COLAX ADJUNCTIONS AND LAX-IDEMPOTENT PSEUDOMONADS

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Abstract. We prove a generalization of a theorem of Bunge and Gray about forming colax adjunctions out of relative Kan extensions and apply it to the study of the Kleisli 2-category for a lax-idempotent pseudomonad. For instance, we establish the weak completeness of the Kleisli 2-category and describe colax change-of-base adjunctions between Kleisli 2-categories. Our approach covers such examples as the bicategory of small profunctors and the 2-category of lax triangles in a 2-category. The duals of our results provide lax analogues of classical results in two-dimensional monad theory: for instance, establishing the weak cocompleteness of the 2-category of strict algebras and lax morphisms and the existence of colax change-of-base adjunctions.

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1. Introduction

The primary motivation for this paper is to develop **lax analogues** of classical results in two-dimensional algebra, in particular two-dimensional monad theory as studied in [2]. The examples commonly studied in this area include 2-categories of categories with structure and pseudo morphisms between them – functors that preserve the structure up to coherent isomorphism. For instance categories equipped with a class of colimits and colimit-preserving functors, or monoidal categories and monoidal functors. Such 2-categories can be described as the 2-category T-Alg of T-algebras and pseudo-T-morphisms for a 2-monad T. Various results have been proven in [2] about T-algebras and pseudo-T-morphisms, for instance their bicocompleteness or the existence of change-of-base biadjunctions between 2-categories of algebras and pseudo-morphisms for two different 2-monads S, T.

On the other hand, there are fewer known results about 2-categories of categories with structure and lax morphisms between them. These still include interesting examples, for instance categories equipped with a class of colimits and all functors between them, or monoidal categories and lax monoidal functors. They can also be described using 2-monads, this time as the 2-category T-Alg$_l$ of T-algebras and lax T-algebra morphisms. While limits in T-Alg$_l$ have been well-understood ([15], [18]), not much has been proven about colimits. This was for a good reason: 2-colimits or even bicolimits often do not exist in those 2-categories. Our task in this paper is to suitably weaken the notion of a bicolimit and show that 2-categories of lax morphisms are in fact cocomplete in this weak sense. Another task we have is to establish change-of-base theorems for algebras and lax morphisms. Again, the notion that works for pseudo-morphisms – biadjunctions – will have to be replaced by a weaker one – colax adjunctions.

The 2-category T-Alg of algebras and pseudo-morphisms can often be described as the Kleisli 2-category for a certain pseudo-idempotent 2-comonad. A key observation to be made is that many statements and proofs about T-Alg in papers [2], [3] are very formal and are in fact true for any pseudo-idempotent 2-comonad on a 2-category. They also easily dualize to pseudo-idempotent 2-monads. Since we are interested in the lax world, we are naturally led to the study of Kleisli 2-categories for lax-idempotent pseudomonads, using the formalism of left Kan pseudomonads [19]. The usage of pseudomonads instead of 2-monads will allow us to consider a wider array of examples such as the small presheaf pseudomonad, and lets us prove that the bicategory PROF of locally small categories and small profunctors is weakly complete in the sense of the previous paragraph.

As mentioned, colax adjunctions are inevitable when working with lax morphisms. The definition of a (co)lax adjunction is hard to work with because it contains a large amount of data. Our first main result, Theorem 3.3, shows that a left colax adjoint F to a pseudofunctor U can be more conveniently given by a collection of 1-cells $y_A : A \to UFA$ satisfying certain "relative U-left Kan extension" conditions. This is an extension of the work of Bunge and Gray ([4], [8]) where this has been proven for the case when U is a 2-functor. A result of this kind is similar to how left Kan pseudomonads provide a more convenient description of lax-idempotent pseudomonads. We will use this theorem to obtain results on colax adjunctions involving the Kleisli 2-category for a lax-idempotent pseudomonad (Theorem 4.15), and the dual of *this* result will be used to obtain results on colax adjunctions involving T-algebras and lax T-morphisms for a 2-monad (Theorem 5.8).

The paper is organized as follows. In Section 2 we recall the necessary concepts that we will need in this paper. With the small exception of *left Kan 2-monads*, everything here is well-known.

In Section 3 we prove the generalization of Bunge's and Gray's results on colax adjunctions to the setting of pseudofunctors: we show that there is a correspondence between left colax adjoint pseudofunctors to a pseudofunctor U and collections of 1-cells $y_A : A \to UFA$ satisfying the aforementioned relative U-left Kan extension conditions (Theorem 3.5).

In Section 4 we first give (an essentially folklore) characterization of algebras for a lax-idempotent pseudomonad in terms of the existence of certain adjoints (Proposition 4.9). We then use this characterization and the generalized Bunge's and Gray's result to prove that when given a lax-idempotent pseudomonad D on $\mathcal{K}$, any left biadjoint $\mathcal{K} \to \mathcal{L}$ that factorizes through the Kleisli 2-category $\mathcal{K}_D$ gives rise to a colax left adjoint $\mathcal{K}_D \to \mathcal{L}$ (Theorem 4.15). We list various applications, for instance the weak completeness of $\mathcal{K}_D$ (Theorem 4.32) provided that $\mathcal{K}$ is bicomplete, or that there is a canonical colax adjunction between $\mathcal{K}_D$ and the 2-category of pseudo-D-algebras (Corollary 4.17).

In Section 5 we spell out what these results in particular say about the 2-category T-Alg$_l$ of strict algebras and lax morphisms for a 2-monad T. This includes the aforementioned colax base-of-change theorem (Corollary 5.10) as well as the weak cocompleteness result for T-Alg$_l$ (Theorem 5.11).

Prerequisities: We assume the reader is familiar with 2-monads and pseudomonads and their pseudo and strict algebras. We also assume the familiarity with lax-idempotent pseudomonads.

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2. Background

2.1. **Colax functors and transformations.** In this text we will primarily use the **colax** versions of concepts such as lax functors, lax transformations. The motivation for this is that we are building on the work of [4] which uses colax structures, as opposed to lax ones[1].

Definition 2.1. Let $\mathcal{A}, \mathcal{B}$ be 2-categories. A *colax functor* $F : \mathcal{A} \to \mathcal{B}$ consists of:

- A function $F_0 : \text{ob } \mathcal{A} \to \text{ob } \mathcal{B}$,
- for every pair $A.B$ of objects of $\mathcal{A}$ a functor $F_{A,B} : \mathcal{A}(A,B) \to \mathcal{B}(FA, FB)$,
- for every composable pair (f, g) of morphisms in $\mathcal{A}$ a 2-cell (*associator*) $\gamma_{f,g} : F(g \circ f) \Rightarrow Fg \circ Ff$,
- for every object $A \in \mathcal{A}$ a 2-cell (*unitor*) $\iota_A : F1_A \Rightarrow 1_{FA}$,

subject to associativity and unit axioms, see for instance [9, Definition 4.1.2]. If γ and ι go in the other direction, we obtain the notion of a *lax functor*. In case γ, ι are invertible, this is called a *pseudofunctor*.

[1] In [4], colax natural transformations are referred to as *lax*. In this paper we are following the modern terminology.

For simplicity, we will always use the letters γ, ι for the associator and the unitor of a colax functor, and always omit the index for any of its components.

Definition 2.2. Given 2-categories $\mathcal{A}, \mathcal{B}$ and two pseudofunctors $F, G : \mathcal{A} \to \mathcal{B}$, a *colax natural transformation* $\alpha : F \Rightarrow G$ consists of the following data:

- For every $A \in \mathcal{A}$ a 1-cell $\alpha_A : FA \to GA$,
- For every $f : A \to B \in \mathcal{A}$ a 2-cell:

$$\begin{tikzcd} FA \arrow[r, "\alpha_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\alpha_B"'] \arrow[ur, phantom, "\Uparrow \alpha_f"] & GB \end{tikzcd}$$

These must satisfy certain unit, composition, local naturality conditions, see [9, Definition 4.3.1]. If the 2-cells α_f go in the other direction, this is referred to as a *lax natural transformation.* If α_f is invertible for all morphisms f, α is called a *pseudo-natural transformation.* If the α_f's are the identities, we use the term *2-natural transformation.*

Definition 2.3. Given two pseudonatural transformations α, β between pseudofunctors $F, G : \mathcal{K} \to \mathcal{L}$, a *modification* $\Gamma : \alpha \to \beta$ consists of a 2-cell $\Gamma_A : \alpha_A \Rightarrow \beta_A$ for every object $A \in \mathcal{K}$, subject to the modification axiom for each 1-cell in $\mathcal{K}$, see [9, Definition 4.4.1].

Example 2.4. Given an endofunctor $T : \mathcal{A} \to \mathcal{A}$, any colax natural transformation $c : T \Rightarrow 1_{\mathcal{A}}$ induces a modification $(cc) : c \circ Tc \to c \circ cT$, whose component at $A \in \mathcal{A}$ is given by:

$$(cc)_A := c_{c_A} : c_A \circ Tc_A \Rightarrow c_A \circ c_{TA}.$$

Remark 2.5. Pseudofunctors preserve colax natural transformations. If $H : \mathcal{C} \to \mathcal{D}$ is a pseudofunctor and $\alpha : F \Rightarrow G : \mathcal{B} \to \mathcal{C}$ is colax natural, there is an induced colax natural transformation $H\alpha$ whose 1-cell component at A is $H\alpha_A$ and whose 2-cell component at a morphism $f : A \to B$ is the following composite 2-cell that we denote by $(H\alpha)_f$:

$$\begin{tikzcd}[row sep=large, column sep=large] FA \arrow[r, "H\alpha_A"] \arrow[d, "HFf"'] \arrow[dr, bend right] \arrow[dr, bend left] & GA \arrow[d, "HGf"] \\ FB \arrow[r, "H\alpha_B"'] \arrow[ur, phantom, "\Uparrow \gamma \quad \Uparrow H\alpha_f \quad \Uparrow \gamma^{-1}"] & GB \end{tikzcd}$$

2.2. **Colax adjunctions.** Lax adjunctions, also called *quasi-adjunctions* in [8, I,7.1] are a categorification of adjunctions between functors where the unit and the counit are replaced by lax natural transformations, and the triangle identities are replaced by modifications. As in the previous section, we will use the dual notion – colax adjunctions.

Definition 2.6. A *colax adjunction* consists of two pseudofunctors $U : \mathcal{D} \to \mathcal{C}$ and $F : \mathcal{C} \to \mathcal{D}$, two colax natural transformations $\eta : 1 \Rightarrow UF$ and $\epsilon : FU \Rightarrow 1$ and two modifications:

$$
\begin{tikzcd}
F \arrow[r, "F\eta"] \arrow[dr, equal] & FUF \arrow[d, "\epsilon F"] \arrow[dl, phantom, "\Psi\,\Swarrow"] \\
 & F
\end{tikzcd}
\qquad
\begin{tikzcd}
U \arrow[r, "\eta U"] \arrow[dr, equal] & UFU \arrow[d, "U\epsilon"] \arrow[dl, phantom, "\Phi\,\Nearrow"] \\
 & U
\end{tikzcd}
$$

Before stating the axioms required, let us fix a convention: we will use the symbol $U\Psi$ to denote the modification obtained from Ψ by not just applying U, but also by pre- and post-composing it with the associator and the unitor for U so that its domain and codomain are $U\epsilon F \circ UF\eta$, 1_{UF}. Let us use the same convention for $F\Phi$. The axioms are the *swallowtail identities*, which assert that the two composite modifications below are the identities on η and ϵ:

$$
\begin{tikzcd}
1_{\mathcal{C}} \arrow[r, "\eta"] \arrow[d, "\eta"'] & UF \arrow[d, "UF\eta"] \arrow[ddr, bend left, equal] \arrow[dl, phantom, "\eta\eta\,\Nearrow"] & \\
UF \arrow[r, "\eta UF"'] \arrow[drr, bend right, equal] & UFUF \arrow[dr, "U\epsilon F"] \arrow[r, phantom, "\Nearrow U\Psi"] \arrow[d, phantom, "\Nearrow \Phi F"] & \\
 & \, & UF
\end{tikzcd}
\qquad
\begin{tikzcd}
FU \arrow[dr, "F\eta U"] \arrow[drr, bend left, equal] \arrow[ddr, bend right, equal] & & \\
 & FUFU \arrow[r, "FU\epsilon"] \arrow[d, "\epsilon_{FU}"] \arrow[ur, phantom, "F\Phi\,\Swarrow"] \arrow[l, phantom, "\Psi U\,\Swarrow"] & FU \arrow[d, "\epsilon"] \arrow[dl, phantom, "\Swarrow\,\epsilon\epsilon"] \\
 & FU \arrow[r, "\epsilon"'] & 1_{\mathcal{D}}
\end{tikzcd}
$$

Notation 2.7. We will denote a colax adjunction as follows and say that F is a *left colax adjoint* to U:

$$
(\Psi, \Phi) : (\epsilon, \eta) : \qquad
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\pm"] & \mathcal{D} \arrow[l, bend left, "U"]
\end{tikzcd}
$$

There are several important variations or special cases:

- if ϵ, η are lax natural, Ψ, Φ go in the other directions and an appropriate dual of the swallowtail identities holds, we will call it a *lax adjunction*,
- in case that ϵ, η are pseudonatural transformations and Ψ, Φ are isomorphisms, we will use the term *biadjunction*.
- if U, F are 2-functors, ϵ, η are 2-natural and Ψ, Φ are the identities, we will call this a *2-adjunction*.

Since the last two cases are the more usual notion, we will use the usual symbol $\dashv$ instead of $\dashv\!\!\vdash$ for them.

Remark 2.8. Contrary to the case of biadjunctions, left colax adjoints are not unique up to an equivalence, not even when U is a 2-functor, η is 2-natural and Ψ, Φ are the identities. An example will be given in Remark 4.31.

2.3. **Lax-idempotent and left Kan pseudomonads.** The notion of a lax-idempotent pseudomonad (see [19, Section 2]) contains a large amount of data and axioms. A major simplification can be achieved if one works with *left Kan pseudomonads* instead. In this section we recall all the basic definitions and mention the equivalence of left Kan pseudomonads and lax-idempotent pseudomonads. We also define a special class of left Kan pseudomonads that we call *left Kan 2-monads* – this is the obvious strict version of the notion.

Definition 2.9. A *left Kan pseudomonad* ([19]) (D, y) on a 2-category $\mathcal{K}$ consists of:

- A function $D : \mathrm{ob}\ \mathcal{K} \to \mathrm{ob}\ \mathcal{K}$,
- For every $A \in \mathcal{K}$ a 1-cell $y_A : A \to DA$ called its *unit*,
- For every 1-cell $f : A \to DB$ a left Kan extension of f along y_B such that the accompanying 2-cell is invertible:

(1)
$$\begin{tikzcd} A \arrow[r, "y_A"] \arrow[dr, "f"'] & DA \arrow[d, "f^{\mathbb{D}}"] \arrow[dl, phantom, "\Uparrow \mathbb{D}_f"] \\ & DB \end{tikzcd}$$

These are subject to the axioms:

- For every $A \in \mathcal{K}$, the identity 2-cell 1_{y_B} on y_B exhibits 1_{DA} as a left Kan extension of y_A along y_A:
- for every $g : B \to DC$, $f : A \to DB$, $g^{\mathbb{D}}$ preserves the left Kan extension (1).

Definition 2.10. A *pseudo-D-algebra* consists of an object $C \in \mathcal{K}$ together with a mapping that sends every 1-cell $f : B \to C$ to the left Kan extension of f along y_B such that the accompanying 2-cell is invertible:

(2)
$$\begin{tikzcd} B \arrow[r, "y_B"] \arrow[dr, "f"'] & DB \arrow[d, "f^{\mathbb{C}}"] \arrow[dl, phantom, "\Uparrow \mathbb{C}_f"] \\ & C \end{tikzcd}$$

and such that for every $f : A \to DB$, $g^{\mathbb{B}}$ preserves the left Kan extension (1).

A *D-pseudomorphism* $h : B \to A$ between pseudo-D-algebras C, X is a 1-cell $h : C \to X$ that preserves the Left Kan extension (2). A *pseudo-D-algebra 2-cell* $\alpha : h \Rightarrow h' : B \to A$ is just a 2-cell in $\mathcal{K}$. All this data assembles into a 2-category that we denote by Ps-D-Alg.

Definition 2.11. By the *Kleisli 2-category* $\mathcal{K}_D$ associated to the left Kan pseudomonad (D, y) we mean the full sub-2-category of Ps-D-Alg spanned by *free D-algebras*, that is, algebras whose underlying object is of form DA for some object $A \in \mathcal{K}$ and the extension operation is given by $(-)^{\mathbb{D}}$.

Remark 2.12. We may also define the *Kleisli bicategory* associated to a left Kan pseudomonad (D, y), where objects are the objects in $\mathcal{K}$ and a morphism $A \rightsquigarrow B$ in $\mathcal{K}_D$ corresponds to a morphism $A \to DB$ in $\mathcal{K}$. The unit is given by the unit of the pseudomonad, while the composition is defined using the extension operation:

$$A \overset{f}{\rightsquigarrow} B \overset{g}{\rightsquigarrow} C \qquad \mapsto \qquad A \overset{g^{\mathbb{D}} \circ f}{\rightsquigarrow} B$$

Denote this bicategory by $\mathrm{Kl}(D)$. It is routine to verify that there is a pseudofunctor $N : \mathrm{Kl}(D) \to \mathcal{K}_D$ sending the Kleisli morphism $f : A \rightsquigarrow B$ to $f^{\mathbb{D}} : DA \to DB$ and that it is a biequivalence of bicategories. In this paper we will for the most part use the 2-category presentation since it is easier to work with.

Proposition 2.13. There is a "free-forgetful" biadjunction given as follows:

$$(\Psi, \Phi) : (p, q) : \qquad \begin{tikzcd} \mathcal{K} \arrow[rr, bend right, "J_D"'] \arrow[rr, phantom, "\top"] & & \mathcal{K}_D \arrow[ll, bend right, "U_D"'] \end{tikzcd}$$

- The right biadjoint U_D is the forgetful 2-functor sending an algebra to its underlying object,
- the left biadjoint is a normal pseudofunctor sending:
$$(f : A \to B) \mapsto ((y_B f)^{\mathbb{D}} : DA \to DB),$$
- the counit $p : J_D U_D \Rightarrow 1$ evaluated at the object DA is the following algebra homomorphism:
$$p_{DA} := (1_{DA})^{\mathbb{D}} : D^2 A \to DA,$$
With its pseudonaturality square at an algebra morphism h being the canonical isomorphism between $1^{\mathbb{D}}_{DB} Dh$ and $h 1^{\mathbb{D}}_{DA}$, as both are the left Kan extensions of h along y_{DA}.
- the unit is given by the unit of the left Kan pseudomonad $y : 1 \Rightarrow U_D J_D$, with the pseudonaturality square at a morphism $h : A \to B$ being given by the canonical isomorphism:
$$\begin{tikzcd} A \arrow[r, "y_A"] \arrow[d, "h"'] \arrow[dr, phantom, "\Uparrow \mathbb{D}_{y_B h}"] & DA \arrow[d, "Dh"] \\ B \arrow[r, "y_B"'] & DB \end{tikzcd}$$
- the components of the modifications are given by the canonical isomorphisms:
$$\Psi : pJ_D \circ J_D y \cong 1_{J_D},$$
$$\Phi : 1_{U_D} \cong U_D p \circ y U_D.$$

This biadjunction is moreover *lax-idempotent*, meaning the following:

Proposition 2.14. There exist (non-invertible) modifications Γ, Θ that serve as the unit and the counit of the following adjunctions:

$$(\Phi^{-1}, \Gamma) : U_D p \dashv y U_D,$$
$$(\Theta, \Psi^{-1}) : J_D y \dashv p J_D.$$

Proof. The 2-cell $\Gamma_{DA} : 1_{D^2A} \Rightarrow y_{DA} \circ p_{DA}$ is the unique solution to the following equation:

$$
\begin{tikzcd}
& D^2A \arrow[dd, equal] \arrow[dr, "p_{DA}"] & \\
DA \arrow[ur, "y_{DA}"] \arrow[dr, "y_{DA}"'] & {} \arrow[r, Rightarrow, "\exists!"] & DA \arrow[dl, "y_{DA}"] \\
& D^2A &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& D^2A \arrow[dr, "p_{DA}"] & \\
DA \arrow[ur, "y_{DA}"] \arrow[rr, equal] \arrow[dr, "y_{DA}"'] & {} \arrow[u, Rightarrow, "\Phi_{DA}"] & DA \arrow[dl, "y_{DA}"] \\
& D^2A &
\end{tikzcd}
$$

This also proves the first triangle identity. The proof of the second triangle identity is done by pre-composing by y_A and using the appropriate universal properties. By doctrinal adjunction, the collection of adjunctions $(\Phi_{DA}^{-1}, \Gamma_{DA}) : p_{DA} \dashv y_{DA}$ lifts to give the claimed adjunction of pseudonatural transformations. The other adjunction is proven in an analogous way. □

Theorem 2.15. There is a correspondence between:

- left Kan pseudomonads (D, y) on $\mathcal{K}$,
- lax-idempotent pseudomonads (D, m, y) on $\mathcal{K}$.

Moreover, the left Kan pseudomonad and lax-idempotent pseudomonad corresponding to one another have biequivalent 2-categories of algebras, and this biequivalence commutes with the forgetful 2-functors to $\mathcal{K}$.

Proof. For the full proof see [19, 4.1, 4.2], here we sketch only the bits relevant for this paper. Given a left Kan pseudomonad (D, y), the lax-idempotent pseudomonad is given by a normal pseudofunctor $D : \mathcal{K} \to \mathcal{K}$ with action on 1-cells and 2-cells given by $U_D \circ J_D$ from Proposition 2.13. The components of the unit y become pseudonatural with the pseudonaturality square given by the Kan extension 2-cell:

$$
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & DA \arrow[d, "(y_B f)^{\mathbb{D}} =: Df"] \\
B \arrow[r, "y_B"'] & DB
\end{tikzcd}
$$

The multiplication at $A \in \mathcal{K}$ is given by the morphism p_{DA} again as in Proposition 2.13. On the other hand, given a lax-idempotent pseudomonad (D, m, y), the left Kan extension of $f : A \to DB$ along $y_A : A \to DA$ is given by the composite of the pseudonaturality 2-cell y_f and the pseudomonad unitor 2-cell:

$$
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow y_f"] & DA \arrow[d, "Df"] \\
DB \arrow[r, "y_{DB}"] \arrow[dr, equal] \arrow[r, phantom, shift right=3ex, "\Downarrow\cong"] & D^2B \arrow[d, "m_B"] \\
& DB
\end{tikzcd}
$$

□

In the rest of the paper we will use the terms "left Kan pseudomonads" and "lax-idempotent pseudomonads" interchangeably.

Remark 2.16 (Duals)**.** A lax-idempotent pseudomonad T on a 2-category $\mathcal{K}$ is equivalently:

- a colax-idempotent pseudomonad T^{co} on $\mathcal{K}^{co}$,
- a colax-idempotent pseudo-comonad T^{op} on $\mathcal{K}^{op}$,
- a lax-idempotent pseudo-comonad T^{coop} on $\mathcal{K}^{coop}$.

2.4. **Left Kan 2-monads.** There is a class of lax-idempotent pseudomonads that will play a role: the ones for which the pseudomonad is actually a *2-monad*. We will show that these correspond to what we call *left Kan 2-monads*.

Definition 2.17. A left Kan pseudomonad (D, y) is a *left Kan 2-monad* if:

- $\mathbb{D}_f$ is the identity 2-cell for every 1-cell $f : B \to DA$, meaning that $f^{\mathbb{D}} \circ y_B = f$,
- $g^{\mathbb{D}} f^{\mathbb{D}} = (g^{\mathbb{D}} f)^{\mathbb{D}}$,
- $y_A^{\mathbb{D}} = 1_{DA}$.

Notice that in case of left Kan 2-monads, the biadjunction from Proposition 2.13 becomes a 2-adjunction. Let us also note the following:

Proposition 2.18. The correspondence from Theorem 2.15 restricts to the correspondence between left Kan 2-monads (D, y) and lax-idempotent 2-monads (D, m, i).

Proof. "⇒": Let (D, y) be a left Kan 2-monad. As we outlined in the proof of Theorem 2.15, the pseudofunctor D is defined as this left Kan extension:

$$\begin{tikzcd} A \arrow[r, "y_A"] \arrow[d, "f"'] & DA \arrow[d, "(y_B f)^{\mathbb{D}} =: Df"] \\ B \arrow[r, "y_B"'] & DB \end{tikzcd}$$

If $(f : A \to B, g : B \to C)$ is a composable pair of morphisms, we have:

$$D(gf) = (y_C g f)^{\mathbb{D}} = ((y_C g)^{\mathbb{D}} y_B f)^{\mathbb{D}} = (y_C g)^{\mathbb{D}} (y_B f)^{\mathbb{D}} = Dg Df$$

Also, $D1_A = y_A^{\mathbb{D}} = 1_{DA}$ so D is a 2-functor. This also makes y a 2-natural transformation since the pseudo-naturality square is the identity. Next, the pseudo-naturality square for the multiplication $m : D^2 \Rightarrow D$ is also the identity since both of the triangles below commute:

$$\begin{tikzcd} D^2 A \arrow[r, "1_{DA}^{\mathbb{D}}"] \arrow[d, "D^2 f"'] \arrow[dr, "(Df)^{\mathbb{D}}" description] & DA \arrow[d, "Df"] \\ D^2 B \arrow[r, "1_{DB}^{\mathbb{D}}"'] & DB \end{tikzcd}$$

"⇐": If (D, m, i) is a lax-idempotent 2-monad, the corresponding left Kan extension in the proof of Theorem 2.15 has the 2-cell component equal to the identity. The other identities in Definition 2.17 are shown by a straightforward manipulation using the 2-monad identities. □

2.5. **Examples.**

Example 2.19. Given a locally small category $\mathcal{A}$, denote by $P\mathcal{A}$ the full subcategory of $[\mathcal{A}^{op}, \mathrm{Set}]$ spanned by *small* presheaves, that is, presheaves that are small colimits of representables. The assignment $\mathcal{A} \mapsto P\mathcal{A}$ defines a left Kan pseudomonad on the (large) 2-category CAT of locally small categories, with the unit $y_{\mathcal{A}} : \mathcal{A} \to P\mathcal{A}$ being given by the Yoneda embedding and the extension operation being given by ordinary left Kan extension along $y_{\mathcal{A}}$. These are guaranteed to exist because of the cocompleteness of $P\mathcal{B}$; and since $y_{\mathcal{A}}$ is fully faithful, the accompanying 2-cell is invertible):

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "y_A"] \arrow[dr, "F"'] \arrow[r, phantom, shift right=6ex, "\Longrightarrow"] & P\mathcal{A} \arrow[d, "\mathrm{Lan}_{y_A} F"] \\
& P\mathcal{B}
\end{tikzcd}$$

A pseudo-P-algebra is precisely a cocomplete category, and pseudo-P-morphisms are cocontinuous functors. The Kleisli 2-category CAT_P thus has presheaf categories as objects and cocontinuous functors as morphisms. In fact, it can be seen to be biequivalent to the bicategory PROF whose objects are locally small categories and whose morphisms $\mathcal{A} \rightsquigarrow \mathcal{B}$ are small profunctors $H : \mathcal{B}^{op} \times \mathcal{A} \to \mathrm{Set}$. Here we call a profunctor $H : \mathcal{B}^{op} \times \mathcal{A} \to \mathrm{Set}$ *small* if for every $a \in \mathcal{A}$, the presheaf $H(-, a) : \mathcal{B}^{op} \to \mathrm{Set}$ is small (belongs to $P\mathcal{B}$).

Under this identification, the left biadjoint from the Kleisli biadjunction in Proposition 2.13 $P : \mathrm{CAT} \to \mathrm{PROF}$ sends functor $f : \mathcal{A} \to \mathcal{B}$ to the profunctor:

$$\mathcal{B}(-, f-) : \mathcal{B}^{op} \times \mathcal{A} \to \mathrm{Set}.$$

We remark that alternatively there is also a 2-monad presentation for this pseudomonad that uses inaccessible cardinals, see [13, Chapter 7].

Example 2.20. Let $\mathcal{K}$ be a 2-category with comma objects and fix an object $C \in \mathcal{K}$. There is a 2-monad P on $\mathcal{K}/C$ that sends a morphism $f : A \to B$ to the morphism $\pi_f : Pg \to C$ which is a projection of the following comma object in $\mathcal{K}$:

$$\begin{tikzcd}
Pf \arrow[r, "\rho_f"] \arrow[d, "\pi_f"'] \arrow[dr, phantom, "\Downarrow\chi"] & A \arrow[d, "f"] \\
C \arrow[r, equal] & C
\end{tikzcd} \tag{3}$$

This 2-monad is known to be colax-idempotent with its algebras being fibrations in $\mathcal{K}$ ([22, Proposition 9]). Its Kleisli 2-category can be presented as having the objects functors with codomain B, while a morphism $F \rightsquigarrow G$ is a 1-cell $\theta : A \to Pg$ making the triangle below left commute:

$$\begin{tikzcd}
A \arrow[r, "\theta"] \arrow[dr, "f"'] & Pg \arrow[r, "\rho_g"] \arrow[d, "\pi_g"'] \arrow[dr, phantom, "\Downarrow\chi"] & B \arrow[d, "g"] \\
& C \arrow[r, equal] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\alpha"] & B \arrow[d, "g"] \\
C \arrow[r, equal] & C
\end{tikzcd}$$

From the definition of the comma object, this corresponds to pairs (u, α) of a 1-cell $u : A \to B$ and a 2-cell $\alpha : gu \Rightarrow f$ as portrayed above right.

In other words, the Kleisli 2-category for this 2-monad is isomorphic to the *colax slice 2-category* $\mathcal{K}//C$[2]. Under this identification, we have a 2-adjunction:

$$\begin{tikzcd} \mathcal{K}/C \arrow[rr, bend right, "J"'] \arrow[rr, phantom, "\top"] & & \mathcal{K}//C \arrow[ll, bend right, "U"'] \end{tikzcd}$$

The left 2-adjoint is the canonical inclusion, the right 2-adjoint sends an object $f : A \to C$ to the comma object projection $\pi_f : Pf \to C$. The counit $p : JU \Rightarrow 1_{\mathcal{K}//C}$ evaluated at an object $f : A \to C$ is the colax commutative triangle $(\rho_f, \chi) : \pi_f \to f$ from (3).

In the remainder of this section we recall a class of lax-idempotent 2-comonads that come from two-dimensional monad theory. Recall the 2-categories T-Alg$_s$, T-Alg, T-Alg$_l$ of strict algebras and strict, pseudo and lax morphisms for a 2-monad T from [2, 1.2]. Also recall the notions of a *codescent object* and a *lax codescent object* from [14, Page 228].

Definition 2.21. Let T be a 2-monad on a 2-category $\mathcal{K}$ and let (A, a) be a strict T-algebra. By its *resolution*, denoted $\mathrm{Res}(A, a)$, we mean the following diagram in T-Alg$_s$:

$$\begin{tikzcd} T^3A \arrow[r, shift left=2, "m_{T^2A}"] \arrow[r, "Tm_{TA}"] \arrow[r, shift right=2, "T^2a"'] & T^2A \arrow[r, shift left=2, "m_A"] \arrow[r, shift right=2, "Ta"'] & TA \arrow[l, "Ti_A"'] \end{tikzcd}$$

Theorem 2.22. Let T be a 2-monad on a 2-category $\mathcal{K}$ and assume the 2-category T-Alg$_s$ admits lax codescent objects of resolutions of strict algebras. Then the inclusion 2-functor T-Alg$_s$ $\to$ T-Alg$_l$ admits a left 2-adjoint. Similarly, assume the 2-category T-Alg$_s$ admits codescent objects of resolutions of strict algebras. Then the inclusion 2-functor T-Alg$_s$ $\to$ T-Alg admits a left 2-adjoint:

$$\begin{tikzcd} \text{T-Alg}_s \arrow[rr, hook, bend right, "J"'] \arrow[rr, phantom, "\perp"] & & \text{T-Alg}_l \arrow[ll, dashed, bend right, "(-)'"'] \end{tikzcd} \qquad \begin{tikzcd} \text{T-Alg}_s \arrow[rr, hook, bend right, "J_p"'] \arrow[rr, phantom, "\perp"] & & \text{T-Alg} \arrow[ll, dashed, bend right, "(-)^\dagger"'] \end{tikzcd}$$

In the first case, the value of a left 2-adjoint at a T-algebra (A, a) is given by the lax codescent object of the diagram $\mathrm{Res}(A, a)$ in T-Alg$_s$. In the second case, codescent object is used.

Proof. See Lemma 3.2 and Theorem 2.6 in [14]. □

Remark 2.23. The assumptions of Theorem 2.22 are satisfied whenever the base 2-category $\mathcal{K}$ is cocomplete and T is finitary (preserves filtered colimits). This is because the codescent objects of a resolution of a strict algebra is *reflexive*, and so is a filtered colimit by [14, Proposition 4.3].

---

[2]This 2-category can also be presented as the 2-category of strict coalgebras and lax morphisms for the 2-comonad $(-) \times C$, see [6, Chapter 5].

Definition 2.24. We denote by Q_l and Q_p the 2-comonads generated by the 2-adjunctions in the above theorem and call them the *lax morphism classifier 2-comonad* and the *pseudo morphism classifier 2-comonad*.

It is easy to see that T-Alg$_l$ is isomorphic to (T-Alg$_s$)$_{Q_l}$, the Kleisli 2-category for the 2-comonad Q_l. Similarly, T-Alg $\cong$ (T-Alg$_s$)$_{Q_p}$.

Proposition 2.25. Let T be a 2-monad on a 2-category $\mathcal{K}$ such that the left 2-adjoints to the inclusions T-Alg$_s \hookrightarrow$ T-Alg, T-Alg$_s \hookrightarrow$ T-Alg$_l$ exist. Then:

- If $\mathcal{K}$ admits oplax limits of arrows, Q_l is lax-idempotent.
- If $\mathcal{K}$ admits pseudo limits of arrows, Q_p is pseudo-idempotent.

Proof. See [18, Lemma 2.5]. □

Proposition 2.26. There is a morphism of 2-comonads $Q_l \to Q_p$.

Proof. Denote the units of the adjunctions in Theorem 2.22 by $p_A : A \rightsquigarrow A'$ and $p_A^\dagger : A \rightsquigarrow A^\dagger$ respectively. Since $p_A^\dagger$ is a pseudo-morphism, it is in particular a lax morphism and thus there exists a unique strict T-algebra morphism $\theta_A : A' \to A^\dagger$ making the diagram commute:

$$\begin{tikzcd} & A' \arrow[d, dashed, "\exists! \theta_A"] \\ A \arrow[ur, squiggly, "p_A"] \arrow[r, squiggly, "p_A^\dagger"'] & A^\dagger \end{tikzcd}$$

Using the universal property of A', it is readily seen that the maps θ_A's assemble into a morphism of 2-comonads. □

3. Relative Kan extensions and colax adjunctions

In [4], Bunge introduced the notion of a relative Kan extensions with respect to a 2-functor U and showed that for a collection $y_A : A \to UFA$ of 1-cells that admit these extensions (and satisfy certain coherence conditions), there is an induced left colax adjoint F to U, where F is a colax functor ([4, Theorem 4.1]). She also proves a partial converse to this result ([4, Theorem 4.3]). Note that at the same time these results also appeared in Gray's work ([8, I,7.8.]).

In this section, we generalize these results to the case where U is a pseudofunctor and, on the other hand, refine it by identifying conditions under which the colax left adjoint F is actually a pseudofunctor. This enables us to describe, in Theorem 3.5, a symmetric relationship between U-extensions and colax adjunctions. We will see an application of these results to the settings of lax-idempotent pseudomonads in Section 4.

Definition 3.1. Let $U : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor, $y_A : A \to UFA$, $f : A \to UB$ 1-cells of $\mathcal{D}$. The *left U-extension* of f along y_A is a pair (f', ψ_f) with the property that for any pair (g, α) pictured below, there is a unique 2-cell $\theta : f' \Rightarrow g$ such that the following 2-cells are equal:

$$
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[ddr, "f"'] & UFA \arrow[dd, "Uf'"'] \arrow[dd, bend left=50, "Ug"] \arrow[dd, phantom, shift left=3, "\overset{U\theta}{\Rightarrow}"] \\
 & \arrow[ul, phantom, "\Uparrow \psi_f"] \\
 & UB
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[ddr, "f"'] & UFA \arrow[dd, "Ug"] \\
 & \arrow[ul, phantom, "\Uparrow \alpha"] \\
 & UB
\end{tikzcd}
$$

Definition 3.2. Let $U : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor. We say that a collection of 1-cells $y_A : A \to UFA$ for each object $A \in \mathcal{C}$ are *coherently closed for U-extensions* if:

- for every $f : A \to UB$ we have a choice of an U-extension $(f^{\mathbb{D}}, \mathbb{D}_f)$,
- the following composite 2-cell exhibits $1_Y^{\mathbb{D}} \circ (y_Y f)^{\mathbb{D}}$ as the left U-extension of f along y_X:

$$
\begin{tikzcd}
X \arrow[rr, "y_X"] \arrow[d, "f"'] & & UFX \arrow[d, "U(y_{UY}f)^{\mathbb{D}}"] \arrow[dd, bend left=60, "U(1_{UY}^{\mathbb{D}} \circ (y_{UY}f)^{\mathbb{D}})"] \\
UY \arrow[rr, "y_{UY}"] \arrow[drr, equal] \arrow[r, phantom, shift left=4, "\Uparrow \mathbb{D}_{y_{UY}f}"] \arrow[r, phantom, shift right=4, "\Uparrow \mathbb{D}_{1_{UY}}"] & & UFUY \arrow[d, "U1_{UY}^{\mathbb{D}}"] \arrow[r, phantom, "\overset{\gamma^{-1}}{\Longrightarrow}"] & {} \\
 & & UY
\end{tikzcd} \tag{4}
$$

Theorem 3.3. Let $U : \mathcal{C} \to \mathcal{D}$ be a pseudofunctor and $y_A : A \to UFA$ a collection of 1-cells coherently closed for U-extensions. Then:

- the mapping $A \mapsto FA$ can be extended to a colax functor $F : \mathcal{D} \to \mathcal{C}$,
- y can be extended to a colax natural transformation $1_{\mathcal{D}} \Rightarrow UF$,
- there exists a colax-natural transformation $\epsilon : FU \Rightarrow 1_{\mathcal{C}}$ and a modification $\Phi : 1_U \to U\epsilon \circ yU$.

  Assume moreover the *composition* and *unit* axioms for U-extensions: the diagram below left is a U-extension of y_A along y_A, and the diagram below right is the U-extension of $y_C g f$ along y_A:

$$
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[dr, "y_A"'] & UFA \arrow[d, equal, bend right=30] \arrow[d, bend left=60, "U1_{FA}"] \arrow[d, phantom, shift left=2, "\overset{\iota^{-1}}{\Longrightarrow}"] \\
 & UFA
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "y_A"] \arrow[d, "f"'] & & UFA \arrow[d, "U(y_B f)^{\mathbb{D}}"] \arrow[dd, bend left=60, "U((y_C g)^{\mathbb{D}}(y_B f)^{\mathbb{D}})"] \\
B \arrow[rr, "y_B"] \arrow[d, "g"'] \arrow[r, phantom, shift left=4, "\Uparrow \mathbb{D}"] \arrow[r, phantom, shift right=4, "\Uparrow \mathbb{D}"] & & UFB \arrow[d, "U(y_C g)^{\mathbb{D}}"] \arrow[r, phantom, "\overset{\gamma^{-1}}{\Rightarrow}"] & {} \\
C \arrow[rr, "y_C"'] & & UFC
\end{tikzcd} \tag{5}
$$

  Then:
- F is a pseudofunctor,

- there is an invertible modification $\Psi : \epsilon F \circ Fy \to 1_F$ and all this data give a colax adjunction:

$$(\Psi, \Phi) : (\epsilon, y) : F \dashv\!\!\!\!\dashv U : \mathcal{C} \to \mathcal{D}.$$

Proof. Denote by $(f^{\mathbb{D}}, \mathbb{D})$ the choice of a U-extension of $f : A \to UB$. Define the colax functor $F : \mathcal{D} \to \mathcal{C}$ on a morphism $f : A \to B$ as the following U-extension:

$$\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & UFA \arrow[d, "UFf"] \\
B \arrow[r, "y_B"'] & UFB
\end{tikzcd}$$

Define the action of F on a 2-cell α as the unique 2-cell making the following equal:

$$\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & UFA \arrow[d, "UFf"'] \arrow[d, bend left=60, "UFg"] \arrow[d, phantom, shift left=3ex, "\overset{U(\exists!)}{\Longrightarrow}"] \\
B \arrow[r, "y_B"'] & UFB
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "g"] \arrow[d, bend right=60, "f"'] \arrow[d, phantom, shift right=3ex, "\overset{\alpha}{\Longrightarrow}"] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & UFA \arrow[d, "UFg"] \\
B \arrow[r, "y_B"'] & UFB
\end{tikzcd}
\tag{6}$$

The above equation makes y locally natural. The associator $\gamma' : F(gf) \Rightarrow Fg \circ Ff$ and the unitor $\iota' : F1_A \Rightarrow 1_{FA}$ for F are given as the unique 2-cells satisfying these equations:

$$\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "gf"'] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & UFA \arrow[d, "UF(gf)"'] \arrow[d, bend left=60, "U(Fg \circ Ff)"] \arrow[d, phantom, shift left=3ex, "\overset{U(\exists!)}{\Rightarrow}"] \\
C \arrow[r, "y_C"'] & UFC
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & UFA \arrow[d, "UFf"] \arrow[dd, bend left=60, "U(Fg \circ Ff)"] \\
B \arrow[r, "y_B"] \arrow[d, "g"'] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & UFB \arrow[d, "UFg"] \arrow[r, phantom, "\overset{\gamma^{-1}}{\Longrightarrow}"] & {} \\
C \arrow[r, "y_C"'] & UFC
\end{tikzcd}
\tag{7}$$

$$\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, equal] \arrow[dr, phantom, "\Uparrow \mathbb{D}"] & UFA \arrow[d, "UF1_A"'] \arrow[d, bend left=60, "U1_{FA}"] \arrow[d, phantom, shift left=3ex, "\overset{U(\exists!)}{\Rightarrow}"] \\
A \arrow[r, "y_A"'] & UFA
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, equal] & UFA \arrow[d, equal] \arrow[d, bend left=60, "U1_{FA}"] \arrow[d, phantom, shift left=3ex, "\overset{\iota^{-1}}{\Rightarrow}"] \\
A \arrow[r, "y_A"'] & UFA
\end{tikzcd}
\tag{8}$$

The colax functor axioms for F follow from those of U and can be readily proven using the universal property of U-extensions. The above equations also make y into a colax-natural transformation $y : 1 \Rightarrow UF$. Next, define $\epsilon_B : FUB \to B$ and Φ_B as the U-extension of the identity on UB along y_{UB}:

$$\begin{tikzcd}
UB \arrow[r, "y_{UB}"] \arrow[dr, equal] \arrow[r, phantom, shift right=4, "\Uparrow\Phi_B"] & UFUB \arrow[d, "U\epsilon_B"] \\
& UB
\end{tikzcd}$$

The colax naturality square for ϵ at a 1-cell $h : B \to C$ is the unique 2-cell ϵ_h making the 2-cells below equal (it is guaranteed to uniquely exist because of the coherence for U-extensions):

$$\begin{tikzcd}
UB \arrow[rr, "y_{UB}"] \arrow[d, "Uh"'] \arrow[dr, phantom, "\mathbb{D}\Nearrow"] & & UFUB \arrow[dl, "UFUh"'] \arrow[d, bend right, "U(\epsilon_C\circ FUh)"'] \arrow[d, bend left=60, "U(h\circ\epsilon_B)"] \arrow[d, phantom, "\overset{U\epsilon_h}{\Longrightarrow}"] \\
UC \arrow[r, "y_{UC}"] \arrow[drr, equal, bend right] \arrow[dr, phantom, "\Phi_C\Uparrow"] & UFUC \arrow[dr, "U\epsilon_C"'] \arrow[r, phantom, "\overset{\gamma^{-1}}{\Longrightarrow}"] & {} \\
& & UC
\end{tikzcd} \tag{9}
\qquad
\begin{tikzcd}
UB \arrow[r, "y_{UB}"] \arrow[d, "Uh"'] \arrow[dr, equal, bend right] \arrow[r, phantom, shift right=6, "\Phi_B\Nearrow"] & UFUB \arrow[d, "U\epsilon_B"] \arrow[dd, bend left=60, "U(h\circ\epsilon_B)"] \\
UC \arrow[dr, equal, bend right] & UB \arrow[d, "Uh"] \arrow[r, phantom, "\overset{\gamma^{-1}}{\Longrightarrow}"] & {} \\
& UC
\end{tikzcd}$$

This also makes Φ into a modification $1_U \to U\epsilon \circ yU$. Let us now consider the additional assumptions. It is clear that F will be a pseudofunctor. Define $\Psi_A : \epsilon_{FA} \circ Fy_A \Rightarrow 1_{FA}$ as the unique 2-cell making the two 2-cells below equal:

$$\begin{tikzcd}
A \arrow[rr, "y_A"] \arrow[d, "y_A"'] \arrow[dr, phantom, "\mathbb{D}\Nearrow"] & & UFA \arrow[dl, "UFy_A"'] \arrow[d, "U(\epsilon_{FA}\circ Fy_A)"'] \arrow[d, bend left=60, "U1_{FA}"] \arrow[d, phantom, shift left=3, "\overset{U(\exists!)}{\Rightarrow}"] \\
UFA \arrow[r, "y_{UFA}"] \arrow[drr, equal, bend right] \arrow[dr, phantom, "\Phi_{FA}\Uparrow"] & UFUFA \arrow[dr, "U\epsilon_{FA}"'] \arrow[r, phantom, "\overset{\gamma^{-1}}{\Longrightarrow}"] & {} \\
& & UFA
\end{tikzcd} \tag{10}
\qquad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "y_A"'] & UFA \arrow[dd, equal] \arrow[dd, bend left=60, "U1_{FA}"] \arrow[dd, phantom, shift left=3, "\overset{\iota^{-1}}{\Rightarrow}"] \\
UFA \arrow[dr, equal, bend right] & \\
& UFA
\end{tikzcd}$$

By the assumption, Ψ_A is invertible. This equality also proves the first swallowtail identity. What remains to prove is the following:

- ϵ is colax-natural,
- Φ is a modification,
- the second swallowtail identity.

These are all straightforward computations and we will prove them in the Appendix as Lemma A.1. □

Theorem 3.4. Let $(\Psi, \Phi) : (\epsilon, y) : F \dashv U : \mathcal{C} \to \mathcal{D}$ be a colax adjunction between pseudofunctors in which Ψ is invertible. Then:

- the components of the unit $y_A : A \to UFA$ are coherently closed for U-extensions,
- the unit and composition axioms (5) for U-extensions hold.

Proof. Notice first that we have the following adjunction:

$$\begin{tikzcd} \mathcal{C}(FA,B) \arrow[rr, bend left, "(\eta_A)^{*}\circ U"] \arrow[rr, phantom, "\top"] & & \mathcal{D}(A,UB) \arrow[ll, bend left, "(\epsilon_B)_{*}\circ F"] \end{tikzcd}$$

The counit and unit 2-cells evaluated at $h : FA \to B$ and $g : A \to UB$ are given as follows:

$$\begin{tikzcd} FA \arrow[d, "Fy_A"] \arrow[drr, equal] \arrow[dd, bend right=80, "F(Uhy_A)"'] & & \\ FUFA \arrow[rr, "\epsilon_{FA}"] \arrow[d, "FUh"] \arrow[r, phantom, "\Psi_A\Uparrow"] \arrow[u, phantom, "\overset{\gamma}{\Rightarrow}"'] & & FA \arrow[d, "h"] \\ FUB \arrow[rr, "\epsilon_B"'] \arrow[urr, phantom, "\Uparrow\epsilon_h"] & & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "y_A"] \arrow[d, "g"'] \arrow[drr, phantom, "\Uparrow y_g"] & & UFA \arrow[d, "UFg"] \arrow[dd, bend left=80, "U(\epsilon_B Fg)"] \\ UB \arrow[rr, "y_{UB}"] \arrow[drr, equal] \arrow[drr, phantom, "\Uparrow\Phi_B"] & & UFUB \arrow[d, "U\epsilon_B"] \arrow[u, phantom, "\overset{\gamma^{-1}}{\Rightarrow}"] \\ & & UB \end{tikzcd}$$

The triangle identities essentially follow from the swallowtail identities of the colax adjunction and we omit the proof for them. Denote by $\mathbb{D}_g$ the unit of this adjunction evaluated at $g : A \to UB$ and denote $g^{\mathbb{D}} := \epsilon_B Fg$. By definition, the pair $(g^{\mathbb{D}}, \mathbb{D}_g)$ is the left U-extension of g along y_A. Next, notice that for $f : A \to B$, the invertible 2-cell:

$$\daleth_f := \Psi_B Ff \circ \epsilon_{FB}\gamma^{-1}_{f,y_B} : \epsilon_{FB}F(y_B f) \Rightarrow Ff : FA \to FB,$$

satisfies the following equality (this again follows from a swallowtail identity):

$$\begin{tikzcd} A \arrow[rr, "y_A"] \arrow[d, "f"'] \arrow[drr, phantom, "\Uparrow\mathbb{D}_f"] & & UFA \arrow[d, "U(y_B f)^{\mathbb{D}}"'] \arrow[d, bend left=60, "UFf"] \arrow[d, phantom, "\overset{U\daleth_f}{\Rightarrow}", shift left=3] \\ B \arrow[rr, "y_B"'] & & UFB \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[rr, "y_A"] \arrow[d, "f"'] \arrow[drr, phantom, "\Uparrow y_f"] & & UFA \arrow[d, "UFf"] \\ B \arrow[rr, "y_B"'] & & UFB \end{tikzcd}$$

This proves that (UFf, y_f) is also a U-extension of $y_B f$ along y_A. Next, notice that for an object B, the following invertible 2-cell:

$$\Xi_B := \epsilon_{1_B} \circ \epsilon_B F\iota^{-1} : \epsilon_B F1_{UB} \Rightarrow \epsilon_B,$$

satisfies this equality:

$$\begin{tikzcd} UB \arrow[r, "y_{UB}"] \arrow[dr, equal] \arrow[dr, phantom, "\overset{\mathbb{D}_{1_{UB}}}{\Rightarrow}"] & UFUB \arrow[d, "U1^{\mathbb{D}}_{UB}"'] \arrow[d, bend left=60, "U\epsilon_B"] \arrow[d, phantom, "\overset{U\Xi_B}{\Rightarrow}", shift left=3] \\ & UB \end{tikzcd} \quad = \quad \begin{tikzcd} UB \arrow[r, "y_{UB}"] \arrow[dr, equal] \arrow[dr, phantom, "\overset{\Phi_B}{\Rightarrow}"] & UFUB \arrow[d, "U\epsilon_B"] \\ & UB \end{tikzcd}$$

This proves that $(U\epsilon_B, \Phi_B)$ is also a U-extension of 1_{UB} along y_{UB}. Using these two isomorphisms of U-extensions, it is clear that the composite 2-cell (4) in Definition 3.2 is a

U-extension if and only if the pair $(f^{\mathbb{D}}, \mathbb{D}_f)$ is a U-extension - which it is, as we have proven. We thus have that the collection $y_A : A \to UFA$ is coherently closed for U-extensions.

Let us now prove the composition and unit axioms (5). The proof that the pair $(1_{FA}, \iota^{-1} y_A)$ is a U-extension follows immediately from the fact that it is isomorphic to the U-extension $(y_A^{\mathbb{D}}, \mathbb{D}_{y_A})$ via the modification Ψ_A (this is the first swallowtail identity):

$$
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[dr, "y_A"'] & UFA \arrow[d, equal] \arrow[d, bend left=60, "U1_{FA}"] \arrow[d, phantom, shift left=3ex, "\overset{\iota^{-1}}{\Rightarrow}"] \\
& UFA
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[dr, "y_A"'] \arrow[dr, phantom, shift left=2ex, "\overset{\mathbb{D}_{y_A}}{\Longrightarrow}"] & UFA \arrow[d, "Uy_A^{\mathbb{D}}"] \arrow[d, bend left=70, "U1_{FA}"] \arrow[d, phantom, shift left=4ex, "\overset{U\Psi_A}{\Longrightarrow}"] \\
& UFA
\end{tikzcd}
$$

Again by using the isomorphism above, the question whether the 2-cell below right is a U-extension is equivalent to asking whether the 2-cell below left is a U-extension:

$$
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] & UFA \arrow[d, "UFf"] \arrow[dd, bend left=80, "UF(gf)"] \\
B \arrow[r, "y_B"] \arrow[d, "g"'] \arrow[ur, phantom, "\Uparrow y_f"] & UFB \arrow[d, "UFg"] \arrow[r, phantom, "\overset{\gamma}{\Longrightarrow}"] & {} \\
C \arrow[r, "y_C"'] \arrow[ur, phantom, "\Uparrow y_g"] & UFC
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[d, "f"'] & UFA \arrow[d, "U(y_B f)^{\mathbb{D}}"] \arrow[dd, bend left=80, "U((y_C g)^{\mathbb{D}}(y_B f)^{\mathbb{D}})"] \\
B \arrow[r, "y_B"] \arrow[d, "g"'] \arrow[ur, phantom, "\Uparrow \mathbb{D}_f"] & UFB \arrow[d, "U(y_C g)^{\mathbb{D}}"] \arrow[r, phantom, "\overset{\gamma}{\Longrightarrow}"] & {} \\
C \arrow[r, "y_C"] \arrow[ur, phantom, "\Uparrow \mathbb{D}_g"] & UFC
\end{tikzcd}
$$

But this 2-cell equals y_{gf} and is thus a U-extension by what we have proven above.

□

Theorem 3.5. Fix a pseudofunctor $U : \mathcal{D} \to \mathcal{C}$ between 2-categories. The following are equivalent for a collection of 1-cells $\{y_A : A \to UFA\}$ with $A \in \mathcal{C}$:

- the collection y_A is coherently closed for U-extensions and satisfies composition and unit axioms (5),
- there is a colax adjunction $(\Psi, \Phi) : (\epsilon, \eta) : F \dashv\!\!| U$ for which Ψ is invertible, F is a pseudofunctor and the 1-cell component of the unit at each $A \in \mathcal{C}$ equals y_A.

Remark 3.6. In the above theorem, we do not have a one-to-one correspondence; instead, there is a suitable "equivalence" between these two concepts. Starting with coherent U-extensions $(f^{\mathbb{D}}, \mathbb{D}_f)$ of f along y_A, producing a colax adjunction and then going back to U-extensions gives the U-extension $(\epsilon_B Ff, \gamma^{-1} y_A \circ U\epsilon_B y_f \circ \Phi_B f)$, which in general will not be equal to $(f^{\mathbb{D}}, \mathbb{D}_f)$ (but will be canonically isomorphic to it). Similarly, starting with left colax adjoint F, going to U-extensions and back only gives a pseudofunctor isomorphic to F.

In our applications to two-dimensional monad theory, we will encounter this very special case of U-extensions:

Definition 3.7. Let $U : \mathcal{C} \to \mathcal{D}$ be a 2-functor. We will say that a collection of 1-cells $y_A : A \to UFA$ is *strictly closed for U-extensions* if:

- for every $f : A \to UB$ there is a U-extension $(f^{\mathbb{D}}, 1_f)$ along y_A with the 2-cell component being the identity,
- $y_A^{\mathbb{D}} = 1_{FA}$,
- for $f : X \to Y$, $g : Y \to Z$ we have $Ff \circ Fg = F(fg)$, where we denote $Ff := = (y_Y \circ f)^{\mathbb{D}}$,
- for $f : A \to UB$ we have $\epsilon_Y \circ Ff = f^{\mathbb{D}}$, where we denote $\epsilon_Y := (1_Y)^{\mathbb{D}}$.

Remark 3.8. It is clear from the proof of Theorem 3.3 that a collection strictly closed for U-extension gives rise to a colax adjunction $(\epsilon, y) : F \dashv\!\vdash U$ for which:

- y is a 2-natural transformation,
- F is a 2-functor,
- the modifications Φ, Ψ are the identities.

(This will in general not be a 2-adjunction because ϵ will only be colax natural.)

4. On the Kleisli 2-category for a left Kan pseudomonad

This section is devoted to studying the Kleisli 2-category for a general left Kan pseudomonad (D, y) on a 2-category $\mathcal{K}$.

In 4.1 we prove a result characterizing the pseudo-D-algebra structure on an object in terms of the existence of certain adjoints (Theorem 4.9).

In 4.2 we use this result and Theorem 3.3 to prove that any left biadjoint $\mathcal{K} \to \mathcal{L}$ that factorizes through the Kleisli 2-category gives rise to a lax left adjoint $\mathcal{K}_D \to \mathcal{L}$. We list several applications, one of which is the assertion that there is a canonical colax adjunction between EM and Kleisli 2-categories for left Kan pseudomonads.

Another application is given in 4.3 where we define *coreflector-limits*, the aforementioned lax analogue of bilimits, and list elementary examples. The main result here is Theorem 4.32 which asserts that whenever the base 2-category $\mathcal{K}$ admits J-indexed bilimits, the Kleisli 2-category for a left Kan pseudomonad on $\mathcal{K}$ will admit them as coreflector-limits.

First, let us recall the following terminology:

Definition 4.1. Let the following be an adjunction in a 2-category $\mathcal{K}$:

$$(\epsilon, \eta) : \quad \begin{tikzcd} A \arrow[r, bend right, "f"'] \arrow[r, phantom, "\top"] & B \arrow[l, bend right, "u"'] \end{tikzcd}$$

- If the counit ϵ is invertible, call f a *reflector* and u a *reflection-inclusion.* In this case f. In case the counit is the identity, f is called a *lali* (*left adjoint-left inverse*) and u a *rari* (*right adjoint-right inverse*).
- if the unit η is invertible, call f a *coreflection-inclusion* and u a *coreflector.* In case the unit is the identity, f is called a *lari* and u a *rali.*

Remark 4.2 (Duals)**.** A morphism f is a reflector (a lali) in $\mathcal{K}$ if and only if:

- it is a reflection-inclusion (a rari) in $\mathcal{K}^{op}$,
- it is a coreflector (a rali) in $\mathcal{K}^{co}$,
- it is a coreflection-inclusion (a lari) in $\mathcal{K}^{coop}$.

4.1. A characterization of algebras.

Definition 4.3. Let $F : \mathcal{K} \to \mathcal{L}$ be a pseudofunctor. We will call a morphism $f : A \to B$ in $\mathcal{K}$ an F*-coreflector* if Ff is a coreflector in the 2-category $\mathcal{L}$. Similarly for the other variants from Definition 4.1.

Example 4.4. Let P : CAT $\to$ PROF be the canonical inclusion pseudofunctor. In Example 4.12 below we will show that a functor $f : \mathcal{A} \to \mathcal{B}$ between locally small categories is a P-coreflection-inclusion if and only if it is fully faithful and satisfies a certain smallness condition.

Example 4.5. Consider the lax morphism classifier 2-comonad Q_l associated to a 2-monad T on a 2-category $\mathcal{K}$. Denote by $J : \text{T-Alg}_s \to \text{T-Alg}_l$ the canonical inclusion to the Kleisli 2-category and by $U : \text{T-Alg}_s \to \mathcal{K}$ the forgetful 2-functor. Notice that by virtue of doctrinal adjunction [11], a strict algebra morphism is a J-reflector if and only if it is a U-reflector, that is, the underlying morphism in $\mathcal{K}$ is a reflector.

Remark 4.6. Given a lax-idempotent pseudomonad P on a 2-category $\mathcal{K}$, 1-cells in $\mathcal{K}$ that are P-left adjoints have been studied in the literature ([5], [1]) under the name of P*-admissible* 1-cells.

The following lemma is the left Kan pseudomonad version of [20, Theorem 3.4]:

Lemma 4.7. Let (D, y) be a left Kan pseudomonad on $\mathcal{K}$. Denote by $D : \mathcal{K} \to \mathcal{K}$ the corresponding endo-pseudofunctor and by $J_D : \mathcal{K} \to \mathcal{K}_D$ the inclusion to the Kleisli 2-category. The following are equivalent for a 1-cell $f : B \to C$:

- f is a D-coreflection-inclusion,
- f is a J_D-coreflection-inclusion.

Proof. "(1) $\Rightarrow$ (2)" follows from [5, Proposition 1.3]: namely, the right adjoint to Df in $\mathcal{K}$ is actually a D-algebra homomorphism and thus is an adjoint in $\mathcal{K}_D$. "(2) $\Rightarrow$ (1)" is obvious because we have the forgetful 2-functor $U_D : \mathcal{K}_D \to \mathcal{K}$ that satisfies $D = U_D J_D$. $\square$

Lemma 4.8. The following holds in a 2-category $\mathcal{K}$:

- Let $f \dashv u : B \to A$ be an adjunction with unit η and let $(\mathbb{D}, g^{\mathbb{D}})$ be the left Kan extension of $g : A' \to C$ along $y : A' \to A$. Then the diagram below left exhibits $g^{\mathbb{D}}u$ as the left Kan extension of g along fy:

$$\begin{tikzcd}
B \arrow[dr, "u"] & & \\
A \arrow[u, "f"] \arrow[r, equal] \arrow[u, phantom, shift left=4ex, "\Uparrow\eta"] & A \arrow[dr, "g^{\mathbb{D}}"] & \\
A' \arrow[u, "y"] \arrow[u, phantom, shift left=4ex, "\Uparrow\mathbb{D}"] \arrow[rr, "g"'] & & C
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[ddr, bend left, "h"] & \\
B \arrow[u, "k"] \arrow[dr, bend left, "hk"] & \\
B' \arrow[u, "i"] \arrow[u, phantom, shift left=4ex, "\Uparrow\alpha"] \arrow[r, "f"'] & D
\end{tikzcd}$$

- In the diagram above right, suppose that the top and outer diagrams are left Kan extensions. If all left Kan extensions along k exist and have invertible unit, then α is a left Kan extension of f along j.

Proof. The first point follows by composing the following bijections. For a 1-cell $h : B \to C$, the first one is given by the adjunction $f \dashv u$, the second one is given by the definition of $g^{\mathbb{D}}$:

$$\mathcal{K}(B,C)(g^{\mathbb{D}}u, h) \cong \mathcal{K}(A,C)(g^{\mathbb{D}}, hf) \cong \mathcal{K}(A',C)(g, hfy).$$

In the second point, assume we have a 2-cell β as pictured below, and we want to find a unique 2-cell solving this equation:

$$\begin{tikzcd} B \arrow[dr, "hk" description] \arrow[dr, bend left=60, "l"] \arrow[dr, phantom, bend left=30, "\Nearrow ?"] & \\ B' \arrow[u, "\iota"] \arrow[r, "f"'] \arrow[ur, phantom, near start, "\Nearrow \alpha"] & D \end{tikzcd} \quad = \quad \begin{tikzcd} B \arrow[dr, bend left=40, "l"] & \\ B' \arrow[u, "\iota"] \arrow[r, "f"'] \arrow[ur, phantom, "\Nearrow \beta"] & D \end{tikzcd} \tag{11}$$

First note that we have a unique 2-cell θ making the following diagram equal (here $l^{\mathbb{A}}$ is the left Kan extension of l along k that exists by assumption):

$$\begin{tikzcd} C \arrow[ddr, bend left=40, "l^{\mathbb{A}}"] \arrow[ddr, "h" description] \arrow[ddr, phantom, bend left=20, "\Nearrow \theta'"] & \\ B \arrow[u, "k"] \arrow[dr, "hk" description] & \\ B' \arrow[u, "\iota"] \arrow[r, "f"'] \arrow[ur, phantom, near start, "\Uparrow \alpha"] & D \end{tikzcd} \quad = \quad \begin{tikzcd} C \arrow[ddr, bend left=40, "l^{\mathbb{A}}"] & \\ B \arrow[u, "k"] \arrow[dr, "l" description] \arrow[ur, phantom, "\mathbb{A}\Uparrow"] & \\ B' \arrow[u, "\iota"] \arrow[r, "f"'] \arrow[ur, phantom, near start, "\beta\Uparrow"] & D \end{tikzcd}$$

Clearly, $\theta := \mathbb{A}^{-1} \circ \theta' k$ solves the equation (11), giving us **the existence** part of the proof. To show **the uniqueness**, let ϕ be a different 2-cell solving (11). Note that there exists a unique 2-cell ϕ' solving the following:

$$\begin{tikzcd} C \arrow[ddr, bend left=40, "l^{\mathbb{A}}"] & \\ B \arrow[u, "k"] \arrow[dr, bend left, "l" description] \arrow[dr, "hk"'] \arrow[ur, phantom, "\mathbb{A}\Nearrow"] \arrow[dr, phantom, bend left=10, "\phi\Nearrow"] & \\ & D \end{tikzcd} \quad = \quad \begin{tikzcd} C \arrow[ddr, bend left=40, "l^{\mathbb{A}}"] \arrow[ddr, "h" description] \arrow[ddr, phantom, bend left=20, "\Nearrow \phi'"] & \\ B \arrow[u, "k"] \arrow[dr, "hk"'] & \\ & D \end{tikzcd}$$

Pre-pasting this with α and using the diagram above this one, we see that $\phi' = \theta'$. From this we obtain:

$$\mathbb{A}^{-1} \circ \theta' k = \mathbb{A}^{-1} \circ \phi' k = \mathbb{A}^{-1} \circ \mathbb{A} \circ \phi = \phi.$$

□

Proposition 4.9. Let (D, y) be a left Kan pseudomonad on a 2-category $\mathcal{K}$. Denote by J_D the inclusion to the Kleisli 2-category and by D the endo-pseudofunctor associated to the left Kan pseudomonad. The following are equivalent for an object $A \in \mathcal{K}$:

(1) A admits the structure of a pseudo-D-algebra,

(2) for every object $B \in \mathcal{K}$, the left Kan extension of a 1-cell $B \to A$ along $y_B : B \to DB$ exists and has invertible unit. In other words, $\mathcal{K}(-, A) : \mathcal{K}^{op} \to \mathrm{Cat}$ sends each y_B to a coreflector,
(3) y_A admits a reflector (left adjoint with invertible counit),
(4) $\mathcal{K}(-, A)$ sends J_D-coreflection-inclusions in $\mathcal{K}$ to coreflectors in Cat,
(5) $\mathcal{K}(-, A)$ sends D-coreflection-inclusions in $\mathcal{K}$ to coreflectors in Cat.

Proof. The equivalence "(1) $\Leftrightarrow$ (3)" is well known, for lax-idempotent 2-monads this has been done for example in [10, Proposition 1.1.13], but the same argument works for lax-idempotent pseudomonads as well. "(1) $\Rightarrow$ (2)" is obvious.

For "(2) $\Rightarrow$ (3)", denote by $(a : DA \to A, \mathbb{A})$ the left Kan extension of 1_A along y_A. Because the identity 2-cell on DA exibits 1_{DA} as the left Kan extension of y_A along y_A, there exists a unique 2-cell η making these 2-cells equal:

$$\begin{tikzcd} & DA \arrow[dr, "a"] \arrow[d, phantom, "\Uparrow\mathbb{A}"] & & \\ A \arrow[ur, "y_A"] \arrow[rr, equal] & {} & A \arrow[r, "y_A"'] & DA \end{tikzcd} \qquad \begin{tikzcd} & & A \arrow[dr, "y_A"] \arrow[d, phantom, "\eta\Uparrow"] & \\ A \arrow[r, "y_A"] & DA \arrow[ur, "a"] \arrow[rr, equal] & {} & DA \end{tikzcd}$$

We will now show that $(\mathbb{A}^{-1}, \eta) : a \dashv y_A$ is an adjunction. The triangle identity $y_A\mathbb{A}^{-1} \circ \eta y_A = 1_{y_A}$ is guaranteed by the above formula – let us prove the other one:

$$\mathbb{A}^{-1}a \circ a\eta = 1_a.$$

Because a is the left Kan extension along y_A, it suffices to prove that both sides of this equation become equal after pre-composing them with y_A. It then becomes:

$$\mathbb{A}^{-1}ay_A \circ a\eta y_A = \mathbb{A}^{-1}ay_A \circ ay_A\mathbb{A} = \mathbb{A}^{-1}ay_A \circ \mathbb{A}ay_A = 1_{ay_A}.$$

"(4) $\Leftrightarrow$ (5)" follows from Lemma 4.7 and "(5) $\Rightarrow$ (2)" is obvious since y_B is a D-coreflection-inclusion.

We will now prove "(2) $\Rightarrow$ (5)". Let $f : B \to C$ such that there is an adjunction in $\mathcal{K}_D$ where the **unit** η is invertible:

$$(\epsilon, \eta) : \qquad \begin{tikzcd} DB \arrow[rr, bend right, "Df"'] \arrow[rr, phantom, "\top"] & & DC \arrow[ll, bend right, "r"'] \end{tikzcd}$$

We wish to show that the functor $f^* : \mathcal{K}(C, A) \to \mathcal{K}(B, A)$ has a left adjoint with invertible unit. We will define this left adjoint by the following formula:

$$L : (g : B \to A) \mapsto (g^{\mathbb{A}} \circ r \circ y_C : C \to A)$$

Define the component of the unit $\widetilde{\eta}$ at $g : B \to A$ as the following composite 2-cell:

$$\begin{tikzcd} C \arrow[rr, "y_C"] & & DC \arrow[dr, "r"] \arrow[d, phantom, "\eta\Uparrow"] & & \\ B \arrow[u, "f"] \arrow[r, "y_B"] \arrow[ur, phantom, "\mathbb{D}^{-1}\Uparrow"] \arrow[rrrr, bend right, "g"'] \arrow[rrrr, phantom, bend right=15, "\mathbb{A}\Uparrow"] & DB \arrow[ur, "Df"] \arrow[rr, equal] & {} & DB \arrow[r, "g^{\mathbb{A}}"] & A \end{tikzcd}$$

We wish to show that this has the universal property of the unit, in other words, $g^{\mathbb{A}} \circ r \circ y_C$ is the left Kan extension of $g : B \to A$ along $f : B \to C$.

By Lemma 4.8 **point 1**, $g^{\mathbb{A}} \circ r$ is the left Kan extension of g along $Df \circ y_B$. Equivalently it is a left Kan extension of g along $y_C f$ with the 2-cell component given by the composite 2-cell above. Since $g^{\mathbb{A}} r$ is a D-morphism, $g^{\mathbb{A}} r$ (with the identity 2-cell component) is the left Kan extension of $g^{\mathbb{A}} r y_C$ along y_C. By Lemma 4.8 **point 2**, for $h := g^{\mathbb{A}} r$, $k := y_C$, $i := f$, $f := g$ and α the 2-cell above, the result follows.

□

Remark 4.10. Using the terminology of [7, Definition 1.2], in Theorem 4.9, the equivalence "(1) $\Leftrightarrow$ (4)" says that an object A is a pseudo-D-algebra if and only if it is *left Kan injective* with respect to the class of 1-cells given by J_D-coreflection-inclusions.

Let us also note that a version of "(1) $\Rightarrow$ (5)" for D-left adjoints in Theorem 4.9 has already been proven in [5, Proposition 1.5].

Remark 4.11. Given a left Kan 2-monad (D, y), a pseudo-D-algebra C will be said to be *normal* if the left Kan extension 2-cell $\mathbb{C}_f$ in Definition 2.10 is the identity for all 1-cells f. Notice that a variation of Proposition 4.9 may be proven for normal pseudo-D-algebras, where we replace all invertible 2-cells by identities, for instance replace a "reflector" by a "lali".

In the remainder of this section we will demonstrate Proposition 4.9 on the case of small presheaf pseudomonad from Example 2.19. An application to the lax morphism classifier 2-comonads will be described in Section 5.

Example 4.12. Consider the small presheaf pseudomonad P on CAT. Note that if we pass to a bigger universe and use the bicategory PROF of locally small categories and **all** profunctors, for any functor $f : \mathcal{A} \to \mathcal{B}$, the small profunctor $Pf = \mathcal{B}(-, f-) : \mathcal{B}^{op} \times \times \mathcal{A} \to \mathrm{Set}$ has a right adjoint:

$$\begin{tikzcd} \mathcal{A} \arrow[rr, bend right, "\mathcal{B}(-,f-)"'] \arrow[rr, phantom, "\top"] & & \mathcal{B} \arrow[ll, bend right, "\mathcal{B}(f-,-)"'] \end{tikzcd}$$

We will call a functor $f : \mathcal{A} \to \mathcal{B}$ *small* if the right adjoint is also a small profunctor (belongs to Prof). Clearly, this happens if and only if Pf has a right adjoint in PROF.

Next, note that the unit of the adjunction is a collection of functions for every pair $(a', a'') \in \mathcal{A}^{op} \times \mathcal{A}$ like this:

$$\mathcal{A}(a', a'') \to \int^{b \in \mathcal{B}} \mathcal{B}(fa', b) \times \mathcal{B}(b, fa''),$$
$$(\theta : a' \to a'') \mapsto [1_{fa'}, f(\theta)].$$

As is readily seen, the unit is invertible if and only if f is fully faithful. So a functor $f : \mathcal{A} \to \mathcal{B}$ is a P-coreflection-inclusion if and only if it is fully faithful and small. The precomposition functor $f^* : \mathrm{CAT}(\mathcal{B}, \mathcal{C}) \to \mathrm{CAT}(\mathcal{A}, \mathcal{C})$ is a coreflector if and only if left Kan extensions along f exist in $\mathcal{C}$. Theorem 4.9 for the small presheaf pseudomonad now gives a folklore result: a category $\mathcal{C}$ is cocomplete if and only if left Kan extensions along small fully faithful functors exist in $\mathcal{C}$.

4.2. Colax adjunctions out of the Kleisli 2-category.

Proposition 4.13. Let (D, y) be a left Kan pseudomonad on a 2-category $\mathcal{K}$ and assume there are pseudofunctors G, H and a biadjunction as pictured below:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "J_D"'] & \mathcal{K}_D \arrow[r, "G"'] \arrow[phantom, u, "\top"] & \mathcal{L} \arrow[ll, bend right=30, "H"']
\end{tikzcd}$$

Then for every object $L \in \mathcal{L}$, the object HL admits the structure of a pseudo-D-algebra.

Proof. By Proposition 4.9, it suffices to show that $\mathcal{K}(-, HL) : \mathcal{K}^{op} \to \mathrm{Cat}$ sends J_D-coreflection-inclusions to coreflectors. Notice that we have the following pseudo-natural equivalence:

$$\mathcal{K}(-, HL) \simeq \mathcal{L}(GJ_D-, L) = \mathcal{L}(G-, L) \circ J_D.$$

Now, by definition, J_D sends J_D-coreflection-inclusions to coreflection-inclusions. Since $\mathcal{L}(G-, L)$ is a (contravariant) pseudofunctor, it sends coreflection-inclusions to coreflectors. We thus obtain the result. □

Remark 4.14. Going through the proof of Proposition 4.9 for the case of HL, we see that the algebra multiplication map $h_L : DHL \to HL$ (the reflector of the morphism $y_{HL} : HL \to DHL$) is given by the following composite:

$$DHL \xrightarrow{c_{DHL}} HGD^2HL \xrightarrow{HGp_{DHL}} HGDHL \xrightarrow{Hs_L} HL$$

Also, the counit of the adjunction $h_L \dashv y_{HL}$, an invertible 2-cell $\epsilon_L : h_L y_{HL} \Rightarrow 1_{HL}$, is given by the following:

$$\begin{tikzcd}
 & DHL \arrow[r, "c_{DHL}"] \arrow[d, phantom, "\Downarrow c_{y_{HL}}"] & HGD^2HL \arrow[r, "HGp_{DHL}"] \arrow[d, phantom, "\Downarrow (HG\Psi)_{HL}"] & HGDHL \arrow[dr, "Hs_L"] & \\
HL \arrow[ur, "y_{HL}"] \arrow[r, "c_{HL}"'] \arrow[rrrr, bend right, equal] \arrow[rrrr, phantom, bend right=15, "\Downarrow \tau_L^{-1}"] & HGDHL \arrow[ur, "HGDy_{HL}"] \arrow[urr, equal, bend right] & & & HL
\end{tikzcd}$$

Theorem 4.15 (The main colax adjunction theorem). Let (D, y) be a left Kan pseudomonad on a 2-category $\mathcal{K}$. Any biadjunction whose left adjoint factorizes through the Kleisli 2-category $\mathcal{K}_D$ induces a colax adjunction pictured below:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, "J_D"'] & \mathcal{K}_D \arrow[r, "G"'] \arrow[phantom, u, "\top"] & \mathcal{L} \arrow[ll, bend right=30, "H"'] & \rightsquigarrow & \mathcal{K}_D \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\mp"] & & \mathcal{L} \arrow[ll, bend right, "J_D H"']
\end{tikzcd}$$

Proof. Denote the unit, counit and the modifications of the biadjunction as follows:

$$\begin{aligned} s &: GJ_D H \Rightarrow 1_{\mathcal{L}}, & \sigma &: sGJ_D \circ GJ_D c \cong 1_{GJ_D}, \\ c &: 1_{\mathcal{K}} \Rightarrow HGJ_D, & \tau &: 1_H \cong Hs \circ cH. \end{aligned}$$

We will show that the components of the counit $s_L : GDHL \to L$ are coherently closed for G-lifts. By (the dual of) Theorem 3.3, there is a right colax adjoint to G. We will prove that it is isomorphic to $J_D H$.

Let us first prove the following: given a 1-cell $l : GDA \to L$ in $\mathcal{L}$, **any** pair (Dl', λ) where $l' : A \to HL$ is a 1-cell and λ is an invertible 2-cell as pictured below exhibits Dl' as the right G-lift of l along s_L:

$$\begin{tikzcd} L & GDHL \arrow[l, "s_L"'] \arrow[dl, phantom, "\lambda\Leftarrow"] \\ & GDA \arrow[ul, "l"] \arrow[u, "GDl'"'] \end{tikzcd}$$

By Theorem 4.13, HL has the structure of a D-algebra. Denoting its multiplication map by h_L as in Remark 4.14, we have the following composite adjunction with invertible counit:

$$\begin{tikzcd}[column sep=large] \mathcal{K}(A, HL) \arrow[r, "(y_{HL})_*"'] \arrow[rr, bend right, "J_D"', ""{name=J}] & \mathcal{K}(A, DHL) \arrow[l, bend right, "(h_L)_*"'] \arrow[l, phantom, shift right=3, "\perp"] \arrow[r, "p_{DHL} J_D(-)"'] \arrow[d, phantom, shift right=1, "\cong"] \arrow[u, phantom, "\ddot{\parallel}"] & \mathcal{K}_D(DA, DHL) \arrow[l, bend right, "U_D(-) y_A"'] \arrow[l, phantom, shift right=3, "\simeq"] \arrow[ll, bend right=50, "(-)^{\#}"'] \end{tikzcd} \tag{12}$$

Notice that there is an isomorphism:

$$\sqsupset : s_L G J_D (-)^{\#} \cong s_L G(-) : \mathcal{K}_D(DA, DHL) \to \mathcal{L}(GDA, L),$$

with the component at $f : DA \to DHL$ being given by the 2-cell[3]:

(13)
$$
\begin{tikzcd}
GD^2HL \arrow[r, "GDc_{DHL}"] \arrow[rrr, bend left, "GDh_L"] \arrow[dr, equal] \arrow[dr, phantom, shift left=3, "\Downarrow \sigma_{DHL}"] & GDHGD^2HL \arrow[r, "GDHGp_{DHL}"] \arrow[d, "s_{GD^2HL}"'] \arrow[dr, phantom, "\Downarrow s_{Gp_{DHL}}"] & GDHGDHL \arrow[r, "GDHs_L"] \arrow[d, "s_{GDHL}"'] \arrow[dr, phantom, "\Downarrow s_{s_L}"] & GDHL \arrow[d, "s_L"] \\
GD^2A \arrow[u, "GDf"] \arrow[dr, "Gp_{DA}"] & GD^2HL \arrow[r, "Gp_{DHL}"] \arrow[d, phantom, "\Downarrow (Gp)_f"] & GDHL \arrow[r, "s_L"'] & L \\
& GDA \arrow[ur, "Gf"'] & & \\
GDA \arrow[uu, "GDy_A"] \arrow[ur, equal] \arrow[u, phantom, shift left=6, "(G\Psi)_A \Downarrow"] & & &
\end{tikzcd}
$$

We have the following chain of bijections:

$$
\begin{aligned}
\mathcal{K}_D(DA, DHL)(f, Dl') &\overset{(A)}{\cong} \mathcal{K}(A, HL)(f^{\#}, l') \\
&\overset{(B)}{\cong} \mathcal{L}(GDA, L)(s_L \circ GDf^{\#}, s_L \circ GDl') \\
&\overset{(C)}{\cong} \mathcal{L}(GDA, L)(s_L \circ Gf, s_L \circ GDl') \\
&\overset{(D)}{\cong} \mathcal{L}(GDA, L)(s_L \circ Gf, l)
\end{aligned}
$$

The bijection **(A)** follows from the adjunction (12) above. **(B)** is given by the action on morphisms of the following functor:

(14)
$$s_L \circ GJ_D(-) : \mathcal{K}(A, HL) \to \mathcal{L}(GDA, L).$$

This functor is (by assumption) an equivalence – in particular it is fully faithful. **(C)** is given by the pre-composition with $\sqsupset_f^{-1}$ and **(D)** is given by the post-composition with λ. To conclude that (Dl', λ) is a G-lift, it has to be shown that the composite bijection is given by the assignment $\alpha \mapsto \lambda \circ s_L G\alpha$. Equivalently, the composite of the first three bijections is the assignment $\alpha \mapsto s_L G\alpha$. We prove this fact in the appendix as Lemma A.2.

The pair (Dl', λ) is thus a G-lift. Because the functor (14) is essentially surjective, such a G-lift is guaranteed to always exist. Make now a choice of a lift for every $l : GDA \to L$ and denote it by $(Dl^{\mathbb{L}}, \mathbb{L})$. To prove that our choice is coherently closed for G-lifts, the

[3]We have not shown it in this diagram, but the 2-cell has to be pre-composed with the associators for the pseudofunctor GD so that its source really equals $s_L GDf^{\#}$.

following 2-cell needs to be shown to be a G-lift of $l : GDA \to L$ along s_L:

$$\begin{tikzcd}
L & GDHL \arrow[l, "s_L"'] & \\
GDA \arrow[u, "l"] \arrow[ur, phantom, "\Downarrow \mathbb{L}"] \arrow[dr, phantom, "\Downarrow \mathbb{L}"] & GDHGDA \arrow[l, "s_{GDA}"'] \arrow[u, "GD(ls_{GDA})^{\mathbb{L}}"'] \arrow[r, phantom, "\overset{\gamma}{\Longrightarrow}"] & {} \\
& GDA \arrow[ul, equal] \arrow[u, "GD1_A^{\mathbb{L}}"'] \arrow[uu, bend right=60, "G(D(ls_{GDA})^{\mathbb{L}} \circ D1_A^{\mathbb{L}})"'] &
\end{tikzcd}$$

But this follows from the what we have shown at the beginning since this composite 2-cell is invertible and the 1-cell component of the proposed G-lift is (isomorphic to) Dh for a 1-cell h in $\mathcal{K}$. For the same reasons, the unit and composition axioms in the assumptions of Theorem 3.3 are satisfied.

We thus have a right colax adjoint to $G : \mathcal{K}_D \to \mathcal{L}$, let us denote it by $R : \mathcal{L} \to \mathcal{K}_D$. Since the pseudonaturality square of the counit s is a G-lift (this again follows from what we have proven at the beginning of the proof), for any 1-cell $l : L \to K$ there exists a unique invertible 2-cell $\delta_l : Rl \Rightarrow J_D H l$ making the following diagrams equal:

$$\begin{tikzcd}
L & GDHL \arrow[l, "s_L"'] \arrow[d, bend left=50, "GRl"] \\
K \arrow[u, "l"] \arrow[ur, phantom, "\Downarrow s_l"] & GDHK \arrow[l, "s_L"] \arrow[u, "GDHl"] \arrow[u, phantom, shift left=4, "\overset{G\delta_l}{\Leftarrow}"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
L & GDHL \arrow[l, "s_L"'] \\
K \arrow[u, "l"] \arrow[ur, phantom, "\Downarrow \mathbb{L}"] & GDHK \arrow[l, "s_K"] \arrow[u, "GRl"']
\end{tikzcd}$$

It is now routine to verify that this data gives an invertible icon $\delta : R \Rightarrow J_D H$ (which is an isomorphism in $\mathrm{Psd}[\mathcal{L}, \mathcal{K}_D]$), proving that the pseudofunctor $J_D H$ is right colax adjoint to $G : \mathcal{K}_D \to \mathcal{L}$ as well. $\square$

Our first application will be the following:

Corollary 4.16. Given a left Kan pseudomonad (D, y) on a 2-category $\mathcal{K}$, the biadjunction between the base 2-category and the Kleisli 2-category induces a colax adjunction on the Kleisli 2-category:

$$\begin{tikzcd}
\mathcal{K} \arrow[r, bend right, "J_D"'] \arrow[r, phantom, "\top"] & \mathcal{K}_D \arrow[l, bend right, "F_D"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\mathcal{K}_D \arrow[r, bend right, equal] \arrow[r, phantom, "\overline{\top}"] & \mathcal{K}_D \arrow[l, bend right, "J_D F_D"']
\end{tikzcd}$$

The following is a categorification of the fact that for an idempotent monad, the Kleisli and EM-categories are equivalent:

Corollary 4.17. Given a left Kan pseudomonad (D, y), the associated free-forgetful biadjunction induces a colax adjunction between the Kleisli 2-category and the 2-category of algebras:

$$\begin{tikzcd} \mathcal{K} \arrow[rr, bend right, "F^D"'] \arrow[rr, phantom, "\top"] && \text{Ps-}D\text{-Alg} \arrow[ll, bend right, "U^D"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathcal{K}_D \arrow[rr, hook, bend right] \arrow[rr, phantom, "\mp"] && \text{Ps-}D\text{-Alg} \arrow[ll, bend right, "J_D \circ U^D"'] \end{tikzcd}$$

The following is a change-of-base-style theorem:

Corollary 4.18. Let D be a lax-idempotent pseudomonad on a 2-category $\mathcal{K}$ and T be a pseudomonad on a 2-category $\mathcal{L}$. Assume that:

- there is a biadjunction between the base 2-categories:

$$\begin{tikzcd} \mathcal{K} \arrow[rr, bend right, "L"'] \arrow[rr, phantom, "\top"] && \mathcal{L} \arrow[ll, bend right, "R"'] \end{tikzcd}$$

- the left biadjoint admits an extension to the Kleisli 2-categories:

$$\begin{tikzcd} \mathcal{K} \arrow[r, "L"] \arrow[d, "J_D"'] & \mathcal{K} \arrow[d, "J_T"] \\ \mathcal{K}_D \arrow[r, "L^{\#}"'] & \mathcal{L}_T \end{tikzcd}$$

Then there is an induced colax adjunction between the Kleisli 2-categories:

$$\begin{tikzcd} \mathcal{K}_D \arrow[rr, bend right, "L^{\#}"'] \arrow[rr, phantom, "\mp"] && \mathcal{L}_T \arrow[ll, bend right] \end{tikzcd}$$

Proof. Composing the Kleisli biadjunction with the $L \dashv R$ biadjunction we obtain the following the following biadjunction on which we can apply the theorem:

$$\begin{tikzcd} \mathcal{K} \arrow[rr, bend right, "L"'] \arrow[rr, phantom, "\top"] \arrow[dr, bend right, "J_D"'] && \mathcal{L} \arrow[ll, bend right, "R"'] \arrow[rr, bend right, "T"'] \arrow[rr, phantom, "\top"] && \mathcal{L}_T \arrow[ll, bend right, "F_T"'] \\ & \mathcal{K}_D \arrow[urrr, bend right, "L^{\#}"'] \end{tikzcd}$$

$\square$

Example 4.19. Consider a 2-category $\mathcal{K}$ with comma objects and pullbacks and take for D the fibration 2-monad on $\mathcal{K}/C$ and T the fibration 2-monad on $\mathcal{K}/D$. For any 1-cell

$k : C \to D$ gives a 2-functor $k_* : \mathcal{K}/C \to \mathcal{K}/D$ with a right 2-adjoint k^* given by pulling back. The 2-functor k_* clearly extends to the colax slice 2-categories, hence giving rise to a **lax** adjunction between the colax slices:

$$\begin{tikzcd} \mathcal{K}//C \arrow[rr, bend right, "k_*"'] \arrow[rr, phantom, "\top"] & & \mathcal{K}//D \arrow[ll, bend right, "D\circ F_D\circ k^*"'] \end{tikzcd}$$

Example 4.20. In the next section (Corollary 5.10) we will see how, when given a morphism of 2-monads $\theta : S \to T$, this gives rise to a colax adjunction between T-Alg_l and S-Alg_l.

Remark 4.21 (Left Kan 2-monads)**.** Assume that (D, y) is a left Kan 2-monad and that we have the same starting biadjunction as in Theorem 4.15, except now the modifications σ, τ are the identities and the counit s is 2-natural. Going through the proof, note that $s_L \circ GJ_D(-) : \mathcal{K}(A, HL) \to \mathcal{L}(GDA, L)$ is an isomorphism of categories: for each $l : GDA \to L$ there is a **unique** $l^{\mathbb{L}} : A \to HL$ such that $s_L \circ GDl^{\mathbb{L}} = l$. Because $J_D : \mathcal{K} \to \mathcal{K}_D$ is now a 2-functor and because of the uniqueness of each $l^{\mathbb{L}}$, the collection $s_L : GDHL \to L$ is strictly closed for G-lifts (dual of Definition 3.7). By Remark 3.8 we obtain a colax adjunction for which the modifications are the identities and the counit s is 2-natural.

4.3. **Coreflector-limits.**

Definition 4.22. Let $\mathcal{K}$ be a 2-category and $F : \mathcal{J} \to \mathcal{K}$, $W : \mathcal{J} \to \mathrm{Cat}$ 2-functors. A *coreflector-limit* of F weighted by W is given by an object $L \in \mathcal{K}$ and a 2-natural transformation $\lambda : W \Rightarrow \mathcal{K}(L, F-)$ with the property that for every $A \in \mathcal{K}$, the *canonical comparison* functor

$$\kappa_A : \mathcal{K}(A, L) \to [\mathcal{J}, \mathrm{Cat}](W, \mathcal{K}(A, F?)),$$
$$\kappa_A : (\theta : A \to L) \mapsto (\mathcal{K}(\theta, F?) \circ \lambda),$$

is a coreflector in Cat. *Coreflector-colimits* in $\mathcal{K}$ are defined as coreflector-limits in $\mathcal{K}^{op}$. Analogously, we say that λ is an ***X**-limit* if κ_A is in class $\mathbf{X}$ of functors for every A.

Remark 4.23. Because the maps $\kappa_A : \mathcal{K}(A, L) \to [\mathcal{J}, \mathrm{Cat}](W, \mathcal{K}(A, F?))$ together form a 2-natural transformation $\kappa : \mathcal{K}(-, L) \Rightarrow [\mathcal{J}, \mathrm{Cat}](W, \mathcal{K}(-, F?))$, by [21, Theorem 1] the above definition is equivalent to requiring that κ is a coreflector in the 2-category $\mathrm{Colax}[\mathcal{K}, \mathrm{Cat}]$ (of 2-functors $\mathcal{K} \to \mathrm{Cat}$, colax natural transformations and modifications).

Remark 4.24 (Enriched weakness)**.** The notion of a coreflector-colimit is a special case of an *enriched weak colimit* in the sense of [17, Section 4]. The enriching category $\mathcal{V}$ is equal to Cat with the class $\mathcal{E}$ being functors that are coreflectors. In [17], the authors studied coreflector-colimits for which κ is actually a *retract equivalence* – meaning that the unit of the adjunction is the identity and the counit is invertible.

Remark 4.25. Conical (left-adjoint)-limits of 2-functors have first been introduced [8, I,7.9.1] under the name *quasi-limits*[4].

Remark 4.26 (Ordinary weakness)**.** Notice that every rali- and lali-limit cone λ is a *weak limit* of F weighted by W. What this means is that given a different cone $\mu : W \Rightarrow \mathcal{K}(A, F-
-)$, the left adjoint L_A to κ_A gives a comparison map $L_A\mu : A \to L$ such that:

$$\mu = \mathcal{K}(L\mu, F-) \circ \lambda.$$

This is like the definition of a 2-limit except that there is no uniqueness requirement. It is not the case that every weak limit is a rali-limit. This is because if the 2-category $\mathcal{K}$ is locally discrete, the notion of rali-limit coincides with an ordinary limit and not a weak limit.

Example 4.27. An object I in a 2-category $\mathcal{K}$ is lali-initial if the unique functor into the terminal category admits a left adjoint for every object $A \in \mathcal{K}$:

$$\begin{tikzcd} \mathcal{K}(I,A) \arrow[rr, bend right, "!"'] \arrow[rr, phantom, "\perp"] & & * \arrow[ll, bend right] \end{tikzcd}$$

Clearly, this happens if and only if the hom-category $\mathcal{K}(I, A)$ has an initial object for every $A \in \mathcal{K}$. For a particular example, consider the 2-category MonCat_l of monoidal categories and lax monoidal functors. The terminal monoidal category $*$ is lali-initial because for every monoidal category $\mathbb{A}$ we have an isomorphism between $\mathrm{MonCat}_l(*, \mathbb{A})$ and the category $\mathrm{Mon}(\mathbb{A})$ of monoids in $\mathbb{A}$, and this category has an initial object given by the monoidal unit of $\mathbb{A}$.

Example 4.28. Consider a 2-category $\mathcal{K}$ with a zero object $0 \in \mathcal{K}$ and with a further property that the zero morphism $0_{A,B} : A \to B$ is the initial object in $\mathcal{K}(A, B)$ for every pair of objects A, B. Then 0 is a conical lali-colimit of any 2-functor $F : \mathcal{J} \to \mathcal{K}$. This is because in the definition of a lali-colimit:

$$\begin{tikzcd} \mathcal{K}(0,A) \arrow[rr, bend right] \arrow[rr, phantom, "\top"] & & \mathrm{Cocone}(A,F) \arrow[ll, bend right] \end{tikzcd}$$

we have $\mathcal{K}(0, A) \cong *$, and so the question becomes whether the category of cocones of F with apex A has an initial object. But it does and it is given by the cocone whose components are the zero morphisms. This for instance applies to the poset-enriched categories Rel of sets and relations and Par of sets and partial functions.

Example 4.29. Let $\mathcal{K}$ be a 2-category with pullbacks and comma objects and consider the slice 2-category $\mathcal{K}/C$ and the colax slice 2-category $\mathcal{K}//C$ from Example 2.20. It is known

[4]In fact, the definition in [8] is stronger than ours because it requires the existence of a 2-functor picking the limits that is right lax adjoint to the constant embedding 2-functor $\mathcal{K} \to \mathcal{K}^{\mathcal{J}}$.

that the product of two objects f_1, f_2 in $\mathcal{K}/C$ is the diagonal in the pullback square of f_1, f_2 in $\mathcal{K}$:

$$\begin{tikzcd}
 & L \arrow[dl] \arrow[dr] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
A_1 \arrow[dr, "f_1"'] & & A_2 \arrow[dl, "f_2"] \\
 & C &
\end{tikzcd}$$

One guess would be that this becomes a weak product in $\mathcal{K}//C$ after applying the inclusion 2-functor $\mathcal{K}/C \hookrightarrow \mathcal{K}//C$, but that would be a wrong guess. To calculate the weak product of f_1, f_2 in $\mathcal{K}//C$, we first calculate the product of the comma object projections for f_1, f_2 (using the notation from Example 2.20) in $\mathcal{K}/C$ as pictured below left:

$$\begin{tikzcd}
 & L \arrow[dl, "\tau_1"'] \arrow[dr, "\tau_2"] \arrow[dd, phantom, very near start, "\lrcorner" rotate=-45] & \\
Pf_1 \arrow[dr, "\pi_{f_1}"'] & & Pf_2 \arrow[dl, "\pi_{f_2}"] \\
 & C &
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r, "\tau_i"] \arrow[dr, "l"'] & Pf_i \arrow[r, "\rho_i"] \arrow[d, "\pi_{f_i}"] \arrow[dr, phantom, "\Downarrow \chi_i"] & A_1 \arrow[d, "f_i"] \\
 & C \arrow[r, equal] & C
\end{tikzcd}$$

Denote $l := \pi_{f_1} \circ \tau_1$. The claim now is that the object $l \in \mathcal{K}//C$ together with the colax triangles $(\rho_i \tau_i, \chi_i \tau_i) : l \to f_i$ (here χ_i is the comma object square pictured above right) is the lali-product of f_1, f_2 in $\mathcal{K}//C$. We will establish why this is the case after we prove the main theorem in this section.

Example 4.30. *Bilimits* are a special case of coreflector-limits where κ_A is an equivalence for every $A \in \mathcal{K}$. In case κ_A is an isomorphism for every $A \in \mathcal{K}$, this is the notion of an ordinary *2-limit*.

Remark 4.31 (Uniqueness of rali-limits)**.** Rali-limits are **not** unique up to an equivalence. It is not even the case that given two rali-limit objects L_1, L_2, there exists a left (or right) adjoint 1-cell $L_1 \to L_2$. For a particular example, consider again the poset-enriched category Par of sets and partial functions. The empty set $\varnothing$ is the terminal object in Par, in particular it is rali-terminal. The singleton set $*$ is rali-terminal: the ordered set $\mathrm{Par}(A, *)$ has a maximal element given by the unique total function $! : A \to *$. They are also "normalized" in the sense that $1_\varnothing$ and 1_* are terminal objects in the hom ordered sets they belong to.

Since this 2-category is poset-enriched, equivalent objects would be isomorphic, and an isomorphism in Par has to be a total function. Thus $*$ and $\varnothing$ can not be equivalent in Par. Moreover, it can be seen that there is no left adjoint 1-cell $* \to \varnothing$.

We may now also give an example of two non-equivalent left colax adjoints that we have promised in Remark 2.8. It can be seen that given a 2-category $\mathcal{K}$, the 2-functor $* \to \mathcal{K}$ picking an object L is a colax left adjoint to the unique 2-functor $\mathcal{K} \to *$ if and only if L is rali-initial. In this colax adjunction, the modification Ψ is invertible if and only if the 1-cell 1_L is the initial object of $\mathcal{K}(L, L)$. Based on above paragraphs, the unique 2-functor $\mathrm{Par}^{op} \to *$ has two left colax adjoints that are not equivalent.

Theorem 4.32. Assume $\mathcal{K}$ is a 2-category that admits weighted J-indexed bilimits and assume D is a lax-idempotent pseudomonad on $\mathcal{K}$. Then, the Kleisli 2-category $\mathcal{K}_D$ admits weighted J-indexed coreflector-limits.

Proof. Let $G : \mathcal{P} \to \mathcal{K}_D$ be a 2-functor, write $\widetilde{G}$ for the 2-functor $DA \mapsto \mathcal{K}_D(DA, G?)$. Denoting again by U_D the canonical pseudofunctor $\mathcal{K}_D \to \mathcal{K}$, there is a biadjunction where the right biadjoint sends a weight W to the bilimit of $U_D G : \mathcal{P} \to \mathcal{K}$ weighted by W:

$$\begin{tikzcd} \mathcal{K} \arrow[r, hook, "J_D"'] & \mathcal{K}_D \arrow[r, "\widetilde{G}"'] \arrow[l, phantom, bend right=40, "\top"] & \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}]^{op} \arrow[ll, bend right=30, "\{-, U_D G\}"'] \end{tikzcd}$$

This is because of the following equivalences that are pseudo-natural with respect to $W \in [\mathcal{P}, \mathrm{Cat}]$ and $A \in \mathcal{K}$:

$$\begin{aligned} \mathcal{K}(A, \{W, U_D G\}) &\simeq \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}(A, U_D G-)) \\ &\simeq \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}_D(DA, G-)) \\ &= \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}]^{op}(\mathcal{K}_D(DA, G-), W). \end{aligned}$$

Transferring the identity across those equivalences, we see that the **counit** of the biadjunction is the composite pseudonatural transformation:

$$W \xRightarrow{\lambda_W} \mathcal{K}(\{W, U_D G\}, U_D G-) \xRightarrow{(p_{G-})_* \circ J_D} \mathcal{K}_D(D\{W, U_D G\}, G-),$$

where λ_W is the W-weighted bilimit cone for $U_D G : \mathcal{P} \to \mathcal{K}$. By Theorem 4.15 this induces a colax adjunction with the same counit:

$$\begin{tikzcd} \mathcal{K}_D \arrow[r, "\widetilde{G}"'] \arrow[r, phantom, bend left=40, "\mp"] & \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}]^{op} \arrow[l, bend right=50, "J_D\{-, U_D G\}"'] \end{tikzcd}$$

Notice that from the beginning of the proof of Theorem 3.4 it can be seen that there is an adjunction on hom categories described below whose unit is invertible:

$$\begin{tikzcd} \mathcal{K}_D(B, J_D\{W, U_D G\}) \arrow[rr, bend right=15, "\theta \mapsto \mathcal{K}_D(\theta, G-) \circ (p_{G-})_* \circ J_D \circ \lambda_W"'] \arrow[rr, phantom, "\top"] & & \mathrm{Psd}[\mathcal{P}, \mathrm{Cat}](W, \mathcal{K}_D(B, G-)) \arrow[ll, bend right=15] \end{tikzcd}$$

This exhibits the pseudonatural transformation $(p_{G-})_* \circ J_D \circ \lambda_W$ as the coreflector-limit of G weighted by W. $\square$

Example 4.33. The bicategory PROF of locally small categories and small profunctors is coreflector-complete. This is because by Example 2.19, it is a Kleisli bicategory for a left Kan pseudomonad on a complete 2-category CAT.

Example 4.34. The proof of Theorem 4.32 gives a concrete way to compute limits. Consider the **colax**-idempotent pseudomonad from Example 2.20. We can see that the process of computing **lali**-product of two objects $(f_1 : A_1 \to C, f_2 : A_2 \to C)$ in the colax-slice 2-category $\mathcal{K}//C$ agrees with the process described in Example 4.29.

Remark 4.35. Given a left Kan 2-monad (D, y), going through the proof of Theorem 4.32 (and considering Remark 4.21) we may now replace Psd$[\mathcal{P}, \mathrm{Cat}]$ by $[\mathcal{P}, \mathrm{Cat}]$ and the result can be changed to the claim that $\mathcal{K}_D$ admits J-indexed lali-limit whenever $\mathcal{K}$ admits them as 2-limits.

We end the section with introducing the concept of preservation of weak limits:

Definition 4.36. We say that a pseudofunctor $H : \mathcal{K} \to \mathcal{L}$ *preserves* **X**-limits (where **X** is any of the classes of morphisms in Definition 4.2 for the case of $\mathcal{K} = \mathrm{Cat}$) if, whenever $\lambda : W \Rightarrow \mathcal{K}(L, F-)$ exhibits L as a **X**-limit of $F : \mathcal{J} \to \mathcal{K}$ weighted by $W : \mathcal{J} \to \mathrm{Cat}$, the composite pictured below is an **X**-limit of HF weighted by W:

$$W \xRightarrow{\lambda} \mathcal{K}(L, F-) \xRightarrow{H} \mathcal{L}(HL, HF-)$$

Example 4.37. In case $\mathcal{K}, \mathcal{L}$ admit comma objects, their preservation as rari-limits has been studied in [24, Definition 7.1] where it has been called *preservation of lax pullbacks up to a right adjoint section*. For instance, given a finitely complete 2-category $\mathcal{K}$, the 2-functor $(-) \times Z$ has this property for any $Z \in \mathcal{K}$ (see [24, Example 7.3]). In Weber's later work [23, 6.1 THEOREM], the class of *familial* functors have been shown to preserve comma objects as lari-limits.

5. Applications to two-dimensional monad theory

Definition 5.1. Let T be a 2-monad on a 2-category $\mathcal{K}$. We will say that it satisfies **Property L** if the inclusion T-Alg$_s \hookrightarrow$ T-Alg$_l$ admits a left 2-adjoint and the corresponding lax-morphism classifier 2-comonad Q_l on T-Alg$_s$ is lax-idempotent.

By Theorem 2.22, Proposition 2.25, a 2-monad T on $\mathcal{K}$ will have this property when $\mathcal{K}$ admits oplax limits of an arrow and T-Alg$_s$ is sufficiently cocomplete (admits lax codescent objects).

To apply the (appropriate dual of the) results developed in Section 4.2 to the lax-idempotent 2-comonad Q_l, notice that (with the hint of the lists in Remark 2.16 and 4.2) this amounts to "going" from $\mathcal{K}$ to $\mathcal{K}^{coop}$. For instance "coreflection-inclusion" gets replaced by "reflector".

5.1. Lax flexibility.

For this section, recall the notions of *semiflexible* and *flexible* algebras for a 2-monad T from [2, Remark 4.5, page 23]. By [3, Proposition 1], a T-algebra (A, a) is semi-flexible if and only if it admits the structure of a pseudo-Q_p-coalgebra. A *pie* T-algebra was then defined to be a T-algebra that admits a strict Q_p-coalgebra structure. This motivates us to define:

Definition 5.2. Let T be a 2-monad on a 2-category $\mathcal{K}$ that satisfies **Property L**. A T-algebra (A, a) is said to be:

- *lax-semiflexible* if it admits a pseudo-Q_l-coalgebra structure.
- *lax-flexible* if it admits a normal pseudo-Q_l-coalgebra structure.
- *lax-pie* if it admits a strict Q_l-coalgebra structure.

Remark 5.3. Every lax-$\mathbf{Y}$ T-algebra is $\mathbf{Y}$, where $\mathbf{Y} \in \{$flexible, semiflexible, pie$\}$. This is because of the fact that by Proposition 2.26 there is an induced 2-functor from pseudo-Q_l-coalgebras to pseudo-Q_p-coalgebras that commutes with the 2-functors that forget the coalgebra structure (and thus keeps the T-algebra structure intact).

Example 5.4. In Corollary 5.9 we will see that every free T-algebra is lax-flexible; this is a strengthening of the fact that every free T-algebra is flexible ([2, Corollary 5.6]).

Example 5.5. Fix a category $\mathcal{J}$ and consider the 2-monad T on $[\mathrm{ob}\ \mathcal{J}, \mathrm{Cat}]$ whose algebras are weights (2-functors) $\mathcal{J} \to \mathrm{Cat}$. Weights that index *lax limits* are precisely the wights that are cofree-Q_l-coalgebras, i.e. those of the form $Q_l W$ (see [12, Chapter 5]). Since a lax limit is in general not a pseudo-limit [2, Remark 5.5], not every pie algebra is lax-pie.

Following Example 4.5 and Remark 4.11, the application of (the dual of) Proposition 4.9 to the lax-idempotent 2-comonad Q_l provides a lax version of [3, Theorem 20 a)]. It reads as:

Theorem 5.6. Let T be a 2-monad on a 2-category $\mathcal{K}$ satisfying **Property L** and denote by $U : \text{T-Alg}_s \to \mathcal{K}$ the forgetful 2-functor. A T-algebra is lax-semiflexible, semiflexible, if and only if, respectively:

- $\text{T-Alg}_s((A,a),-) : \text{T-Alg}_s \to \mathrm{Cat}$ sends U-reflectors to reflectors in Cat.
- $\text{T-Alg}_s((A,a),-) : \text{T-Alg}_s \to \mathrm{Cat}$ sends U-lalis to lalis in Cat.

Remark 5.7. In a future work we will study lax-pie T-algebras for a 2-monad T. Using a comonadicity theorem, it can be shown that when T is a 2-monad of form $\mathrm{Cat}(T')$ for a cartesian monad T' on a category $\mathcal{E}$ with pullbacks, lax-pie T-algebras are equivalent to T'-multicategories.

5.2. **Colax adjunctions and lali-cocompleteness of lax morphisms.** Considering Remark 4.21, the application of Theorem 4.13 and Theorem 4.15 for the 2-comonad Q_l reads as:

Theorem 5.8. Let T be a 2-monad satisfying **Property L**. Any 2-adjunction below left induces a colax adjunction below right:

$$
\begin{tikzcd}
\text{T-Alg}_s \arrow[r, "J"'] \arrow[rr, phantom, bend left=30, "\perp"] & \text{T-Alg}_l \arrow[r, "G"'] & \mathcal{L} \arrow[ll, bend right=30, "H"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\text{T-Alg}_l \arrow[r, "JH"'] \arrow[r, phantom, bend left=30, "\perp"] & \mathcal{L} \arrow[l, bend right=50, "G"']
\end{tikzcd}
$$

Moreover, for every $L \in \mathcal{L}$, the T-algebra HL is lax-flexible.

Corollary 5.9. The free-forgetful adjunction for a 2-monad T on a 2-category $\mathcal{K}$ satisfying **Property L** induces a colax adjunction between T-Alg_l and $\mathcal{K}$. In particular, every free T-algebra is lax-flexible.

$$
\begin{tikzcd}
\text{T-Alg}_s \arrow[r, "J"'] \arrow[rr, phantom, bend left=30, "\perp"] & \text{T-Alg}_l \arrow[r, "U"'] & \mathcal{K} \arrow[ll, bend right=30, "F^T"']
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
\text{T-Alg}_l \arrow[r, bend right=50, "U"'] \arrow[r, phantom, "\perp"] & \mathcal{K} \arrow[l, bend right=50, "JF^T"']
\end{tikzcd}
$$

Corollary 5.10. Let T, S be two 2-monads on a 2-category $\mathcal{K}$ satisfying **Property L** and let $\theta : S \to T$ be a strict monad morphism. Assume that the induced 2-functor $\theta^* : \text{T-Alg}_s \to S\text{-Alg}_s$ admits a left 2-adjoint θ_* (this is the case when $\mathcal{K}$ is complete and cocomplete and T is finitary, see [2, Theorem 3.9]). Then there is an induced colax adjunction between T-Alg_l and S-Alg_l:

$$\begin{tikzcd}
\text{T-Alg}_s \arrow[rr, phantom, "\top"] \arrow[rr, bend right, "\theta^*"'] \arrow[drr, bend right, "J"'] & & S\text{-Alg}_s \arrow[ll, bend right, "\theta_*"'] \arrow[rr, phantom, "\top"] \arrow[rr, bend right, "J"'] & & S\text{-Alg}_l \arrow[ll, bend right, "(-)'"'] & \rightsquigarrow & \text{T-Alg}_l \arrow[r, phantom, "\overline{\top}"] \arrow[r, bend right, "\theta^*"'] & S\text{-Alg}_l \arrow[l, bend right] \\
& & \text{T-Alg}_l \arrow[urr, bend right, "\theta^*"'] & & & & &
\end{tikzcd}$$

The following shows lali-cocompleteness of T-Alg_l:

Theorem 5.11. Let T be a 2-monad on a 2-category $\mathcal{K}$ that admits oplax limits of an arrow. Assume that T-Alg_s is cocomplete (in particular T satisfies **Property L**). Then T-Alg_l is lali-cocomplete.

Proof. This follows from (the dual of) Remark 4.35. □

Remark 5.12. By Remark 4.26, this in particular shows that T-Alg_l is weakly cocomplete.

Corollary 5.13. The following 2-categories are lali-cocomplete:
1. for a category $\mathcal{J}$, the 2-category $\text{Lax}[\mathcal{J}, \text{Cat}]$ of 2-functors $\mathcal{J} \to \text{Cat}$, lax-natural transformations and modifications,
2. the 2-category of monoidal categories and lax-monoidal functors and its symmetric/braided variants,
3. the 2-category of small 2-categories, lax functors, and icons,
4. for a set Φ of small categories, the 2-category $\Phi\text{-Colim}_l$ of small categories that admit a choice of J-indexed colimits for $J \in \Phi$ and **all** functors between them.

Proof. Each of these is a 2-category of form T-Alg_l, where $\mathcal{K}$ is a complete and cocomplete 2-category and T is one of the following 2-monads:
1. the 2-monad T on $[\text{ob}\, \mathcal{J}, \text{Cat}]$ given by the left Kan extension along $\text{ob}\, \mathcal{J} \to \mathcal{J}$ followed by restriction, see [2, 6.6],
2. the 2-monad on Cat for monoidal categories, see [16, 5.5],
3. the 2-category 2-monad T on the 2-category Cat-Gph of Cat-enriched graphs, see [3, 3.3],
4. the 2-monad T described in [13, Theorem 6.1] whose strict T-morphisms are functors that preserve the choices of Φ-colimits. Lax T-morphisms are all functors because this 2-monad is lax-idempotent by [13, Theorem 6.3].

□

Remark 5.14. There is also a dual version for the 2-category T-Alg_c of T-algebras and **colax** T-morphisms. If T-Alg_s is sufficiently cocomplete, there exists an induced 2-comonad Q_c (the *colax morphism classifier 2-comonad*) and if $\mathcal{K}$ admits lax limits of arrows, Q_c is colax-idempotent. If T-Alg_s is cocomplete, T-Alg_c can be seen to be rali-cocomplete.

Appendix A. Auxiliary lemmas

Lemma A.1. In the proof of Theorem 3.3:

- ϵ is colax-natural,
- Φ is a modification,
- the second swallowtail identity.

Proof. In this proof, we will reference the defining equations for $\gamma', \iota', F\alpha, \epsilon, \Psi$ above the equals sign. In the unlabeled equations we use the middle-four interchange rule combined with the pseudofunctor laws.

ϵ **is colax-natural**: The **composition axiom** amounts to proving the equality of the following 2-cells:

$$\begin{tikzcd}
FUB \arrow[r, equal] \arrow[dd, "FU(gh)"'] & FUB \arrow[r, equal] \arrow[dd, "F(Ug\circ Uh)"] & FUB \arrow[r, "\epsilon_B"] \arrow[d, "FUh"] & B \arrow[d, "h"] \\
{} \arrow[r, phantom, "\xRightarrow{F\gamma}"] & {} \arrow[r, phantom, "\xRightarrow{\gamma'}"] & FUC \arrow[r, "\epsilon_C"] \arrow[d, "FUg"] \arrow[u, phantom, "\Uparrow\epsilon_h", shift left=6] & C \arrow[d, "g"] \\
FUD \arrow[r, equal] & FUD \arrow[r, equal] & FUD \arrow[r, "\epsilon_D"'] \arrow[u, phantom, "\Uparrow\epsilon_g", shift left=6] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
FUB \arrow[rr, "\epsilon_B"] \arrow[dd, "FU(gh)"'] & & B \arrow[d, "h"] \\
& \Uparrow\epsilon_{gh} & C \arrow[d, "g"] \\
FUD \arrow[rr, "\epsilon_D"'] & & D
\end{tikzcd}$$

It is enough to prove these after applying $U(-) \circ \gamma y_{UB} \circ U\epsilon_D \mathbb{D} \circ \Phi_D U(gh)$ on both sides. We then have:

$$\begin{aligned}
&U(LHS) \circ \gamma^{-1} y_{UB} \circ U\epsilon_D \mathbb{D}_{y_{UC}U(gh)} \circ \Phi_D U(gh) = \\
&= U(g\epsilon_h \circ \epsilon_g FUh \circ \epsilon_D \gamma') y_{UB} \circ \gamma^{-1} y_{UB} \circ U\epsilon_D UF\gamma y_{UB} \circ U\epsilon_D \mathbb{D}_{y_{UC}U(gh)} \circ \Phi_D U(gh) \\
&\overset{(6)}{=} U(g\epsilon_h \circ \epsilon_g FUh \circ \epsilon_D \gamma') y_{UB} \circ \gamma^{-1} y_{UB} \circ U\epsilon_D \mathbb{D}_{y_{UC}UgUh} \circ \Phi_D UgUh \circ \gamma \\
&= U(g\epsilon_h \circ \epsilon_g FUh) y_{UB} \circ \gamma^{-1} y_{UB} \circ U\epsilon_D U\gamma' y_{UB} \circ U\epsilon_D \mathbb{D}_{y_{UC}UgUh} \circ \Phi_D UgUh \circ \gamma \\
&\overset{(7)}{=} U(g\epsilon_h \circ \epsilon_g FUh) y_{UB} \circ \gamma^{-1} y_{UB} \circ U\epsilon_D \gamma^{-1} y_{UB} \circ U\epsilon_g UFUg \mathbb{D}_{y_{UC}Uh} \circ \\
&\quad \circ U\epsilon_D \mathbb{D}_{y_{UD}Ug} Uh \circ \Phi_D UgUh \circ \gamma \\
&= U(g\epsilon_h) y_{UB} \circ \gamma^{-1} y_{UB} \circ U\epsilon_g UFUh y_{UB} \circ \gamma^{-1} UFUh y_{UB} \circ U\epsilon_D UFUg \mathbb{D}_{y_{UC}Uh} \circ \\
&\quad \circ U\epsilon_D \mathbb{D}_{y_{UD}Ug} Uh \circ \Phi_D UgUh \circ \gamma \\
&\overset{(9)}{=} U(g\epsilon_h) y_{UB} \circ \gamma^{-1} y_{UB} \circ \gamma^{-1} UFUh y_{UB} \circ UgU\epsilon_B \mathbb{D}_{y_{UC}Uh} \circ Ug\Phi_B Uh \circ \gamma \\
&= \gamma^{-1} y_{UB} \circ UgU\epsilon_h y_{UB} \circ Ug\gamma^{-1} y_{UB} \circ UgU\epsilon_B \mathbb{D}_{y_{UC}Uh} \circ Ug\Phi_B Uh \circ \gamma \\
&\overset{(9)}{=} \gamma^{-1} y_{UB} \circ Ug\gamma^{-1} y_{UB} \circ UgUh\Phi_B \circ \gamma \\
&= \gamma^{-1} y_{UB} \circ \gamma^{-1} U\epsilon_B y_{UB} \circ UgUh\Phi_B \circ \gamma \\
&\overset{(9)}{=} U(RHS) \circ \gamma^{-1} y_{UB} \circ U\epsilon_D \mathbb{D}_{y_{UC}U(gh)} \circ \Phi_D U(gh).
\end{aligned}$$

The **unit axiom** for ϵ amounts to showing that:

$$
\begin{tikzcd}
FUB \arrow[r, "\epsilon_B"] \arrow[d, "FU1_B"'] \arrow[dr, phantom, "\Uparrow \epsilon_{1_B}"] & B \arrow[d, equal] \\
FUB \arrow[r, "\epsilon_B"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
FUB \arrow[r, "F1_{UB}" description] \arrow[r, bend left=60, equal] \arrow[r, bend right=60, "FU1_B"'] \arrow[r, phantom, bend left=30, "\Uparrow \iota'"] \arrow[r, phantom, bend right=30, "\Uparrow F\iota"] & FUB \arrow[r, "\epsilon_B"] & B
\end{tikzcd}
$$

It suffices to prove that these 2-cells are equal after applying the 2-cell $U(-) \circ \gamma^{-1} y_{UB} \circ U\epsilon_D \mathbb{D}_{y_{UB}U1_B} \circ \Phi_D U1_B$ on both sides. This is done as follows:

$$
\begin{tikzcd}
U\epsilon_B UFU1_B y_{UB} \arrow[r, "\gamma^{-1} y_{UB}"] \arrow[dr, "U\epsilon_B UF\iota y_{UB}"] & U(\epsilon_B FU1_B) y_{UB} \arrow[rr, "U(\epsilon_B F\iota) y_{UB}"] & & U(\epsilon_B F1_{UB}) y_{UB} \arrow[d, "U(\epsilon_B \iota') y_{UB}"] \\
& U\epsilon_B UF1_B y_{UB} \arrow[urr, bend left=10, "\gamma^{-1} y_{UB}"] \arrow[r, "U\epsilon_B U\iota' y_{UB}"] & U\epsilon_B U1_{FUB} y_{UB} \arrow[r, "\gamma^{-1} y_{UB}"] \arrow[dr, phantom, "(8)"] & U\epsilon_B y_{UB} \\
U\epsilon_B y_{UB} U1_B \arrow[uu, "U\epsilon_B \mathbb{D}_{y_{UB}U1_B}"] \arrow[rrr, "U\epsilon_B y_{UB} \iota" description] \arrow[ur, phantom, "(6)"] & & & U\epsilon_B y_{UB} \arrow[u, "1"'] \arrow[ul, "U\epsilon_B \iota^{-1} y_{UB}"'] \arrow[ull, "U\epsilon_B \mathbb{D}_{y_{UB}}"] \\
& U1_B U\epsilon_B y_{UB} \arrow[urr, "\gamma^{-1} y_{UB}"] \arrow[r, phantom, "(9)"] & & \\
U1_B \arrow[uu, "\Phi_B U1_B"] \arrow[ur, "U1_B \Phi_B"] \arrow[r, "\Phi_B U1_B"'] & U\epsilon_B y_{UB} U1_B \arrow[r, "U\epsilon_B \mathbb{D}_{y_{UB}U1_B}"'] & U\epsilon_B UFU1_B y_{UB} \arrow[r, "\gamma^{-1} y_{UB}"'] & U(\epsilon_B \circ FU1_B) y_{UB} \arrow[uu, "U\epsilon_{1_B} y_{UB}"']
\end{tikzcd}
$$

The **local naturality** for ϵ amounts to showing that the 2-cells below are equal:

$$
\begin{tikzcd}
FUB \arrow[r, equal] \arrow[d, "FUh"'] \arrow[dr, phantom, "\overset{}{\Longrightarrow}_{FU\alpha}"] & FUB \arrow[r, "\epsilon_B"] \arrow[d, "FUk"] \arrow[dr, phantom, "\epsilon_k \Uparrow"] & B \arrow[d, "k"] \\
FUC \arrow[r, equal] & FUC \arrow[r, "\epsilon_C"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
FUB \arrow[r, "\epsilon_B"] \arrow[d, "FUh"'] \arrow[dr, phantom, "\Uparrow \epsilon_h"] & B \arrow[r, equal] \arrow[d, "h"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & B \arrow[d, "k"] \\
FUC \arrow[r, "\epsilon_C"'] & C \arrow[r, equal] & C
\end{tikzcd}
$$

An analogous approach will be done here as well, this time pre-composing with the 2-cell $U(-) \circ \gamma^{-1} y_{UB} \circ U\epsilon_D \mathbb{D}_{y_{UC}Uh} \circ \Phi_D Uh$:

$$
\begin{tikzcd}
Uh \arrow[rr, "\Phi_C Uh"] \arrow[dr, "U\alpha" description] \arrow[ddr, "Uh\Phi_B"'] \arrow[dd, "\Phi_D Uh"'] & & U\epsilon_C y_{UC} Uh \arrow[r, "U\epsilon_C \mathbb{D}_{y_{UC}Uh}"] \arrow[d, "U\epsilon_C y_{UC} U\alpha"'] \arrow[dr, phantom, "(6)"] & U\epsilon_C UFUh y_{UB} \arrow[r, "\gamma^{-1} y_{UB}"] \arrow[d, "U\epsilon_C UFU\alpha y_{UB}"] & U(\epsilon_C FUh) y_{UB} \arrow[d, "U(\epsilon_C FU\alpha) y_{UB}"] \\
& Uk \arrow[r, "\Phi_C Uk"] \arrow[drr, "Uk\Phi_B" description] & U\epsilon_C y_{UC} Uk \arrow[r, "U\epsilon_D \mathbb{D}_{y_{UC}Uk}"'] & U\epsilon_C UFUk y_{UB} \arrow[r, "\gamma^{-1} y_{UB}"'] \arrow[d, phantom, "(9)"] & U(\epsilon_C FUk) y_{UB} \arrow[dd, "U\epsilon_k y_{UB}"] \\
U\epsilon_C y_{UC} Uh \arrow[d, "U\epsilon_D \mathbb{D}_{y_{UC}Uh}"'] \arrow[r, phantom, "(9)"] & Uh U\epsilon_B y_{UB} \arrow[rr, "U\alpha U\epsilon_B y_{UB}"'] \arrow[dr, "\gamma^{-1} y_{UB}" description] & & Uk U\epsilon_B y_{UB} \arrow[dr, "\gamma^{-1} y_{UB}" description] & \\
U\epsilon_C UFUh y_{UB} \arrow[r, "\gamma^{-1} y_{UB}"'] & U(\epsilon_C FUh) \arrow[r, "U\epsilon_h y_{UB}"'] & U(h\epsilon_B) y_{UB} \arrow[rr, "U(\alpha\epsilon_B) y_{UB}"'] & & U(k\epsilon_B) y_{UB}
\end{tikzcd}
$$

Ψ **is a modification**: This amounts to showing that these 2-cells are equal:

$$
\begin{tikzcd}
FA \arrow[r, "Fy_A"] \arrow[rr, bend left=40, equal] \arrow[dr] \arrow[dr, bend right=50, "F(y_Bf)"'] \arrow[dr, phantom, bend right=25, "\Uparrow Fy_f"] & FUFA \arrow[r, "\epsilon_{FA}"] \arrow[d, "FUFf"] \arrow[dl, phantom, "\gamma' \Uparrow"] \arrow[r, phantom, bend left=30, "\Uparrow \Psi_A"] \arrow[dr, phantom, "\Uparrow \epsilon_{Ff}"] & FA \arrow[d, "Ff"] \\
& FUFB \arrow[r, "\epsilon_{FB}"'] & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[rr, bend left=30, equal] \arrow[dr, bend right=50, "F(y_Bf)"'] \arrow[dr, phantom, "\gamma' \Uparrow"] & FB \arrow[d, "Fy_B"] \arrow[dr, bend left=20, equal] \arrow[dr, phantom, "\Uparrow \Psi_B"] & FA \arrow[d, "Ff"] \\
& FUFB \arrow[r, "\epsilon_{FB}"'] & FB
\end{tikzcd}
$$

This time we precompose both sides with $U(-) \circ \gamma^{-1} y_{UB} \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}y_Bf} \circ \Phi_{FB}y_Bf$ to obtain:

$$
\begin{aligned}
&U(LHS)y_A \circ \gamma^{-1}y_{UB} \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}y_Bf} \circ \Phi_{FB}y_Bf = \\
&= U(Ff\Psi_A \circ \epsilon_{Ff}Fy_A)y_A \circ \gamma^{-1}y_A \circ U\epsilon_{FB}U\gamma' y_A \circ U\epsilon_{FB}UF\mathbb{D}_{y_Bf}y_A \circ \\
&\quad \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}y_Bf} \circ \Phi_{FB}y_Bf \\
&\overset{(6)}{=} U(Ff\Psi_A \circ \epsilon_{Ff}Fy_A)y_A \circ \gamma^{-1}y_A \circ U\epsilon_{FB}U\gamma' y_A \circ U\epsilon_{FB}\mathbb{D}_{UFfy_A} \circ \\
&\quad \circ U\epsilon_{FB}y_{UFB}\mathbb{D}_{y_Bf} \circ \Phi_{FB}y_Bf \\
&\overset{(7)}{=} U(Ff\Psi_A \circ \epsilon_{Ff}Fy_A)y_A \circ \gamma^{-1}y_A \circ U\epsilon_{FB}\gamma^{-1}y_A \circ U\epsilon_{FB}UFUFf\mathbb{D}_{y_{UFA}y_A} \circ \\
&\quad \circ U\epsilon_{FB}UFUFf\mathbb{D}_{y_{UFB}UFf}y_A \circ U\epsilon_{FB}y_{UFB}\mathbb{D}_{y_Bf} \circ \Phi_{FB}y_Bf \\
&= U(Ff\Psi_A)y_A \circ \gamma^{-1}y_A \circ U\epsilon_{Ff}UFy_Ay_A \circ U(\epsilon_{FB}FUFf)\mathbb{D}_{y_{UFA}y_A} \circ \\
&\quad \circ \gamma^{-1}y_{UFA}y_A \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}UFf}y_A \circ \Phi_{FB}UFfy_A \circ \mathbb{D}_{y_Bf} \\
&\overset{(9)}{=} U(Ff\Psi_A)y_A \circ \gamma^{-1}y_A \circ \gamma^{-1}UFy_Ay_A \circ UFfU\epsilon_{FA}\mathbb{D}_{y_{UFA}y_A} \circ \\
&\quad \circ UFf\Phi_Ay_A \circ \mathbb{D}_{y_Bf} \\
&= \gamma^{-1}y_A \circ UFfU\Psi_Ay_A \circ UFf\gamma^{-1}y_A \circ UFfU\epsilon_{FA}\mathbb{D}_{y_{UFA}y_A} \circ \\
&\quad \circ UFf\Phi_Ay_A \circ \mathbb{D}_{y_Bf} \\
&\overset{(10)}{=} \gamma^{-1}y_A \circ UFf\iota^{-1}y_A \circ \mathbb{D}_{y_Bf} \\
&\overset{(10)}{=} \gamma^{-1}y_A \circ U\Psi_BUFfy_A \circ U\epsilon_{FB}\gamma^{-1}y_A \circ U\epsilon_{FB}UFy_B\mathbb{D}_{y_Bf} \circ \\
&\quad \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}y_B}f \circ \Phi_{FB}y_Bf \\
&= U(\Psi_BFf)y_A \circ \gamma^{-1}y_A \circ U\epsilon_{FB}\gamma^{-1}y_A \circ U\epsilon_{FB}UFy_B\mathbb{D}_{y_Bf} \circ \\
&\quad \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}y_B}f \circ \Phi_{FB}y_Bf \\
&\overset{(7)}{=} U(\Psi_BFf)y_A \circ \gamma^{-1}y_A \circ U\epsilon_{FB}U\gamma' y_A \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}y_Bf} \circ \Phi_{FB}y_Bf \\
&= U(RHS)y_A \circ \gamma^{-1}y_{UB} \circ U\epsilon_{FB}\mathbb{D}_{y_{UFB}y_Bf} \circ \Phi_{FB}y_Bf.
\end{aligned}
$$

The second swallowtail identity: This amounts to showing the following equality, which we will again do by an appropriate pre-composition:

$$
\begin{tikzcd}[column sep=large, row sep=large]
FUB \arrow[dr, "Fy_{UB}"] \arrow[drr, equal, bend left=20] \arrow[ddr, "F(U\epsilon_B y_{UB})" near end] \arrow[ddr, bend right=40, "F1_{UB}"'] \arrow[dr, phantom, shift left=6ex, "\overset{\Psi_{UB}}{\Longrightarrow}"] \arrow[dr, phantom, shift right=3ex, "\overset{\gamma'}{\Longrightarrow}"] \arrow[ddr, phantom, shift right=5ex, "\underset{(F\Phi)_B}{\Longrightarrow}"] & & \\
 & FUFU \arrow[r, "\epsilon_{FUB}"] \arrow[d, "FU\epsilon_B"'] & FUB \arrow[d, "\epsilon_B"] \\
 & FUB \arrow[r, "\epsilon_B"'] \arrow[ur, equal, "\epsilon_{\epsilon_B}"] & B
\end{tikzcd}
\qquad = \qquad \epsilon_B \iota
$$

$$
\begin{aligned}
&U(LHS)y_{UB} \circ \gamma^{-1}y_{UB} \circ U\epsilon_B\mathbb{D}_{y_{UB}} \circ \Phi_B = \\
&= U(\epsilon_B\Psi_{UB} \circ \epsilon_{\epsilon_B}Fy_{UB})y_{UB} \circ \gamma^{-1}y_{UB} \circ U\epsilon_B U\gamma' y_{UB} \circ U\epsilon_B UF\Phi_B y_{UB} \circ \\
&\quad \circ U\epsilon_B\mathbb{D}_{y_{UB}} \circ \Phi_B \\
&\overset{(6)}{=} U(\epsilon_B\Psi_{UB} \circ \epsilon_{\epsilon_B}Fy_{UB})y_{UB} \circ \gamma^{-1}y_{UB} \circ U\epsilon_B U\gamma' y_{UB} \circ U\epsilon_B\mathbb{D}_{y_{UB}U\epsilon_B y_{UB}} \circ \\
&\quad \circ \Phi_B U\epsilon_B y_{UB} \circ \Phi_B \\
&\overset{(7)}{=} U(\epsilon_B\Psi_{UB} \circ \epsilon_{\epsilon_B}Fy_{UB})y_{UB} \circ \gamma^{-1}y_{UB} \circ U\epsilon_B\gamma^{-1}y_{UB} \circ \\
&\quad \circ U\epsilon_B UFU\epsilon_B\mathbb{D}_{y_{UFUB}y_{UB}} \circ U\epsilon_B\mathbb{D}_{y_{UB}U\epsilon_B}y_{UB} \circ \Phi_B U\epsilon_B y_{UB} \circ \Phi_B \\
&= U(\epsilon_B\Psi_{UB})y_{UB} \circ \gamma^{-1}y_{UB} \circ U\epsilon_{\epsilon_B}UFy_{UB}y_{UB} \circ \gamma^{-1}UFy_{UB}y_{UB} \circ \\
&\quad \circ U\epsilon_B UFU\epsilon_B\mathbb{D}_{y_{UFUB}y_{UB}} \circ U\epsilon_B\mathbb{D}_{y_{UB}\epsilon_B}y_{UB} \circ U\epsilon_B y_{UB}\Phi_B \circ \Phi_B \\
&\overset{(9)}{=} U(\epsilon_B\Psi_{UB})y_{UB} \circ \gamma^{-1}y_{UB} \circ \gamma^{-1}UFy_{UB}y_{UB} \circ U\epsilon_B U\epsilon_{FUB}\mathbb{D} \circ \\
&\quad \circ U\epsilon_B\Phi_B y_{UB} \circ \Phi_B \\
&= \gamma^{-1}y_{UB} \circ U\epsilon_B U\Psi_B y_{UB} \circ U\epsilon_B U\epsilon_{FUB}\mathbb{D}_{y_{UFUB}y_{UB}} \circ U\epsilon_B\Phi_B y_{UB} \circ \Phi_B \\
&\overset{(10)}{=} \gamma^{-1}y_{UB} \circ U\epsilon_B\iota^{-1}y_{UB} \circ \Phi_B \\
&\overset{(8)}{=} \gamma^{-1}y_{UB} \circ U\epsilon_B U\iota' y_{UB} \circ U\epsilon_B\mathbb{D}_{y_{UB}} \circ \Phi_B \\
&= U(RHS)y_{UB} \circ \gamma^{-1}y_{UB} \circ U\epsilon_B\mathbb{D}_{y_{UB}} \circ \Phi_B.
\end{aligned}
$$

□

Lemma A.2. The composite bijection **(A)+(B)+(C)** in the proof of Theorem 4.15:

$$\mathcal{K}_D(DA, DHL)(f, Dl') \cong \mathcal{L}(GDA, L)(s_L \circ Gf, s_L \circ GDl'),$$

is given by the assignment:

$$\alpha \mapsto s_L G\alpha.$$

Proof. Because of the swallowtail identity for the biadjunction in Proposition 2.13, it can be seen that the counit of the adjunction (12) evaluated at $f : A \to HL$ is equal to the following (using notation from Remark 4.14):

$$
\begin{tikzcd}
A \arrow[r, "y_A"] \arrow[dr, "f"'] & DA \arrow[r, "Df"] \arrow[d, phantom, "\Downarrow y_f^{-1}"] & DHL \arrow[r, "h_L"] \arrow[d, phantom, "\Downarrow \epsilon_L"] & HL \\
& HL \arrow[ur, "y_{HL}"] \arrow[urr, equal, bend right] & &
\end{tikzcd}
$$

The composite bijection **(A)+(B)+(C)** is thus the assignment:

$$\alpha \mapsto s_L \circ GD(\epsilon_L f \circ h_L y_{Dl'}^{-1} \circ h_L \alpha y_A) \circ \beth_f^{-1}.$$

Unwrapping the definitions of variables $h_L, \epsilon_L, \beth_f^{-1}$, what we need to show that the composite 2-cell below equals $s_L G\alpha$ (note that we use the same convention for the modifications on which a pseudofunctor is applied as in Definition 2.6):

$$
\begin{tikzcd}
& & & GDA \arrow[dr, "Gf"] & & \\
& & & GD^2HL \arrow[r, "Gp_{DHL}"] & GDHL \arrow[r, "s_L"] & L \\
GDA \arrow[r, "GDy_A"'] \arrow[urrr, equal, bend left] \arrow[dr, bend right, "GDl'"'] & GD^2A \arrow[uurr, "Gp_{DA}"] \arrow[r, bend left, "GDf"] \arrow[r, bend right, "GD^2l'"'] \arrow[r, phantom, "\Downarrow GD\alpha"] \arrow[ur, phantom, "\Downarrow (G\Psi)_A^{-1}"] & GD^2HL \arrow[r, "GDc_{DHL}"'] \arrow[ur, equal] \arrow[u, phantom, "\Downarrow (Gp)_f^{-1}"] & GDHGD^2HL \arrow[r] \arrow[u, "s_{GD^2HL}"] \arrow[ul, phantom, "\Downarrow \sigma_{DHL}^{-1}"] \arrow[ur, phantom, "\Downarrow s_{Gp_{DHL}}^{-1}"] & GDHGDHL \arrow[r, "GDHs_L"'] \arrow[u, "s_{GDHL}"] \arrow[ur, phantom, "\Downarrow s_{s_L}^{-1}"] & GDHL \arrow[u, "s_L"'] \\
& GDHL \arrow[r] \arrow[ur, "GDy_{HL}"] \arrow[u, phantom, "\Downarrow (GDy)_{l'}"] \arrow[urrrr, equal, bend right] \arrow[urrrr, phantom, "\Downarrow (GD\tau)_L^{-1}"'] & GDHGDHL \arrow[ur, "GDHGDy_{HL}"] \arrow[urr, equal, bend right] \arrow[u, phantom, "\Downarrow (GDc)_{y_{HL}}"] \arrow[ur, phantom, "\Downarrow (GDHG\Psi)_{HL}"'] & & &
\end{tikzcd}
$$

The diagram below proves this equality:

$$
\begin{tikzcd}
s_LGf \arrow[r, "s_LGf(G\Psi)_A^{-1}"] \arrow[d, "s_LG\alpha"'] \arrow[dr, phantom, "(*)"] & s_LGfGp_{DA}GDy_A \arrow[rr, "s_L(Gp)_f^{-1}GDy_A"] \arrow[d, "s_LG\alpha Gp_{DA}GDy_A"] \arrow[drr, phantom, "(a)"] & & s_LGp_{DHL}GDfGDy_A \arrow[d, "s_LGp_{DHL}GD\alpha GDy_A"] \\
s_LGDl' \arrow[r, "s_LGDl'(G\Psi)_A^{-1}"] \arrow[d, "1"'] \arrow[drrr, "s_L(G\Psi)_{HL}^{-1}GDl'"] & s_LGDl'Gp_{DA}GDy_A \arrow[rr, "s_L(Gp)_{Dl}^{-1}GDy_A"] \arrow[rr, phantom, "(b)"'] & & s_LGp_{DHL}GD^2l'GDy_A \arrow[d, "s_LGp_{DHL}GDy_{l'}^{-1}"] \\
s_LGDl' \arrow[dd, "1"'] \arrow[ddddr, "s_L\sigma_{HL}^{-1}GDl'"] \arrow[drr, phantom, "(*)"] & & & s_LGp_{DHL}GDy_{HL}GDl' \arrow[lll, "s_L(G\Psi)_{HL}GDl'"] \arrow[dll, "s_LGp_{DHL}GDy_{HL}\sigma_{HL}^{-1}GDl'"'] \arrow[d, "s_LGp_{DHL}\sigma_{DHL}^{-1}GDy_{HL}GDl'"] \\
& s_LGp_{DHL}GDy_{HL}s_{GDHL}GDc_{HL}GDl' \arrow[dd, "s_L(G\Psi)_{HL}s_{GDHL}GDc_{HL}GDl'"] & (e) & s_LGp_{DHL}s_{GD^2HL}GDc_{DHL}GDy_{HL}GDl' \arrow[d, "s_LGp_{DHL}s_{GD^2HL}GDc_{y_{HL}}GDl'"] \\
s_LGDl' & & & s_LGp_{DHL}s_{GD^2HL}GDHGDy_{HL}GDc_{HL}GDl' \arrow[ull, "s_LGp_{DHL}s_{GDy_{HL}}GDc_{HL}GDl'"] \arrow[d, "s_Ls_{Gp_{DHL}}^{-1}GDHGDy_{HL}GDc_{HL}GDl'"] \arrow[dll, phantom, "(d)"] \\
s_LGDHs_LGDc_{HL}GDl' \arrow[u, "s_L(GD\tau)_L^{-1}GDl'"] & s_Ls_{GDHL}GDc_{HL}GDl' \arrow[l, "s_{s_L}^{-1}GDc_{HL}GDl'"] \arrow[ul, "s_L\sigma_{HL}GDl'"] \arrow[l, phantom, "(c)", shift left=4] & & s_Ls_{GDHL}GDHGp_{DHL}GDHGDy_{HL}GDc_{HL}GDl' \arrow[ll, "s_Ls_{GDHL}(GDHG\Psi)_{HL}GDc_{HL}GDl'"]
\end{tikzcd}
$$

In this diagram:

- (a) is the local naturality of $(Gp)^{-1}$,
- (b) is the modification axiom for $(G\Psi)^{-1}$
- (c) is the swallowtail identity for (s, c),

- (d) is the equation derived from the local naturality of s,
- (e) is the modification axiom for σ^{-1},
- $(*)$'s are the middle-four-interchange laws.

□

References

[1] Nathanael Arkor, Ivan Di Liberti, and Fosco Loregian. Adjoint functor theorems for lax-idempotent pseudomonads. *ArXiv preprint arXiv:2306.10389*, 2023.

[2] Robert Blackwell, Gregory M Kelly, and A John Power. Two-dimensional monad theory. *Journal of Pure and Applied Algebra*, 59(1):1–41, 1989.

[3] John Bourke and Richard Garner. On semiflexible, flexible and pie algebras. *Journal of Pure and Applied Algebra*, 217(2):293–321, 2013.

[4] Marta Bunge. Coherent extensions and relational algebras. *Transactions of the American Mathematical Society*, 197:355–390, 1974.

[5] Marta Bunge and Jonathon Funk. On a bicomma object condition for KZ-doctrines. *Journal of Pure and Applied Algebra*, 143(1-3):69–105, 1999.

[6] Maria Manuel Clementino and Fernando Lucatelli Nunes. Lax comma 2-categories and admissible 2-functors. *ArXiv preprint arXiv:2002.03132*, 2020.

[7] Ivan Di Liberti, Gabriele Lobbia, and Lurdes Sousa. KZ-pseudomonads and Kan injectivity. *ArXiv preprint arXiv:2211.00380*, 2022.

[8] John Walker Gray. *Formal category theory: adjointness for 2-categories*, volume 391. Springer, 2006.

[9] Niles Johnson and Donald Yau. *2-dimensional categories*. Oxford University Press, USA, 2021.

[10] Peter T Johnstone. *Sketches of an elephant: a topos theory compendium: volume 2*, volume 2. Oxford University Press, 2002.

[11] G Max Kelly. Doctrinal adjunction. In *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, pages 257–280. Springer, 2006.

[12] Gregory Maxwell Kelly. Elementary observations on 2-categorical limits. *Bulletin of the Australian Mathematical Society*, 39(2):301–317, 1989.

[13] Gregory Maxwell Kelly and Stephen Lack. On the monadicity of categories with chosen colimits. *Theory and Applications of Categories*, 7(7):148–170, 2000.

[14] Stephen Lack. Codescent objects and coherence. *Journal of Pure and Applied Algebra*, 175(1-3):223–241, 2002.

[15] Stephen Lack. Limits for lax morphisms. *Applied Categorical Structures*, 13:189–203, 2005.

[16] Stephen Lack. A 2-categories companion. In *Towards Higher Categories*, pages 105–191. Springer, 2009.

[17] Stephen Lack and Jiří Rosickỳ. Enriched weakness. *Journal of Pure and Applied Algebra*, 216(8-9):1807–1822, 2012.

[18] Stephen Lack and Michael Shulman. Enhanced 2-categories and limits for lax morphisms. *Advances in Mathematics*, 229(1):294–356, 2012.

[19] Francisco Marmolejo and Richard James Wood. Kan extensions and lax idempotent pseudomonads. *Theory and Applications of Categories*, 26(1):1–29, 2012.

[20] Robert Paré. Three easy pieces: Imaginary seminar talks in honour of Bob Rosebrugh. *Theory and Applications of Categories*, 36(6):171–200, 2021.

[21] Ross Street. Two constructions on lax functors. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 13(3):217–264, 1972.

[22] Ross Street. Fibrations and Yoneda's lemma in a 2-category. In *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, pages 104–133. Springer, 2006.

[23] Mark Weber. Familial 2-functors and parametric right adjoints. *Theory and Applications of Categories*, 18(22):665–732, 2007.

[24] Mark Weber. Yoneda structures from 2-toposes. *Applied Categorical Structures*, 15:259–323, 2007.