# SOME REMARKS ON PERIODIC GRADINGS 

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#### Abstract

Let $\mathfrak{q}$ be a finite-dimensional Lie algebra, $\vartheta \in \operatorname{Aut}(\mathfrak{q})$ a finite order automorphism, and $\mathfrak{q}_{0}$ the subalgebra of fixed points of $\vartheta$. Using $\vartheta$ one can construct a pencil $\mathcal{P}$ of compatible Poisson brackets on $\mathcal{S}(\mathfrak{q})$, and thereby a 'large' Poisson-commutative subalgebra $Z(\mathfrak{q}, \vartheta)$ of $\mathcal{S}(\mathfrak{q})^{q_{0}}$. In this article, we study one particular bracket $\{,\}_{\infty} \in \mathcal{P}$ and the related Poisson centre $\mathcal{Z}_{\infty}$. It is shown that $\mathcal{Z}_{\infty}$ is a polynomial ring, if $\mathfrak{q}$ is reductive.


## Contents

Introduction ..... 1

1. Preliminaries on Poisson brackets and polynomial contractions ..... 3
2. Properties of $\mathfrak{q}_{(\infty)}$ and of the Poisson centre $\mathcal{Z}_{\infty} \subset \mathcal{S}\left(\mathfrak{q}_{(\infty)}\right)$ ..... 8
3. The reductive case ..... 9
4. Properties of $\mathcal{Z}_{\infty}$ in the reductive case ..... 14
References ..... 20

## Introduction

The ground field $\mathbb{k}$ is algebraically closed and $\operatorname{char}(\mathbb{k})=0$. Let $\mathfrak{q}=(\mathfrak{q},[]$,$) be a finite-$ dimensional algebraic Lie algebra, i.e., $\mathfrak{q}=\operatorname{Lie} Q$, where $Q$ is a connected affine algebraic group. The dual space $\mathfrak{q}^{*}$ is a Poisson variety, i.e., the algebra of polynomial functions on $\mathfrak{q}^{*}, \mathbb{k}\left[\mathfrak{q}^{*}\right] \simeq \mathcal{S}(\mathfrak{q})$, is equipped with the Lie-Poisson bracket $\{$,$\} . Here \{x, y\}=[x, y]$ for $x, y \in \mathfrak{q}$. Poisson-commutative subalgebras of $\mathbb{k}\left[\mathfrak{q}^{*}\right]$ are important tools for the study of geometry of the coadjoint action of $Q$ and representation theory of $\mathfrak{q}$.
0.1. There is a well-known method, the Lenard-Magri scheme, for constructing "large" Poisson-commutative subalgebras of $\mathbb{k}\left[\mathfrak{q}^{*}\right]$, which is related to compatible Poisson brackets, see e.g. [GZ00]. Two Poisson brackets $\{,\}^{\prime}$ and $\{,\}^{\prime \prime}$ are said to be compatible, if any linear combination $\{,\}_{a, b}:=a\{,\}^{\prime}+b\{,\}^{\prime \prime}$ with $a, b \in \mathbb{k}$ is a Poisson bracket. Then one defines a certain dense open subset $\Omega_{\mathrm{reg}} \subset \mathbb{k}^{2}$ that corresponds to the regular brackets in the pencil $\mathcal{P}=\left\{\{,\}_{a, b} \mid(a, b) \in \mathbb{k}^{2}\right\}$. Let $\mathcal{Z}_{a, b} \subset \mathcal{S}(\mathfrak{q})$ denote the Poisson centre of $\left(\mathcal{S}(\mathfrak{q}),\{,\}_{a, b}\right)$. Then the subalgebra $\mathcal{Z} \subset \mathcal{S}(\mathfrak{q})$ generated by $\mathcal{Z}_{a, b}$ with $(a, b) \in \Omega_{\text {reg }}$ is Poisson-commutative

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w.r.t. $\{,\}^{\prime}$ and $\{,\}^{\prime \prime}$. An obvious first step is to take the initial Lie-Poisson bracket $\{$, as $\{,\}^{\prime}$. The rest depends on a clever choice of $\{,\}^{\prime \prime}$.
0.2. Let $\vartheta$ be an automorphism of $\mathfrak{q}$ of finite order $m$. Then $\mathfrak{q}$ is equipped with a $\mathbb{Z}_{m^{-}}{ }^{-}$ grading $\mathfrak{q}=\bigoplus_{i=0}^{m-1} \mathfrak{q}_{i}$, where each $\mathfrak{q}_{i}$ is an eigenspace of $\vartheta$. One can naturally construct a compatible Poisson bracket $\{,\}^{\prime \prime}$ associated with the grading [PY, PY21]. In this case, all Poisson brackets in $\mathcal{P}$ are linear and there are two lines $l_{1}, l_{2} \subset \mathbb{k}^{2}$ such that $\Omega=$ $\mathbb{k}^{2} \backslash\left(l_{1} \cup l_{2}\right) \subset \Omega_{\mathrm{reg}}$ and the Lie algebras corresponding to $(a, b) \in \Omega$ are isomorphic to $\mathfrak{q}$. The lines $l_{1}$ and $l_{2}$ give rise to new Lie algebras, denoted $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$. These new algebras are different contractions of $\mathfrak{q}$. A definition and basic properties of contractions are discussed in Section 1.2. Let ind $\mathfrak{q}$ denote the index of $\mathfrak{q}$ (see Section 1.1). Then we have ind $\mathfrak{q} \leqslant \operatorname{ind} \mathfrak{q}_{(t)}$ for $t \in\{0, \infty\}$. Let $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}} \simeq \mathbb{k}\left[\mathfrak{q}^{*}\right]^{Q}$ be the ring of symmetric invariants of $\mathfrak{q}$, i.e., the Poisson centre of $\mathcal{S}(\mathfrak{q})$. Let further $\mathcal{Z}=\mathcal{Z}(\mathfrak{q}, \vartheta)$ be the Poisson-commutative subalgebra associated with $\mathcal{P}$. Many features of $\mathcal{Z}$ depend on the properties of $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$.

In [PY21, PY23], we have studied the ring $\mathcal{Z}_{0}=\mathcal{S}\left(\mathfrak{q}_{(0)}\right)^{\mathfrak{q}_{(0)}}$ in case $\mathfrak{q}=\mathfrak{g}$ is reductive. For some $\mathfrak{g}$ such that ind $\mathfrak{g}_{(0)}=r k \mathfrak{g}$, this is a polynomial ring with rk $\mathfrak{g}$ generators. However, there are exceptions even if $m=2$ [Y17]. Partial results were obtained for $\mathcal{Z}_{\infty}=\mathcal{S}\left(\mathfrak{q}_{(\infty)}\right)^{\mathfrak{q}_{(\infty)}}$. Namely, if $\mathfrak{g}$ is reductive and $\vartheta$ is an inner automorphism, then $\mathcal{Z}_{\infty}=\mathcal{S}\left(\mathfrak{g}_{0}\right)$ [PY21].

In Section 1, we collect basic facts on the coadjoint action and symmetric invariants. Explicit descriptions of the algebras $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$ are presented in Section 1.3.
0.3. If ind $\mathfrak{q}_{(\infty)}>$ ind $\mathfrak{q}$, then $\mathcal{Z}_{\infty}$ does not have to be Poisson-commutative. Our first result states that $\left\{\mathcal{Z}_{\infty}^{\mathfrak{q}_{0}}, \mathcal{Z}_{\infty}^{\mathfrak{q}_{0}}\right\}=0$, if $\mathfrak{q}_{0}^{*}$ contains a regular in $\mathfrak{q}^{*}$ element. Furthermore, under that assumption, the algebra $\operatorname{alg}\left\langle\mathcal{Z}, \mathcal{Z}_{\infty}^{q_{0}}\right\rangle$, generated by $\mathcal{Z}$ and $\mathcal{Z}_{\infty}^{q_{0}}$, is still Poissoncommutative, see Theorem 2.3. Both statements have applications related to the current algebra $\mathfrak{q}[t]$. Namely, one can construct a large Poisson-commutative subalgebra of $\mathcal{S}\left(\mathfrak{q}[t]^{\vartheta}\right)$ following the ideas of [PY21, Sect. 8]. However, our new approach works for several non-reductive Lie algebras and does not require the assumption that ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$, which is imposed in [PY21]. The construction will appear in a forthcoming paper.

Section 3 contains a brief summary of Kac's classification of finite order automorphisms for a semisimple $\mathfrak{g}$ [Ka69]. In particular, we describe a relation between the roots of $\mathfrak{g}$ and of $\mathfrak{g}^{\sigma}$, where $\sigma$ is a diagram automorphism of $\mathfrak{g}$. Then in Section 3.3, we state main results of [PY21] on generators of $Z(\mathfrak{g}, \vartheta)$. Many crucial properties of $Z(\mathfrak{g}, \vartheta)$, including its transcendence degree, depend on the equality ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$, see e.g. [PY21, Thm 3.10] or Theorem 3.6 here. Conjecture 3.1 in [PY23] states that ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$ for all $\mathfrak{g}$ and all $\vartheta$. In Section 3.4, we recollect several instances, where the equality holds, and provide a few new positive examples, see Theorem 3.10.

In the more favourable reductive case, we prove that $\mathcal{Z}_{\infty}$ and $\mathcal{Z}_{\infty}^{\mathrm{g}_{0}}$ are always polynomial rings and describe their generators explicitly, see Section 4. We consider also the non-reductive Lie algebra $\tilde{\mathfrak{g}}=\mathfrak{g}_{0} \ltimes \mathfrak{g}_{(\infty)}$ and its coadjoint representation. It is shown that ind $\tilde{\mathfrak{g}}=\mathrm{rk} \mathfrak{g}+\mathrm{rk} \mathfrak{g}_{0}$ and that $\tilde{\mathfrak{g}}$ has the codim-2 property, see Theorem 4.4. Then by Theorem $4.5, \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial ring with ind $\tilde{\mathfrak{g}}$ generators. These generators are described explicitly.

Non-reductive Lie algebras $\mathfrak{q}$ such that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ is a polynomial ring with ind $\mathfrak{q}$ generators attract a lot of attention, see e.g. [J06, PPY, P07, P09, Y14, CM16, FL18]. A quest for this type of algebras continues. Many examples that are found so far are related to particular simple Lie algebras. For instance, assertions of [J06] and [PPY] hold in full generality only for $\mathfrak{s l}_{n}$ and $\mathfrak{s p}_{2 n}$. Note that our results on $\mathfrak{g}_{(\infty)}$ and $\mathfrak{g}$ are independent of the type of $\mathfrak{g}$ and apply to all finite order automorphisms. We prove also that $\tilde{\mathfrak{g}}$ is a Lie algebra of Kostant type in the terminology of [Y14].

The Lie algebra $\tilde{\mathfrak{g}}$ is quadratic, i.e., there is a $\tilde{\mathfrak{g}}$-invariant non-degenerate symmetric bilinear form on $\tilde{\mathfrak{g}}$. This implies that the adjoint and coadjoint representations of $\tilde{\mathfrak{g}}$ are isomorphic, see e.g. [P09, Sect. 1.1]. In [P09, Sect. 4], invariants of the adjoint action of $\tilde{\mathfrak{g}}$ are studied. In the notation of that paper, $\tilde{\mathfrak{g}}=\mathfrak{g}\langle m+1\rangle_{0}$. A more general object $\mathfrak{g}\langle n m+1\rangle_{0}$ of [P09] can be also interpreted in our context.

Suppose that $\mathfrak{g}=\mathfrak{h}^{\oplus n}$ is a sum of $n$ copies of a reductive Lie algebra $\mathfrak{h}$ and $\vartheta$ is a composition of a finite order automorphism $\tilde{\vartheta} \in \operatorname{Aut}(\mathfrak{h})$ and a cyclic permutation of the summands. Formally we have
$(0 \cdot 1) \vartheta\left(\left(x_{1}, x_{2}, \ldots, x_{n}\right)\right)=\left(x_{n}, \tilde{\vartheta}\left(x_{1}\right), x_{2}, \ldots, x_{n-1}\right)$ for $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in \mathfrak{h} \oplus \mathfrak{h} \oplus \ldots \oplus \mathfrak{h}$.
Then $\tilde{\mathfrak{g}}$ associated with $\vartheta$ is equal to $\mathfrak{h}\langle n m+1\rangle_{0}$. In [P09, Thm 4.1(ii)], it is shown that $\mathbb{k}\left[\mathfrak{g}\langle n m+1\rangle_{0}\right]^{\mathfrak{g}\{n m+1\rangle_{0}}$ is a polynomial algebra of Krull dimension $n \mathrm{rk} \mathfrak{g}+\mathrm{rk} \mathfrak{g}_{0}$, if $\mathfrak{g}_{0}$ contains a regular nilpotent element of $\mathfrak{g}$. Here we have no assumptions on $\mathfrak{g}_{0}$, i.e., we show that [P09, Thm 4.1(ii)] holds for all $\mathfrak{g}$ and $\vartheta$.

Our general reference for semisimple Lie groups and algebras is [Lie3].

## 1. Preliminaries on Poisson brackets and polynomial contractions

Let $Q$ be a connected affine algebraic group with Lie algebra $\mathfrak{q}$. The symmetric algebra of $\mathfrak{q}$ over $\mathbb{k}$ is $\mathbb{N}_{0}$-graded, i.e., $\mathcal{S}(\mathfrak{q})=\bigoplus_{i \geqslant 0} \delta^{i}(\mathfrak{q})$. It is identified with the algebra of polynomial functions on the dual space $\mathfrak{q}^{*}$.
1.1. The coadjoint representation. The group $Q$ acts on $\mathfrak{q}^{*}$ via the coadjoint representation and then $\mathrm{ad}^{*}: \mathfrak{q} \rightarrow \operatorname{GL}\left(\mathfrak{q}^{*}\right)$ is the coadjoint representation of $\mathfrak{q}$. The algebra of $Q$-invariant polynomial functions on $\mathfrak{q}^{*}$ is denoted by $\mathcal{S}(\mathfrak{q})^{Q}$ or $\mathbb{k}\left[\mathfrak{q}^{*}\right]^{Q}$. Write $\mathbb{k}\left(\mathfrak{q}^{*}\right)^{Q}$ for the field of $Q$-invariant rational functions on $\mathfrak{q}^{*}$.

Let $\mathfrak{q}^{\xi}=\left\{x \in \mathfrak{q} \mid \operatorname{ad}^{*}(x) \cdot \xi=0\right\}$ be the stabiliser in $\mathfrak{q}$ of $\xi \in \mathfrak{q}^{*}$. The index of $\mathfrak{q}$,
ind $\mathfrak{q}$, is the minimal codimension of $Q$-orbits in $\mathfrak{q}^{*}$. Equivalently, ind $\mathfrak{q}=\min _{\xi \in \mathfrak{q}^{*}} \operatorname{dim} \mathfrak{q}^{\xi}$. By the Rosenlicht theorem (see [Sp89, IV.2]), one also has ind $\mathfrak{q}=\operatorname{tr} . \operatorname{deg} \mathbb{k}\left(\mathfrak{q}^{*}\right)^{Q}$. Set $\boldsymbol{b}(\mathfrak{q})=(\operatorname{dim} \mathfrak{q}+\operatorname{ind} \mathfrak{q}) / 2$. Since the $Q$-orbits in $\mathfrak{q}^{*}$ are even-dimensional, $\boldsymbol{b}(\mathfrak{q})$ is an integer. If $\mathfrak{q}$ is reductive, then ind $\mathfrak{q}=\operatorname{rkq}$ and $\boldsymbol{b}(\mathfrak{q})$ equals the dimension of a Borel subalgebra.

The Lie-Poisson bracket on $\mathcal{S}(\mathfrak{q})$ is defined on $\mathcal{S}^{1}(\mathfrak{q})=\mathfrak{q}$ by $\{x, y\}:=[x, y]$. It is then extended to higher degrees via the Leibniz rule. Hence $\mathcal{S}(\mathfrak{q})$ has the usual associativecommutative structure and additional Poisson structure. Whenever we refer to subalgebras of $\mathcal{S}(\mathfrak{q})$, we always mean the associative-commutative structure. Then a subalgebra $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})$ is said to be Poisson-commutative, if $\{H, F\}=0$ for all $H, F \in \mathcal{A}$. It is well known that if $\mathcal{A}$ is Poisson-commutative, then $\operatorname{tr} . \operatorname{deg} \mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{q})$, see e.g. [Vi90, 0.2]. More generally, suppose that $\mathfrak{h} \subset \mathfrak{q}$ is a Lie subalgebra and $\mathcal{A} \subset \mathcal{S}(\mathfrak{q})^{\mathfrak{h}}$ is Poisson-commutative. Then $\operatorname{tr} . \operatorname{deg} \mathcal{A} \leqslant \boldsymbol{b}(\mathfrak{q})-\boldsymbol{b}(\mathfrak{h})+$ ind $\mathfrak{h}$, see [MY19, Prop. 1.1].

The centre of the Poisson algebra $(\mathcal{S}(\mathfrak{q}),\{\}$,$) is$

$$
\mathcal{Z}(\mathfrak{q}):=\{H \in \mathcal{S}(\mathfrak{q}) \mid\{H, F\}=0 \forall F \in \mathcal{S}(\mathfrak{q})\}=\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}=\mathbb{k}\left[\mathfrak{q}^{*}\right]^{\mathfrak{q}}=\mathbb{k}\left[\mathfrak{q}^{*}\right]^{Q} .
$$

Since the quotient field of $\mathbb{k}\left[\mathfrak{q}^{*}\right]^{Q}$ is contained in $\mathbb{k}\left(\mathfrak{q}^{*}\right)^{Q}$, we deduce from the Rosenlicht theorem that

$$
\operatorname{tr} \cdot \operatorname{deg}\left(\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}\right) \leqslant \operatorname{ind} \mathfrak{q} .
$$

The set of regular elements of $\mathfrak{q}^{*}$ is

$$
\mathfrak{q}_{\text {reg }}^{*}=\left\{\eta \in \mathfrak{q}^{*} \mid \operatorname{dim} \mathfrak{q}^{\eta}=\operatorname{ind} \mathfrak{q}\right\}=\left\{\eta \in \mathfrak{q}^{*} \mid \operatorname{dim} Q \cdot \eta \text { is maximal }\right\} .
$$

It is a dense open subset of $\mathfrak{q}^{*}$. Set $\mathfrak{q}_{\text {sing }}^{*}=\mathfrak{q}^{*} \backslash \mathfrak{q}_{\text {reg }}^{*}$. We say that $\mathfrak{q}$ has the codim- $n$ property if codim $\mathfrak{q}_{\text {sing }}^{*} \geqslant n$. The codim-2 property is going to be most important for us.

For $\gamma \in \mathfrak{q}^{*}$, let $\hat{\gamma}$ be the skew-symmetric bilinear form on $\mathfrak{q}$ defined by $\hat{\gamma}(\xi, \eta)=\gamma([\xi, \eta])$ for $\xi, \eta \in \mathfrak{q}$. It follows that $\operatorname{ker} \hat{\gamma}=\mathfrak{q}^{\gamma}$. The 2-form $\hat{\gamma}$ is related to the Poisson tensor (bivector) $\pi$ of the Lie-Poisson bracket $\{$,$\} as follows.$

Let $d H$ denote the differential of $H \in \mathcal{S}(\mathfrak{q})=\mathbb{k}\left[\mathfrak{q}^{*}\right]$. Then $\pi$ is defined by the formula $\pi(d H \wedge d F)=\{H, F\}$ for $H, F \in \mathcal{S}(\mathfrak{q})$. Then $\pi(\gamma)\left(d_{\gamma} H \wedge d_{\gamma} F\right)=\{H, F\}(\gamma)$ and therefore $\hat{\gamma}=\pi(\gamma)$. In this terms, ind $\mathfrak{q}=\operatorname{dim} \mathfrak{q}-\operatorname{rk} \pi$, where $\operatorname{rk} \pi=\max _{\gamma \in \mathfrak{q}^{*}} \operatorname{rk} \pi(\gamma)$.

For a subalgebra $A \subset \mathcal{S}(\mathfrak{q})$ and $\gamma \in \mathfrak{q}^{*}$, set $d_{\gamma} A=\left\langle d_{\gamma} F \mid F \in A\right\rangle_{\mathbb{k}}$. By the definition of $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$, we have

$$
d_{\gamma} \mathcal{S}(\mathfrak{q})^{\mathfrak{q}} \subset \operatorname{ker} \pi(\gamma)
$$

for each $\gamma \in \mathfrak{q}^{*}$.
1.2. Contractions and invariants. We refer to [Lie3, Ch.7, §2] for basic facts on contractions of Lie algebras. In this article, we consider contractions of the following form. Let $\mathbb{k}^{\star}=\mathbb{k} \backslash\{0\}$ be the multiplicative group of $\mathbb{k}$ and $\varphi: \mathbb{k}^{\star} \rightarrow \mathrm{GL}(\mathfrak{q}), s \mapsto \varphi_{s}$, a polynomial representation. That is, the matrix entries of $\varphi_{s}: \mathfrak{q} \rightarrow \mathfrak{q}$ are polynomials in $s$ w.r.t. some
(any) basis of $\mathfrak{q}$. Define a new Lie algebra structure on the vector space $\mathfrak{q}$ and associated Lie-Poisson bracket by

$$
[x, y]_{(s)}=\{x, y\}_{(s)}:=\varphi_{s}^{-1}\left[\varphi_{s}(x), \varphi_{s}(y)\right], x, y \in \mathfrak{q}, s \in \mathbb{k}^{\star}
$$

All the algebras $\left(\mathfrak{q},[,]_{(s)}\right)$ are isomorphic and $\left(\mathfrak{q},[,]_{(1)}\right)$ is the initial Lie algebra $\mathfrak{q}$. The induced $\mathbb{k}^{\star}$-action on the variety of structure constants is not necessarily polynomial, i.e., $\lim _{s \rightarrow 0}[x, y]_{(s)}$ may not exist for all $x, y \in \mathfrak{q}$. Whenever such a limit exists, we obtain a new linear Poisson bracket, denoted $\{,\}_{0}$, and thereby a new Lie algebra $\mathfrak{q}_{(0)}$, which is said to be a contraction of $\mathfrak{q}$. If we wish to stress that this construction is determined by $\varphi$, then we write $\{x, y\}_{(\varphi, s)}$ for the bracket in (1-4) and say that $\mathfrak{q}_{(0)}=\mathfrak{q}_{(0, \varphi)}$ is the $\varphi$-contraction of $\mathfrak{q}$ or is the zero limit of $\mathfrak{q}$ w.r.t. $\varphi$. A criterion for the existence of $\mathfrak{q}_{(0)}$ can be given in terms of Lie brackets of the $\varphi$-eigenspaces in $\mathfrak{q}$, see [Y17, Sect. 4]. We identify all algebras $\mathfrak{q}_{(s)}$ and $\mathfrak{q}_{(0)}$ as vector spaces. The semi-continuity of index implies that ind $\mathfrak{q}_{(0)} \geqslant$ ind $\mathfrak{q}$.

The map $\varphi_{s}, s \in \mathbb{k}^{\star}$, is naturally extended to an invertible transformation of $\mathcal{S}^{j}(\mathfrak{q})$, which we also denote by $\varphi_{s}$. The resulting graded map $\varphi_{s}: \mathcal{S}(\mathfrak{q}) \rightarrow \mathcal{S}(\mathfrak{q})$ is nothing but the comorphism associated with $s \in \mathbb{k}^{\star}$ and the dual representation $\varphi^{*}: \mathbb{k}^{\star} \rightarrow \operatorname{GL}\left(\mathfrak{q}^{*}\right)$. Since $\mathcal{S}^{j}(\mathfrak{q})$ has a basis that consists of $\varphi\left(\mathbb{k}^{\star}\right)$-eigenvectors, any $F \in \mathcal{S}^{j}(\mathfrak{q})$ can be written as $F=\sum_{i \geqslant 0} F_{i}$, where the sum is finite and $\varphi_{s}\left(F_{i}\right)=s^{i} F_{i} \in \mathcal{S}^{j}(\mathfrak{q})$. Let $F^{\bullet}$ denote the non-zero component $F_{i}$ with maximal $i$.

Proposition 1.1 ([Y14, Lemma 3.3]). If $F \in \mathcal{Z}(\mathfrak{q})$ and $\mathfrak{q}_{(0)}$ exists, then $F^{\bullet} \in \mathcal{Z}\left(\mathfrak{q}_{(0)}\right)$.
1.3. Periodic gradings of Lie algebras and related compatible brackets. Let $\vartheta \in \operatorname{Aut}(\mathfrak{q})$ be a Lie algebra automorphism of finite order $m \geqslant 2$ and $\zeta=\sqrt[m]{1}$ a primitive root of unity. Write also $\operatorname{ord}(\vartheta)$ for the order of $\vartheta$. If $\mathfrak{q}_{i}$ is the $\zeta^{i}$-eigenspace of $\vartheta, i \in \mathbb{Z}_{m}$, then the direct sum $\mathfrak{q}=\bigoplus_{i \in \mathbb{Z}_{m}} \mathfrak{q}_{i}$ is a periodic grading or $\mathbb{Z}_{m}$-grading of $\mathfrak{q}$. The latter means that $\left[\mathfrak{q}_{i}, \mathfrak{q}_{j}\right] \subset \mathfrak{q}_{i+j}$ for all $i, j \in \mathbb{Z}_{m}$. Here $\mathfrak{q}_{0}=\mathfrak{q}^{\vartheta}$ is the fixed-point subalgebra for $\vartheta$ and each $\mathfrak{q}_{i}$ is a $\mathfrak{q}_{0}$-module.

We choose $\{0,1, \ldots, m-1\} \subset \mathbb{Z}$ as a fixed set of representatives for $\mathbb{Z}_{m}=\mathbb{Z} / m \mathbb{Z}$. Under this convention, we have $\mathfrak{q}=\mathfrak{q}_{0} \oplus \mathfrak{q}_{1} \oplus \ldots \oplus \mathfrak{q}_{m-1}$ and

$$
\left[\mathfrak{q}_{i}, \mathfrak{q}_{j}\right] \subset \begin{cases}\mathfrak{q}_{i+j}, & \text { if } i+j \leqslant m-1 \\ \mathfrak{q}_{i+j-m}, & \text { if } i+j \geqslant m\end{cases}
$$

This is needed below, when we consider $\mathbb{Z}$-graded contractions of $\mathfrak{q}$ associated with $\vartheta$.
The presence of $\vartheta$ allows us to split the Lie-Poisson bracket on $\mathfrak{q}^{*}$ into a sum of two compatible Poisson brackets. Consider the polynomial representation $\varphi: \mathbb{k}^{\star} \rightarrow \mathrm{GL}(\mathfrak{q})$ such that $\varphi_{s}(x)=s^{j} x$ for $x \in \mathfrak{q}_{j}$. As in Section 1.2, this defines a family of linear Poisson brackets on $\mathcal{S}(\mathfrak{q})$ parametrised by $s \in \mathbb{k}^{\star}$, see (1•4).

Below we outline some results of [PY21]:
(i) There is a limit $\lim _{s \rightarrow 0}\{x, y\}_{(s)}=:\{x, y\}_{0}$, which is a linear Poisson bracket on $\mathcal{S}(\mathfrak{q})$.
(ii) The difference $\{\}-,\{,\}_{0}=:\{,\}_{\infty}$ is a linear Poisson bracket on $\mathcal{S}(\mathfrak{q})$, which is obtained as the zero limit w.r.t. the polynomial representation $\psi: \mathbb{k}^{\star} \rightarrow \mathrm{GL}(\mathfrak{q})$ such that $\psi_{s}=s^{m} \cdot \varphi_{s^{-1}}=s^{m} \cdot \varphi_{s}^{-1}, s \in \mathbb{k}^{\star}$. In other words, $\{,\}_{\infty}=\lim _{s \rightarrow 0}\{,\}_{(\psi, s)}$.
(iii) For any $\vartheta \in \operatorname{Aut}(\mathfrak{q})$ of finite order, the Poisson brackets $\{,\}_{0}$ and $\{,\}_{\infty}$ are compatible, and the corresponding pencil contains the initial Lie-Poisson bracket.

Set

$$
\{,\}_{t}=\{,\}_{0}+t\{,\}_{\infty}
$$

where $t \in \mathbb{P}:=\mathbb{k} \cup\{\infty\}$ and the value $t=\infty$ corresponds to the bracket $\{,\}_{\infty}$. Let $\mathfrak{q}_{(t)}$ stand for the Lie algebra corresponding to $\{,\}_{t}$. All these Lie algebras have the same underlying vector space.

Proposition 1.2 ([PY21, Prop. 2.3]). The Lie algebras $\mathfrak{q}_{(0)}$ and $\mathfrak{q}_{(\infty)}$ are $\mathbb{N}_{0}$-graded. More precisely, if $\mathfrak{r}[i]$ stands for the component of grade $i \in \mathbb{N}_{0}$ in an $\mathbb{N}_{0}$-graded Lie algebra $\mathfrak{r}$, then

$$
\mathfrak{q}_{(0)}[i]=\left\{\begin{array}{ll}
\mathfrak{q}_{i} & \text { for } i=0,1, \ldots, m-1 \\
0 & \text { otherwise }
\end{array}, \mathfrak{q}_{(\infty)}[i]= \begin{cases}\mathfrak{q}_{m-i} & \text { for } i=1,2, \ldots, m \\
0 & \text { otherwise }\end{cases}\right.
$$

In particular, $\mathfrak{q}_{(\infty)}$ is nilpotent and the subspace $\mathfrak{q}_{0}$, which is the highest grade component of $\mathfrak{q}_{(\infty)}$, belongs to the centre of $\mathfrak{q}_{(\infty)}$.

Suppose that $t \in \mathbb{k}^{\star}$. Then $\{,\}_{t}=\{,\}_{(s)}$, where $s^{m}=t$. The algebras $\mathfrak{q}_{(t)}$ with $t \in \mathbb{k}^{\star}$ are isomorphic and they have one and the same index. We say that $t \in \mathbb{P}$ is regular if ind $\mathfrak{q}_{(t)}=$ ind $\mathfrak{q}$ and write $\mathbb{P}_{\text {reg }}$ for the set of regular values. Then $\mathbb{P}_{\text {sing }}:=\mathbb{P} \backslash \mathbb{P}_{\text {reg }} \subset\{0, \infty\}$ is the set of singular values.

Let $Q_{0} \subset Q$ be the connected subgroup of $Q$ with Lie $Q_{0}=\mathfrak{q}_{0}$. It is easy to see that $Q_{0}$ is an algebraic group. Hence there are connected algebraic groups $Q_{(t)}$ such that $\mathfrak{q}_{(t)}=$ Lie $Q_{(t)}$ for each $t \in \mathbb{P}$.

Let $\mathcal{Z}_{t}$ be the centre of the Poisson algebra $\left(\mathcal{S}(\mathfrak{q}),\{,\}_{t}\right)$. In particular, $\mathcal{Z}_{1}=\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$. If $t=s^{m} \in \mathbb{k}^{\star}$, then $\mathcal{Z}_{t}=\varphi_{s}^{-1}\left(\mathcal{Z}_{1}\right)$. By Eq. (1•1), we have tr.deg $\mathcal{Z}_{t} \leqslant \operatorname{ind} \mathfrak{q}_{(t)}$. In [PY21], we have studied the subalgebra $Z \subset \mathcal{S}(\mathfrak{q})$ generated by the centres $\mathcal{Z}_{t}$ with $t \in \mathbb{P}_{\text {reg }}$, i.e.,

$$
\mathcal{Z}=\mathcal{Z}(\mathfrak{q}, \vartheta)=\operatorname{alg}\left\langle\mathcal{Z}_{t} \mid t \in \mathbb{P}_{\mathrm{reg}}\right\rangle
$$

By a general property of compatible brackets, the algebra $Z$ is Poisson-commutative w.r.t. all brackets $\{,\}_{t}$ with $t \in \mathbb{P}$, cf. [PY, Sect. 2]. Note that the Lie subalgebra $\mathfrak{q}_{0} \subset \mathfrak{q}=\mathfrak{q}_{(1)}$ is also the same Lie subalgebra in any $\mathfrak{q}_{(t)}$ with $t \neq \infty$ (cf. Proposition 1.2 for $\mathfrak{q}_{(0)}$ ). Therefore,

$$
\begin{equation*}
\mathcal{Z}_{t} \subset \mathcal{S}(\mathfrak{q})^{q_{0}} \text { for } t \neq \infty \tag{1.6}
\end{equation*}
$$

Convention. We think of $\mathfrak{q}^{*}$ as the dual space for any Lie algebra $\mathfrak{q}_{(t)}$ and sometimes omit the subscript ' $(t)^{\prime}$ in $\mathfrak{q}_{(t)}^{*}$. However, if $\xi \in \mathfrak{q}^{*}$, then the stabiliser of $\xi$ with respect to the coadjoint representation of $\mathfrak{q}_{(t)}$ is denoted by $\mathfrak{q}_{(t)}^{\xi}$. Set $\mathfrak{q}_{\infty, \text { reg }}^{*}:=\left(\mathfrak{q}_{(\infty)}^{*}\right)_{\text {reg }}$.
1.4. Good generating systems. Let $\mathfrak{g}$ be a reductive Lie algebra. Consider a contraction $\mathfrak{g}_{(0)}$ of $\mathfrak{g}$ given by $\left\{\varphi_{s} \mid s \in \mathbb{k}^{\star}\right\}$. Let $\left\{x_{1}, \ldots, x_{\operatorname{dim} \mathfrak{g}}\right\}$ be basis of $\mathfrak{g}$ and $\omega=x_{1} \wedge \ldots \wedge x_{\operatorname{dim} \mathfrak{g}}$ a volume form. Then $\varphi(\omega)=s^{D} \omega$, where $D$ is a non-negative integer. We set $D_{\varphi}:=D$.

The Poisson centre $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is a polynomial algebra of Krull dimension $l=$ rk $\mathfrak{g}$ and ind $\mathfrak{g}=l$. Hence one has now the equality in Eq. (1-1). Let $\left\{H_{1}, \ldots, H_{l}\right\}$ be a set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and $d_{i}=\operatorname{deg} H_{i}$. Then $\sum_{i=1}^{l} d_{i}=\boldsymbol{b}(\mathfrak{g})$. As above, each $H_{j}$ decomposes as $H_{j}=\sum_{i \geqslant 0} H_{j, i}$, where $\varphi_{s}\left(H_{j}\right)=\sum_{i \geqslant 0} s^{i} H_{j, i}$. The polynomials $H_{j, i}$ are called bi-homogeneous components of $H_{j}$. By definition, the $\varphi$-degree of
 of $H_{j}$ with maximal $\varphi$-degree. We set $\operatorname{deg}_{\varphi} H_{j}=\operatorname{deg}_{\varphi} H_{j}^{\bullet}$ and $d_{j}^{\bullet}=\operatorname{deg}_{\varphi} H_{j}^{\bullet}$.

Definition 1.3. Let us say that $H_{1}, \ldots, H_{l}$ is a good generating system in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ (g.g.s. for short) for $\varphi$, if $H_{1}^{\bullet}, \ldots, H_{l}^{\bullet}$ are algebraically independent. Then we also say that $\varphi$ admits a g.g.s.

The property of being 'good' really depends on a generating system. The importance of g.g.s. is manifestly seen in the following result.

Theorem 1.4 ([Y14, Theorem 3.8]). Let $H_{1}, \ldots, H_{l}$ be an arbitrary set of homogeneous algebraically independent generators of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Then
(i) $\sum_{j=1}^{l} \operatorname{deg}_{\varphi} H_{j} \geqslant D_{\varphi}$;
(ii) $H_{1}, \ldots, H_{l}$ is a g.g.s. if and only if $\sum_{j=1}^{l} \operatorname{deg}_{\varphi} H_{j}=D_{\varphi}$;
(iii) if $\mathfrak{g}_{(0)}$ has the codim-2 property, ind $\mathfrak{g}_{(0)}=l$, and $H_{1}, \ldots, H_{l}$ is a g.g.s., then $\mathcal{Z}_{0}=$ $\mathcal{S}\left(\mathfrak{g}_{(0)}\right)^{\mathfrak{g}_{(0)}}$ is a polynomial algebra freely generated by $H_{1}^{\bullet}, \ldots, H_{l}^{\bullet}$ and

$$
\left\{\xi \in \mathfrak{g}^{*} \mid d_{\xi} H_{1}^{\bullet} \wedge \ldots \wedge d_{\xi} H_{l}^{\bullet}=0\right\}=\left(\mathfrak{g}_{(0)}\right)_{\text {sing }}^{*}
$$

Let $F_{1}, \ldots, F_{N} \in \mathbb{k}\left[x_{i} \mid 1 \leqslant i \leqslant n\right]=\mathbb{k}\left[\mathbb{A}^{n}\right]$ be algebraically independent homogeneous polynomials. Set

$$
\mathcal{J}\left(F_{1}, \ldots, F_{N}\right):=\left\{x \in \mathbb{A}^{n} \mid d_{x} F_{1} \wedge \ldots \wedge d_{x} F_{N}=0\right\}
$$

Then $\mathcal{J}\left(F_{1}, \ldots, F_{N}\right)$ is a proper closed subset of $\mathbb{A}^{n}$. An open subset of $\mathbb{A}^{n}$ is said to be big, if its complement does not contain divisors. Thereby the differentials $d F_{i}$ with $1 \leqslant i \leqslant N$ are linearly independent on a big open subset if and only if $\operatorname{dim} \mathcal{J}\left(F_{1}, \ldots, F_{N}\right) \leqslant n-2$.

By the Kostant regularity criterion for $\mathfrak{g}, \mathcal{J}\left(H_{1}, \ldots, H_{l}\right)=\mathfrak{g}_{\text {sing }}^{*}$, see [K63, Theorem 9]. By [K63], $\mathfrak{g}$ has the codim-3 property, i.e., $\operatorname{dim} \mathfrak{g}_{\text {sing }}^{*} \leqslant \operatorname{dim} \mathfrak{g}-3$.

Part (iii) of Theorem 1.4 states that the Kostant regularity criterion holds for $\mathfrak{g}_{(0)}$. One of the ingredients of the proof, which will be used in this paper as well, is the following statement.

Theorem 1.5 ([PPY, Theorem 1.1]). Let $F_{1}, \ldots, F_{N}$ be as above. If $\operatorname{dim} \mathcal{J}\left(F_{1}, \ldots, F_{N}\right) \leqslant n-2$, then $\mathcal{F}=\mathbb{k}\left[F_{j} \mid 1 \leqslant j \leqslant N\right]$ is an algebraically closed subalgebra of $\mathbb{k}\left[\mathbb{A}^{n}\right]$, i.e., if $H \in \mathbb{k}\left[\mathbb{A}^{n}\right]$ is algebraic over the field $\mathbb{k}\left(F_{1}, \ldots, F_{N}\right)$, then $H \in \mathcal{F}$.

## 2. Properties of $\mathfrak{q}_{(\infty)}$ And of the Poisson centre $\mathcal{Z}_{\infty} \subset \mathcal{S}\left(\mathfrak{q}_{(\infty)}\right)$

By Proposition 1.2, $\mathfrak{q}_{(\infty)}$ is a nilpotent $\mathbb{N}$-graded Lie algebra. Recall also that the subspace $\mathfrak{q}_{0}$ belongs to the centre of $\mathfrak{q}_{(\infty)}$. Let $\pi_{t}$ be the Poisson tensor of $\mathfrak{q}_{(t)}$. We identify $\mathfrak{q}_{0}^{*}$ with the annihilator $\operatorname{Ann}\left(\bigoplus_{i=1}^{m-1} \mathfrak{q}_{i}\right) \subset \mathfrak{q}^{*}$ and regard it as a subspace of $\mathfrak{q}^{*}$.

For any $\xi \in \mathfrak{q}_{0}^{*} \subset \mathfrak{q}^{*}$, we have

$$
\mathfrak{q}_{(\infty)}^{\xi}=\mathfrak{q}_{0} \oplus \bigoplus_{j \geqslant 1} \mathfrak{q}_{j}^{\xi}, \quad \text { where } \quad \mathfrak{q}_{j}^{\xi}=\left\{y \in \mathfrak{q}_{j} \mid \xi\left(\left[y, \mathfrak{q}_{m-j}\right]\right)=0\right\}
$$

Furthermore, $\mathfrak{q}^{\xi}=\mathfrak{q}_{0}^{\xi} \oplus \bigoplus_{j \geqslant 1} \mathfrak{q}_{j}^{\xi}$. If this $\xi$ is regular in $\mathfrak{q}^{*}$, then $\xi \in\left(\mathfrak{q}_{0}^{*}\right)_{\text {reg }}$.
Theorem 2.1. Suppose that $\mathfrak{q}_{0}^{*} \cap \mathfrak{q}_{\text {reg }}^{*} \neq \varnothing$. Then ind $\mathfrak{q}_{(\infty)}=\operatorname{dim} \mathfrak{q}_{0}+\operatorname{ind} \mathfrak{q}-\operatorname{ind} \mathfrak{q}_{0}$.
Proof. (1) Take any $\xi \in \mathfrak{q}_{0}^{*} \cap \mathfrak{q}_{\text {reg }}^{*}$. Then $\mathfrak{q}_{(\infty)}^{\xi}=\mathfrak{q}^{\xi}+\mathfrak{q}_{0}$. Furthermore $\mathfrak{q}^{\xi} \cap \mathfrak{q}_{0}=\mathfrak{q}_{0}^{\xi}$. As we have explained above, $\xi \in\left(\mathfrak{q}_{0}^{*}\right)_{\text {reg }}$. Thereby $\operatorname{dim} \mathfrak{q}_{0}^{\xi}=\operatorname{ind} \mathfrak{q}_{0}$ and hence $\operatorname{dim} \mathfrak{q}_{(\infty)}^{\xi}=$ ind $\mathfrak{q}+\operatorname{dim} \mathfrak{q}_{0}-\operatorname{ind} \mathfrak{q}_{0}$. This leads to ind $\mathfrak{q}_{(\infty)} \leqslant \operatorname{ind} \mathfrak{q}+\operatorname{dim} \mathfrak{q}_{0}-\operatorname{ind} \mathfrak{q}_{0}$.
(2) Let us prove the opposite inequality. Take any $\xi \in \mathfrak{q}_{\text {reg }}^{*}$ such that $\bar{\xi}=\left.\xi\right|_{\mathfrak{q}_{0}} \in\left(\mathfrak{q}_{0}^{*}\right)_{\text {reg }}$. Note that there is a non-empty open subset consisting of suitable elements. For all but finitely many $t$, we have $\operatorname{dim} \mathfrak{q}_{(t)}^{\xi}=$ ind $\mathfrak{q}$. Hence $\mathfrak{v}=\lim _{t \rightarrow \infty} \mathfrak{q}_{(t)}^{\xi}$ is a well-defined subspace of $\mathfrak{q}$ of dimension ind $\mathfrak{q}$. If $t \neq \infty$, then $\left.\pi_{t}(\xi)\right|_{\mathfrak{q}_{0} \times \mathfrak{q}}=\left.\pi_{1}(\xi)\right|_{\mathfrak{q}_{0} \times \mathfrak{q}}$ and $\pi_{1}(\xi)\left(\mathfrak{q}_{0}, \mathfrak{q}_{(t)}^{\xi}\right)=$ $\pi_{t}(\xi)\left(\mathfrak{q}_{0}, \mathfrak{q}_{(t)}^{\xi}\right)=0$. Therefore

$$
\pi_{1}(\xi)\left(\mathfrak{q}_{0}, \mathfrak{v}\right)=0
$$

and $\mathfrak{v} \cap \mathfrak{q}_{0} \subset \mathfrak{q}_{0}^{\bar{\xi}}$. By the construction, $\mathfrak{v} \subset \mathfrak{q}_{(\infty)}^{\xi}$. Thus

$$
\operatorname{dim} \mathfrak{q}_{(\infty)}^{\xi} \geqslant \operatorname{dim}\left(\mathfrak{v}+\mathfrak{q}_{0}\right)=\operatorname{dim} \mathfrak{v}+\operatorname{dim} \mathfrak{q}_{0}-\operatorname{dim}\left(\mathfrak{q}_{0} \cap \mathfrak{v}\right) \geqslant \operatorname{ind} \mathfrak{q}+\operatorname{dim} \mathfrak{q}_{0}-\operatorname{ind} \mathfrak{q}_{0}
$$

This finishes the proof, since the inequality holds on a non-empty open subset.
The assumption $\mathfrak{q}_{0}^{*} \cap \mathfrak{q}_{\text {reg }}^{*} \neq \varnothing$ is satisfied in the reductive case, since the reductive subalgebra $\mathfrak{g}_{0}=\mathfrak{g}^{\vartheta}$ contains regular semisimple elements of $\mathfrak{g}$, see e.g. [Ka83, §8.8]. Thus, we obtain a new proof of [PY21, Theorem 3.2].

Corollary 2.2 ([PY21]). Suppose that $\mathfrak{q}=\mathfrak{g}$ is reductive. Then one has $\infty \in \mathbb{P}_{\text {reg }}$ if and only if $\operatorname{dim} \mathfrak{g}_{0}=\operatorname{rk} \mathfrak{g}_{0}$, i.e., $\mathfrak{g}_{0}$ is an abelian subalgebra of $\mathfrak{g}$.

Recall that $\mathfrak{q}_{0} \subset \mathcal{Z}_{\infty}$. Thereby $\mathcal{Z}_{\infty}$ is not Poisson-commutative, unless $\mathfrak{q}_{0}$ is commutative. However, if $\left[\mathfrak{q}_{0}, \mathfrak{q}_{0}\right]=0$ and $\mathfrak{q}_{0}^{*} \cap \mathfrak{q}_{\text {reg }}^{*} \neq \varnothing$, then ind $\mathfrak{q}_{0}=\operatorname{dim} \mathfrak{q}_{0}$ and $\{,\}_{\infty}$ is a regular structure in the pencil spanned by $\{$,$\} and \{,\}_{0}$, see Theorem 2.1. In this case, $\mathcal{Z}_{\infty} \subset \mathcal{Z}(\mathfrak{q}, \vartheta)$. Thereby a description of $\mathcal{Z}_{\infty}$ is desirable.

The group $Q_{0}$ acts on $\mathfrak{q}_{(\infty)}$. Hence one may consider $\mathcal{Z}_{\infty}^{q_{0}} \subset \mathcal{Z}_{\infty}$.
Theorem 2.3. Assume that $\mathfrak{q}_{0}^{*} \cap \mathfrak{q}_{\text {reg }}^{*} \neq \varnothing$.
(i) We have $\left\{\mathcal{Z}_{\infty}^{q_{0}}, \mathcal{Z}_{\infty}^{\text {q. }_{0}}\right\}=0$.
(ii) The algebra $\operatorname{alg}\left\langle\mathcal{Z}, \mathcal{Z}_{\infty}^{q_{0}}\right\rangle$ is still Poisson-commutative.
(iii) We have also tr. $\operatorname{deg} \mathcal{Z}_{\infty}^{\mathfrak{q}_{0}} \leqslant$ ind $\mathfrak{q}$.

Proof. Take $\xi \in \mathfrak{q}_{\text {reg }}^{*} \cap \mathfrak{q}_{\infty, \text { reg }}^{*}$ such that $\bar{\xi}=\left.\xi\right|_{\mathfrak{q}_{0}} \in\left(\mathfrak{q}_{0}^{*}\right)_{\text {reg. }}$. Note that there is a non-empty open subset consisting of suitable elements. Following the proof of Theorem 2.1, set $\mathfrak{v}=$ $\lim _{t \rightarrow \infty} \mathfrak{q}_{(t)}^{\xi}$. Recall that $\mathfrak{v}+\mathfrak{q}_{0} \subset \mathfrak{q}_{(\infty)}^{\xi}$ and that $\operatorname{dim}\left(\mathfrak{v}+\mathfrak{q}_{0}\right) \geqslant \operatorname{ind} \mathfrak{q}_{(\infty)}$, see (2•3). Since $\xi \in \mathfrak{q}_{\infty, \text { reg }}^{*}$ there is the equality $\mathfrak{q}_{(\infty)}^{\xi}=\mathfrak{v}+\mathfrak{q}_{0}$.

Next $d_{\xi} \mathcal{Z}_{\infty}^{q_{0}} \subset \mathfrak{q}_{(\infty)}^{\xi}$ by $(1 \cdot 3)$ and $\pi_{1}(\xi)\left(d_{\xi} \mathcal{Z}_{\infty}^{q_{0}}, \mathfrak{q}_{0}\right)=0$, since $\mathcal{Z}_{\infty}^{q_{0}}$ consists of $\mathfrak{q}_{0}$-invariants. In the proof of Theorem 2.1, we have established that $\pi_{1}(\xi)\left(\mathfrak{v}, \mathfrak{q}_{0}\right)=0$, see (2.2). Suppose that $y \in \mathfrak{q}_{0}$ and $\pi_{1}(\xi)\left(y, \mathfrak{q}_{0}\right)=0$. Then $y \in \mathfrak{q}_{0}^{\bar{\xi}}$. In particular, $\boldsymbol{d}_{\xi} \mathcal{Z}_{\infty}^{\mathfrak{q}_{0}} \subset \mathfrak{v}+\mathfrak{q}_{0}^{\bar{\xi}}$ and $\mathfrak{v} \cap \mathfrak{q}_{0} \subset \mathfrak{q}_{0}^{\bar{\xi}}$. By the dimension reasons, $\mathfrak{v} \cap \mathfrak{q}_{0}=\mathfrak{q}_{0}^{\bar{\xi}}$. Thus $d_{\xi} \mathcal{Z}_{\infty}^{\mathfrak{q}_{0}} \subset \mathfrak{v}+\mathfrak{q}_{0}^{\bar{\xi}} \subset \mathfrak{v}$. The inclusion proves the inequality tr. $\operatorname{deg} \mathcal{Z}_{\infty}^{\mathfrak{q}_{0}} \leqslant \operatorname{dim} \mathfrak{v}=$ ind $\mathfrak{q}$.

For almost all $t \in \mathbb{P}$, we have $\operatorname{dim} \mathfrak{q}_{(t)}^{\xi}=\operatorname{ind} \mathfrak{q}$. Hence $\mathfrak{v}$ is a subspace of

$$
L(\xi):=\sum_{t: \operatorname{rk} \pi_{t}(\xi)=\operatorname{dim} \mathfrak{q}-\operatorname{ind} \mathfrak{q}} \mathfrak{q}_{(t)}^{\xi}
$$

and the latter is known to be isotropic w.r.t. $\pi_{t}(\xi)$ for any $t$, see e.g. [PY08, Appendix]. Then, in particular, any $F \in\left\{\mathcal{Z}_{\infty}^{\mathfrak{q}_{0}}, \mathcal{Z}_{\infty}^{\mathfrak{q}_{0}}\right\}$ vanishes at $\xi$, and, since $\xi$ is generic, the first claim is settled.

If $\infty \in \mathbb{P}_{\text {reg }}$, then $\mathcal{Z}_{\infty}^{\text {q }_{0}} \subset \mathcal{Z}_{\infty} \subset \mathcal{Z}$ and part (ii) is clear. Suppose that $\infty \in \mathbb{P}_{\text {sing }}$. Then $z \subset \operatorname{alg}\left\langle\mathcal{Z}_{t} \mid t \neq \infty\right\rangle$. For any $t \neq \infty$, the brackets $\{,\}_{\infty}$ and $\{,\}_{t}$ span

$$
\mathcal{P}=\left\{a\{,\}_{0}+b\{,\}_{\infty} \mid(a, b) \in \mathbb{k}^{2}\right\}
$$

and $\left\{\mathcal{Z}_{\infty}, \mathcal{Z}_{t}\right\}_{\infty}=\left\{\mathcal{Z}_{\infty}, \mathcal{Z}_{t}\right\}_{t}=0$. Thereby $\left\{\mathcal{Z}_{\infty}, \mathcal{Z}_{t}\right\}=0$ for each $t \neq \infty$. This finishes the proof.

## 3. The reductive case

In most of this section, we recollect known results about automorphisms of reductive Lie algebras and properties of $\mathcal{Z}(\mathfrak{g}, \vartheta)$. Statements of Section 3.2 are crucial for the proof of Theorem 4.4. Theorem 3.10 on the index of $\mathfrak{g}_{(0)}$ is a new result.
3.1. The Kac diagram of a finite order automorphism. We describe briefly Kac's classification of finite order automorphisms in the semisimple case [Ka69].

A pair $(\mathfrak{g}, \vartheta)$ is decomposable, if $\mathfrak{g}$ is a direct sum of two non-trivial $\vartheta$-stable ideals. Otherwise $(\mathfrak{g}, \vartheta)$ is said to be indecomposable. Classification of finite order automorphism readily reduces to the indecomposable case. The centre of $\mathfrak{g}$ is always a $\vartheta$-stable ideal and automorphisms of an abelian Lie algebra have no particular significance (in our context). Therefore assume that $\mathfrak{g}$ is semisimple.

Suppose that $\mathfrak{g}$ is not simple and $(\mathfrak{g}, \vartheta)$ is indecomposable. Then $\mathfrak{g}=\mathfrak{h}^{\oplus n}$ is a sum of $n$ copies of a simple Lie algebra $\mathfrak{h}$ and $\vartheta$ is a composition of an automorphism of $\mathfrak{h}$ and a cyclic permutation of the summands.

Below we assume that $\mathfrak{g}$ is simple. By a result of R. Steinberg [St68, Theorem 7.5], every semisimple automorphism of $\mathfrak{g}$ fixes a Borel subalgebra of $\mathfrak{g}$ and a Cartan subalgebra thereof. Let $\mathfrak{b}$ be a $\vartheta$-stable Borel subalgebra and $\mathfrak{t} \subset \mathfrak{b}$ a $\vartheta$-stable Cartan subalgebra. This yields a $\vartheta$-stable triangular decomposition $\mathfrak{g}=\mathfrak{u}^{-} \oplus \mathfrak{t} \oplus \mathfrak{u}$, where $\mathfrak{u}=[\mathfrak{b}, \mathfrak{b}]$. Let $\Delta=\Delta(\mathfrak{g})$ be the set of roots of $\mathfrak{g}$ related to $\mathfrak{t}, \Delta^{+}$the set of positive roots corresponding to $\mathfrak{u}$, and $\Pi \subset \Delta^{+}$the set of simple roots. Let $\mathfrak{g}_{\gamma}$ be the root space for $\gamma \in \Delta$. Hence $\mathfrak{u}=\bigoplus_{\gamma \in \Delta^{+}} \mathfrak{g}_{\gamma}$. Let $e_{\gamma} \in \mathfrak{g}_{\gamma}$ be a non-zero root vector.

Clearly, $\vartheta$ induces a permutation of $\Pi$, which is an automorphism of the Dynkin diagram, and $\vartheta$ is inner if and only if this permutation is trivial. Accordingly, $\vartheta$ can be written as a product $\sigma \circ \vartheta^{\prime}$, where $\vartheta^{\prime}$ is inner and $\sigma$ is the so-called diagram automorphism of $\mathfrak{g}$. We refer to $[K a 83, \S 8.2]$ for an explicit construction and properties of $\sigma$. In particular, $\sigma$ depends only on the connected component of $\operatorname{Aut}(\mathfrak{g})$ that contains $\vartheta$ and $\operatorname{ord}(\sigma)$ equals the order of the corresponding permutation of $\Pi$.
— The case of an inner $\vartheta$ :
Set $\Pi=\left\{\alpha_{1}, \ldots, \alpha_{l}\right\}$ and let $\delta=\sum_{i=1}^{l} n_{i} \alpha_{i}$ be the highest root in $\Delta^{+}$. An inner periodic automorphism with $\mathfrak{t} \subset \mathfrak{g}_{0}$ is determined by an (l+1)-tuple of non-negative integers, Kac labels, $\boldsymbol{p}=\left(p_{0}, p_{1}, \ldots, p_{l}\right)$ such that $\operatorname{gcd}\left(p_{0}, \ldots, p_{l}\right)=1$ and $\boldsymbol{p} \neq(0, \ldots, 0)$. Set $m:=$ $p_{0}+\sum_{i=1}^{l} n_{i} p_{i}$ and let $\overline{p_{i}}$ denote the unique representative of $\{0,1, \ldots, m-1\}$ such that $p_{i} \equiv \overline{p_{i}}(\bmod m)$. The $\mathbb{Z}_{m}$-grading $\mathfrak{g}=\bigoplus_{i=0}^{m-1} \mathfrak{g}_{i}$ corresponding to $\vartheta=\vartheta(\boldsymbol{p})$ is defined by the conditions that

$$
\mathfrak{g}_{\alpha_{i}} \subset \mathfrak{g}_{\overline{p_{i}}} \text { for } i=1, \ldots, l, \mathfrak{g}_{-\delta} \subset \mathfrak{g}_{\overline{p_{0}}}, \text { and } \mathfrak{t} \subset \mathfrak{g}_{0}
$$

The Kac diagram $\mathcal{K}(\vartheta)$ of $\vartheta=\vartheta(\boldsymbol{p})$ is the affine (= extended) Dynkin diagram of $\mathfrak{g}, \tilde{\mathcal{D}}(\mathfrak{g})$, equipped with the labels $p_{0}, p_{1}, \ldots, p_{l}$. In $\mathcal{K}(\vartheta)$, the $i$-th node of the usual Dynkin diagram $\mathcal{D}(\mathfrak{g})$ represents $\alpha_{i}$ and the extra node represents $-\delta$. It is convenient to assume that $\alpha_{0}=-\delta$ and $n_{0}=1$. Then $(l+1)$-tuple $\left(n_{0}, n_{1}, \ldots, n_{l}\right)$ yields the coefficients of a linear dependence for $\alpha_{0}, \alpha_{1}, \ldots, \alpha_{l}$.
— The case of an outer $\vartheta$ :
Let $\sigma$ be the diagram automorphism of $\mathfrak{g}$ related to $\vartheta$. The order of a nontrivial diagram automorphism is either 2 or 3 , there 3 is possible only for $\mathfrak{g}$ of type $D_{4}$. Therefore, $\sigma$ defines either a $\mathbb{Z}_{2}$ - or $\mathbb{Z}_{3}$-grading of $\mathfrak{g}$, say $\mathfrak{g}=\mathfrak{g}^{\sigma} \oplus \mathfrak{g}_{1}^{(\sigma)}$ or $\mathfrak{g}=\mathfrak{g}^{\sigma} \oplus \mathfrak{g}_{1}^{(\sigma)} \oplus \mathfrak{g}_{2}^{(\sigma)}$.

The Kac diagrams of outer periodic automorphisms are supported on the twisted affine Dynkin diagrams of index 2 and 3, see [Vi76, §8] and [Lie3, Table3]. Such a diagram has $r+1$ nodes, where $r=\mathrm{rk} \mathfrak{g}^{\sigma}$, certain $r$ nodes comprise the Dynkin diagram of the simple Lie algebra $\mathfrak{g}^{\sigma}$, and the additional node represents the lowest weight $-\delta_{1}$ of the
$\mathfrak{g}^{\sigma}$-module $\mathfrak{g}_{1}^{(\sigma)}$. Write $\delta_{1}=\sum_{i=1}^{r} a_{i}^{\prime} \nu_{i}$, where the elements $\nu_{i}$ are the simple roots of $\mathfrak{g}^{\sigma}$, and set $a_{0}^{\prime}=1$. Then the $(r+1)$-tuple $\left(a_{0}^{\prime}, a_{1}^{\prime}, \ldots, a_{r}^{\prime}\right)$ yields coefficients of linear dependence for $-\delta_{1}, \nu_{1}, \ldots, \nu_{r}$.

Let $\boldsymbol{p}=\left(p_{0}, p_{1}, \ldots, p_{r}\right)$ be an $(r+1)$-tuple of non-negative integers, Kac labels, such that $\boldsymbol{p} \neq(0,0, \ldots, 0)$ and $\operatorname{gcd}\left(p_{0}, p_{1}, \ldots, p_{r}\right)=1$. The Kac diagram of $\vartheta=\vartheta(\boldsymbol{p})$ is the required twisted affine diagram equipped with the labels $\left(p_{0}, p_{1}, \ldots, p_{r}\right)$ over the nodes. Then $m=\operatorname{ord}(\vartheta(\boldsymbol{p}))=\operatorname{ord}(\sigma) \cdot \sum_{i=0}^{r} a_{i}^{\prime} p_{i}$.
3.2. Relation between roots of $\mathfrak{g}$ and $\mathfrak{g}^{\sigma}$. Let $\sigma$ be a diagram automorphism of $\mathfrak{g}$ associated with $\mathfrak{b}$ and $\mathfrak{t} \subset \mathfrak{b}$. Then $\sigma(\mathfrak{b})=\mathfrak{b}$ and $\sigma(\mathfrak{t})=\mathfrak{t}$ by the construction. Let $\Delta_{\mathfrak{g}^{\sigma}}^{+}$be the set of positive roots of $\mathfrak{g}^{\sigma}$ associated with $\left(\mathfrak{b}^{\sigma}, \mathfrak{t}_{0}\right)$, where $\mathfrak{t}_{0}=\mathfrak{t}^{\sigma}$.

Take any $\alpha \in \Delta^{+}$. If $\sigma(\alpha) \neq \alpha$, then the restriction $\bar{\alpha}=\left.\alpha\right|_{\mathfrak{t}_{0}}$ is a positive root of $\mathfrak{g}^{\sigma}$. In any case, $\bar{\alpha}=\left.\sigma(\alpha)\right|_{\mathfrak{t}_{0}}$. Suppose $\beta \in \Delta_{\mathfrak{g}^{\sigma}}$. Then there is a non-zero root vector $x_{\beta} \in \mathfrak{g}^{\sigma}$. We can write $x_{\beta}=\sum_{\alpha \in \Delta} b_{\alpha} e_{\alpha}$ with $b_{\alpha} \in \mathbb{k}$. Then $\left.\alpha\right|_{\mathfrak{t}_{0}}=\beta$, whenever $b_{\alpha} \neq 0$. In particular, $\Delta_{\mathfrak{g}^{\sigma}}$ is contained in $\left.\Delta\right|_{\mathrm{t}_{0}}$.

Let $\langle\sigma\rangle \subset$ Aut $(\mathfrak{g})$ be a subgroup generated by $\sigma$. In all types except $\mathrm{A}_{2 n}$, the restriction of roots from $\mathfrak{t}$ to $\mathfrak{t}_{0}$ produces bijections between $\langle\sigma\rangle$-orbits on $\Delta^{+}$and $\Delta_{\mathfrak{g}^{\sigma}}^{+}$. If $\mathfrak{g}=\mathfrak{s l}_{2 n+1}$, then the situation is slightly different. These are well-known facts, nevertheless we give a brieft explanation below. Set $\mathfrak{s}_{\alpha}=\mathfrak{g}_{\alpha} \oplus \mathfrak{g}_{-\alpha} \oplus\left[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}\right]$.

Lemma 3.1 (cf. [Ka83, §8.2]). Suppose $\operatorname{ord}(\sigma)=3$. Then there is a bijection between $\langle\sigma\rangle$-orbits on $\Delta^{+}$and $\Delta_{\mathfrak{g}^{\sigma}}^{+}$.

Proof. Here $\mathfrak{g}$ is of type $\mathrm{D}_{4}$. There are six $\langle\sigma\rangle$-orbits on $\Delta^{+}$. Three of these orbits have three elements, namely

$$
\left\{\alpha_{1}, \alpha_{3}, \alpha_{4}\right\},\left\{\alpha_{1}+\alpha_{2}, \alpha_{2}+\alpha_{3}, \alpha_{2}+\alpha_{4}\right\},\left\{\alpha_{1}+\alpha_{2}+\alpha_{3}, \alpha_{1}+\alpha_{2}+\alpha_{4}, \alpha_{2}+\alpha_{3}+\alpha_{4}\right\}
$$

The other three consist of fixed points: $\left\{\alpha_{2}\right\},\left\{\alpha_{1}+\alpha_{2}+\alpha_{3}+\alpha_{4}\right\},\left\{\alpha_{1}+2 \alpha_{2}+\alpha_{3}+\alpha_{4}\right\}$. The Lie algebra $\mathfrak{g}^{\sigma}$ is of type $G_{2}$ and it has 6 positive roots. Therefore there is nothing more to show.

Lemma 3.2. Suppose $\operatorname{ord}(\sigma)=2$. Let $x_{\mu}=\sum_{\alpha \in \Delta} b_{\alpha} e_{\alpha} \in\left(\mathfrak{b} \cap \mathfrak{g}^{\sigma}\right)$ with $b_{\alpha} \in \mathbb{k}$ be a non-zero root vector of $\mathfrak{g}^{\sigma}$. Fix one $\alpha \in \Delta^{+}$such that $b_{\alpha} \neq 0$. Suppose further that $\bar{\beta}=\left.\beta\right|_{\mathfrak{t}_{0}} \in \mathbb{k} \bar{\alpha}$ for some $\beta \in \Delta^{+}$that does not belong to the $\langle\sigma\rangle$-orbit of $\alpha$. Then $\beta=\alpha+\sigma(\alpha)$.

Proof. Since $\sigma\left(x_{\mu}\right)=x_{\mu}$, we have $\left(\mathfrak{s}_{\alpha}+\sigma\left(\mathfrak{s}_{\alpha}\right)\right)^{\sigma} \simeq \mathfrak{s l}_{2}$, and if $\sigma(\alpha)=\alpha$, then $\left.\sigma\right|_{\mathfrak{g}_{\alpha}}=$ id.
Let $h \in \mathfrak{t}_{0}$ be a subregular element of $\mathfrak{g}^{\sigma}$ such that $\bar{\alpha}(h)=0$ and $\gamma(h) \neq 0$, whenever $\gamma \in \Delta$ and $\left.\gamma\right|_{\mathfrak{t}_{0}} \notin \mathbb{k} \bar{\alpha}$. Consider $\mathfrak{g}^{h}$. On the one hand, we have $\left(\mathfrak{g}^{h}\right)^{\sigma}=\left(\mathfrak{g}^{\sigma}\right)^{h}=\mathfrak{s l}_{2}+\mathfrak{t}_{0}$. On the other hand, if $\sigma(\beta) \neq \beta$, then

$$
\left(\mathfrak{g}^{h}\right)^{\sigma} \supset\left(\mathfrak{s}_{\alpha}+\mathfrak{s}_{\sigma(\alpha)}\right)^{\sigma} \oplus\left(\mathfrak{s}_{\beta}+\mathfrak{s}_{\sigma(\beta)}\right)^{\sigma} \simeq \mathfrak{s l}_{2} \oplus \mathfrak{s l}_{2}
$$

which is a contradiction. This shows that $\sigma(\beta)=\beta$.

By the same reason as above, $\left.\sigma\right|_{\mathfrak{g}_{\beta}} \neq \mathrm{id}$. It follows that $\left.\sigma\right|_{\mathfrak{g}_{\beta}}=\left.\sigma\right|_{\mathfrak{g}_{-\beta}}=-\mathrm{id}$, since $\operatorname{ord}(\sigma)=2$. Then $\left[\mathfrak{g}_{\beta}, \mathfrak{g}_{-\beta}\right] \subset \mathfrak{t}_{0}$. Let $h_{\beta} \in\left[\mathfrak{g}_{\beta}, \mathfrak{g}_{-\beta}\right]$ be such that $\beta\left(h_{\beta}\right)=2$. Since $h_{\beta} \in \mathfrak{t}_{0}$ and $\bar{\beta} \in \mathbb{k} \bar{\alpha}$, we have $\alpha\left(h_{\beta}\right) \neq 0$. In view of this, $\left[\mathfrak{s}_{\beta}, \mathfrak{s}_{\alpha}\right] \neq 0$.

Set $\mathfrak{f}=\left[\mathfrak{g}^{h}, \mathfrak{g}^{h}\right]$. Then $\mathfrak{s}_{\alpha}+\mathfrak{s}_{\beta} \subset \mathfrak{f}$. Furthermore, if $\gamma \in \Delta$ is a root of $\mathfrak{f}$, then $\bar{\gamma}=\left.\gamma\right|_{\mathfrak{t}_{0}}$ belongs to $\mathbb{k} \bar{\alpha}$. Hence, $\operatorname{rk} \mathfrak{f} \geqslant 2$ and $\mathfrak{f}$ is simple. By the construction, $\mathfrak{f}^{\sigma}=\mathfrak{s l}_{2}+\tilde{\mathfrak{t}}$, where $\tilde{\mathfrak{t}} \subset \mathfrak{t}_{0}$ is a maximal torus of $\mathfrak{f}^{\sigma}$. The involution $\sigma$ induces an automorphism of $\mathfrak{f}$ of order 2 . It cannot be inner, because the restrictions of $\alpha$ and $\beta$ to $\tilde{\mathfrak{t}}$ coincide. From the description of outer involutions, we deduce that $\mathrm{rk} \mathfrak{f} \leqslant 2$. Therefore the only possibility for $\left(\mathfrak{f}, \mathfrak{f}^{\sigma}\right)$ is the pair $\left(\mathfrak{s l}_{3}, \mathfrak{S o}_{3}\right)$. Here $\beta=\alpha_{1}+\alpha_{2}$ with $\alpha_{1}, \alpha_{2}$ being simple roots of $\mathfrak{s l}_{3}$ and $\sigma\left(\alpha_{1}\right)=\alpha_{2}$. Since $e_{\beta}=\left[e_{\alpha_{1}}, e_{\alpha_{2}}\right]$ in $\mathfrak{s l}_{3}$, up to a suitable normalisation, we have also $e_{\beta}=\left[e_{\alpha}, e_{\sigma(\alpha)}\right]$ in $\mathfrak{g}$ and $\beta=\alpha+\sigma(\alpha)$ in $\Delta$.
3.3. Properties and generators of algebras $\mathcal{Z}(\mathfrak{g}, \vartheta)$. From now on, $G$ is a connected semisimple algebraic group and $\mathfrak{g}=\operatorname{Lie} G$. We consider $\vartheta \in \operatorname{Aut}(\mathfrak{g})$ of order $m \geqslant 2$ and freely use the previous notation and results, with $\mathfrak{q}$ being replaced by $\mathfrak{g}$. In particular,

$$
\mathfrak{g}=\mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \oplus \ldots \oplus \mathfrak{g}_{m-1}
$$

where $\{0,1, \ldots, m-1\}$ is the fixed set of representatives for $\mathbb{Z}_{m}$, and $G_{0}$ is the connected subgroup of $G$ with Lie $G_{0}=\mathfrak{g}_{0}$. Then $\mathfrak{g}_{(t)}$ is a family of Lie algebras parameterised by $t \in \mathbb{P}=\mathbb{k} \cup\{\infty\}$, where the algebras $\mathfrak{g}_{(t)}$ with $t \in \mathbb{k}^{\star}$ are isomorphic to $\mathfrak{g}=\mathfrak{g}_{(1)}$, while $\mathfrak{g}_{(0)}$ and $\mathfrak{g}_{(\infty)}$ are different $\mathbb{N}_{0}$-graded contractions of $\mathfrak{g}$.

Note that $\mathfrak{g}_{0}$ is a reductive Lie algebra. Let $\kappa$ be the Killing form on $\mathfrak{g}$. We identify $\mathfrak{g}$ and $\mathfrak{g}_{0}$ with their duals via $\kappa$. Moreover, since $\kappa\left(\mathfrak{g}_{i}, \mathfrak{g}_{j}\right)=0$ if $i+j \notin\{0, m\}$, the dual space of $\mathfrak{g}_{j}, \mathfrak{g}_{j}^{*}$, can be identified with $\mathfrak{g}_{m-j}$. We identify also $\mathfrak{t}$ with $\mathfrak{t}^{*}$ and $\mathfrak{t}_{0}$ with $\mathfrak{t}_{0}^{*}$. Set $\mathcal{Z}=\mathcal{Z}(\mathfrak{g}, \vartheta)$.

Theorem 3.3 ([PY21]). Suppose that ind $\mathfrak{g}_{(0)}=\operatorname{rk} \mathfrak{g}$. Then $\operatorname{tr} . \operatorname{deg} \mathcal{Z}=\boldsymbol{b}(\mathfrak{g}, \vartheta):=\boldsymbol{b}(\mathfrak{g})-\boldsymbol{b}\left(\mathfrak{g}_{0}\right)+$ rk $\mathfrak{g}_{0}$.

Note that $\mathcal{Z} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}_{0}}$ by [PY21, Eq. (3.6)]. Thereby tr.deg $Z \leqslant \boldsymbol{b}(\mathfrak{g}, \vartheta)$, see [MY19, Prop. 1.1]. Thus, if ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$, then $\operatorname{tr} . \operatorname{deg} Z$ takes the maximal possible value. Note also that $\boldsymbol{b}(\mathfrak{g}, \vartheta)=\boldsymbol{b}(\mathfrak{g})$ if and only if $\left[\mathfrak{g}_{0}, \mathfrak{g}_{0}\right]=0$.

Since $\vartheta$ acts on $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, there is a generating set $\left\{H_{1}, \ldots, H_{l}\right\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ consisting of $\vartheta$ eigenvectors. Then $\vartheta\left(H_{i}\right)=\zeta^{r_{i}} H_{i}$ with $0 \leqslant r_{i}<m$. The integers $r_{i}$ depend only on the connected component of $\operatorname{Aut}(\mathfrak{g})$ that contains $\vartheta$, and if $a$ is the order of $\vartheta \operatorname{in} \operatorname{Aut}(\mathfrak{g}) / \operatorname{lnt}(\mathfrak{g})$, then $\zeta^{a r_{i}}=1$. Therefore, if $\mathfrak{g}$ is simple, then $\zeta^{r_{i}}= \pm 1$ for all types but $\mathrm{D}_{4}$.

Recall from Section 1.3 that $\mathfrak{g}_{(0)}=\mathfrak{g}_{(0, \varphi)}=: \mathfrak{g}_{(0, \vartheta)}$ is a contraction of $\mathfrak{g}$ defined by $\varphi$. Below we freely use notation of Sections 1.2, 1.4.

Lemma 3.4 ([PY21]). For any $\vartheta \in \operatorname{Aut}(\mathfrak{g})$ of order $m$, we have
(1) $\vartheta\left(H_{j}\right)=H_{j}$ if and only if $d_{j}^{\bullet} \in m \mathbb{Z}$;
(2) $\sum_{j=1}^{l} r_{j}=\frac{1}{2} m\left(\mathrm{rkg}-\mathrm{rk} \mathfrak{g}_{0}\right)$;
(3) $\mathrm{rk} \mathfrak{g}_{0}=\#\left\{j \mid \vartheta\left(H_{j}\right)=H_{j}\right\}$.

Lemma 3.5. For each $j$, the restriction of $H_{j}$ to $\mathfrak{g}_{0}^{*}$ is non-zero if and only if $\vartheta\left(H_{j}\right)=H_{j}$.
Proof. If $\left.H_{j}\right|_{\mathfrak{g}_{0}^{*}} \neq 0$, then the lowest $\varphi$-component $H_{j, 0} \in \mathcal{S}\left(\mathfrak{g}_{0}\right)$ is non-zero and hence we have $\vartheta\left(H_{j}\right)=H_{j}$.

Consider some $x \in \mathfrak{t}_{0}^{*} \cap \mathfrak{g}_{\text {reg }}^{*}$, which exists by [Ka83, §8.8], and apply Kostant's regularity criterion $\left[K 63\right.$, Theorem 9] to $x$. According to this criterion, $\left\langle d_{x} H_{i} \mid 1 \leqslant i \leqslant l\right\rangle_{\mathbb{k}}=\mathfrak{g}^{x}=\mathfrak{t}$. Here $d_{x} H_{i} \in \mathfrak{g}_{0}^{x}=\mathfrak{t}_{0}$ if and only if $\vartheta\left(H_{i}\right)=H_{i}$. In view of Lemma 3.4(3), we have $d_{x} H_{i} \neq 0$ for each $i$ such that $r_{i}=0$. Then also $H_{i, 0} \neq 0$, whenever $r_{i}=0$.

Theorem 3.6 ([PY21]). Suppose that $\vartheta \in \operatorname{Aut}(\mathfrak{g})$ admits a g.g.s. and ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$. Then
(i) $z_{x}:=\operatorname{alg}\left\langle H_{j, i} \mid 1 \leqslant j \leqslant l, 0 \leqslant i \leqslant d_{j}^{\bullet}\right\rangle \subset z$ is a polynomial Poisson-commutative subalgebra of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}_{0}}$ having the maximal transcendence degree.
(ii) More precisely, if $H_{1}, \ldots, H_{l}$ is a g.g.s. that consists of $\vartheta$-eigenvectors, then $z_{x}$ is freely generated by the non-zero bi-homogeneous components of all $H_{j}$.

A precise relationship between $Z$ and $Z_{x}$ depends on further properties of $\vartheta$. Two complementary assertion are given below.

Corollary 3.7 ([PY21]). In addition to the hypotheses of Theorem 3.6, suppose that $\mathfrak{g}_{(0)}$ has the codim-2 property and $\mathfrak{g}_{0}=\mathfrak{g}^{\vartheta}$ is not abelian. Then $\mathcal{Z}=\mathcal{Z}_{\times}$is the polynomial algebra freely generated by all non-zero bi-homogeneous components $H_{j, i}$.

Corollary 3.8 ([PY21]). In addition to the hypotheses of Theorem 3.6, suppose that $\mathfrak{g}_{(0)}$ has the codim-2 property, $\vartheta$ is inner, and $\mathfrak{g}_{0}=\mathfrak{g}^{\vartheta}$ is abelian. Then $\mathcal{Z}_{\infty}=\mathcal{S}\left(\mathfrak{g}_{0}\right)$ and $\mathcal{Z}=\operatorname{alg}\left\langle\mathcal{Z}_{\times}, \mathfrak{g}_{0}\right\rangle$ is a polynomial algebra.
3.4. The equality for the index of $\mathfrak{g}_{(0)}$. The equality ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$ is very important in our context. If it holds, then $Z$ can be extended to a Poisson-commutative subalgebra of the maximal possible transcendence degree $\boldsymbol{b}(\mathfrak{g})$ in the same way as in [PY, Sect. 6.2].

- It holds if ord $(\vartheta)$ is two or three [P07, PY23].
- It holds if $\mathfrak{g}$ is either $\mathfrak{s o}_{n}$ or of type $\mathrm{G}_{2}$ [PY23].
- It holds if $\mathfrak{g}_{1} \cap \mathfrak{g}_{\mathrm{reg}} \neq \varnothing$ by [P09, Prop. 5.3].

Below we will see a few more positive examples.
Let $(\mathfrak{g}, \vartheta)$ be an indecomposable pair, where $\mathfrak{g}=\mathfrak{h}^{\oplus n}$, the algebra $\mathfrak{h}$ is simple, and $\vartheta$ corresponds to $\tilde{\vartheta} \in \operatorname{Aut}(\mathfrak{h})$. Note that $\tilde{\vartheta}$ may be trivial. We remark also that $\mathfrak{g}_{0} \simeq \mathfrak{h}^{\tilde{\vartheta}}$.

Let $\mathfrak{h}_{(0)}$ be the contraction of $\mathfrak{h}$ associated with $\tilde{\vartheta}$.
Lemma 3.9 ([PY21, Lemma 8.1]). If ind $\mathfrak{h}_{(0)}=r k \mathfrak{h}$, then ind $\mathfrak{g}_{(0)}=r k \mathfrak{g}$. If there is a g.g.s. for $\tilde{\vartheta}$ in $\mathcal{S}(\mathfrak{h})$, then there is a g.g.s. for $\vartheta$ in $\mathcal{S}(\mathfrak{g})$.

Theorem 3.10. Suppose that either $\mathfrak{g}_{0}=\mathfrak{t}_{0} \subset \mathfrak{t}$ or $\mathfrak{g}$ is simple and among the Kac labels of $\vartheta$, see Section 3.1 for the definition, only $p_{0}$ is zero. Then ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$.

Proof. Making use of Lemma 3.9, we may assume that $\mathfrak{g}$ is simple. If $\mathfrak{g}_{0}=\mathfrak{t}_{0}$, then each Kac label of $\vartheta$ is non-zero, see [Vi76, Prop.17] or e.g. [PY23, Sect. 2.4]. Let $\vartheta_{1}$ be the automorphism obtained from $\vartheta$ by changing each non-zero Kac label of $\vartheta$ to 1 . Then $\vartheta_{1}$ may be different from $\vartheta$ and it may define a different $\mathbb{Z}_{m}$-grading of $\mathfrak{g}$. However, the Lie brackets [, $]_{(0)}$ defined by $\vartheta$ and $\vartheta_{1}$ coincide [PY23]. Thus we may suppose that each non-zero Kac label of $\vartheta$ is 1 . By our assumptions on $\vartheta$, only $p_{0}$ may not be equal to 1 .

Let $\Pi$ be a set of the simple roots of $\mathfrak{g}$ chosen in the same way as in Section 3.1. Let $e_{i} \in \mathfrak{g}_{\alpha_{i}}$ be a non-zero root vector corresponding to a simple root $\alpha_{i} \in \Pi$. If $\vartheta$ is inner, then $\boldsymbol{e}=\sum_{i=1}^{l} e_{i} \in \mathfrak{g}_{1}$ is a regular nilpotent element in $\mathfrak{g}$. Thereby ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$ by [P09, Prop.5.3].

Suppose that $\vartheta$ is outer. Let $\sigma$ be the diagram automorphism of $\mathfrak{g}$ associated with $\vartheta$ and $\Pi^{\prime}=\left\{\nu_{1}, \ldots, \nu_{r}\right\}$ the set of simple roots of $\mathfrak{g}^{\sigma}$ used in Section 3.1. Then there is a bijection between $\Pi^{\prime}$ and the set of $\langle\sigma\rangle$-orbits on $\Pi$. Namely, $\nu_{j}$ corresponds to the $\langle\sigma\rangle$-orbit $\langle\sigma\rangle \cdot \alpha_{i}$ of $\alpha_{i} \in \Pi$, if $\nu_{j}=\left.\alpha_{i}\right|_{\mathrm{t}_{0}}$. There is a normalisation of the root vectors $e_{i}=e_{\alpha_{i}}$ such that $e_{j}^{\prime}=\sum_{\alpha \in\langle\sigma\rangle \cdot \alpha_{i}} e_{\alpha}$ is a simple root vector of $\mathfrak{g}^{\sigma}$ of the weight $\nu_{j}=\left.\alpha_{i}\right|_{\mathfrak{t}_{0}}$ for each $j$ [Ka83, $\S 8.2$ ]. If the Kac label $p_{j}$ is 1 , then $e_{j}^{\prime} \in \mathfrak{g}_{1}$ according to the description of $\mathfrak{g}_{1}$ given in [Vi76, Prop. 17]. Since $p_{j}=1$ for any $j \geqslant 1$, we obtain $\boldsymbol{e}=\sum_{i=1}^{l} e_{i} \in \mathfrak{g}_{1}$. Then again by [P09, Prop. 5.3], we have ind $\mathfrak{g}_{(0)}=\operatorname{rk} \mathfrak{g}$.

The case $\mathfrak{g}_{0}=\mathfrak{t}_{0}$ is of particular importance, because tr. $\operatorname{deg} Z(\mathfrak{g}, \vartheta)=\boldsymbol{b}(\mathfrak{g})$ here. If $\mathfrak{g}_{0}=\mathfrak{t}$, then $Z(\mathfrak{g}, \vartheta)$ coincides with the Poisson-commutative subalgebra $\mathcal{Z}\left(\mathfrak{b}, \mathfrak{u}^{-}\right)$constructed in [ $P Y^{\prime}$ ], see [PY23, Example 4.5]. In particular, this algebra is known to be maximal [ $P Y^{\prime}$, Theorem 5.5]. If $\mathfrak{g}_{0}=\mathfrak{t}_{0}$ is a proper subspace of $\mathfrak{t}$, then we obtain a less studied subalgebra. We conjecture that it is still maximal.

## 4. Properties of $\mathcal{Z}_{\infty}$ In the reductive case

In order to understand $\mathcal{Z}_{\infty}^{\mathfrak{g}_{0}}$, we study symmetric invariants of $\tilde{\mathfrak{g}}=\mathfrak{g}_{0} \ltimes \mathfrak{g}_{(\infty)}$. The Lie algebra $\tilde{\mathfrak{g}}$ is a contraction of $\mathfrak{g}_{0} \oplus \mathfrak{g}$ associated with a map $\tilde{\varphi}: \mathbb{k}^{\star} \rightarrow \operatorname{GL}\left(\mathfrak{g}_{0} \oplus \mathfrak{g}\right)$, where $\tilde{\varphi}(s)=\tilde{\varphi}_{s}$. In order to define $\tilde{\varphi}_{s}$, we consider a vector space decomposition

$$
\mathfrak{g}_{0} \oplus \mathfrak{g}=\mathfrak{g}_{0}^{\mathrm{d}} \oplus \mathfrak{g}_{m-1} \oplus \ldots \oplus \mathfrak{g}_{2} \oplus \mathfrak{g}_{1} \oplus \mathfrak{g}_{0}^{\mathrm{ab}}
$$

where the first summand $\mathfrak{g}_{0}^{\mathrm{d}}$ is embedded diagonally into $\mathfrak{g}_{0} \oplus \mathfrak{g}$ and the last summand $\mathfrak{g}_{0}^{\text {ab }}$ is embedded anti-diagonally. Then set $\left.\tilde{\varphi}_{s}\right|_{\mathfrak{g}_{0}^{d}}=\mathrm{id},\left.\tilde{\varphi}_{s}\right|_{\mathfrak{g}_{m-j}}=s^{j} \mathrm{id}$, and finally $\left.\tilde{\varphi}_{s}\right|_{\mathfrak{g}_{0}^{\mathrm{ab}}}=s^{m} \mathrm{id}$. If we consider $\xi \in\left(\mathfrak{g}_{0}^{\text {ab }}\right)^{*}$ as an element of $\mathfrak{g}^{*}$ and of $\tilde{\mathfrak{g}}^{*}$, then $\tilde{\mathfrak{g}}^{\xi}=\mathfrak{g}_{0}^{\xi} \oplus \mathfrak{g}^{\xi}$ as a vector space. If $\xi$ is regular in $\mathfrak{g}^{*}$, then $\operatorname{dim} \tilde{\mathfrak{g}}^{\xi}=\operatorname{ind} \mathfrak{g}_{0}+\operatorname{ind} \mathfrak{g}$, cf. (2•1). Then, in view of [Ka83, §8.8], which states that $\mathfrak{g}_{0}^{*} \cap \mathfrak{g}_{\text {reg }}^{*} \neq \varnothing$, we have ind $\tilde{\mathfrak{g}} \leqslant$ ind $\mathfrak{g}_{0}+$ ind $\mathfrak{g}$. Since $\tilde{\mathfrak{g}}$ is a contraction of $\mathfrak{g}_{0} \oplus \mathfrak{g}$, there is the equality, cf. [P09, Sect. 6.2],

$$
\operatorname{ind} \tilde{\mathfrak{g}}=\operatorname{ind}\left(\mathfrak{g}_{0} \oplus \mathfrak{g}\right)=\operatorname{rk} \mathfrak{g}_{0}+\operatorname{rk} \mathfrak{g}
$$

Note that $\vartheta$ is also an automorphism of $\mathfrak{g}_{0} \oplus \mathfrak{g}$, acting on $\mathfrak{g}_{0} \oplus \mathfrak{g}_{0}$ trivially, as well as of $\tilde{\mathfrak{g}}$. However, the contraction defined by $\tilde{\varphi}$ is not directly related to $\vartheta$.
4.1. Small rank examples. Let $\vartheta$ be the outer automorphism of $\mathfrak{s l}_{3}$ of order 4 with the Kac labels $\boldsymbol{p}=(0,1)$. Then the corresponding $\mathbb{Z}_{4}$-grading of $\mathfrak{g}=\mathfrak{s l}_{3}$ looks as follows

$$
\mathfrak{g}=\mathfrak{s l}_{2} \oplus \mathbb{k}_{I}^{2} \oplus \mathfrak{t}_{1} \oplus \mathbb{k}_{I I}^{2}
$$

where $\operatorname{dim} \mathbb{k}_{I}^{2}=\operatorname{dim} \mathbb{k}_{I I}^{2}=2$ and $\mathfrak{t}_{1} \subset \mathfrak{t}$ is one-dimensional.
Let $E_{i j} \in \mathfrak{g l}_{n}$ be elementary matrices (matrix units).
Lemma 4.1. Suppose $\tilde{\mathfrak{g}}=\mathfrak{g}_{0} \ltimes \mathfrak{g}_{(\infty)}$ corresponds to the pair $(\mathfrak{g}, \vartheta)$ described above. Let $e \in\left(\mathfrak{g}_{0}^{\text {ab }}\right)^{*}$ be a non-zero nilpotent element. Then $e+y \in \tilde{\mathfrak{g}}_{\text {reg }}^{*}$ for a generic $y \in \mathfrak{g}_{1}^{*}$.

Proof. We may suppose that $\mathfrak{g}_{0}=\mathfrak{s l}_{2}$ is embedded into $\mathfrak{g}$ as $\left\langle E_{11}-E_{33}, E_{13}, E_{31}\right\rangle_{\mathfrak{k}}$. Then $\mathbb{k}_{I}^{2}=\left\langle E_{12}-E_{23}, E_{21}+E_{32}\right\rangle_{k^{\prime}} \mathbb{k}_{I I}^{2}=\left\langle E_{12}+E_{23}, E_{21}-E_{32}\right\rangle_{k^{\prime}} \mathfrak{t}=\mathbb{k}\left(E_{11}-2 E_{22}+E_{33}\right)$, and $e$ is identified with $\kappa\left(E_{13},\right)$. As $y$ we choose $\kappa\left(E_{21}-E_{32},\right)$. Set $\gamma=y+e$ and let $\pi$ be the Poisson tensor of $\tilde{\mathfrak{g}}$. We identify $\mathfrak{g}_{0}^{\mathrm{d}}$ and $\mathfrak{g}_{0}^{\text {ab }}$ with $\mathfrak{s l}_{2}$ in a natural way.

Observe that $\pi(\gamma)\left(\mathfrak{t}_{1}, \mathbb{k}\left(E_{12}+E_{23}\right)\right) \neq 0$ and that

$$
\pi(\gamma)\left(\mathfrak{t}_{1}+\mathbb{k}\left(E_{12}+E_{23}\right), \mathfrak{g}_{0}^{\mathrm{d}}+\mathbb{k}_{I}^{2}+\mathbb{k}\left(E_{21}-E_{32}\right)+\mathfrak{g}_{0}^{\mathrm{ab}}\right)=0
$$

Therefore $\tilde{\mathfrak{g}}^{\gamma} \subset \mathfrak{g}_{0}^{\mathrm{d}}+\mathbb{k}_{I}^{2}+\mathbb{k}\left(E_{21}-E_{32}\right)+\mathfrak{g}_{0}^{\text {ab }}$. Clearly $E_{13} \in \mathfrak{g}_{0}^{\text {ab }}$ belongs to $\tilde{\mathfrak{g}}^{\gamma}$. Now suppose that $x=x_{0}+x_{1}+x_{3}+x_{4}$ with $x_{0} \in \mathfrak{g}_{0}^{\mathrm{d}}, x_{1} \in \mathbb{k}\left(E_{21}-E_{32}\right), x_{3} \in \mathbb{k}_{I}^{2}, x_{4} \in \mathfrak{g}_{0}^{\text {ab }}$ belongs to $\tilde{\mathfrak{g}}^{\gamma}$. We may assume that $x_{4} \in \mathbb{k} E_{31}+\mathbb{k}\left(E_{11}-E_{33}\right)$. Then

$$
\begin{array}{r}
\pi(\gamma)\left(x, \mathbb{k}_{I}^{2}\right)=\pi(\gamma)\left(x_{0}+x_{1}, \mathbb{k}_{I}^{2}\right)=0, \pi(\gamma)\left(x, \mathfrak{g}_{0}^{\mathrm{ab}}\right)=\pi(\gamma)\left(x_{0}, \mathfrak{g}_{0}^{\mathrm{ab}}\right)=0, \quad \text { and } \\
\pi(\gamma)\left(x, \mathfrak{g}_{0}^{\mathrm{d}}\right)=\pi(\gamma)\left(x_{3}+x_{4}, \mathfrak{g}_{0}^{\mathrm{d}}\right)=0 .
\end{array}
$$

Thereby $x_{0} \in \mathbb{k} E_{13}$, which implies $x_{0}+x_{1} \in \mathbb{k}\left(E_{13}+E_{21}-E_{32}\right)$. Since $\pi(\gamma)\left(x_{4}, E_{13}\right)=0$ and $\pi(\gamma)\left(E_{21}+E_{32}, E_{13}\right) \neq 0$ for $E_{13} \in \mathfrak{g}_{0}^{\mathrm{d}}$, we obtain $x_{3} \in \mathbb{k}\left(E_{12}-E_{23}\right)$. Looking at the commutators with $E_{11}-E_{33} \in \mathfrak{g}_{0}^{\mathrm{d}}$, we obtain $x_{3}+x_{4} \in \mathbb{k}\left(E_{12}-E_{23}+E_{31}\right)$.

We colnclude that $\operatorname{dim} \tilde{\mathfrak{g}}^{\gamma} \leqslant 3$. It cannot be smaller than $3=\operatorname{ind} \tilde{\mathfrak{g}}$, see (4•1). Thus $\gamma$ is a regular point and the same holds for the elements of a non-empty open subset of $\mathfrak{g}_{1}^{*}$.

Next we generalise Lemma 4.1 to periodic automorphisms of order $4 n$ of direct sums $\mathfrak{s l}_{3}^{\oplus n}$. Let now $\tilde{\vartheta}$ be the outer automorphism of $\mathfrak{h}=\mathfrak{s l}_{3}$ of order 4 with the Kac labels $(1,0)$. Set $\mathfrak{g}=\mathfrak{h}^{\oplus n}$ and let $\vartheta \in \operatorname{Aut}(\mathfrak{g})$ be obtained from $\tilde{\vartheta}$ and a cyclic permutation of the summands of $\mathfrak{g}$, see ( $0 \cdot 1$ ).

Lemma 4.2. Suppose $\tilde{\mathfrak{g}}=\mathfrak{g}_{0} \ltimes \mathfrak{g}_{(\infty)}$ corresponds to the pair $(\mathfrak{g}, \vartheta)$ described above. Let $e \in\left(\mathfrak{g}_{0}^{\text {ab }}\right)^{*}$ be a non-zero nilpotent element. Then $e+y \in \tilde{\mathfrak{g}}_{\text {reg }}^{*}$ for a generic $y \in \mathfrak{g}_{1}^{*}$.

Proof. It is convenient to identify $\mathfrak{g}_{0}^{\text {d }}$ with $\mathfrak{h}_{0}, \mathfrak{g}_{0}^{\text {ab }}$ with $\mathfrak{h}_{0} \bar{t}^{4 n}$, each $\mathfrak{g}_{4 k+i}$, where $0 \leqslant i \leqslant 3$, with $\mathfrak{h}_{i} \bar{t}^{4 n-4 k-i}$, assuming that $\bar{t}^{4 n+1}=0$. Then $\left[x \bar{t}^{j}, y \bar{t}^{u}\right]=[x, y] \bar{t}^{j+u}$ in $\tilde{\mathfrak{g}}$ for $\tilde{\vartheta}$-eigenvectors $x, y \in \mathfrak{h}$.

We understand $e$ as a linear function such that $e\left(E_{13} \bar{t}^{4 n}\right)=e\left(\left(E_{11}-E_{33}\right) \bar{t}^{4 n}\right)=0$ and $e\left(E_{31} \bar{t}^{4 n}\right)=1$. Modelling the proof of Lemma 4.1, take $y$ such that $y\left(\left(E_{12}-E_{23}\right) \bar{t}^{4 n-1}\right)=2$ and $y\left(\left(E_{21}+E_{32}\right) \bar{t}^{4 n-1}\right)=0$. Set $\gamma=e+y$. Note that $\gamma\left(\left[x \bar{t}^{j}, y \bar{t}^{u}\right]\right)=0$, if $j+u \notin\{4 n-1,4 n\}$. Iterating computations in the spirit of (4•2), (4•3), (4•4), we obtain

$$
\tilde{\mathfrak{g}}^{\gamma}=\left\langle E_{13} \bar{t}^{4 n}, E_{13} \bar{t}^{4 k}+\left(E_{21}-E_{32}\right) \bar{t}^{4 k+1},\left(E_{12}-E_{23}\right) \bar{t}^{4 k+3}+E_{31} \bar{t}^{4 k+4} \mid 0 \leqslant k<n\right\rangle_{\mathbb{k}} .
$$

In particular, $\operatorname{dim} \tilde{\mathfrak{g}}^{\gamma}=1+2 n=\operatorname{ind} \tilde{\mathfrak{g}}$ and $\gamma \in \tilde{\mathfrak{g}}_{\text {reg }}^{*}$. The same holds for the elements of a non-empty open subset of $\mathfrak{g}_{1}^{*}$.
4.2. General computations. Let us begin with a technical lemma.

Lemma 4.3. Let $\mathfrak{q}$ be a finite-dimensional Lie algebra. Suppose we have $\eta \in \mathfrak{q}^{*}$ and $\bar{y} \in\left(\mathfrak{q}^{\eta}\right)^{*}$ such that $\operatorname{dim}\left(\mathfrak{q}^{\eta}\right)^{\bar{y}}=\operatorname{ind} \mathfrak{q}$. Then for any extension $y$ of $\bar{y}$ to a linear function on $\mathfrak{q}$, there is $s \in \mathbb{k}^{\star}$ such that $s \eta+y \in \mathfrak{q}_{\text {reg }}^{*}$.
Proof. Write $\mathfrak{q}=\mathfrak{m} \oplus \mathfrak{q}^{\eta}$, where $\hat{\eta}$ is non-degenerate on $\mathfrak{m}$. Let $\mathfrak{m}_{1} \subset \mathfrak{q}^{\eta}$ be a subspace of dimension $\operatorname{dim} \mathfrak{q}^{\eta}-\operatorname{ind} \mathfrak{q}$ such that $\hat{\bar{y}}$ is non-degenerate on $\mathfrak{m}_{1}$. By the construction $\operatorname{dim}\left(\mathfrak{m} \oplus \mathfrak{m}_{1}\right)=\operatorname{dim} \mathfrak{q}$ - ind $\mathfrak{q}$. We extend $\bar{y}$ to a linear function $y$ on $\mathfrak{q}$ in some way, which is of no importance. Choose bases in $\mathfrak{m}, \mathfrak{m}_{1}$, take their union, and let $M=\left(\begin{array}{cc}s A+C & B \\ -B^{t} & A_{1}\end{array}\right)$ be the matrix of $s \hat{\eta}+\hat{y}$, where $s \in \mathbb{k}$, in this basis; here $A$ is the matrix of $\left.\hat{\eta}\right|_{\mathfrak{m}}$ and $A_{1}$ is the matrix of $\left.\hat{\bar{y}}\right|_{\mathfrak{m}_{1}}$. Then

$$
\operatorname{det}(M)=s^{\operatorname{dim} m} \operatorname{det}(A) \operatorname{det}\left(A_{1}\right)+(\text { terms of smaller degree in } s) .
$$

Since $\operatorname{det}(A) \operatorname{det}\left(A_{1}\right) \neq 0$, for almost all $s \in \mathbb{k}$, we have $\operatorname{det}(M) \neq 0$. Whenever this happens, $\operatorname{rk}(s \hat{\eta}+\hat{y}) \geqslant \operatorname{dim} \mathfrak{q}-\operatorname{ind} \mathfrak{q}$ and $s \eta+y \in \mathfrak{q}_{\text {reg }}^{*}$.

Theorem 4.4. The algebra $\mathfrak{g}$ has the codim-2 property.
Proof. The subalgebra $\mathfrak{g}_{0}$ is reductive and it contains a semisimple element $x$ that is regular in $\mathfrak{g}$, see e.g. [Ka83, §8.8]. Thus, there is $\eta \in\left(\mathfrak{g}_{0}^{\text {ab }}\right)^{*} \subset \tilde{\mathfrak{g}}^{*}$ that corresponds to a regular element of $\mathfrak{g}$ and here $\operatorname{dim} \tilde{\mathfrak{g}}^{\eta}=\operatorname{rk} \mathfrak{g}+\operatorname{rk} \mathfrak{g}_{0}=$ ind $\tilde{\mathfrak{g}}$, cf. (2•1).

Take now $\xi \in \tilde{\mathfrak{g}}^{*}$ such that $\bar{\xi}=\left.\xi\right|_{\mathfrak{g}_{0}^{\text {ab }}}$ corresponds to a regular element of $\mathfrak{g}$. Note that $\lim _{s \rightarrow 0} s^{m} \tilde{\varphi}_{s}(\xi)=\bar{\xi}$. Since $\bar{\xi} \in \tilde{\mathfrak{g}}_{\text {reg }}^{*}$, we have $\varphi_{s}(\xi) \in \tilde{\mathfrak{g}}_{\text {reg }}^{*}$ for almost all $s$. Each map $\tilde{\varphi}_{s}: \tilde{\mathfrak{g}} \rightarrow \tilde{\mathfrak{g}}$ is an automorphism of the Lie algebra $\tilde{\mathfrak{g}}$. Thus $\xi \in \tilde{\mathfrak{g}}_{\text {reg }}^{*}$.

Assume that $D \subset \tilde{\mathfrak{g}}_{\text {sing }}^{*}$ is a divisor in $\tilde{\mathfrak{g}}^{*}$. Then $\{\bar{\xi} \mid \xi \in D\} \subset\left(\mathfrak{g}_{0}^{\text {ab }}\right)^{*}$ lies in a proper closed subset of $\left(\mathfrak{g}_{0}^{\mathrm{ab}}\right)^{*}$. Thereby $D=D_{0} \times \operatorname{Ann}\left(\mathfrak{g}_{0}^{\text {ab }}\right)$ for some divisor $D_{0} \subset\left(\mathfrak{g}_{0}^{\mathrm{ab}}\right)^{*}$. Since $\tilde{\mathfrak{g}}_{\text {sing }}^{*}$ is a $G_{0}$-stable subset of $\tilde{\mathfrak{g}}^{*}$, the divisor $D_{0}$ is $G_{0}$-stable as well. Since $\tilde{\mathfrak{g}}_{\text {sing }}^{*}$ is a conical subset, $D_{0}$ is the zero set of a homogeneous polynomial.

A generic fibre of the categorical quotient $\varpi:\left(\mathfrak{g}_{0}^{\mathrm{ab}}\right)^{*} \rightarrow\left(\mathfrak{g}_{0}^{\mathrm{ab}}\right)^{*} / / G_{0}$ consists of a single $G_{0^{-}}$ orbit. Since $D_{0}$ is $G_{0}$-stable, it follows that $\operatorname{dim} \overline{\varpi\left(D_{0}\right)}=\mathrm{rk} \mathfrak{g}_{0}-1$. Then also $\varpi^{-1}(y) \subset D_{0}$ for any $y \in \varpi\left(D_{0}\right)$, because each fibre of $\varpi$ is irreducible.

The intersection $D_{0} \cap \mathfrak{t}_{0}^{*}$ does not contain regular in $\mathfrak{g}$ elements, thereby it is a subspace of codimension 1 , the zero set of $\bar{\alpha}=\left.\alpha\right|_{\mathfrak{t}_{0}}$ for some $\alpha \in \Delta^{+}$. Let $\eta \in D_{0} \cap \mathfrak{t}_{0}^{*}$ be a generic point. Then $\eta$ is either regular or subregular in $\mathfrak{g}_{0}$. Furthermore, if $\beta \in \Delta$ and $\beta(\eta)=0$, then $\bar{\beta}=\left.\beta\right|_{\mathfrak{t}_{0}} \in \mathbb{k} \bar{\alpha}$. Set $\mathfrak{f}=\left[\mathfrak{g}^{\eta}, \mathfrak{g}^{\eta}\right]$. Note that $\mathfrak{f}$ is semisimple and that the pair $\left(\mathfrak{f}, \mathfrak{f}^{\vartheta}\right)$ is indecomposable.
$(\diamond)$ Case 1. Suppose that $\eta$ is not regular in $\mathfrak{g}_{0}$. Then necessarily $\left[\mathfrak{g}_{0}^{\eta}, \mathfrak{g}_{0}^{\eta}\right]=\mathfrak{s l}_{2}$. Let $e \in\left[\mathfrak{g}_{0}^{\eta}, \mathfrak{g}_{0}^{\eta}\right]$ be a non-zero nilpotent element, which we regard also as a point in $\left(\mathfrak{g}_{0}^{\text {ab }}\right)^{*}$. Then $\eta+e \in D_{0}$, since $\varpi(\eta+e)=\varpi(\eta)$. Suppose $\left\{\beta \in \Delta^{+} \mid \beta(\eta)=0\right\}=\left\{\vartheta^{k}(\alpha) \mid k \geqslant 0\right\}$. Then $\vartheta^{k}(\alpha)+\vartheta^{k^{\prime}}(\alpha)$ with $k, k^{\prime} \geqslant 0$ is never a root of $\mathfrak{g}$, the subalgebra $\mathfrak{f}$ is a direct sum of copies of $\mathfrak{s l}_{2}$, the nilpotent element $e$ is regular in $\mathfrak{f}$, and $\eta+e \in \mathfrak{g}_{\text {reg }}^{*}$. This is a contradiction.

Suppose that there is no equality for the sets above. Then $\left.\vartheta\right|_{\mathfrak{f}}$ is constructed from an outer automorphism $\tilde{\vartheta}$ of a simple Lie algebra $\mathfrak{h}$. From Section 3.2 we know that $\mathfrak{h}=\mathfrak{s l}_{3}$. Next $\mathfrak{h}^{\tilde{\vartheta}} \simeq \mathfrak{s l}_{2}$. The fixed points subalgebra can be embedded in two different ways, as $\mathfrak{s o}_{3}$ or as $\left\langle E_{11}-E_{33}, E_{13}, E_{31}\right\rangle_{\mathfrak{k}}$. For the first embedding, $e$ is still regular in $\mathfrak{f}$, a contradiction.

Suppose that $\mathfrak{h}^{\tilde{\vartheta}}=\left\langle E_{11}-E_{33}, E_{13}, E_{31}\right\rangle_{\mathfrak{k}}$. The twisted affine Dynkin digram of type $\mathrm{A}_{2}$ has two nodes. Our choice of $\mathfrak{h}^{\tilde{\vartheta}}$ implies that $\tilde{\vartheta}$ is the automorphism considered in Lemma 4.1. Then by Lemma 4.2, there is $y \in \mathfrak{f}_{1}^{*} \simeq \mathfrak{f}_{i} \subset \mathfrak{g}_{j}$, where $0<j<m$ and $i$ depends on $\left.\vartheta\right|_{\mathfrak{f}}$, such that $e+y \in \tilde{f}_{\text {reg }}^{*}$. Note that $s \eta+e+y \in D$ for any $s \in \mathbb{k}^{\star}$.

The vector spaces $\tilde{\mathfrak{g}}^{\eta}$ and $\left(\mathfrak{g}_{0} \oplus \mathfrak{g}\right)^{\eta}$ coinside. We have $\tilde{\mathfrak{g}}^{\eta}=\tilde{\mathfrak{f}}+\mathfrak{t}_{0}+\mathfrak{t}$, where $\tilde{\mathfrak{g}}^{\eta}$ and $\tilde{\mathfrak{f}}$ are contractions of $\mathfrak{g}^{\eta}$ and $\mathfrak{f} \oplus \mathfrak{f}^{\vartheta}$, respectively, given by the restrictions of $\tilde{\varphi}$. We extend $e$ and $y$ to linear functions on $\tilde{\mathfrak{g}}^{\eta}$ and $\tilde{\mathfrak{g}}$ keeping the same symbols for the extensions. Then

$$
\operatorname{dim}\left(\tilde{\mathfrak{g}}^{\eta}\right)^{e+y}=\left(\operatorname{rk} \mathfrak{g}+\operatorname{rk} \mathfrak{g}_{0}\right)-\left(\operatorname{rk} \mathfrak{f}+\operatorname{rk} \mathfrak{f}_{0}\right)+\operatorname{ind} \tilde{\mathfrak{f}}=\operatorname{ind} \tilde{\mathfrak{g}} .
$$

By Lemma 4.3, $s \eta+e+y \in \tilde{\mathfrak{g}}_{\text {reg }}^{*}$ for some $s \in \mathbb{k}^{\star}$, a contradiction.
$(\diamond)$ Case 2. Suppose now that $\eta$ is regular in $\mathfrak{g}_{0}$. Then $\mathfrak{f}_{0} \subset \mathfrak{t}_{0}$. The quotient of $\tilde{\mathfrak{f}}$ by $\left(\mathfrak{t}_{0}^{\mathrm{ab}} \cap \tilde{\mathfrak{f}}\right) \subset \mathfrak{g}_{0}^{\text {ab }}$ is the contraction $\mathfrak{f}_{(0), \vartheta^{-1}}$ associated with the restriction of $\vartheta^{-1}$ to $\mathfrak{f}$. For $y \in \tilde{\mathfrak{f}}^{*}$ such that $y\left(\mathfrak{t}_{0}^{\mathrm{ab}}\right)=0$, we have $\tilde{\mathfrak{f}}^{y}=\mathfrak{f}_{\left(0, \vartheta^{-1}\right)}^{y} \oplus\left(\mathfrak{t}_{0}^{\mathrm{ab}} \cap \tilde{\mathfrak{f}}\right)$. By Theorem 3.10, ind $\mathfrak{f}_{(0), \vartheta^{-1}}=\operatorname{rkf}$. Therefore there is $y \in \tilde{\mathfrak{g}}^{*}$ such that $y\left(\mathfrak{t}_{0}^{\mathrm{ab}}\right)=0$ and $\operatorname{dim} \tilde{\mathfrak{f}}^{\bar{y}}=\operatorname{rk} \mathfrak{f}+\operatorname{rk} \mathfrak{f}_{0}$ for $\bar{y}=\left.y\right|_{\tilde{\mathfrak{f}}}$. Repeating the argument of (4.5) and using again Lemma 4.3, we conclude that $s \eta+y \in \tilde{\mathfrak{g}}_{\text {reg }}$ for some $s \in \mathbb{K}^{\star}$. This final contradiction shows that there is no divisor $D$.
4.3. Symmetric invariants. Set $r=\mathrm{rk} \mathfrak{g}_{0}$ and choose a set of homogeneous generators $F_{1}, \ldots, F_{r} \in \mathcal{S}\left(\mathfrak{g}_{0}\right)^{\mathfrak{g}_{0}}$. As above, let $\mathfrak{g}_{0}^{\text {ab }}$ stand for the abelian ideal of $\tilde{\mathfrak{g}}$, which is isomorphic to $\mathfrak{g}_{0}$ as a $\mathfrak{g}_{0}$-module. Let further $\left\{H_{j} \mid 1 \leqslant j \leqslant l\right\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ be a generating set consisting of homogeneous polynomials that are $\vartheta$-eigenvectors. Assume that $\vartheta\left(H_{j}\right)=H_{j}$ if and only
if $j \leqslant r$, cf. Lemma 3.4. Unless stated otherwise, we use upper bullets for the highest $\tilde{\varphi}$-components of $H \in \mathcal{S}\left(\mathfrak{g}_{0} \oplus \mathfrak{g}\right)$.

Theorem 4.5. There is a g.g.s. $F_{i}, \tilde{H}_{i}, H_{j}$ with $1 \leqslant i \leqslant r<j \leqslant l$ for the contraction $\mathfrak{g}_{0} \oplus \mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ defined by $\tilde{\varphi}$. Furthermore, for $i \leqslant r, \tilde{H}_{i} \in H_{i}+\mathcal{S}\left(\mathfrak{g}_{0}\right)^{\mathfrak{g}_{0}}$, where $\mathfrak{g}_{0}$ is a direct summand of $\mathfrak{g}_{0} \oplus \mathfrak{g}$, and the ring $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by $\left\{F_{i}^{\bullet}, \tilde{H}_{i}^{\bullet}, H_{j}^{\bullet} \mid 1 \leqslant i \leqslant r<j \leqslant l\right\}$.

Proof. First we compute the number $D_{\tilde{\varphi}}$ using the properties of the grading on $\tilde{\mathfrak{g}}$ :

$$
D_{\tilde{\varphi}}=m \operatorname{dim} \mathfrak{g}_{0}+\sum_{j=1}^{m-1}(m-j) \operatorname{dim} \mathfrak{g}_{j}=m \operatorname{dim} \mathfrak{g}_{0}+\frac{m}{2}\left(\operatorname{dim} \mathfrak{g}-\operatorname{dim} \mathfrak{g}_{0}\right)=\frac{m}{2}\left(\operatorname{dim} \mathfrak{g}+\operatorname{dim} \mathfrak{g}_{0}\right) .
$$

Since $\mathfrak{g}_{0}^{\text {ab }}$ is embedded anti-diagonally in $\mathfrak{g}_{0} \oplus \mathfrak{g}$, we have $\operatorname{deg}_{\tilde{\varphi}} F_{i}^{\bullet}=m \operatorname{deg} F_{i}$.
Suppose that $\vartheta\left(H_{j}\right)=H_{j}$. Then the restriction of $H_{j}$ from $\mathfrak{g}^{*}$ to $\mathfrak{g}_{0}^{*} \subset \mathfrak{g}^{*}$ is non-zero by Lemma 3.5. Thus $H_{j}^{\bullet} \in \mathcal{S}\left(\mathfrak{g}_{0}^{\mathrm{ab}}\right)^{\mathfrak{g}_{0}}$. Hence there is a homogeneous polynomial $\tilde{H}_{j}$ in $H_{j}+\mathcal{S}\left(\mathfrak{g}_{0}\right)^{\mathfrak{g}_{0}}$, where $\mathfrak{g}_{0}$ is a direct summand of $\mathfrak{g}_{0} \oplus \mathfrak{g}$, such that $\tilde{d}_{j}=\operatorname{deg}_{\tilde{\varphi}} \tilde{H}_{j}^{\bullet}<m \operatorname{deg} H_{j}$. We have $\vartheta\left(\tilde{H}_{j}^{\bullet}\right)=\zeta^{-\tilde{d}_{j}^{\bullet}} \tilde{H}_{j}^{\bullet}$ and at the same time $\vartheta\left(\tilde{H}_{j}^{\bullet}\right)=\tilde{H}_{j}^{\bullet}$. Thereby $\tilde{d}_{j}^{\bullet} \leqslant m \operatorname{deg} H_{j}-m$.

Suppose now that $\vartheta\left(H_{j}\right) \neq H_{j}$, i.e., $\vartheta\left(H_{j}\right)=\zeta^{r_{j}} H_{j}$ with $0<r_{j}<m$. Here $H_{j}^{\bullet} \notin \mathcal{S}\left(\mathfrak{g}_{0}^{\text {ab }}\right)$ and $\operatorname{deg}_{\tilde{\varphi}} \tilde{H}_{j}^{\bullet} \leqslant m \operatorname{deg} H_{j}-r_{j}$. According to Lemma 3.4, $\sum_{j=1}^{l} r_{j}=\frac{1}{2} m\left(\mathrm{rkg}-\mathrm{rk} \mathfrak{g}_{0}\right)$. Thus

$$
\begin{align*}
& \sum_{i=1}^{r} \operatorname{deg}_{\tilde{\varphi}} F_{i}^{\bullet}+\sum_{j=1}^{r} \operatorname{deg}_{\tilde{\varphi}} \tilde{H}_{j}^{\bullet}+\sum_{j=r+1}^{l} \operatorname{deg}_{\tilde{\varphi}} H_{j}^{\bullet} \leqslant \\
& \quad \leqslant m \boldsymbol{b}\left(\mathfrak{g}_{0}\right)+m \boldsymbol{b}(\mathfrak{g})-m \mathrm{rk} \mathfrak{g}_{0}-\frac{1}{2} m\left(\mathrm{rk} \mathfrak{g}-\operatorname{rk} \mathfrak{g}_{0}\right)=\frac{m}{2}\left(\operatorname{dim} \mathfrak{g}+\operatorname{dim} \mathfrak{g}_{0}\right)=D_{\tilde{\varphi}}
\end{align*}
$$

By Theorem 1.4(i), we have the equality in (4.6) and the polynomials $F_{i}, \tilde{H}_{i}, H_{j}$ form a good generating system for $\tilde{\varphi}$. We obtain also

$$
\operatorname{deg}_{\tilde{\varphi}} H_{j}^{\bullet}=m \operatorname{deg} H_{j}-r_{j} \quad \text { if } \quad 0<r_{j}<m
$$

By Theorem 4.4, the Lie algebra $\tilde{\mathfrak{g}}$ has the codim-2 property. Then Theorem 1.4(iii) states that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by $F_{i}^{\bullet}, \tilde{H}_{i}^{\bullet}, H_{j}^{\bullet}$ with $1 \leqslant i \leqslant r<j \leqslant l$.

Recall that $H_{j}=\sum_{i \geqslant 0} H_{j, i}$, where $\varphi_{s}\left(H_{j}\right)=\sum_{i \geqslant 0} s^{i} H_{j, i}$. Let $H_{j, \bullet}$ be the the non-zero bi-homogeneous component of $H_{j}$ with the minimal $\varphi$-degree. We regard each $H_{j, i}$ as an element of $\mathcal{S}\left(\mathfrak{g}_{(\infty)}\right) \subset \mathcal{S}(\tilde{\mathfrak{g}})$ identifying $\mathfrak{g}$ with the subspace

$$
\mathfrak{g}_{(\infty)}=\mathfrak{g}_{m-1} \oplus \ldots \oplus \mathfrak{g}_{1} \oplus \mathfrak{g}_{0}^{\mathrm{ab}} \subset \tilde{\mathfrak{g}}
$$

Note that $\operatorname{deg}_{\tilde{\varphi}} H_{j, i}=m \operatorname{deg} H_{j}-i$.
Theorem 4.6. The algebra $\mathcal{Z}_{\infty}^{\mathfrak{g}_{0}}$ is freely generated by $F_{i}^{\bullet} \in \mathcal{S}\left(\mathfrak{g}_{0}^{\text {ab }}\right)$ with $1 \leqslant i \leqslant r$ and the set $\left\{H_{j, \bullet} \mid r<j \leqslant l\right\}$, where $H_{j, \bullet}=H_{j}^{\bullet}$ for each $j$.

Proof. For any $H_{j} \in \mathcal{S}(\mathfrak{g}) \subset \mathcal{S}\left(\mathfrak{g}_{0} \oplus \mathfrak{g}\right)$, the highest $\tilde{\varphi}$-component $H_{j}^{\bullet} \in \mathcal{S}(\tilde{\mathfrak{g}})$ is obtained by taking first $H_{j, \bullet} \in \mathcal{S}(\mathfrak{g})$ and then replacing in it each element of $\mathfrak{g}_{0}$ by its copy in $\mathfrak{g}_{0}^{\text {ab }}$. This is exactly how we understand $H_{j, \bullet} \in \mathcal{S}(\tilde{\mathfrak{g}})$.

By the construction and Proposition 1.1,

$$
\mathscr{H}:=\left\{F_{i}^{\bullet}, H_{j}^{\bullet} \mid 1 \leqslant i \leqslant r, r<j \leqslant l\right\} \subset \mathcal{Z}_{\infty}^{\mathfrak{g}_{0}} .
$$

These homogeneous polynomials are algebraically independent, see Theorem 4.5. In view of Theorem 2.3(iii), we have tr. $\operatorname{deg} \mathcal{Z}_{\infty}^{\mathfrak{g}_{0}}=\operatorname{rk} \mathfrak{g}=l$. By Theorems 1.4(iii), 4.4, 4.5, $\tilde{\mathfrak{g}}$ satisfies the Kostant regularity criterion and the differentials of the generators $F_{i}^{\bullet}, \tilde{H}_{i}^{\bullet}, H_{j}^{\bullet}$ with $1 \leqslant i \leqslant r<j \leqslant l$ are linearly independent on a big open subset. Thus also

$$
\operatorname{dim} \mathcal{J}(\mathscr{H}) \leqslant \operatorname{dim} \mathfrak{g}-2
$$

where $\mathcal{J}(\mathscr{H}) \subset \mathfrak{g}_{(\infty)}^{*}$. By Theorem 1.5, the subalgebra generated by $\mathscr{H}$ is algebraically closed in $\mathcal{S}\left(\mathfrak{g}_{(\infty)}\right)$ and hence it coincides with $\mathcal{Z}_{\infty}^{\mathfrak{g}_{0}}$.

If $\vartheta$ is inner, then $\mathcal{Z}_{\infty}=\mathcal{S}\left(\mathfrak{g}_{0}\right)$ [PY21] and $\mathcal{Z}_{\infty}^{\mathfrak{g}_{0}}=\mathcal{S}\left(\mathfrak{g}_{0}\right)^{\mathfrak{g}_{0}}$. One does not need any heavy machinery in that case. If $\vartheta$ is outer, the situation is different. We have seen this in Theorem 4.6 and are going to obtain a description of $\mathcal{Z}_{\infty}$ next.

Proposition 4.7. If $\vartheta$ is outer, then a basis $\left\{y_{1}, \ldots, y_{R}\right\}$ of $\mathfrak{g}_{0}$ together with the lowest $\varphi$-components $H_{j, \bullet}$, where $r<j \leqslant l$, freely generate $\mathcal{Z}_{\infty}$.

Proof. We check that the differentials $d y_{i}$ and $d H_{j, \bullet}$ of the proposed generators are linearly independent on a big open subset of $\mathfrak{g}^{*}$. For $\xi \in \mathfrak{g}^{*}$, set $\bar{\xi}=\left.\xi\right|_{\mathfrak{g}_{0}}$. Note that $\bar{\xi} \in\left(\mathfrak{g}_{0}^{*}\right)_{\text {reg }}$ for all $\xi$ in a big open subset of $\mathfrak{g}^{*}$. Let $\varpi_{\xi}: T_{\xi}^{*} \mathfrak{g}^{*} \rightarrow T_{\xi}^{*}\left(G_{0} \cdot \xi\right)$ be the restriction map. Since $H_{j, \bullet}$ is a $G_{0}$-invariant, we have $\varpi_{\xi}\left(d_{\xi} H_{j, \bullet}\right)=0$ for each $j$. Note that $\operatorname{ker}\left(\left.\varpi_{\xi}\right|_{\mathfrak{g}_{0}}\right)=\mathfrak{g}_{0}^{\bar{\xi}}$.

Suppose that $\bar{\xi} \in\left(\mathfrak{g}_{0}^{*}\right)_{\text {reg }}$. Then $\operatorname{dim} \varpi_{\xi}\left(\mathfrak{g}_{0}\right)=\operatorname{dim} \mathfrak{g}_{0}-\mathrm{rk} \mathfrak{g}_{0}$. By the Kostant regularity criterion [K63, Theorem 9] applied to $\mathfrak{g}_{0}$, we have

$$
\left\langle d_{\xi} F_{i}^{\bullet} \mid 1 \leqslant i \leqslant r\right\rangle_{\mathfrak{k}}=\mathfrak{g}_{0}^{\bar{\xi}}
$$

Suppose now that $\xi \notin \mathcal{J}(\mathscr{H})$. This additional restriction still leaves us a big open subset of suitable elements, see (4•8). Then

$$
\operatorname{dim}\left(\mathfrak{g}_{0}^{\bar{\xi}}+\left\langle\boldsymbol{d}_{\xi} H_{j, \bullet} \mid r<j \leqslant l\right\rangle_{\mathfrak{k}}\right)=l .
$$

Thus, on a big open subset, the dimension of $V_{\xi}=\left\langle d_{\xi} y_{i}, d_{\xi} H_{j, \bullet} \mid 1 \leqslant i \leqslant R, r<j \leqslant l\right\rangle_{\mathbb{k}}$ is equal to the sum of $\operatorname{dim} \varpi_{\xi}\left(V_{\xi}\right)=\operatorname{dim} \mathfrak{g}_{0}-\operatorname{rk} \mathfrak{g}_{0}$ and $l=\operatorname{dim} \operatorname{ker}\left(\left.\varpi_{\xi}\right|_{V_{\xi}}\right)$, i.e., $\operatorname{dim} V_{\xi}=$ $R+(l-r)$. This number is equal to the number of generators. By Theorem 1.5, the subalgebra

$$
\operatorname{alg}\left\langle y_{i}, H_{j, \bullet} \mid 1 \leqslant i \leqslant R, r<j \leqslant l\right\rangle
$$

is algebraically closed in $\mathcal{S}\left(\mathfrak{g}_{(\infty)}\right)$. Furthermore,

$$
\operatorname{alg}\left\langle y_{i}, H_{j, \bullet} \mid 1 \leqslant i \leqslant R, r<j \leqslant l\right\rangle \subset \mathcal{Z}_{\infty} \text { and } \operatorname{tr} \cdot \operatorname{deg} \mathcal{Z}_{\infty} \leqslant \operatorname{ind} \mathfrak{g}_{(\infty)}=R+l-r
$$

see Theorem 2.1 for the last equality. Thereby $\operatorname{alg}\left\langle y_{i}, H_{j, \bullet} \mid 1 \leqslant i \leqslant R, r<j \leqslant l\right\rangle=\mathcal{Z}_{\infty}$.
In [PY21, Sect. 5], we have considered a larger Poisson-commutative subalgebra $\tilde{\mathcal{Z}}:=$ $\operatorname{alg}\left\langle Z, \mathcal{S}\left(\mathfrak{g}_{0}\right)^{\mathfrak{g}_{0}}\right\rangle$.

Corollary 4.8. Suppose that ind $\mathfrak{g}_{(0)}=\mathrm{rk} \mathfrak{g}$ and that $\mathcal{S}\left(\mathfrak{g}_{(0)}\right)^{\mathfrak{g}_{(0)}}=\mathbb{k}\left[H_{1}^{\bullet}, \ldots, H_{l}^{\bullet}\right]$, where each $H_{j}$ is homogeneous, $\vartheta\left(H_{j}\right) \in \mathbb{k} H_{j}$, and the polynomials $H_{j}^{\bullet \bullet}$ are highest $\varphi$-components.

- If $\infty \in \mathbb{P}_{\text {reg }}$, then $\operatorname{alg}\left\langle\mathcal{Z}, \mathcal{Z}_{\infty}^{\mathfrak{g}_{0}}\right\rangle=\operatorname{alg}\left\langle\mathcal{Z}, \mathcal{Z}_{\infty}\right\rangle=\tilde{\mathcal{Z}}=\mathcal{Z}=\operatorname{alg}\left\langle\mathcal{Z}_{x}, \mathfrak{g}_{0}\right\rangle ;$
- if $\infty \notin \mathbb{P}_{\text {reg }}$, then $\operatorname{alg}\left\langle\mathcal{Z}, \mathcal{Z}_{\infty}^{\mathfrak{g}_{0}}\right\rangle=\tilde{\mathcal{Z}}=\operatorname{alg}\left\langle\mathcal{Z}_{x}, \mathcal{S}\left(\mathfrak{g}_{0}\right)^{\mathfrak{g}_{0}}\right\rangle$.

In both cases, $\tilde{z}$ is a free (polynomial) algebra.
Proof. Our assumptions on $\mathcal{S}\left(\mathfrak{g}_{(0)}\right)^{\mathfrak{g}_{(0)}}$ imply that $\mathcal{Z}_{0} \subset \mathcal{Z}_{\times}$. Furthermore, $\left\{H_{1}, \ldots, H_{l}\right\}$ is a g.g.s. for $\vartheta$. Then $\mathcal{Z}_{x}$ is a polynomial algebra by Theorem 3.6. It has $\boldsymbol{b}(\mathfrak{g}, \vartheta)$ algebraically independent generators $H_{j, i}$. Exactly $r=\mathrm{rk} \mathfrak{g}_{0}$ of these generators belong to $\mathcal{S}\left(\mathfrak{g}_{0}\right)$, see Lemmas 3.4,3.5. If we replace them with algebraically independent generators of $\mathcal{S}\left(\mathfrak{g}_{0}\right)^{\mathfrak{g}_{0}}$, the new algebra is still polynomial. This finishes the proof in view of Theorem 4.6.

The first statement of Corollary 4.8 generalises Corollary 3.8 to outer automorphisms.

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