Spectrum of Hybrid Charmonium, Bottomonium and B_c Mesons by Power Series Method

Nosheen Akbar^{*}, Zeeshan Ali[‡], Saadia Arshad[‡]

* Department of Physics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan.

‡ Department of Mathematics, COMSATS University Islamabad, Lahore Campus, Lahore, Pakistan.

Abstract

Power series method (PSM) is revisited to find the masses of S, P, and D states of conventional charmonium $(c\overline{c})$, bottomonium $(b\overline{b})$, and B_c $(\overline{b}c)$ mesons by assuming the solution of N-dimensional radial Schrodinger equation in series form. An extension in the potential model is proposed by fitting it with lattice data to study the hybrid mesons. The proposed potential model is used to find the masses of hybrid $c\overline{c}$, $b\overline{b}$, and $\overline{b}c$ mesons by applying the power series method. Calculated results are compared with theoretical findings and available experimental data. Our results can be helpful for the investigation of newly experimentally discovered charmonium, bottomonium, and B_c states.

1. Introduction

Mesons are hadronic particles made up of a quark and an antiquark pair bounded by a strong gluonic field. When this gluonic field comes in the excited state, mesons are named as "Hybrid Mesons". Many states of conventional and hybrid mesons have been observed in the experiments. The study of these experimentally discovered states of mesons becomes more interesting for physicists. In the investigation of the conventional and hybrid mesons, Schrödinger equation (SE) plays the most important role. Schrödinger equation is a differential equation which is mostly used to investigate the atomic structure of matter.

To study mesons, SE is solved by different methods like Exact-Analytical Iteration Method [1], Numerov's Discretization Method [2], Fourier grid Hamiltonian method [2], Nikiforov-Uvarov

^{*}e mail: nosheenakbar@cuilahore.edu.pk

method [3, 4, 5], Exact Quantization Rule technique [6], Laplace Transform method [7, 8], series expansion method [9]. In this work, SE is solved by the power series method [10] to find the masses of conventional and hybrid mesons for different J^{PC} states. This method has the advantage that it can be applied to the equations containing the arbitrary coefficients and boundary conditions.

To study the mesons, different potential models are used. Yukawa potential is adopted in ref. [9] to obtain the approximation solution of SE for energy eigenvalues using the series expansion method. Trigonometric Rosen-Morse Potential (TRMP) is employed in ref. [10] to calculate the mass spectra(MS) of heavy and heavy-light meson in the framework of N-radial fractional Schrödinger equation (SE) by using the generalized fractional extended Nikiforov–Uvarov method (GFD-ENU). Cornell plus inverse quadratic potential is used in ref. [11] to find the energy spectrum of $c\bar{c}$ and $b\bar{b}$ at finite and zero temperatures. In ref. [10], Cornell plus quadratic plus inverse quadratic potential is used to find the mass of bottomonium, charmonium, and B_c mesons by applying the power series method. In present work, we revisit the work of ref. [10] and then extend this work to find the spectrum of hybrid bottomonium, charmonium and B_c mesons by modifying the potential with the addition of a potential term ($A(1 - Br^2)$) whose parameters are determined by fitting this term with the lattice data [12]. Earlier, this additional term is suggested in the Gaussian form in ref. [13], but extension in gaussian form is not suitable for the power series method.

Hybrid mesons have been studied by different methods like the bag model [14], the Lattice QCD [15, 16, 17, 18], the flux tube model [19, 20, 21], and QCD sum rules [22, 23, 24, 25, 26]. We compare our calculated results with available theoretical and experimental results.

The paper is organized as follows: In section 2, the N dimensional radial Schrodinger equation is described for mesons along with the series solution. Potential models for conventional and hybrid mesons are defined. Energy expressions derived for conventional as well as hybrid mesons are also described in this section. Section 3 describes the method for finding the values of parameters of the potential models (for conventional and hybrid mesons) for the calculation of masses of charmonium, bottomonium, and B_c mesons. In section 4, the results are discussed in detail.

2. Solution of Schrodinger Equation and Energy of Mesons by Power Series Method

N-dimensional radial SE for two interacting particles can be written as

$$\left[\frac{d^2}{dr^2} + \frac{N-1}{r}\frac{d}{dr} - \frac{L(L+N-2)}{r^2} + 2\mu(E-V(r))\right]R(r) = 0,$$
(1)

where L is the angular momentum quantum number, N denotes dimension, and μ is the reduced mass of the two interacting particles. In this work, these two interacting particles are quark and antiquark, so reduced mass can be defined as $\mu = \frac{m_q m_{\overline{q}}}{m_q + m_{\overline{q}}}$ with m_q and $m_{\overline{q}}$ are the quark and anti-quark masses respectively. E is the energy eigenvalues corresponding to the radial eigenfunctions R(r) and it is approximated in ref. [27, 28] as:

$$R(r) = e^{(-\alpha r^2 - \beta r)} F(r), \qquad (2)$$

where F(r) is considered in the series form as

$$F(r) = \sum_{n=0}^{\infty} a_n r^{\frac{3n}{2} + L}.$$
 (3)

 a_n is the expansion coefficient. Values of parameters, α , β depend on the potential model (V(r)). Details of the potential V(r) are discussed in next sections 2.1. For the study of hybrid mesons, the potential model used for conventional mesons is extended (discussed below in section 2.2. in detail). Due to the change in the potential model, values of parameters of the radial wave function for hybrids are changed and can be written as:

$$R(r) = e^{(-\alpha' r^2 - \beta' r)} F(r).$$

$$\tag{4}$$

 $\alpha, \beta, \alpha', \beta'$ are defined below in section 2.1 and 2.2.

2.1. Potential model for conventional meson

In eq.(1), V(r) represents the potential of interacting particles. For conventional mesons, the potential is modelled [10, 28] as:

$$V(r) = ar^{2} + br - \frac{c}{r} + \frac{d}{r^{2}}.$$
(5)

To find the energy of mesons, we applied the power series method as used in ref.[10]. From eq.(1), eq.(3)-eq.(5), we obtained the following expressions similar to ref. [28].

$$a = \frac{2\alpha^2}{\mu}, \quad \Rightarrow \quad \alpha = \sqrt{\frac{a\mu}{2}},$$
 (6)

$$b = \frac{2\alpha\beta}{\mu}, \quad \Rightarrow \quad \beta = \frac{b\mu}{2\alpha},$$
(7)

$$c = \frac{-\beta(3n+2L+2) - (N-1)\beta}{2\mu}, \quad \Rightarrow \quad \beta = \frac{-2\mu c}{3n+2L+N+1},$$
(8)

$$d = \frac{L(L+N+2) - 2(N-1)(3n+2L+2) - 4L(L+N+2)}{8\mu}$$
(9)

and

$$E = \frac{\alpha(3n + 2L + N)}{\mu} - \frac{\beta^2}{2\mu}$$
(10)

Eq.(10) can also be written as

$$E = \frac{a}{2\mu}(3n + 2L + N) - \frac{b^2}{4a}$$
(11)

2.2. Potential model for hybrid meson

To study the hybrid mesons, above mentioned potential model (defined in eq.5) is modified as:

$$V(r) = ar^{2} + br - \frac{c}{r} + \frac{d}{r^{2}} + A(1 - Br^{2}).$$
(12)

Here, the additional term $(A(1-Br^2))$ is suggested for the potential energy difference between conventional and hybrid mesons. The parameters A and B are found by fitting $(A(1-Br^2))$ with the lattice simulation data obtained from Fig. 3 of ref.[12] for the potential energy differences (v_d) between the ground and excited states. The best fit is obtained for A = 1.40498GeV and $B = 0.016824GeV^3$. χ^2 is calculated by using the following formula:

$$\chi^2 = \frac{\sum_{i=1}^{n} [\upsilon_d(r_i) - A(1 - Br_i^2)]}{\sum_{i=1}^{n} (\upsilon_d(r_i))^2},$$
(13)

In the flux tube model [19], $\frac{\pi}{r}$ is used for the excited part of the quark-antiquark potential. Few models in the Gaussian form for the potential energy differences between ground and excited states are suggested in ref. [13]. A comparison of χ^2 for these models with our newly used potential model is reported in Table 1. It is observed that our newly suggested potential model has χ^2 less than π/r .

With the modified potential model for hybrid mesons defined in eq.(12), SE can be written

		Parameters			
$ans \ddot{a}tz$	χ^2	A	В	c	
		GeV			
$A(1 - Br^2)$	0.1459	1.40498	$0.016824 \ GeV^2$	-	
π/r	0.2305	-	-	-	
$A \times \exp(-Br^2)$	0.0857	1.8139	$0.0657 \ GeV^2$	-	
$\frac{c}{r} + A \times \exp(-Br)$	0.0012	1.2448	$0.1771~{\rm GeV}$	0.3583	

Table 1: χ^2 for the lattice data(for potential difference) with models $(V_g(r))$ with best fit parameter's values.

as:

$$\begin{split} \Sigma_{n=0}^{\infty} a_n \bigg[\bigg(\beta'^2 + 4\alpha'^2 r^2 + 4\alpha'\beta' r - 2\alpha' + \frac{N-1}{r} (-2\alpha' r - \beta') - \frac{L(L+N-2)}{r^2} + 2\mu E - 2\mu A (1-Br^2) \bigg) r^{\frac{3n}{2}+L} \\ + \bigg((-2\beta - 4\alpha r) (\frac{3n+2L}{2}) + \frac{N-1}{r} (\frac{3n+2L}{2}) + 2\mu c \bigg) r^{\frac{3n}{2}+L-1} + \bigg(\frac{(3n+2L-2)(3n+2L)}{4} - 2\mu d \bigg) r^{\frac{3n}{2}+L-2} \\ - 2\mu a r^{\frac{3n}{2}+L+2} - 2\mu b r^{\frac{3n}{2}+L+1} \bigg] = 0 \quad (14) \end{split}$$

equating each coefficient of r to zero, we obtained the following relations:

$$E = A + \frac{\alpha'}{\mu} (3n + 2L + N) - \frac{{\beta'}^2}{2\mu}$$
(15)

$$\alpha' = \sqrt{\frac{\mu(a - AB)}{2}},\tag{16}$$

$$\beta' = \frac{\mu b}{2\alpha'}, \qquad \beta' = \frac{-2\mu c}{3n + 2L + N + 1}$$
 (17)

3. Parameters of Potential Model and Mass spectra of mesons

3.1. Conventional mesons

Mass of the mesons can be calculated by adding the constituent quark masses in the energy (E) defined in eq.(11), i.e;

$$M = m_q + m_{\overline{q}} + \frac{a}{2\mu}(3n + 2L + N) - \frac{b^2}{4a}.$$
 (18)

This equation helps in finding the parameters $(\alpha, \beta, a, b, c, d)$ by taking M equal to the experimental mass for a particular meson $(c\overline{c}, b\overline{b} \text{ and } b\overline{c})$.

To find the potential parameters a and b, charm and bottom quark mass is taken equal to 1.48 GeV and 4.75 GeV respectively. In case of charmonium, two algebraic equations are obtained by substituting the experimental values of 1S and 2D mass in eq.(18). These two equations are solved for finding the value of parameter a and b. Substitution of these values of a and b in eqs.(6,7), α and β are calculated. With this calculated value of β , eq.(8) gives the value of parameter c.

For bottomonium, same steps are repeated to find a, b, α, β and c by inserting experimental values of M for 1S and 3P in eq.(18). For B_c , two algebraic equations are obtained by inserting experimental values of M for 1S and 2S in eq.(18).

Substituting the values of parameters for each sector in eq.(18), we find the mass of different states of mesons by varying quantum numbers (n, L) and taking N=3.

3.2. Hybrid mesons

Mass of hybrid mesons is calculated by the following relation:

$$M = m_q + m_{\overline{q}} + A + \frac{\alpha'}{\mu} (3n + 2L + N) - \frac{\beta'^2}{2\mu}.$$
 (19)

The parameters α' and β' are obtained from eqs.(16,17) where a, b, and c are the parameters of the conventional meson potential model. By taking N=3, and using the calculated values of all parameters, masses are calculated for different excited states of hybrid meson by varying n, L.

4. Results and Discussion

In the present work, we derive the expressions for the energy of conventional and hybrid mesons by solving the N-dimensional radical Schrödinger equation using the power series technique. By using these energy expressions, masses of conventional and hybrid charmonium, bottomonium, and B_c mesons are calculated for ground, radial, and orbital excited states. Our calculated masses of conventional $c\bar{c}$, $b\bar{b}$, and $\bar{b}c$ mesons are reported in Tables 2-4 along with the experimental and other theoretical calculated masses.

Masses of hybrid $c\bar{c}$, $b\bar{b}$, and $\bar{b}c$ mesons are reported in Tables 5,6 along with the masses calculated by others by different methods. By observing the results reported in Tables 2-6, it is concluded that mass is increasing toward higher states. It is also observed that hybrid mesons are heavier than the normal mesons for the same quantum numbers (n,L). Masses of hybrid charmonium for lowest and first excited J^{PC} states are compared with others theoretical work in Tables 8,9. Comparison of hybrid bottomonium and B_c for the lowest J^{PC} states is given in Tables 10,11. Here, $J = L \oplus S$, P is the parity and C is the charge quantum numbers defined as $P = (-1)^{L+1}$ and $C = (-1)^{L+S}$. L and S are the total angular momentum and the spin angular momentum quantum numbers for meson. For hybrid mesons P and C are defined as

States	This work	[28]	[13]	[29]	$\operatorname{Exp}[30]$
1S	3.0974	3.068	3.09	3.09	3.0969 ± 0.000006
2S	3.5528	3.663	3.6718	3.672	3.6861 ± 0.000025
3S	4.0081	4.258	4.0716	4.072	-
4S	4.4635	4.852	4.406	4.406	-
5S	4.9189	-	4.7038	-	-
6S	4.9189	-	4.9769	-	-
1P	3.401	3.464	3.4245	3.424	3.41475 ± 0.00031
2P	3.8564	4.059	3.8523	3.852	-
3P	4.3117	-	4.2017	4.202	-
4P	4.7671	-	4.5092	-	-
5P	5.2225	-	4.7894	-	-
1D	3.7046	3.861	3.7850	3.785	3.77313 ± 0.00035
2D	4.1599	-	4.1415	4.142	4.159 ± 0.00021
3D	4.6153	-	4.4547	-	-
4D	5.0707	-	4.7395	-	-

Table 2: Mass spectra of $c\overline{c}$ in GeV with a=0.0341 GeV³ and b = 0.20826 GeV²

Table 3: Mass spectra of $b\overline{b}$ in GeV with a=0.0936GeV³ and b= 0.4151GeV²

States	This work	[28]	[32]	[31]	$\operatorname{Exp}[30]$
1S	9.4609	9.46	9.5299	9.459	9.4603 ± 0.00026
2S	9.8820	10.023	10.010	10.004	10.023 ± 0.00031
3S	10.3032	10.585	10.295	10.354	10.3552 ± 0.0005
4S	10.7243	11.148	10.5244	10.663	10.5794 ± 0.0012
5S	11.1454	-	10.7251	10.875	10.8852 ± 0.00016
6S	11.5665	-	10.9074	-	-
1P	9.7416	9.8354	9.9326	9.896	9.91221 ± 0.00026
2P	10.1628	10.398	10.2245	10.261	10.26865 ± 0.00022
3P	10.5839	-	10.4585	10.549	10.524 ± 0.0008
4P	11.005	-	10.6627	10.797	-
5P	11.4262	-	10.8478	-	-
1D	10.0224	10.210	10.1389	10.155	$10.1637\ \pm 0.0014$
2D	10.4435	-	10.3799	10.455	-
3D	10.8647	-	10.5892	10.711	-
4D	11.2858	-	10.7782	10.939	-

 $P = \epsilon(-1)^{L+\Lambda+1}$, and $C = \epsilon \eta(-1)^{L+\Lambda+S}$ with $\epsilon, \eta = \pm 1$. where $\Lambda = 0$ for conventional mesons and $\Lambda = 1$ for hybrid mesons [36]. In the present work, spin interactions are not incorporated in the potential model so the charge quantum number (C) remains unchanged for the same value of L with S = 0, 1. Following are a few observations obtained from the tables for hybrid mesons:

States	This work	[28]	[35]	[33]	$\operatorname{Exp}[30]$	Lattice [34]
1S	6.2745	6.277	6.2749	6.332	6.2749 ± 0.008	$6.280 \pm 0.030 \pm 0.190$
2S	6.8415	6.4963	6.841	6.881	6.842 ± 0.004	$6.960{\pm}0.080$
3S	7.4084	6.7148	7.197	7.235	-	
4S	7.9754	6.9333	7.488	-	-	
5S	8.5423	-	-	-	-	
6S	9.1092	-	-	-	-	
1P	6.6525	6.4234	6.753	6.734	-	$6.783 {\pm} 0.030$
2P	7.2194	6.6419	7.111	7.126	-	-
3P	7.7864	-	7.406	-	-	-
4P	8.3533	-	-	-	-	-
5P	9.4872	-	-	-	-	-
1D	7.0304	6.569	6.998	7.072	-	-
2D	7.5974	-	7.302	-	-	-
3D	8.1643	-	7.57	-	-	-
4D	8.7313	-	-	-	-	-

Table 4: Mass spectra of B_c mesons in GeV with a=0.0806GeV³ and b= 0.4104GeV²

Hybrid Charmonium

Hybrid charmonium states with $J^{PC} = 0^{++}, 1^{+-}, 0^{--}, 1^{-+}$ have the same mass equal to 4.2992 GeV which is the lowest calculated mass for hybrid charmonium. This value of lowest mass is comparable to the numerically reported lowest mass in ref. [13] with $J^{PC} = 0^{--}, 0^{++}$. In Lattice QCD [15], the lowest mass is 4.189 GeV with $J^{PC} = 1^{--}$. Recently, ref.[37] reported the lowest mass value equal to 3.56 with $J^{PC} = 0^{-+}$. From these observations, we conclude that our prediction for the lowest mass is much closer to the predictions of refs.[15, 38].

Hybrid Bottomonium

For hybrid bottomonium, the states with $J^{PC} = 0^{++}, 1^{+-}, 0^{--}, 1^{-+}$ have mass equal to 10.8088 GeV which is the lowest calculated mass. This value of lowest mass is comparable to the numerically reported lowest mass in ref. [32] with $J^{PC} = 0^{--}, 0^{++}$. In refs. [24, 39], the lowest mass is 9.68 GeV with $J^{PC} = 0^{-+}$ while in ref.[40], the lowest mass is 10.5 GeV with $J^{PC} = 1^{++}$. Ref.[37] also reported the lowest mass value equal to 9.68 with $J^{PC} = 0^{+-}$.

Hybrid B_c

In case of hybrid B_c mesons, the lowest mass is calculated as equal to 7.3724 GeV with $J^P = 1^-$. Refs. [35, 26] also found the 1^- state with the lowest mass equal to 7.422 and 6.83 respectively. Ref.[37] reported the lowest state mass as 6.63 GeV, which is much smaller than our calculated mass.

	hybrid cha	armonium			hybrid bo	otmonium	
States	This work	NR[38]	Rel. [38]	States	This work	NR [32]	Rel. [32]
1S	4.2992	4.1063	4.1707	1S	10.8088	10.7747	10.8079
2S	4.5515	4.4084	4.4837	2S	11.1729	10.9211	10.928
3S	4.8037	4.6855	4.7614	3S	11.537	11.0664	11.048
4S	5.056	4.9438	5.0132	4S	11.9011	11.2086	11.1662
5S	5.3082	5.1876	5.2448	5S	12.2652	11.3469	11.2817
6S	5.5604	5.4197	5.4602	6S	12.6293	11.4814	11.394
1P	4.4674	4.2464	4.3203	1P	11.0516	10.8366	10.8569
2P	4.7196	4.5264	4.6070	2P	11.4157	10.9857	10.9856
3P	4.9719	4.7875	4.8659	3P	11.7797	11.1372	11.1097
4P	5.2214	5.0338	5.1034	4P	12.1438	11.2795	11.2293
5P	5.4764	5.2682	5.3236	5P	12.5079	11.5503	11.4566
1D	4.6356	4.4232	4.4320	1D	11.2943	10.9063	10,9127
2D	4.8878	4.6955	4.6892	2D	11.6584	11.0591	11.0415
3D	5.14	4.9402	4.966	3D	12.0225	11.2054	11.165
4D	5.3923	5.1912	5.2034	4D	12.3866	11.3463	11.2837

Table 5: Mass spectra of hybrid Charmonium and hybrid botmonium in (GeV). Masses are rounded to 4 decimal places

Table 6: Mass spectra of hybrid Bc mesons in (GeV). Our calculated masses are rounded to 4 decimal places

States	This work	[35]	[26]
B_c^h 1S	7.3724	7.415	$6.90 \pm 0.12 \pm 0.09$ or $7.37 \pm 0.12 \pm 0.07 \pm 0.12$
B_c^h 2S	7.849	7.646	-
$B_c^h 3S$	8.3256	7.866	-
B_c^h 4S	8.8022	8.075	-
$B_c^h 5S$	9.2788	-	-
$B_c^h 6S$	9.7555	-	-
$B_c^h 1 P$	7.6901	7.547	$7.15 \pm 0.08 \pm 0.05 \pm 0.09$ or $7.67 \pm 0.07 \pm 0.02 \pm 0.12$
B^h_c 2P	8.1668	7.776	-
B_c^h 3P	8.6434	7.990	-
B_c^h 4P	9.12	-	-
B_c^h 5P	10.073	-	-
B_c^h 1D	8.0079	7.663	-
B_c^h 2D	8.4845	7.886	-
B_c^h 3D	8.9611	8.095	-
B_c^h 4D	9.4377	-	-

Table 7: J ² States							
L	\mathbf{S}	J^{PC} for conventional	J^{PC} for Hybrid				
0	0, 1	$0^{-+}, 1^{}$	$0^{++}, 1^{+-}$				
			$0^{}, 1^{-+}$				
1	0, 1	$0^{++}, 1^{+-}, 1^{++}, 2^{++}$	$0^{-+}, 1^{}, 1^{-+}, 2^{-+}$				
			$0^{+-}, 1^{++}, 1^{+-}, 2^{+-}$				
2	0, 1	$2^{-+}, 1^{}, 2^{}, 3^{}$	$2^{++}, 1^{+-}, 2^{+-}, 3^{+-}$				
			$2^{}, 1^{-+}, 2^{-+}, 3^{-+}$				

Table 7: J^{PC} states

Table 8: Lowest Mass J^{PC} states of Hybrid Charmonium

States	This work	[38]	[19]	Lattice QCD $[15]$	[37]
1^{+-}	4.2992	4.1063	4.19	-	$4.21 {\pm} 0.15$
1^{-+}	4.2992	4.1063	4.19	4.213	$3.93 {\pm} 0.10$
0^{++}	4.2992	4.0802	-	-	$4.53 {\pm} 0.06$
$0^{}$	4.2992	4.0802	-	-	$4.63 {\pm} 0.14$
1	4.4674	4.2678	4.19	4.189	$4.12 {\pm} 0.11$
1^{++}	4.4674	4.2678	4.19	-	$4.15 {\pm} 0.09$
0^{-+}	4.4674	4.2464	4.19	4.920	$3.56{\pm}0.09$
0^{+-}	4.4674	4.2464	4.19	4.35	$4.06 {\pm} 0.12$
2^{-+}	4.4674	4.2739	4.19	4.3	-
2^{+-}	4.4674	4.2739	4.19	4.4	-
3^{+-}	4.6356	4.4197	-	-	-
3^{-+}	4.6356	4.4197	-	4.7	-

Table 9: Mass of first excited J^{PC} states of Hybrid Charmonium

State	\mathbf{s}	This work	[38]	Lattice QCD [15]
1+-		4.5515	4.4084	-
1-+		4.5515	4.4080	-
0^{-+}		4.7196	4.5264	-
0^{+-}		4.7196	4.5264	-
2^{-+}		4.7196	4.5264	-
2^{+-}		4.7196	4.5653	4.505

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States	This work	[32]	QCD sum rules $[39][24]$	[40]	[37]
0	10.8088	10.8069	$11.48 {\pm} 0.75$	10.66	$10.51 {\pm} 0.03$
0^{++}	10.8088	10.8069	$11.20 {\pm} 0.48$	-	$10.57{\pm}0.08$
1^{+-}	10.8088	10.8079	$10.70 {\pm} 0.53$	-	$10.46 {\pm} 0.06$
1^{-+}	10.8088	10.8079	$9.79 {\pm} 0.22$	10.80	$9.85{\pm}0.11$
1	11.0516	10.8561	$9.7 {\pm} 0.12$	-	-
1^{++}	11.0516	10.8561	$11.09 {\pm} 0.60$	10.50	$10.41{\pm}0.18$
0^{-+}	11.0516	10.8534	$9.68 {\pm} 0.29$	-	$10.55 {\pm} 0.10$
0^{+-}	11.0516	10.8534	$10.17 {\pm} 0.22$	10.68	$9.68 {\pm} 0.20$
2^{-+}	11.0516	10.8569	$9.93 {\pm} 0.21$	-	$10.12 {\pm} 0.06$
2^{+-}	11.0516	10.8569	-	-	
3^{+-}	11.2943	10.9127	-	-	-
3^{-+}	11.2943	10.9127	-	-	-
2^{++}	11.2943	10.9125	$10.64{\pm}0.03$		-
$2^{}$	11.2943	10.9125	-		-

Table 10: Masses of Hybrid Bottomonium for different J^{PC} states with lowest mass

Table 11: Masses of Hybrid B_c for different J^{PC} states with lowest mass

States	P.W	[35]	QCD [26]	[37]
1-	7.3724	7.422	$6.83{\pm}0.08{\pm}0.01{\pm}0.07$	$6.63 {\pm} 0.14$
1^{+}	7.3724	7.422	$7.7{\pm}0.06{\pm}0.05{\pm}0.13$	$7.17 {\pm} 0.09$
0^{-}	7.3724	7.415	$6.90{\pm}0.12{\pm}0.01{\pm}0.09$	$6.65{\pm}0.08$
0^{-}	7.3724	7.415	$7.73{\pm}0.12{\pm}0.07{\pm}0.12$	$7.03 {\pm} 0.12$
2^{-}	7.6901	7.547	$7.15{\pm}0.08{\pm}0.05{\pm}0.09$	-
2^{+}	7.6901	7.547	$7.67{\pm}0.07{\pm}0.02{\pm}0.09$	-

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