NEGATIVE CONTACT SURGERY ON LEGENDRIAN NON-SIMPLE KNOTS

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ABSTRACT. We prove that for any pair of Legendrian representatives of the Chekanov-Eliashberg twist knots with different LOSS invariants, any negative rational contact r-surgery with $r \neq -1$ always gives rise to different contact 3-manifolds distinguished by their contact invariants. This gives the first examples of pairs of Legendrian knots with the same classical invariants but distinct contact r-surgeries for all negative rational number r. We also generalize the statement from the twist knots to a certain families of two bridge knots.

1. INTRODUCTION

In [Etn08] Etnyre first asked the question on whether Legendrian surgery, i.e. contact (-1)-surgery, on distinct Legendrian knots in the standard tight contact structure on S^3 always produces distinct contact manifolds, and especially whether this is the case for the Chekanov-Eliashberg twist knots E_n (Figure 1a). Later using linearized contact homology Bourgeois-Ekholm-Eliashberg showed that Legendrian surgery on max-tb, non Legendrian isotopic representatives of the twist knots E_n give different contact 3-manifolds [BEE12]. However it is not known whether Legendrian surgery on the stabilized Legendrian twist knots (or equivalently contact negative integer surgery on non-stabilized twist knots) gives different contact 3-manifolds or not. The Bourgeois-Ekholm-Eliashberg argument does not directly apply in those cases, since the Legendrian DGA vanishes for stabilized Legendrians [Che02].

On the other hand, one can consider invariants from the Heegaard Floer theory, namely the contact invariant and the LOSS invariant, for contact 3-manifolds and Legendrian knots respectively. Since the LOSS invariant is unchanged under negative stabilization of the Legendrian knot[LOSS09], it makes the calculation of Legendrian surgery on negative stabilized knot possible.

In [OS10a] Ozsváth and Stipsicz showed that for the Eliashberg–Chekanov twist knot E_n with n > 3 and odd, there are $\lceil \frac{n}{4} \rceil$ different Legendrian representatives of E_n in (S^3, ξ_{std}) with the maximal Thurston-Bennequin number 1 and rotation number 0 that have different LOSS invariants. Moreover the classification of Legendrian and transverse twsit knots by Etnyre-Ng-Vértesi [ENV13] implies that the ones Ozsváth and Stipsicz found are all the Legendrian representatives of E_n that could have different LOSS invariant. Those twist knots will be the key objects for this paper.



FIGURE 1. The numbers in boxes indicate half-twists (positive for the right-handed twists and negative for the left-handed twists). The diagram on the left is the Eliashberg–Chekanov twist knot E_n . The diagram on the right depicts the knot E(m, n), where taking m = 1 recovers E_n .

Recall that in order for the contact r-surgery to be well-defined, a convention for choosing a stabilization is required. Throughout the paper we only consider contact structure that corresponds to stabilizations being all negative (see Section 3.1 for more details). Under the above convention we show the following.

Theorem 1.1. Let L_1 and L_2 be two Legendrian representatives of E_n in (S^3, ξ_{std}) with n > 3 odd that have same tb = 1 and rot = 0 but different LOSS invariants. Then for any negative rational number $r \neq -1$, contact r surgeries on L_1 and L_2 result in contact manifolds with different Heegaard Floer contact invariants.

It is interesting to point out that when r = -1, Lisca and Stipsicz [LS06] showed that for any max to representative of E_n , regardless of the LOSS invariant, the contact (-1)-surgery always results in manifolds with the same contact invariant. However on the other hand, Bourgeois-Ekholm-Eliashberg showed that Legendrian surgeries on the max to representatives of E_n yield different contact 3-manifolds [BEE12]. Combining with our result, we have the following.

Corollary 1.2. There exist arbitrarily large (finite) families of Legendrian knots in (S^3, ξ_{std}) such that knots in each family have the same smooth knot type, tb, and rot, but different contact r-surgery (up to contact isotopy) for any given rational number r < 0.

When we take the 4-dimensional perspective of Legendrian surgery, i.e. as the attachment of Stein handles, then the above corollary immediately implies the following.

Corollary 1.3. For any integer $N \ge 2$, there exist infinitely many examples of smooth 4-manifolds X that each admit N inequivalent Stein structures, all inducing isomorphic Spin^c structures on X. The Stein structures are distinguished by the contact invariants on the contact boundary.

Remark 1.4. For $N \ge 2$, the 4-manifolds are obtained by attaching two handles along E_{4N-3} (or E_{4N-1}) with any smooth framing less than 0. In [KOU17] Karakurt-Oba-Ukida gives infinitely many examples of contractible manifolds each admitting N = 2 inequivalent Stein structures with the same Spin^c structure, but do not address the case $N \ge 3$. On the other hand the results from Bourgeois-Ekholm-Eliashberg [BEE12] yield examples for all $N \ge 2$, but not infinite families (their examples cannot be distinguished by the contact invariants).

It is also worth mentioning that for positive integer contact surgery the resulting contact invariant is determined by the classical invariants of the Legendrian knots and the underlying contact manifolds [Wan23a, Corollary 1.6] (see also [Gol15]), and it is natural to ask whether the same is true for negative surgery. Theorem 1.1 gives negative answer to this question.

Corollary 1.5. The contact invariant of negative contact surgery is not necessarily determined by the classical invariants of the Legendrian knot.

Using the naturality of LOSS invariant and the Legendrian representatives of E_n the first author found families of two bridge knots with the same tb and rot but different LOSS invariants:

Theorem 1.6. [[Wan23a] Theorem 5.2] Let m, n be positive odd integers with n > 3. The knot E(m, n) (Figure 1b) has at least $\lceil \frac{n}{4} \rceil$ Legendrian representatives that have tb = m and rot = 0 that have different LOSS invariants.

When m = 1 this gives back the twist knot E_n and the corresponding Legendrian representatives are the ones from [OS10a]. We have a theorem analogous to Theorem 1.1 for these representatives of E(m, n).

Theorem 1.7. Let L_1 and L_2 be two different Legendrian representatives of E(m, n)from Theorem 1.6. Then for any negative rational number $r \neq -m$, contact r-surgery on L_1 and L_2 gives non contact-isotopic manifolds with different contact invariant.

Organization. In Section 2 and 3 we review the preliminaries for the Heegaard Floer homology and contact surgeries. After collecting computational results in Section 4, we prove Theorem 1.1 in Section 5. We prove Theorem 1.7 in Section 6.

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2. Heegaard Floer Homology Preliminaries

We provide preliminaries for Heegaard Floer homology in this section. The goal is to introduce the dual knot surgery formula in Section 2.3.

2.1. Heegaard Floer invariants for three-manifolds and knots. Heegaard Floer homology is defined by Ozsváth and Szabó in [OS04b]. For a closed oriented 3manifold Y with a fixed basepoint z, they associate an invariant $CF^{\circ}(Y)$ with four different flavors $\circ = \wedge, +, -$ and ∞ , called the *Heegaard Floer chain complex* (we will in general suppress the basepoint from the notation). The generators for $CF^{\circ}(Y)$ are given by the intersections of two Lagrangian tori in the symmetric product of the Heegaard surface and the differentials are given by count of certain holomorphic disks between generators. The invariant $\widehat{CF}(Y)$ is a chain complex over \mathbb{F} , where $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ is the field with two elements; invariants $CF^{-}(Y)$ and $CF^{+}(Y)$ are chain complexes over the ring $\mathbb{F}[U]$ and $CF^{\infty}(Y)$ is a chain complex over the ring $\mathbb{F}[U, U^{-1}]$; each complex admits a Maslov grading. By taking the homology of $CF^{\circ}(Y)$, we obtain the modules $HF^{\circ}(Y)$, called the *Heegaard Floer homology of* Y.

Given a knot $K \subset Y$, defined by Ozsváth and Szabó in [OS04a], the Heegaard Floer complexes admit a refinement to a knot invariant. For each pair (Y, K), by adding a basepoint w which encodes the knot information, one imposes an (i, j)double-filtration over the original chain complex $CF^{\infty}(Y)$. The resulting chain complex is graded and doubly-filtered; denoted by $CFK^{\infty}(Y, K)$, this is called the *full knot Floer chain complex*, since other versions of the knot invariants can be obtained as a sub/quotient complex of it. The complex $CFK^{\infty}(Y, K)$ can be viewed as formed by formal elements $x = [\mathbf{x}, i, j]$, where \mathbf{x} denotes an intersection point of the Lagrangian tori, and the filtration level (i, j) can be seen as the *coordinate* of x over an (i, j)-plane.

There are a couple of other versions of the knot complexes that we will use. Define

$$\widehat{\operatorname{CFK}}(Y,K) = \{ [\mathbf{x}, i, j] \in \operatorname{CFK}^{\infty}(Y,K) \mid j = 0 \}$$

$$\operatorname{CFK}_{s}^{-}(Y,K) = \{ [\mathbf{x}, i, j] \in \operatorname{CFK}^{\infty}(Y,K) \mid i \leq 0, j = s \}$$

with the differentials inherited from $\operatorname{CFK}^{\infty}(Y, K)$. Let $\operatorname{HFK}^{-}_{s}(Y, K)$ and $\widehat{\operatorname{HFK}}(Y, K)$ be the homology of $\widehat{\operatorname{CFK}}(Y, K)$ and $\operatorname{CFK}^{-}_{s}(Y, K)$ respectively and define $\operatorname{HFK}^{-}(Y, K) = \bigoplus_{s \in \mathbb{Z}} \operatorname{HFK}^{-}_{s}(Y, K)$. The above notations follow the conventions in [OS10a].

The group of Spin^c structures of a closed 3-manifold Y admits a non-canonical isomorphism $\operatorname{Spin}^c(Y) \cong H_1(Y; Z)$. The Heegaard Floer chain complexes and knot Floer chain complexes split over the Spin^c structures. Namely,

$$\operatorname{CF}^{\circ}(Y) = \bigoplus_{\mathfrak{s} \in \operatorname{Spin}^{c}(Y)} \operatorname{CF}^{\circ}(Y, \mathfrak{s}) \qquad \qquad \operatorname{CFK}^{\circ}(Y, K) = \bigoplus_{\mathfrak{s} \in \operatorname{Spin}^{c}(Y)} \operatorname{CFK}^{\circ}(Y, K, \mathfrak{s}).$$

2.1.1. Stong invariance and naturality. In [OS04b], Ozsváth and Szabó proved that the Heegaard Floer chain complex $HF^{\circ}(Y)$ is an invariant of the three-manifold Y

up to graded isomorphism. In [JTZ21], it was proved that $HF^{\circ}(Y, z)$ is an invariant of the based three-manifold (Y, z) in the following strong sense. In fact, $HF^{\circ}(Y, z)$ is a well-defined group, not just an isomorphism class of groups. Furthermore, isotopic diffeomorphisms induce identical maps on $HF^{\circ}(Y, z)$.

Similarly, $HFK^{-}(Y, K)$ and HFK(Y, K) are invariants of the pair (Y, K) up to graded isomorphism by [OS04a]. It was proved in [JTZ21] that in fact they are invariants of the based pair (Y, K, z, w) in the above strong sense.

The strong Heegaard invariance is crucial in our arguments. Viewing the Heegaard Floer invariants as actual groups allows us to distinguish certain generators inside it (that might have been indistinguishable up to isomorphism). We adopt this perspective thoughout the paper, specifically in the proofs in Section 5 and 6.

2.2. Rational surgery and mapping cone formula. For a null-homologous knot $K \subset Y$ where Y is a rational homology sphere, Ozsváth and Szabó defined in [OS11] an algorithm that computes $CF^{\infty}(Y_r(K))$ with $r \in \mathbb{Q}$ using the input of the knot Floer complex $CFK^{\infty}(Y, K)$. We will now describe this algorithm.

Let p, q be a pair of coprime integers such that q > 0. Given $C = CFK^{\infty}(Y, K)$, a finitely generated, graded chain complex over the ring $\mathbb{F}[U, U^{-1}]$ with (i, j) doublefiltration. For $t \in \mathbb{Z}$, let $(t, A_s(C))$ and $(t, B_s(C))$ both denote a copy of C and set $s = \lfloor t/q \rfloor$. Define $v_t^{\infty} : (t, A_s(C)) \to (t, B_s(C))$ to be the identity map and $h_t^{\infty} : (t, A_s(C)) \to (t+p, B_{s'}(C))$ with $s' = \lfloor (t+p)/q \rfloor$ to be the composition $U^s \circ flip$, where $flip : C \to C$ denotes the "flip map". When Y is an L-space, flip is the reflection along the line i = j. For $i \in \mathbb{Z}/p\mathbb{Z}$, define $(\mathcal{X}_{p/q}^{\infty}(C), i)$ to be the mapping cone of

(2.1)
$$\bigoplus_{\substack{t=-(g-1)q\\t\equiv i \mod |p|}}^{gq-1} (t, A_s(C)) \xrightarrow{v_t^{\infty} + h_t^{\infty}} \bigoplus_{\substack{t=-(g-1)q-1+p\\t\equiv i \mod |p|}}^{gq-1} (t, B_s(C)),$$

where g = g(K). Denote also $\mathcal{X}_{p/q}^{\infty}(C) = \bigoplus_{i \in \mathbb{Z}/p\mathbb{Z}} (\mathcal{X}^{\infty}(C), i)$. When the surgery coefficient is clear, we suppress p/q from the notations and simply write $\mathcal{X}^{\infty}(C)$ for the simplicity. When $C = \operatorname{CFK}^{\infty}(Y, K)$, we can also write $\mathcal{X}^{\infty}(Y, K)$ to specify the manifold-knot pair.

Theorem 2.1. [OS11] For each $i \in \text{Spin}^{c}(Y_{p/q}) \cong \mathbb{Z}/p\mathbb{Z}$, the complex $\text{CF}^{\infty}(Y_{p/q}(K), i)$ is chain homotopy equivalent to $(\mathcal{X}_{p/q}^{\infty}(Y, K), i)$.

When q = 1, namely for the integer framed surgery on K, we have q = 1 and we can simply write A_s and B_s in the place of $(t, A_s(C))$ and $(t, B_s(C))$. In the case of *n*-surgery for some $n \in \mathbb{Z}$, $\mathcal{X}^{\infty}(C)$ is the mapping cone of

(2.2)
$$\bigoplus_{s=-g+1}^{g-1} A_s(C) \xrightarrow{v_s^{\infty} + h_s^{\infty}} \bigoplus_{s=-g+n+1}^{g-1} B_s(C),$$

which similarly splits over Spin^c structures. The same algorithm works for the hat version as well. Recall that there is an \mathcal{I} -filtration over $\mathcal{X}^{\infty}(C)$, where for $[\mathbf{x}, i, j] \in (t, A_s(C))$,

$$\mathcal{I}([\mathbf{x}, i, j]) = \max\{i, j - s\}$$

and for $[\mathbf{x}, i, j] \in (t, B_s(C)),$

 $\mathcal{I}([\mathbf{x}, i, j]) = i.$

If we define $\widehat{\mathcal{X}}(C) = \mathcal{X}^{\infty}(C)|_{\mathcal{I}=0}$, then the complex $\widehat{\mathrm{CF}}(Y_{p/q}(K))$ is chain homotopy equivalent to $\widehat{\mathcal{X}}(Y, K)$. We denote $\hat{A}_s = A_s|_{\mathcal{I}=0}$, $\hat{B}_s = B_s|_{\mathcal{I}=0}$ and denote the induced maps by \hat{v}_s and \hat{h}_s respectively.

2.3. Dual knot surgery formula. In [HL24], Hedden and Levine defined a refinement of the above construction, which for $n \neq 0$ outputs $\operatorname{CFK}^{\infty}(S_n^3(K), \mu)$, the knot Floer chain complex of the meridian (dual knot) of K in the surgery. In fact this construction extends to rational surgeries on knots inside rational homology spheres. For the sake of the simplicity, we record here only the case of integer surgery on knots in S^3 . Following the previous conventions, define $\mathcal{X}_K^{\infty}(C)$ to be the mapping cone of

(2.3)
$$\bigoplus_{s=-g+1}^{g} A_s(C) \xrightarrow{v_s^{\infty} + h_s^{\infty}} \bigoplus_{s=-g+n+1}^{g} B_s(C).$$

and define the double-filtration $(\mathcal{I}, \mathcal{J})$ and the Maslov grading over $\mathcal{X}_{K}^{\infty}(C)$ as follows.

For
$$[\mathbf{x}, i, j] \in A_s(C)$$
,
(2.4) $\mathcal{I}([\mathbf{x}, i, j]) = \max\{i, j - s\}$

(2.5)
$$\mathcal{J}([\mathbf{x}, i, j]) = \max\{i - 1, j - s\} + \frac{2s + n - 1}{2n}$$

(2.6)
$$\operatorname{gr}([\mathbf{x}, i, j]) = \widetilde{\operatorname{gr}}([\mathbf{x}, i, j]) + \frac{(2s-n)^2}{4n} + \frac{2-3\operatorname{sign}(n)}{4}$$

and for $[\mathbf{x}, i, j] \in B_s(C)$,

(2.7)
$$\mathcal{I}([\mathbf{x}, i, j]) = i$$

(2.8)
$$\mathcal{J}([\mathbf{x}, i, j]) = i - 1 + \frac{2s + n - 1}{2n}$$

(2.9)
$$\operatorname{gr}([\mathbf{x}, i, j]) = \widetilde{\operatorname{gr}}([\mathbf{x}, i, j]) + \frac{(2s-n)^2}{4n} + \frac{-2 - 3\operatorname{sign}(n)}{4}$$

where $\widetilde{\text{gr}}$ indicates the Maslov grading in the original complex. Collapsing the \mathcal{J} filtration in $\mathcal{X}_{K}^{\infty}(C)$ recovers the complex $\mathcal{X}^{\infty}(C)$ in the original construction by
Ozsváth and Szabó. In other words, $\mathcal{X}_{K}^{\infty}(C)$ is chain homotopy equivalent to $\mathcal{X}^{\infty}(C)$ as unfiltered chain complex.

Theorem 2.2. [HL24] The complex $CFK^{\infty}(S_n^3(K), \mu)$ is filtered chain homotopy equivalent to $\mathcal{X}_K^{\infty}(CFK^{\infty}(S^3, K))$, where μ is the image of the meridian of K in the surgery.

3. Contact surgery

In this section we talk about the preliminaries on contact surgery, contact invariant and LOSS invariant. We will state the naturality theorems of those invariants in Section 3.3, which will play a crucial role in the proofs.

3.1. Contact surgery and DGS algorithm. Given an oriented Legendrian knot L in a contact 3 manifold (Y, ξ) there exist a contact framing defined by a vector field along L that is always transverse to the 2-plane field ξ . In [DG01] Ding and Geiges define a notion of contact r-surgery on L which, for a choice of rational number r, gives rise to another contact 3 manifold $(Y', \xi_r(L))$. (In general there are choices involve to completely determine the resulting contact structure for a general contact rational r-surgery, and in this paper we only consider the one correspond to choices are all negative stabilizations.) In [DGS04] Ding, Geiges and Stipsicz prove the following theorems.

Theorem 3.1. Every (closed, orientable) contact 3 manifold (Y,ξ) can be obtained via contact (± 1) -surgery on a Legendrian link in (S^3, ξ_{std}) .

Moreover they describe an algorithm to transform any rational r-surgery on a Legendrian knot L to a sequence of (± 1) -surgeries on some (stabilizations of) push-offs of L. The procedure can be divided into the following two theorems that correspond to the two cases when r < 0, and r > 0 respectively.

Theorem 3.2 (DGS algorithm for r < 0[DGS04]). Given a Legendrian knot L in (Y,ξ) . Let $0 > -x/y = r \in \mathbb{Q}$ be a contact surgery coefficient with the continued fraction

(3.1)
$$-\frac{x}{y} = [a_1 + 1, a_2, ..., a_\ell]^- = a_1 + 1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_\ell}}}$$

where each $a_i \leq -2$. Then any contact (x/y)-surgery on L can be described as contact surgery along a link $L_1 \cup ... \cup L_l$, where

- L_1 is obtained from a Legendrian push-off of L by stabilizing $|a_1 + 2|$ times.
- L_j is obtained from a Legendrian push-off of L_{j-1} by stabilizing $|a_j+2|$ times, for $j \ge 2$.
- The contact surgery coefficients are -1 on each L_i .

Theorem 3.3 (DGS algorithm for r > 0[DGS04]). Given a Legendrian knot L in (Y,ξ) . Let $0 < x/y = r \in \mathbb{Q}$ be a contact surgery coefficient. Let $e \in \mathbb{Z}$ be the minimal

positive integer such that $\frac{x}{y-ex} < 0$, with the continued fraction

(3.2)
$$\frac{x}{y - ex} = [a_1, a_2, \dots, a_\ell] = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_\ell}}}$$

where each $a_j \leq -2$. Then any contact (x/y)-surgery on L can be described as contact surgery along a link $(L_0^1 \cup L_0^2 \cup ... \cup L_0^e) \cup L_1 \cup ... \cup L_l$, where

- $L_0^1, ..., L_0^e$ are Legendrian push-offs of L.
- L_1 is obtained from a Legendrian push-off of L_0^e by stabilizing $|a_1 + 1|$ times.
- L_j is obtained from a Legendrian push-off of L_{j-1} by stabilizing $|a_j+2|$ times, for $j \ge 2$.
- The contact surgery coefficients are +1 on each L_0^i and -1 on each L_i .

The choices we mentioned in the beginning of the section correspond to the choices of stabilization for each L_j , each of which can be either positive or negative. As we mentioned in the introduction we only consider the case when the stabilizations are all being negative. The most important cases we are interested in are the negative (-n)-surgery and positive $(\frac{n+1}{n})$ -surgery (we will consider general -x/y later). If we follow the above two algorithms carefully and only consider negative stabilization, the -n contact surgery on a Legendrian knot L is the same as doing contact (-1)-surgery along the n-1 times negative stabilized L, and the $\frac{n+1}{n}$ contact surgery on L is the same as doing contact surgery along link $L \cup L_1$ where L_1 is the n times negative stabilization of a Legendrian push-off of L with contact +1 on L, and -1 on L_1 .

3.2. Contact invariants and LOSS invariants. Given a contact 3-manifold (Y, ξ) Ozsváth-Szabó[OS05] and later Honda-Kazez-Matić [HKM09] showed that (Y, ξ) determines a distinguished element $c(\xi) \in \widehat{\mathrm{HF}}(-Y)$, called the Heegaard Floer contact invariant. Subsequently, for a Legendrian knot L in (Y, ξ) , Lisca-Ozsváth-Stipsicz-Szabó defined the "LOSS invariant" $\mathfrak{L}(L) \in \mathrm{HFK}^{-}(-Y, L)$ [LOSS09]. We refer the reader to the above papers for precise definitions.

For any 3-manifold Y and a knot K in Y there is a natural chain map

$$g: \mathrm{CFK}^-(Y, K, \mathfrak{t}) \to \widehat{\mathrm{CF}}(Y, \mathfrak{t})$$

by setting U = 1. The map

$$G: \mathrm{HFK}^{-}(Y, K, \mathfrak{t}) \to \mathrm{HF}(Y, \mathfrak{t})$$

is the homology map induced by g. The LOSS invariant is related to the contact invariant in the following way.

Lemma 3.4. [[LOSS09]] Let L be an oriented null-homologous Legendrian knot in a contact 3-manifold (Y, ξ) , then the map

(3.3)
$$G: \mathrm{HFK}^{-}(-Y, L, \mathfrak{t}) \to \mathrm{HF}(-Y, \mathfrak{t})$$

has the property that

$$G(\mathfrak{L}(L)) = c(\xi).$$

Another important property of the LOSS invariant is that it is unchanged under negative stabilization.

Theorem 3.5 ([LOSS09]). Suppose that L is an oriented Legendrian knot and denote the negative stabilizations of L as L^- Then, $\mathfrak{L}(L^-) = \mathfrak{L}(L)$.

3.3. Naturality of contact invariant and LOSS invariant under contact surgery.

Theorem 3.6. [MT18, Theorem 1.1] Let L be an oriented null-homologous Legendrian knot in a contact rational homology sphere (Y,ξ) with non-vanishing contact invariant $c(\xi)$, let $0 < x/y \in \mathbb{Q}$ be the contact surgery coefficient and $p/q = \operatorname{tb}(L) + x/y$ be the corresponding smooth surgery coefficient. Let $W : Y \# - L(q,r) \to Y_{p/q}(L)$ be the corresponding rational surgery cobordism, where p = mq - r, and consider $\xi_{x/y}(L)$ on $Y_{p/q}(L)$ (When we write smooth surgery on a Legendrian knot L we meant to view L as it smooth knot type).

(1) There exist a Spin^c structure \mathfrak{s} on W and a generator $\tilde{c} \in \operatorname{HF}(L(q,r))$ such that the homomorphism

$$F_{-W,\mathfrak{s}}: \widehat{\mathrm{HF}}(-Y \# L(q,r)) \to \widehat{\mathrm{HF}}(-Y_{p/q}(L))$$

induced by W with its orientation reversed satisfies

$$F_{-W,\mathfrak{s}}(c(\xi)\otimes\tilde{c})=c(\xi_{x/y}(L)).$$

(2) Moreover if both ξ and $\xi_{x/y}(L)$ have torsion first Chern class. Then \mathfrak{s} has the property that

$$\pm \langle c_1(\mathfrak{s}), [\tilde{Z}] \rangle = p + (\operatorname{rot}(L) - \operatorname{tb}(L))q - 1$$

where Z is a Seifert surface for L and $[\tilde{Z}]$ is the result of capping off Z with q parallel copy of the core of the handle in W.

We will analyse the map $F_{-W,\mathfrak{s}}$ in the above theorem to see when this map is injective. More specifically, we will study the corresponding map using the mapping cone formula given by the following proposition.

Proposition 3.7. ([MT18][Corollary 1.5]) Let L be an oriented Legendrian knot in a contact integer homology sphere (Y,ξ) , fix $0 < x/y \in \mathbb{Q}$ to be a contact surgery coefficient corresponding to smooth surgery coefficient $p/q = \operatorname{tb}(L) + x/y$. Let W : $Y \# - L(q, r) \to Y_{p/q}(L)$ be the corresponding rational surgery cobordism, where p =mq - r. Then the contact invariant $c(\xi_{x/y}(L)) \in \widehat{\operatorname{HF}}(-Y_{p/q}(L))$ is equal (up to conjugation of the Spin^c structure on the cobordism) to the image of $c(\xi)$ in homology of the map given by inclusion

$$(t, \hat{B}_s) \hookrightarrow \widehat{\mathcal{X}}_{-p/q}(-Y, L),$$

where we view $c(\xi)$ is represented by element (t, \hat{B}_s) under the identification between (t, \hat{B}_s) and $\widehat{CF}(Y)$, and t satisfies

$$2t = (\operatorname{rot}(L) - \operatorname{tb}(L) + 1)q - 2.$$

Remark 3.8. Mark-Tosun only state the above proposition for $(S^3, \xi_s td)$, but their proofs extend naturally for the situation when Y is an integer homology sphere, and even for null-homologous knot in rational homology sphere.

Note that in both Theorem 3.6 and Proposition 3.7 the Spin^c structures are only determined up to conjugation. However, such sign (conjugation) ambiguity can be removed, given that we are doing positive integer contact surgery on (rationally) null-homologous Legendrian knot, together with both Y and the manifold after surgery Y' are rational homology spheres due to the following technical result. This will end up playing an important role in the proof of Theorems 1.1 and 1.7.

Theorem 3.9. [Wan23a, Theorem 6.3] Let L be an oriented rationally null-homologous Legendrian knot in a contact rational homology sphere (Y, ξ) with non-vanishing contact invariant $c(\xi)$. Let $0 < n \in \mathbb{Z}$ be the contact surgery coefficient, Y' be the manifold after +n contact surgery on L, and let $W : Y \to Y'$ be the corresponding surgery cobordism, and consider $\xi_n^-(L)$ on Y'. There exist a Spin^c structure \mathfrak{s} on Wsuch that the homomorphism

$$F_{-W,\mathfrak{s}}:\widehat{\mathrm{HF}}(-Y)\to\widehat{\mathrm{HF}}(-Y')$$

induced by W with its orientation reversed satisfies

$$F_{-W,\mathfrak{s}}(c(\xi)) = c(\xi_n^-(L)).$$

Moreover, if Y' is also a rational homology sphere. Then the Spin^c structure \mathfrak{s} has the property that

$$\langle c_1(\mathfrak{s}), [\tilde{F}] \rangle = y(rot_{\mathbb{Q}}(L) + n - 1)$$

where y is the order of [L], F is a rational Seifert surface for L and \tilde{F} is the "capped off" surface of F.

We also have the parallel naturality theorem for the LOSS invariant which will also be used later.

Theorem 3.10. [OS10a, Theorem 1.1] Let $L, S \in (Y, \xi)$ be two disjoint oriented Legendrian knots in the contact 3-manifold (Y, ξ) with L null-homologous. Let (Y', ξ_1) denote the 3-manifold with the associated contact structure obtained by performing contact (+1)-surgery along S, and let L' denote the oriented Legendrian knot which is the image of L in (Y', ξ_1) . Moreover suppose that L' is null-homologous in Y'. Let W be the 2-handle cobordism from Y to Y' induced by the surgery, and let

$$F_{-W,\mathfrak{s}}$$
: HFK⁻($-Y, L$) \rightarrow HFK⁻($-Y', L'$)

be the homomorphism in knot Floer homology induced by -W, the cobordism with reversed orientation, for \mathfrak{s} a Spin^c structure on -W. Then

(1) there is a unique choice of \mathfrak{s} for which

$$F_{-W,\mathfrak{s}}(\mathfrak{L}(L)) = \mathfrak{L}(L')$$

holds, and for any other Spin^c structure \mathfrak{s} the map $F_{-W,\mathfrak{s}}$ is trivial on $\mathfrak{L}(L)$.

(2) [Wan23a, Proposition 1.4] If S is null-homologous and both Y and Y' are rational homology sphere then \mathfrak{s} has the property that

$$\langle c_1(\mathfrak{s}), [\tilde{Z}] \rangle = \operatorname{rot}(S)$$

where Z is a Seifert surface for S and \tilde{Z} is the result of capping off Z with the core of the handle in W.

4. A FILTERED MAPPING CONE COMPUTATION

In this section we perform the computations used in the proofs of the main theorems.

The knot E_n has the two-bridge notation [-2, n + 1], which can be viewed as the numerical closure of the rational tangle $-2 + \frac{1}{n+1} = -\frac{2n+1}{n+1}$. We will be interested in the (knot Floer of) mirror $-E_n$, which has the rational tangle number $\frac{2n+1}{n+1}$. For two-bridge knots, $\widehat{\text{HFK}}$ and Alexander polynomials are determined by the rational tangle number (see for example [BS34, Ras02]). Together with the classification theorem for thin knots [OS03, Pet13], we compute

$$\operatorname{CFK}^{\infty}(-E_n) \cong C \oplus \left(\bigoplus_{i=1}^{\frac{n-1}{2}} D_i\right)$$

where C is isomorphic to $CFK^{\infty}(-T_{2,3})$ and each D_i is a length-one box summand (compare with the homology $HFK^{-}(-E_n)$ and $\widehat{HFK}(-E_n)$ calculated in[OS10b]). We pick the basis as follows. Let x, y, z be the generators of C over $\mathbb{F}[U]$, such that in the (i, j)-coordinate of CFK^{∞}, they have coordinate (0, 1), (0, 0) and (1, 0), with Malov grading 2, 1 and 2 respectively. The differentials are given by

$$\partial x = \partial y = z.$$

Each D_i is isomorphic to D, generated by a, b, c and d over $\mathbb{F}[U]$, with coordinate (1, 1), (0, 1), (1, 0) and (0, 0) and Maslov grading 3, 2, 2 and 1 respectively. he differentials are given by

 $\partial a = b + c$ $\partial b = \partial c = d.$

FIGURE 2. On the left, the knot Floer complex of
$$-E_5 = 7_2$$
. On the right, the knot Floer complex of its dual knot, $CFK^{\infty}(S^3_{+1}(-E_5), \mu)$.

Proposition 4.1. The dual knot complex of $-E_n$ is given by

$$\operatorname{CFK}^{\infty}(S^3_{+1}(-E_n),\mu) \cong O \oplus \left(\bigoplus_{i=1}^{\frac{n+1}{2}} H_i\right) \oplus \left(\bigoplus_{i=1}^{\frac{n+1}{2}} V_i\right)$$

where O is the complex with a single generator, supported in (0,0) coordinate. Each H_i is generated by x_i^h, y_i^h in (0,0) and (1,0) coordinate respectively, with the differential $\partial y_i^h = x_i^h$. Each H_i is generated by x_i^v, y_i^v in (0,0) and (0,1) coordinate respectively, with the differential $\partial y_i^v = x_i^v$.

Proof. We first compute $\mathcal{X}_{K}^{\infty}(C)$ with surgery coefficient n = 1, which by (2.3) is the mapping cone of

(4.1)
$$A_0(C) \oplus A_1(C) \xrightarrow{v_1+h_0} B_1(C).$$

Recall that $A_0(C)$, $A_1(C)$ and $B_1(C)$ are each isomorphic to a copy of C. We denote the generators in $A_s(C)$ by $x^{(s)}, y^{(s)}, z^{(s)}$ for s = 0, 1 and the generators in $B_1(C)$ by x', y', z'. The mapping cone differentials are given by

$$v_1(x^{(1)}) = x'$$
 $v_1(y^{(1)}) = y'$ $v_1(z^{(1)}) = z'$
 $h_0(x^{(0)}) = y'$ $h_0(y^{(0)}) = x'$ $h_0(z^{(0)}) = z'.$

Note that in the $(\mathcal{I}, \mathcal{J})$ -filtration defined by (2.4), (2.5), (2.7) and (2.8), x' and z' are in the same $(\mathcal{I}, \mathcal{J})$ -coordinate. Therefore quotienting out $\{x', \partial x'\}$ yields a chain homotopy equivalent complex. Similarly we can quotient out $\{x^{(0)}, \partial x^{(0)}\}$. The resulting complex is generated by $\{y^{(0)}, z^{(0)}, x^{(0)} + y^{(1)}, x^{(1)}, z^{(1)}\}$ over $\mathbb{F}[U]$. We can further simplify the complex by a filtered change of basis to $\{y^{(0)}, z^{(0)}, x^{(0)} + y^{(0)} + x^{(1)} + y^{(1)}, x^{(1)}, z^{(1)}\}$, with $(\mathcal{I}, \mathcal{J})$ -filtration (1, 0), (0, 0), (1, 1), (0, 1), (0, 0) respectively and differntials

$$\partial y^{(0)} = z^{(0)}$$
$$\partial x^{(1)} = z^{(1)}.$$

Next we compute $\mathcal{X}_{K}^{\infty}(D)$ for $D \cong D_{i}, 1 \leq i \leq \frac{n-1}{2}$. By (2.3) this is the mapping cone of

(4.2)
$$A_0(D) \oplus A_1(D) \xrightarrow{v_1+h_0} B_1(D).$$

We adopt the similar notations and denote the generators in $A_s(D)$ by $a^{(s)}, b^{(s)}, c^{(s)}, d^{(s)}$ for s = 0, 1 and the generators in $B_1(D)$ by a', b', c', d'. Note that a', c' and b', d' are pairwise in the same $(\mathcal{I}, \mathcal{J})$ -coordinate, and therefore the mapping cone is homotopy equivalent to $A_0(D) \oplus A_1(D)$. We can further quotient out $\{a^{(0)}\partial a^{(0)}\}$ and $\{a^{(1)}\partial a^{(1)}\}$. The resulting complex is generated by $\{c^{(0)}, d^{(0)}, b^{(1)}, d^{(1)}\}$ with $(\mathcal{I}, \mathcal{J})$ filtration (1, 0), (0, 0), (0, 1), (0, 0) respectively and differentials

$$\partial c^{(0)} = d^{(0)}$$
$$\partial b^{(1)} = d^{(1)}$$

We have

$$CFK^{\infty}(S^{3}_{+1}(-E_{n}),\mu) \cong \mathcal{X}_{K}^{\infty}(CFK^{\infty}(S^{3},-E_{n}))$$
$$\cong \mathcal{X}_{K}^{\infty}(C \oplus (\bigoplus_{i=1}^{\frac{n-1}{2}} D_{i}))$$
$$= \mathcal{X}_{K}^{\infty}(C) \oplus (\bigoplus_{i=1}^{\frac{n-1}{2}} \mathcal{X}_{K}^{\infty}(D_{i})).$$

The result follows from the previous computation. In particular, each D_i contributes a copy of V_i and H_i summand, and C contributes a copy of V_i and H_i summand and the O summand.

Corollary 4.2. The map

$$G: \mathrm{HFK}^{-}(S^{3}_{+1}(-E_{n}), \mu) \to \widehat{\mathrm{HF}}(S^{3}_{+1}(-E_{n})),$$

is injective in the top Alexander grading.

Proof. Using the notation from Proposition 4.1, the only generators supported in the top Alexander grading of $\mathrm{HFK}^{-}(S^{3}_{+1}(-E_{n}),\mu)$ are y^{v}_{i} . When s < 0, each $U^{1-s}y^{v}_{i} \in C\{i \leq 0, j = s\} \cong \widehat{\mathrm{CF}}(S^{3}_{+1}(-E_{n}))$ survives into the homology. The result follows. \Box

5. Proof of Theorem 1.1

We will prove the theorem by considering different cases depending on the contact surgery framing r. Recall that $\xi_r(L)$ denotes the contact structure obtained by contact r-surgery on the Legendrian knot L.

5.1. Case for r = -2. This is the most essential case and the starting point.

Theorem 5.1. Let L_1 and L_2 be two Legendrian representatives of E_n with n > 3and odd that have same tb = 1 and rot = 0 but different LOSS invariants. Then (-2)-contact surgery on L_1 and L_2 gives contact manifolds that have different contact invariants.

The point of doing the (-2)-contact surgery is that smoothly we are doing (-1)surgery, which makes the resulting manifold an integer homology sphere. Thus all
knot will null-homologous. The idea of proving Theorem 5.1 is by first showing
the LOSS invariants of the induced Legendrian push-offs of L_1 and L_2 are distinct
and then use the fact that the map (3.3) is injective on the top Alexander grading
(therefore the LOSS invariants of L_1 and L_2 are necessarily distinct as well).

As we pointed out earlier, since we only consider negative stabilization, (-2)contact surgery on L_i is the same as (-1)-contact surgery on L_i^{-1} , the once negativelystabilized L_i for i = 1, 2. Identifying $\xi_{-2}(L_i)$ with $\xi_{-1}(L_i^{-1})$, in the following proof we
denote by $(Y_i, \xi_{-2}(L_i))$ the manifold with the associated contact structure obtained
from -1 contact surgery on L_i^{-1} . Smoothly $Y_i \cong S_{-1}^3(E_n)$ for i = 1, 2.

Proof of Theorem 5.1. Let P_i be the Legendrian push-off of L_i^{-1} in S^3 , and denote by P'_i the induced Legendrian knot in Y_i . See Figure 3. Apply Theorem 3.10 to the pair (P'_i, P''_i) in $(Y_i, \xi_{-2}(L_i))$, where P''_i is induced by another Legendrian push-off of L_i^{-1} in $(Y_i, \xi_{-2}(L_i))$. As contact (-1)-surgery on L_i^{-1} and (+1)-surgery on P''_i cancel, we recover (S^3, P_i) and obtain maps

(5.1)
$$F_{-W_i,\mathfrak{s}_i} : \mathrm{HFK}^-(-Y_i, P_i') \to \mathrm{HFK}^-(-S^3, P_i)$$



FIGURE 3. Denoted by L_i^{-1} is the once negatively-stabilized L_i and P_i its push-off. Then (-1)-contact surgery on L_i^{-1} is the same as (-2)-contact surgery on L_i .

with the property that $F_{-W_i,\mathfrak{s}_i}(\mathfrak{L}(P'_i)) = \mathfrak{L}(P_i)$, for i = 1, 2 respectively, where W_i is the two-handle cobordism and \mathfrak{s}_i is given by Theorem 3.10 (2).

Since smoothly L_1 and L_2 are isotopic, $-W_1$ and $-W_2$ are diffeomorphic. Moreover, P'_1 and P'_2 have the same rotation number (because we are doing the same surgery on Legendrian knots with the same classical invariant), therefore $\mathfrak{s}_1 = \mathfrak{s}_2$. It follows that both F_{-W_i,\mathfrak{s}_i} are given by

(5.2)
$$F_{-W,\mathfrak{s}} : \mathrm{HFK}^{-}(-S^{3}_{-1}(E_{n}), P') \to \mathrm{HFK}^{-}(-S^{3}, P)$$

with the property that $F_{-W,\mathfrak{s}}(\mathfrak{L}(P'_i)) = \mathfrak{L}(P_i)$, for i = 1, 2, where $W = W_i$, $\mathfrak{s} = \mathfrak{s}_i$, and P', P are smooth representative of P'_i, P_i respectively. Since $\mathfrak{L}(L_1) \neq \mathfrak{L}(L_2)$ in HFK⁻($-S^3, P$), and LOSS invariant is unchanged under negative stabilization (Theorem 3.5), we have $\mathfrak{L}(P_1) \neq \mathfrak{L}(P_2)$ in HFK⁻($-S^3, P$). It follows that $\mathfrak{L}(P'_1) \neq \mathfrak{L}(P'_2)$.

Now consider the map

(5.3)
$$G: \mathrm{HFK}^{-}(-S^{3}_{-1}(E_{n}), P') \to \widehat{\mathrm{HF}}(-S^{3}_{-1}(E_{n})).$$

According to [DG09, Proposition 2] the Legendrian push-off P'_i in the Legendrian surgery is smoothly isotopic to the dual knot of $L_i = E_n$, and $-S^3_{-1}(E_n) \cong S^3_{+1}(-E_n)$. Thus the above G map is the same as

(5.4)
$$G: \mathrm{HFK}^{-}(S^{3}_{+1}(-E_{n}), \mu) \to \widehat{\mathrm{HF}}(S^{3}_{+1}(-E_{n})),$$

where μ is the meridian of the surgery knot $-E_n$. Using [LOSS09, Lemma 6.6] it is easy to calculate that $\operatorname{tb}(P'_i) = 0$ and $\operatorname{rot}(P'_i) = -1$, thus by [OS10a, Theorem 1.6] the Alexander grading of $\mathfrak{L}(P'_i)$ equals 1 for all i = 1, 2. By Corollary 4.2, G is injective in the top Alexander grading 1. Therefore by Lemma 3.4, we conclude that $c(\xi_{-2}(L_1)) \neq c(\xi_{-2}(L_2))$.

Remark 5.2. As discussed in Section 2.1.1, the last step of the argument uses the naturality of Heegaard Floer homology [JTZ21].

5.2. Case for r = -2 - k, $k \in \mathbb{Z}_{>0}$. In this subsection we prove the following.

Theorem 5.3. Let L_1 and L_2 be two Legendrian representatives of E_n with n > 3 odd, that have same tb = 1 and rot = 0 but different LOSS invariants. Then (-2-k) contact surgery for $k \in \mathbb{Z}_{>0}$ on L_1 and L_2 gives contact manifolds that have different contact invariants.

Let us start with a lemma that relates the surgery to negative stabilization.

Lemma 5.4. The two Legendrian arcs e_1 and e_2 depicted in Figure 4 in the corresponding local contact surgery diagram are Legendrian isotopic.



FIGURE 4. The number n in the second diagram indicates that there are n zigzags (i.e. n negative-stabilizations). We call the surgery Legendrian in the first diagram the standard Legendrian meridian of e_1 (or more precisely, the knot which e_1 belongs to).

Proof. We demonstrate the equivalence by a sequence of contact surgery and Stein handle move shown in Figure 5. \Box

Observe that using the above lemma we can infer that if L is a Legendrian knot in some (Y, ξ) , then contact $(\frac{n+1}{n})$ -surgery on the standard Legendrian meridian of L send L to L^{-n} . Therefore we can view Legendrian surgery on L^{-n} as Legendrian surgery on L followed by $(\frac{n+1}{n})$ -surgery on the standard Legendrian meridian of L^{-n} . Now we are ready to proof the Theorem 5.3

Proof of Theorem 5.3. To make the argument clear, we use the same notation as in the proof of Theorem 5.1. We again let P_i be the Legendrian push off of L_i^{-1} , $(Y_i, \xi_{-1}(L_i^{-1}))$ be the contact manifold obtained by Legendrian surgery on L_i^{-1} , P'_i be the induced Legendrian knot of P_i in $(Y_i, \xi_{-1}(L_i^{-1}))$, and $(Y'_i, \xi_{-2-k}(L_i))$ be the contact manifold obtained by contact (-2 - k)-surgery on L_i

We first view contact (-2 - k)-surgery on L_i as Legendrian surgery on L_i^{-k-1} . Then by the observation above, Legendrian surgery on L_i^{-k-1} is the same as Legendrian surgery on L_i^{-1} followed by contact $(\frac{k+1}{k})$ -surgery on the standard Legendrian meridian of L_i^{-1} . In other words we can view the contact manifold $(Y'_i, \xi_{-2-k}(L_i))$ obtained by doing contact $(\frac{k+1}{k})$ -surgery on the standard Legendrian meridian of P_i in





FIGURE 5. The equivalence between diagrams follows from the same process as the proof of Lemma 5.1 in [Wan23a]. For the reader's convenience, we spell out the proof. From **a** to **b** is a Legendrian Reidemeister move; from \mathbf{b} to \mathbf{c} we isotopy the Legendrian meridian from bottom to top using [DG09, Figure 13-15]; from \mathbf{c} to \mathbf{d} we use the DGS algorithm [DGS04] to change the $(\frac{n+1}{n})$ -contact surgery to a sequence of (+1) and (-1)-contact surgeries, and we use negative stabilizations for all push-offs as the assumption; from \mathbf{d} to \mathbf{e} we use [DG09, Theorem 4] to identify the surgery diagram with the handle diagram; from **e** to **f** we slide the red curve over the (-1)-framed 2-handle using [DG09, Proposition 1]; from **f** to **g** we cancel out the (-1)-framed 2-handle with the 1-handle; from \mathbf{g} to \mathbf{h} we perform a Legendrian Reidemeister move to get rid of the extra crossing.



FIGURE 6. By lemma 5.4, this two contact surgery diagram are equivalent.

contact manifold $(Y_i, \xi_{-1}(L_i^{-1}))$, moreover by [DG09, Proposition 2] again, this standard Legendrian meridian is Legendrian isotopic to P'_i in $(Y_i, \xi_{-1}(L_i^{-1}))$. Recall that $\operatorname{tb}(P'_i) = 0$ and $\operatorname{rot}(P'_i) = -1$, implies that smoothly we are also doing $(\frac{k+1}{k})$ -surgery on P'_i . We also notice and denote $Y = Y_i = S^3_{-1}(L_i)$ is a homology sphere, and $Y'_i = Y_{k+1}(P'_i)$ is a rational homology sphere.

Thus by the naturality theorem 3.6 we obtain maps

$$F_{-W_i,\mathfrak{s}_i}:\widehat{\mathrm{HF}}(-Y\#L(k,-1))\to\widehat{\mathrm{HF}}(-(Y_{\frac{k+1}{k}}(P'_i)))$$

with the properties

$$F_{-W_i,\mathfrak{s}_i}(c(\xi_{-2}(L_i))\otimes\tilde{c})=c(\xi_{-2-k}(L_i)).$$

Since the contact invariants are all torsion, we also have

$$\pm \langle c_1(\mathfrak{s}_i), [\tilde{Z}] \rangle = p + (\operatorname{rot}(L) - \operatorname{tb}(L))q - 1 = k + 1 + (-1 - 0)k - 1 = 0,$$

which implies $\mathfrak{s}_1 = \mathfrak{s}_2$. Clearly W_1 is diffeomorphic to W_2 , so F_{-W_i,\mathfrak{s}_i} , i = 1, 2 above are again given by a single map

$$F_{-W,\mathfrak{s}}: \widehat{\mathrm{HF}}(-Y \# L(k,-1)) \to \widehat{\mathrm{HF}}(-(Y_{\frac{k+1}{k}}(P')))$$

with the property that for each i = 1, 2

$$F_{-W,\mathfrak{s}}(c(\xi_{-2}(L_i))\otimes\tilde{c})=c(\xi_{-2-k}(L_i)),$$

where $W = W_i$, $\mathfrak{s} = \mathfrak{s}_i$ and P' is the smooth representative of P'_i .

The goal now is to show this map $F_{-W,\mathfrak{s}}$ is injective. By Proposition 3.7, we only need to show that

(5.5)
$$(t, \hat{B}_s) \hookrightarrow \widehat{\mathcal{X}}_{-(k+1)/k}(-Y, P')$$

is injective in homology, where 2t = (-1 - 0 + 1)k - 2, so t = -1. (Proposition 3.7 only determines \mathfrak{s} up to conjugation. However here we have $\langle c_1(\mathfrak{s}), [\tilde{Z}] \rangle = 0$, so \mathfrak{s} is self-conjugate.)

The mapping cone corresponding to Spin^{c} structure -1 is given by



The genus of P' is 1 as it has the same knot complement as E_n , so the map \hat{v}_s (resp. \hat{h}_s) is an isomorphism for all s > 0 (resp. s < 0). By a standard truncation argument we see that

$$(-1, \hat{B}_{-1}) \hookrightarrow (\widehat{\mathcal{X}}_{-(k+1)/k}(-Y, P'), -1)$$

in fact induces an isomorphism in the homology. It follows that $c(\xi_{-2-k}(L_1)) \neq c(\xi_{-2-k}(L_2))$.

5.3. Case for general rational r < 0. To obtain the general result we need to use Theorem 3.2, thus for r < 0 we write

(5.7)
$$r = -\frac{x}{y} = [a_1 + 1, a_2, ..., a_\ell] = a_1 + 1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_\ell}}}$$

where each $a_j \leq -2$. Aside from the DGS algorithm we also need the following proposition, which essentially states that the contact (-1)-surgery (Legendrian surgery) preserve the distinction between contact invariants.

Proposition 5.5 ([Wan23b] Theorem 1.1). Let ξ^1 and ξ^2 be two contact structures on a 3-manifold Y. Take any smooth knot K in Y. Let L_i , i = 1, 2 be a Legendrian representative of K in ξ^i . Then

(5.8)
$$c(\xi^1) \neq c(\xi^2) \text{ implies } c(\xi^{-1}_{-1}(L_1)) \neq c(\xi^{-1}_{-1}(L_2)).$$

Lemma 5.6. Let $r = [a_1, a_2, ..., a_\ell] < 0$ where each $a_j \leq -2$ but not all are equal to -2. Let L_1 and L_2 be two Legendrian representatives of E_n with n > 3 odd, that have same tb = 1 and rot = 0 but different LOSS invariants. Then contact r-surgery on L_1 and L_2 gives contact manifolds that have different contact invariants.

Proof. The previous two subsections already prove the case when $\ell = 1$ so we assume $\ell > 1$. Suppose $a_t < -2$ for some $t \in \{1, 2, ..., \ell\}$. By DGS algorithm for r < 0 (Theorem 3.2), contact r-surgery on L_i for i = 1, 2 is the same as a sequence of contact

(-1)-surgeries on $L_i^{(j)}$ for $j = 1, 2, ..., \ell$, where each $L_i^{(j)}$ is a Lengendrian push-off of L_i , with $|a_j + 2|$ negative-stabilizations. See Figure 7. If we denote $(Y_i^{(t)}, \xi_{-1}(L_i^{(t)}))$ to be the contact 3-manifold obtained by contact (-1)-surgery on $L_i^{(t)}$, we can we view contact r-surgery on L_i as obtained by first doing contact (-1)-surgery on $L_i^{(t)}$, then the rest contact (-1)-surgery in $(Y_i^{(t)}, \xi_{-1}(L_i^{(t)}))$. Since $a_t < -2$, $L_i^{(t)}$ is negative stabilized at least once, thus by Theorem 5.3 we have $c(\xi_{-1}(L_1^{(t)})) \neq c(\xi_{-1}(L_2^{(t)}))$. Then by applying Proposition 5.5 repeatedly on a sequence of contact (-1)-surgeries we infer that $c(\xi_r(L_1)) \neq c(\xi_r(L_2))$.



FIGURE 7. The contact *r*-surgery on L_i for i = 1, 2 according to DGS algorithm. The surgery coefficient is for contact surgery. Each component $L_i^{(j)}$ is negatively stabilized $|a_i + 2|$ times.

Now we are left with one case where $r = [a_1 + 1, ..., a_\ell]$ and all $a_j = -2$. This will be included in the proof of Theorem 1.1.

Proof of Theorem 1.1. First we note that $a_j = -2$ for all $j = 1, 2, ..., \ell$ if and only if $r = -\frac{1}{\ell}$, thus as we discussed above, the only remaining case is when $r = -\frac{1}{\ell}$. We let $r' = r - 1 = -\frac{1}{\ell} - 1$ and we denote $(Y, \xi_r(L_i))$ and $(Y', \xi_{r'}(L_i))$ to be the contact 3-manifold obtained by taking contact $r = -\frac{1}{\ell}$ and $r' = -\frac{1}{\ell} - 1$ surgeries on L_i respectively. Then by applying Lemma 5.4 to the standard Legendrian meridian of L_i , we can view $(Y', \xi_{r'}(L_i))$ as obtained from contact (+2)-surgery on the standard Legendrian meridian of L_i in $(Y, \xi_r(L_i))$. Moreover when $\ell > 1$, both Y and Y' are rational homology spheres $(tb(L_i) = 1)$ thus by (1) in Theorem 3.6 where the contact surgery is +2 we obtain maps

$$F_{-W_i,\mathfrak{s}_i}: \widehat{\mathrm{HF}}(-Y) \to \widehat{\mathrm{HF}}(-Y')$$

with the properties

$$F_{-W_i,\mathfrak{s}_i}(c(\xi_r(L_i))) = c(\xi_{r'}(L_i)).$$

 L_i have the same smooth knot type, tb and rot implies their corresponding standard Legendrian meridians have the same classical invariants (even if they are just rationally null-homologous). Thus by (Theorem 3.9) $\mathfrak{s}_1 = \mathfrak{s}_2$ which implies two maps F_{-W_i,\mathfrak{s}_i} are equivalent. Notice that r' is not of the form of $-\frac{1}{k}$, thus by Theorem 5.3 we have $c(\xi_{r'}(L_1)) \neq c(\xi_{r'}(L_2))$ which implies $c(\xi_r(L_1)) \neq c(\xi_r(L_2))$.



FIGURE 8. contact $(-\frac{1}{\ell}-1)$ -surgery on L_i is obtained by contact +2surgery on the standard Legendrian meridian of $(-\frac{1}{\ell})$ -surgery on L_i

Remark 5.7. An important prerequisite for applying Theorem 3.6 is that the starting manifold needs to be a rational homology sphere, and this is also the reason why we are not able to obtain different contact invariants on Legendrian surgery of L_i . Although not explicitly written in the paper [MT18], it has been pointed out by Baldwin [Bal] that for the proof of [Bal13, Proposition 2.3] the manifold being a rational homology sphere is a necessary condition.

6. Result for Legendrian knot E(m, n)

In this section we prove Theorem 1.7. Recall from [Wan23a, Theorem 5.2], the Legendrian representatives L_1 and L_2 of E(m, n) with different LOSS invariants are obtained by adding m-1 half positive-twist to the Legendrian representatives L'_1 and L'_2 of E_n , where L'_1 and L'_2 have different LOSS invariants (see the top left of Figure 9). As a strategy adopted in the proof of [Wan23a, Theorem 1.6], one can "undo" a full-twist by performing a contact (+2)-surgery on a standard Legendrian unknot, at the cost of adding two zig-zags (depicted in the top row of Figure 9). Therefore by performing a sequence of (m-1)/2 many contact (+2)-surgeries, one obtains L'_i with (m-1) negative stabilizations.

Proof of Theorem 1.7. First fix a negative rational contact surgery coefficient r, and assume m > 1 and odd. Denote $(Y, \xi_r(L_i))$ to be the contact 3-manifold obtained



FIGURE 9. Diagram for the proof of Theorem 1.7. Top row shows the case of one Legendrian representative of E(m, n) when m = 3.

by taking contact r-surgery on L_i , and $(Y', \xi_{r-m+1}(L'_i))$ to be the contact 3-manifold obtained by taking contact (r-m+1)-surgery on L'_i (recall that contact (r-m+1)surgery on L'_i is the same as contact r-surgery on (m-1) negative stabilized L'_i).

As we described above, by undoing the full twist using contact (+2)-surgery on a sequence of standard Legendrian unknots, we will bring the Legendrian knot L_i back to L'_i with m-1 negative stabilization. Thus after applying contact r-surgery on L_i we have the following maps

$$F_{-W_i,\mathfrak{s}_i}: \widehat{\mathrm{HF}}(-Y) \to \widehat{\mathrm{HF}}(-Y').$$

Moreover since $tb(L_i) = m$ for both i = 1, 2, when $r \neq -m$, both Y and Y' are rational homology spheres. Therefore the naturality of contact invariant, i.e. Theorem 3.6 applies and we get

$$F_{-W_i,\mathfrak{s}_i}(c(\xi_r(L_i))) = c(\xi_{r-m+1}(L'_i)).$$

Again since both L_i share the same smooth knot type, tb and rot, both W_i are diffeomorphic and by Theorem 3.9 \mathfrak{s}_i are identical. Therefore F_{-W_i,\mathfrak{s}_i} are given by the same map for i = 1, 2. Since $r - m + 1 \neq 1$, by Theorem 1.1 we have $c(\xi_{r-m+1}(L'_1)) \neq c(\xi_{r-m+1}(L'_2))$. Combining the above, we infer $c(\xi_r(L_1)) \neq c(\xi_r(L_2))$.

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 \square

Remark 6.1. An alternative approach is by following the proof outline in Section 5, replacing E_n by E(m, n) at each instance. This method produces an analogous result to Theorem 5.1, but fails at the second step, when extending to (-2 - k)-surgery. The main reason is that the genus of E(m, n) is greater than 1, and increases when m increase, so the truncation of the mapping cone (5.6) becomes more complicated. Thus we are unable to conclude the injectivity which is necessary for the arguments.

References

- [Bal] John Baldwin, Personal communication.
- [Bal13] John A. Baldwin, Capping off open books and the Ozsváth-Szabó contact invariant, J. Symplectic Geom. 11 (2013), no. 4, 525–561. MR 3117058
- [BEE12] Frédéric Bourgeois, Tobias Ekholm, and Yasha Eliashberg, Effect of Legendrian surgery, Geom. Topol. 16 (2012), no. 1, 301–389, With an appendix by Sheel Ganatra and Maksim Maydanskiy. MR 2916289
- [BS34] Carl Bankwitz and Hans Georg Schumann, Über viergeflechte, Abh. Math. Sem. Univ. Hamburg 10 (1934), no. 1, 263–284. MR 3069630
- [Che02] Yuri Chekanov, Differential algebra of Legendrian links, Invent. Math. 150 (2002), no. 3, 441–483. MR 1946550
- [DG01] Fan Ding and Hansjörg Geiges, Symplectic fillability of tight contact structures on torus bundles, Algebr. Geom. Topol. 1 (2001), 153–172. MR 1823497
- [DG09] _____, Handle moves in contact surgery diagrams, J. Topol. 2 (2009), no. 1, 105–122. MR 2499439
- [DGS04] Fan Ding, Hansjörg Geiges, and András I. Stipsicz, Surgery diagrams for contact 3manifolds, Turkish J. Math. 28 (2004), no. 1, 41–74. MR 2056760
- [ENV13] John B. Etnyre, Lenhard L. Ng, and Vera Vértesi, Legendrian and transverse twist knots, J. Eur. Math. Soc. (JEMS) 15 (2013), no. 3, 969–995. MR 3085098
- [Etn08] John B. Etnyre, On contact surgery, Proc. Amer. Math. Soc. 136 (2008), no. 9, 3355–3362. MR 2407103
- [Gol15] Marco Golla, Ozsváth-Szabó invariants of contact surgeries, Geom. Topol. 19 (2015), no. 1, 171–235. MR 3318750
- [HKM09] Ko Honda, William H. Kazez, and Gordana Matić, On the contact class in Heegaard Floer homology, J. Differential Geom. 83 (2009), no. 2, 289–311. MR 2577470
- [HL24] Matthew Hedden and Adam Simon Levine, A surgery formula for knot Floer homology, Quantum Topol. 15 (2024), no. 2, 229–336. MR 4725826
- [JTZ21] András Juhász, Dylan Thurston, and Ian Zemke, Naturality and mapping class groups in Heegard Floer homology, Mem. Amer. Math. Soc. 273 (2021), no. 1338, v+174. MR 4337438
- [KOU17] Çağrı Karakurt, Takahiro Oba, and Takuya Ukida, Planar Lefschetz fibrations and Stein structures with distinct Ozsváth-Szabó invariants on corks, Topology Appl. 221 (2017), 630–637. MR 3624490

- [LOSS09] Paolo Lisca, Peter Ozsváth, András I. Stipsicz, and Zoltán Szabó, Heegaard Floer invariants of Legendrian knots in contact three-manifolds, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 6, 1307–1363. MR 2557137
- [LS06] Paolo Lisca and András I. Stipsicz, Notes on the contact Ozsváth-Szabó invariants, Pacific J. Math. 228 (2006), no. 2, 277–295. MR 2274521
- [MT18] Thomas E. Mark and Bülent Tosun, Naturality of Heegaard Floer invariants under positive rational contact surgery, J. Differential Geom. 110 (2018), no. 2, 281–344. MR 3861812
- [OS03] Peter Ozsváth and Zoltán Szabó, Heegaard Floer homology and alternating knots, Geom. Topol. 7 (2003), 225–254. MR 1988285
- [OS04a] _____, Holomorphic disks and knot invariants, Adv. Math. **186** (2004), no. 1, 58–116. MR 2065507
- [OS04b] _____, Holomorphic disks and topological invariants for closed three-manifolds, Ann. of Math. (2) **159** (2004), no. 3, 1027–1158. MR 2113019
- [OS05] _____, Heegaard Floer homology and contact structures, Duke Math. J. 129 (2005), no. 1, 39–61. MR 2153455
- [OS10a] Peter Ozsváth and András I. Stipsicz, Contact surgeries and the transverse invariant in knot Floer homology, J. Inst. Math. Jussieu 9 (2010), no. 3, 601–632. MR 2650809
- [OS10b] _____, Contact surgeries and the transverse invariant in knot Floer homology, J. Inst. Math. Jussieu 9 (2010), no. 3, 601–632. MR 2650809
- [OS11] Peter S. Ozsváth and Zoltán Szabó, Knot Floer homology and rational surgeries, Algebr. Geom. Topol. 11 (2011), no. 1, 1–68. MR 2764036
- [Pet13] Ina Petkova, Cables of thin knots and bordered Heegaard Floer homology, Quantum Topol. 4 (2013), no. 4, 377–409. MR 3134023
- [Ras02] Jacob Andrew Rasmussen, Floer homology of surgeries on two-bridge knots, Algebr. Geom. Topol. 2 (2002), 757–789. MR 1928176
- [Wan23a] Shunyu Wan, Naturality of legendrian loss invariant under positive contact surgery, arXiv preprint arXiv:2306.09516 (2023).
- [Wan23b] _____, Tight contact structures on some families of small seifert fiber spaces, arXiv preprint arXiv:2311.10171 (2023).

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