rfd_R M is guaranteed by [6, Theorem 1.1]. Hence, using [41, Proposition 8.52], we conclude $pd_R M = \sup\{pd_{R\mathfrak{m}} M\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Max}(R)\} \leq \operatorname{Rfd}_R M < \infty$.

(2) Fix $\mathfrak{m} \in \operatorname{Max}(R)$. By hypothesis, $\mathfrak{m}S \neq S$, hence there exists $\mathfrak{n} \in \operatorname{Max}(S)$ such that $\mathfrak{m}S \subseteq \mathfrak{n}$, and in addition there exists $h \geq 0$ such that $\operatorname{Ext}_R^h(M, S/\mathfrak{n}) = 0$. As S is module-finite over R, we get that S/\mathfrak{n} is a direct sum of finitely many copies of R/\mathfrak{m} , which yields $\operatorname{Ext}_R^h(M, R/\mathfrak{m}) = 0$ by [41, Proposition 7.22]. The rest of the proof is similar to the one given in (1).

(3) By [41, Proposition 7.21], we have $\operatorname{Ext}_R^i(R/\mathfrak{m},M)=0$ for each $\mathfrak{m}\in\operatorname{Max}(R)$. Since $i>\dim M\geq \operatorname{depth}_{R_\mathfrak{m}}M_\mathfrak{m}$ for each $\mathfrak{m}\in\operatorname{Max}(R)\cap\operatorname{Supp}_R(M)$, we can localize the above vanishing condition and use [40, II. Theorem 2] in order to get $\operatorname{id}_{R_\mathfrak{m}}M_\mathfrak{m}< i$. Therefore, for every R-module L, we have

$$\operatorname{Ext}_R^{\geq i}(L,M)_{\mathfrak{m}} \cong \operatorname{Ext}_{R_{\mathfrak{m}}}^{\geq i}(L_{\mathfrak{m}},M_{\mathfrak{m}}) = 0 \quad \text{for every} \quad \mathfrak{m} \in \operatorname{Max}(R).$$

Thus, $\operatorname{Ext}_R^{\geq i}(L,M) = 0$ for every R-module L, so that $\operatorname{id}_R M < \infty$ and, more precisely, $\operatorname{id}_R M < i$ by [11, Proposition 3.1.10]. Finally, if $\mathfrak{p} \in \operatorname{Supp}_R(M)$, then the nonzero $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ has finite injective dimension and hence $R_{\mathfrak{p}}$ must be Cohen–Macaulay by the well-known Bass' Theorem.

(4) As in (3), we have $\operatorname{Ext}_R^{\gg 0}(R/\mathfrak{m},M)=0$ for each $\mathfrak{m}\in\operatorname{Max}(R)$. Localizing this vanishing condition and using [11, Proposition 3.1.14], we get $\operatorname{id}_{R_\mathfrak{m}}M_\mathfrak{m}<\infty$. Thus, $\operatorname{id}_{R_\mathfrak{m}}M_\mathfrak{m}=\operatorname{depth}R_\mathfrak{m}\leq \operatorname{dim}R$ for each $\mathfrak{m}\in\operatorname{Max}(R)$, where the equality follows by [11, Theorem 3.1.17]. Hence, for every R-module L, we have $\operatorname{Ext}_R^{>\dim R}(L,M)_\mathfrak{m}\cong\operatorname{Ext}_{R_\mathfrak{m}}^{>\dim R}(L_\mathfrak{m},M_\mathfrak{m})=0$ for every $\mathfrak{m}\in\operatorname{Max}(R)$. Thus,

$$\operatorname{Ext}_{R}^{>\dim R}(L,M)=0$$
 for every *R*-module *L*,

and so [11, Proposition 3.1.10] yields $id_R M \le \dim R < \infty$. The last assertion follows from (3).

Before we proceed with further preparation for a theorem on finiteness of projective dimension, we record the following byproduct which gives a criterion for an interesting new global bound on injective dimension (in the local case, the bound is attained).

Corollary 2. Let $R \to S$ be a ring homomorphism, with S not necessarily Noetherian, such that $\mathfrak{m}S \neq S$ for every $\mathfrak{m} \in \operatorname{Max}(R)$. Let M be a finite R-module with $\dim M < \infty$. If, for each $\mathfrak{n} \in \operatorname{Max}(S)$, we have $\operatorname{Ext}_R^{\dim M+1}(S/\mathfrak{n},M)=0$, then

$$id_R M \leq \dim M$$

and R is locally Cohen-Macaulay on $\operatorname{Supp}_R(M)$. In particular, if R is local then $\dim M = \operatorname{id}_R M = \operatorname{depth} R$.

Proof. Apply Proposition 1(3) with $i = \dim M + 1$. If R is local then $\dim M \le \operatorname{id}_R M = \operatorname{depth} R$ (see [11, Theorem 3.1.17]), and hence the assertion follows.

Below we furnish simple instances of a ring homomorphism $R \to S$ satisfying $\mathfrak{m}S \neq S$ for every $\mathfrak{m} \in \operatorname{Max}(R)$.

Example 3. (1) Let $R \to S$ be a finite morphism of rings and assume (R, \mathfrak{m}) is local. Then, $\mathfrak{m}S \neq S$ is automatic by Nakayama's lemma.

- (2) Let $R \to S$ be a faithfully flat map. Then, $S/\mathfrak{m}S \cong S \otimes_R R/\mathfrak{m} \neq 0$ implies $\mathfrak{m}S \neq S$ for all $\mathfrak{m} \in Max(R)$.
- (3) Let $R \subseteq S$ be an integral extension. Then, each $\mathfrak{m} \in \operatorname{Max}(R)$ is the contraction of a maximal ideal of S, hence $\mathfrak{m}S \neq S$.
- (4) Assume $R \subseteq S$ is a ring extension such that R is a retraction of a given ring S, i.e., there is an R-linear map $S \to R$ whose restriction to R is the identity map (note R is also Noetherian; see, e.g., [32, Exercise 5.27]). Then, $\mathfrak{m}S \neq S$ for all $\mathfrak{m} \in \operatorname{Max}(R)$.

The next lemma is an elementary fact that will be useful in the sequel.

Lemma 4. Let R be a ring. Let $\{T_n\}$ be a covariant homological delta-functor, $\{F^n\}$ a covariant cohomological delta-functor, and $\{G^n\}$ a contravariant cohomological delta-functor, all from the category of R-modules to itself. Let $s,h \geq 0$ be integers, and N,X_0,\ldots,X_h be R-modules fitting into an exact sequence

$$0 \to X_h \to \cdots \to X_0 \to N \to 0$$
.

The following assertions hold true:

- (1) If $T_{s+h-i+1 \le j \le s+h+1}(X_i) = 0$ for all i = 0, ..., h, then $T_{s+h+1}(N) = 0$.
- (2) If $F^{s+1 \le j \le s+i+1}(X_i) = 0$ for all i = 0, ..., h, then $F^{s+1}(N) = 0$.

(3) If
$$G^{s+h-i+1} \le j \le s+h+1(X_i) = 0$$
 for all $i = 0, ..., h$, then $G^{s+h+1}(N) = 0$.

Proof. The case where h = 0 is clear, so we assume $h \ge 1$. We break apart the given exact sequence into short exact sequences

$$0 \rightarrow N_i \rightarrow X_{i-1} \rightarrow N_{i-1} \rightarrow 0$$
,

for $1 \le i \le h$ and *R*-modules N_i , with $N_h = X_h$ and $N_0 = N$.

- (1) Applying $\{T_n\}$ to the short exact sequence above for i=h, we obtain $T_{s+2 \le j \le s+h+1}(N_{h-1})=0$. If h=1, we are done, otherwise applying $\{T_n\}$ to the short exact sequence for i=h-1, we get $T_{s+3 \le j \le s+h+1}(N_{h-2})=0$. Continuing similarly, we have $T_{s+h+1}(N_0)=0$, which completes the proof since $N_0=N$.
- (2) Applying $\{F^n\}$ to the short exact sequence above for i=h, we get $F^{s+1\leq j\leq s+h}(N_{h-1})=0$. If h=1, we are done, otherwise applying $\{F^n\}$ to the short exact sequence for i=h-1, we obtain $F^{s+1\leq j\leq s+h-1}(N_{h-2})=0$. Continuing this way, we finally get $F^{s+1}(N_0)=0$, which completes the proof since $N_0=N$.
 - (3) The proof is similar to that of (1).

Following standard terminology, we say that a module over a ring S is (maximal) Cohen–Macaulay if it is locally (maximal) Cohen–Macaulay everywhere on Spec(R).

Theorem 5. Let $R \to S$ be a ring map such that $\mathfrak{m}S \neq S$ for each $\mathfrak{m} \in Max(R)$. Let M be a finite R-module. Assume that S is Cohen–Macaulay and that any one of the following conditions holds true:

- (1) $\operatorname{Tor}_{\gg 0}^R(M,X) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.
- (2) There exist (uniform) integers $s,h \ge 0$ such that $\operatorname{Tor}_{s+1 \le j \le s+h+1}^R(M,X) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.
- (3) S is module-finite over R and, in addition, $\operatorname{Ext}_R^{\gg 0}(M,X) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.
- (4) S is module-finite over R and, in addition, there exist (uniform) integers $s,h \geq 0$ such that $\operatorname{Ext}_R^{s+1 \leq j \leq s+h+1}(M,X) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.

Then, $pd_R M < ∞$.

Proof. For items (1) and (2) (resp. item (3) and (4)), it suffices to prove that for each $n \in Max(S)$ there exists an integer $h' \ge 0$ such that

$$\operatorname{Tor}_{h'}^R(M, S/\mathfrak{n}) = 0$$

(resp. $\operatorname{Ext}_R^{h'}(M,S/\mathfrak{n})=0$), by Proposition 1(1) (resp. Proposition 1(2)). So, let $\mathfrak{n}\in\operatorname{Max}(S)$ and set $t=\operatorname{depth} S_\mathfrak{n}(=\dim S_\mathfrak{n})$. If Z stands for the t-th syzygy S-module of S/\mathfrak{n} , then by [41, Proposition 8.5] we obtain that $Z_\mathfrak{q}$ is a free $S_\mathfrak{q}$ -module for each $\mathfrak{q}\in\operatorname{Spec}(S)\setminus\{\mathfrak{n}\}$, while for $\mathfrak{q}=\mathfrak{n}$ the $S_\mathfrak{n}$ -module $Z_\mathfrak{n}$ is maximal Cohen–Macaulay by [11, Exercise 1.3.7]. This proves that Z is maximal Cohen–Macaulay and also locally free on $\operatorname{Spec}(S)\setminus\operatorname{Max}(S)$. Note that S obviously satisfies this property as well. In other words, there exists an exact sequence

$$(5.1) 0 \to X_t \to X_{t-1} \to \cdots \to X_0 \to S/\mathfrak{n} \to 0$$

where each X_i is a maximal Cohen–Macaulay S-module that is locally free on $Spec(S) \setminus Max(S)$. Therefore, the existence of the desired integer h' follows by $d\acute{e}calage$ for items (1) and (3), and by applying Lemma 4 with $T_*(-) := Tor_*^R(M, -)$ for item (2) and $F^*(-) := Ext_R^*(M, -)$ for item (4).

Remark 6. In Theorem 5, if we suppose S is regular, we may clearly replace all the modules X with S itself. Then, in this case, we note the following:

(i) In item (1), S is a test R-module in the sense of [13]. In particular, item (1) was proved in [13, Proposition 2.4] if $R \to S$ is a finite local ring map.

(ii) In item (3), if R is a Cohen-Macaulay local ring with a canonical module ω_R and S is maximal Cohen-Macaulay over R, then $\operatorname{Hom}_R(S, \omega_R)$ is a test R-module by [13, Proposition 3.6].

Theorem 7. Let $R \to S$ be a ring map such that $\mathfrak{m}S \neq S$ for each $\mathfrak{m} \in \operatorname{Max}(R)$. Let M be a finite R-module. Assume that S is Cohen–Macaulay and that any one of the following conditions holds true:

- (1) $\dim R < \infty$ and $\operatorname{Ext}_R^{\gg 0}(X, M) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.
- (2) $\dim S = h < \infty$, $\dim M < \infty$, and there exists a (uniform) integer $s \ge 0$ with $s + h + 1 > \dim M$ such that $\operatorname{Ext}_R^{s+1 \le j \le s + h + 1}(X, M) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.

Then, $id_R M < \infty$ *and* R *is locally Cohen–Macaulay on* $Supp_R(M)$.

Proof. As in the proof of Theorem 5, given $\mathfrak{n} \in \operatorname{Max}(S)$ there exists an exact sequence (5.1) with $t = \operatorname{depth} S_{\mathfrak{n}} = \dim S_{\mathfrak{n}}$ (< h in item (2)). Item (1) follows from Proposition 1(4), and item (2) from Proposition 1(3) along with Lemma 4(3) by letting $G^*(-) := \operatorname{Ext}_R^*(-,M)$.

Remark 8. As in Remark 6, suppose S is regular so that the X's can be replaced with S. If in addition R admits a dualizing complex (e.g., if R is a quotient of a Gorenstein ring of finite Krull dimension), then Theorem T(1) follows from Remark T(1) follows from Remark T(1) and T(1) representation of T(2).

Example 9. Let R be a reduced ring of dimension 1. Let S be the integral closure of R in the total quotient ring Q(R) (note S is also Noetherian, by the Krull-Akizuki theorem). Since S is reduced and integrally closed in Q(R) = Q(S), we get that S is normal (see, e.g., [27, Corollary 2.1.13]). As dim S = 1 and each localization of S is a normal domain, we obtain that S is regular. Finally, by Example 3(3), $mS \neq S$ for every maximal ideal m of R. Therefore, R, S is a pair as in items (1), (2) and (3) of Theorem 5 and as in Theorem 7. If moreover R is an analytically unramified local ring, then S is also module-finite over R, hence the scenario of Theorem 5(4) also applies.

Example 10. We give another instance where $R \to S$ is finite. As recalled in Example 3(4), any algebra retraction $R \subseteq S$ satisfies $\mathfrak{m}S \neq S$ for every $\mathfrak{m} \in \operatorname{Max}(R)$. In particular, we can take

$$R = S^G$$
 for a finite $G \le Aut(S)$

whose order is invertible in S. If moreover S is a domain, then S is module-finite over R; see [32, Exercise 5.27, Exercise 5.28, and Proposition 5.4].

Example 11. Suppose *S* is Cohen–Macaulay and let *T* be a polynomial or power series ring over *S* in the variables $X_1, ..., X_n$. Let *J* be an ideal of *T* such that $(X_1, ..., X_n)J$ is contained in the Jacobson radical of *T*. Let $R = T/(X_1, ..., X_n)J$. Then,

$$R \to R/(X_1,\ldots,X_n)R \cong S$$

are rings as in Theorem 5 and Theorem 7.

3. DUALS HAVING FINITE HOMOLOGICAL DIMENSIONS, AND FREENESS

3.1. **Some preliminaries.** We denote by G-dim $_R$ the Gorenstein dimension over a ring R (see [34, Definition 12 and Definition 16]). An R-module M is totally reflexive if G-dim $_RM = 0$. Given finite R-modules M, N, we write $M \approx N$ to mean $M \oplus F \cong N \oplus G$ for some projective R-modules F, G (see [34, Definition 3]). In this case, M and N are said to be stably isomorphic. Note that if N is projective (resp. totally reflexive) and $M \approx N$, then M is also projective (resp. totally reflexive). Let TrM stand for the Auslander transpose of M (see [34, Definition 2]). Recall $M \approx TrTrM$ (see [34, Remark (3) following Proposition 4]). It follows that M is totally reflexive if and only if TrM is totally reflexive. Also, note TrM fits into an exact sequence

$$(11.1) 0 \rightarrow M^* \rightarrow F_0 \rightarrow F_1 \rightarrow \text{Tr} M \rightarrow 0,$$

where F_0 , F_1 are free R-modules and $M^* = \operatorname{Hom}_R(M, R)$ is the (algebraic) dual of M. In case R possesses a canonical module ω_R , we use the notation $M^{\dagger} = \operatorname{Hom}_R(M, \omega_R)$, the canonical dual of M. Thus, $M^{\dagger} \cong M^*$ whenever R is a Gorenstein local ring.

Next, recall that a finite *R*-module *M* is said to satisfy (\widetilde{S}_n) , for a given integer $n \ge 0$, if

$$\operatorname{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq \min\{n, \operatorname{depth} R_{\mathfrak{p}}\} \quad \text{for all} \quad \mathfrak{p} \in \operatorname{Spec}(R).$$

Since the depth of the zero module is set to be ∞ by a widely accepted convention, then in order to check that M satisfies (\widetilde{S}_n) it suffices to only consider primes in $\operatorname{Supp}_R(M)$. Clearly, if M satisfies (\widetilde{S}_n) then it also satisfies (\widetilde{S}_m) for all m < n. If n = 0, the condition (\widetilde{S}_0) trivially holds. For completeness, if $n \ge 1$, one (cohomological) criterion is as follows. If M is n-torsionless (i.e., $\operatorname{Ext}_R^i(\operatorname{Tr} M, R) = 0$ for all $i = 1, \ldots, n$), then M satisfies (\widetilde{S}_n) (see [34, Proposition 11] or [12, Propositions (16.30), (16.31) and (16.32)]).

3.2. A key proposition and first corollaries. The complete intersection dimension of a finite module M over a local ring R is written $\operatorname{Cl-dim}_R M$. It is related to classical homological dimensions by means of the inequalities

$$\operatorname{\mathsf{G-dim}}_R M \leq \operatorname{\mathsf{CI-dim}}_R M \leq \operatorname{\mathsf{pd}}_R M.$$

We refer to [5] for the theory. For any $n \ge 0$, the *n*-th syzygy module of M over R is denoted $\operatorname{Syz}_R^n M$, with $\operatorname{Syz}_R^0 M = M$.

Convention 12. Whenever convenient, we denote by H-dim $_RM$ any of the homological dimensions pd_RM , CI-dim $_RM$ or G-dim $_RM$.

Our first result in this section is the following.

Proposition 13. Let R be a local ring of depth t and M be a finite R-module. Let r be an integer with $0 \le r \le t$ such that M satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(M, R) = 0$. If H-dim $_R M^* < \infty$, then H-dim $_R M = 0$.

Proof. First, if r < t then, as $\operatorname{Ext}_R^{1 \le i \le t - r}(\operatorname{Tr}\operatorname{Tr}M, R) = 0$ (recall M and $\operatorname{Tr}\operatorname{Tr}M$ are stably isomorphic), the module $\operatorname{Tr}M$ is (t-r)-torsionless, hence $\operatorname{depth}_R\operatorname{Tr}M \ge t - r$ (see [34, Proposition 11(c)]), and the case r = t is trivial. Now notice that M^* is stably isomorphic with $\operatorname{Syz}_R^2\operatorname{Tr}M$. So, in any of the possibilities for $\operatorname{H-dim}_R$, we obtain $\operatorname{G-dim}_R\operatorname{Tr}M < \infty$. Also, by [50, Theorem 5.8(1)], we get (if $r \ge 1$)

$$\operatorname{Ext}_R^{1 \le i \le r}(\operatorname{Tr} M, R) = 0.$$

Next, we give a proof for each choice of H-dim $_R$.

- (1) The case $\operatorname{H-dim}_R M^* = \operatorname{pd}_R M^*$. Since $M^* \approx \operatorname{Syz}_R^2 \operatorname{Tr} M$, we have $\operatorname{pd}_R \operatorname{Tr} M < \infty$. By the Auslander–Buchsbaum formula, $\operatorname{pd}_R \operatorname{Tr} M = t \operatorname{depth}_R \operatorname{Tr} M \le r$. Now, by [36, p. 154, Lemma 1(iii)] we get $\operatorname{pd}_R \operatorname{Tr} M = 0$, i.e., $\operatorname{Tr} M$ is R-free. Since M is stably isomorphic to $\operatorname{Tr} \operatorname{Tr} M$, it follows that M must be free as well.
 - (2) The case H-dim $_R M^* = G$ -dim $_R M^*$. By the Auslander–Bridger formula,

$$G-\dim_R \operatorname{Tr} M = t - \operatorname{depth}_R \operatorname{Tr} M \le r.$$

Now, if $r \ge 1$, $\operatorname{Ext}_R^{1 \le i \le r}(\operatorname{Tr} M, R) = 0$ implies $\operatorname{\mathsf{G-dim}}_R \operatorname{\mathsf{Tr}} M = 0$ (see [15, 1.2.7(iii)]), i.e., $\operatorname{\mathsf{Tr}} M$ is totally reflexive and therefore M has the same property. The case r = 0 is clear.

(3) The case $\operatorname{H-dim}_R M^* = \operatorname{Cl-dim}_R M^*$. By [5, Theorem 1.4] and item (2) above, we obtain that M is totally reflexive, and hence so is M^* . Hence, $\operatorname{G-dim}_R M^* = 0$. By [5, Theorem 1.4], we then get

$$\mathsf{CI}\text{-}\mathsf{dim}_R M^* = \mathsf{G}\text{-}\mathsf{dim}_R M^* = 0.$$

Now, applying [9, Lemma 3.5] we derive Cl-dim_R $M^{**} = 0$. Since M is reflexive, we are done.

Remark 14. When t = 0 (which forces r = 0, and note every module satisfies (\widetilde{S}_0)), our Proposition 13 immediately gives that if M^* is free (resp. totally reflexive), then M is also free (resp. totally reflexive). Moreover, if M satisfies (\widetilde{S}_t) and $\operatorname{pd}_R M^* < \infty$ then M must be free. Compare it with [17, Theorem 3.10].

Remark 15. It is customary to say that a finite R-module M is an *ideal-module* if M^* is free; this mimics the standard fact that $I^* \cong R$ whenever I is an ideal with grade $I \geq 2$. Thus, as a consequence of Proposition 13 with t = r, we obtain that a non-free ideal-module over a local ring of depth t cannot satisfy (\widetilde{S}_t) .

Corollary 16. Let R be a local ring of depth t and M a finite R-module such that $H\text{-}dim_R M^* < \infty$. Let N be a non-zero finite R-module such that $\text{Ext}_R^{1 \le i \le t-1}(N,M) = 0$. The following assertions hold true:

- (1) If depth_R Hom_R(N,M) $\geq t$, then H-dim_RM = 0.
- (2) If $\operatorname{depth}_R N \ge t$ and $\operatorname{Hom}_R(N,M)$ is free, then both M and N^* are free.

Proof. Item (2) follows directly from (1) together with [17, Corollary 3.7]. To prove (1), notice that by Proposition 13 we are done once M satisfies (\widetilde{S}_t) , which we claim to hold. Even more, we will show depth $_R M \ge t$. To this end, let m denote the maximal ideal of R and consider the spectral sequence (see [22, Proposition 2.1])

$$E_2^{i,j} = H^i_{\mathfrak{m}}(\operatorname{Ext}^j_R(N,M)) \Rightarrow H^{i+j}_{\mathfrak{m}}(N,M),$$

where $H_{\mathfrak{m}}^{i+j}(N,M)$ stands for the (i+j)-th generalized local cohomology module of the pair N,M (see [24], also [44]). Since

$$E_2^{i,0} = 0 \quad \text{for all} \quad i < t, \quad \text{and} \quad E_2^{i,j} = 0 \quad \text{for all} \quad j = 1, \dots, t-1,$$

we conclude by convergence that $H^i_{\mathfrak{m}}(N,M) = 0$ whenever i < t, and therefore $\operatorname{depth}_R M \ge t$ by [44, Theorem 2.3].

Question 17. Does Corollary 16(2) remain true if the freeness condition on $\operatorname{Hom}_R(N,M)$ is replaced with $\operatorname{pd}_R\operatorname{Hom}_R(N,M)<\infty$? What if N=M? A particular positive answer for this question will be given later in Proposition 23.

Remark 18. We record the following immediate byproduct of the proof of Corollary 16 (more precisely, from the spectral sequence argument along with [44, Theorem 2.3]). Let R be a local ring and M, N non-zero finite R-modules, and set $s := \operatorname{depth}_R \operatorname{Hom}_R(N, M)$. If either s = 1 or $s \ge 2$ and $\operatorname{Ext}_R^{1 \le i \le s-1}(N, M) = 0$, then $\operatorname{depth}_R M > s$.

For the proposition below, given a finite module M over a local ring R, we denote by $\operatorname{cx}_R M$ and $\operatorname{curv}_R M$ the complexity and the curvature of M, respectively. For the definitions and properties, see [2, 4.2]. It should be noticed that item (4) of the next proposition is an extension of Proposition 13.

Proposition 19. Let R be a local ring of depth t, M a finite R-module, and r an integer with $0 \le r \le t$. If r < t, suppose $\operatorname{Ext}_R^{1 \le i \le t - r}(M, R) = 0$. Let $N \ne 0$ be a finite R-module satisfying $\operatorname{pd}_R N \le t - r$. The following assertions hold true:

- (1) $\operatorname{cx}_R \operatorname{Hom}_R(M,N) = \operatorname{cx}_R M^*$
- (2) $\operatorname{curv}_R \operatorname{Hom}_R(M,N) = \operatorname{curv}_R M^*$.
- (3) $\operatorname{H-dim}_R \operatorname{Hom}_R(M,N) = \operatorname{H-dim}_R M^* + \operatorname{pd}_R N$.
- (4) If M satisfies (S_r) and H-dim_R $Hom_R(M,N) < \infty$, then H-dim_R M = 0.

Proof. First, if r = t then N is free, and hence items (1), (2), and (3) follow easily, whereas (4) is a consequence of Proposition 13.

So, we can consider r < t. Since $\operatorname{Ext}_R^{1 \le i \le t - r}(\operatorname{Tr}\operatorname{Tr} M, R) = 0$, the module $\operatorname{Tr} M$ is (t - r)-torsionfree and hence $\operatorname{Tr} M$ is stably isomorphic to $\Omega^{t-r}X$ for some R-module X (see [34, Proposition 11(b)]). Now, as $\operatorname{pd}_R N \le t - r$, we must have

$$\operatorname{Tor}_{>0}^R(\operatorname{Tr} M,N) \cong \operatorname{Tor}_{>t-r}^R(X,N) = 0.$$

By [1, Theorem 2.8(b)], we then obtain $M^* \otimes_R N \cong \operatorname{Hom}_R(M,N)$. Furthermore, $\operatorname{Tor}_{>0}^R(M^*,N) \cong \operatorname{Tor}_{>2}^R(\operatorname{Tr} M,N) = 0$, hence

$$M^* \otimes_R N \cong M^* \otimes_R^{\mathbf{L}} N$$
.

Also, $cx_R(N) = 0 = curv_R(N)$ (see [2, Remark 4.2.3]).

Now, assertions (1) and (2) follow from [2, Proposition 4.2.4(6)]. Assertion (3) follows from [15, Theorem (A.7.6)], [42, Theorem 3.11] and [28, Theorem 5.1]] – in the last reference, we make R = S = T and take into account that M^* and N are Tor-independent, as noticed above – and, along with Proposition 13, it implies assertion (4).

Remark 20. It is possible to avoid the explicit requirement of the Serre-type condition (\tilde{S}_r) by replacing it with different hypotheses. Indeed, using [34, Proposition 11] or [12, Proposition (16.30) and Proposition (16.31)], all the assertions in Proposition 13 as well as Proposition 19(4) are seen to be valid whenever M is r-torsionless and TrM is (t-r)-torsionless.

Proposition 19(4) allows us to extend Corollary 16 as follows.

Corollary 21. Let R be a local ring of depth t, M a finite R-module, and r an integer with $0 \le r \le t$ such that, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(M, R) = 0$. Suppose there exists a finite R-module $N \ne 0$ such that $\operatorname{pd}_R N \le t - r$ and $\operatorname{H-dim}_R \operatorname{Hom}_R(M, N) < \infty$. Also, assume that there exists a finite R-module $N' \ne 0$ satisfying $\operatorname{Ext}_R^{1 \le i \le t - 1}(N', M) = 0$. The following assertions hold true:

- (1) If depth_R Hom_R(N', M) $\geq t$, then H-dim_RM = 0.
- (2) If depth_R $N' \ge t$ and $\operatorname{Hom}_R(N', M)$ is free, then M and $(N')^*$ are free.

Proof. We provide two proofs of (1). For the first one, note Proposition 19(3) ensures that H-dim_R M^* < ∞ and therefore Corollary 16(1) applies. As to the second proof, from Remark 18 we derive depth_R $M \ge t$, so M satisfies (\widetilde{S}_t) and, consequently, (\widetilde{S}_r) . Thus, Proposition 19(4) applies. Item (2) follows directly from Corollary 16(2).

Corollary 22. Let R be a local ring of depth t and M a finite R-module such that H-dim $_R M^* < \infty$ and $\operatorname{Ext}_R^{1 \le i \le t-1}(M,M) = 0$. If $\operatorname{depth}_R \operatorname{Hom}_R(M,M) \ge t$, then H-dim $_R M = 0$. Moreover, if $\operatorname{Hom}_R(M,M)$ is free, then so is M.

Proof. Apply Corollary 21 with N' = M, N = R, and r = t.

Proposition 23. Let R be a local ring of depth t, M a finite R-module, and r an integer with $0 \le r \le t$ such that M satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(M, R) = 0$. Suppose there exists a finite R-module $N \ne 0$ such that $\operatorname{pd}_R N \le t - r$ and $\operatorname{H-dim}_R \operatorname{Hom}_R(M, N) < \infty$. Also, assume that there exists a finite R-module $N' \ne 0$ satisfying $\operatorname{Hom}_R(N', M) \ne 0$ and $\operatorname{Ext}_R^{1 \le i \le t - 1}(N', M) = 0$. If $\operatorname{pd}_R \operatorname{Hom}_R(N', M) < \infty$, then both M and N' are free.

Proof. By Proposition 19(4), H-dim_R M = 0. For any choice of H-dim_R (see Convention 12) we obtain that M is totally reflexive. Thus we apply [19, Theorem 6.6] to conclude that M is free. In particular, depth_R M = t and so [19, Lemma 3.1(1)(i)] guarantees that

$$\operatorname{depth}_R \operatorname{Hom}_R(N', M) \geq t$$
,

while the Auslander-Buchsbaum formula yields that $\operatorname{Hom}_R(N',M)$ is free. So, write $M=R^m$ and $\operatorname{Hom}_R(N',M)=R^h$. It follows that $R^h=\operatorname{Hom}_R(N',M)\simeq (N')^m$ and thus N' is also free.

Remark 24. In virtue of [19, Theorem 5.18], we can suppose in Proposition 23 the condition

$$\operatorname{pd}_R \operatorname{Hom}_R(C, \operatorname{Hom}_R(N', M)) < \infty,$$

with *C* a semidualizing *R*-module, instead of $pd_R Hom_R(N', M) < \infty$.

Corollary 25. Let R be a local ring of depth t, M a finite R-module such that H-dim $_R M^* < \infty$, $\operatorname{pd}_R \operatorname{Hom}_R(M,M) < \infty$, and $\operatorname{Ext}_R^{1 \le i \le t-1}(M,M) = 0$. Suppose there exists an integer r with $0 \le r \le t$ such that M satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t-r}(M,R) = 0$. Then, M is free.

Proof. Take N = R and N' = M in Proposition 23.

The case r = 0 of Corollary 25 gives the following contribution to the celebrated Auslander-Reiten conjecture (see [19], [26] and their references on the subject), whose commutative local version predicts that a finite module M over a local ring R must be free if

$$\operatorname{Ext}_R^{>0}(M,M\oplus R)=0.$$

Corollary 26. The Auslander-Reiten conjecture is true if $G-\dim_R M^* < \infty$ and $pd_R \operatorname{Hom}_R(M,M) < \infty$.

Here it should be mentioned that the conjecture holds true provided that $G\text{-}dim_R M < \infty$ and $pd_R \operatorname{Hom}_R(M,M) < \infty$ (see [19, Corollary 6.9(2)]).

In another byproduct, we immediately retrieve the following old result of Vasconcelos.

Corollary 27. ([48, Theorem 3.1]) Let R be a one-dimensional Gorenstein local ring and M a finite R-module. If $Hom_R(M,M)$ is free, then M is free.

The theory developed in Section 2 enables us to produce freeness criteria for M from the vanishing of (co)homology modules of $\operatorname{Hom}_R(M,N)$ via ring homomorphisms. We gather them together in the following result.

Corollary 28. Let $(R, \mathfrak{m}) \to S$ be a ring map such that $\mathfrak{m}S \neq S$. Let t be the depth of R and r an integer with $0 \le r \le t$. Let M be a finite R-module satisfying (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(M, R) = 0$. In addition, let $N \ne 0$ be a finite R-module such that $\operatorname{pd}_R N \le t - r$. Assume that any one of the following conditions holds true:

- (1) For each $\mathfrak{n} \in \operatorname{Max}(S)$, there is an integer $h \geq 0$ (possibly depending on \mathfrak{n}) such that $\operatorname{Tor}_h^R(\operatorname{Hom}_R(M,N),S/\mathfrak{n})=0$.
- (2) S is module-finite over R and, for each $n \in Max(S)$, there is an integer $h \ge 0$ (possibly depending on n) such that $Ext_R^h(Hom_R(M,N),S/n) = 0$.
- (3) S is Cohen–Macaulay and $\operatorname{Tor}_{\gg 0}^R(\operatorname{Hom}_R(M,N),X)=0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S)\setminus\operatorname{Max}(S)$.
- (4) S is Cohen–Macaulay and there exist integers $s,h \ge 0$ such that $\operatorname{Tor}_{s+1 \le j \le s+h+1}^R(\operatorname{Hom}_R(M,N),X) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.
- (5) *S is Cohen–Macaulay, S is module-finite over R, and* $\operatorname{Ext}_R^{\gg 0}(M^*,X) = 0$ *for every maximal Cohen–Macaulay S-module X which is locally free on* $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.
- (6) S is Cohen–Macaulay, S is module-finite over R, and there exist integers $s,h \ge 0$ such that $\operatorname{Ext}_R^{s+1 \le j \le s+h+1}(\operatorname{Hom}_R(M,N),X) = 0$ for every maximal Cohen–Macaulay S-module X which is locally free on $\operatorname{Spec}(S) \setminus \operatorname{Max}(S)$.

Then, M is free.

Proof. Apply Proposition 1(1),(2) for items (1) and (2) respectively, and Theorem 5 for items (3)-(6) in order to conclude that $pd_R \operatorname{Hom}_R(M,N) < \infty$. Finally, we apply Proposition 19(4).

It is worth recording the following special situation.

Corollary 29. Let R be a d-dimensional Cohen-Macaulay local ring and M a maximal Cohen-Macaulay R-module. Let $N \neq 0$ be a finite R-module and r an integer with $0 \leq r \leq d$ such that $\operatorname{pd}_R N \leq d - r$ and, if r < d, $\operatorname{Ext}_R^{1 \leq i \leq d - r}(M, R) = 0$. If $\operatorname{H-dim}_R \operatorname{Hom}_R(M, N) < \infty$, then $\operatorname{H-dim}_R M = 0$.

Proof. Over a Cohen-Macaulay local ring, any maximal Cohen-Macaulay module satisfies (\widetilde{S}_n) for all $n \ge 0$. Now the result follows from Proposition 19(4).

In the Gorenstein case, Corollary 29 gives the following result.

Corollary 30. Let R be a Gorenstein local ring and M a maximal Cohen-Macaulay R-module. If H-dim $_R Hom_R(M,N) < \infty$ (which is automatic in the G-dim $_R$ case), where $N \neq 0$ is a finite R-module with $pd_R N < \infty$, then H-dim $_R M = 0$.

We shall return to the Gorenstein property (as a target) in Section 4.

Corollary 31. Let (R, \mathfrak{m}) be a local ring of depth t, M a finite R-module, and r an integer with $0 \le r \le t$ such that M satisfies (\widetilde{S}_r) (e.g., take r = t if R is Cohen-Macaulay and M is maximal Cohen-Macaulay) and, if r < t, $\operatorname{Ext}_R^{1 \le t \le t-r}(M,R) = 0$. In addition, suppose there is a ring map $R \to S$, where S is regular and $\mathfrak{m}S \ne S$, and a finite R-module $N \ne 0$ such that $\operatorname{pd}_R N \le t-r$ and

$$\operatorname{Tor}_{i}^{R}(\operatorname{Hom}_{R}(M,N),S)=0$$
 for all $i\gg 0$.

Then, M is free.

Proof. By Remark 6(i), we must have $pd_R Hom_R(M,N) < \infty$. Now we apply Proposition 19(4).

Next, we consider the class of Golod local rings that are not hypersurface rings. This includes, for instance, Cohen-Macaulay non-Gorenstein local rings with minimal multiplicity, such as for example the ring $k[[x,y,z]]/(y^2-xz,x^2y-z^2,x^3-yz)$, where x,y,z are formal indeterminates over a field k.

Corollary 32. Let R be a Golod local ring of depth t which is not a hypersurface ring, M a finite R-module, and r an integer with $0 \le r \le t$ such that M satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le t \le t - r}(M, R) = 0$. If there exists a finite R-module $N \ne 0$ satisfying $\operatorname{pd}_R N \le t - r$ and $\operatorname{G-dim}_R \operatorname{Hom}_R(M, N) < \infty$, then M is free.

Proof. By Proposition 19(4), M must be totally reflexive. But a local ring R as in the statement is known to have the property that every totally reflexive R-module is necessarily free (see [7, Examples 3.5(2)]). In particular, M is free.

To close the subsection, let R be a local ring of prime characteristic p > 0 and let $F: R \to R$ be the Frobenius map $F(a) = a^p$, for all $a \in R$. For a positive integer e, we can consider the e-th iteration of F, i.e., the assignment

$$F^e: a \mapsto a^{p^e}$$
,

which defines on R a new R-module structure. Such a module is denoted $R^{(e)}$, which is known to play an important role in the theory; e.g., a classical result of Kunz states that $R^{(1)}$ is R-flat if and only if R is regular. Also, recall that R is F-finite if F^e is a finite map, which means $R^{(e)}$ is a finite R-module, for some (or equivalently, for every) $e \ge 1$. This property holds for fundamental classes of rings, for instance, when R is a complete local ring with perfect residue field or a localization of an affine algebra over a perfect field (see, e.g., [11, p. 398]).

Corollary 33. Let R be an F-finite local ring of prime characteristic p > 0 and depth t, M a finite R-module, and r an integer with $0 \le r \le t$ such that M satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le t \le t - r}(M, R) = 0$. Let $N \ne 0$ be a finite R-module such that $\operatorname{pd}_R N \le t - r$. Suppose any one of the following assertions:

- (1) $\operatorname{Ext}_{R}^{i}(\operatorname{Hom}_{R}(M,N),R^{(e)})=0$ for all $i=\ell,\ldots,\ell+t$ (for some integer $\ell\geq 1$) and infinitely many e.
- (2) $\operatorname{Tor}_{j}^{R}(\operatorname{Hom}_{R}(M,N),R^{(e)})=0$ for all $j\gg 0$ and $p^{e}\gg 0$. Then, M is free.

Proof. Assume that (1) (resp. (2)) holds. Then, by virtue of [38, Corollary 2.4] (resp. [45, Theorem 4.5(1)]), we deduce that $pd_R Hom_R(M, N) < \infty$. Now Proposition 19(4) finishes the proof.

More on the positive characteristic case will be provided in Subsection 4.3.

4. THE GORENSTEIN PROPERTY, AND MORE ON FREENESS

This section is mainly concerned with criteria for the Gorensteiness of Cohen-Macaulay local rings admitting a canonical module, and for the freeness of finite modules over such rings. We will also present criteria for regular and complete intersection local rings.

4.1. **A general criterion.** Our first result in this part makes use of an iteration between the algebraic and the canonical duals of a given module, and requires the resulting module to have finite projective dimension.

Proposition 34. Let R be a d-dimensional Cohen-Macaulay local ring possessing a canonical module. Let M be a finite R-module such that $M^* \neq 0$ and $\operatorname{pd}_R(M^*)^{\dagger} < \infty$. The following assertions hold true:

- (1) If $d \le 1$, then M^{**} is free.
- (2) If $\operatorname{Ext}_R^j(M,R) = 0$ for all j = 1, ..., d when $d \ge 1$, or for all j > 0 when d = 0, then M is free. In either case, R is Gorenstein.

Proof. (1) Note depth_R $M^* \ge \min\{2, \text{depth } R\} = \min\{2, d\} = d$. Thus, the *R*-module M^* is maximal Cohen-Macaulay, and hence so is its canonical dual $(M^*)^\dagger \ne 0$. As $\operatorname{pd}_R(M^*)^\dagger < \infty$, we obtain $(M^*)^\dagger \cong R^{\oplus b}$ for some integer $b \ge 1$. Being M^* maximal Cohen-Macaulay, we deduce that

$$M^* \cong (M^*)^{\dagger\dagger} \cong \omega_R^{\oplus b},$$

where ω_R stands for the canonical module of R. Consequently, we have an exact sequence $0 \to \omega_R^{\oplus b} \to F_0 \to F_1$ for some finite free R-modules F_0, F_1 . This, in turn, gives a short exact sequence

$$(34.1) 0 \to \omega_R^{\oplus b} \to F_0 \to X \to 0,$$

for an R-submodule $X \subset F_1$. As $d \leq 1$, we get that X is necessarily maximal Cohen-Macaulay, and hence $\operatorname{Ext}^1_R(X,\omega_R)=0$, which implies that the sequence (34.1) splits. Thus, $\omega_R^{\oplus b}$ is a direct summand of F_0 , and so ω_R itself is a direct summand of F_0 . It follows that ω_R is free, i.e., R is Gorenstein. For the freeness of the bidual of M, we now have $(M^*)^{\dagger} \cong M^{**}$, which must be maximal Cohen-Macaulay, hence free because $\operatorname{pd}_R M^{**} < \infty$.

(2) It follows from [26, Theorem 2.1(i)] (also [26, Remark 2.2(i)] in the Artinian case) that

$$(M^*)^{\dagger} \cong M \otimes_R \omega_R$$

and that this module is maximal Cohen-Macaulay, hence free. Therefore, both ω_R and M must be free, whence the result.

Corollary 35. Let R be a Cohen–Macaulay local ring of dimension $d \le 1$ possessing a canonical module. If $M \ne 0$ is a torsionless finite R-module such that $\operatorname{pd}_R(M^*)^{\dagger} < \infty$, then M is free.

Proof. Proposition 34(1) ensures that M^{**} is free and R is Gorenstein. Now, [34, Theorem 17(b)] yields

$$\operatorname{Ext}_{R}^{2}(\operatorname{Tr}M,R)=0,$$

and therefore *M* is reflexive by [34, Proposition 5].

Corollary 35 motivates us to ask the following.

Question 36. Let R be a Cohen–Macaulay local ring possessing a canonical module, and let s be a positive integer such that $\dim R \le s$. If $M \ne 0$ is an s-torsionless finite R-module such that $\operatorname{pd}_R(M^*)^{\dagger} < \infty$, then must R be Gorenstein and M be free?

Remark 37. For R as above, assume in addition that R is locally Gorenstein on its punctured spectrum and that M is maximal Cohen-Macaulay with a rank. Then, by [23, Lemma 2.1], the condition $\operatorname{Ext}_R^j(M,R) = 0$ for all $j = 1, \ldots, d$ (present in Proposition 34(2) in case $d \ge 1$) is equivalent to $M \otimes_R \omega_R$ being maximal Cohen-Macaulay.

Remark 38. Maximal Cohen-Macaulay modules of finite injective dimension are said to be *Gorenstein* modules. Now for completeness we record the following statement, which again assumes no constraint on the dimension of the ring. Let R be as above, $M^* \neq 0$ and $pd_R(M^*)^{\dagger} < \infty$. If

$$\operatorname{Ext}_{R}^{i}(M,R) = 0$$
 for all $i = 1, ..., \max\{1, d-2\},\$

then M^* is Gorenstein. Indeed, applying [29, Lemma, p. 2763] we deduce that M^* is maximal Cohen-Macaulay, and exactly as in the proof of Proposition 34(1) there is an integer $b \ge 1$ such that $M^* \cong \omega_R^{\oplus b}$, which is a Gorenstein module.

4.2. The case of the (anti)canonical module. Now we focus on the case where $M = \omega_R$. Note $M^* = \omega_R^*$ is the anticanonical module of R, which (as mentioned in the introduction) is also of recognized significance.

We begin by invoking part of a motivating question from [26].

Question 39. ([26, Question 5.24]) Let R be a Cohen-Macaulay local ring with canonical module ω_R . If $\mathsf{pd}_R \, \omega_R^* < \infty$ or $\mathsf{pd}_R (\omega_R^*)^\dagger < \infty$, must R be Gorenstein?

Because ω_R is maximal Cohen-Macaulay, the first half of this question is immediately seen to have an affirmative answer by Corollary 29. Alternatively, it also follows from some results of [26]; indeed, since $pd_R \omega_R^* < \infty$ we obtain $pd_R \operatorname{Tr} \omega_R < \infty$ and so

$$\operatorname{Ext}_{R}^{d+1}(\operatorname{Tr}\omega_{R},R)=0.$$

If d = 0, we apply [26, Corollary 3.21] with $M = \omega_R$. Inductively, we may suppose that ω_R is a vector bundle (i.e., R is locally Gorenstein on its punctured spectrum), so that [26, Corollary 3.27] applies.

We are able, however, to provide a much stronger statement with $\operatorname{H-dim}_R\operatorname{Hom}_R(\omega_R,N)<\infty$ (along with some extra hypotheses), for a suitable finite R-module $N\neq 0$, in place of $\operatorname{pd}_R\omega_R^*<\infty$. Note this extends even the case $\operatorname{G-dim}_R\omega_R^*<\infty$. In addition, we settle the second half of Question 39 in dimension at most 1. Below in Corollary 42 we record such facts, but first we need the following auxiliary lemma, which is probably known to the experts.

Lemma 40. A local ring R is Gorenstein if and only if R admits a non-zero finite module M such that $G-\dim_R M < \infty$ and $\operatorname{id}_R M < \infty$.

Proof. If *R* is Gorenstein, then we can simply take M = R. To prove the converse, let $M \neq 0$ be a finite *R*-module with G-dim_{*R*} M and id_{*R*} M both finite. Since G-dim_{*R*} $M < \infty$, we can use Gorenstein dimension approximation (see [16, Lemma 2.17]) to guarantee the existence of a short exact sequence

$$(40.1) 0 \rightarrow M \rightarrow X \rightarrow Y \rightarrow 0,$$

where $\operatorname{pd}_R X < \infty$ and Y is totally reflexive. Hence, either Y = 0 or (by the Auslander-Bridger formula) $\operatorname{depth}_R Y = \operatorname{depth} R$. It now follows from Ischebeck's theorem (see [11, 3.1.24]) that $\operatorname{Ext}_R^1(Y,M) = 0$, which implies that the sequence (40.1) splits, i.e., $X \cong M \oplus Y$. Therefore, since $\operatorname{pd}_R X < \infty$, we get $\operatorname{pd}_R M < \infty$. Now, being $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ both finite, we can apply [21, Corollary 4.4] to conclude that R is Gorenstein.

Corollary 41. Let R be a local ring of depth t, and r an integer with $0 \le r \le t$. Let M be a non-zero finite R-module satisfying (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(M, R) = 0$. If $\operatorname{id}_R M^* < \infty$ and there exists a finite R-module $N \ne 0$ such that $\operatorname{pd}_R N \le t - r$ and $\operatorname{H-dim}_R \operatorname{Hom}_R(M, N) < \infty$, then R is Gorenstein and M is free.

Proof. First, Proposition 19(4) gives H-dim_R M=0. In particular, $M^* \neq 0$ and G-dim_R $M^* < \infty$, and so by Lemma 40 we conclude that R must be Gorenstein. In particular, $pd_R M^* < \infty$ by [11, Exercise 3.1.25] and therefore Proposition 13 ensures that M is free. (Instead of applying Proposition 13, we can also argue directly by noticing that M^* is also totally reflexive, hence free, and so is M.)

Corollary 42. Let R be a d-dimensional Cohen-Macaulay local ring admitting a canonical module ω_R . Assume any one of the following situations:

- (1) $d \leq 1$ and $\operatorname{pd}_R(\omega_R^*)^{\dagger} < \infty$.
- (2) $d \geq 2$, $\operatorname{pd}_R(\omega_R^*)^{\dagger} < \infty$ and $\operatorname{Ext}_R^j(\omega_R, R) = 0$ for all $j = 1, \ldots, d$. (3) There exist an integer r with $0 \leq r \leq d$ and a finite R-module $N \neq 0$ such that $\operatorname{pd}_R N \leq t r$, $\operatorname{\mathsf{H-dim}}_R \operatorname{\mathsf{Hom}}_R(\omega_R, N) < \infty \ and, \ in \ case \ r < d, \ \operatorname{\mathsf{Ext}}_R^{1 \le i \le d-r}(\omega_R, R) = 0.$

Then, R is Gorenstein.

Proof. Assertions (1) and (2) follow readily from Proposition 34 with $M = \omega_R$. Now let us assume (3). By Corollary 29 with $M = \omega_R$ (which is maximal Cohen-Macaulay), we obtain that

$$\operatorname{\mathsf{G-dim}}_R \omega_R = \operatorname{\mathsf{H-dim}}_R \omega_R = 0 < \infty.$$

Since moreover $id_R \omega_R < \infty$, we conclude by Lemma 40 that R is Gorenstein, as needed.

Remark 43. For completeness, it is worth recalling the following fact shown in [23, Corollary 2.2] as a partial solution to the famous Tachikawa conjecture (see [26, Conjecture 1.2]). Let R be as above and in addition suppose $R_{\mathfrak{p}}$ is Gorenstein for every $\mathfrak{p} \in \operatorname{Spec} R$ with height $\mathfrak{p} = 0$. If $\operatorname{Ext}_R^l(\omega_R, R) = 0$ for all i = 1, ..., d, then R is Gorenstein. There is also the question (see [26, Question 4.7]), which can be regarded as a dual version of Tachikawa's conjecture, as to whether a Cohen-Macaulay local ring R of dimension $d \ge 1$ with canonical module ω_R must be Gorenstein if

$$\operatorname{Ext}_{R}^{j}(\omega_{R}^{*},R)=0$$
 for all $j>0$.

4.3. Positive characteristic. In this part, we stick to the preparation given for the statement of Corollary 33 in order to provide two more results in the prime characteristic setting.

Corollary 44. Let R be an F-finite local ring of prime characteristic and depth t. If there exist an integer r with $0 \le r \le t$ and a finite R-module $N \ne 0$ satisfying $pd_R N \le t - r$, G-dim $_R Hom_R(R^{(e)}, N) < \infty$ for some $e \ge 1$, and, in case r < t, $\operatorname{Ext}_{R}^{1 \le i \le t-r}(R^{(e)}, R) = 0$, then R is Gorenstein.

Proof. As Frobenius push-forward localizes, we can write $\operatorname{depth}_{R_{\mathfrak{p}}}(R^{(e)})_{\mathfrak{p}} = \operatorname{depth}_{R_{\mathfrak{p}}}R^{(e)}_{\mathfrak{p}} = \operatorname{depth}_{R_{\mathfrak{p}}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, where the last equality follows from the fact that a sequence $\{\xi_1, \ldots, \xi_t\} \subset \mathfrak{p} R_{\mathfrak{p}}$ is an $R^{(e)}$ -sequence if and only if

$$\{\xi_1^{p^e},\ldots,\xi_t^{p^e}\}$$

is an *R*-sequence, with $p = \operatorname{char} R$. It follows that $R^{(e)}$ satisfies (\widetilde{S}_n) for every n. Now, Proposition 19(4) gives G-dim $_R R^{(e)} = 0 < \infty$, and we are done by [45, Theorem 6.2].

Corollary 45. Let R be an F-finite local ring of prime characteristic and depth t. Let r be an integer with $0 \le r \le t$. The following assertions hold true:

- (1) If there exists a finite R-module $N \neq 0$ such that $pd_R N \leq t r$ and $pd_R Hom_R(R^{(e)}, N) < \infty$ for some e > 0, and if $\operatorname{Ext}_R^{1 \le t \le t-r}(R^{(e)}, R) = 0$ in case r < t, then R is regular.
- (2) If there exists a finite R-module $N \neq 0$ such that $\mathrm{id}_R N < \infty$ and $\mathrm{id}_R \mathrm{Hom}_R(R^{(e)}, N) < \infty$ for some e > 0, then R is regular.
- (3) If there exists a finite R-module $N \neq 0$ such that $pd_R N \leq t r$ and $Cl-dim_R Hom_R(R^{(e)}, N) < \infty$ for some e > 0, and if $\operatorname{Ext}_R^{1 \le t - r}(R^{(e)}, R) = 0$ in case r < t, then R is a complete intersection ring.

Proof. We shall use the fact (noticed in the proof of Corollary 44) that $\operatorname{depth}_R R^{(e)} = \operatorname{depth} R$ and consequently $R^{(e)}$ satisfies (\widetilde{S}_n) for every n.

- (1) By Proposition 19(4), we have $\operatorname{pd}_R R^{(e)} = 0 < \infty$. Now, since the Frobenius map F: $R \to R$ is a contracting endomorphism, we can apply [3, Theorem 1.1] to conclude that R is regular.
- (2) Since $id_R N < \infty$, we can make use of a well-known result of Ischebeck (see, e.g., [11, Exercise 3.1.24]), which yields $\operatorname{Ext}_R^{>0}(R^{(e)}, N) = 0$, and so $\mathbf{R}\operatorname{Hom}_R(R^{(e)}, N) \cong \operatorname{Hom}_R(R^{(e)}, N)$. Thus,

$$\operatorname{id}_R \operatorname{Hom}_R(R^{(e)}, N) = \operatorname{id}_R \operatorname{\mathbf{R}} \operatorname{Hom}_R(R^{(e)}, N) = \operatorname{pd}_R R^{(e)} + \operatorname{id}_R N,$$

where the last equality follows by [15, Theorem (A.7.7), (A.7.4.1) and (A.7.5.1)]. Thus, $pd_R R^{(e)} < \infty$ and we are done by [3, Theorem 1.1].

(3) By Proposition 19(4), we have $\operatorname{CI-dim}_R R^{(e)} = 0 < \infty$ and therefore $\operatorname{G-dim}_R R^{(e)}$ must be finite as well. By [45, Theorem 6.2], we obtain that R is Gorenstein. Note that, because R is F-finite, there is an equality between $\operatorname{CI-dim}_R R^{(e)}$ and the complete intersection flat dimension $\operatorname{CI-fd}_R R^{(e)}$ of $R^{(e)}$ (see [43, Definition 2.4]). Consequently, $\operatorname{CI-fd}_R R^{(e)} < \infty$. Now let $\operatorname{CI*-id}_R R^{(e)}$ denote the upper complete intersection injective dimension of $R^{(e)}$ (see [43, Definition 2.6]). Since R is Gorenstein, we can apply [43, Corollary 5.7] to get

$$Cl*-id_R R^{(e)} < \infty.$$

By [43, Remark 2.8], the complete intersection injective dimension $Cl-id_R R^{(e)}$ must also be finite. Finally, by [43, Theorem C, p. 2595, or Corollary 6.7], the ring R is necessarily a complete intersection.

Remark 46. An observation on the case e = 1 is that in [10, Proposition 1] it was proved that R must be a complete intersection ring if $\operatorname{CI-dim}_R R^{(1)} < \infty$ (without taking the R-dual). Furthermore, we can ask whether Corollary 45 remains valid if (in items (1) and (3)) the algebraic duals are replaced with canonical duals, if ω_R exists.

4.4. **Further criteria.** We close the section with new criteria for the regular, complete intersection (e.g., hypersurface) and Gorenstein properties of local rings. Following [18, Definition 2.1], a finite module N over a local ring (R, \mathfrak{m}, k) is said to be *strongly rigid* provided that $pd_R M < \infty$ whenever M is a finite R-module with

$$\operatorname{Tor}_{i}^{R}(M,N) = 0$$
 for some $i \ge 1$.

For instance, if N is either the R-module $k = R/\mathfrak{m}$ or any integrally closed \mathfrak{m} -primary ideal of R, or if N has infinite projective dimension over the local ring $R = (\mathbb{Z}/p\mathbb{Z})[[x,y,z]]/(xy-z^2)$ for a prime number $p \geq 3$, then N is a strongly rigid R-module.

Corollary 47. Let R be a local ring of depth t and let r be an integer with $0 \le r \le t$. Let N be a strongly rigid R-module such that $\operatorname{Ext}_R^{r+1 \le i \le t}(N,R) = 0$ if r < t, and let $N' \ne 0$ be a finite R-module such that $\operatorname{pd}_R N' \le t - r$. The following assertions hold true:

- (1) If $pd_R Hom_R(Syz_R^r N, N') < \infty$, then R is regular.
- (2) If $CI-\dim_R Hom_R(Syz_R^rN,N') < \infty$, then R is a complete intersection ring.
- (3) If G-dim $_R$ Hom $_R$ (Syz $_R^rN$, N') $< \infty$, then R is Gorenstein.
- (4) If $G-\dim_R \operatorname{Hom}_R(\operatorname{Syz}_R^r N, N') < \infty$ and R is Golod, then R is a hypersurface ring.

Proof. First, a couple of observations that will be useful to the proof. The *R*-module $\operatorname{Syz}_R^r M$ satisfies (\widetilde{S}_r) for any given finite *R*-module *M* (see [11, Exercise 1.3.7]). In particular, if r = t then $\operatorname{Syz}_R^t M$ satisfies (\widetilde{S}_t) . Otherwise, if r < t, the hypothesis $\operatorname{Ext}_R^{r+1 \le i \le t}(N, R) = 0$ (along with induction) implies

$$\operatorname{Ext}_{R}^{1 \le i \le t-r}(\operatorname{Syz}_{R}^{r}N, R) = 0.$$

(1) By Proposition 19(4), we conclude that $\operatorname{Syz}_R^r N$ is free, hence $\operatorname{pd}_R N < \infty$. Therefore, $\operatorname{Tor}_i^R(k,N) = 0$ for all $i \gg 0$. Because N is strongly rigid, this forces $\operatorname{pd}_R k < \infty$, i.e., R is regular.

- (2) By Proposition 19(4), we obtain that $Cl\text{-}dim_R \operatorname{Syz}_R^r N = 0$, hence $Cl\text{-}dim_R N < \infty$ by [5, Lemma (1.9)]. Now, applying [46, Corollary 3.4.4] (and observing that strongly rigid modules are *test* modules), we conclude that R is a complete intersection.
- (3) By Proposition 19(4), the module $\operatorname{Syz}_R^r N$ must be totally reflexive. Therefore, $\operatorname{G-dim}_R N < \infty$ and hence [14, Corollary 3.9] forces R to be Gorenstein.
- (4) According to (3) above, R is Gorenstein. Now we use the fact that Gorenstein Golod local rings are necessarily hypersurface rings (see [2, Remark after Proposition 5.2.5]).

Taking r = t and N = k (the residue class field of R) in the corollary above, we immediately derive the following result.

Corollary 48. *Let R be a local ring with residue field k and depth t. The following assertions hold true:*

- (1) If $pd_R(Syz_R^t k)^* < \infty$, then R is regular.
- (2) If $CI-dim_R(Syz_R^t k)^* < \infty$, then R is a complete intersection ring.
- (3) If $G-\dim_R(\operatorname{Syz}_R^t k)^* < \infty$, then R is Gorenstein.
- (4) If $G-\dim_R(\operatorname{Syz}_R^t k)^* < \infty$ and R is Golod, then R is a hypersurface ring.

Example 49. The results above can be used to detect instances of (dual) modules having infinite Gorenstein dimension. A plentiful source of examples comes from the class consisting of almost complete intersection rings that are not complete intersection rings. Indeed, consider for instance the local ring $R = k[[x_1, \ldots, x_m]]/I$, where x_1, \ldots, x_m are formal indeterminates over a field k and l is an ideal of height k which is minimally generated by k 1 elements. By [30, Corollary 1.2], k cannot be Gorenstein. So, assuming for simplicity that k is Cohen-Macaulay, Corollary 48(3) yields

$$G-\dim_R(\operatorname{Syz}_R^{m-h}k)^* = \infty.$$

5. APPLICATIONS TO SOME CONJECTURES

In this last part we consider some long-held conjectures involving certain modules such as derivation modules, differential modules, and normal modules, which are needless to say important entities in commutative algebra and algebraic geometry.

5.1. Strong Zariski-Lipman conjecture on derivation modules. Also dubbed Herzog-Vasconcelos-Zariski-Lipman conjecture, it predicts that R must be regular if $pd_R Der_k(R) < \infty$, where R is either

(49.1)
$$k[x_1,...,x_m]_{\mathfrak{q}}/I \quad (\mathfrak{q} \in \operatorname{Spec} k[x_1,...,x_m]) \quad \text{or} \quad k[[x_1,...,x_m]]/I,$$

with I a proper radical ideal and x_1, \ldots, x_m indeterminates over a field k of characteristic 0 (see the survey [25]). As usual, $\operatorname{Der}_k(R)$ stands for the module of k-derivations of R, i.e., the additive maps $R \to R$ that vanish on k and satisfy Leibniz rule (more generally, given any R-module N we can consider the module $\operatorname{Der}_k(R,N)$ formed by the k-derivations of R with values in N). Now recall that, in both situations, R admits a universally finite k-differential module, which is designed to be a finite R-module, denoted by $\Omega_{R/k}$. In the first case, $\Omega_{R/k}$ is just the module of Kähler differentials of R over k. See [31] for the general theory.

Our contribution is as follows (notice that it substantially improves [26, Corollary 5.8(iii)]).

Corollary 50. Let R be as in (49.1) and write $t = \operatorname{depth} R$. Then, the strong Zariski-Lipman conjecture holds true if there exists an integer r with $0 \le r \le t$ such that $\Omega_{R/k}$ satisfies (\widetilde{S}_r) and, if r < t,

$$\operatorname{Ext}_R^{1 \leq i \leq t-r}(\Omega_{R/k}, R) = 0.$$

In particular, the conjecture is true when R is Cohen-Macaulay and $\Omega_{R/k}$ is maximal Cohen-Macaulay.

Proof. Recall that $\Omega_{R/k}^* \cong \operatorname{Der}_k(R)$ (see, e.g., [36, p. 192]). Now we apply Proposition 13 to obtain that $\Omega_{R/k}$ is free. But this is equivalent to R being regular – for a proof of this fact in case $R = k[x_1, \dots, x_m]_{\mathfrak{q}}/I$ (resp. $R = k[x_1, \dots, x_m]/I$), see [31, Theorem 7.2] (resp. [31, Theorem 14.1]).

Remark 51. Maintain the above setup and notations, and recall that in fact $\operatorname{Hom}_R(\Omega_{R/k}, N) \cong \operatorname{Der}_k(R, N)$ for any R-module N (see [36, p. 192]). Now, using Proposition 19(4), our Corollary 50 is immediately seen to admit the following generalization. Let r be an integer with $0 \le r \le t$ such that $\Omega_{R/k}$ satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(\Omega_{R/k}, R) = 0$. Suppose there exists a finite R-module $N \ne 0$ satisfying $\operatorname{pd}_R N \le t - r$ and

$$\operatorname{pd}_R\operatorname{Der}_k(R,N)<\infty.$$

Then, R is regular (to retrieve Corollary 50, take N = R). It is also worth pointing out that a similar argument can be used to generalize Corollary 53 below.

The above remark suggests a possible generalization of the conjecture, as follows.

Question 52. Let *R* be as in (49.1). Suppose there exists a finite *R*-module $N \neq 0$ satisfying $pd_R N < \infty$ and $pd_R Der_k(R,N) < \infty$. Is it true that *R* must be regular?

It is well-known that the condition $\operatorname{pd}_R\operatorname{Der}_k(R)<\infty$ forces R to be a normal domain, which in particular settles the conjecture in the case of curves. A major case is that of quasi-homogeneous complete intersections with isolated singularities (see [25, Theorem 2.4]). Notice furthermore that if $t\leq 2$ then, since $\operatorname{Der}_k(R)$ is a dual, the conjecture is easily seen to be equivalent to the original version of the Zariski-Lipman conjecture, which says that R is regular if $\operatorname{Der}_k(R)$ is free. The critical open case of the latter is when R is Cohen-Macaulay (in fact, Gorenstein) of dimension 2; so, as a consequence of Corollary 50, we obtain that it holds true provided that depth $\Omega_{R/k}=2$.

Let us mention that there is also the following related conjecture (see [37, Conjecture 3.12], also [26, Conjecture 5.10]). If R is as in (49.1) and

$$\operatorname{\mathsf{G-dim}}_R\operatorname{\mathsf{Der}}_k(R)<\infty$$
 (resp. $\operatorname{\mathsf{CI-dim}}_R\operatorname{\mathsf{Der}}_k(R)<\infty$),

then R is a Gorenstein ring (resp. a complete intersection ring). In this regard, we have the following result.

Corollary 53. Let R be as in (49.1) and write $t = \operatorname{depth} R$. Suppose there exists an integer r with $0 \le r \le t$ such that $\Omega_{R/k}$ satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(\Omega_{R/k}, R) = 0$ (e.g., if R is Cohen-Macaulay and $\Omega_{R/k}$ is maximal Cohen-Macaulay). The following assertions hold:

- (1) If G-dim $_R$ $Der_k(R) < \infty$, then $\Omega_{R/k}$ is totally reflexive. If in addition R is Golod, then R is a hypersurface ring.
- (2) If $\operatorname{CI-dim}_R \operatorname{Der}_k(R) < \infty$ and either $\operatorname{Ext}_R^{2j}(\operatorname{Der}_k(R),\operatorname{Der}_k(R)) = 0$ or $\operatorname{Ext}_R^{2j}(\Omega_{R/k},\Omega_{R/k}) = 0$ for some integer $j \ge 1$, then R is regular.

Proof. (1) The first part follows by Proposition 13. For the second part, assume by way of contradiction that R is not a hypersurface ring. In particular, R cannot be regular. On the other hand, Corollary 32 forces $\Omega_{R/k}$ to be free, which as we recalled in the proof of Corollary 50 is equivalent to R being regular, a contradiction.

(2) Let us first consider the case where $\operatorname{Ext}_R^{2j}(\operatorname{Der}_k(R),\operatorname{Der}_k(R))=0$. Since $\operatorname{Cl-dim}_R\operatorname{Der}_k(R)<\infty$, we can apply [4, Theorem 4.2] to get $\operatorname{pd}_R\operatorname{Der}_k(R)<\infty$, and the result follows by Corollary 50. Next, suppose $\operatorname{Ext}_R^{2j}(\Omega_{R/k},\Omega_{R/k})=0$. Notice that Corollary 13 yields

$$\mathsf{CI}\text{-}\mathsf{dim}_R\Omega_{R/k}=0.$$

Using [4, Theorem 4.2] once again, we obtain $\operatorname{pd}_R \Omega_{R/k} < \infty$, hence $\operatorname{pd}_R \Omega_{R/k} = \operatorname{CI-dim}_R \Omega_{R/k} = 0$ and therefore R is regular.

5.2. Berger's conjecture on differential modules. Pick R as in (49.1). The 1963 Berger's conjecture (see [8]) asserts that R must be regular if dim R = 1 and $\Omega_{R/k}$ is torsionfree. We refer to [25] for further information about this problem.

Corollary 54. Let R be as in (49.1). Then, Berger's conjecture holds true if

$$\operatorname{pd}_R\operatorname{Der}_k(R)^{\dagger}<\infty.$$

Proof. In the present setup, being a reduced local ring of dimension 1, R is Cohen-Macaulay. Its differential module $\Omega_{R/k}$, being torsionfree by hypothesis and possessing a generic rank (which for completeness is equal to dim R=1), must also be torsionless (see [11, Exercise 1.4.18]). In addition, we have $(\Omega_{R/k}^*)^{\dagger} \cong \operatorname{Der}_k(R)^{\dagger}$. Now we are in a position to apply Corollary 35 with $M=\Omega_{R/k}$ to conclude that $\Omega_{R/k}$ is free; as recalled in the proof of Corollary 50, this means that R is regular.

Remark 55. A few comments are in order. Berger's conjecture is known to be true if $pd_R Der_k(R)$ < ∞. This is because in this case the local ring R must be normal, and hence, being one-dimensional, necessarily regular. However, and inspired by Corollary 54 above, we wonder whether the conjecture holds true provided that

$$\operatorname{pd}_R\operatorname{Der}_k(R)^*<\infty$$
,

where we recall that the module $\operatorname{Der}_k(R)^*$ (the bidual of $\Omega_{R/k}$) has been considered in the literature and is dubbed *module of Zariski differentials of R over k* (see, e.g., [39]).

5.3. **Vasconcelos' conjecture on normal modules.** In this last subsection, let R = S/I where S is a local ring and I is an ideal with $pd_SI < \infty$ (typically, S is taken regular). Note this setting is far more general than (49.1). There is a conjecture by Vasconcelos (see [49, p. 373]) which states that I must be generated by a regular sequence if $pd_R N_R < \infty$, where N_R is the normal module of R, i.e.,

$$N_R = \text{Hom}_R(I/I^2, R) = (I/I^2)^*.$$

Corollary 56. Let R be as above, and write $t = \operatorname{depth} R$. Then, Vasconcelos' conjecture holds true if there exists an integer r with $0 \le r \le t$ such that I/I^2 satisfies (\widetilde{S}_r) and, if r < t,

$$\operatorname{Ext}_{R}^{1 \le i \le t - r}(I/I^{2}, R) = 0.$$

In particular, the conjecture is true when R is Cohen-Macaulay and I/I^2 is maximal Cohen-Macaulay.

Proof. By Proposition 13, we obtain that I/I^2 is free. Since $pd_S I < \infty$, this forces I to be generated by an S-sequence (see [47]).

Remark 57. Maintain the above setup and notations. Using Proposition 19(4), our Corollary 56 is immediately seen to admit the following generalization. Let r be an integer with $0 \le r \le t$ such that I/I^2 satisfies (\widetilde{S}_r) and, if r < t, $\operatorname{Ext}_R^{1 \le i \le t - r}(I/I^2, R) = 0$. Suppose there exists a finite R-module $N \ne 0$ satisfying $\operatorname{pd}_R N \le t - r$ and

$$\operatorname{pd}_R \operatorname{Hom}_R(I/I^2, N) < \infty.$$

Then, *I* is generated by an *S*-sequence (to retrieve Corollary 56, pick N = R).

This remark seems to suggest the following potential generalization of Vasconcelos' conjecture.

Question 58. Let R be as above. Suppose there exists a finite R-module $N \neq 0$ satisfying $\operatorname{pd}_R N < \infty$ and $\operatorname{pd}_R \operatorname{Hom}_R(I/I^2, N) < \infty$. Is it true that R must be a complete intersection ring?

Next, we exemplify the Cohen-Macaulay case of Corollary 56 in its contrapositive form, that is, we proceed to illustrate the property

$$\operatorname{pd}_R \operatorname{N}_R = \infty$$
.

It is worth mentioning that in this case the normal module can benefit, in particular, from the well-established theory of infinite free resolutions (see [2]), which includes, e.g., the investigation of eventual periodicity of resolutions as well as connections to complexity, curvature, and other numerical invariants of modules.

Example 59. Let k be a field and $R = S/I = k[x, y, z]_{(x,y,z)}/I$, where

$$I = (x^{\ell}, xy^{\ell-2}z, y^{\ell-1}z), \quad \ell \ge 3.$$

Note R is a (non-Gorenstein) Cohen-Macaulay almost complete intersection local ring. Moreover, S/I^2 has the same feature; this has been observed in [35, Proposition 2.5] by means of the theory of Buchberger graphs, but here we provide a much simpler argument. It suffices to notice that, for any given $\ell \geq 3$, a minimal free resolution of I^2 is given by $0 \to S^5 \xrightarrow{\varphi} S^6 \to I^2 \to 0$, where

$$oldsymbol{arphi} = \left(egin{array}{ccccc} 0 & 0 & 0 & 0 & -y^{\ell-2}z \ 0 & 0 & -y & 0 & x^{\ell-1} \ 0 & 0 & x & -y^{\ell-3}z & 0 \ 0 & -y & 0 & x^{\ell-2} & 0 \ -y & x & 0 & 0 & 0 \ x & 0 & 0 & 0 & 0 \end{array}
ight),$$

which, by the classical Hilbert-Burch theorem, yields the claim. Thus, by the short exact sequence

$$0 \rightarrow I/I^2 \rightarrow S/I^2 \rightarrow S/I \rightarrow 0$$

we deduce that the conormal module I/I^2 is maximal Cohen-Macaulay. Consequently, Corollary 56 gives $pd_R N_R = \infty$.

Example 60. Let c be an integer with $5 \le c \le 10$, and let J be the homogeneous ideal of the standard graded polynomial ring $\mathbb{Q}[x_0, \dots, x_c]$ defining a set of

$$1+c+\left\lceil \frac{c(c-1)}{6} \right\rceil$$

general points in c-dimensional projective space over $\mathbb Q$ (here, as usual, $\lceil q \rceil$ denotes the smallest integer which is bigger than, or equal to, a given number $q \in \mathbb Q$). So, e.g., if c = 5 then J defines 10 general points in $\mathbb P^5_{\mathbb Q}$. Now consider the regular local ring $S = \mathbb Q[x_0,\dots,x_c]_{(x_0,\dots,x_c)}$ and the ideal I = JS. Then, according to [33, paragraph after Conjecture 7.2], the quotient R = S/I must be a non-Gorenstein (hence I is not generated by an S-sequence) Cohen-Macaulay local ring and the R-module I/I^2 is (necessarily maximal) Cohen-Macaulay. By virtue of Corollary 56, we deduce $\operatorname{pd}_R N_R = \infty$.

We close the paper by establishing the following result, related to the module of differentials. Recall that a finite R-module M is almost Cohen-Macaulay if depth $_R M \ge \dim R - 1$.

Corollary 61. Let R be as in (49.1), and suppose R is Cohen-Macaulay. Then, Vasconcelos' conjecture holds true if $\Omega_{R/k}$ is almost Cohen-Macaulay.

Proof. Since I is radical and char k = 0, the so-called conormal sequence takes the form

$$(61.1) 0 \rightarrow I/I^{(2)} \rightarrow R^m \rightarrow \Omega_{R/k} \rightarrow 0,$$

where $I^{(2)}$ denotes the second symbolic power of I (i.e., the ideal formed by the $g \in S$ such that $gf \in I^2$ for some R-regular element f). This exact sequence shows, by means of the standard depth lemma, that $I/I^{(2)}$ is maximal Cohen-Macaulay since $\Omega_{R/k}$ is almost Cohen-Macaulay. On the other hand, dualizing the short exact sequence

$$0 \to I^{(2)}/I^2 \to I/I^2 \to I/I^{(2)} \to 0$$

and using that $I^{(2)}/I^2$ is in fact the *R*-torsion of I/I^2 , we obtain

$$(I/I^{(2)})^* \cong (I/I^2)^* = N_R.$$

Now, applying Corollary 29, we deduce that $I/I^{(2)}$ is free. By the sequence (61.1), this implies $\operatorname{pd}_R\Omega_{R/k}\leq 1$, which according to [49, Remark (b), p. 374] forces I to be generated by a regular sequence, as needed.

Clearly, $\Omega_{R/k}$ is almost Cohen-Macaulay (with R as in Corollary 61) if, for example, dim R=2 and $\Omega_{R/k}$ is torsionfree, or if dim R=3 and $\Omega_{R/k}$ is reflexive. In particular, the latter situation suggests the question as to when $\Omega_{R/k}$ is reflexive, and we refer to [39, Remark 4, p. 10] for a list of instances where this property holds.

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