Aspects of Modular Flavor Symmetries*

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Abstract

Modular flavor symmetries refers to scenarios in which fermion masses respect modular symmetries. Such scenarios have been studied in the bottom-up approach and have an explicit realization in string theory. They rely on the remarkable properties of vector-valued modular forms.

1 Introduction

The standard model (SM) is remarkably successful in summarizing our current understanding of particle physics. Currently experiments do not give us unambiguous hints for how particle physics beyond the SM may look like. At the same time, the SM does not provide us with a theory of flavor, i.e. rather than explaining the patterns of fermion masses it only describes them by introducing more than 20 continuous parameters.

2 Theories of flavor

Developing theories of flavor has been a longstanding theme in beyond the SM physics. Two main schemes emerged:

- 1. mass hierarchies from Froggatt-Nielsen [1] (FN) and Randall-Sundrum [2] (RS) models,
- 2. flavor structures from non-Abelian discrete flavor symmetries [3].

While the former can convincingly explain hierarchies, they typically introduce numerous extra parameters e.g. in the coefficients of terms in the Lagrange density. On the other hand, non-Abelian discrete symmetries allow one to avoid relative coefficients. For instance, consider the alternating group of order 4, A_4 . At leading order the neutrino mass matrix reads [4]

$$m_{\nu} = \frac{v_{u}^{2}}{\Lambda} \begin{pmatrix} 2v_{1} & -v_{3} & -v_{2} \\ -v_{3} & 2v_{2} & -v_{1} \\ -v_{2} & -v_{1} & 2v_{3} \end{pmatrix}, \tag{1}$$

where the v_i denote the components of the vacuum expectation value (VEV) of a triplet flavon, and v_u stands for the VEV of the u-type Higgs of the minimal supersymmetric standard model (MSSM). This means that the structure of the neutrino mass matrix is fixed by the A_4 symmetry. However, it is nontrivial to explain VEV patterns in which all v_i are nonvanishing and their ratios are hierarchical. This conundrum sometimes gets referred to as VEV alignment problem (see e.g. [5] for a discussion).

More recently, a new scheme emerged: modular flavor symmetries [6]. While the underlying symmetries have infinitely many elements, this approach hosts non-Abelian finite groups and naturally gives rise to hierarchies (cf. Figure 1).

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¹It is beyond the scope, and page limit, of these proceedings to fully summarize all related activities, see e.g. [7–11] for reviews and more references.

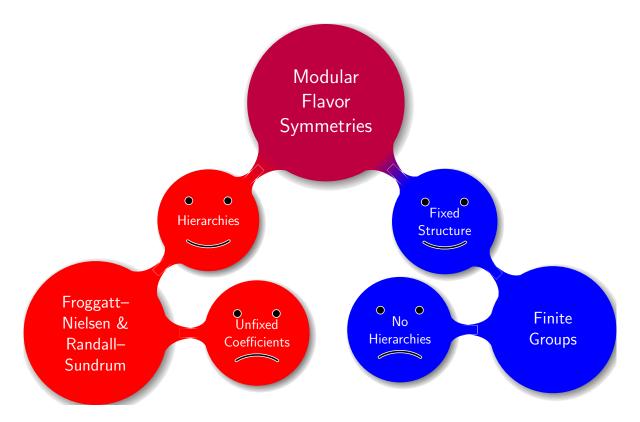


Figure 1: Modular flavor symmetries give rise to the attractive features of FN and RS models as well as non-Abelian flavor symmetries while avoiding their problematic aspects.

3 Modular invariance and flavor

Modular symmetries are, in a way, the symmetries of tori. A torus defines a lattice, and modular transformations are transformations of the lattice basis vectors which are such that the new basis vectors span the same lattice. These transformations form the infinite group

$$SL(2,\mathbb{Z}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; a,b,c,d \in \mathbb{Z} \land ad - bc = 1 \right\}.$$
 (2)

The so-called half-period ratio τ transforms as

$$\tau \xrightarrow{\gamma} \frac{a \tau + b}{c \tau + d}$$
, (3)

and 'sees' only the group $\Gamma := SL(2, \mathbb{Z})/\mathbb{Z}_2$, where the nontrivial \mathbb{Z}_2 element is given by diag(-1, -1). Models of flavor are often based on the so-called congruence subgroups

$$\Gamma(N) = \left\{ \gamma \in \Gamma \; ; \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\} \; . \tag{4}$$

The quotients $\Gamma_N := \Gamma/\Gamma(N)$ are finite groups, for instance $\Gamma_3 \simeq A_4$. The important ingredient to tackle the flavor problem are the so-called vector-valued modular forms [12] (VVMF), which, by definition, transform under (3) as

$$f_i(\gamma \tau) = (c\tau + d)^k \left[\rho_n(\gamma) \right]_{ii} f_i(\tau) . \tag{5}$$

Here, the f_i furnish an n-dimensional representation under Γ_N , $\rho_n(\gamma)$ is a representation matrix, and k denotes the modular weight. The f_i are further holomorphic functions of τ and do not possess poles. Crucially, these requirements make the VVMF unique.

This can be used to construct theories of flavor in which Yukawa couplings and mass matrices are modular forms. For instance, in the Feruglio model [6] the pendant of (1), i.e. neutrino mass matrix, is proportional to

$$m^{(\nu)} \propto \mathcal{M}_{\nu}(\tau) = \begin{pmatrix} 2Y_1(\tau) & -Y_3(\tau) & -Y_2(\tau) \\ -Y_3(\tau) & 2Y_2(\tau) & -Y_1(\tau) \\ -Y_2(\tau) & -Y_1(\tau) & 2Y_3(\tau) \end{pmatrix}. \tag{6}$$

The proportionality factors contain the VEV of the so-called u-type Higgs of the MSSM, the see-saw scale Λ as well as some factors from the Kähler metric which depend on τ and $\bar{\tau}$. Crucially, the Y_i are known modular functions, which can be written as [12]

$$\begin{pmatrix} Y_{1}(\tau) \\ Y_{2}(\tau) \\ Y_{3}(\tau) \end{pmatrix} = \begin{pmatrix} (X_{2}(\tau))^{2} \\ \sqrt{2}X_{1}(\tau)X_{2}(\tau) \\ -(X_{1}(\tau))^{2} \end{pmatrix} \quad \text{where} \begin{cases} X_{1}(\tau) := 3\sqrt{2} \frac{\eta^{3}(3\tau)}{\eta(\tau)} \\ X_{2}(\tau) := -3\frac{\eta^{3}(3\tau)}{\eta(\tau)} - \frac{\eta^{3}(\tau/3)}{\eta(\tau)} \end{cases} . \tag{7}$$

Here, η denotes the Dedekind η -function. For largish $t := \text{Im } \tau$,

$$|X_1(it)| \xrightarrow{t \text{ large}} |X_1(i)| e^{-2\pi(t-1)/3} \simeq 0.5234 e^{-2\pi(t-1)/3}$$
, (8)

$$|X_2(it)| \xrightarrow{t \text{ large}} 1$$
. (9)

This reveals that the entries of *Y* in (7) can accommodate hierarchies. Somewhat ironically, though, it turns out that the best-fit point of this model is at $\tau \simeq i$. In this model, 3 parameters predict 9 observables,

$$\begin{array}{c}
\Lambda \\
\text{Re } \tau \\
\text{Im } \tau
\end{array}
\right\} \xrightarrow{\text{predict}}
\begin{cases}
3 \text{ mass eigenvalues } m_i, \\
3 \text{ mixing angles } \theta_{ij}, \\
3 \text{ phases (1 Dirac & 2 Majorana)}.
\end{cases} (10)$$

This means that this model is dramatically overconstrained. It is still remarkably consistent with data, and can be made fully consistent by introducing one, admittedly ad hoc, parameter [17].

4 Modular invariant holomorphic observables

The predictive power of modular flavor symmetries relies on three key ingredients,

- modular covariance/invariance (cf. Section 3),
 - $\widehat{\mathbf{Z}}$ meromorphy, i.e. the couplings do not depend on $\bar{\tau}$, and
 - \bowtie the couplings remain finite for all values of τ .

In the respective models, these requirements fix the superpotential. It turns out that, in specific cases, they directly fix observables [18]. This is true e.g. in the Feruglio model [6], where [18]

$$I_{ij}(\tau) := \frac{m_{ii}^{(\nu)}(\tau, \bar{\tau}) \, m_{jj}^{(\nu)}(\tau, \bar{\tau})}{\left(m_{ij}^{(\nu)}(\tau, \bar{\tau})\right)^2} = \frac{\left(\mathcal{M}_{\nu}(\tau)\right)_{ii} \left(\mathcal{M}_{\nu}(\tau)\right)_{jj}}{\left(\mathcal{M}_{\nu}(\tau)\right)_{ij}^2} \tag{11}$$

with $m^{(v)}$ from (6) are meromorphic modular invariant functions, i.e. fulfill requirements and \mathfrak{F} . The I_{ij} are known to be renormalization group (RG) invariant [19]. At the same time, the I_{ij} are functions that depend only on the physical neutrino masses, mixing angles and phases. Two combinations of the I_{ij} even fulfill \mathfrak{S} . This allows one to obtain a large amount of information of the model directly from the theory of modular forms.

 $^{^2}$ Generally, modular symmetries give rise to linearly realized symmetries at certain critical points [13–15], which lead to characteristic patterns of mass matrices. Small departures of τ from the critical values can be related to supersymmetry (SUSY) breaking [16].

5 Origin of modular flavor symmetries

While modular flavor symmetries have been proposed in the bottom-up approach, Yukawa couplings are known to be modular forms in certain explicit string models [20–23], including the so-called toroidal orbifolds (see [24] for a recent review). The latter are, as their name suggests, based on tori and, therefore, it is not too surprising that they exhibit modular symmetries. Crucially, this means that modular flavor symmetries emerge from a consistent scheme of quantum gravity. More recently, these symmetries have been explored further, which has led to the so-called eclectic scheme [25–28], which unifies modular symmetries, traditional family symmetries, R symmetries and outer automorphisms like the \mathcal{CP} transformation under one umbrella. Modular flavor symmetries arise also from magnetized tori [29–31].

Given that modular flavor symmetries have a ultraviolet (UV) completion, it is instructive to compare bottom-up and top-down models [32]. Both approaches use VVMF to describe fermion masses. In explicit string models, the modular weights of quarks and leptons (and other matter fields) are small, indeed often fractional, whereas in phenomenological models they are sometimes taken to be rather large, thus allowing for several contractions to contribute to the fermion mass matrices. This introduces several freely adjusted parameters in the bottom-up approach, giving rise to models that reproduce data, something that this is much harder in the top-down approach.

6 Some open questions and challenges

6.1 Problems with kinetic terms

While modular flavor symmetries constrain the superpotential very strongly, the non-holomorphic Kähler potential is far less restricted. In fact, the effective field theory (EFT) expansion of the Kähler potential [33]

$$K = \alpha_0 \left(-i \tau + i \bar{\tau} \right)^{-1} \left(\overline{L} L \right)_1 + \sum_{k=1}^7 \alpha_k \left(-i \tau + i \bar{\tau} \right) \left(Y L \overline{Y} \overline{L} \right)_{1,k} + \dots$$
 (12)

contains infinitely many terms beyond the so-called minimal term proportional to α_0 . Even moderate values of the $\alpha_{i>0}$ coefficients lead to changes of the predicted values of observables which exceed the experimental error bars by far [33]. It should be stressed that higher-order terms in the Kähler potential are essential for the viability of the model as the soft SUSY breaking terms rely on them. Furthermore, attempts to suppress them completely in UV complete models appear hard, if not impossible [34]. Nonetheless, a proof-of-principle solution to the problem in which the impact of these terms does not exceed the experimental uncertainties exists [35], but more research in this direction will be needed to arrive at a fully convincing picture. Note that these extra terms can also be used in order to improve the fit of a given model [36], though at the expense of introducing more parameters and thus reducing the predictive power of the scheme.

6.2 Is SUSY indispensable?

A key ingredient of modular flavor models so far is holomorphy, (?), which is implemented by imposing SUSY. Given the absence of clear signals for low-energy SUSY, one may ask whether it is possible to define a nonsupersymmetric version of modular flavor symmetries. It has been pointed out [29,31] that in certain scenarios low-energy SUSY may not be crucial to have mass matrices described by modular forms. However, to date there is no complete model illustrating these points.

7 Outlook

Modular flavor symmetries are a rather new and exciting scheme allowing us to tackle the flavor problem. This scheme may constitute the most concrete way of figuring out what completes the SM in the UV. Models in this realm can be highly predictive, and can be tested in the foreseeable future. It remains a challenge for the future to build predictive models which are fully consistent with observation and in which the theoretical uncertainties are smaller than the experimental ones.

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References

- [1] C. D. Froggatt and H. B. Nielsen, "HIERARCHY OF QUARK MASSES, CABIBBO ANGLES AND CP VIOLATION," *Nucl. Phys.* **B147** (1979) 277.
- [2] L. Randall and R. Sundrum, "A Large mass hierarchy from a small extra dimension," *Phys. Rev. Lett.* **83** (1999) 3370–3373, arXiv:hep-ph/9905221.
- [3] D. B. Kaplan and M. Schmaltz, "Flavor unification and discrete nonAbelian symmetries," *Phys. Rev. D* **49** (1994) 3741–3750, arXiv:hep-ph/9311281.
- [4] G. Altarelli and F. Feruglio, "Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions," *Nucl. Phys.* **B720** (2005) 64–88, arXiv:hep-ph/0504165 [hep-ph].
- [5] M. Holthausen and M. A. Schmidt, "Natural Vacuum Alignment from Group Theory: The Minimal Case," *JHEP* **01** (2012) 126, arXiv:1111.1730 [hep-ph].
- [6] F. Feruglio, "Are neutrino masses modular forms?," arXiv:1706.08749 [hep-ph].
- [7] F. Feruglio and A. Romanino, "Lepton flavor symmetries," *Rev. Mod. Phys.* **93** no. 1, (2021) 015007, arXiv:1912.06028 [hep-ph].
- [8] Y. Almumin, M.-C. Chen, M. Cheng, V. Knapp-Perez, Y. Li, A. Mondol, S. Ramos-Sanchez, M. Ratz, and S. Shukla, "Neutrino Flavor Model Building and the Origins of Flavor and Violation," *Universe* 9 no. 12, (2023) 512, arXiv:2204.08668 [hep-ph].
- [9] T. Kobayashi and M. Tanimoto, "Modular flavor symmetric models," 7, 2023. arXiv:2307.03384 [hep-ph].
- [10] G.-J. Ding and S. F. King, "Neutrino Mass and Mixing with Modular Symmetry," arXiv:2311.09282 [hep-ph].
- [11] G.-J. Ding and J. W. F. Valle, "The symmetry approach to quark and lepton masses and mixing," arXiv:2402.16963 [hep-ph].
- [12] X.-G. Liu and G.-J. Ding, "Modular flavor symmetry and vector-valued modular forms," *JHEP* **03** (2022) 123, arXiv:2112.14761 [hep-ph].
- [13] F. Feruglio, "Universal Predictions of Modular Invariant Flavor Models near the Self-Dual Point," *Phys. Rev. Lett.* **130** no. 10, (2023) 101801, arXiv:2211.00659 [hep-ph].
- [14] F. Feruglio, "Fermion masses, critical behavior and universality," *JHEP* **03** (2023) 236, arXiv:2302.11580 [hep-ph].
- [15] G.-J. Ding, F. Feruglio, and X.-G. Liu, "Universal predictions of Siegel modular invariant theories near the fixed points," arXiv:2402.14915 [hep-ph].
- [16] V. Knapp-Perez, X.-G. Liu, H. P. Nilles, S. Ramos-Sanchez, and M. Ratz, "Matter matters in moduli fixing and modular flavor symmetries," *Phys. Lett. B* **844** (2023) 138106, arXiv:2304.14437 [hep-th].
- [17] J. C. Criado and F. Feruglio, "Modular Invariance Faces Precision Neutrino Data," *SciPost Phys.* **5** no. 5, (2018) 042, arXiv:1807.01125 [hep-ph].

- [18] M.-C. Chen, X. Li, X.-G. Liu, O. Medina, and M. Ratz, "Modular invariant holomorphic observables," *Phys. Lett. B* **852** (2024) 138600, arXiv:2401.04738 [hep-ph].
- [19] S. Chang and T.-K. Kuo, "Renormalization invariants of the neutrino mass matrix," *Phys. Rev. D* **66** (2002) 111302, arXiv:hep-ph/0205147.
- [20] E. J. Chun, J. Mas, J. Lauer, and H. P. Nilles, "Duality and Landau–Ginzburg Models," *Phys. Lett. B* 233 (1989) 141–146.
- [21] J. Erler, D. Jungnickel, M. Spaliński, and S. Stieberger, "Higher twisted sector couplings of \mathbb{Z}_N orbifolds," *Nucl. Phys.* **B397** (1993) 379–416, hep-th/9207049.
- [22] F. Quevedo, "Lectures on superstring phenomenology," arXiv:hep-th/9603074 [hep-th]. [AIP Conf. Proc.359,202(1996)].
- [23] S. Groot Nibbelink and P. K. S. Vaudrevange, "T-duality orbifolds of heterotic Narain compactifications," *JHEP* **04** (2017) 030, arXiv:1703.05323 [hep-th].
- [24] S. Ramos-Sanchez and M. Ratz, "Heterotic Orbifold Models," arXiv: 2401.03125 [hep-th].
- [25] A. Baur, H. P. Nilles, A. Trautner, and P. K. S. Vaudrevange, "A String Theory of Flavor and *CP*," *Nucl. Phys. B* **947** (2019) 114737, arXiv:1908.00805 [hep-th].
- [26] H. P. Nilles, S. Ramos–Sánchez, and P. K. S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory I. Basic results," *Phys. Lett. B* **808** (2020) 135615, arXiv:2006.03059 [hep-th].
- [27] H. P. Nilles, S. Ramos–Sánchez, and P. K. S. Vaudrevange, "Eclectic flavor scheme from ten-dimensional string theory II detailed technical analysis," *Nucl. Phys. B* **966** (2021) 115367, arXiv:2010.13798 [hep-th].
- [28] A. Baur, M. Kade, H. P. Nilles, S. Ramos-Sanchez, and P. K. S. Vaudrevange, "The eclectic flavor symmetry of the Z₂ orbifold," *JHEP* **02** (2021) 018, arXiv:2008.07534 [hep-th].
- [29] D. Cremades, L. E. Ibáñez, and F. Marchesano, "Computing Yukawa couplings from magnetized extra dimensions," *JHEP* **05** (2004) 079, arXiv:hep-th/0404229.
- [30] S. Kikuchi, T. Kobayashi, and H. Uchida, "Modular flavor symmetries of three-generation modes on magnetized toroidal orbifolds," arXiv:2101.00826 [hep-th].
- [31] Y. Almumin, M.-C. Chen, V. Knapp-Pérez, S. Ramos-Sánchez, M. Ratz, and S. Shukla, "Metaplectic Flavor Symmetries from Magnetized Tori," *JHEP* **05** (2021) 078, arXiv:2102.11286 [hep-th].
- [32] H. P. Nilles and S. Ramos-Sanchez, "Flavor's Delight," *Entropy* **26** (2024) 355, arXiv:2404.16933 [hep-th].
- [33] M.-C. Chen, S. Ramos-Sánchez, and M. Ratz, "A note on the predictions of models with modular flavor symmetries," *Phys. Lett. B* **801** (2020) 135153, arXiv:1909.06910 [hep-ph].
- [34] S. Kachru, L. McAllister, and R. Sundrum, "Sequestering in String Theory," *JHEP* **10** (2007) 013, arXiv:hep-th/0703105.
- [35] M.-C. Chen, V. Knapp–Pérez, M. Ramos-Hamud, S. Ramos–Sánchez, M. Ratz, and S. Shukla, "Quasi–eclectic modular flavor symmetries," *Phys. Lett. B* **824** (2022) 136843, arXiv:2108.02240 [hep-ph].
- [36] A. Baur, H. P. Nilles, S. Ramos-Sanchez, A. Trautner, and P. K. S. Vaudrevange, "The first string-derived eclectic flavor model with realistic phenomenology," *JHEP* **09** (2022) 224, arXiv:2207.10677 [hep-ph].