VARIATIONAL INEQUALITIES FOR THE ORNSTEIN-UHLENBECK SEMIGROUP: THE HIGHER-DIMENSIONAL CASE

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ABSTRACT. We study the ϱ -th order variation seminorm of a general Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t)_{t>0}$ in \mathbb{R}^n , taken with respect to t. We prove that this seminorm defines an operator of weak type (1,1) with respect to the invariant measure when $\varrho > 2$. For large t, one has an enhanced version of the standard weak-type (1,1) bound. For small t, the proof hinges on vector-valued Calderón–Zygmund techniques in the local region, and on the fact that the t derivative of the integral kernel of \mathcal{H}_t in the global region has a bounded number of zeros in (0,1]. A counterexample is given for $\varrho = 2$; in fact, we prove that the second order variation seminorm of $(\mathcal{H}_t)_{t>0}$, and therefore also the ϱ -th order variation seminorm for any $\varrho \in [1,2)$, is not of strong nor weak type (p,p) for any $p \in [1,\infty)$ with respect to the invariant measure.

1. Introduction

In this paper we prove the weak type (1,1) for the variation operator of a general Ornstein–Uhlenbeck semigroup $(\mathcal{H}_t)_{t>0}$ in \mathbb{R}^n for any $n \geq 1$.

Recently the authors proved this result in dimension one [15], answering a question asked by Almeida et al. in [2, p. 31]. In this article we provide an answer to the question in [2] covering the higher–dimensional case. The methods we use are different from those of [15], which seem hard to adapt to the case n > 1.

The proof relies on extensive application of properties of the Ornstein-Uhlenbeck semigroup and its integral kernel $K_t(x, u)$ (known as the Mehler kernel), recently proved by the authors in a series of papers concerning various issues arising from harmonic analysis (see [10, 11, 12, 13, 14, 15] and also the brief introductory summary [16]). We also apply vector-valued Calderón-Zygmund theory. As far as we know, the first application of this theory in this context was in [17] (see also [42, 21], where [17] has its roots).

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Let $(T_t)_{t>0}$ be a family of bounded operators between spaces of functions defined on a measure space (X, μ) . The ϱ -th order variation operator of $(T_t)_{t>0}$ on an interval $I \subset \mathbb{R}_+$ is defined, for $1 \leq \varrho < \infty$ and suitable f, by

$$||T_t f(x)||_{v(\varrho),I} = \sup \left(\sum_{i=1}^N |T_{t_i} f(x) - T_{t_{i-1}} f(x)|^{\varrho}\right)^{1/\varrho}, \quad x \in X,$$
 (1.1)

where the supremum is taken over all finite, increasing sequences $(t_i)_0^N$ of points in I. In the last fifty years, variational inequalities, stating that the $L^p(\mu)$ norm of (1.1) is bounded by a constant times the $L^p(\mu)$ norm of f, have been widely investigated. The first bounds, due to D. Lépingle [30] for bounded martingales and to V. F. Gaposhkin [18, 19] and J. Bourgain [7] for ergodic averages, have been generalized in various directions. We refer to [15] for a brief description of recent results with a focus on harmonic analysis; see [3, 9, 22, 23, 25, 26, 27, 20, 4, 5, 6, 1, 31, 32]. More recently, the study of oscillatory and jump inequalities, in addition to variational inequalities, began to develop from both an analytic and a number—theoretic perspective; we refer in particular to [8, 33, 34, 35, 36, 37, 38]. For an overview of the connections with ergodic theory, analytic number theory, and harmonic analysis, see in particular [28].

In this paper the family $(T_t)_{t>0}$ in (1.1) will always be the Ornstein-Uhlenbeck semigroup $(\mathcal{H}_t)_{t>0}$ in \mathbb{R}^n . This is the semigroup generated by the elliptic operator

$$\mathcal{L} = \frac{1}{2} \operatorname{tr}(Q\nabla^2) + \langle Bx, \nabla \rangle, \tag{1.2}$$

called the Ornstein-Uhlenbeck operator. Here ∇ is the gradient and ∇^2 the Hessian matrix. Moreover, Q is a real, symmetric and positive definite $n \times n$ matrix, called the covariance of \mathcal{L} , and B is a real $n \times n$ matrix whose eigenvalues have negative real parts; B indicates the drift of \mathcal{L} . In Section 2 we will provide explicit expressions for \mathcal{H}_t , seen as an integral operator with a kernel $K_t(x, u)$. It is well known that there exists an invariant measure under the action of \mathcal{H}_t , unique up to a positive factor. This measure, denoted by γ_{∞} , is basic in Ornstein-Uhlenbeck theory; it is described explicitly in Section 2.

In 2001 Jones and Reinhold [23] proved that for $\varrho > 2$ the variation operator of any symmetric diffusion semigroup is L^p bounded for $1 . Ten years later Le Merdy and Xu [29] extended this result to a nonsymmetric context. In fact, Corollary 4.5 in [29] applies to <math>\mathcal{H}_t$ (see [2, p. 31] for a discussion). It says that the operator given by $f \mapsto \|\mathcal{H}_t f(x)\|_{v(\varrho),\mathbb{R}_+}$, is bounded on $L^p(\gamma_\infty)$ for $1 and <math>\varrho > 2$. Our result gives the corresponding weak type (1,1), as follows.

Theorem 1.1. For each $\varrho > 2$ the operator that maps $f \in L^1(\gamma_\infty)$ to the function

$$\|\mathcal{H}_t f(x)\|_{v(\varrho),\mathbb{R}_+}, \quad x \in \mathbb{R}^n,$$

where the $v(\varrho)$ seminorm is taken in the variable t, is of weak type (1,1) with respect to the measure γ_{∞} .

In other words, the inequality

$$\gamma_{\infty}\{x \in \mathbb{R}^n : \|\mathcal{H}_t f(x)\|_{v(\varrho),\mathbb{R}_+} > \alpha\} \le \frac{C}{\alpha} \|f\|_{L^1(\gamma_{\infty})}, \qquad \alpha > 0, \tag{1.3}$$

holds for some C > 0 and all functions $f \in L^1(\gamma_{\infty})$.

The pointwise estimates of the Mehler kernel and its derivative needed to prove Theorem 1.1 are quite different for small and large t. Thus we are led to distinguish between the variation in the interval $0 < t \le 1$ and that in $1 \le t < \infty$.

The variation in $[1, \infty)$ is treated following a geometric approach recently developed by the authors in [10, 11, 12, 14], which relies on a system of polar-like coordinates. Proposition 4.3 is actually an enhanced version of (1.3) with \mathbb{R}_+ replaced by $[1, \infty)$.

The study of the variation in (0,1] is more delicate and requires a further distinction between local and global parts of the Ornstein-Uhlenbeck semigroup operator. In the Ornstein-Uhlenbeck setting, the local part is usually defined by the relation $|x-u| \lesssim 1/(1+|x|)$ between the two variables of the Mehler kernel. This splitting was first introduced by Muckenhoupt [39] in one dimension, and by the third author [43] in higher dimension. It has since been widely used in the literature, in particular in [40]. In this paper, Pérez shows that the local parts of many operators related to the Ornstein-Uhlenbeck semigroup behave precisely like the corresponding classical operators in Euclidean space.

The reason for this definition of the local part is that it makes the density of the measure γ_{∞} have the same order of magnitude at the two points x and u. But this remains true if x and u belong to a suitable elliptic ring of width roughly 1/(1+|x|). In this paper, we will define the local part by splitting \mathbb{R}^n into rings of this type. This makes the arguments more explicit; for details see Section 5. The expression "local part" is not quite adequate here, since x and u may be far apart, but we prefer to keep it, since this part plays the same role in the proofs as it does in earlier work.

To prove (1.3) for the global part and $0 < t \le 1$, we estimate its kernel using a method from [14]. It is based on the observation that the number of zeros in (0,1] of the function $t \mapsto \partial_t K_t(x, u)$ is bounded, uniformly in $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$.

The argument for the local part and $0 < t \le 1$ is based on vector–valued Calderón–Zygmund techniques for singular integrals, and requires a transition to Lebesgue measure.

The parameter ϱ deserves more attention. Some of our results hold for $\varrho \geq 1$ as well. This is the case of the variational bounds in $[1, \infty)$ and also in (0, 1] for the global part; we refer to Theorem 4.3 and Theorem 6.1, respectively. What forces the restriction $\varrho > 2$ in Theorem 1.1 is the variational bound for the local part of \mathcal{H}_t when $t \in (0, 1]$; see Theorem 7.2, where we apply results from [29] holding only for $\varrho > 2$. In the last section, we provide a counterexample showing that the condition $\varrho > 2$ is necessary in Theorem 1.1.

1.1. Structure of the paper. We gather some known facts about the Ornstein–Uhlenbeck semigroup and its integral kernel K_t in Section 2; in particular, we give pointwise bounds for K_t and its time derivative $\dot{K}_t = \partial_t K_t$. In Section 3 some basic properties of the variation seminorm are discussed; here we also introduce a few

reductions which will simplify the proof of Theorem 1.1. Section 4 is devoted to the proof of the weak type inequality (1.3) with \mathbb{R}_+ replaced by $[1, +\infty)$. The proof of (1.3) for (0,1] is carried out in Sections 5, 6 and 7. More precisely, in Section 5 we describe our localization procedure and the local and global parts, and Section 6 deals with the global part. The local part is treated in Section 7. In Section 8 we conclude by showing that Theorem 1.1 does not hold for $\rho = 2$.

1.2. **Notation.** We will use the symbols $0 < c, C < \infty$ to denote constants, not necessarily equal at different occurrences. These constants will depend only on n, Q and B, unless otherwise explicitly stated. If a and b are positive quantities, $a \lesssim b$ or equivalently $b \gtrsim a$ means $a \leq Cb$. If both $a \lesssim b$ and $b \lesssim a$ hold, we shall write $a \simeq b$. By \mathbb{N} we denote the set of all nonnegative integers.

We write $K_t = \partial_t K_t$, that is, we adopt the dot notation for differentiation with respect to the time variable t.

2. The Mehler Kernel

The Ornstein-Uhlenbeck semigroup may be formally written as $\mathcal{H}_t = e^{t\mathcal{L}}$, t > 0, with \mathcal{L} given by (1.2). In order to give more explicit expressions for this semigroup, we introduce the positive definite, symmetric matrices

$$Q_t = \int_0^t e^{sB} Q e^{sB^*} ds, \qquad 0 < t \le +\infty,$$
 (2.1)

and the normalized Gaussian measures γ_t in \mathbb{R}^n , with $t \in (0, +\infty]$, whose densities with respect to Lebesgue measure dx are given by

$$x \mapsto (2\pi)^{-\frac{n}{2}} (\det Q_t)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\langle Q_t^{-1}x, x\rangle\right).$$

Here the case $t = +\infty$ gives the invariant measure γ_{∞} that appears already in the introduction.

Kolmogorov's formula states that for all bounded and continuous functions in \mathbb{R}^n one has

$$\mathcal{H}_t f(x) = \int f(e^{tB}x - y) \, d\gamma_t(y) \,, \qquad x \in \mathbb{R}^n, \ t > 0.$$
 (2.2)

Starting from (2.2), one may write \mathcal{H}_t as an integral operator with a (Mehler) kernel $K_t(x, u)$. We point out that the term kernel in this paper refers to integration with respect to our basic measure γ_{∞} , with an exception in Section 7. In fact, we saw in [11] that for each $f \in L^1(\gamma_{\infty})$ and all t > 0 one has

$$\mathcal{H}_t f(x) = \int K_t(x, u) f(u) d\gamma_\infty(u), \qquad x \in \mathbb{R}^n, \tag{2.3}$$

where for $x, u \in \mathbb{R}^n$ and t > 0 the Mehler kernel K_t is given by

$$K_t(x,u) = \left(\frac{\det Q_{\infty}}{\det Q_t}\right)^{1/2} e^{R(x)} \exp\left[-\frac{1}{2}\left\langle \left(Q_t^{-1} - Q_{\infty}^{-1}\right)(u - D_t x), u - D_t x\right\rangle\right]. \tag{2.4}$$

Here R(x) is the quadratic form

$$R(x) = \frac{1}{2} \langle Q_{\infty}^{-1} x, x \rangle, \qquad x \in \mathbb{R}^n,$$

and

$$D_t = Q_{\infty} e^{-tB^*} Q_{\infty}^{-1}, \tag{2.5}$$

which is a one-parameter group, defined for all $t \in \mathbb{R}$.

It will be convenient to introduce another norm on \mathbb{R}^n by

$$|x|_Q := |Q_{\infty}^{-1/2} x|, \qquad x \in \mathbb{R}^n.$$

Then $|x|_Q \simeq |x|$, and $R(x) = |x|_Q^2/2$, $x \in \mathbb{R}^n$. Observe that the density of γ_∞ is proportional to $e^{-R(x)}$.

2.1. Pointwise estimates for the Mehler kernel. We shall repeatedly use the following pointwise estimate of the Mehler kernel. For $0 < t \le 1$ and all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$ one has

$$K_t(x,u) \lesssim \frac{e^{R(x)}}{t^{n/2}} \exp\left(-c\frac{|u-D_t x|^2}{t}\right). \tag{2.6}$$

This is proved in [11, (3.4)].

We shall need estimates for the time derivative of the Mehler kernel as well. For an explicit expression of \dot{K}_t , the reader is referred to [14, Lemma 4.2]; we only recall here the following pointwise bounds for \dot{K}_t (see [14, (5.5) and (5.4)]):

$$|\dot{K}_t(x,u)| \lesssim e^{R(x)} t^{-n/2} \exp\left(-c \frac{|u - D_t x|^2}{t}\right) (t^{-1} + |x|t^{-1/2}),$$
 (2.7)

for $0 < t \le 1$, and

$$|\dot{K}_t(x,u)| \lesssim e^{R(x)} \exp\left(-c|D_{-t}u - x|^2\right) \left(|D_{-t}u| + e^{-ct}\right)$$
 (2.8)

for $t \geq 1$. Both formulae hold for all $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$.

3. The variation seminorm

To state some basic properties of the variation seminorm, we define it for a generic continuous real-valued function ϕ defined in an interval I, by setting for any $1 \le \varrho < \infty$

$$\|\phi\|_{v(\varrho),I} = \sup\left(\sum_{i=1}^{N} |\phi(t_i) - \phi(t_{i-1})|^{\varrho}\right)^{1/\varrho}.$$
 (3.1)

As in (1.1), the supremum is taken over all finite, increasing sequences $(t_i)_0^N$ of points in I. This seminorm vanishes only for constant functions.

Observe that the seminorm $\|.\|_{v(\varrho),I}$ is decreasing in ϱ for $1 \leq \varrho < \infty$. It is also subadditive in I, in the following sense. Take an inner point τ of I and set $I_+ = I \cap [\tau, +\infty)$ and $I_- = I \cap (-\infty, \tau]$. Then for $1 \leq \varrho < \infty$ and any φ

$$\|\phi\|_{v(\varrho),I} \le \|\phi\|_{v(\varrho),I_+} + \|\phi\|_{v(\varrho),I_-}.$$

If $\phi \in C^1(I)$ and $\phi' \in L^1(I)$, then for $1 \le \varrho < \infty$

$$\|\phi\|_{v(\varrho),I} \le \int_{I} |\phi'(t)| \, dt, \tag{3.2}$$

see [15, Lemma 2.1].

In the last section, we will consider a discrete version of the variation, obtained by replacing I in the definition (3.1) by a set \mathcal{I} which is the intersection of \mathbb{N}_+ and an interval, and where ϕ is defined. For $\rho = 2$ one has the simple estimate

$$\|\phi\|_{v(2),\mathcal{I}} \lesssim \left(\sum_{\ell \in \mathcal{I}} \phi(\ell)^2\right)^{1/2}.$$
 (3.3)

For future convenience, we note that a combination of (3.2) and [14, Lemma 5.3] yields that for any $\varrho \in [1, \infty)$, any interval $I \subset \mathbb{R}_+$ and all $x \in \mathbb{R}^n$

$$\|\mathcal{H}_t f(x)\|_{v(\varrho),I} \le \int_I \left| \frac{\partial}{\partial t} \int K_t(x,u) f(u) \, d\gamma_\infty(u) \right| \, dt = \int_I \left| \int \dot{K}_t(x,u) f(u) \, d\gamma_\infty(u) \right| \, dt$$

$$\le \int \int_I \left| \dot{K}_t(x,u) \right| \, dt \, |f(u)| \, d\gamma_\infty(u), \qquad f \in L^1(\gamma_\infty). \tag{3.4}$$

3.1. **Simplifications.** By means of a few observations, it is possible to simplify the proof of Theorem 1.1.

First of all, when proving the inequality (1.3) one may take f such that $||f||_{L^1(\gamma_\infty)} = 1$. As a consequence, α in the same estimate may be assumed large, for instance $\alpha > 2$, since γ_∞ is finite.

Moreover, as already observed in the study of the weak-type (1,1) of operators related to the Ornstein-Uhlenbeck semigroup (see [11, Section 5]), when proving (1.3) we mostly need to take into account only points x belonging to the ellipsoidal annulus

$$C_{\alpha} = \left\{ x \in \mathbb{R}^n : \frac{1}{2} \log \alpha \le R(x) \le 2 \log \alpha \right\}.$$

Indeed, the unbounded component of the complement of C_{α} can always be neglected, since $\gamma_{\infty}(\{x \in \mathbb{R}^n : R(x) > 2 \log \alpha\}) \lesssim 1/\alpha$.

For the variation of \mathcal{H}_t in $[1, +\infty)$, the bounded component is negligible as well. Indeed, we see from [14, (5.4) and Lemma 5.1] that

$$\int_{1}^{\infty} |\dot{K}_{t}(x, u)| dt \lesssim e^{R(x)}.$$

Combined with (3.4), where $I = [1, \infty)$ and f is normalized in $L^1(\gamma_{\infty})$, this leads to

$$\|\mathcal{H}_t f(x)\|_{v(\varrho),[1,+\infty)} \lesssim e^{R(x)} \lesssim \sqrt{\alpha}$$

for $R(x) < \frac{1}{2} \log \alpha$. Taking α suitable large, we will have $\|\mathcal{H}_t f(x)\|_{v(\varrho),[1,+\infty)} < \alpha$ in the bounded component of the complement of \mathcal{C}_{α} .

4. The variation of
$$\mathcal{H}_t f(x)$$
 in $[1, +\infty)$

We first introduce the adapted polar coordinates from [11]. Fix $\beta > 0$ and consider the ellipsoid

$$E_{\beta} = \{x \in \mathbb{R}^n : R(x) = \beta\}.$$

As a consequence of [11, formula (4.3)], the map $s \mapsto R(D_s z)$ is strictly increasing for each $0 \neq z \in \mathbb{R}^n$. Thus we may write any $x \in \mathbb{R}^n$, $x \neq 0$, uniquely as

$$x = D_s \tilde{x}$$
,

for some $\tilde{x} \in E_{\beta}$ and $s \in \mathbb{R}$, and s and \tilde{x} are our polar coordinates of x.

We recall two results, previously proved by the authors, which will be essential in our arguments. The following lemma is [12, Lemma 5.1] with $\sigma = 1$; the factor 1/4 occurring in [12] is replaced by a generic $\delta > 0$, which causes no problem.

Lemma 4.1. [12] Let $\delta > 0$. For $x, u \in \mathbb{R}^n$, one has

$$\int_{1}^{+\infty} \exp\left(-\delta |D_{-t} u - x|^2\right) |D_{-t} u| dt \lesssim 1,$$

where the implicit constant may depend on δ , n, Q and B.

Lemma 4.2. [12, Lemma 7.2], [11, Proposition 6.1] Let $\delta > 0$ and $\alpha > 2$, and assume f normalized in $L^1(\gamma_{\infty})$. Then

$$\gamma_{\infty} \Big\{ x = D_s \, \tilde{x} \in \mathcal{C}_{\alpha} : e^{R(x)} \int \exp\left(-\delta \left| \tilde{x} - \tilde{u} \right|^2\right) |f(u)| \, d\gamma_{\infty}(u) > \alpha \Big\} \lesssim \frac{1}{\alpha \sqrt{\log \alpha}}.$$

Here the polar coordinates are defined with $\beta = \log \alpha$, the implicit constant is as in Lemma 4.1, and $\sigma = 1$ in [12, Lemma 7.2].

We can now deduce the bound for the variation operator in $[1, +\infty)$.

Theorem 4.3. Let $1 \leq \varrho < \infty$. The operator that maps $f \in L^1(\gamma_\infty)$ to the function $\|\mathcal{H}_t f(x)\|_{v(\varrho),[1,+\infty)}, \quad x \in \mathbb{R}^n,$

is of weak type (1,1) with respect to the measure γ_{∞} . In fact, the following stronger result holds: if $||f||_{L^1(\gamma_{\infty})} = 1$, then

$$\gamma_{\infty} \left\{ x \in \mathbb{R}^n : \|H_t f(x)\|_{v(\varrho),[1,\infty)} > \alpha \right\} \lesssim \frac{1}{\alpha \sqrt{\log \alpha}}, \qquad \alpha > 2.$$
 (4.1)

Proof. This proof is similar to that of Proposition 6.1 in [14]; we sketch it for the sake of completeness. Let f be normalized in $L^1(\gamma_\infty)$. In the light of the considerations made in Subsection 3.1, it suffices to consider α large and $x \in \mathcal{C}_{\alpha}$. Using polar coordinates with $\beta = \log \alpha$ for $x \in \mathcal{C}_{\alpha}$ and $u \neq 0$, we write $x = D_s \tilde{x}$ and $u = D_\sigma \tilde{u}$, with $s, \sigma \in \mathbb{R}$. Since $R(x) \geq \beta/2$, [11, Lemma 4.3(i)] yields

$$|D_{-t} u - x| = |D_{\sigma - t} \tilde{u} - D_s \tilde{x}| \gtrsim |\tilde{x} - \tilde{u}|, \qquad x \in \mathcal{C}_{\alpha}.$$

By reducing the value of c in the exponential factor in (2.8), we arrive at

$$|\dot{K}_t(x,u)| \lesssim e^{R(x)} \exp\left(-c|\tilde{x}-\tilde{u}|^2\right) \exp\left(-c|D_{-t}u-x|^2\right) (|D_{-t}u|+e^{-ct}), \qquad t > 1.$$

An application of Lemma 4.1 leads to

$$\int_{1}^{\infty} |\dot{K}_{t}(x,u)| dt \lesssim e^{R(x)} \exp\left(-c \left|\tilde{x} - \tilde{u}\right|^{2}\right), \qquad x \in \mathcal{C}_{\alpha},$$

and then (3.4) yields

$$\|\mathcal{H}_t f(x)\|_{v(\rho),[1,\infty)} \lesssim e^{R(x)} \exp\left(-c\left|\tilde{x}-\tilde{u}\right|^2\right).$$

Finally, Lemma 4.2 implies (4.1), and the weak type (1,1) also follows.

Remark 4.4. Theorem 4.3 says that for the variation in $[1, \infty)$ the standard weak type (1,1) estimate is enhanced by a logarithmic factor. This phenomenon was observed in the one-dimensional case in [15].

5. Local versus global region

From now on, we shall focus on the variation of \mathcal{H}_t in the interval (0,1]. This case requires a further distinction between the local and the global parts of \mathcal{H}_t . We start describing the localization procedure.

5.1. Splitting of \mathbb{R}^n into rings. Let

$$R_i = \{x \in \mathbb{R}^n : j \le R(x) \le j+1\}, \qquad j = 0, 1, \dots$$

These rings cover \mathbb{R}^n and are pairwise disjoint except for boundaries. We also set $R_j = \emptyset$ if j < 0.

The width of R_j , defined as the $|\cdot|_Q$ distance between the two components of its boundary, is for $j \geq 1$

$$\sqrt{2}\left(\sqrt{j+1} - \sqrt{j}\right) = \frac{\sqrt{2}}{\sqrt{j+1} + \sqrt{j}} \in \left(\frac{1}{2\sqrt{j}}, \frac{\sqrt{2}}{\sqrt{j}}\right),\tag{5.1}$$

as easily verified.

We take a sequence of smooth non-negative functions $(r_j)_{j\in\mathbb{N}}$ satisfying $\sum_{j=0}^{\infty} r_j(x) = 1$ for all x, with supp $r_j \subseteq R_j \cup R_{j+1}$. Further, we may choose them such that

$$\left|\nabla r_j(x)\right| \lesssim 1 + |x|. \tag{5.2}$$

We also introduce slightly larger functions $\widetilde{r}_j \in \mathcal{C}_0^{\infty}(\mathbb{R}^n)$ taking values in [0,1] such that $\widetilde{r}_j = 1$ in $\bigcup_{\nu=j-1}^{j+2} R_{\nu}$ and supp $\widetilde{r}_j \subset \bigcup_{\nu=j-2}^{j+3} R_{\nu}$. They also satisfy

$$\left|\nabla \widetilde{r}_j(x)\right| \lesssim 1 + |x|. \tag{5.3}$$

Then the supports of these $\widetilde{r_j}$ have bounded overlap, and

$$e^{-R(x)} \simeq e^{-j}$$
 for $x \in \operatorname{supp} \widetilde{r_j}$. (5.4)

5.2. The splitting of \mathcal{H}_t . We can now split the operator $\mathcal{H}_t f$ in a local and a global part, in a way adapted to the rings. The local part is defined by

$$\mathcal{H}_t^{\text{loc}} f(x) := \sum_{j=0}^{\infty} \widetilde{r}_j(x) \mathcal{H}_t(fr_j)(x)$$
$$= \sum_{j=0}^{\infty} \widetilde{r}_j(x) \int_{\mathbb{R}^n} K_t(x, u) f(u) r_j(u) d\gamma_{\infty}(u).$$

This sum is locally finite and thus well defined for any $f \in L^1(\gamma_\infty)$, because of the bounded overlap of the sets supp $\widetilde{r_i}$.

Setting

$$\eta(x, u) = \sum_{j=0}^{\infty} \widetilde{r_j}(x) \, r_j(u),$$

we get

$$\mathcal{H}_t^{\text{loc}} f(x) = \int_{\mathbb{R}^n} K_t(x, u) \, \eta(x, u) \, f(u) \, d\gamma_{\infty}(u), \tag{5.5}$$

and $0 \le \eta(x,u) \le 1$ for all $x,u \in \mathbb{R}^n$. Moreover, if supp $f \subset R_j \cup R_{j+1}$, then the support of $\mathcal{H}_t^{\text{loc}} f$ is contained in $\bigcup_{\nu=j-2}^{j+4} R_{\nu}$.

The global part of \mathcal{H}_t is defined by $\mathcal{H}_t^{\text{glob}} = \mathcal{H}_t - \mathcal{H}_t^{\text{loc}}$, or equivalently

$$\mathcal{H}_t^{\text{glob}} f(x) = \int_{\mathbb{R}^n} K_t(x, u) \left(1 - \eta(x, u)\right) f(u) \, d\gamma_{\infty}(u), \qquad x \in \mathbb{R}^n.$$

The next two lemmas give sufficient conditions for the kernel $K_t(x, u)$ $(1 - \eta(x, u))$ of the global part to vanish.

Lemma 5.1. Let $j \in \{0, 1, ...\}$. If $x, u \in R_j \cup R_{j+1}$, then $\eta(x, u) = 1$.

Proof. When $u \in R_j \cup R_{j+1}$ the function $r_i(u)$ can be nonzero only for $i \in \{j-1, j, j+1\}$, and then $r_{j-1}(u) + r_j(u) + r_{j+1}(u) = 1$.

Also, for $x \in R_j \cup R_{j+1}$ one has $\widetilde{r}_{j-1}(x) = \widetilde{r}_j(x) = \widetilde{r}_{j+1}(x) = 1$. This implies that $\eta(x,u) = \sum_{\nu=j-1}^{j+1} \widetilde{r}_{\nu}(x) r_{\nu}(u) = 1$.

Lemma 5.2. If

$$|x - u|_Q \le \frac{1}{2(1 + |x|_Q)},$$

then $\eta(x,u)=1$.

Proof. Suppose $x \in R_j$. Then $|x|_Q \ge \sqrt{2j}$ and we can write

$$|x - u|_Q \le \frac{1}{2(1 + |x|_Q)} \le \frac{1}{2(1 + \sqrt{j})} \le \frac{1}{2\sqrt{j+1}}.$$

From this and (5.1), we see that $|x - u|_Q$ is less than the widths of R_{j-1} and R_{j+1} (only that of R_{j+1} if j = 0 or j = 1). Since $x \in R_j$, this means that both x and u must be in $R_{j-1} \cup R_j$ or $R_j \cup R_{j+1}$. From Lemma 5.1 we then get the assertion. \square

We will need an estimate for the gradient of $\eta(x, u)$. If a point (x, u) is in the support of η , then x and u are both in the support of some \tilde{r}_j . It follows that $1 + |u| \simeq 1 + |x|$, and (5.2) and (5.3) then imply that

$$|\nabla_x \eta(x, u)| + |\nabla_u \eta(x, u)| \lesssim 1 + |x|. \tag{5.6}$$

6. The variation of the global part

The result of this section is the following.

Theorem 6.1. For each $\varrho \geq 1$ the operator that maps $f \in L^1(\gamma_\infty)$ to the function

$$\|\mathcal{H}_t^{\text{glob}} f(x)\|_{v(\varrho),(0,1]}, \quad x \in \mathbb{R}^n,$$

is of weak type (1,1) with respect to the measure γ_{∞} .

Proof. This proof follows the pattern from that of [14, Proposition 10.2]. In particular, we will need the following result concerning a maximal operator.

Theorem 6.2. The maximal operator defined by

$$S_0^{\text{glob}} f(x) = \int \sup_{0 < t \le 1} K_t(x, u) \left(1 - \eta(x, u) \right) |f(u)| \, d\gamma_\infty(u)$$

is of weak type (1,1) with respect to the invariant measure γ_{∞} .

This statement seems to coincide with that of [14, Theorem 10.1], but our $\eta(x, u)$ is not the same as that in [14]. However, by tracing the proof in [14], one sees that what matters is only the implication

$$\eta(x,u) < 1$$
 \Rightarrow $|x - u|_Q > \frac{1}{2(1+|x|_Q)},$

which is a consequence of our Lemma 5.2. In this way, Theorem 6.2 follows.

We will also need [14, Proposition 9.1]. It says that for any $(x, u) \in \mathbb{R}^n \times \mathbb{R}^n$, the number of zeros N(x, u) in (0, 1] of the map $t \mapsto \dot{K}_t(x, u)$ is bounded by a positive integer \bar{N} that depends only on n and B.

As in [14, Proposition 10.2], we denote these zeros by $t_1(x, u) < \cdots < t_{N(x,u)}(x, u)$, and set $t_0(x, u) = 0$, $t_{N(x,u)+1}(x, u) = 1$. Since $K_t(x, u)$ vanishes at t = 0 if $x \neq u$, it follows from the fundamental theorem of calculus that

$$\int_{0}^{1} \left| \dot{K}_{t}(x, u) \right| dt = \sum_{i=0}^{N(x, u)} \left| \int_{t_{i}(x, u)}^{t_{i+1}(x, u)} \dot{K}_{t}(x, u) dt \right|$$

$$= \sum_{i=0}^{N(x, u)} \left| K_{t_{i+1}(x, u)}(x, u) - K_{t_{i}(x, u)}(x, u) \right|$$

$$\leq 2 \sum_{i=0}^{N(x, u)+1} K_{t_{i}(x, u)}(x, u) \lesssim \bar{N} \sup_{0 < t \leq 1} K_{t}(x, u).$$

By (3.4), which remains valid with the extra factor $1 - \eta(x, u)$, we obtain

$$\|\mathcal{H}_t^{\text{glob}} f(x)\|_{v(\varrho),(0,1]} \lesssim \int \sup_{0 < t \leq 1} K_t(x,u) (1 - \eta(x,u) |f(u)| d\gamma_\infty(u).$$

Now Theorem 6.2 implies the assertion of Theorem 6.1.

7. The local part for small t

In this section, we prove the following result.

Theorem 7.1. For each $\varrho > 2$ the operator that maps $f \in L^1(\gamma_\infty)$ to the function

$$\|\mathcal{H}_t^{\mathrm{loc}}f(x)\|_{v(\rho),(0,1]}, \quad x \in \mathbb{R}^n,$$

is of weak type (1,1) with respect to the measure γ_{∞} .

The proof relies on vector-valued Calderón–Zygmund theory.

7.1. **Preparations.** Part of our notation in this section follows that of [17]. Let Θ be the set of all finite, increasing sequences in (0,1] written

$$\underline{\varepsilon} = (\varepsilon_i)_0^N$$

for some $N = N(\underline{\varepsilon}) \in \mathbb{N}_+$. We define F as the vector space of all functions $g : \mathbb{N} \times \Theta \to \mathbb{R}$ such that $g(i,\underline{\varepsilon}) = 0$ for $i > N(\underline{\varepsilon})$. For $\varrho \in [1,\infty)$ the subspace $F_\varrho \subset F$ is defined to consist of those $g \in F$ for which the mixed seminorm

$$||g||_{F_{\varrho}} := \sup_{\underline{\varepsilon} \in \Theta} \left(\sum_{i=1}^{N(\underline{\varepsilon})} |g(i,\underline{\varepsilon})|^{\varrho} \right)^{1/\varrho}$$

is finite. Notice that this is a seminorm because $g(0,\underline{\varepsilon})$ does not appear here.

We shall work with the operator V that maps real-valued functions $f \in L^1(\gamma_\infty)$ to F-valued functions defined in \mathbb{R}^n and is given by

$$V f(x)(i,\underline{\varepsilon}) = \mathcal{H}_{\varepsilon_i} f(x) - \mathcal{H}_{\varepsilon_{i-1}} f(x), \qquad i = 1, \dots, N(\underline{\varepsilon}), \ \underline{\varepsilon} \in \Theta,$$

and $Vf(x)(0,\underline{\varepsilon}) = 0$. Similarly, we set

$$V^{\text{loc}}f(x)(i,\underline{\varepsilon}) = \mathcal{H}^{\text{loc}}_{\varepsilon_i}f(x) - \mathcal{H}^{\text{loc}}_{\varepsilon_{i-1}}f(x), \qquad i = 1, \dots, N(\underline{\varepsilon}), \tag{7.1}$$

and analogously for $V^{\text{glob}}f$.

Then for $\rho \in [1, \infty)$

$$||Vf(x)||_{F_{\varrho}} = ||\mathcal{H}_t f(x)||_{v(\varrho),(0,1]}, \tag{7.2}$$

and similar equalities hold with superscripts loc or glob.

Theorem 7.1 can now be rewritten in the following equivalent way.

Theorem 7.2. For each $\varrho > 2$ the operator that maps $f \in L^1(\gamma_\infty)$ to the function

$$\mathbb{R}^n \ni x \mapsto \|V^{\mathrm{loc}}f(x)\|_{F_{\mathcal{Q}}}$$

is of weak type (1,1) with respect to the measure γ_{∞} .

The advantage of Theorem 7.2 is that the proof can be based on vector-valued Calderón–Zygmund theory. To apply this machinery, we will pass to Lebesgue measure, and set

$$\widetilde{V}^{\text{loc}} g(x) = e^{-R(x)} V^{\text{loc}} (g(\cdot) e^{R(\cdot)})(x). \tag{7.3}$$

Lebesgue measure will be written either dx or du.

The following proposition clarifies the connection between $\widetilde{V}^{\mathrm{loc}}$ and V^{loc} .

Proposition 7.3. Let $\varrho > 2$. If the operator that maps $g \in L^1(du)$ to the function

$$\left\| \widetilde{V}^{\text{loc}} g(x) \right\|_{F_o}, \quad x \in \mathbb{R}^n,$$

is of weak type (1,1) with respect to Lebesgue measure, then the operator that maps $f \in L^1(\gamma_\infty)$ to the function

$$||V^{\mathrm{loc}}f(x)||_{F_a}, \quad x \in \mathbb{R}^n,$$

is of weak type (1,1) with respect to γ_{∞} .

Proof. We have

$$\|V^{\text{loc}}f(x)\|_{F_{\varrho}} = \left\| \sum_{j=0}^{\infty} \widetilde{r_{j}}(x) V(fr_{j})(x) \right\|_{F_{\varrho}}$$

$$\leq \left\| \sum_{j=0}^{\infty} \widetilde{r_{j}}(x) V^{\text{glob}}(fr_{j})(x) \right\|_{F_{\varrho}} + \left\| \sum_{j=0}^{\infty} \widetilde{r_{j}}(x) V^{\text{loc}}(fr_{j})(x) \right\|_{F_{\varrho}} = I + II.$$

Since the $\widetilde{r_j}$ have supports with bounded overlap, the sums here are uniformly locally finite. It follows that

$$||I||_{L^{1,\infty}(\gamma_{\infty})} \lesssim \sum_{j=0}^{\infty} ||\widetilde{r}_j(x)||V^{\text{glob}}(fr_j)(x)||_{F_{\varrho}} ||_{L^{1,\infty}(\gamma_{\infty})}.$$

The F_{ϱ} quasinorm here equals $\|\mathcal{H}_t^{\text{glob}}f(x)\|_{v(\varrho),(0,1]}$. After estimating the factor $\widetilde{r}_j(x)$ by 1, we can apply Theorem 6.1 and get

$$||I||_{L^{1,\infty}(\gamma_{\infty})} \lesssim \sum_{j=0}^{\infty} ||fr_j||_{L^1(\gamma_{\infty})} = ||f||_{L^1(\gamma_{\infty})}.$$

We estimate the $L^{1,\infty}(\gamma_{\infty})$ quasinorm of II similarly, but then apply the definition of \widetilde{V}^{loc} . This gives

$$\begin{aligned} \|II\|_{L^{1,\infty}(\gamma_{\infty})} &\lesssim \sum_{j=0}^{\infty} \|\widetilde{r}_{j}(x)\|V^{\operatorname{loc}}(fr_{j})(x)\|_{F_{\varrho}}\|_{L^{1,\infty}(\gamma_{\infty})} \\ &= \sum_{j=0}^{\infty} \|\widetilde{r}_{j}(x)e^{R(x)}\|\widetilde{V}^{\operatorname{loc}}(fr_{j}e^{-R(\cdot)})(x)\|_{F_{\varrho}}\|_{L^{1,\infty}(\gamma_{\infty})} \\ &\simeq \sum_{j=0}^{\infty} \|\widetilde{r}_{j}(x)\|\widetilde{V}^{\operatorname{loc}}(fr_{j}e^{-R(\cdot)})(x)\|_{F_{\varrho}}\|_{L^{1,\infty}(dx)}; \end{aligned}$$

in the last step here, we switched from γ_{∞} to Lebesgue measure. This was possible because of (5.4). Next, we estimate $\tilde{r}_j(x)$ by 1 and use the hypothesis of the proposition, getting

$$||II||_{L^{1,\infty}(\gamma_{\infty})} \lesssim \sum_{j=0}^{\infty} ||fr_j e^{-R(\cdot)}||_{L^1(du)} \simeq ||f||_{L^1(\gamma_{\infty})}.$$

The proposition is proved.

7.2. Vector-valued Calderón-Zygmund operators. This theory will be applied to the operator \tilde{V}^{loc} . The following proposition gives the strong (p, p) bound needed.

Proposition 7.4. For any $\varrho > 2$ and any $1 , the operator that maps <math>g \in L^p(du)$ to

$$\left\| \widetilde{V}^{loc} g(x) \right\|_{F_{\varrho}}, \qquad x \in \mathbb{R}^{n},$$

is bounded from $L^p(du)$ to $L^p(dx)$.

Proof. As a consequence of the definitions of V^{loc} and V^{loc} and the bounded overlap of the $\widetilde{r_i}$, one has

$$\left\| \left\| \widetilde{V}^{\text{loc}} g(x) \right\|_{F_{\varrho}} \right\|_{L^{p}(dx)}^{p} = \int \left\| \sum_{j=0}^{\infty} \widetilde{r}_{j}(x) e^{-R(x)} V \left(g \, r_{j} \, e^{R(\cdot)} \right) (x) \right\|_{F_{\varrho}}^{p} dx$$

$$\lesssim \sum_{j=0}^{\infty} \int \left\| \widetilde{r}_{j}(x) e^{-R(x)} V \left(g \, r_{j} \, e^{R(\cdot)} \right) (x) \right\|_{F_{\varrho}}^{p} dx$$

$$\simeq \sum_{j=0}^{\infty} e^{-jp} \int \left\| \widetilde{r}_{j}(x) V \left(g \, r_{j} \, e^{R(\cdot)} \right) (x) \right\|_{F_{\varrho}}^{p} dx$$

$$\simeq \sum_{j=0}^{\infty} e^{j(1-p)} \int \left\| \widetilde{r}_{j}(x) V \left(g \, r_{j} \, e^{R(\cdot)} \right) (x) \right\|_{F_{\varrho}}^{p} d\gamma_{\infty}(x).$$

Notice that we also used (5.4) here. In the last expression, we delete the factor $\widetilde{r}_{j}(x)$ and observe that the resulting F_{ϱ} seminorm equals $\|\mathcal{H}_{t}\left(g\,r_{j}\,e^{R(\cdot)}\right)(x)\|_{v(\varrho),(0,1]}$.

As mentioned in the introduction, the variation operator for \mathcal{H}_t is bounded on $L^p(\gamma_\infty)$ with $1 , for <math>\varrho > 2$ [29]. We therefore obtain

$$\left\| \left\| \widetilde{V}^{\text{loc}} g(x) \right\|_{F_{\varrho}} \right\|_{L^{p}(dx)}^{p} \lesssim \sum_{j=0}^{\infty} e^{j(1-p)} \int \left\| \mathcal{H}_{t} \left(g \, r_{j} \, e^{R(\cdot)} \right) (x) \right\|_{v(\varrho),(0,1]}^{p} d\gamma_{\infty}(x)$$

$$\lesssim \sum_{j=0}^{\infty} e^{j(1-p)} \left\| g \, r_{j} \, e^{R(\cdot)} \right\|_{L^{p}(\gamma_{\infty})}^{p} \simeq \sum_{j=0}^{\infty} \left\| g \, r_{j} \right\|_{L^{p}(du)}^{p} \leq \left\| g \right\|_{L^{p}(du)}^{p},$$

proving the assertion.

We shall now prove the appropriate standard estimates for the vector-valued kernel of $\widetilde{V}^{\mathrm{loc}}.$

From (7.1) and (5.5), we see that V^{loc} is an integral operator with an F-valued kernel $M^{\text{loc}}(x, u)$ given by

$$M^{\text{loc}}(x,u)(i,\underline{\varepsilon}) = (K_{\varepsilon_i}(x,u) - K_{\varepsilon_{i-1}}(x,u)) \eta(x,u), \qquad i = 1,\dots, N(\underline{\varepsilon}).$$

This means that for $f \in L^1(\gamma_\infty)$

$$V^{\mathrm{loc}}f(x) = \int M^{\mathrm{loc}}(x, u) f(u) \, d\gamma_{\infty}(u).$$

We will need the kernel $\widetilde{M}^{\mathrm{loc}}(x,u)$ of $\widetilde{V}^{\mathrm{loc}}$, for integration against Lebesgue measure in the sense that

$$\widetilde{V}^{\mathrm{loc}} f(x) = \int \widetilde{M}^{\mathrm{loc}}(x, u) f(u) du$$

for suitable f. From (7.3) we get

$$\widetilde{M}^{\text{loc}}(x,u)(i,\underline{\varepsilon}) = e^{-R(x)} M^{\text{loc}}(x,u)(i,\underline{\varepsilon}) = e^{-R(x)} \left(K_{\varepsilon_{i+1}}(x,u) - K_{\varepsilon_i}(x,u) \right) \eta(x,u),$$

$$i = 1, \dots, N(\varepsilon). \quad (7.4)$$

In analogy with (7.2), this implies

$$\|\widetilde{M}^{\text{loc}}(x,y)\|_{F_{\varrho}} = e^{-R(x)} \|K_{t}(x,u)\eta(x,u)\|_{v(\varrho),(0,1]}$$

$$= e^{-R(x)} \eta(x,u) \|K_{t}(x,u)\|_{v(\varrho),(0,1]}, \tag{7.5}$$

where the variation is meant with respect to t.

We will need an auxiliary result.

Proposition 7.5. Let $p, r \ge 0$ with p+r/2 > 1. Assume that $\eta(x, u) \ne 0$ and $x \ne u$. Then for $\delta > 0$

$$\int_0^1 t^{-p} \exp\left(-\delta \frac{|u - D_t x|^2}{t}\right) |x|^r dt \lesssim C |u - x|^{-2p - r + 2}.$$

Here the constant C may depend on δ , p and r, in addition to n, Q and B.

Proof. The statement of this proposition is similar to that of [14, Proposition 8.3], which is based on [12, Lemma 8.1]. But our function η is not the same as there, and we must verify that for $\eta(x, u) \neq 0$ and 0 < t < 1

$$\frac{|D_t x - u|^2}{t} > c \frac{|x - u|^2}{t} + ct|x|^2 - C.$$
 (7.6)

Once this is established, we can follow the argument for [12, Lemma 8.1] and finish the proof.

To verify (7.6), assume first that $t < c_0|x - u|/|x|$ for a small $c_0 > 0$ to be chosen. Then [12, Lemma 2.3] implies $|D_t x - x| \le Ct|x| < Cc_0|x - u|$ and thus

$$|D_t x - u| \ge |x - u| - |D_t x - x| > |x - u| - Cc_0|x - u|.$$

So with $c_0 > 0$ small enough we get

$$|D_t x - u| > \frac{1}{2} |x - u| \simeq |x - u| + t|x|.$$

To obtain (7.6) in this case, take squares and divide by t.

In the opposite case $c_0|x-u|/|x| \le t < 1$, we use [11, Lemma 4.1] to conclude that

$$\frac{\partial}{\partial t} |D_t x|_Q = \frac{\partial}{\partial t} \sqrt{2R(D_t x)} \simeq |D_t x|_Q \simeq |x|_Q.$$

Integration yields

$$|D_t x|_Q - |x|_Q \simeq t|x|_Q \simeq t|x|_Q + |x - u|.$$

Thus

$$|D_t x - u|_Q \ge |D_t x|_Q - |u|_Q = |D_t x|_Q - |x|_Q + |x|_Q - |u|_Q$$

$$\ge ct|x|_Q + c|x - u| - ||x|_Q - |u|_Q|.$$
(7.7)

Since $\eta(x,u) > 0$, there exists a $j \in \mathbb{N}$ such that such that $\widetilde{r}_j(x) > 0$ and $r_j(u) > 0$. Thus $||x|_Q - |u|_Q| \lesssim 1/(1+\sqrt{j}) \simeq \frac{1}{1+|x|}$, and by squaring (7.7) we get

$$|D_t x - u|_Q^2 \ge ct^2 |x|_Q^2 + c|x - u|^2 - C(t|x|_Q + |x - u|)/(1 + |x|)$$

$$\ge ct^2 |x|_Q^2 + c|x - u|^2 - Ct.$$

This implies (7.6).

Proposition 7.6. For all (x, u) such that $x \neq u$ and $\eta(x, u) \neq 0$, the following estimates hold:

(a)
$$\|\widetilde{M}^{loc}(x,u)\|_{F_o} \lesssim |x-u|^{-n};$$

(b)
$$\|\nabla_x \widetilde{M}^{loc}(x, u)\|_{F_{\varrho}} \lesssim |x - u|^{-n-1};$$

(c)
$$\|\nabla_u \widetilde{M}^{loc}(x, u)\|_{F_0} \lesssim |x - u|^{-n-1}$$
,

with implicit constants depending on n, B and Q.

Proof. (a) Starting from (7.5), we use (3.2) and then (2.7) to get

$$\|\widetilde{M}^{\text{loc}}(x,u)\|_{F_{\varrho}} = e^{-R(x)} \eta(x,u) \| K_{t}(x,u)\|_{v(\varrho),(0,1]} \le e^{-R(x)} \int_{0}^{1} |\dot{K}_{t}(x,u)| dt$$

$$\lesssim \int_{0}^{1} t^{-\frac{n}{2}} \exp\left(-c \frac{|u - D_{t} x|^{2}}{t}\right) \left(\frac{1}{t} + \frac{|x|}{\sqrt{t}}\right) dt.$$

Since $\eta(x, u) \neq 0$, Proposition 7.5 tells us that the last integral is bounded by $C|u-x|^{-n}$, and (a) follows.

(b) Differentiating (7.4) with respect to x_{ℓ} , we obtain as in (7.5)

$$\|\partial_{x_{\ell}} \widetilde{M}^{\text{loc}}(x, u)\|_{F_{\varrho}}$$

$$\leq \|\partial_{x_{\ell}} \left(e^{-R(x)} \eta(x, u) \right) K_{t}(x, u)\|_{v(\varrho), (0, 1]} + e^{-R(x)} \eta(x, u) \|\partial_{x_{\ell}} K_{t}(x, u)\|_{v(\varrho), (0, 1]}.$$
(7.8)

One has $|\partial_{x_{\ell}} e^{-R(x)}| \lesssim e^{-R(x)} |x|$, and by means of (5.6) we see that

$$\left| \partial_{x_{\ell}} \left(e^{-R(x)} \eta(x, u) \right) \right| \lesssim e^{-R(x)} \left(1 + |x| \right) \lesssim e^{-R(x)} \left(t^{-1/2} + |x| \right)$$
 (7.9)

for 0 < t < 1.

We now use (3.2) as in (a) to estimate the first term in (7.8) as follows

$$\int_{0}^{1} \left| \partial_{x_{\ell}} \left(e^{-R(x)} \eta(x, u) \right) \right| \left| \dot{K}_{t}(x, u) \right| dt$$

$$\lesssim \int_{0}^{1} \left(t^{-1/2} + |x| \right) t^{-\frac{n}{2}} \exp \left(-c \frac{|u - D_{t} x|^{2}}{t} \right) \left(\frac{1}{t} + \frac{|x|}{\sqrt{t}} \right) dt,$$

where the inequality comes from (7.9) and (2.7). Because of Proposition 7.5, these quantities are at most constant times $|x-u|^{-n-1}$.

It remains to deal with the last term in (7.8). This requires an estimate of $\partial_t \partial_{x_\ell} K_t(x, u)$, and we write

$$\partial_{x_{\ell}} K_t(x, u) = K_t(x, u) P_{\ell}(t, x, u), \qquad t > 0,$$

where

$$P_{\ell}(t, x, u) = \langle Q_{\infty}^{-1} x, e_{\ell} \rangle + \langle Q_{t}^{-1} e^{tB} e_{\ell}, u - D_{t} x \rangle;$$

see [12, Lemma 4.1]. Thus

$$\partial_t \partial_{x_\ell} K_t(x, u) = \dot{K}_t(x, u) P_\ell(t, x, u) + K_t(x, u) \dot{P}_\ell(t, x, u),$$

and from (3.2) we have

$$\|\partial_{x_{\ell}} K_{t}(x,u)\|_{v(\varrho),(0,1]} \lesssim \int_{0}^{1} |\dot{K}_{t}(x,u)| P_{\ell}(t,x,u)| + |K_{t}(x,u)\dot{P}_{\ell}(t,x,u)| dt.$$
 (7.10)

Using the facts that $\partial Q_t^{-1}/\partial t = -Q_t^{-1} e^{tB} Q e^{tB^*} Q_t^{-1}$ and $\dot{D}_t = -Q_\infty B^* Q_\infty^{-1} D_t$, taken from [14, Lemma 4.1], we find that

$$\dot{P}_{\ell}(t, x, u) = \left\langle Q_{t}^{-1} e^{tB} Q e^{tB^{*}} Q_{t}^{-1} e^{tB} e_{\ell}, u - D_{t} x \right\rangle + \left\langle Q_{t}^{-1} B e^{tB} e_{\ell}, u - D_{t} x \right\rangle + \left\langle Q_{t}^{-1} e^{tB} e_{\ell}, Q_{\infty} B^{*} Q_{\infty}^{-1} D_{t} x \right\rangle.$$

Since $||Q_t^{-1}|| \lesssim 1 + 1/t$ (see [11, Lemma 3.2(ii)]), we obtain for 0 < t < 1

$$|P_{\ell}(t, x, u)| \lesssim \frac{|u - D_t x|}{t} + |x|$$
 (7.11)

and

$$\left|\dot{P}_{\ell}(t, x, u)\right| \lesssim \frac{|u - D_t x|}{t^2} + \frac{|x|}{t}.$$
 (7.12)

To estimate the integrals in (7.10), we use (7.11) and (7.12) together with (2.7) and (2.6). By reducing the constants c in the factors $\exp(-c|D_t x - u|^2/t)$ occurring in (2.7) and (2.6), we can eliminate factors $|D_t x - u|/\sqrt{t}$ in (7.11) and (7.12). It follows that the last term in (7.8) is no larger than constant times

$$\int_{0}^{1} t^{-\frac{n}{2}} \exp\left(-c \frac{|u - D_{t} x|^{2}}{t}\right) \left(\frac{1}{t} + \frac{|x|}{\sqrt{t}}\right) \left(\frac{1}{\sqrt{t}} + |x|\right) dt + \int_{0}^{1} t^{-\frac{n}{2}} \exp\left(-c \frac{|u - D_{t} x|^{2}}{t}\right) \left(\frac{1}{t^{3/2}} + \frac{|x|}{t}\right) dt$$

Proposition 7.5 implies that this quantity is controlled by $|x-u|^{-n-1}$, and this ends part (b).

(c) This part is similar to (b). In (7.8) and in the treatment of the first term there, we need only replace $\partial_{x_{\ell}}$ by $\partial_{u_{\ell}}$.

From [12, Lemma 4.1] we have

$$\partial_{u_{\ell}} K_t(x, u) = -K_t(x, u) R_{\ell}(t, x, u), \qquad t > 0,$$

where

$$R_{\ell}(t, x, u) = \left\langle Q_t^{-1} e^{tB} \left(D_{-t} u - x \right), e_{\ell} \right\rangle.$$

Further,

$$|R_{\ell}(t,x,u)| \lesssim \frac{|D_{-t}u-x|}{t} \simeq \frac{|u-D_{t}x|}{t},$$

since $t \in (0, 1)$, and

$$\begin{split} \dot{R}_{\ell}(t,x,u) &= -\left\langle Q_{t}^{-1} \, e^{tB} \, Q \, e^{tB^{*}} \, Q_{t}^{-1} e^{tB} \, (D_{-t} \, u - x), \, e_{\ell} \right\rangle \\ &+ \left\langle Q_{t}^{-1} B e^{tB} \, (D_{-t} \, u - x), \, e_{\ell} \right\rangle + \left\langle Q_{t}^{-1} e^{tB} \, Q_{\infty} \, B^{*} \, Q_{\infty}^{-1} \, D_{-t} \, u, \, e_{\ell} \right\rangle. \end{split}$$

It follows that

$$|\dot{R}_{\ell}(t,x,u)| \lesssim \frac{|u-D_t x|}{t^2} + \frac{|D_{-t} u - x| + |x|}{t} \lesssim \frac{|u-D_t x|}{t^2} + \frac{|x|}{t}.$$

We can now mimic the argument for the last term in (7.8) in (b), and complete the proof of Proposition 7.6.

Now Propositions 7.6 and 7.4 yield Theorems 7.2 and 7.1.

Finally, Theorem 1.1 follows from Theorems 4.3, 6.1 and 7.1.

8. A COUNTEREXAMPLE

In this section we prove that the condition $\varrho > 2$ is necessary in Theorem 7.1 and therefore also in Theorem 1.1. The proof of the main result, Theorem 8.1, falls naturally into several parts and will occupy the entire section.

Theorem 8.1. The variation operator

$$f \mapsto \|\mathcal{H}_t f(x)\|_{v(2),\mathbb{R}_+}, \qquad x \in \mathbb{R}^n,$$

is not of strong nor weak type (p, p) with respect to γ_{∞} , for any $p \in [1, \infty)$.

8.1. **Approximation of the kernel.** In the proof of this result, we only consider $t \in (0,1]$, and f will be supported in a compact set. Moreover, $\mathcal{H}_t f$ will only be considered at points x in another compact set. For the L^p norms of f and those of the variation of $\mathcal{H}_t f$, we can therefore use Lebesgue measure instead of γ_{∞} . In the integral (2.3) defining $\mathcal{H}_t f(x)$, we can write du instead of $d\gamma_{\infty}(u)$. This also allows us to delete the factors $(\det Q_{\infty})^{1/2}$ and $e^{-R(x)}$ in the expression (2.4) for the Mehler kernel. What remains is the kernel

$$\widetilde{K}_t(x, u) = (\det Q_t)^{-1/2} \exp \left[-\frac{1}{2} \left\langle \left(Q_t^{-1} - Q_{\infty}^{-1} \right) (u - D_t x), u - D_t x \right\rangle \right],$$

and we will approximate this kernel. The $\mathcal{O}(\cdot)$ symbol will be used for scalars, vectors and matrices and is for $t \to 0$. The implicit constants involved may depend on the compact sets mentioned, as well as on Q and B.

Since $t \leq 1$, the definition (2.1) implies $Q_t = Qt + \mathcal{O}(t^2)$, and then

$$Q_t^{-1} - Q_{\infty}^{-1} = Q^{-1}t^{-1} + \mathcal{O}(1).$$

Further, $D_t x = x + \mathcal{O}(t)$ because of (2.5). Since x and u stay bounded, it follows that

$$\langle \left(Q_t^{-1} - Q_\infty^{-1} \right) (u - D_t x), u - D_t x \rangle$$

$$= \langle Q^{-1} t^{-1} (u - x + \mathcal{O}(t)), u - x + \mathcal{O}(t) \rangle + \mathcal{O}(|x - u| + t)$$

$$= \langle Q^{-1} t^{-1} (u - x), u - x \rangle + \mathcal{O}(|x - u| + t). \tag{8.1}$$

We also observe that det $Q_t = t^n \det Q(1 + \mathcal{O}(t))$, so that

$$(\det Q_t)^{-1/2} = (\det Q)^{-1/2} t^{-n/2} (1 + \mathcal{O}(t)).$$
 (8.2)

This makes it natural to approximate $\widetilde{K}_t(x,u)$ by the simple convolution kernel $K_t^{\rm c}(x-u)$, where

$$K_t^{c}(y) = (\det Q)^{-1/2} t^{-n/2} \exp\left(-\frac{1}{2} t^{-1} |Q^{-1/2}y|^2\right),$$
 (8.3)

and we let

$$\mathcal{H}_t^{\mathbf{c}} f(x) = K_t^{\mathbf{c}} * f(x).$$

Our plan for proving Theorem 8.1 is to find functions f which give a counterexample for the variation of the operator \mathcal{H}_t^c . The construction is based on a probabilistic result for the one-dimensional torus due to Qian [41]. We first verify that the approximation described above is good enough to let us work with the kernel $K_t^c(x-u)$ instead of $\widetilde{K}_t(x,u)$. In this process, we consider a discrete variation, limiting t to the sequence $2^{-2\ell}$, $\ell \in \mathbb{N}_+$.

8.2. The difference operator. Let Δ_t be the operator defined by the kernel $\widetilde{K}_t(x,u) - K_t^c(x-u)$.

Proposition 8.2. Let C_1 and C_2 be compact subsets of \mathbb{R}^n . If $f \in L^2(\mathbb{R}^n)$ has support contained in C_1 , then

$$\left\| \|\Delta_{2^{-2\ell}} f\|_{v(2),\mathbb{N}_+} \right\|_{L^2(C_2)} \lesssim \|f\|_{L^2(C_1)}.$$

Here the implicit constants may depend on C_1 and C_2 , in addition to Q and B.

Proof. By means of (8.1) and (8.2), we get for $x \in C_2$ and $u \in C_1$

$$\widetilde{K}_{t}(x,u)
= (\det Q)^{-1/2} t^{-n/2} (1 + \mathcal{O}(t)) \exp\left(-\frac{1}{2} t^{-1} \left\langle Q^{-1}(u-x), u-x \right\rangle + \mathcal{O}(|x-u|+t)\right)
= K_{t}^{c}(x-u) (1 + \mathcal{O}(t)) \exp\left(\mathcal{O}(|x-u|+t)\right)
= K_{t}^{c}(x-u) + K_{t}^{c}(x-u) \mathcal{O}(|x-u|+t).$$

In the last term here, we may replace |x-u| by $t^{1/2}$, if we reduce the factor 1/2 in the exponential factor in the expression (8.3) for K_t^c . Thus

$$|\widetilde{K}_t(x,u) - K_t^c(x-u)| \lesssim t^{1/2} t^{-n/2} \exp\left(-ct^{-1}|Q^{-1/2}(x-u)|^2\right).$$

With f supported in C_1 and $x \in C_2$, we thus have

$$|\Delta_t f(x)| \lesssim t^{1/2} \int f(u) t^{-n/2} \exp\left(-ct^{-1}|Q^{-1/2}(x-u)|^2\right) du.$$

The integral here is the convolution of f and an integrable kernel, and hence

$$\|\Delta_t f\|_{L^2(C_2)} \lesssim t^{1/2} \|f\|_{L^2(C_1)}.$$

Letting $t = 2^{-2\ell}$, squaring and summing, we obtain

$$\left\| \left(\sum_{\ell=1}^{\infty} (\Delta_{2^{-2\ell}} f)^2 \right) \right\|_{L^2(C_2)} \lesssim \|f\|_{L^2(C_1)}.$$

Because of (3.3), this completes the proof.

8.3. A counterexample for K_t^c . For convenience we normalize K_t^c by multiplying by $(2\pi)^{-n/2}$. Then we make an orthogonal change of variables in x and in u which diagonalizes the matrix Q. By also scaling the coordinates, we can then replace Q by the identity matrix. The new variables will be called x' and u', and we replace f by g defined by f(u) = g(u'), with equivalent L^p norms. In the new coordinates, the relevant kernel is a tensor product

$$K_t^*(y') = \prod_{i=1}^n J_t(y_i')$$

where J_t is the standard one-dimensional gaussian kernel

$$J_t(y_i') = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2} t^{-1} |y_i'|^2\right)$$

and the operator is given by $\mathcal{H}_t^* g = K_t^* * g$.

We will choose g as a tensor product $g(u') = \prod_{i=1}^n g_i(u'_i)$, so that

$$\mathcal{H}_t^* g(x') = \prod_{i=1}^n J_t * g_j(x_i'),$$

with the convolutions taken in \mathbb{R} . In the sequel, we will consider the values of \mathcal{H}_t^*g only at points x' in the cube $[0,1]^n$, and each g_i will have support in [-1,2]. This determines the compact sets C_1 and C_2 , via the change of variables. For $i=2,\ldots,n$, we simply choose $g_i=\chi_{[-1,2]}$, and observe that for these i

$$J_t * g_i(x_i') = 1 + \mathcal{O}(t), \qquad x_i' \in [0, 1].$$

It follows that

$$1 - \prod_{i=2}^{n} J_t * g_i = \mathcal{O}(t) \quad \text{in} \quad [0, 1]^{n-1}.$$
 (8.4)

The construction of g_1 requires more care. In the rest of this subsection, we will write simply x, u, y instead of x'_1 , u'_1 , y'_1 in one dimension. As already mentioned, we restrict t to the set $\{2^{-2\ell}, \ell = 1, 2, ...\}$, and define operators

$$A_{\ell} g_1 = J_{2^{-2\ell}} * g_1, \qquad \ell = 1, 2, \dots$$

Notice that $J_{2^{-2\ell}}(y) = (2\pi)^{-1/2} 2^{\ell} \exp\left(-\frac{1}{2} |2^{\ell}y|^2\right)$.

We will consider the variation in ℓ with ℓ ranging over the set

$$\mathcal{I}_N := \{ \ell \in \mathbb{Z} : 2N < \ell \le 3N \},$$

for large $N \in \mathbb{N}$.

To choose our function g_1 , we use the Rademacher functions, supported in [0,1] and given by

$$r_k = \sum_{j=1}^{2^{k-1}} \left(\chi_{((2j-2)2^{-k}, (2j-1)2^{-k})} - \chi_{((2j-1)2^{-k}, 2j2^{-k})} \right), \qquad k = 1, 2, \dots$$

We identify [0,1) with the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, considered as a probability space with Lebesgue measure. Then the r_k are independent Bernoulli random variables, taking the values ± 1 with equal probability 1/2.

In \mathbb{R} we define functions coinciding with the r_k in (0,1) and supported in [-1,2], by setting for $k=1,2,\ldots$

$$q_k(u) = r_k(u+1) + r_k(u) + r_k(u-1), \qquad u \in \mathbb{R}.$$

The function g_1 will be

$$g_N := \sum_{k \in \mathcal{I}_N} q_k.$$

Notice that the notation g_N is not in conflict with g_i , i = 1, ..., n, since N is large. Then for 1

$$||g_N||_{L^p(\mathbb{R})} \simeq ||g_N||_{L^p(\mathbb{T})} \simeq \sqrt{N}, \tag{8.5}$$

due to Khinchine's inequality. We also observe the trivial estimate

$$|g_N(x)| \le N, \qquad x \in \mathbb{R}. \tag{8.6}$$

The following is the main result of the present subsection.

Proposition 8.3. For some constant c > 0, the Lebesque measure of the set

$$\left\{ x \in (0,1) : \|A_{\ell} g_N(x)\|_{v(2),\mathcal{I}_N} > c \sqrt{N \log \log N} \right\}$$

tends to 1 as $N \to \infty$. Here the variation is taken in ℓ .

Proof. Starting with A_{ℓ} , we will change operator in several steps, until we arrive at an operator to which Qian's result in [41] applies.

Step 1 will take us from A_{ℓ} to the mean value operator D_{ℓ} , $\ell = 1, 2, \ldots$, given by

$$D_{\ell}f(x) = 2^{\ell-1} \int_{x-2^{-\ell}}^{x+2^{-\ell}} f(y) \, dy.$$

This definition works equally well in \mathbb{R} and in \mathbb{T} , and we will write $D_{\ell}^{\mathbb{R}}$ and $D_{\ell}^{\mathbb{T}}$ to distinguish the two.

Jones and Wang [27, Section 2] have introduced an equivalence relation \sim_p between sequences of operators acting on functions defined on the torus. We define a similar relation for operators P_{ℓ} and Q_{ℓ} , $\ell \in \mathbb{N}_+$, acting on functions in \mathbb{R} , with p=2. By $(P_{\ell}) \sim_2 (Q_{\ell})$ we mean that for any sequence $(\nu_{\ell})_1^{\infty}$, with $|\nu_{\ell}| \leq 1$, and any $f \in L^2(\mathbb{R})$

$$\|\sum_{\ell>1} (P_{\ell}f - Q_{\ell}f) \nu_{\ell}\|_2 \lesssim \|f\|_2.$$

Here the norm is that in $L^2(\mathbb{R})$. As shown in [27, Theorem 2.5], this implies that

$$\left\| \left(\sum_{\ell \ge 1} |P_{\ell} f - Q_{\ell} f|^2 \right)^{1/2} \right\|_2 \lesssim \|f\|_2,$$

which by (3.3) yields

$$\left\| \|P_{\ell}f - Q_{\ell}f\|_{v(2),\mathbb{N}_{+}} \right\|_{2} \lesssim \|f\|_{2}. \tag{8.7}$$

Lemma 8.4. We have

$$(A_{\ell}) \sim_2 (D_{\ell}^{\mathbb{R}}).$$

Proof. We adapt the proof of [27, Lemma 2.8] to \mathbb{R} . Observe first that A_{ℓ} and $D_{\ell}^{\mathbb{R}}$ are convolution operators with kernels $J_{2^{-2\ell}} = (2\pi)^{-1/2} \, 2^{\ell} \, \exp\left(-\frac{1}{2} \, \left(2^{\ell} \, y\right)^2\right)$ and $\chi_{\ell}(y) = 2^{\ell-1} \, \chi \, \left(2^{\ell} \, y\right)$, respectively, χ denoting the characteristic function of the interval (-1,1). We use Fourier transforms, defined by $\mathcal{F} f(\xi) = \hat{f}(\xi) = \int f(x) e^{-2\pi i x \, \xi} \, dx$, and the Plancherel theorem, to deduce

$$\| \sum_{\ell \ge 1} (A_{\ell} f - D_{\ell}^{\mathbb{R}} f) \nu_{\ell} \|_{2}^{2} = \int_{\mathbb{R}} \Big| \sum_{\ell \ge 1} \mathcal{F}(A_{\ell} f - D_{\ell}^{\mathbb{R}} f)(\xi) \nu_{\ell} \Big|^{2} d\xi$$

$$\leq \int_{\mathbb{R}} |\hat{f}(\xi)|^{2} \Big(\sum_{\ell \ge 1} |\widehat{J}_{2^{-2\ell}}(\xi) - \widehat{\chi}_{\ell}(\xi)| \Big)^{2} d\xi.$$

It is thus enough to prove that

$$\sum_{\ell>1} \left| \widehat{J_{2^{-2\ell}}}(\xi) - \widehat{\chi_{\ell}}(\xi) \right| \lesssim 1. \tag{8.8}$$

The Fourier transform of the kernel $J_{2^{-2\ell}}$ is

$$\widehat{J_{2^{-2\ell}}}(\xi) = \exp\left(-2\pi^2(2^{-\ell}\xi)^2\right),$$

so that $|1 - \widehat{J_{2^{-2\ell}}}(\xi)| \lesssim |2^{-\ell}\xi|^2$ for $|2^{-2\ell}\xi| \leq 1$ and $|\widehat{J_{2^{-\ell}}}(\xi)| \lesssim |2^{-\ell}\xi|^{-2}$ for $|2^{-\ell}\xi| \geq 1$. The Fourier transform

$$\widehat{\chi_{\ell}}(\xi) = \frac{\sin(2\pi 2^{-\ell}\xi)}{2\pi 2^{-\ell}\xi}$$

similarly satisfies $|1-\widehat{\chi}_{\ell}(\xi)| \lesssim |2^{-\ell}\xi|^2$ for $|2^{-\ell}\xi| \leq 1$ but only $|\widehat{\chi}_{\ell}(\xi)| \lesssim |2^{-\ell}\xi|^{-1}$ for $|2^{-\ell}\xi| \geq 1$. From this (8.8) and the lemma follow.

To $A_{\ell}(g_N)(x)$ and $D_{\ell}^{\mathbb{R}}(g_N)(x)$ we now apply the triangle inequality for the variation seminorm and write for $x \in \mathbb{R}$

$$||A_{\ell}(g_N)(x)||_{v(2),\mathcal{I}_N} \ge ||D_{\ell}^{\mathbb{R}}(g_N)(x)||_{v(2),\mathcal{I}_N} - h_1(x),$$
 (8.9)

where

$$h_1(x) = \|A_{\ell}(g_N)(x) - D_{\ell}^{\mathbb{R}}(g_N)(x)\|_{v(2),\mathcal{I}_N}.$$

From Lemma 8.4, (8.7) and (8.5), we conclude that $||h_1||_{L^2(\mathbb{R})} \lesssim \sqrt{N}$.

Step 2 consists in passing from \mathbb{R} to \mathbb{T} . Observe that for any point $x \in (0,1)$, also considered as a point in \mathbb{T} , one has $D_{\ell}^{\mathbb{R}}(q_k)(x) = D_{\ell}^{\mathbb{T}}(r_k)(x)$. Summing in k, we get for these x

$$D_{\ell}^{\mathbb{R}}(g_N)(x) = D_{\ell}^{\mathbb{T}}(T_N)(x),$$

where the function

$$T_N = \sum_{k \in \mathcal{I}_N} r_k$$

is defined in \mathbb{T} . From (8.5) we conclude

$$||T_N||_{L^2(\mathbb{T})} \simeq \sqrt{N}. \tag{8.10}$$

For Step 3, we let E_{ℓ} for $\ell = 1, 2, \ldots$ denote the conditional expectation operator which replaces an integrable function defined in \mathbb{T} by its mean value in each dyadic subinterval $((j-1)2^{-\ell}, j2^{-\ell}) \subset [0,1]$. Then

$$E_{\ell}(T_N) = \sum_{k \in \mathcal{I}_N, \, k \le \ell} r_k \tag{8.11}$$

for $\ell \in \mathcal{I}_N$, so that the $E_{\ell}(T_N)$ form a finite martingale. Step 3 aims at replacing $D_{\ell}^{\mathbb{T}}$ by E_{ℓ} .

Jones and Rosenblatt prove in [24, Remark 4] that for functions $f \in L^2(\mathbb{T})$

$$\left\| \left(\sum_{\ell=1}^{\infty} |D_{\ell}^{\mathbb{T}} f - E_{\ell} f|^{2} \right)^{1/2} \right\|_{2} \lesssim \|f\|_{2}.$$

Here we refer to the norm in $L^2(\mathbb{T})$. These authors use a one-sided, right mean value operator (see [24, p.528]), but since our $D_{\ell}^{\mathbb{T}}$ is the mean of the right and left operators, this is no problem.

Letting here $f = T_N$, we can apply (3.3) and then (8.10), to get

$$\left\| \left\| D_{\ell}^{\mathbb{T}}(T_N) - E_{\ell}(T_N) \right\|_{v(2), \mathcal{I}_N} \right\|_2 \le \left\| \left(\sum_{\ell \in \mathcal{I}_N} |D_{\ell}^{\mathbb{T}}(T_N) - E_{\ell}(T_N)|^2 \right)^{1/2} \right\|_2 \lesssim \|T_N\|_2 \simeq \sqrt{N}.$$

The triangle inequality for the seminorm $\|\cdot\|_{v(2),\mathcal{I}_N}$ now implies that for $x\in\mathbb{T}$

$$||D_{\ell}^{\mathbb{T}}(T_N)(x)||_{v(2),\mathcal{I}_N} \ge ||E_{\ell}(T_N)(x)||_{v(2),\mathcal{I}_N} - h_2(x), \tag{8.12}$$

with

$$||h_2||_{L^2(\mathbb{T})} \lesssim \sqrt{N}. \tag{8.13}$$

In **Step 4** we observe that the r_k with $k \in \mathcal{I}_N$ are N independent random variables fulfilling the hypotheses of Qian [41, Theorem 2.2]. Because of (8.11), this result tells us that the probability, i.e., the Lebesgue measure, of the set

$$\left\{ x \in \mathbb{T} : \|E_{\ell}(T_N)(x)\|_{v(2),\mathcal{J}_N} > c\sqrt{N\log\log N} \right\}$$

tends to 1 as $N \to \infty$, for some c > 0.

Finally, **Step 5** combines the result of Step 4 with the preceding steps, in reverse order. Considering Step 3, we see that (8.13) and Chebyshev's inequality imply that $|h_2(x)|$ is much smaller than $\sqrt{N \log \log N}$ except on a subset of \mathbb{T} whose measure tends to 0 as $N \to \infty$. Then (8.12) implies that the measure of the set

$$\left\{ x \in \mathbb{T} : \|D_{\ell}^{\mathbb{T}}(T_N)(x)\|_{v(2),\mathcal{I}_N} > c\sqrt{N\log\log N} \right\}$$

tends to 1 as $N \to \infty$, for some c > 0.

Now Step 2 says that this set is the same as

$$\left\{ x \in [0,1) : \|D_{\ell}^{\mathbb{R}}(g_N)(x)\|_{v(2),\mathcal{I}_N} > c\sqrt{N \log \log N} \right\}.$$

As for Step 1, we can apply (8.9) with $x \in [0,1)$ and argue as we did for Step 3. Thus the set

$$\left\{ x \in [0,1) : \|A_{\ell}(g_N)(x)\|_{v(2),\mathcal{I}_N} > c\sqrt{N\log\log N} \right\}.$$

also has measure tending to 1, for some c > 0.

The proof of Proposition 8.3 is complete.

8.4. End of proof of Theorem 8.1. The function $g = \bigotimes_{i=1}^n g_i$, where $g_1 = g_N$, satisfies for $x' \in [0,1]^n$

$$\mathcal{H}_{t}^{*} g(x') = J_{t} * g_{N}(x'_{1}) \prod_{i=2}^{n} J_{t} * g_{i}(x'_{1})$$

$$= J_{t} * g_{N}(x'_{1}) - J_{t} * g_{1}(x'_{1}) \left(1 - \prod_{i=2}^{n} J_{t} * g_{i}(x'_{i})\right)$$

To estimate the second term here, we apply (8.6) to its first factor and (8.4) to the second factor. The second term is thus no larger than constant times Nt, which is dominated by 2^{-N} if $t = 2^{-2\ell}$ with $\ell \in \mathcal{I}_N$. From (3.3) we see that the seminorm $\|\cdot\|_{v(2),\mathcal{I}_N}$ of the second term is no larger than $N^{1/2}2^{-N}$, since the cardinality of \mathcal{I}_N is N. Then we apply Proposition 8.3 to the first term, and conclude that the statement of this proposition also holds for $\mathcal{H}_t^* g(x')$.

The same is then true for $\mathcal{H}_t^c f(x)$, where f(u) = g(u') as before, and x is now in some compact set. Finally, Proposition 8.2 allows us to replace \mathcal{H}_t^c by the operator defined by the kernel $\widetilde{K}_t(x,u)$, and Theorem 8.1 follows.

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