

# COMPATIBLE WEAK FACTORIZATION SYSTEMS AND MODEL STRUCTURES

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**ABSTRACT.** In this paper the concept of compatible weak factorization systems in general categories is introduced as a counterpart of compatible complete cotorsion pairs in abelian categories. We describe a method to construct model structures on general categories via two compatible weak factorization systems satisfying certain conditions, and hence generalize a very useful result by Gillespie for abelian model structures. As particular examples, we show that weak factorizations systems associated to some classical model structures (for example, the Kan-Quillen model structure on  $\mathbf{sSet}$ ) satisfy these conditions.

## INTRODUCTION

To study various homotopy theories on a category  $\mathbf{E}$  via a uniform and axiomatic approach, Quillen [15] introduced *model structures*, which are triples  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  of classes of morphisms such that both  $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are weak factorization systems, and  $\mathcal{W}$  satisfies the 2-out-of-3 property. This notion plays a central role in modern homotopy theory, and people try to find various methods to construct model structures on  $\mathbf{E}$ .

In the case that  $\mathbf{E}$  is an abelian category, people are more interested in *abelian model structures* compatible with the abelian structure of  $\mathbf{E}$  (see Hovey [11]), that is, cofibrations are monomorphisms with cofibrant cokernels and fibrations coincide with epimorphisms with fibrant kernels. The celebrated Hovey's correspondence provides a bijective correspondence between abelian model structures on  $\mathbf{E}$  and Hovey triples  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  of classes of objects in  $\mathbf{E}$  such that both  $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are complete cotorsion pairs, and  $\mathcal{W}$  is thick. Therefore, constructions of abelian models structures on  $\mathbf{E}$  are equivalent to constructions of Hovey triples, which in general are simpler. Furthermore, by introducing compatible cotorsion pairs, Gillespie proves the following result, which provides a more convenient way to construct Hovey triples; see [9, Theorem 1.1].

**Theorem (Gillespie).** *Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be compatible and complete hereditary cotorsion pairs in an abelian category  $\mathbf{E}$ . Then there is a subcategory  $\mathcal{W}$  such*

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that  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  forms a Hovey triple. Moreover,  $\mathcal{W}$  can be described as:

$$\begin{aligned} \mathcal{W} &= \{M \mid \text{there is a s.e.s. } 0 \rightarrow M \rightarrow A \rightarrow B \rightarrow 0 \text{ with } A \in \tilde{\mathcal{F}} \text{ and } B \in \tilde{\mathcal{C}}\} \\ &= \{M \mid \text{there is a s.e.s. } 0 \rightarrow A' \rightarrow B' \rightarrow M \rightarrow 0 \text{ with } A' \in \tilde{\mathcal{F}} \text{ and } B' \in \tilde{\mathcal{C}}\}. \end{aligned}$$

Recall from [9] that two complete cotorsion pairs  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  in  $\mathbf{E}$  are *compatible* if  $\tilde{\mathcal{C}} \subseteq \mathcal{C}$  (or equivalently  $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ ) and  $\tilde{\mathcal{C}} \cap \mathcal{F} = \mathcal{C} \cap \tilde{\mathcal{F}}$ .

The main aim of this paper is to generalize the above theorem, describing a method to construct model structures (might not be abelian) on general categories. Note that the key result used to establish Hovey's correspondence is that a pair  $(\mathcal{C}, \mathcal{F})$  of classes of objects in an abelian category  $\mathbf{E}$  is a complete cotorsion pair if and only if the pair  $(\text{Mon}(\mathcal{C}), \text{Epi}(\mathcal{F}))$  forms a weak factorization system, where

$$\begin{aligned} \text{Mon}(\mathcal{C}) &= \{\alpha \mid \alpha \text{ is a monomorphism with } \text{Coker}(\alpha) \in \mathcal{C}\}, \text{ and} \\ \text{Epi}(\mathcal{F}) &= \{\alpha \mid \alpha \text{ is an epimorphism with } \text{Ker}(\alpha) \in \mathcal{F}\}. \end{aligned}$$

It is reasonable to consider weak factorization systems in general categories which take the role of complete cotorsion pairs in abelian categories. Therefore, we need to find appropriate conditions on weak factorization systems in general categories which are analogues of the hereditary and compatible conditions for complete cotorsion pairs in abelian categories.

Recall from Joyal [12, Definition C.0.20] that a class  $\mathcal{C}$  of morphisms in a category  $\mathbf{E}$  satisfies the *left cancellation property* if

$$\beta\alpha \in \mathcal{C} \text{ and } \beta \in \mathcal{C} \Rightarrow \alpha \in \mathcal{C}.$$

Dually, a class  $\mathcal{F}$  of morphisms in  $\mathbf{E}$  satisfies the *right cancellation property* if

$$\beta\alpha \in \mathcal{F} \text{ and } \alpha \in \mathcal{F} \Rightarrow \beta \in \mathcal{F}.$$

The following theorem tells us that these cancellation properties on weak factorization systems properly generalize the hereditary condition on cotorsion pairs since they coincide in the abelian situation.

**Theorem A.** *Let  $(\mathcal{C}, \mathcal{F})$  be a pair of classes of objects in an abelian category  $\mathbf{E}$ . Then the following statements are equivalent.*

- (i)  $(\mathcal{C}, \mathcal{F})$  is a complete and hereditary cotorsion pair.
- (ii)  $(\text{Mon}(\mathcal{C}), \text{Epi}(\mathcal{F}))$  is a weak factorization system such that  $\text{Mon}(\mathcal{C})$  has the left cancellation property and/or  $\text{Epi}(\mathcal{F})$  has the right cancellation property.

We say that two weak factorization systems  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  in a category  $\mathbf{E}$  are *compatible* if the following conditions hold:

- $\tilde{\mathcal{C}} \subseteq \mathcal{C}$  (or equivalently,  $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ );
- given composable morphisms  $\alpha$  and  $\beta$  in  $\mathcal{F}$ , if two of the three morphisms  $\alpha$ ,  $\beta$  and  $\beta\alpha$  are in  $\tilde{\mathcal{F}}$ , then so is the third one;
- given  $c \in \tilde{\mathcal{C}}$  and  $f \in \mathcal{F}$ , if  $fc \in \tilde{\mathcal{C}}$ , then  $f \in \tilde{\mathcal{F}}$ .

As asserted by the following theorem, this compatible condition on weak factorization systems indeed generalizes the one for cotorsion pairs.

**Theorem B.** *Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two pairs of classes of objects in an abelian category  $\mathbf{E}$ . Then the following statements are equivalent.*

- (i)  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  are compatible and complete cotorsion pairs.

- (ii)  $(\text{Mon}(\mathcal{C}), \text{Epi}(\tilde{\mathcal{F}}))$  and  $(\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\mathcal{F}))$  are compatible weak factorization systems.

The following result as a generalization of Gillespie theorem ([9, Theorem 1.1]) provides a method to construct model structures on general categories via compatible weak factorization systems satisfying certain properties. To extend the application of this theorem, we use a slightly more general definition of model structures without assuming that  $\mathbf{E}$  is bicomplete (see Definition 2.1), and say that a model structure is *hereditary* if both the class of cofibrations and the class of trivial cofibrations satisfy the left cancellation property.

**Theorem C.** *Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two compatible weak factorization systems in a category  $\mathbf{E}$  satisfying the following properties:*

- (1)  $\mathbf{E}$  has pushouts along morphisms in  $\mathcal{C}$  and pullbacks along morphisms in  $\mathcal{F}$ ;
- (2)  $(\tilde{\mathcal{C}}, \mathcal{F})$  satisfies the Frobenius property;
- (3) both  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$  satisfy the left cancellation property.

*Then  $(\mathcal{C}, \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}, \mathcal{F})$  forms a hereditary model structure on  $\mathbf{E}$ , where*

$$\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}} = \{\alpha \mid \alpha \text{ can be decomposed as } \alpha = \tilde{f}\tilde{c} \text{ with } \tilde{c} \in \tilde{\mathcal{C}} \text{ and } \tilde{f} \in \tilde{\mathcal{F}}\}.$$

Recall from van den Berg and Garner [17] that a weak factorization system  $(\mathcal{C}, \mathcal{F})$  satisfies the *Frobenius property* if  $\mathbf{E}$  has pullbacks along morphisms in  $\mathcal{C}$  or  $\mathcal{F}$ , and the morphisms in  $\mathcal{C}$  are preserved under pullback along morphisms in  $\mathcal{F}$ . Since weak factorization systems induced by complete cotorsion pairs automatically satisfy this property (see Proposition 1.15), it follows from Theorems A and B that the weak factorization systems  $(\text{Mon}(\mathcal{C}), \text{Epi}(\tilde{\mathcal{F}}))$  and  $(\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\mathcal{F}))$  induced by compatible complete hereditary cotorsion pairs  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  in an abelian category  $\mathbf{E}$  satisfy all conditions specified in Theorem C. We mention that there are non-abelian examples: in Section 3 we consider the classical and constructive Kan-Quillen model structures on the category  $\mathbf{sSet}$  of simplicial sets and the standard projective model structure on the category  $\mathbf{Ch}_{\geq 0}(R)$  of nonnegative chain complexes of modules over a ring  $R$ , and show that the weak factorization systems associated to these model structures satisfy all conditions specified in Theorem C.

## 1. WEAK FACTORIZATION SYSTEMS AND COTORSION PAIRS

In this section we introduce the compatible condition on weak factorization systems, which generalizes the one for cotorsion pairs, and prove Theorems A and B advertised in Introduction.

**Setup.** Throughout this section we let  $\mathbf{E}$  denote a category and  $\mathbf{A}$  denote an abelian category.

**1.1 Lifting property.** Let  $l : A \rightarrow B$  and  $r : C \rightarrow D$  be two morphisms in  $\mathbf{E}$ . Recall that  $l$  has the *left lifting property* with respect to  $r$  (or  $r$  has the *right lifting property* with respect to  $l$ ) if for every pair of morphisms  $f : A \rightarrow C$  and  $g : B \rightarrow D$  such that  $rf = gl$ , there exists a morphism  $t : B \rightarrow C$  such that  $f = tl$  and  $g = rt$ ,

that is, the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow l & \nearrow e & \downarrow r \\ B & \xrightarrow{g} & D \end{array}$$

For a class  $\mathcal{C}$  of morphisms in  $\mathbf{E}$ , denote by  $\mathcal{C}^\square$  the class of morphisms  $r$  in  $\mathbf{E}$  having the right lifting property with respect to all morphisms  $l \in \mathcal{C}$ . The class  ${}^\square\mathcal{C}$  is defined dually.

The next definition of weak factorization systems was given by Bousfield in [2].

**1.2 Definition.** A pair  $(\mathcal{C}, \mathcal{F})$  of classes of morphisms in  $\mathbf{E}$  is called a *weak factorization system* if  $\mathcal{C}^\square = \mathcal{F}$  and  ${}^\square\mathcal{F} = \mathcal{C}$ , and every morphism  $\alpha$  in  $\mathbf{E}$  can be decomposed as  $\alpha = fc$  with  $c \in \mathcal{C}$  and  $f \in \mathcal{F}$ .

**1.3 Remark.** Let  $(\mathcal{C}, \mathcal{F})$  be a weak factorization system in  $\mathbf{E}$ . Then

- (a) the classes  $\mathcal{C}$  and  $\mathcal{F}$  are closed under compositions and retracts, and contain the isomorphisms;
- (b) if  $\mathbf{E}$  has pushouts along morphisms in  $\mathcal{C}$ , then  $\mathcal{C}$  is closed under pushouts;
- (c) if  $\mathbf{E}$  has pullbacks along morphisms in  $\mathcal{F}$ , then  $\mathcal{F}$  is closed under pullbacks;
- (d)  $\mathcal{C} \cap \mathcal{F}$  is the class of isomorphisms.

For details, see [12, Propositions D.1.2 and D.1.3].

**1.4 Cotorsion pairs.** A pair  $(\mathbf{C}, \mathbf{F})$  of classes of objects in  $\mathbf{A}$  is called a *cotorsion pair* if  $\mathbf{C}^\perp = \mathbf{F}$  and  ${}^\perp\mathbf{F} = \mathbf{C}$  where

$$\begin{aligned} \mathbf{C}^\perp &= \{M \in \mathbf{A} \mid \text{Ext}_\mathbf{A}^1(C, M) = 0 \text{ for all objects } C \in \mathbf{C}\}, \text{ and} \\ {}^\perp\mathbf{F} &= \{M \in \mathbf{A} \mid \text{Ext}_\mathbf{A}^1(M, D) = 0 \text{ for all objects } D \in \mathbf{F}\}. \end{aligned}$$

Following Enochs and Jenda [5], a cotorsion pair  $(\mathbf{C}, \mathbf{F})$  is said to be *complete* if for any object  $M$  in  $\mathbf{A}$ , there exist short exact sequences  $0 \rightarrow D \rightarrow C \rightarrow M \rightarrow 0$  and  $0 \rightarrow M \rightarrow D' \rightarrow C' \rightarrow 0$  in  $\mathbf{A}$  with  $D, D' \in \mathbf{F}$  and  $C, C' \in \mathbf{C}$ . A cotorsion pair  $(\mathbf{C}, \mathbf{F})$  is called *hereditary* if  $\mathbf{C}$  is closed under kernels of epimorphisms and  $\mathbf{F}$  is closed under cokernels of monomorphisms.

Recall that for a class  $\mathbf{C}$  of objects in  $\mathbf{A}$ ,

$$\begin{aligned} \text{Mon}(\mathbf{C}) &= \{\alpha \mid \alpha \text{ is a monomorphism with } \text{Coker}(\alpha) \in \mathbf{C}\}, \text{ and} \\ \text{Epi}(\mathbf{C}) &= \{\alpha \mid \alpha \text{ is an epimorphism with } \text{Ker}(\alpha) \in \mathbf{C}\}. \end{aligned}$$

The next result is essentially due to Hovey [11]; see also Positselski and Šťovíček [14, Theorem 2.4].

**1.5 Theorem.** A pair  $(\mathbf{C}, \mathbf{F})$  of classes of objects in  $\mathbf{A}$  is a complete cotorsion pair if and only if  $(\text{Mon}(\mathbf{C}), \text{Epi}(\mathbf{F}))$  is a weak factorization system.

We describe a few auxiliary results before proving Theorem A and Theorem B advertised in Introduction. Recall from Joyal [12, Definition C.0.20] that a class  $\mathcal{C}$  of morphisms in a category  $\mathbf{E}$  satisfies the *left cancellation property* if

$$\beta\alpha \in \mathcal{C} \text{ and } \beta \in \mathcal{C} \Rightarrow \alpha \in \mathcal{C}.$$

Dually, a class  $\mathcal{F}$  of morphisms in  $\mathbf{E}$  satisfies the *right cancellation property* if

$$\beta\alpha \in \mathcal{F} \text{ and } \alpha \in \mathcal{F} \Rightarrow \beta \in \mathcal{F}.$$

**1.6 Lemma.** *Let  $\mathcal{C}$  be a class of objects in  $\mathbf{A}$ . Then the following are equivalent.*

- (i)  $\mathcal{C}$  is closed under kernels of epimorphisms.
- (ii)  $\text{Mon}(\mathcal{C})$  satisfies the left cancellation property.

**Proof.** (i)  $\Rightarrow$  (ii). Let  $\alpha : X \rightarrow Y$  and  $\beta : Y \rightarrow Z$  be morphisms in  $\mathbf{A}$  such that both  $\beta\alpha$  and  $\beta$  are in  $\text{Mon}(\mathcal{C})$ . In particular,  $\beta\alpha$  is a monomorphism, so is  $\alpha$ . Thus we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{\alpha} & Y & \longrightarrow & \text{Coker}(\alpha) \longrightarrow 0 \\ & & \parallel & & \downarrow \beta & & \downarrow \\ 0 & \longrightarrow & X & \xrightarrow{\beta\alpha} & Z & \longrightarrow & \text{Coker}(\beta\alpha) \longrightarrow 0, \end{array}$$

which induces a short exact sequence

$$0 \rightarrow \text{Coker}(\alpha) \rightarrow \text{Coker}(\beta\alpha) \rightarrow \text{Coker}(\beta) \rightarrow 0.$$

by the Snake Lemma. Since  $\mathcal{C}$  is closed under the kernels of epimorphisms by the assumption, and both  $\text{Coker}(\beta\alpha)$  and  $\text{Coker}(\beta)$  are in  $\mathcal{C}$ , it follows that  $\text{Coker}(\alpha) \in \mathcal{C}$  as well, and hence  $\alpha$  is in  $\text{Mon}(\mathcal{C})$ .

(ii)  $\Rightarrow$  (i). Let  $0 \rightarrow X \xrightarrow{\beta} C' \rightarrow C \rightarrow 0$  be an exact sequence in  $\mathbf{A}$  with  $C'$  and  $C$  in  $\mathcal{C}$ . Let  $\alpha$  be the zero morphism from  $0$  to  $X$ . Then  $\beta\alpha$  is in  $\text{Mon}(\mathcal{C})$ , and so is  $\alpha$  as  $\beta$  is in  $\text{Mon}(\mathcal{C})$  clearly. Thus  $X$  is contained in  $\mathcal{C}$ .  $\square$

The next result can be prove dually.

**1.7 Lemma.** *Let  $\mathcal{F}$  be a class of objects in  $\mathbf{A}$ . Then the following are equivalent.*

- (i)  $\mathcal{F}$  is closed under cokernels of monomorphisms.
- (ii)  $\text{Epi}(\mathcal{F})$  satisfies the right cancellation property.

**1.8 Lemma.** *Let  $(\mathcal{C}, \mathcal{F})$  be a pair of classes of objects in  $\mathbf{A}$  such that  $(\text{Mon}(\mathcal{C}), \text{Epi}(\mathcal{F}))$  is a weak factorization system. Then  $\text{Mon}(\mathcal{C})$  satisfies the left cancellation property if and only if  $\text{Epi}(\mathcal{F})$  satisfies the right cancellation property.*

**Proof.** We only prove the “only if” part as the “if” part is a dual statement. Since  $\text{Mon}(\mathcal{C})$  satisfies the left cancellation property, by Lemma 1.6,  $\mathcal{C}$  is closed under kernels of epimorphisms. Since  $(\mathcal{C}, \mathcal{F})$  is a complete cotorsion pair by Theorem 1.5,  $\mathcal{F}$  is closed under the cokernels of monomorphisms; see Becker [1, Corollary 1.1.12]. Thus  $\text{Epi}(\mathcal{F})$  satisfies the right cancellation property by Lemma 1.7.  $\square$

The next result can be proved in a way similar to the proof of Lemma 1.6.

**1.9 Lemma.** *Let  $\mathcal{C}$  be a class of objects in  $\mathbf{A}$ . Then the following are equivalent.*

- (i)  $\mathcal{C}$  is closed under extensions.
- (ii)  $\text{Mon}(\mathcal{C})$  and/or  $\text{Epi}(\mathcal{C})$  are closed under compositions.

**1.10 Theorem.** *Let  $(\mathcal{C}, \mathcal{F})$  be a pair of classes of objects in  $\mathbf{A}$ . Then the following are equivalent.*

- (i)  $(\mathcal{C}, \mathcal{F})$  is a complete and hereditary cotorsion pair.

- (ii)  $(\text{Mon}(\mathbf{C}), \text{Epi}(\mathbf{F}))$  is a weak factorization system such that  $\text{Mon}(\mathbf{C})$  satisfies the left cancellation property and/or  $\text{Epi}(\mathbf{F})$  satisfies the right cancellation property.

**Proof.** The equivalence follows immediately from Theorem 1.5 and Lemmas 1.6, 1.7, 1.8 and 1.9.  $\square$

In the following we introduce the compatible condition on weak factorization systems.

**1.11 Definition.** Two weak factorization systems  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  in  $\mathbf{E}$  are called *compatible* if the following conditions hold:

- (CP1)  $\tilde{\mathcal{C}} \subseteq \mathcal{C}$  (or equivalently,  $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ );
- (CP2) given composable morphisms  $\alpha$  and  $\beta$  in  $\mathcal{F}$ , if two of the three morphisms  $\alpha$ ,  $\beta$  and  $\beta\alpha$  are in  $\tilde{\mathcal{F}}$ , then so is the third one;
- (CP3) given  $c \in \tilde{\mathcal{C}}$  and  $f \in \mathcal{F}$ , if  $fc \in \tilde{\mathcal{C}}$ , then  $f \in \tilde{\mathcal{F}}$ .

We mention that the condition (CP3) was called the span property by Sattler in [16, Definition 2.3].

Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two weak factorization systems in  $\mathbf{E}$ . We define

$$\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}} = \{\alpha \mid \alpha \text{ can be decomposed as } \alpha = \tilde{f}\tilde{c} \text{ with } \tilde{c} \in \tilde{\mathcal{C}} \text{ and } \tilde{f} \in \tilde{\mathcal{F}}\}.$$

The next result, first proved by Sattler [16, Lemma 2.1], will be used frequently in the sequel. For the convenience of the reader, we include a detailed proof.

**1.12 Lemma.** Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two weak factorization systems in  $\mathbf{E}$  satisfying the condition (CP1). Then  $\tilde{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\tilde{\mathcal{F}} = \mathcal{F} \cap \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .

**Proof.** We only prove the first equality as the second one can be proved similarly.

It is clear that  $\tilde{\mathcal{C}} \subseteq \mathcal{C} \cap \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , so it suffices to show the inclusion of the other direction. Take a morphism  $\alpha : X \rightarrow Y$  in  $\mathcal{C} \cap \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ . For a morphism  $f : A \rightarrow B$  in  $\mathcal{F}$  and two morphisms  $\lambda : X \rightarrow A$  and  $\mu : Y \rightarrow B$  such that  $f\lambda = \mu\alpha$ , since  $\alpha$  is in  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , we can find a morphism  $\tilde{c} : X \rightarrow C$  in  $\tilde{\mathcal{C}}$  and a morphism  $\tilde{f} : C \rightarrow Y$  in  $\tilde{\mathcal{F}}$  such that  $\alpha = \tilde{f}\tilde{c}$ , as shown in the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & A \\ \alpha \downarrow & \searrow \tilde{c} & \downarrow f \\ & C & \\ \tilde{f} \swarrow & & \\ Y & \xrightarrow{\mu} & B. \end{array}$$

Since  $\tilde{c}$  has the left lifting property with respect to  $f$ , one can find a morphism  $h : C \rightarrow A$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\lambda} & A \\ \alpha \downarrow & \searrow \tilde{c} & \downarrow f \\ & C & \nearrow h \\ \tilde{f} \swarrow & & \\ Y & \xrightarrow{\mu} & B, \end{array}$$

in which the left triangle gives rises to the following commutative square:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{c}} & C \\ \alpha \downarrow & & \downarrow \tilde{f} \\ Y & \xlongequal{\quad} & Y. \end{array}$$

Since  $\alpha$  has the left lifting property with respect to  $\tilde{f}$ , there is a morphism  $h' : Y \rightarrow C$  such that  $h'\alpha = \tilde{c}$  and  $\tilde{f}h' = \text{id}_Y$ . Thus  $hh'\alpha = h\tilde{c} = \lambda$  and  $fhh' = \mu\tilde{f}h' = \mu$ , so  $\alpha$  has the left lifting property with respect to  $f$ , and hence belongs to  ${}^\square\mathcal{F} = \tilde{\mathcal{C}}$ . Consequently, one has  $\mathcal{C} \cap \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}} \subseteq \tilde{\mathcal{C}}$ .  $\square$

**1.13 Lemma.** *Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two compatible weak factorization systems in  $\mathbf{E}$ , and suppose that the composite  $g = fh$  is contained in  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ . One has:*

- (a) *if  $\mathbf{E}$  has pullbacks along morphisms in  $\tilde{\mathcal{F}}$  and  $f \in \tilde{\mathcal{F}}$ , then  $h \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ ;*
- (b) *if  $\mathbf{E}$  has pushouts along morphisms in  $\tilde{\mathcal{C}}$  and  $h \in \tilde{\mathcal{C}}$ , then  $f \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .*

**Proof.** We only prove the first statement as the second one is a formal dual.

Since  $(\mathcal{C}, \tilde{\mathcal{F}})$  is a weak factorization system,  $h$  can be written as a composite of a morphism  $h_1$  in  $\mathcal{C}$  followed by a morphism  $h_2$  in  $\tilde{\mathcal{F}}$ . Then  $g = fh_2h_1$  with  $fh_2$  being contained in  $\tilde{\mathcal{F}}$ . If the conclusion holds for  $h_1$ , that is,  $h_1 \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , then  $h_1 \in \tilde{\mathcal{C}}$  by Lemma 1.12, so  $h = h_2h_1 \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  as well. Therefore, we only need to deal with the special case that  $h$  is contained in  $\mathcal{C}$ .

Write  $h : X \rightarrow Z$  and  $f : Z \rightarrow Y$ . Since  $g = fh$  is contained in  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , by definition, there is a morphism  $\tilde{c} : X \rightarrow Z'$  in  $\tilde{\mathcal{C}}$  and a morphism  $\tilde{f} : Z' \rightarrow Y$  in  $\tilde{\mathcal{F}}$  such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ \tilde{c} \downarrow & & \downarrow f \\ Z' & \xrightarrow{\tilde{f}} & Y. \end{array}$$

Now consider the following pullback diagram:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}_2} & Z \\ \tilde{f}_1 \downarrow \lrcorner & & \downarrow f \\ Z' & \xrightarrow{\tilde{f}} & Y. \end{array}$$

Then one has  $\tilde{f}_1 \in \tilde{\mathcal{F}}$  and  $\tilde{f}_2 \in \tilde{\mathcal{F}}$ ; see Remark 1.3(c). By the universal property of pullbacks, there is a morphism  $\alpha : X \rightarrow P$  such that  $\tilde{f}_1\alpha = \tilde{c}$  and  $\tilde{f}_2\alpha = h$ . Since  $(\tilde{\mathcal{C}}, \mathcal{F})$  is a weak factorization system, there is a morphism  $\tilde{c}' : X \rightarrow P'$  in  $\tilde{\mathcal{C}}$  and a morphism  $f' : P' \rightarrow P$  in  $\mathcal{F}$  such that  $\alpha = f'\tilde{c}'$ . Putting these pieces of information

together, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
 & & & & Z' & & \\
 & & & \nearrow \tilde{c} & \uparrow \tilde{f}_1 & \searrow \tilde{f} & \\
 X & \xrightarrow{\tilde{c}'} & P' & \xrightarrow{f'} & P & & Y \\
 & \searrow h & & & \downarrow \tilde{f}_2 & \nearrow f & \\
 & & & & Z & & 
 \end{array}$$

Since  $\tilde{f}_1 f' \tilde{c}' = \tilde{c}$  is contained in  $\tilde{\mathcal{C}}$ , one deduces that  $\tilde{f}_1 f'$  belongs to  $\tilde{\mathcal{F}}$  by the condition (CP3), so  $f' \in \tilde{\mathcal{F}}$  by the condition (CP2). Consequently,  $h = (\tilde{f}_2 f') \tilde{c}'$  is contained in  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .  $\square$

Recall from [9] that two complete cotorsion pairs  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  in  $\mathbf{A}$  are *compatible* if  $\tilde{\mathcal{C}} \subseteq \mathcal{C}$  (or equivalently  $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ ) and  $\mathcal{C} \cap \tilde{\mathcal{F}} = \tilde{\mathcal{C}} \cap \mathcal{F}$ . The next result shows the compatible condition on weak factorization systems generalizes the one for cotorsion pairs.

**1.14 Theorem.** *Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two pairs of classes of objects in  $\mathbf{A}$ . Then the following are equivalent.*

- (i)  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  are compatible and complete cotorsion pairs.
- (ii)  $(\text{Mon}(\mathcal{C}), \text{Epi}(\tilde{\mathcal{F}}))$  and  $(\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\mathcal{F}))$  are compatible weak factorization systems.

**Proof.** (i)  $\implies$  (ii). By Theorem 1.5,  $(\text{Mon}(\mathcal{C}), \text{Epi}(\tilde{\mathcal{F}}))$  and  $(\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\mathcal{F}))$  are two weak factorization systems. It remains to show that they are compatible. By [9, Theorem 1.1], there is a class  $\mathcal{W}$  of objects in  $\mathbf{A}$  satisfying the 2-out-of-3 property<sup>1</sup> such that  $\tilde{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$  and  $\tilde{\mathcal{F}} = \mathcal{F} \cap \mathcal{W}$ .

The condition (CP1) holds clearly. Now let  $\alpha : X \rightarrow Y$  and  $\beta : Y \rightarrow Z$  be two morphisms in  $\text{Epi}(\mathcal{F})$ . Then  $\beta\alpha$  is an epimorphism inducing the following commutative diagram of short exact sequence:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Ker}(\alpha) & \longrightarrow & X & \xrightarrow{\alpha} & Y \longrightarrow 0 \\
 & & \downarrow & & \parallel & & \downarrow \beta \\
 0 & \longrightarrow & \text{Ker}(\beta\alpha) & \longrightarrow & X & \xrightarrow{\beta\alpha} & Z \longrightarrow 0,
 \end{array}$$

which gives rise to another short exact

$$0 \rightarrow \text{Ker}(\alpha) \rightarrow \text{Ker}(\beta\alpha) \rightarrow \text{Ker}(\beta) \rightarrow 0.$$

Thus the condition (CP2) holds as  $\mathcal{W}$  satisfies the 2-out-of-3 property.

To prove the condition (CP3), let  $c : A \rightarrow B$  be in  $\text{Mon}(\tilde{\mathcal{C}})$  and  $f : B \rightarrow C$  be in  $\text{Epi}(\mathcal{F})$  such that  $g = fc$  is in  $\text{Mon}(\tilde{\mathcal{C}})$ . Then  $c$  and  $g$  are monomorphisms with  $\text{Coker}(c)$  and  $\text{Coker}(g)$  in  $\tilde{\mathcal{C}}$ , and  $f$  is an epimorphism with  $\text{Ker}(f)$  in  $\mathcal{F}$ . The

<sup>1</sup>That is, if  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is an exact sequence in  $\mathbf{A}$  then two of the three objects  $M'$ ,  $M$  and  $M''$  are in  $\mathcal{W}$  imply the third one is in  $\mathcal{W}$ .



following commutative diagram of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{c} & B & \longrightarrow & \text{Coker}(c) \longrightarrow 0 \\
 & & \parallel & & \downarrow f & & \downarrow \\
 0 & \longrightarrow & A & \xrightarrow{g} & C & \longrightarrow & \text{Coker}(g) \longrightarrow 0.
 \end{array}$$

then induces another short exact sequence

$$0 \rightarrow \text{Ker}(f) \rightarrow \text{Coker}(c) \rightarrow \text{Coker}(g) \rightarrow 0.$$

Since both  $\text{Coker}(c)$  and  $\text{Coker}(g)$  are in  $\tilde{\mathcal{C}} \subseteq \mathcal{W}$ , and the class  $\mathcal{W}$  satisfies the 2-out-of-3 property, it follows that  $\text{Ker}(f)$  belongs to  $\mathcal{W}$ . Thus  $\text{Ker}(f)$  is in  $\tilde{\mathcal{F}}$ .

(ii)  $\implies$  (i). By Theorem 1.5,  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  are two complete cotorsion pairs in  $\mathbf{E}$ . If  $X$  is contained in  $\tilde{\mathcal{C}}$ , then the morphism  $0 \rightarrow X$  is contained in  $\text{Mon}(\tilde{\mathcal{C}})$  and hence in  $\text{Mon}(\mathcal{C})$ , so  $X \in \mathcal{C}$  which yields that  $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ .

Now we show that  $\tilde{\mathcal{C}} \cap \mathcal{F} \subseteq \mathcal{C} \cap \tilde{\mathcal{F}}$ . Let  $M$  be an object in  $\tilde{\mathcal{C}} \cap \mathcal{F}$ . It is clear that  $0 \rightarrow M$  is in  $\text{Mon}(\tilde{\mathcal{C}})$  and  $M \rightarrow 0$  is in  $\text{Epi}(\mathcal{F})$ . Since  $(\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\mathcal{F}))$  and  $(\text{Mon}(\mathcal{C}), \text{Epi}(\tilde{\mathcal{F}}))$  are two compatible weak factorization systems in  $\mathbf{E}$ , it follows from the condition (CP3) that  $M \rightarrow 0$  is in  $\text{Epi}(\tilde{\mathcal{F}})$  as  $0 \rightarrow 0$  is in  $\text{Mon}(\tilde{\mathcal{C}})$ . Therefore,  $M$  is contained in  $\tilde{\mathcal{F}}$ . But  $M$  is also contained in  $\tilde{\mathcal{C}} \subseteq \mathcal{C}$ , so it belongs to  $\mathcal{C} \cap \tilde{\mathcal{F}}$ .

We next prove  $\mathcal{C} \cap \tilde{\mathcal{F}} \subseteq \tilde{\mathcal{C}} \cap \mathcal{F}$ . Let  $N$  be an object in  $\mathcal{C} \cap \tilde{\mathcal{F}}$ . Then  $0 \rightarrow N$  is contained in  $\text{Mon}(\mathcal{C})$  and  $N \rightarrow 0$  is contained in  $\text{Epi}(\tilde{\mathcal{F}})$ . It follows from Lemma 1.13 that the morphism  $0 \rightarrow N$  is in  $\mathcal{W}_{\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\tilde{\mathcal{F}})}$  since  $0 \rightarrow 0$  belongs to  $\mathcal{W}_{\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\tilde{\mathcal{F}})}$  and  $N \rightarrow 0$  belongs to  $\text{Epi}(\tilde{\mathcal{F}})$ . Consequently, the morphism  $0 \rightarrow N$  is contained in  $\mathcal{W}_{\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\tilde{\mathcal{F}})} \cap \text{Mon}(\mathcal{C}) = \text{Mon}(\tilde{\mathcal{C}})$  by Lemma 1.12, so  $N \in \tilde{\mathcal{C}}$ . On the other hand, because  $N \in \tilde{\mathcal{F}} = \mathcal{C}^\perp \subseteq \tilde{\mathcal{C}}^\perp = \mathcal{F}$ , one has  $N \in \tilde{\mathcal{C}} \cap \mathcal{F}$ . Therefore,  $\tilde{\mathcal{C}} \cap \mathcal{F} \subseteq \mathcal{C} \cap \tilde{\mathcal{F}}$ .

We have shown that  $\mathcal{C} \cap \tilde{\mathcal{F}} = \tilde{\mathcal{C}} \cap \mathcal{F}$ , so the cotorsion pairs  $(\tilde{\mathcal{C}}, \mathcal{F})$  and  $(\mathcal{C}, \tilde{\mathcal{F}})$  are compatible.  $\square$

Recall from [17] that a weak factorization system  $(\mathcal{C}, \mathcal{F})$  in  $\mathbf{E}$  satisfies the *Frobenius property* if  $\mathbf{E}$  has pullbacks along morphisms in  $\mathcal{C}$  or  $\mathcal{F}$ , and the morphisms in  $\mathcal{C}$  are preserved under pullback along morphisms in  $\mathcal{F}$ . Frobenius's name is invoked here, because there is a connection between the Frobenius property for a weak factorization system and Lawvere's Frobenius condition [13]; see Clementino, Giuli and Tholen [3] for an explanation.

We mention that the Frobenius property automatically holds for weak factorization systems induced by complete cotorsion pairs, that is, if  $(\mathcal{C}, \mathcal{F})$  is a complete cotorsion pair in  $\mathbf{A}$  then the weak factorization system  $(\text{Mon}(\mathcal{C}), \text{Epi}(\mathcal{F}))$  satisfies the Frobenius property. This fact is an immediate consequence of the next result.

**1.15 Proposition.** *Let  $\mathcal{C}$  be a class of objects in  $\mathbf{A}$ . One has:*

- (a) *the morphisms in  $\text{Mon}(\mathcal{C})$  are preserved under pullback along epimorphisms;*
- (b) *the morphisms in  $\text{Epi}(\mathcal{C})$  are preserved under pushout along monomorphisms.*

**Proof.** We only prove (a); the statement (b) is proved dually.

Take two morphisms  $\alpha : X \rightarrow Z$  and  $\beta : Y \rightarrow Z$  with  $\alpha \in \mathbf{Mon}(\mathbf{C})$  and  $\beta$  an epimorphism. We deduce from the following pullback diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{\alpha'} & Y & \longrightarrow & \mathrm{Coker}(\alpha) \longrightarrow 0 \\ & & \downarrow \ulcorner & & \downarrow \beta & & \parallel \\ 0 & \longrightarrow & X & \xrightarrow{\alpha} & Z & \longrightarrow & \mathrm{Coker}(\alpha) \longrightarrow 0. \end{array}$$

that  $\alpha'$  is contained in  $\mathbf{Mon}(\mathbf{C})$  as well, and the conclusion follows.  $\square$

## 2. FROM WEAK FACTORIZATION SYSTEMS TO MODEL STRUCTURES

In this section, we describe a method to construct model structures on general categories via two compatible weak factorization systems satisfying certain conditions and prove Theorem C advertised in Introduction, which generalizes a very useful result by Gillespie for abelian model structures.

Throughout this section, let  $\mathbf{E}$  denote a category. We begin with the following definition of model structures, which is a slight generalization of the usual one; see Gambino, Henry, Sattler and Szumilo [6].

**2.1 Definition.** A *model structure* on  $\mathbf{E}$  is a triple  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  of classes of morphisms satisfying:

- (a)  $\mathbf{E}$  has pushouts along morphisms in  $\mathcal{C}$  and pullbacks along morphisms in  $\mathcal{F}$ ;
- (b)  $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are weak factorization systems;
- (c)  $\mathcal{W}$  satisfies the 2-out-of-3 property, i.e., for composable morphisms  $\alpha$  and  $\beta$ , if two of the three morphisms  $\alpha$ ,  $\beta$  and  $\beta\alpha$  are in  $\mathcal{W}$ , then so is the third one.

Morphisms in  $\mathcal{C}$  (resp.,  $\mathcal{W}$ ,  $\mathcal{F}$ ) are called *cofibrations* (resp., *weak equivalences*, *fibrations*). Morphisms in  $\mathcal{C} \cap \mathcal{W}$  (resp.,  $\mathcal{F} \cap \mathcal{W}$ ) are called *trivial cofibrations* (resp., *trivial fibrations*). Recall that a model structure is *hereditary* if both the class of cofibrations and the class of trivial cofibrations satisfy the left cancellation property.

**2.2 Remark.** With respect to this definition, one can show that  $\mathcal{W}$  is closed under retracts; see the proof of [12, Proposition E.1.3]. One can also show that the model structure is completely determined by two of its three classes of morphisms. When  $\mathbf{E}$  is finitely complete and cocomplete, the above definition is equivalent to the classical one in the sense of Quillen [15].

Given a model structure  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  on  $\mathbf{E}$ , it is easy to check that  $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$  and  $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$  are two compatible weak factorization systems in  $\mathbf{E}$ . In the following, we prove a partial converse statement, that is, under some mild conditions two compatible weak factorization systems in  $\mathbf{E}$  induce a model structure on  $\mathbf{E}$ .

**2.3 Lemma.** Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two compatible weak factorization systems in  $\mathbf{E}$ . Suppose that  $(\tilde{\mathcal{C}}, \mathcal{F})$  satisfies the Frobenius property, and  $\mathcal{C}$  satisfies the left cancellation property. If  $\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\beta \in \tilde{\mathcal{C}}$ , then  $\beta\alpha$  (if it is defined) is in  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .

**Proof.** Write  $\alpha : X \rightarrow Y$  and  $\beta : Y \rightarrow Z$ . Since  $(\mathcal{C}, \tilde{\mathcal{F}})$  is a weak factorization system in  $\mathbf{E}$ , there exist a morphism  $c : X \rightarrow T$  in  $\mathcal{C}$  and a morphism  $\tilde{f} : T \rightarrow Z$  in  $\tilde{\mathcal{F}}$  such that  $\beta\alpha = \tilde{f}c$ . It remains to show that  $c$  is contained in  $\tilde{\mathcal{C}}$ .

Consider the pullback

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}'} & Y \\ \tilde{c} \downarrow & \ulcorner & \downarrow \beta \\ T & \xrightarrow{\tilde{f}} & Z. \end{array}$$

Then  $\tilde{f}'$  is contained in  $\tilde{\mathcal{F}}$  as so is  $\tilde{f}$ . It follows from the Frobenius property of the pair  $(\tilde{\mathcal{C}}, \tilde{\mathcal{F}})$  and Lemma 1.12 that  $\tilde{c}$  is in  $\tilde{\mathcal{C}}$ . By the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow c & \nearrow \tau & \downarrow \beta \\ & P & \\ \downarrow \tilde{c} & \nearrow \tilde{f}' & \\ T & \xrightarrow{\tilde{f}} & Z. \end{array}$$

as well as the universal property of pullbacks, there is a morphism  $\tau : X \rightarrow P$  such that  $\tilde{f}'\tau = \alpha$  and  $\tilde{c}\tau = c$ , so it suffices to show that  $\tau$  is contained in  $\tilde{\mathcal{C}}$ . Clearly,  $\tau$  belongs to  $\mathcal{C}$  as  $\mathcal{C}$  satisfies the left cancellation property. Moreover, since  $\tilde{f}'\tau = \alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , one has  $\tau \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  by Lemma 1.13, and hence  $\tau \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}} \cap \mathcal{C} = \tilde{\mathcal{C}}$  by Lemma 1.12.  $\square$

To prove that  $(\mathcal{C}, \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}, \mathcal{F})$  is a model structure on  $\mathbf{E}$ , it suffices to show that  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  satisfies the 2-out-of-3 property, which is established in the following three lemmas.

**2.4 Lemma.** *Under the same assumptions as specified in Lemma 2.3, if  $\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\beta \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , then  $\beta\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  as well.*

**Proof.** Write  $\alpha : X \rightarrow Y$  and  $\beta : Y \rightarrow Z$ . Then we can find morphisms  $\tilde{c} : X \rightarrow X'$  and  $\tilde{c}' : Y \rightarrow Y'$  in  $\tilde{\mathcal{C}}$  together with morphisms  $\tilde{f} : X' \rightarrow Y$  and  $\tilde{f}' : Y' \rightarrow Z$  in  $\tilde{\mathcal{F}}$  such that  $\alpha = \tilde{f}\tilde{c}$  and  $\beta = \tilde{f}'\tilde{c}'$ . By Lemma 2.3, one has  $\tilde{c}'\tilde{f} \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , so there are morphisms  $\tilde{c}'' : X' \rightarrow T$  in  $\tilde{\mathcal{C}}$  and  $\tilde{f}'' : T \rightarrow Y'$  in  $\tilde{\mathcal{F}}$  satisfying  $\tilde{c}'\tilde{f} = \tilde{f}''\tilde{c}''$ . Consequently,  $\beta\alpha = \tilde{f}'\tilde{f}''\tilde{c}''\tilde{c}$  is contained in  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .  $\square$

**2.5 Lemma.** *Under the same assumptions as specified in Lemma 2.3, if  $\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\beta\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , then  $\beta \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .*

**Proof.** Write  $\alpha : X \rightarrow Y$  and  $\beta : Y \rightarrow Z$ . Since  $(\tilde{\mathcal{C}}, \mathcal{F})$  is a weak factorization system in  $\mathbf{E}$ , there is a decomposition  $\beta = f'\tilde{c}'$  with  $\tilde{c}' : Y \rightarrow Y'$  in  $\tilde{\mathcal{C}}$  and  $f' : Y' \rightarrow Z$  in  $\mathcal{F}$ . It suffices to show that  $f'$  is contained in  $\tilde{\mathcal{F}}$ .

Since  $\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , one gets a decomposition  $\alpha = \tilde{f}\tilde{c}$  with  $\tilde{c} : X \rightarrow X'$  in  $\tilde{\mathcal{C}}$  and  $\tilde{f} : X' \rightarrow Y$  in  $\tilde{\mathcal{F}}$ . By Lemmas 1.12 and 2.4,  $\tilde{c}'\tilde{f}$  is in  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , so it also has a decomposition  $\tilde{c}'\tilde{f} = \tilde{f}''\tilde{c}''$  with  $\tilde{c}'' : X' \rightarrow T$  in  $\tilde{\mathcal{C}}$  and  $\tilde{f}'' : T \rightarrow Y'$  in  $\tilde{\mathcal{F}}$ . Consequently, one has

$$\beta\alpha = f'\tilde{c}'\tilde{f}\tilde{c} = f'\tilde{f}''\tilde{c}''\tilde{c}.$$

Since  $\beta\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\tilde{c}''\tilde{c} \in \tilde{\mathcal{C}}$ , one has  $f'\tilde{f}'' \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  by Lemma 1.13, so  $f'\tilde{f}''$  is in  $\tilde{\mathcal{F}}$  by Lemma 1.12. Condition (CP2) tells us that  $f' \in \tilde{\mathcal{F}}$  as desired.  $\square$

**2.6 Lemma.** *Under the same assumptions as specified in Lemma 2.3 and the extra condition that  $\tilde{\mathcal{C}}$  satisfies the left cancellation property, if  $\beta \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\beta\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ , then  $\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .*

**Proof.** The proof is very similar to that of the previous one, so we only give a sketch to illustrate the application of the additional condition that  $\tilde{\mathcal{C}}$  satisfies the left cancellation property. Write  $\alpha : X \rightarrow Y$  and  $\beta : Y \rightarrow Z$ . Decompose  $\alpha = \tilde{f}c$  with  $c : X \rightarrow X'$  in  $\mathcal{C}$  and  $\tilde{f} : X' \rightarrow Y$  in  $\tilde{\mathcal{F}}$ . It remains to show that  $c \in \tilde{\mathcal{C}}$ . As we did in the previous proof, there is a decomposition  $\beta\alpha = \tilde{f}'\tilde{f}''\tilde{c}''c$ . Since  $\beta\alpha \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\tilde{f}'\tilde{f}'' \in \tilde{\mathcal{F}}$ , one has  $\tilde{c}''c \in \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  by Lemma 1.13, so  $\tilde{c}''c \in \tilde{\mathcal{C}}$  by Lemma 1.12. Now the left cancellation property of  $\tilde{\mathcal{C}}$  tells us that  $c \in \tilde{\mathcal{C}}$ .  $\square$

We are now in a position to give the main result in this section, which is a generalization of Gillespie theorem ([9, Theorem 1.1]).

**2.7 Theorem.** *Let  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  be two compatible weak factorization systems in  $\mathbf{E}$  satisfying the following properties:*

- (1)  *$\mathbf{E}$  has pushouts along morphisms in  $\mathcal{C}$  and pullbacks along morphisms in  $\mathcal{F}$ ;*
- (2)  *$(\tilde{\mathcal{C}}, \mathcal{F})$  satisfies the Frobenius property;*
- (3) *both  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$  satisfy the left cancellation property.*

*Then  $(\mathcal{C}, \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}, \mathcal{F})$  forms a hereditary model structure on  $\mathbf{E}$ , where*

$$\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}} = \{\alpha \mid \alpha \text{ can be decomposed as } \alpha = \tilde{f}\tilde{c} \text{ with } \tilde{c} \in \tilde{\mathcal{C}} \text{ and } \tilde{f} \in \tilde{\mathcal{F}}\}.$$

**Proof.** By Lemma 1.12,  $(\mathcal{C}, \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}} \cap \mathcal{F})$  and  $(\mathcal{C} \cap \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}, \mathcal{F})$  are weak factorization systems in  $\mathbf{E}$ . Moreover,  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  satisfies the 2-out-of-3 property by Lemmas 2.4, 2.5 and 2.6. Thus  $(\mathcal{C}, \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}, \mathcal{F})$  forms a model structure in  $\mathbf{E}$ , and furthermore it is hereditary by the assumption.  $\square$

**2.8 Remark.** In Theorem 2.7, the class  $\mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  in the model structure  $(\mathcal{C}, \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}, \mathcal{F})$  is unique. Indeed, if  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  is another model structure, then  $\mathcal{C} \cap \mathcal{W} = \mathcal{C} \cap \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$  and  $\mathcal{W} \cap \mathcal{F} = \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}} \cap \mathcal{F}$ . It is easy to check that  $\mathcal{W} = \mathcal{W}_{\tilde{\mathcal{C}}, \tilde{\mathcal{F}}}$ .

### 3. EXAMPLES OF COMPATIBLE WEAK FACTORIZATION SYSTEMS

It follows from Theorems 1.10 and 1.14 and Proposition 1.15 that if  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  are compatible complete hereditary cotorsion pairs in an abelian category  $\mathbf{A}$  then the induced weak factorization systems  $(\text{Mon}(\mathcal{C}), \text{Epi}(\tilde{\mathcal{F}}))$  and  $(\text{Mon}(\tilde{\mathcal{C}}), \text{Epi}(\mathcal{F}))$  are compatible satisfying all conditions specified in Theorem 2.7. In this section we list some examples of compatible weak factorization systems associated to non-abelian model structures: the classical and constructive Kan-Quillen model structures on the category  $\mathbf{sSet}$  of simplicial sets and the standard projective model structure on the category  $\mathbf{Ch}_{\geq 0}(R)$  of nonnegative chain complexes of modules over a ring  $R$ , and show that the weak factorization systems associated to these model structures satisfy all conditions specified in Theorem 2.7.

**3.1. Classical Kan-Quillen model structure.** Throughout this subsection we let  $\mathbf{sSet}$  denote the category of simplicial sets, defined as usual to be the category of presheaves over the simplex category  $\Delta$ . Let  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  be the Kan-Quillen model structure on  $\mathbf{sSet}$ ; see [15]. Explicitly,

- *cofibrations* (morphisms in  $\mathcal{C}$ ) are monomorphisms, i.e., morphisms  $f : X \rightarrow Y$  in  $\mathbf{sSet}$  such that  $f_k : X_k \rightarrow Y_k$  is an injection of sets for each  $k \in \mathbb{N}$ ;
- *weak equivalences* (morphisms in  $\mathcal{W}$ ) are morphisms  $f : X \rightarrow Y$  in  $\mathbf{sSet}$  whose geometric realization  $|f|$  is a weak homotopy equivalence of topological spaces;
- *fibrations* (morphisms in  $\mathcal{F}$ ) are the Kan fibrations, i.e., morphisms  $f : X \rightarrow Y$  in  $\mathbf{sSet}$  that have the right lifting property with respect to all horn inclusions.

In the following we let  $\tilde{\mathcal{F}} = \mathcal{F} \cap \mathcal{W}$  and  $\tilde{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$ .

**3.1 Proposition.** *The weak factorization systems  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  satisfy all conditions specified in Theorem 2.7.*

**Proof.** The compatible condition is clear since the condition (CP1) holds obviously, and the conditions (CP2) and (CP3) follows from the 2-out-of-3 property of  $\mathcal{W}$ . The condition (1) in Theorem 2.7 automatically holds since  $\mathbf{sSet}$  is bicomplete. It is clear that  $\mathcal{C}$  and  $\tilde{\mathcal{C}}$  satisfy the left cancellation property. Finally, it follows from Gambino and Sattler [7, Theorem 4.8] that the weak factorization system  $(\tilde{\mathcal{C}}, \mathcal{F})$  satisfies the Frobenius property.  $\square$

**3.2. Constructive Kan-Quillen model structure.** The original proofs of the existence of the classical Kan-Quillen model structure on  $\mathbf{sSet}$  use the law of excluded middle (EM) and the axiom of choice (AC), which are not valid in constructive mathematics. Recently, a constructively valid model structure on  $\mathbf{sSet}$  was given by Henry [10] and Gambino, Sattler and Szumilo [8], which coincides with the classical Kan-Quillen model structure once (EM) and (AC) are assumed.

For the convenience of the reader we include some details on the constructive Kan-Quillen model structure on  $\mathbf{sSet}$ ; for more details, please refer to [8]. A map  $i : A \rightarrow B$  of sets is called a *decidable inclusion* if there is a map  $j : C \rightarrow B$  such that  $i$  and  $j$  exhibit  $B$  as a coproduct of  $A$  and  $C$ , that is, the diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & C \\ \downarrow & & \downarrow j \\ A & \xrightarrow{i} & B. \end{array}$$

is a pushout and  $B \cong A \sqcup C$ . The following lemma asserts that the class of decidable inclusions satisfies the left cancellation property, which is trivially true while assuming (EM) as in this case decidable inclusions are precisely injections.

**3.2 Lemma.** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be two maps of sets, and suppose that both  $\beta$  and  $\beta\alpha$  are decidable inclusions. Then  $\alpha$  is also a decidable inclusion.*

*Proof.* By definition, we have two pushout diagrams which are also pullback diagrams by [8, Lemma 2.1.1]:

$$\begin{array}{ccc} \emptyset & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\beta\alpha} & C \cong A \sqcup A' \end{array} \quad \begin{array}{ccc} \emptyset & \longrightarrow & B' \\ \downarrow & & \downarrow \beta' \\ B & \xrightarrow{\beta} & C \cong B \sqcup B'. \end{array}$$

Note that the map  $\beta' : B' \rightarrow C$  is also decidable inclusion.

Now consider the intersection  $A \cap B'$ , which is given by the pullback

$$\begin{array}{ccc} A \cap B' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\beta\alpha} & C \end{array}$$

We then have a commutative diagram where the right square is a pullback

$$\begin{array}{ccccc} & & A \cap B' & & \\ & \swarrow & \vdots & \searrow & \\ A & & \emptyset & & B' \\ & \searrow \alpha & \downarrow & \xrightarrow{\beta} & \downarrow \beta' \\ & & B & & C \end{array}$$

Consequently,  $A \cap B' = \emptyset$  as there is a map from the intersection to the empty set. Thus the union  $A \cup B'$ , given by the following pushout, is actually a coproduct

$$\begin{array}{ccc} A \cap B' = \emptyset & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \cup B' \cong A \sqcup B'. \end{array}$$

By [8, Lemma 2.1.7], the map  $A \sqcup B' \rightarrow C$  is again a decidable inclusion. Consider the following diagram

$$\begin{array}{ccc} A & \longrightarrow & A \sqcup B' \\ \downarrow \alpha & & \downarrow \\ B & \longrightarrow & C \cong B \sqcup B' \end{array}$$

which is clearly a pullback. Therefore, it follows from [8, Lemma 2.1.4] that  $\alpha$  is a decidable inclusion.  $\square$

Let  $I$  (resp.,  $J$ ) be the class of boundary inclusions (resp., horn inclusions) in  $\mathbf{sSet}$ . A morphism in  $\mathbf{sSet}$  is a *trivial fibration* (resp., *Kan fibration*) if it has the right lifting property with respect to  $I$  (resp.,  $J$ ). A morphism in  $\mathbf{sSet}$  is a *trivial cofibration* (resp., *cofibration*) if it has the left lifting property with respect to Kan fibrations (resp., trivial fibrations). A simplicial set  $X$  is called *cofibrant* if the morphism  $\emptyset \rightarrow X$  is a cofibration. Let  $\mathbf{sSet}_{\text{cof}}$  denote the full subcategory of  $\mathbf{sSet}$  consisting of cofibrant simplicial sets, and set

- $\mathcal{C}_{\text{cof}}$ : the class of morphisms in  $\mathbf{sSet}_{\text{cof}}$  that are cofibrations;
- $\tilde{\mathcal{C}}_{\text{cof}}$ : the class of morphisms in  $\mathbf{sSet}_{\text{cof}}$  that are trivial cofibrations;

- $\mathcal{F}_{\text{cof}}$ : the class of morphisms in  $\mathbf{sSet}_{\text{cof}}$  that are Kan fibrations;
- $\tilde{\mathcal{F}}_{\text{cof}}$ : the class of morphisms in  $\mathbf{sSet}_{\text{cof}}$  that are trivial fibrations;
- $\mathcal{W}_{\text{cof}}$ : the class of morphisms in  $\mathbf{sSet}_{\text{cof}}$  that are weak homotopy equivalences in the sense of [8, 3.1].

It follows from [8, Proposition 2.2.7] that the pairs  $(\mathcal{C}_{\text{cof}}, \tilde{\mathcal{F}}_{\text{cof}})$  and  $(\tilde{\mathcal{C}}_{\text{cof}}, \mathcal{F}_{\text{cof}})$  are weak factorization systems in  $\mathbf{sSet}_{\text{cof}}$ , and furthermore  $\tilde{\mathcal{F}}_{\text{cof}} = \mathcal{F}_{\text{cof}} \cap \mathcal{W}_{\text{cof}}$  and  $\tilde{\mathcal{C}}_{\text{cof}} = \mathcal{C}_{\text{cof}} \cap \mathcal{W}_{\text{cof}}$  by [8, Propositions 3.6.3 and 3.6.4]. We have the next result.

**3.3 Proposition.** *The weak factorization systems  $(\mathcal{C}_{\text{cof}}, \tilde{\mathcal{F}}_{\text{cof}})$  and  $(\tilde{\mathcal{C}}_{\text{cof}}, \mathcal{F}_{\text{cof}})$  satisfy all conditions specified in Theorem 2.7.*

**Proof.** The compatible condition is clear since the condition (CP1) holds obviously, and the conditions (CP2) and (CP3) follows from the 2-out-of-3 property of  $\mathcal{W}_{\text{cof}}$ ; see [8, Lemma 3.1.5]. The condition (1) in Theorem 2.7 automatically holds.

Now we check the left cancellation property. Note that if  $\mathcal{C}_{\text{cof}}$  has this property, then so does  $\tilde{\mathcal{C}}_{\text{cof}}$ . Therefore, we only need to show that  $\mathcal{C}_{\text{cof}}$  has left cancellation property. Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be two morphisms in  $\mathbf{sSet}_{\text{cof}}$  such that both  $g$  and  $gf$  are cofibrations. By [8, Corollary 2.4.6], for each  $k \in \mathbb{N}$ ,  $g_k : Y_k \rightarrow Z_k$  and  $(gf)_k = g_k f_k : X_k \rightarrow Z_k$  are decidable inclusions, so it follows from Lemma 3.2 that each  $f_k : X_k \rightarrow Y_k$  is also a decidable inclusion. Applying [8, Corollary 2.4.6] again, we conclude that  $f : X \rightarrow Y$  is a cofibration.

Finally, it follows from [8, Proposition 4.1.6] that the weak factorization system  $(\tilde{\mathcal{C}}_{\text{cof}}, \mathcal{F}_{\text{cof}})$  satisfies the Frobenius property.  $\square$

**3.3. Projective model structure.** Throughout this subsection all  $R$ -modules are left modules. An object  $X = \{X_k\}_{k \geq 0}$  in  $\mathbf{Ch}_{\geq 0}(R)$  is called *acyclic* if the homology groups  $H_k(X)$  vanish for  $k \geq 0$ . Let  $(\mathcal{C}, \mathcal{W}, \mathcal{F})$  be the standard projective model structure on  $\mathbf{Ch}_{\geq 0}(R)$ ; see Dwyer and Spaliński [4]. Explicitly,

- *cofibrations* (morphisms in  $\mathcal{C}$ ) are morphisms  $f : X \rightarrow Y$  in  $\mathbf{Ch}_{\geq 0}(R)$  such that  $f_k : X_k \rightarrow Y_k$  is a monomorphism whose cokernel is a projective  $R$ -module for  $k \geq 0$ ;
- *weak equivalences* (morphisms in  $\mathcal{W}$ ) are morphisms  $f : X \rightarrow Y$  which induce isomorphisms  $H_k(X) \cong H_k(Y)$  for  $k \geq 0$ ;
- *fibrations* (morphisms in  $\mathcal{F}$ ) are morphisms  $f$  such that all  $f_k : X_k \rightarrow Y_k$  are epimorphisms for  $k > 0$ .

Although  $\mathbf{Ch}_{\geq 0}(R)$  is a bicomplete abelian category, this model structure is not abelian since fibrations are required to be epic only in positive degrees.

In the following we let  $\tilde{\mathcal{F}} = \mathcal{F} \cap \mathcal{W}$  and  $\tilde{\mathcal{C}} = \mathcal{C} \cap \mathcal{W}$ .

**3.4 Proposition.** *The weak factorization systems  $(\mathcal{C}, \tilde{\mathcal{F}})$  and  $(\tilde{\mathcal{C}}, \mathcal{F})$  satisfy all conditions specified in Theorem 2.7.*

**Proof.** The compatible condition is clear since the condition (CP1) holds obviously, and the conditions (CP2) and (CP3) follows from the 2-out-of-3 property of  $\mathcal{W}$ . The condition (1) in Theorem 2.7 automatically holds since  $\mathbf{Ch}_{\geq 0}(R)$  is bicomplete.

Now we check the left cancellation property. Note that if  $\mathcal{C}$  has this property, then so does  $\tilde{\mathcal{C}}$ . Therefore, we only need to show that  $\mathcal{C}$  has left cancellation property. Given morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  such that  $gf \in \mathcal{C}$  and  $g \in \mathcal{C}$ , we want to show that  $f$  belongs to  $\mathcal{C}$  as well, that is, for each  $k \geq 0$ ,  $f_k : X_k \rightarrow Y_k$

is a monomorphism (which holds trivially since  $g_k f_k = (gf)_k$  is a monomorphism) such that its cokernel is a projective  $R$ -module. This is also clear. Indeed, applying the Snake Lemma to the commutative diagram of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_k & \xlongequal{\quad} & X_k & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow f_k & & \downarrow g_k f_k & & \downarrow \\ 0 & \longrightarrow & Y_k & \xrightarrow{g_k} & Z_k & \longrightarrow & \text{Coker}(g_k) \longrightarrow 0 \end{array}$$

we obtain a short exact sequence

$$0 \rightarrow \text{Coker}(f_k) \rightarrow \text{Coker}(g_k f_k) \rightarrow \text{Coker}(g_k) \rightarrow 0$$

of  $R$ -modules. Since both  $\text{Coker}(g_k f_k)$  and  $\text{Coker}(g_k)$  are projective, so is  $\text{Coker}(f_k)$ .

Finally we verify the Frobenius property of  $(\tilde{\mathcal{C}}, \mathcal{F})$ . Given a pullback diagram in  $\text{Ch}_{\geq 0}(R)$  with  $f \in \tilde{\mathcal{C}}$  and  $g \in \mathcal{F}$ :

$$\begin{array}{ccc} M & \xrightarrow{p} & N \\ \downarrow \ulcorner & & \downarrow g \\ X & \xrightarrow{f} & Y, \end{array}$$

We want to show that  $p$  belongs to  $\tilde{\mathcal{C}}$ . Since  $p$  is monic as so is  $f$ , it remains to show that  $\text{Coker}(p)$  is acyclic and each  $\text{Coker}(p_k)$  is a projective  $R$ -module for  $k \geq 0$ . We do this by proving that the morphism  $r$  in the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{p} & N & \longrightarrow & \text{Coker}(p) \longrightarrow 0 \\ & & \downarrow & & \downarrow g & & \downarrow r \\ 0 & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & \text{Coker}(f) \longrightarrow 0 \end{array}$$

of short exact sequences is an isomorphism. Indeed, in this case  $\text{Coker}(p) \cong \text{Coker}(f)$  and  $\text{Coker}(f)$  has the desired property since  $f$  is contained in  $\tilde{\mathcal{C}}$ .

For  $k \geq 1$ , the left square in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_k & \xrightarrow{p_k} & N_k & \longrightarrow & \text{Coker}(p_k) \longrightarrow 0 \\ & & \downarrow & & \downarrow g_k & & \downarrow r_k \\ 0 & \longrightarrow & X_k & \xrightarrow{f_k} & Y_k & \longrightarrow & \text{Coker}(f_k) \longrightarrow 0 \end{array}$$



of short exact sequences is both a pullback and a pushout since  $g_k$  is an epimorphism. Thus  $r_k$  is an isomorphism for  $k \geq 1$ . When  $k = 0$ , consider the commutative diagram of  $R$ -modules with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 & \xrightarrow{p_1} & N_1 & \longrightarrow & \text{Coker}(p_1) & \longrightarrow & 0 \\
& & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
& 0 & \longrightarrow & M_0 & \xrightarrow{p_0} & N_0 & \longrightarrow & \text{Coker}(p_0) & \longrightarrow 0 \\
& & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
0 & \longrightarrow & X_1 & \xrightarrow{f_1} & Y_1 & \longrightarrow & \text{Coker}(f_1) & \longrightarrow & 0 \\
& & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
& 0 & \longrightarrow & X_0 & \xrightarrow{f_0} & Y_0 & \longrightarrow & \text{Coker}(f_0) & \longrightarrow 0.
\end{array}$$

Since  $\text{Coker}(f)$  is acyclic, the morphism  $\text{Coker}(f_1) \rightarrow \text{Coker}(f_0)$  is an epimorphism, so  $r_0$  is also epic as  $r_1$  is an isomorphism. On the other hand, since

$$\begin{array}{ccc}
M_0 & \xrightarrow{p_0} & N_0 \\
\downarrow & & \downarrow g_0 \\
X_0 & \xrightarrow{f_0} & Y_0
\end{array}$$

is a pullback diagram,  $r_0$  is a monomorphism as well, and hence an isomorphism. Thus  $r$  is indeed an isomorphism, and our proof is complete.  $\square$

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