Efficient and Adaptive Posterior Sampling Algorithms for Bandits

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Abstract

We study Thompson Sampling-based algorithms for stochastic bandits with bounded rewards. As the existing problem-dependent regret bound for Thompson Sampling with Gaussian priors [Agrawal and Goyal, 2017] is vacuous when $T \leq 288e^{64}$, we derive a more practical bound that tightens the coefficient of the leading term to 1270. Additionally, motivated by large-scale real-world applications that require scalability, adaptive computational resource allocation, and a balance in utility and computation, we propose two parameterized Thompson Sampling-based algorithms: Thompson Sampling with Model Aggregation (TS-MA-α) and Thompson Sampling with Timestamp Duelling (TS-TD-α), where $\alpha \in [0, 1]$ controls the trade-off between utility and computation. Both algorithms achieve $O\left(K \ln^{\alpha+1}(T)/\Delta\right)$ regret bound, where K is the number of arms, T is the finite learning horizon, and Δ denotes the single round performance loss when pulling a sub-optimal arm.

1 INTRODUCTION

We study the learning problem of stochastic multi-armed bandits specified by $(K; p_1, \ldots, p_K)$ where K is the number of arms and p_i is the reward distribution of arm i with its mean denoted by μ_i. In this learning problem, a learner chooses an arm to pull in each round $t = 1, 2, \ldots, T$ without knowledge of the reward distributions. At the end of the round, the learner obtains and observes a reward drawn from the reward distribution associated with the pulled arm. The goal of the learner is to pull arms sequentially to maximize the cumulative reward over T rounds. Since only the pulled arm is observed at the end of each round, the main challenge for solving bandit problems is to balance exploitation and exploration. Exploitation involves pulling arms expected to obtain high rewards based on past experience, whereas exploration involves pulling arms to help the learner better learn the reward distributions or the means of the reward distributions, to make more informed decisions in the future.

Many real-world applications that require this exploitation-vs-exploration balance can be framed as bandit learning problems. For example, in healthcare, bandit algorithms can be used in clinical trials where exploitation is to prescribe known effective treatments and exploration is to prescribe new medicine to discover potentially more effective treatments. In inventory management, bandit algorithms can be used to decide whether to re-order products that sold well (exploitation) or to switch to new products (exploration). In online advertising systems, bandit algorithms can help to decide whether to show users familiar content (exploitation) or to introduce users to new but potentially more interesting content (exploration) to optimize overall satisfaction.

There are two well-studied and empirically successful mechanisms to balance exploitation-vs-exploration in bandit problems. The first approach relies on using the Upper Confidence Bound (UCB) of the estimate of each arm's mean reward [Auer et al., 2002, Audibert et al., 2007, Garivier and Cappé, 2011, Kaufmann et al., 2012a, Lattimore, 2018]. In essence, exploration is thus driven by a bonus term added to the empirical estimate of the mean reward. The bonus depends on the number of observations collected for this arm. The second approach is to add randomness in the estimate of each arm's mean reward [Agrawal and Goyal, 2017, Kaufmann et al., 2012b, Jin et al., 2021, Bian and Jun, 2022, Jin et al., 2022, 2023]. This can be viewed as injecting controllable data-dependent noise into the empirical estimate to ensure exploration. In practice this randomness is added by, in each round, sampling each arm's mean reward estimate from a data-dependent distribution. When this data-dependent distribution is the posterior distribution of arm's mean reward (typically given a simple prior and reward likelihood), this approach is called Thompson Sampling [Kaufmann et al., 2012b, Agrawal and Goyal, 2017], one of the oldest Bayes-inspired randomized learning algorithms.

Thompson Sampling is appealing for practitioners, due to their good empirical performance, and their intuitive exploration mechanism. However, they suffer from two shortcomings. First, the theoretical analysis of Thompson Sampling with Gaussian priors (the most widely applicable, as it applies to any reward distribution with a bounded support) suffers from large constants, making its short learning horizon behavior hard to predict. Second, it requires a new posterior sample *for each arm, at each step*, which can be impractical when the number of arms is large (*e.g.*, in advertising or online recommendation). This paper aims to address these shortcomings with two contributions.

Overview of Contribution 1. We revisit Thompson Sampling with Gaussian priors (Algorithm 2 in Agrawal and Goyal [2017]). Denoting Δ_i the mean reward gap between a sub-optimal arm i and the optimal arm, our new bound is $\sum_{i:\Delta_i>0} 1270 \ln\left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)/\Delta_i + O\left(1/\Delta_i\right)$, which significantly improves the existing $\sum_{i:\Delta_i>0} 288\left(e^{64}+6\right)\ln\left(T\Delta_i^2 + e^{32}\right)/\Delta_i + O\left(1/\Delta_i\right)$ problem-dependent bound [Agrawal and Goyal, 2017]. The improved bound relies on a new technical Lemma 2, which provides an upper bound on the concentration speed of the optimal arm's posterior distribution. Additionally, we justify that Thompson Sampling is also inspired by the philosophy of optimism in the face of uncertainty (OFU).

Overview of Contribution 2. Even when all arms have concentrated posterior distributions, Thompson Sampling-based algorithms [Kaufmann et al., 2012b, Agrawal and Goyal, 2017, Bian and Jun, 2022, Jin et al., 2021, 2022] still draw one random sample for each arm in each round. In total, they require KT data-dependent random samples over T rounds. This exploration mechanism becomes computationally prohibitive at scale, when the number of arms is very large (*e.g.*, in online advertising, recommendation systems). This problem is exacerbated when sampling from the posterior reward distribution is computationally expensive, for example when it relies on Markov chain Monte Carlo (MCMC) methods. This computational cost can prevent the deployment of Thompson Sampling algorithms at scale despite their good empirical performance.

Motivated by the intuition that new samples from the posterior distribution cannot provide much new information when no new data was collected for a given arm, *we propose two parameterized Thompson Sampling-based algorithms that simultaneously achieve good problem-dependent regret bounds and draw fewer data-dependent samples.* Both algorithms achieve a $\sum_{i:\Delta_i>0} O\left(\ln^{\alpha+1}(T)/\Delta_i\right)$ sub-linear regret, but with the potential to draw fewer than KT samples, where parameter $\alpha \in [0,1]$ controls the trade-off between utility (regret) and computation (total number of samples).

Our first algorithm, Thompson Sampling with Model Aggregation (TS-MA-α), is inspired by OFU and motivated by Gaussian anti-concentration bounds. The key idea of TS-MA-α is to draw a batch of $\phi = O\left(T^{0.5(1-\alpha)}\ln^{0.5(3-\alpha)}(T)\right)$ i.i.d. random samples for each arm at once rather than drawing an independent sample in each round. Then, TS-MA-α commits to the best sample among this batch and only draws a new batch when the data-dependent distribution of the given arm changes. TS-MA-α only draws $T\phi$ data-dependent samples and uses $\alpha \in [0,1]$ to control the trade-off between utility (regret) and computation (total number of drawn samples).[1] The key advantage of TS-MA-α is that the total number of drawn samples does not depend on the number of arms K. Since the total amount of data-dependent samples does not depend on K, TS-MA-α is extremely efficient when the number of arms is very large. However, when the number of arms is small, it may not save on computation.

Our second algorithm, Thompson Sampling with Timestamp Duelling (TS-TD-α), is an adaptive switching algorithm between Thompson Sampling and TS-MA-α. Intuitively, we would like to draw data-dependent samples for the optimal arms to preserve good practical performance, and save on data-dependent samples for the sub-optimal arms that are not frequently pulled and hence see little new data. The key idea of TS-TD-α is to aggregate up to ϕ samples for each arm over time instead of drawing them all at once. Given arm i, there are thus two phases in the rounds between two consecutive pulls (the updates of posterior distributions). In the first phase, of up to ϕ rounds, TS-TD-α draws a data-dependent sample for arm i (running Thompson Sampling). The second phase starts after ϕ rounds if arm i wasn't pulled, and TS-TD-α uses the maximum of the ϕ samples accumulated in the first phase (running TS-MA-α). Similarly to TS-MA-α, TS-TD-α also realizes a $\sum_{i:\Delta_i>0} O\left(\ln^{\alpha+1}(T)/\Delta_i\right)$ regret bound, but only draws $\min\left\{O\left(\phi\ln^{1+\alpha}(T)/\Delta_i^2\right), T\phi\right\}$ data-dependent samples in expectation for a sub-optimal arm i. The key advantage of TS-TD-α is that it is extremely efficient when the learning problem has many sub-optimal arms, as it stops drawing data-dependent samples for the sub-optimal arms.

2 LEARNING PROBLEM

As a general rule: in a stochastic multi-arm bandit (MAB) problem, we have an arm set $\mathcal{A}$ of size K and each arm $i \in \mathcal{A}$ is associated with an unknown reward distribution p_i with $[0,1]$ support. At the beginning of each round t, a random reward vector $X(t) := (X_1(t), X_2(t), \ldots, X_K(t))$ is generated, where each $X_i(t) \sim p_i$. Simultaneously, the learner pulls an arm $i_t \in \mathcal{A}$. At the end of the round, the learner observes and obtains $X_{i_t}(t)$, the reward of the pulled arm. Let μ_i denote the mean of reward distribution p_i. Without loss of generality, we assume the first arm is

[1] TS-MA-α only needs to draw T random samples if being implemented efficiently. More details can be found in Section 4.2.

the unique optimal arm, that is, $\mu_1 > \mu_i$ for all $i \neq 1$. Let $\Delta_i := \mu_1 - \mu_i$ denote the mean reward gap, which indicates the single round performance loss when pulling arm i instead of the optimal arm. The goal of the learner is to pull arms sequentially to minimize *pseudo-regret* $\mathcal{R}(T)$, expressed as

$$T\mu_1 - \mathbb{E}\left[\sum_{t=1}^{T} X_{i_t}(t)\right] = \sum_i \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i\right\}\right] \Delta_i,$$

where the expectation is taken over $i_1, i_2, \ldots, i_T$.

3 RELATED WORK

There is a vast literature on MAB algorithms, which we split based on deterministic versus randomized exploration.

Deterministic exploration algorithms (the pulled arms are determined by the history data) are predominantly UCB-based [Auer et al., 2002, Audibert et al., 2007, Garivier and Cappé, 2011, Kaufmann et al., 2012a, Lattimore, 2018]. They rely on the OFU principle, and leverage well-known concentration inequalities to construct a confidence interval for the empirical estimate of each arm. They then use the upper bound of the confidence interval to decide which arm to pull. There also exists non-OFU deterministic algorithms, such as DMED [Honda and Takemura, 2010], IMED [Honda and Takemura, 2015], and an elimination-style algorithm by Auer and Ortner [2010].

Randomized exploration algorithms often rely on Thompson Sampling (TS) [Agrawal and Goyal, 2017, Kaufmann et al., 2012b, Bian and Jun, 2022, Jin et al., 2021, 2022, 2023]. The key idea of TS algorithms is to maintain a posterior on each arm's mean reward updated each time new data is acquired, and use a random sample drawn from this data-dependent distribution to decide which arm to pull. There also are non TS-based MAB algorithms, such as Non-parametric TS [Riou and Honda, 2020] and Generic Dirichlet Sampling [Baudry et al., 2021].[2] All aforementioned algorithms achieve (at least) one of three notions of optimality: asymptotic optimality (i.e., learning horizon T is infinite), worst-case optimality (i.e., minimax optimality), and sub-UCB criteria (problem-dependent optimality with a finite horizon T) [Lattimore, 2018]. This work focuses on deriving problem-dependent regret bounds with a finite learning horizon T to study the sub-UCB property. Problem-dependent regret bounds with a finite learning horizon T can take either the $\sum_{i:\Delta_i>0} \frac{C\ln(T)}{\Delta_i}$ or $\sum_{i:\Delta_i>0} \frac{C\ln(T)\Delta_i}{\mathrm{KL}(\mu_i,\mu_i+\Delta_i)}$ form for the term involving T, where $C \geq 1$ is a universal constant and $\mathrm{KL}(a,b)$ denotes the KL-divergence between two Bernoulli distributions with parameters $a, b \in (0,1)$.

For stochastic bandits with $[0,1]$ rewards, there are two original versions of TS using different conjugate pairs for the prior distribution and reward likelihood function. The first version is TS with Beta priors [Kaufmann et al., 2012b, Agrawal and Goyal, 2017], using a uniform $\mathrm{Beta}(1,1)$ prior and Bernoulli likelihood function. Kaufmann et al. [2012b], Agrawal and Goyal [2017] both present a $\sum_{i:\Delta_i>0} \frac{(1+\varepsilon)\ln(T)\Delta_i}{\mathrm{KL}(\mu_i,\mu_i+\Delta_i)} + C(1/\varepsilon,\mu)$ problem-dependent regret bound, where $\varepsilon > 0$ can be arbitrarily small and $C(1/\varepsilon,\mu)$ is a problem-dependent constant and goes to infinity if $\varepsilon \to 0$. The second version is TS with Gaussian priors [Agrawal and Goyal, 2017], using an improper Gaussian prior $\mathcal{N}(0,\infty)$ and a Gaussian likelihood function. Agrawal and Goyal [2017] derive a $\sum_{i:\Delta_i>0} 288\left(e^{64}+6\right)\ln\left(T\Delta_i^2+e^{32}\right)/\Delta_i + O\left(1/\Delta_i\right)$ problem-dependent regret bound. Since the coefficient for the $\ln(T)$ leading term is at least $288e^{64} \approx 1.8 \times 10^{30}$, this regret bound is vacuous when $T \leq 288e^{64}$. Note that when $T \leq 288e^{64}$, the regret is at most T, and thus, the existing bound does not take the bandit instance $(K;\mu_1,\mu_2,\ldots,\mu_K)$ into account. In this work, we derive a new problem-dependent regret bound with a more acceptable coefficient for the leading term. Our Theorem 1 improves the coefficient for the leading term to 1270. Also, we justify that TS is also an OFU-inspired algorithm.

Very recently, Jin et al. [2023] proposed ϵ-Exploring Thompson Sampling (ϵ-TS), a hybrid algorithm simply blending Thompson Sampling with pure exploitation for some specific reward distributions in the exponential family including Bernoulli, Gaussian, Poisson, and Gamma distributions to save on sampling computation. The core idea of ϵ-TS is, for each individual arm, to apply the normal TS procedure—sample from posterior —with probability $\epsilon \in [1/K, 1]$, and use the empirical mean with probability $1-\epsilon$. It is not hard to see that the expected number of drawn posterior samples is ϵKT. They derive an $O\left(\sqrt{KT\ln(eK\epsilon)}\right)$ worst-case regret bound if we ignore some problem-dependent parameters, and prove that ϵ-TS achieves asymptotic optimality.

The limitation of ϵ-TS is that the derived bounds may not work for other reward distributions. The theoretical analysis of ϵ-TS relies on knowing the true reward distribution family to use the proper conjugate pair. Hence, contrary to TS with Gaussian priors [Agrawal and Goyal, 2017], ϵ-TS (at least its performance guarantees) may not apply to other reward distributions. Further, ϵ-TS is not adaptive and efficient in using sampling resources in the common case of a large number of sub-optimal arms. This is because ϵ-TS simply flips a biased coin to decide whether to draw a posterior sample for each arm or not. It does not take the underlying bandit problem into account. For example, given the same amount of arms, the expected number of posterior samples

[2] In some work, all algorithms with randomized exploration are categorized into Thompson Sampling-based regardless of whether they use data-dependent distributions to model the mean rewards or not. In this work, we only say a randomized algorithm is Thompson Sampling-based if data-dependent distributions are used to model the mean rewards.

stays the same whether the bandit learning problem has one optimal arm or $K-1$ optimal arms. We provide a detailed discussion of ϵ-TS in Section 4.

To use sampling resources more efficiently and adaptively, we propose two TS-based algorithms, TS-MA-α and TS-TD-α, that allocate sampling resources based on whether exploration is necessary. The intuition is that once sub-optimal arm's posterior has concentrated enough, drawing new posterior samples is unnecessary. Both of our proposed algorithms achieve an $O\left(K\ln^{1+\alpha}(T)/\Delta\right)$ problem-dependent regret bound , where $\alpha\in[0,1]$ is a parameter controlling the trade-off between utility (regret) and computation (number of drawn posterior samples).

4 ALGORITHMS

We first present notations specific to this section. Let $n_i(t-1):=\sum_{q=1}^{t-1}\mathbf{1}\left\{i_q=i\right\}$ denote the number of pulls of arm i by the end of round $t-1$. Let $\hat{\mu}_{i,n_i(t-1)}(t-1):=\frac{1}{n_i(t-1)}\sum_{q=1}^{t-1}\mathbf{1}\left\{i_q=i\right\}X_i(q)$ denote the empirical mean of arm i by the end of round $t-1$. We write it as $\hat{\mu}_{i,n_i(t-1)}$ for short. Let $\mathcal{N}\left(\mu,\sigma^2\right)$ denote a Gaussian distribution with mean μ and variance σ^2. Let $c_0:=1/(2\sqrt{2e\pi})$.

4.1 VANILLA THOMPSON SAMPLING

We first revisit Thompson Sampling with Gaussian priors (Algorithm 2 in Agrawal and Goyal [2017]). For ease of presentation, we rename it as Vanilla Thompson Sampling in this work. The idea of Vanilla Thompson Sampling is to draw a posterior sample $\theta_i(t)\sim\mathcal{N}\left(\hat{\mu}_{i,n_i(t-1)},\ 1/n_i(t-1)\right)$ for each arm i in each round t. Then, the learner pulls the arm with the highest posterior sample value, that is, $i_t=\arg\max_{i\in\mathcal{A}}\theta_i(t)$.

Now, we present our new bounds for it.

Theorem 1. *The problem-dependent regret of Algorithm 1 is* $\sum_{i:\Delta_i>0}1270\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)/\Delta_i+182.5/\Delta_i+\Delta_i$. *The worst-case regret is* $O\left(\sqrt{KT\ln(K)}\right)$, *where* $O(\cdot)$ *only hides a universal constant.*

Comparison with bounds in Agrawal and Goyal [2017].

Note that Theorems 1.3 and 1.4 in Agrawal and Goyal [2017] only presented an $O\left(\sqrt{KT\ln(K)}\right)$ worst-case regret upper bound and a matching $\Omega\left(\sqrt{KT\ln(K)}\right)$ regret lower bound for Vanilla Thompson Sampling. They did not state any problem-dependent results in their theorems. We compare our new results with a problem-dependent regret bound embedded in their proof for Theorem 1.3. In page A.18 in Agrawal and Goyal [2017], they derived a $\sum_{i:\Delta_i>0}288\left(e^{64}+6\right)\ln\left(T\Delta_i^2+e^{32}\right)/\Delta_i+O\left(1/\Delta_i\right)$ regret bound. Note that our improved problem-dependent regret bound implies an improved worst-case regret bound, i.e., our $O(\cdot)$ hides a smaller constant than theirs.

Comparison with ϵ-TS. Note that although ϵ-TS reduces to Thompson Sampling when setting $\epsilon=1$, the authors do not revisit and derive new bounds for Vanilla Thompson Sampling as ϵ-TS only works for reward distributions that have conjugate priors. As presented in Table 1 in Jin et al. [2023], ϵ-TS using Gaussian priors only works for Gaussian rewards, whereas Vanilla Thompson Sampling works for any bounded reward distributions. Vanilla Thompson Sampling simply uses Gaussian distributions to model the means of the underlying reward distributions. The true reward distribution and Gaussian prior does not need to form a conjugate pair in Agrawal and Goyal [2017].

The improvement of our problem-dependent regret bound mainly comes from Lemma 2 below, which improves the results shown in Lemma 2.13 in Agrawal and Goyal [2017] significantly. Let $\mathcal{F}_{t-1}=\left\{i_q,X_{i_q}(q),q=1,2,\ldots,t-1\right\}$ collect all the history information by the end of round $t-1$. Let $L_{1,i}:=4\left(\sqrt{2}+\sqrt{3.5}\right)^2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)/\Delta_i^2$.

Lemma 2. *Let* $\tau_s^{(1)}$ *be the round when the* s*-th pull of the optimal arm* 1 *occurs and* $\theta_{1,s}\sim\mathcal{N}\left(\hat{\mu}_{1,s},\frac{1}{s}\right)$. *Then, we have*

$$\mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s}>\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{\tau_s^{(1)}}\right\}}\right]-1=\begin{cases}29, & \forall s\geq 1,\\ \frac{180}{T\Delta_i^2}, & \forall s\geq L_{1,i}.\end{cases}$$

Discussion of Lemma 2. If event $\theta_{1,s}>\mu_1-\Delta_i/2$ occurs, we can view the optimal arm as having a good posterior sample as compared to sub-optimal arm i. Informally, a good posterior sample of the optimal arm indicates the pull of the optimal arm. The pulls of the optimal arm contribute to reducing the variance of the posterior distribution and the width of the confidence intervals. The value of $\mathbb{E}\left[1/\mathbb{P}\left\{\theta_{1,s}>\mu_1-\frac{\Delta_i}{2}\mid\mathcal{F}_{\tau_s^{(1)}}\right\}\right]-1$ quantifies the expected number of independent draws needed before the optimal arm has a good posterior sample. The first case in Lemma 2 says that even if the optimal arm has not been observed enough, i.e., $s<L_{1,i}$, it takes at most 29 draws in expectation before the optimal arm has a good posterior sample. The second case says that after the optimal arm has been observed enough, i.e., $s\geq L_{1,i}$, the expected number of draws before the optimal arm has a good posterior sample is in the order of $\frac{1}{T\Delta_i^2}$. That is also to say, after the optimal arm has been observed enough, it will be pulled very frequently, which preserves the excellent practical performance of Thompson Sampling. If the learning algorithm draws one random sample in each round, this quantity reflects the concentration speed of the optimal arm's posterior distribution. In Lemma 2.13 of Agrawal and Goyal [2017], the derived upper bound on this quantity is $e^{64}+5$ when $s\geq 1$. Our new bound is 29.

Lemma 2 also shows that Thompson Sampling follows the principle of OFU, as do UCB-based algorithms. Note that in each round, the learning algorithm either pulls the optimal arm or a sub-optimal arm. If the optimal arm is pulled, we do not suffer any performance loss and gain information for the optimal arm. The pull of the optimal arm contributes to being closer to having enough observations of the optimal arm and reducing the variance of the posterior distribution. If the learning algorithm pulls a sub-optimal arm, we attain information from the following two perspectives. First, the pull of the sub-optimal arm contributes to shrinking the width of the confidence interval and reducing the variance of the posterior distribution of the pulled sub-optimal arm. Second, the draw of the posterior sample for the optimal arm still provides some information. From Lemma 2 we know that even if the optimal arm has not been observed enough yet, it takes at most a constant number of i.i.d. draws in expectation before the optimal arm is pulled again.

Algorithm 1 Vanilla Thompson Sampling (Algorithm 2 in Agrawal and Goyal [2017])

Ini: Pull each arm i once to initialize n_i and $\hat{\mu}_{i,n_i}$
for $t = K+1, K+2, \cdots$ **do**
  Draw $\theta_i(t) \sim \mathcal{N}\left(\hat{\mu}_{i,n_i}, 1/n_i\right)$ for all $i \in \mathcal{A}$
  Pull arm $i_t \in \arg\max_{i\in\mathcal{A}} \theta_i(t)$ and observe $X_{i_t}(t)$
  Update n_{i_t} and $\hat{\mu}_{i_t,n_{i_t}}$.
end for

4.2 TS-MA-α

Since Vanilla Thompson Sampling draws a random sample for each arm in each round, the total number of posterior samples is KT and the percentage of posterior samples for the optimal arms is exactly m/K if we have m optimal arms. Inspired by the facts that Gaussian distributions have nice anti-concentration bounds (Fact 6) and exploration is unnecessary once sub-optimal arms have been pulled enough, we propose TS-MA-α (Algorithm 2) to save on some sampling computation and allocate computational resources efficiently. TS-MA-α can be viewed as a randomized and parameterized version of UCB1 of Auer et al. [2002] with parameter α controlling the trade-off between utility (regret) and computation (number of drawn posterior samples).

TS-MA-α draws $\phi = O\left(T^{0.5(1-\alpha)}\ln^{0.5(3-\alpha)}(T)\right)$ i.i.d. random samples at once rather than drawing a fresh sample in each round. Let $\theta_i(t) := \max_{h\in[\phi]} \theta^h_{i,n_i(t-1)}$, where $\theta^h_{i,n_i(t-1)} \sim \mathcal{N}\left(\hat{\mu}_{i,n_i(t-1)}, \ln^\alpha(T)/n_i(t-1)\right)$, be the maximum among all these ϕ random samples. Then, the learner uses the best sample $\theta_i(t)$ in the learning and pulls the arm with the highest $\theta_i(t)$, i.e., $i_t = \arg\max \theta_i(t)$. At the end of round t, the learner updates the statistics of the pulled arm i_t. Since the posterior distribution of arm i_t changes, now, the learner draws a new batch of ϕ i.i.d. random samples for arm i_t according to the updated posterior distribution. Note that all $\theta_i(t)$ are analogous to the upper confidence bounds in UCB-based algorithms. The term model aggregation refers to the fact that we aggregate ϕ posterior samples into one by putting all weight over the best posterior sample.

Algorithm 2 TS-MA-α

1: **Input:** Time horizon T and parameter $\alpha \in [0,1]$
2: **Initialization:** Set $\phi = 2T^{0.5(1-\alpha)}\ln^{0.5(3-\alpha)}(T)/c_0$; For each arm i, pull it once to initialize n_i and $\hat{\mu}_{i,n_i}$, set $\theta_i \leftarrow \max_{h\in[\phi]} \theta^h_{i,n_i}$, where each $\theta^h_{i,n_i} \sim \mathcal{N}\left(\hat{\mu}_{i,n_i}, \ln^\alpha(T)/n_i\right)$
3: **for** $t = K+1, K+2, \cdots, T$ **do**
4: Pull arm $i_t \in \arg\max_{i\in\mathcal{A}} \theta_i$ and observe $X_{i_t}(t)$
5: Update n_{i_t}, $\hat{\mu}_{i_t,n_{i_t}}$, and $\theta_{i_t} \leftarrow \max_{h\in[\phi]} \theta^h_{i_t,n_{i_t}}$, where each $\theta^h_{i_t,n_{i_t}} \sim \mathcal{N}\left(\hat{\mu}_{i_t,n_{i_t}}, \ln^\alpha(T)/n_{i_t}\right)$.
6: **end for**

Now, we present regret bounds for TS-MA-α.

Theorem 3. *The problem-dependent regret of Algorithm 2 is* $\sum_{i:\Delta_i>0} O\left(\ln^{\alpha+1}(T)/\Delta_i\right)$, *where* $\alpha \in [0,1]$. *The worst-case regret is* $O\left(\sqrt{KT\ln^{\alpha+1}(T)}\right)$.

Discussion of TS-MA-α and comparison with ϵ-TS. At first glance, Algorithm 2 seems to draw $T\phi = O\left((T\ln(T))^{0.5(3-\alpha)}\right)$ random samples, but it can be implemented only drawing T random samples. In Line 5 in Algorithm 2, we can directly draw θ_{i_t} from the distribution of the maximum of ϕ i.i.d. Gaussian random variables. TS-MA-α is scalable as the total number of drawn random samples does not depend on K. When setting $\alpha = 1$, TS-MA-α achieves a sub-linear $O\left(K\ln^2(T)/\Delta\right)$ regret, but only draws $O(T\ln(T))$ random samples in total even without using the efficient implementation. TS-MA-α is quite efficient when $K \gg O(\ln(T))$. When setting $\alpha = 0$, TS-MA-α achieves the optimal $O\left(K\ln(T)/\Delta\right)$ problem-dependent regret. In addition, TS-MA-α is minimax optimal up to an extra $\sqrt{\ln^{\alpha+1}(T)}$ factor. Furthermore, TS-MA-α works for any reward distribution with bounded support, whereas ϵ-TS only works for some special reward distributions including Gaussian, Bernoulli, Poisson, and Gamma. As compared to ϵ-TS, our proposed TS-MA-α saves on drawing random samples from sub-optimal arms as the expected number of random samples drawn for a sub-optimal arm i is at most $O\left(\phi\ln^{1+\alpha}(T)/\Delta_i^2\right)$, whereas ϵ-TS draws exactly ϵT random samples in expectation for any sub-optimal arm.

Now, we compare our theoretical results with ϵ-TS for the Bernoulli reward case as Bernoulli distributions have bounded support. Theorem 2.1 of Jin et al. [2023] states that the worst-case regret of ϵ-TS is $O\left(\sqrt{KT\log(eK\epsilon)}\right)$ and ϵ-TS is asymptotically optimal. Since they do not present

any finite-time problem-dependent results, we compare TS-MA-α to some inferred problem-dependent regret bounds based on their regret analysis. From their theoretical analysis, the optimal $O\left(\sum_{i:\Delta_i>0}\frac{\ln(T\Delta_i^2)}{\Delta_i}\right)$ problem-dependent regret bound can be inferred by tuning ϵ as a constant. However, this tuning sacrifices the worst-case regret guarantee as ϵ-TS is only worst-case (minimax) optimal when setting $\epsilon = O(1/K)$. In addition, this tuning results in $O(KT)$ posterior samples in total, which may not be more efficient than TS-MA-α.

4.3 TS-TD-α

Algorithm 3 TS-TD-α

1: **Input:** Time horizon T and parameter $\alpha \in [0,1]$
2: **Initialization:** Set $\phi = 2T^{0.5(1-\alpha)}\ln^{0.5(3-\alpha)}(T)/c_0$; For each arm i, pull it once to initialize n_i and $\hat{\mu}_{i,n_i}$, set used sample counter $h_i \leftarrow 0$ and $\psi_i \leftarrow 0$
3: **for** $t = K+1, K+2, \cdots, T$ **do**
4: **for** $i \in \mathcal{A}$ **do**
5: **if** $h_i \leq \phi - 1$ **then**
6: Draw $\theta_i(t) \sim \mathcal{N}\left(\hat{\mu}_{i,n_i}, \frac{\ln^{\alpha}(T)}{n_i}\right)$ %Phase I
 Set $h_i \leftarrow h_i + 1$ and $\psi_i \leftarrow \max\{\psi_i, \theta_i(t)\}$
7: **else**
8: Set $\theta_i(t) \leftarrow \psi_i$ %Phase II
9: **end if**
10: **end for**
11: Pull arm $i_t \in \arg\max_{i\in\mathcal{A}}\theta_i(t)$ and observe $X_{i_t}(t)$
12: Update n_{i_t}, $\hat{\mu}_{i_t,n_{i_t}}$; Reset $h_{i_t} \leftarrow 0$ and $\psi_{i_t} \leftarrow 0$.
13: **end for**

Since TS-MA-α (Algorithm 2) draws $T\phi$ random samples if not being implemented efficiently, it may not save on posterior sampling computation as compared to Vanilla Thompson Sampling (Algorithm 1) for some α, K and T. To further reduce computation, we propose TS-TD-α (Algorithm 3), an adaptive switching algorithm between Algorithm 1 and Algorithm 2 with minor modifications. The intuition behind TS-TD-α is to keep drawing random samples for optimal arms, whereas saving on sampling for sub-optimal arms. The key reason for introducing Thompson Sampling (drawing posterior samples in each round) is, as already discussed in Section 4.1, optimal arms need certain number of draws to have concentrated posterior distributions. On the other hand, we would like to avoid drawing random samples for the sub-optimal arms. As already discussed in Section 4.2, once sub-optimal arms have been observed enough, it is not necessary to draw posterior samples to further explore them.

TS-TD-α allocates $\phi = O\left(T^{0.5(1-\alpha)}\ln^{0.5(3-\alpha)}(T)\right)$ sampling budget for each pull of arm i and separates all rounds between two consecutive pulls of arm i into two phases. Note that the posterior distributions for all the rounds between the two pulls stay the same as there is no new observation from arm i. Drawing posterior samples is only allowed in the first phase (Vanilla Thompson Sampling), whereas the best posterior sample in the first phase will be used among all rounds in the second phase (TS-MA-α). Note that the second phase does not draw any new posterior samples for the given arm i. Since an optimal arm is pulled very frequently, the number of rounds between two consecutive pulls is unlikely to be greater than ϕ, that is, optimal arms mostly stay in the first phase (Line 6). On the other hand, sub-optimal arms are unlikely to be pulled after they have been pulled sufficiently, that is, sub-optimal arms mostly stay in the second phase (Line 8).

Now, we present regret bounds for TS-TD-α.

Theorem 4. *The problem-dependent regret of Algorithm 3 is* $\sum_{i:\Delta_i>0} O\left(\ln^{\alpha+1}(T)/\Delta_i\right)$, *where* $\alpha \in [0,1]$. *The worst-case regret is* $O\left(\sqrt{KT\ln^{\alpha+1}(T)}\right)$.

Discussion of TS-TD-α and comparison with ϵ-TS. When T is large enough, for a sub-optimal arm i, TS-TD-α draws $O\left(T^{0.5(1-\alpha)}\ln^{0.5(\alpha+5)}(T)/\Delta_i^2\right)$ posterior samples in expectation. When setting $\alpha = 0$, TS-TD-α draws $O\left(\sqrt{T}\ln^{2.5}(T)/\Delta_i^2\right)$ posterior samples and achieves the optimal $O\left(K\ln(T)/\Delta\right)$ regret. When setting $\alpha = 1$, TS-TD-α draws $O\left(\ln^3(T)/\Delta_i^2\right)$ posterior samples and achieves an $O\left(K\ln^2(T)/\Delta\right)$ regret. Still, parameter α is used to control the trade-off between utility and sampling computation. The expected number of posterior samples in TS-TD-α is $\sum_{i:\Delta_i>0} O\left(T^{0.5(1-\alpha)}\ln^{0.5(\alpha+5)}(T)/\Delta_i^2\right) + mT$, where m is the total number of optimal arms. When setting a reasonable α, for example, $\alpha > 0.5$, the total number of drawn posterior samples in TS-TD-α for the sub-optimal arms is negligible. In contrast, ϵ-TS draws ϵKT posterior samples in expectation.

Although TS-TD-α and ϵ-TS are both hybrid learning algorithms, fundamentally, they are very different in allocating sampling resources and TS-TD-α is extremely efficient when the number of sub-optimal arms is large. For each arm regardless of being sub-optimal or optimal, ϵ-TS simply flips a biased coin with parameter ϵ to decide whether to draw a posterior sample. It does not consider the situation whether the given arm needs exploration or not. In a sharp contrast, TS-TD-α decides whether to draw posterior samples based on the need for exploration. It draws posterior samples more frequently for good arms, whereas avoiding drawing posterior samples for sub-optimal arms. The advantage of this sophisticated hybrid approach is more sampling resources are allocated to the optimal arms.

5 EXPERIMENTAL RESULTS

In this section, we conduct experiments to compare our TS-MA-α and TS-TD-α with the following state-of-the-art algorithms which pull arm $i_t \in \arg\max_{i\in\mathcal{A}} a_i(t)$, where $a_i(t)$ is the index defined as follows for those algorithms. Note that except Vanilla TS (Gaussian priors) and ϵ-TS (Gaussian priors), all the other compared algorithms are proved to be asymptotically optimal, i.e., achieving a $\sum_{i:\Delta_i>0} \frac{\Delta_i \ln(T)}{\mathrm{KL}(\mu_i,\mu_1)} + o(\ln(T))$ regret when T is sufficiently large. We would like to emphasize that the goal of this work is not to show that our proposed algorithms can empirically perform better than the state-of-the-art algorithms. Instead, we would like to show that designing computationally efficient algorithms from another angles is possible.

- Vanilla TS (Gaussian priors) (Algorithm 1): $a_i(t) \sim \mathcal{N}\left(\hat{\mu}_{i,n_i}, 1/n_i\right)$.
- Vanilla TS (Beta priors) [Agrawal and Goyal, 2017]: $a_i(t) \sim \text{Beta}\left(\hat{\mu}_{i,n_i} \cdot n_i + 1, (1-\hat{\mu}_{i,n_i}) \cdot n_i + 1\right)$.
- ϵ-TS (Gaussian priors) [Jin et al., 2023]: with probability ϵ, draw $a_i(t) \sim \mathcal{N}\left(\hat{\mu}_{i,n_i}, 1/n_i\right)$, whereas with probability $1-\epsilon$, use $a_i(t) = \hat{\mu}_{i,n_i}$. Since ϵ can be any value in $[1/K, 1]$, we choose $\epsilon = \left\{1/K, 1/\sqrt{K}, 1\right\}$. Note that for the choice of $\epsilon = 1$, ϵ-TS (Gaussian priors) boils down to Vanilla TS (Gaussian priors).
- ϵ-TS (Beta priors) [Jin et al., 2023]: with probability ϵ, draw $a_i(t) \sim \text{Beta}\left(\hat{\mu}_{i,n_i} \cdot n_i + 1, (1-\hat{\mu}_{i,n_i}) \cdot n_i + 1\right)$, whereas with probability $1-\epsilon$, use $a_i(t) = \hat{\mu}_{i,n_i}$.
- ExpTS $^+$ [Jin et al., 2022]: with probability $1/K$, draw $a_i(t) \sim P\left(\hat{\mu}_{i,n_i}, n_i\right)$, where $P\left(\mu, n_i\right)$ is a sampling distribution with CDF defined as:

$$F(x) = \begin{cases} 1 - 1/2e^{-(n_i-1)\cdot \mathrm{KL}(a,x)}, & x \geq a, \\ 1/2e^{-(n_i-1)\cdot \mathrm{KL}(a,x)}, & x \leq a, \end{cases}$$

  whereas with probability $1-1/K$, use $a_i(t) = \hat{\mu}_{i,n_i}$.
- KL-UCB++ [Ménard and Garivier, 2017]: $a_i(t) = \sup\left\{ a : \mathrm{KL}\left(\hat{\mu}_{i,n_i}, a\right) \leq \frac{\overline{\ln}\left(\frac{T\left(\overline{\ln}^2\left(\frac{T}{Kn_i}\right)+1\right)}{Kn_i}\right)}{n_i} \right\}$, where $\overline{\ln}(\cdot) := \max\{0, \cdot\}$.
- Similar to Jin et al. [2023], we also plot the asymptotic $\sum_{i:\Delta_i>0} \frac{\Delta_i \ln(T)}{\mathrm{KL}(\mu_i,\mu_1)} + o(\ln(T))$ regret lower bound to study the optimal regret growth rate.

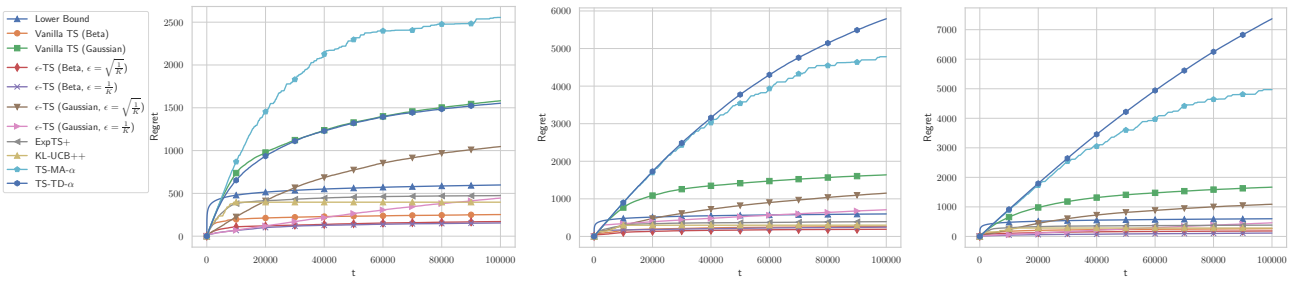
We have in total $K = 20$ arms and each arm is associated with a Bernoulli reward distribution. The mean rewards of the optimal arms are set to 0.9 and the mean rewards of the sub-optimal arms are set to 0.8, i.e., $\Delta_i = 0.1$ for all the sub-optimal arms. We set $\alpha = \{0.0, 0.8, 0.9, 1.0\}$.

Discussions for the empirical performances. We set $T = 10^5$. Fig. 1 reports the results of the regret for all the compared algorithms. Note that when $\alpha = 0$, our proposed TS-TD-α boils down to Vanilla TS (Gaussian priors). It is not surprising that our proposed algorithms cannot beat the state-of-the-art algorithms due to the following reasons. First, all the state-of-the-art algorithms except Vanilla TS (Gaussian priors) and ϵ-TS (Gaussian priors)[3] have been proved to achieve asymptotic optimality, whereas we only prove that our proposed algorithms are problem-dependent optimal. They may not be asymptotically optimal. Second, our proposed algorithms are developed based on Gaussian priors instead of Beta priors, but for bounded reward distributions. Empirically, less exploration leads to better performances. Beta posteriors are in favour of reducing exploration as they have bounded support, and on some occasions, Beta posterior has a much smaller variance than a Gaussian posterior given arm i's statistics $\hat{\mu}_{i,n_i}$ and n_i. Smaller variance makes the learning algorithm explore less.

Fig. 2 reports the total number of drawn posterior samples in TS-TD-α with different values of α to show the adaptivity and efficiency in drawing posterior samples. Note that $T = 10^6$ and $K = 20$. From the results, we can see that when $\alpha = 0$, TS-TD-α basically is Vanilla TS (Gaussian priors). Regardless of the number of optimal arms, the total number of drawn posterior samples is $KT = 2 \times 10^7$. However, for other α value, e.g., $\alpha = 0.8$, given K, the total number of drawn posterior samples decreases with the decrease of the number of the optimal arms. For example, when we have $K/2 = 10$ optimal arms, the total number of posterior samples drawn is around 1.0×10^7 (the blue bars). In contrast, if we only have one optimal arm, the total number of posterior samples drawn is around 0.25×10^7 (the green bars). These experimental results demonstrate that TS-TD-α itself is able to figure out how to use sampling resources wisely and is extremely efficient when the number of sub-optimal arms is very large.

Fig. 3 reports the percentage of data-dependent samples drawn in TS-MA-α for the optimal arms with different numbers of optimal arms and different values of α. This is also to study the adaptive property in drawing data-dependent samples. From the results we can see that almost all the data-dependent samples are for optimal arms (more than 90%) and given K and α, the percentage increases with the increase of the number of the optimal arms. For example, when $\alpha = 0.8$, if there is only one optimal arm, the percentage is 94% (the blue bar), whereas the percentage is around 97% (the green bar) when the number of optimal arms is $K/2 = 10$.

[3] ϵ-TS (Gaussian priors) for Bernoulli rewards may not have theoretical guarantees as Jin et al. [2023] only provide theoretical guarantee for Gaussian priors for Gaussian rewards.



(a) $\alpha = 0$ (b) $\alpha = 0.8$ (c) $\alpha = 1$

Figure 1: Comparison between different algorithms for 20 arms with one optimal arm.

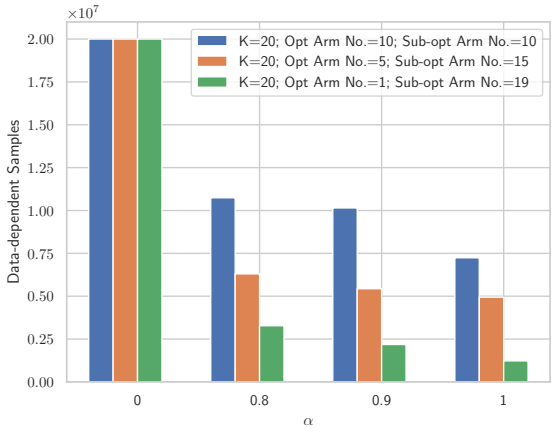


Figure 2: Total number of drawn posterior samples in TS-TD-α.

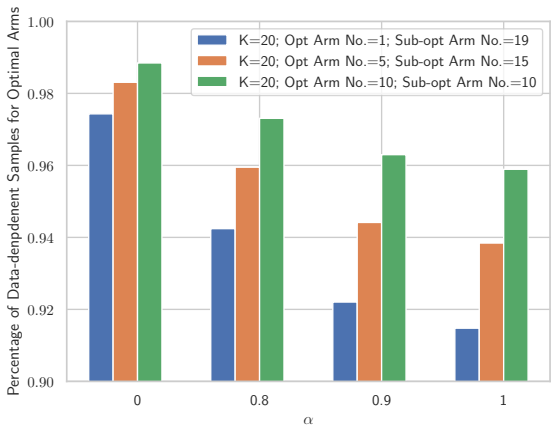


Figure 3: Percentage of data-dependent samples drawn for optimal arms in TS-Ma-α.

6 DISCUSSION AND FUTURE WORK

In this work, we have tackled Thompson Sampling-based algorithms from balancing utility and computation. When $\alpha = 0$, both of our proposed algorithms have the optimal $O(K\ln(T)/\Delta)$ regret bound. The key advantage of TS-MA-α is the scalability, whereas the key advantage of TS-TD-α is the expected number of drawn posterior samples for a sub-optimal arm is $\widetilde{O}\left(\sqrt{T}/\Delta^2\right)$, a rate in the order of $\sqrt{T}$ instead of the classical linear rate in T. When $\alpha = 1$, although TS-TD-α is sub-optimal, the expected number of drawn posterior samples for a sub-optimal arm is further reduced to $O\left(\ln^3(T)/\Delta^2\right)$, a rate only polylogarithmic in T. We believe it is worthwhile to study the fundamental reasons resulting in the savings on posterior samples. We conjecture that it is at the cost of sacrificing learning algorithms' anytime property and the worst-case regret bounds. Note that Vanilla Thompson Sampling is anytime and has an $O\left(\sqrt{KT\ln(K)}\right)$ worst-case regret bound, whereas our proposed algorithms are not anytime and only have an $O\left(\sqrt{KT\ln(T)}\right)$ worst-case regret bound when setting $\alpha = 0$. The conjecture motivates making our proposed algorithms anytime as one of the future work. Since ϵ-TS from Jin et al. [2023] has demonstrated excellent practical performance for some reward distributions in exponential family, another future work is to study the possibility to generalize ϵ-TS to other reward distributions, for example, bounded reward distributions or Sub-Gaussian reward distributions. Jin et al. [2023] proved that ϵ-TS (setting $\epsilon = 1$) with Gaussian priors and rewards is simultaneously minimax optimal and asymptotically optimal, whereas ϵ-TS (setting ϵ as a constant) is simultaneously Sub-UCB, asymptotically optimal, and minimax optimal up to an extra $\sqrt{\ln(K)}$ factor. So far, Gaussian priors for bounded rewards can only be Sub-UCB and minimax optimal up to an extra $\sqrt{\ln(K)}$ factor (our Theorem 1). It would be interesting to see whether Gaussian priors for bounded rewards can achieve Sub-UCB and asymptotic optimality. We are not optimistic about it and conjecture that Gaussian priors for bounded rewards can never achieve asymptotic optimality.

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Efficient and Adaptive Posterior Sampling Algorithms for Bandits (Supplementary Material)

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The appendix is organized as follows.

1. Motivations for our proposed learning algorithms are provided in Appendix A;
2. Useful facts related to this work are provided in Appendix B;
3. Proofs for Theorem 1 are presented in Appendix C;
4. Proofs for Theorem 3 are presented in Appendix D;
5. Proofs for Theorem 4 are presented in Appendix E;
6. Proofs for the worst-case regret bounds are presented in Appendix F .

A MOTIVATIONS

Most bandit learning algorithms only focus on sample efficiency, but for large-scale applications, algorithms' *computational efficiency* and *algorithmic scalability* also play important roles. A learning algorithm that uses fewer posterior samples is computationally efficient because drawing posterior samples and processing the generated samples both consume computational resources. Algorithmic scalability is crucial because scalable algorithms can function steadily when the volume of data is increasing. Take online advertising systems for instance. A platform often receives a changing volume of interactions from users and needs to adjust the decisions frequently to capture users' dynamic preferences to maximize the overall click rate. An algorithm with good scalability can maintain a steady response speed when the volume of user interactions grows. Typically, more computation brings better performance. A good learning algorithm should also be able to balance the trade-off between performance and computational cost. We show, both theoretically and empirically, that our proposed algorithms do not sacrifice a lot in terms of accumulating rewards but consume fewer computational resources. We introduce parameter $\alpha \in [0, 1]$, which allows practitioners to control this trade-off and personalize their algorithms based on their priorities and preferences. We would like to emphasize that although our proposed learning algorithms use fewer posterior samples, provably, the exploration within the learning algorithms is still sufficient. This is because our proposed algorithms adaptively allocate sufficient computation to the optimal arms over time and save on computation from the sub-optimal arms. In the example of online advertising systems, our proposed algorithms can allocate more computational resources to process the portion of data believed to bring good performance to the platform instead of distributing all the available computational resources evenly.

B USEFUL FACTS

Fact 5. *Let $X_1, X_2, \ldots, X_n$ be independent random variables with support $[0, 1]$. Let $\mu_{1:n} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, for any $z > 0$, we have*

$$\mathbb{P}\left\{|\mu_{1:n} - \mathbb{E}[\mu_{1:n}]| \geq z\sqrt{\frac{1}{n}}\right\} \leq 2e^{-2z^2} \quad . \tag{1}$$

Fact 6. *(Concentration and anti-concentration bounds of Gaussian distributions). For a Gaussian distributed random variable Z with mean μ and variance σ^2, for any $z > 0$, we have*

$$\mathbb{P}\{Z > \mu + z\sigma\} \leq \frac{1}{2}e^{-\frac{z^2}{2}}, \quad \mathbb{P}\{Z < \mu - z\sigma\} \leq \frac{1}{2}e^{-\frac{z^2}{2}} \quad , \tag{2}$$

and

$$\mathbb{P}\{Z > \mu + z\sigma\} \geq \frac{1}{\sqrt{2\pi}}\frac{z}{z^2+1}e^{-\frac{z^2}{2}} \quad . \tag{3}$$

Fact 7. *For any fixed $T > e^3$, for any $\alpha \in [0,1]$, we have*

$$\ln^{1-\alpha}(T) \leq (1-\alpha)\ln(T) + 1 \quad . \tag{4}$$

Proof. Let $f(\alpha) = (1-\alpha)\ln(T) + 1 - \ln^{1-\alpha}(T)$. Then, we have

$$f'(\alpha) = -\ln(T) + \ln^{1-\alpha}(T)\ln(\ln(T)) \quad . \tag{5}$$

It is not hard to verify that $f'\left(\frac{\ln(\ln(\ln(T)))}{\ln(\ln(T))}\right) = 0$, and, when $\alpha \in \left[0, \frac{\ln(\ln(\ln(T)))}{\ln(\ln(T))}\right]$, we have $f'(\alpha) \geq 0$, and when $\alpha \in \left[\frac{\ln(\ln(\ln(T)))}{\ln(\ln(T))}, 1\right]$, we have $f'(\alpha) \leq 0$. We also have $f(0) = 1$ and $f(1) = 0$. Therefore, for any $\alpha \in \left[0, \frac{\ln(\ln(\ln(T)))}{\ln(\ln(T))}\right]$, we have $f(\alpha) \geq 1$, and, any $\alpha \in \left[\frac{\ln(\ln(\ln(T)))}{\ln(\ln(T))}, 1\right]$, we have $f(\alpha) \geq 0$. □

C PROOFS FOR THEOREM 1

In this appendix, we present proofs for Lemma 2 and the full proofs for Theorem 1.

C.1 PROOFS FOR LEMMA 2

Re-statement of Lemma 2. Let $L_{1,i} = 4\left(\sqrt{2}+\sqrt{3.5}\right)^2 \ln\left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)/\Delta_i^2$. Let $\tau_s^{(1)}$ be the round when the s-th pull of the optimal arm 1 occurs and $\theta_{1,s} \sim \mathcal{N}\left(\hat{\mu}_{1,s}, \frac{1}{s}\right)$. Then, we have

$$\mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} \mid \mathcal{F}_{\tau_s^{(1)}}\right\}} - 1\right] = \begin{cases} 29, & \forall s \geq 1, \\ \frac{180}{T\Delta_i^2}, & \forall s \geq L_{1,i}. \end{cases}$$

Proof. of Lemma 2: We have two results stated in Lemma 2. The purpose of the first result is to show that even if the optimal arm has not been pulled enough, the regret caused by the underestimation of the optimal arm can still be well controlled. The second result states that after the optimal arm 1 has been pulled enough, the regret caused by the underestimation of the optimal arm is very small.

Similar to the proof of Lemma 2.13 in Agrawal and Goyal [2017], we introduce a geometric random variable in the proof. For any integer $s \geq 1$, we let $\mathcal{G}_{1,s}$ be a geometric random variable denoting the number of consecutive independent trials until event $\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2}$ occurs. Then, we have

$$\mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} \mid \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] = \mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} \mid \hat{\mu}_{1,s}\right\}}\right] = \mathbb{E}\left[\mathbb{E}\left[\mathcal{G}_{1,s} \mid \hat{\mu}_{1,s}\right]\right] = \mathbb{E}\left[\mathcal{G}_{1,s}\right]. \tag{6}$$

To upper bound $\mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s}>\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{\tau_s^{(1)}}\right\}}\right]$, it is sufficient to upper bound $\mathbb{E}\left[\mathcal{G}_{1,s}\right]$. To upper bound $\mathbb{E}\left[\mathcal{G}_{1,s}\right]$, we use the definition of expectation when the random variable has non-negative integers as support. We have

$$\mathbb{E}\left[\mathcal{G}_{1,s}\right] \quad = \quad \sum_{r=0}^{\infty}\mathbb{P}\left\{\mathcal{G}_{1,s}>r\right\} \quad = \quad \sum_{r=0}^{\infty}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right] \quad . \tag{7}$$

For any $s\geq 1$, we claim

$$\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right] \quad \leq \quad \begin{cases} 1, & r\in[0,12] \quad , \\ e^{-\sqrt{\frac{r}{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}+\frac{1}{r}, & r\in[13,100] \quad , \\ e^{-\sqrt{\frac{r}{3\pi}}}+r^{-\frac{4}{3}}, & r\geq 101 \quad . \end{cases} \tag{8}$$

For any $s\geq L_{1,i}=\frac{4\left(\sqrt{2}+\sqrt{3.5}\right)^2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{\Delta_i^2}$, we claim

$$\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right] \quad \leq \quad \begin{cases} 1, & r=0 \quad , \\ \frac{1}{r^2\left(T\Delta_i^2\right)}+\frac{0.5^r}{\left(T\Delta_i^2\right)^r}, & r\in\left[1,\left\lfloor\left(T\Delta_i^2+100^{\frac{1}{3}}\right)^3\right\rfloor\right] \quad , \\ e^{-\sqrt{\frac{r}{3\pi}}}+r^{-\frac{4}{3}}, & r\geq\left\lfloor\left(T\Delta_i^2+100^{\frac{1}{3}}\right)^3\right\rfloor+1 \quad . \end{cases} \tag{9}$$

The proofs for the results shown in (8) and (9) are deferred to the end of this session.

With (8) in hand, for any fixed integer $s\geq 1$, we have

$$\begin{aligned} & \mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s}>\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{\tau_s^{(1)}}\right\}}-1\right] \\ = \quad & \mathbb{E}\left[\mathcal{G}_{1,s}\right]-1 \\ = \quad & \sum_{r=0}^{\infty}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right]-1 \\ = \quad & \sum_{r=0}^{12}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right]+\sum_{r=13}^{100}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right]+\sum_{r=101}^{\infty}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right]-1 \\ \leq \quad & 13+\sum_{r=13}^{100}\left(e^{-\sqrt{\frac{r}{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}+\frac{1}{r}\right)+\sum_{r=101}^{\infty}\left(e^{-\sqrt{\frac{r}{3\pi}}}+\frac{1}{r^{\frac{4}{3}}}\right)-1 \\ \leq \quad & 12+\int_{12}^{100}\left(e^{-\sqrt{\frac{r}{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}+\frac{1}{r}\right)dr+\int_{100}^{\infty}\left(e^{-\sqrt{\frac{r}{3\pi}}}+\frac{1}{r^{\frac{4}{3}}}\right)dr \\ \leq \quad & 12+10.44+2.13+3.1+0.65 \\ \leq \quad & 29 \quad , \end{aligned} \tag{10}$$

which concludes the proof for the first stated result in Lemma 2.

With (9), for any fixed integer $s \geq L_{1,i}$, we have

$$
\begin{aligned}
& \mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}} - 1\right] \\
= & \sum_{r=0}^{\infty} \mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] - 1 \\
\leq & 1 + \sum_{r=1}^{\left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor} \mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] + \sum_{r=\left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor + 1}^{\infty} \mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] - 1 \\
\leq & \sum_{r=1}^{\left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor} \left(\frac{1}{r^2\left(T\Delta_i^2\right)} + \frac{0.5^r}{\left(T\Delta_i^2\right)^r}\right) + \sum_{r=\left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor + 1}^{\infty} \left(e^{-\sqrt{\frac{r}{3\pi}}} + r^{-\frac{4}{3}}\right) \\
\leq & \frac{1}{T\Delta_i^2} + \frac{0.5}{T\Delta_i^2} + \int_1^{+\infty} \left(\frac{1}{r^2\left(T\Delta_i^2\right)} + \frac{1}{\left(2T\Delta_i^2\right)^r}\right) dr + \int_{(T\Delta_i^2)^2}^{\infty} e^{-\frac{\sqrt{r}}{\sqrt{3\pi}}} dr + \int_{(T\Delta_i^2)^3}^{\infty} r^{-\frac{4}{3}} dr \\
\leq^{(a)} & \frac{1}{T\Delta_i^2} + \frac{0.5}{T\Delta_i^2} + \frac{1}{T\Delta_i^2} + \frac{0.5}{T\Delta_i^2} + \frac{6 \cdot 3\pi \cdot \sqrt{3\pi}}{T\Delta_i^2} + \frac{3}{T\Delta_i^2} \\
\leq & \frac{180}{T\Delta_i^2} \quad ,
\end{aligned}
\tag{11}
$$

which concludes the proof for the second stated result in Lemma 2.

Inequality (a) uses the fact that, for any $a, b > 0$, we have $\int_b^{+\infty} e^{-a\sqrt{x}} dx = \frac{2\sqrt{b}}{ae^{a\sqrt{b}}} + \frac{2}{a^2 e^{a\sqrt{b}}} \leq \frac{2\sqrt{b}}{a \cdot \left(1 + a\sqrt{b} + \frac{1}{2}a^2 b\right)} + \frac{2}{a^2 \cdot \left(1 + a\sqrt{b} + \frac{1}{2}a^2 b\right)} \leq \frac{4}{a^3\sqrt{b}} + \frac{2}{a^3\sqrt{b}} = \frac{6}{a^3\sqrt{b}}$, where the first inequality uses $e^x \geq 1 + x + \frac{1}{2}x^2$.

Now, we present the proofs for the results shown in (8) and (9).

Proofs for (8). We express the LHS in (8) as

$$
\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] = 1 - \mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} \leq r \mid \hat{\mu}_{1,s}\right\}\right] = 1 - \underbrace{\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\mathcal{G}_{1,s} \leq r\right\} \mid \hat{\mu}_{1,s}\right]\right]}_{=:\gamma} \quad . \tag{12}
$$

Our goal is to construct a lower bound for γ. Let $\theta_{1,s}^h$ for all $h \in [r]$ be i.i.d. random variables according to $\mathcal{N}\left(\hat{\mu}_{1,s}, \frac{1}{s}\right)$.

When $r \in [0, 12]$, the proof is trivial as $\gamma \geq 0$. Then, we have $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] \leq 1$.

When $r \in [13, 100]$, we introduce $z = \sqrt{\frac{1}{2}\ln(r)} > 0$ and have

$$
\begin{aligned}
\gamma &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\mathcal{G}_{1,s} \leq r\right\} \mid \hat{\mu}_{1,s}\right]\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\max_{h \in [r]} \theta_{1,s}^h > \mu_1 - \frac{\Delta_i}{2}\right\} \mid \hat{\mu}_{1,s}\right]\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \geq \mu_1 - \tfrac{\Delta_i}{2}\right\} \mathbf{1}\left\{\max_{h \in [r]} \theta_{1,s}^h > \hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}}\right\} \mid \hat{\mu}_{1,s}\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}\left\{\hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \geq \mu_1 - \tfrac{\Delta_i}{2}\right\} \cdot \underbrace{\mathbb{P}\left\{\max_{h \in [r]} \theta_{1,s}^h > \hat{\mu}_{1,s} + z\sqrt{\frac{1}{s}} \mid \hat{\mu}_{1,s}\right\}}_{=:\beta}\right] \quad .
\end{aligned}
\tag{13}
$$

We construct a lower bound for β and have

$$\begin{aligned}
\beta &= \mathbb{P}\left\{\max_{h\in[r]}\theta^h_{1,s} > \hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \mid \hat{\mu}_{1,s}\right\} \\
&= 1 - \prod_{h\in[r]}\left(1 - \mathbb{P}\left\{\theta^h_{1,s} > \hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \mid \hat{\mu}_{1,s}\right\}\right) \\
&= 1 - \left(1 - \mathbb{P}\left\{\theta_{1,s} > \hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \mid \hat{\mu}_{1,s}\right\}\right)^r \\
&\geq^{(a)} 1 - \left(1 - \tfrac{1}{\sqrt{2\pi}}\tfrac{\sqrt{\frac{1}{2}\ln(r)}}{\frac{1}{2}\ln(r)+1}e^{-\frac{1}{4}\ln(r)}\right)^r \\
&\geq^{(b)} 1 - e^{-r\cdot\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\frac{1}{2}\ln(r)}}{\frac{1}{2}\ln(r)+1}\cdot r^{-\frac{1}{4}}} \\
&\geq^{(c)} 1 - e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\frac{1}{2}\ln(r)}}{\frac{1}{2}\ln(r)+\frac{1}{\ln(13)}\ln(r)}\cdot r^{\frac{1}{4}}} \\
&= 1 - e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}\frac{\sqrt{r^{\frac{1}{2}}}}{\sqrt{\ln(r)}}} \\
&\geq^{(d)} 1 - e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}} \quad .
\end{aligned} \tag{14}$$

Inequalities (a) and (b) use anti-concentration bounds of Gaussian distributions (Fact 6) and the fact that $e^{-x} \geq 1 - x$. Inequalities (c) and (d) use the facts that when $r \geq 13$, we have $1 \leq \frac{\ln(r)}{\ln(13)}$ and $\sqrt{r^{\frac{1}{2}}} \geq \sqrt{\ln(r)}$.

Now, we have

$$\begin{aligned}
\gamma &\geq \left(1 - e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}\right)\cdot\mathbb{P}\left\{\hat{\mu}_{1,s} + \sqrt{\tfrac{1}{2}\ln(r)}\sqrt{\tfrac{1}{s}} \geq \mu_1 - \tfrac{\Delta_i}{2}\right\} \\
&\geq \left(1 - e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}\right)\cdot\mathbb{P}\left\{\hat{\mu}_{1,s} + \sqrt{\tfrac{1}{2}\ln(r)}\sqrt{\tfrac{1}{s}} \geq \mu_1\right\} \\
&\geq^{(a)} \left(1 - e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}\right)\cdot\left(1 - \tfrac{1}{r}\right) \\
&\geq 1 - \left(e^{-\sqrt{\frac{r}{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}} + \tfrac{1}{r}\right) \quad ,
\end{aligned} \tag{15}$$

where (a) uses Hoeffding's inequality (Fact 5).

Plugging the lower bound of γ into (12) gives $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] \leq e^{-\sqrt{\frac{r}{\pi}}\frac{\ln(13)}{\ln(13)+2}} + \frac{1}{r}$.

When $r \geq 101$, we introduce $z = \sqrt{\frac{2}{3}\ln(r)} > 0$. We still construct the lower bound of γ as

$$\begin{aligned}
\gamma &\geq \mathbb{E}\left[\mathbf{1}\left\{\hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \geq \mu_1 - \tfrac{\Delta_i}{2}\right\}\cdot\mathbb{P}\left\{\max_{h\in[r]}\theta^h_{1,s} > \hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \mid \hat{\mu}_{1,s}\right\}\right] \\
&\geq \left(1 - \left(1 - \tfrac{1}{\sqrt{2\pi}}\tfrac{z}{z^2+1}e^{-0.5z^2}\right)^r\right)\cdot\mathbb{P}\left\{\hat{\mu}_{1,s} + z\sqrt{\tfrac{1}{s}} \geq \mu_1 - \tfrac{\Delta_i}{2}\right\} \\
&\geq \left(1 - \underbrace{e^{-r\cdot\frac{1}{\sqrt{2\pi}}\frac{z}{z^2+1}e^{-0.5z^2}}}_{=:f(r)}\right)\cdot\mathbb{P}\left\{\hat{\mu}_{1,s} + \sqrt{\tfrac{2\ln(r)}{3}}\sqrt{\tfrac{1}{s}} \geq \mu_1\right\} \\
&\geq \left(1 - e^{-\sqrt{\frac{r}{3\pi}}}\right)\cdot\left(1 - \tfrac{1}{r^{\frac{4}{3}}}\right) \\
&\geq 1 - \left(e^{-\sqrt{\frac{r}{3\pi}}} + \tfrac{1}{r^{\frac{4}{3}}}\right) \quad .
\end{aligned} \tag{16}$$

The third inequality in (16) uses the fact that

$$\begin{aligned}
f(r) &= e^{-r\cdot\frac{1}{\sqrt{2\pi}}\frac{z}{z^2+1}e^{-0.5z^2}} \\
&= e^{-r\cdot\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\frac{\ln r}{1.5}}}{\frac{\ln r}{1.5}+1}e^{-0.5\cdot\frac{\ln r}{1.5}}} \\
&\leq^{(a)} e^{-\frac{1}{\sqrt{2\pi}}\frac{\sqrt{1.5\ln r}}{\ln r + 0.5\ln r}\cdot r^{\frac{2}{3}}} \\
&= e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{3\pi}}\sqrt{\frac{r^{\frac{1}{3}}}{\ln r}}} \\
&\leq^{(b)} e^{-\sqrt{\frac{r}{3\pi}}} \quad ,
\end{aligned} \tag{17}$$

where inequality (a) and (b) use the facts that when $r \geq 100$, we have $1.5 < 0.5 \ln r$ and $\sqrt{\frac{r^{\frac{1}{3}}}{\ln r}} \geq 1$.

Plugging the lower bound of γ into (12) gives $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] \leq e^{-\sqrt{\frac{r}{3\pi}}} + \frac{1}{r^{\frac{1}{3}}}$.

Proofs for (9). We still express the LHS in (9) as

$$\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] = 1 - \mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} \leq r \mid \hat{\mu}_{1,s}\right\}\right] = 1 - \underbrace{\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\mathcal{G}_{1,s} \leq r\right\} \mid \hat{\mu}_{1,s}\right]\right]}_{=:\gamma} \quad . \tag{18}$$

When $r = 0$, we have $\gamma \geq 0$, which means $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] \leq 1$.

When $r \in \left[1, \left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor\right]$, we define event $\mathcal{E}_{1,s}^{\mu} := \left\{\mu_1 \leq \hat{\mu}_{1,s} + \sqrt{\frac{0.5\ln\left(r^2\left(T\Delta_i^2+100^{\frac{1}{3}}\right)\right)}{s}}\right\}$. Then, we have

$$\begin{aligned}
\gamma &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\mathcal{G}_{1,s} \leq r\right\} \mid \hat{\mu}_{1,s}\right]\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \mu_1 - \tfrac{\Delta_i}{2}\right\} \mid \hat{\mu}_{1,s}\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \mu_1 - \tfrac{\Delta_i}{2}\right\}\right] \\
&= \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \mu_1 - \tfrac{\Delta_i}{2}\right\}\right] + \mathbb{E}\left[\mathbf{1}\left\{\overline{\mathcal{E}_{1,s}^{\mu}}\right\}\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \mu_1 - \tfrac{\Delta_i}{2}\right\}\right] \\
&\geq^{(a)} \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \hat{\mu}_{1,s} + \sqrt{\frac{0.5\ln\left(r^2\left(T\Delta_i^2+100^{\frac{1}{3}}\right)\right)}{s}} - \tfrac{\Delta_i}{2}\right\}\right] \\
&\geq^{(b)} \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \hat{\mu}_{1,s} + \sqrt{\frac{0.5\ln\left(r^2\left(T\Delta_i^2+100^{\frac{1}{3}}\right)\right)}{s}} - \sqrt{\frac{\left(\sqrt{2}+\sqrt{3.5}\right)^2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{s}}\right\}\right] \\
&\geq^{(c)} \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \hat{\mu}_{1,s} + \sqrt{\frac{0.5\ln\left(\left(T\Delta_i^2+100^{\frac{1}{3}}\right)^7\right)}{s}} - \sqrt{\frac{\left(\sqrt{2}+\sqrt{3.5}\right)^2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{s}}\right\}\right] \\
&= \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\mathbf{1}\left\{\max_{h\in[r]} \theta_{1,s}^h > \hat{\mu}_{1,s} - \sqrt{\frac{2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{s}}\right\}\right] \\
&= \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\cdot\mathbb{P}\left\{\max_{h\in[r]} \theta_{1,s}^h > \hat{\mu}_{1,s} - \sqrt{\frac{2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{s}} \mid \hat{\mu}_{1,s}\right\}\right] \\
&= \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\cdot\left(1 - \mathbb{P}\left\{\max_{h\in[r]} \theta_{1,s}^h \leq \hat{\mu}_{1,s} - \sqrt{\frac{2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{s}} \mid \hat{\mu}_{1,s}\right\}\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\cdot\left(1 - \prod_{h\in[r]}\mathbb{P}\left\{\theta_{1,s}^h \leq \hat{\mu}_{1,s} - \sqrt{\frac{2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{s}} \mid \hat{\mu}_{1,s}\right\}\right)\right] \\
&\geq^{(d)} \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\cdot\left(1 - \frac{0.5^r}{\left(T\Delta_i^2+100^{\frac{1}{3}}\right)^r}\right)\right] \\
&= \left(1 - \frac{0.5^r}{\left(T\Delta_i^2+100^{\frac{1}{3}}\right)^r}\right)\cdot\mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_{1,s}^{\mu}\right\}\right] \\
&\geq^{(e)} \left(1 - \frac{0.5^r}{\left(T\Delta_i^2+100^{\frac{1}{3}}\right)^r}\right)\cdot\left(1 - \frac{1}{r^2\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}\right) \\
&\geq 1 - \frac{1}{r^2\left(T\Delta_i^2+100^{\frac{1}{3}}\right)} - \frac{0.5^r}{\left(T\Delta_i^2+100^{\frac{1}{3}}\right)^r} \\
&\geq 1 - \frac{1}{r^2\left(T\Delta_i^2\right)} - \frac{0.5^r}{\left(T\Delta_i^2\right)^r} \quad .
\end{aligned} \tag{19}$$

We now provide a detailed explanation for some key steps in (19). Inequality (a) uses the fact that if event $\mathcal{E}_{1,s}^{\mu}$ is true, we have $\mu_1 \leq \hat{\mu}_{1,s} + \sqrt{\frac{0.5\ln\left(r^2\left(T\Delta_i^2+100^{\frac{1}{3}}\right)\right)}{s}}$. Recall $L_{1,i} = \left\lceil \frac{4\left(\sqrt{2}+\sqrt{3.5}\right)^2 \ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{\Delta_i^2} \right\rceil$. Inequality (b) uses the fact that $\frac{\Delta_i}{2} \geq \sqrt{\frac{4\left(\sqrt{2}+\sqrt{3.5}\right)^2 \ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{4L_1}} \geq \sqrt{\frac{\left(\sqrt{2}+\sqrt{3.5}\right)^2 \ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{s}}$, when $s \geq L_{1,i}$. Recall $1 \leq r \leq \left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor$. Inequality (c) uses the fact that $r^2 \leq \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^6$. Inequality (d) uses Gaussian concentration bounds (Fact 6) and inequality (e) uses Hoeffding's inequality (Fact 5) giving $\mathbb{P}\left\{\mathcal{E}_{1,s}^{\mu}\right\} \geq 1 - \frac{1}{r^2\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}$.

Plugging the lower bound of γ into (18) gives $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] \leq \frac{1}{r^2\left(T\Delta_i^2\right)} + \frac{0.5^r}{\left(T\Delta_i^2\right)^r}$.

When $r \geq \left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor + 1$, we reuse the result shown in (8) directly. Note that we have $r \geq \left\lfloor \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \right\rfloor + 1 \geq \left(T\Delta_i^2 + 100^{\frac{1}{3}}\right)^3 \geq 101$. □

C.2 PROOFS FOR THEOREM 1

For a sub-optimal arm i such that $\Delta_i \leq \sqrt{\frac{1}{T}}$, we have the total regret of pulling this sub-optimal arm i over T rounds is at most $T\Delta_i \leq \sqrt{T} \leq \frac{1}{\Delta_i}$.

For a sub-optimal arm i such that $\Delta_i > \sqrt{\frac{1}{T}}$, we upper bound its expected number of pulls $\mathbb{E}\left[n_i(T)\right]$ by the end of round T.

Let $\bar{\mu}_{i,n_i(t-1)} := \hat{\mu}_{i,n_i(t-1)} + \sqrt{\frac{2\ln\left(T\Delta_i^2\right)}{n_i(t-1)}}$ be the upper confidence bound.

Let $L_i := \left\lceil \frac{4\left(\sqrt{0.5}+\sqrt{2}\right)^2 \ln\left(T\Delta_i^2\right)}{\Delta_i^2} \right\rceil$.

To decompose the regret, we define $\mathcal{E}_i^{\mu}(t-1) := \left\{ \left|\hat{\mu}_{i,n_i(t-1)} - \mu_i\right| \leq \sqrt{\frac{0.5\ln\left(T\Delta_i^2\right)}{n_i(t-1)}} \right\}$ and $\mathcal{E}_i^{\theta}(t) := \left\{\theta_i(t) \leq \bar{\mu}_{i,n_i(t-1)}\right\}$. Then, we have

$$
\begin{aligned}
\mathbb{E}\left[n_i(T)\right] &\leq L_i + \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, n_i(t-1) \geq L_i\right\}\right] \\
&\leq L_i + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^{\theta}(t), \mathcal{E}_i^{\mu}(t-1), n_i(t-1) \geq L_i\right\}\right]}_{=:\omega_1} \\
&+ \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^{\theta}(t)}, n_i(t-1) \geq L_i\right\}\right]}_{=:\omega_2} + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^{\mu}(t-1)}, n_i(t-1) \geq L_i\right\}\right]}_{=:\omega_3} .
\end{aligned} \tag{20}
$$

Term ω_1 is very similar to Lemma 2.14 in Agrawal and Goyal [2017], which is challenging to derive a tighter upper bound. Terms ω_2 (similar to Lemma 2.16 in Agrawal and Goyal [2017]) and ω_3 (similar to Lemma 2.15 in Agrawal and Goyal [2017]) are not difficult to bound. We use Gaussian concentration inequality (Fact 6) for upper bounding term ω_2 and Hoeffding's inequality (Fact 5) for upper bounding terms ω_3. From Lemma 9 and Lemma 10 we have $\omega_2 \leq \frac{0.5}{\Delta_i^2}$ and $\omega_3 \leq \frac{2}{\Delta_i^2}$.

For ω_1, similar to Lemma 2.8 in Agrawal and Goyal [2017], we have our Lemma 8, a lemma that links the probability of pulling a sub-optimal arm i to the probability of pulling the optimal arm 1 in round t by introducing the upper confidence bound $\bar{\mu}_{i,n_i(t-1)}$. Note that both events $\mathcal{E}_i^{\mu}(t-1)$ and $n_i(t-1) \geq L_i$ are determined by the history information. The value

of $\bar{\mu}_{i,n_i(t-1)}$ is determined by the history information. Also, the distributions for $\theta_j(t)$ for all $j \in \mathcal{A}$ are determined by the history information.

Lemma 8. *For any instantiation F_{t-1} of $\mathcal{F}_{t-1}$, we have*

$$\mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^\theta(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \quad \leq \quad \frac{\mathbb{P}\left\{\theta_1(t) \leq \bar{\mu}_{i,n_i(t-1)} | \mathcal{F}_{t-1} = F_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \bar{\mu}_{i,n_i(t-1)} | \mathcal{F}_{t-1} = F_{t-1}\right\}} \cdot \mathbb{E}\left[\mathbf{1}\left\{i_t = 1, \mathcal{E}_i^\theta(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right]. \tag{21}$$

With Lemma 8 in hand, now, we are ready to upper bound term ω_1. We have

$$\begin{aligned}
\omega_1 &= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^\theta(t), \mathcal{E}_i^\mu(t-1), n_i(t-1) \geq L_i\right\}\right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_i^\mu(t-1), n_i(t-1) \geq L_i\right\} \cdot \underbrace{\mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^\theta(t)\right\} \mid \mathcal{F}_{t-1}\right]}_{\text{LHS in Lemma 8}}\right] \\
&\leq \sum_{t=1}^{T} \mathbb{E}\left[\underbrace{\mathbf{1}\left\{\mathcal{E}_i^\mu(t-1), n_i(t-1) \geq L_i\right\} \frac{\mathbb{P}\left\{\theta_1(t) \leq \bar{\mu}_{i,n_i(t-1)} \mid \mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \bar{\mu}_{i,n_i(t-1)} \mid \mathcal{F}_{t-1}\right\}} \mathbb{E}\left[\mathbf{1}\left\{i_t = 1, \mathcal{E}_i^\theta(t)\right\} \mid \mathcal{F}_{t-1}\right]}_{\text{RHS in Lemma 8}}}_{\eta}\right] \\
&\leq^{(a)} \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_1(t) \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}} \cdot \mathbf{1}\left\{i_t = 1\right\} \mid \mathcal{F}_{t-1}\right]\right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_1(t) \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}} \cdot \mathbf{1}\left\{i_t = 1\right\}\right] \quad .
\end{aligned} \tag{22}$$

Inequality (a) in (22) uses the argument that for a specific F_{t-1} of $\mathcal{F}_{t-1}$ such that either event $\mathcal{E}_i^\mu(t-1)$ or $n_i(t-1) \geq L_i$ does not occur, term η in (22) will be 0. Note that for any F_{t-1}, we have $1 < \frac{1}{\mathbb{P}\left\{\theta_1(t) > \bar{\mu}_{i,n_i(t-1)} | \mathcal{F}_{t-1} = F_{t-1}\right\}} < +\infty$. Recall $L_i = \left\lceil \frac{4\left(\sqrt{0.5}+\sqrt{2}\right)^2 \ln\left(T\Delta_i^2\right)}{\Delta_i^2} \right\rceil$. For any specific F_{t-1} such that both events $\mathcal{E}_i^\mu(t-1)$ and $n_i(t-1) \geq L_i$ occur, we have $\bar{\mu}_{i,n_i(t-1)} = \hat{\mu}_{i,n_i(t-1)} + \sqrt{\frac{2\ln\left(T\Delta_i^2\right)}{n_i(t-1)}} \leq \mu_i + \sqrt{\frac{0.5\ln\left(T\Delta_i^2\right)}{n_i(t-1)}} + \sqrt{\frac{2\ln\left(T\Delta_i^2\right)}{n_i(t-1)}} \leq \mu_i + \sqrt{\frac{0.5\ln\left(T\Delta_i^2\right)}{L_i}} + \sqrt{\frac{2\ln\left(T\Delta_i^2\right)}{L_i}} \leq \mu_i + \frac{\Delta_i}{2} = \mu_1 - \frac{\Delta_i}{2}$, which implies $\frac{\mathbb{P}\left\{\theta_1(t) \leq \bar{\mu}_{i,n_i(t-1)} | \mathcal{F}_{t-1} = F_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \bar{\mu}_{i,n_i(t-1)} | \mathcal{F}_{t-1} = F_{t-1}\right\}} \leq \frac{\mathbb{P}\left\{\theta_1(t) \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1} = F_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1} = F_{t-1}\right\}}$.

Now, we partition all T rounds into multiple intervals based on the arrivals of the observations of the optimal arm 1. Let $\tau_s^{(1)}$ be the round when arm 1 is pulled for the s-th time. Note that this partition in time horizon ensures that the posterior distribution of arm 1 stays the same among all the rounds when $t \in \left\{\tau_s^{(1)} + 1, \ldots, \tau_{s+1}^{(1)}\right\}$. Recall $L_{1,i} = \frac{4\left(\sqrt{2}+\sqrt{3.5}\right)^2 \ln\left(T\Delta_i^2\right)}{\Delta_i^2}$ and $\theta_{1,s} \sim \mathcal{N}\left(\hat{\mu}_{1,s}, \frac{1}{s}\right)$. Then, we have

$$\begin{aligned}
\omega_1 &\leq \sum_{t=1}^{T} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_1(t) \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}} \cdot \mathbf{1}\left\{i_t = 1\right\}\right] \\
&\leq \mathbb{E}\left[\sum_{s=1}^{T} \sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}} \frac{\mathbb{P}\left\{\theta_1(t) \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_1(t) > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}} \cdot \mathbf{1}\left\{i_t = 1\right\}\right] \\
&\leq \sum_{s=1}^{T} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_1\left(\tau_{s+1}^{(1)}\right) \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_{s+1}^{(1)}-1}\right\}}{\mathbb{P}\left\{\theta_1\left(\tau_{s+1}^{(1)}\right) > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_{s+1}^{(1)}-1}\right\}}\right] \\
&\leq \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}} - 1\right] + \sum_{s=L_{1,i}+1}^{T} \mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}} - 1\right] \quad .
\end{aligned} \tag{23}$$

With Lemma 2 in hand, we have

$$\begin{aligned}\omega_1 &\leq \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s}>\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{\tau_s(1)}\right\}}-1\right]+\sum_{s=L_{1,i}+1}^{T}\mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s}>\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{\tau_s(1)}\right\}}-1\right] \\ &\leq 29\cdot L_{1,i}+\sum_{s=1}^{T}\frac{180}{T\Delta_i^2} \\ &\leq \frac{116\left(\sqrt{2}+\sqrt{3.5}\right)^2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{\Delta_i^2}+\frac{180}{\Delta_i^2} \quad .\end{aligned} \tag{24}$$

Plugging the upper bounds of ω_1, ω_2, and ω_3 together in (20), we have

$$\begin{aligned}\mathbb{E}\left[n_i(T)\right] &\leq L_i+\omega_1+\omega_2+\omega_3 \\ &\leq \frac{4\left(\sqrt{0.5}+\sqrt{2}\right)^2\ln\left(T\Delta_i^2\right)}{\Delta_i^2}+1+\frac{116\left(\sqrt{2}+\sqrt{3.5}\right)^2\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{\Delta_i^2}+\frac{180}{\Delta_i^2}+\frac{0.5}{\Delta_i^2}+\frac{2}{\Delta_i^2} \\ &\leq \frac{1270\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{\Delta_i^2}+\frac{182.5}{\Delta_i^2}+1 \quad .\end{aligned} \tag{25}$$

which concludes the proof.

Proofs for the worst-case regret bound. We reuse the arguments shown in Agrawal and Goyal [2017]. Let $\Delta_* := e\sqrt{\frac{K\ln K}{T}}$ be the critical gap. The total regret of pulling any sub-optimal arm i such that $\Delta_i < \Delta_*$ is at most $T\cdot\Delta_* = e\sqrt{KT\ln(K)}$. Now, we consider all the remaining sub-optimal arms i with $\Delta_i > \Delta_*$. For such i, the regret is upper bounded by

$$\Delta_i\mathbb{E}\left[n_i(T)\right] \leq \frac{1270\ln\left(T\Delta_i^2+100^{\frac{1}{3}}\right)}{\Delta_i}+\frac{182.5}{\Delta_i}+\Delta_i,$$

which is decreasing in $\Delta_i \in (\frac{e}{\sqrt{T}}, 1]$. Therefore, for every such i, the regret is bounded by $O\left(\sqrt{\frac{T\ln K}{K}}\right)$. Taking a sum over all sub-optimal arms concludes the proofs.

Proof of Lemma 8. The proof is very similar to the proof of Lemma 2.8 in Agrawal and Goyal [2017]. Note that the value of $\bar{\mu}_{i,n_i(t-1)}$ is determined by the history information.

We have the following two pieces of argument.

The first piece is

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\mathcal{E}_i^\theta(t)\right\}\mid\mathcal{F}_{t-1}=F_{t-1}\right] \\ \leq\ &\mathbb{E}\left[\mathbf{1}\left\{\theta_j(t)\leq\bar{\mu}_{i,n_i(t-1)},\forall j\in\mathcal{A}\right\}\mid\mathcal{F}_{t-1}=F_{t-1}\right] \\ =\ &\mathbb{P}\left\{\theta_1(t)\leq\bar{\mu}_{i,n_i(t-1)}\mid\mathcal{F}_{t-1}=F_{t-1}\right\}\underbrace{\mathbb{P}\left\{\theta_j(t)\leq\bar{\mu}_{i,n_i(t-1)},\forall j\in\mathcal{A}\setminus\{1\}\mid\mathcal{F}_{t-1}=F_{t-1}\right\}}_{>0} \quad .\end{aligned} \tag{26}$$

The second piece is

$$\begin{aligned}&\mathbb{E}\left[\mathbf{1}\left\{i_t=1,\mathcal{E}_i^\theta(t)\right\}\mid\mathcal{F}_{t-1}=F_{t-1}\right] \\ \geq\ &\mathbb{E}\left[\mathbf{1}\left\{\theta_1(t)>\bar{\mu}_{i,n_i(t-1)}\geq\theta_j(t),\forall j\in\mathcal{A}\setminus\{1\}\right\}\mid\mathcal{F}_{t-1}=F_{t-1}\right] \\ =\ &\underbrace{\mathbb{P}\left\{\theta_1(t)>\bar{\mu}_{i,n_i(t-1)}\mid\mathcal{F}_{t-1}=F_{t-1}\right\}}_{>0}\underbrace{\mathbb{P}\left\{\theta_j(t)\leq\bar{\mu}_{i,n_i(t-1)},\forall j\in\mathcal{A}\setminus\{1\}\mid\mathcal{F}_{t-1}=F_{t-1}\right\}}_{>0} \quad .\end{aligned} \tag{27}$$

Combining (26) and (27) concludes the proof.

□

Lemma 9. *We have*

$$\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\overline{\mathcal{E}_i^\theta(t)},n_i(t-1)\geq L_i\right\}\right] \leq \frac{0.5}{\Delta_i^2} \quad . \tag{28}$$

Proof of Lemma 9. This lemma is very similar to Lemma 2.16 in Agrawal and Goyal [2017]. Recall event $\mathcal{E}_i^\theta(t) = \left\{\theta_i(t) \le \hat{\mu}_{i,n_i(t-1)} + \sqrt{\frac{2\ln\left(T\Delta_i^2\right)}{n_i(t-1)}}\right\}$. Let $\tau_s^{(i)}$ be the round when arm i is pulled for the s-th time. Now, we partition all T rounds based on the pulls of arm i. We have

$$\begin{aligned}
& \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\theta(t)}, n_i(t-1) \ge L_i\right\}\right] \\
\le & \sum_{s=L_i}^{T} \mathbb{E}\left[\sum_{t=\tau_s^{(i)}+1}^{\tau_{s+1}^{(i)}} \mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\theta(t)}\right\}\right] \\
\le & \sum_{s=L_i}^{T} \mathbb{E}\left[\mathbf{1}\left\{\overline{\mathcal{E}_i^\theta\left(\tau_{s+1}^{(i)}\right)}\right\}\right] \\
= & \sum_{s=L_i}^{T} \mathbb{E}\left[\mathbf{1}\left\{\theta_{i,s} > \hat{\mu}_{i,s} + \sqrt{\frac{2\ln(T\Delta_i^2)}{s}}\right\}\right] \\
= & \sum_{s=L_i}^{T} \mathbb{E}\left[\underbrace{\mathbb{P}\left\{\theta_{i,s} > \hat{\mu}_{i,s} + \sqrt{\frac{2\ln(T\Delta_i^2)}{s}} \mid \hat{\mu}_{1,s}\right\}}_{\text{Fact 6}}\right] \\
\le & \sum_{s=L_i}^{T} \frac{1}{2} e^{-0.5 \cdot 2T\Delta_i^2} \\
\le & \frac{0.5}{\Delta_i^2} \quad ,
\end{aligned} \tag{29}$$

which concludes the proof. □

Lemma 10. *We have*

$$\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\mu(t-1)}, n_i(t-1) \ge L_i\right\}\right] \le \frac{2}{\Delta_i^2} \quad . \tag{30}$$

Proof of Lemma 10. Recall event $\mathcal{E}_i^\mu(t-1) = \left\{\left|\hat{\mu}_{i,n_i(t-1)} - \mu_i\right| \le \sqrt{\frac{0.5\ln\left(T\Delta_i^2\right)}{n_i(t-1)}}\right\}$. Let $\tau_s^{(i)}$ be the round when arm i is pulled for the s-th time. Now, we partition all T rounds based on the pulls of arm i. We have

$$\begin{aligned}
& \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\mu(t-1)}, n_i(t-1) \ge L_i\right\}\right] \\
\le & \sum_{s=L_i}^{T} \mathbb{E}\left[\sum_{t=\tau_s^{(i)}+1}^{\tau_{s+1}^{(i)}} \mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\mu(t-1)}\right\}\right] \\
\le & \sum_{s=L_i}^{T} \mathbb{E}\left[\mathbf{1}\left\{\overline{\mathcal{E}_i^\mu\left(\tau_{s+1}^{(i)} - 1\right)}\right\}\right] \\
= & \sum_{s=L_i}^{T} \underbrace{\mathbb{P}\left\{\left|\hat{\mu}_{i,s} - \mu_i\right| > \sqrt{\frac{0.5\ln\left(T\Delta_i^2\right)}{s}}\right\}}_{\text{Hoeffding's inequality}} \\
\le & \sum_{s=L_i}^{T} \frac{2}{(T\Delta_i^2)} \\
\le & \frac{2}{\Delta_i^2} \quad ,
\end{aligned} \tag{31}$$

which concludes the proof. □

D PROOFS FOR THEOREM 3

Proof of Theorem 3. For a fixed sub-optimal arm $i \in \mathcal{A}$, we upper bound the expected number of pulls $\mathbb{E}\left[n_i(T)\right]$ by the end of round T. Let $L_i := \left\lceil \frac{\left(\sqrt{0.5}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{\Delta_i^2} \right\rceil$. Let $\theta_i(t) := \max_{h\in[\phi]} \theta_{i,n_i(t-1)}^h$ be the maximum of ϕ i.i.d. posterior samples,

where each $\theta^h_{i,n_i(t-1)} \sim \mathcal{N}\left(\hat{\mu}_{i,n_i(t-1)}, \frac{\ln^\alpha(T)}{n_i(t-1)}\right)$. We decompose the regret and have

$$\begin{aligned}
\mathbb{E}\left[n_i(T)\right] &= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i\right\}\right] \\
&\leq L_i + \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, n_i(t-1) \geq L_i\right\}\right] \\
&\leq L_i + \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \theta_i(t) \geq \theta_1(t), n_i(t-1) \geq L_i\right\}\right] \\
&\leq L_i + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \theta_i(t) \geq \mu_i + \Delta_i, n_i(t-1) \geq L_i\right\}\right]}_{=:\omega_1, \text{over-estimation of sub-optimal arm } i} + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) < \mu_1\right\}\right]}_{=:\omega_2, \text{under-estimation of optimal arm 1}} .
\end{aligned} \tag{32}$$

Over-estimation of the sub-optimal arm i. Define $\mathcal{E}_i^\mu(t-1)$ as the event that $\left|\hat{\mu}_{i,n_i(t-1)} - \mu_i\right| \leq \sqrt{\frac{0.5\ln(T)}{n_i(t-1)}}$. Define $\tau_s^{(i)}$ as the round when arm i is pulled for the s-th time. Based on the definition, we know $\hat{\mu}_{i,n_i(t-1)}$ stays the same for all rounds $t \in \left\{\tau_s^{(i)}+1, \ldots, \tau_{s+1}^{(i)}\right\}$. We have

$$\begin{aligned}
\omega_1 &= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \theta_i(t) \geq \mu_i + \Delta_i, n_i(t-1) \geq L_i\right\}\right] \\
&\leq \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^\mu(t-1), \theta_i(t) \geq \mu_i + \Delta_i, n_i(t-1) \geq L_i\right\}\right] + \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\mu(t-1)}, n_i(t-1) \geq L_i\right\}\right] \\
&\leq \sum_{s=L_i}^{T} \mathbb{E}\left[\sum_{t=\tau_s^{(i)}+1}^{\tau_{s+1}^{(i)}} \mathbf{1}\left\{i_t = i, \mathcal{E}_i^\mu(t-1), \theta_i(t) \geq \mu_i + \Delta_i\right\}\right] + \sum_{s=L_i}^{T} \mathbb{E}\left[\sum_{t=\tau_s^{(i)}+1}^{\tau_{s+1}^{(i)}} \mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\mu(t-1)}\right\}\right] \\
&\leq \sum_{s=L_i}^{T} \mathbb{E}\left[\mathbf{1}\left\{\theta_i\left(\tau_{s+1}^{(i)}\right) \geq \mu_i + \Delta_i, \mathcal{E}_i^\mu\left(\tau_{s+1}^{(i)} - 1\right)\right\}\right] + \sum_{s=L_i}^{T} \mathbb{E}\left[\mathbf{1}\left\{\overline{\mathcal{E}_i^\mu\left(\tau_{s+1}^{(i)} - 1\right)}\right\}\right] \\
&= \sum_{s=L_i}^{T} \mathbb{E}\left[\mathbf{1}\left\{\max_{h\in[\phi]} \theta_{i,s}^h \geq \mu_i + \Delta_i, |\hat{\mu}_{1,s} - \mu_i| \leq \sqrt{\frac{0.5\ln(T)}{s}}\right\}\right] + \sum_{s=L_i}^{T} \underbrace{\mathbb{P}\left\{|\hat{\mu}_{1,s} - \mu_i| > \sqrt{\frac{0.5\ln(T)}{s}}\right\}}_{\leq 2e^{-\ln(T)}, \text{ Fact 5}} \\
&\leq^{(a)} \sum_{s=L_i}^{T} \mathbb{P}\left\{\max_{h\in[\phi]} \theta_{i,s}^h \geq \hat{\mu}_{i,s} - \sqrt{\frac{0.5\ln(T)}{s}} + \Delta_i\right\} + \sum_{t=1}^{T} 2e^{-\ln(T)} \\
&\leq \underbrace{\sum_{s=L_i}^{T} \mathbb{P}\left\{\max_{h\in[\phi]} \theta_{i,s}^h \geq \hat{\mu}_{i,s} - \sqrt{\frac{0.5\ln^{1+\alpha}(T)}{s}} + \Delta_i\right\}}_{I_1} + 2 \quad .
\end{aligned} \tag{33}$$

To upper bound term I_1, we recall $L_i = \left\lceil \frac{\left(\sqrt{0.5}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{\Delta_i^2} \right\rceil$. Then, we have $\Delta_i \geq \sqrt{\frac{\left(\sqrt{0.5}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{L_i}} \geq \sqrt{\frac{\left(\sqrt{0.5}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{s}}$ for any $s \geq L_i$.

Now, we have

$$
\begin{aligned}
I_1 &= \mathbb{P}\left\{\max_{h\in[\phi]}\theta_{i,s}^h \geq \hat{\mu}_{1,s} - \sqrt{\tfrac{0.5\ln^{1+\alpha}(T)}{s}} + \Delta_j\right\} \\
&\leq \mathbb{P}\left\{\max_{h\in[\phi]}\theta_{i,s}^h \geq \hat{\mu}_{1,s} - \sqrt{\tfrac{0.5\ln^{1+\alpha}(T)}{s}} + \sqrt{\tfrac{\left(\sqrt{0.5}+\sqrt{5-\alpha}\right)^2\ln^{1+\alpha}(T)}{s}}\right\} \\
&\leq \sum_{h=1}^{\phi}\underbrace{\mathbb{P}\left\{\theta_{i,s}^h \geq \hat{\mu}_{1,s} + \sqrt{(5-\alpha)\ln(T)}\sqrt{\frac{\ln^{\alpha}(T)}{s}}\right\}}_{\text{Fact 6}} \\
&\leq \phi\cdot\tfrac{1}{2T^{2.5-0.5\alpha}} \\
&= \tfrac{2}{c_0}\cdot\ln^{1.5-0.5\alpha}(T)\cdot T^{0.5(1-\alpha)}\cdot\tfrac{1}{2T^{2.5-0.5\alpha}} \\
&\leq \tfrac{1}{c_0}\cdot\tfrac{1}{T}\cdot\tfrac{\ln^{1.5}(T)}{T} \\
&\leq^{(a)} \tfrac{1}{c_0\cdot T} \quad ,
\end{aligned}
\tag{34}
$$

where inequality (a) uses the fact that $T \geq \ln^{1.5}(T)$, when $T \geq 1$.

By plugging the upper bound of I_1 into (33), we have $\omega_1 \leq \frac{1}{c_0} + 2$.

Under-estimation of the optimal arm 1. Define $\mathcal{E}_1^{\mu}(t-1)$ as the event that $\left|\hat{\mu}_{1,n_1(t-1)} - \mu_1\right| \leq \sqrt{\frac{\ln(T)}{n_1(t-1)}}$. Define $\tau_s^{(1)}$ as the round when arm 1 is pulled for the s-th time. Based on the definition, we know $\hat{\mu}_{1,n_1(t-1)}$ stays the same for all rounds $t \in \left\{\tau_s^{(1)}+1,\ldots,\tau_{s+1}^{(1)}\right\}$. Still, Recall $\phi = \frac{2}{c_0}T^{\frac{1-\alpha}{2}}\ln^{\frac{3-\alpha}{2}}(T)$. We have

$$
\begin{aligned}
\omega_2 &= \sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) < \mu_1\right\}\right] \\
&\leq \sum_{s=1}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}}\mathbf{1}\left\{\theta_1(t) < \mu_1\right\}\right] \\
&\leq \sum_{s=1}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}}\mathbf{1}\left\{\theta_1(t) < \mu_1, \mathcal{E}_1^{\mu}(t-1)\right\}\right] + \sum_{s=1}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}}\mathbf{1}\left\{\overline{\mathcal{E}_1^{\mu}(t-1)}\right\}\right] \\
&\leq \sum_{s=1}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}}\mathbf{1}\left\{\theta_1(t) < \hat{\mu}_{1,s} + \sqrt{\tfrac{\ln(T)}{s}}\right\}\right] + \sum_{s=1}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}}\mathbf{1}\left\{|\hat{\mu}_{1,s} - \mu_1| > \sqrt{\tfrac{\ln(T)}{s}}\right\}\right] \\
&\leq \sum_{s=1}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}}\mathbf{1}\left\{\max_{h\in[\phi]}\theta_{1,s}^h < \hat{\mu}_{1,s} + \sqrt{\tfrac{\ln(T)}{s}}\right\}\right] + \sum_{s=1}^{T}\sum_{t=1}^{T}2e^{-2\ln(T)} \\
&\leq \sum_{s=1}^{T}\mathbb{E}\left[\sum_{t=1}^{T}\mathbb{P}\left\{\max_{h\in[\phi]}\theta_{1,s}^h < \hat{\mu}_{1,s} + \sqrt{\tfrac{\ln(T)}{s}} \mid \hat{\mu}_{1,s}\right\}\right] + 2 \\
&\leq \sum_{s=1}^{T}\sum_{t=1}^{T}\mathbb{E}\left[\prod_{h\in[\phi]}\underbrace{\mathbb{P}\left\{\theta_{1,s}^h < \hat{\mu}_{1,s} + \sqrt{\frac{\ln(T)}{s}} \mid \hat{\mu}_{1,s}\right\}}_{\eta}\right] + 2 \\
&\leq^{(a)} \sum_{s=1}^{T}\sum_{t=1}^{T}\left(1 - c_0\cdot\tfrac{1}{\sqrt{\ln^{(1-\alpha)}(T)\cdot T^{(1-\alpha)}}}\right)^{\phi} + 2 \\
&\leq^{(b)} T^2\cdot e^{-c_0\cdot\frac{1}{\sqrt{\ln^{(1-\alpha)}(T)\cdot T^{(1-\alpha)}}}\cdot\phi} + 2 \\
&\leq 3 \quad ,
\end{aligned}
\tag{35}
$$

where inequality (b) uses the fact $1-x \leq e^{-x}$, where $x = c_0 \cdot \frac{1}{\sqrt{\ln^{(1-\alpha)}(T) \cdot T^{(1-\alpha)}}}$. Inequality (a) in (35) uses the following result, which is

$$\begin{aligned}
\eta &= \mathbb{P}\left\{\theta_{1,s}^h < \hat{\mu}_{1,s} + \sqrt{\tfrac{\ln(T)}{s}} \mid \hat{\mu}_{1,s}\right\} \\
&= 1 - \mathbb{P}\left\{\theta_{1,s}^h \geq \hat{\mu}_{1,s} + \sqrt{\ln^{(1-\alpha)}(T)} \cdot \sqrt{\tfrac{\ln^{\alpha}(T)}{s}} \mid \hat{\mu}_{1,s}\right\} \\
&\leq 1 - \tfrac{1}{\sqrt{2\pi}} \cdot \tfrac{\sqrt{\ln^{(1-\alpha)}(T)}}{\ln^{(1-\alpha)}(T)+1} e^{-0.5 \ln^{(1-\alpha)}(T)} \\
&\leq^{(a)} 1 - \tfrac{1}{2\sqrt{2\pi}} \cdot \tfrac{\sqrt{\ln^{(1-\alpha)}(T)}}{\ln^{(1-\alpha)}(T)} e^{-0.5((1-\alpha)\ln(T)+1)} \\
&= 1 - \tfrac{1}{2\sqrt{2e\pi}} \cdot \tfrac{1}{\sqrt{\ln^{(1-\alpha)}(T) \cdot T^{(1-\alpha)}}} \\
&= 1 - c_0 \cdot \tfrac{1}{\sqrt{\ln^{(1-\alpha)}(T) \cdot T^{(1-\alpha)}}} \quad .
\end{aligned} \tag{36}$$

Inequality (a) in (36) uses Fact 7 and $\ln^{(1-\alpha)}(T) \geq 1$.

By plugging the upper bounds of ω_1 and ω_2 into (35), we have

$$\begin{aligned}
\mathbb{E}\left[n_i(T)\right] &\leq L_i + \omega_1 + \omega_2 \\
&\leq \tfrac{\left(\sqrt{0.5}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{\Delta_i^2} + 1 + \tfrac{1}{c_0} + 2 + 3 \\
&\leq \tfrac{\left(\sqrt{0.5}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{\Delta_i^2} + O(1) \\
&\leq \tfrac{\left(\sqrt{0.5}+\sqrt{5}\right)^2 \ln^{1+\alpha}(T)}{\Delta_i^2} + O(1) \quad ,
\end{aligned} \tag{37}$$

which concludes the proof. □

E PROOFS FOR THEOREM 4

For a fixed sub-optimal arm i, we upper bound the expected number of pulls $\mathbb{E}\left[n_i(T)\right]$ by the of round T. Let $\bar{\mu}_i(t-1) := \hat{\mu}_{i,n_i(t-1)} + \sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{n_i(t-1)}}$ be the upper confidence bound. Define $\mathcal{E}_i^{\mu}(t-1) := \left\{\left|\hat{\mu}_{i,n_i(t-1)} - \mu_i\right| \leq \sqrt{\frac{2\ln(T)}{n_i(t-1)}}\right\}$ and $\mathcal{E}_i^{\theta}(t) := \{\theta_i(t) \leq \bar{\mu}_i(t-1)\}$. Note that $\theta_i(t)$ can be either a fresh posterior sample (arm i is in the first phase in round t) or the best posterior sample based on the previously drawn samples (arm i is in the second phase in round t).

Define $\mathcal{E}_1^{\mu}(t-1) := \left\{\left|\hat{\mu}_{1,n_1(t-1)} - \mu_1\right| \leq \sqrt{\frac{2\ln(T)}{n_1(t-1)}}\right\}$.

Let $L_i := \frac{4\left(\sqrt{2}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{\Delta_i^2}$. Now, we are ready to decompose the regret. We have

$$\begin{aligned}
&\mathbb{E}\left[n_i(T)\right] \\
\leq\ & L_i + \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, n_i(t-1) \geq L_i\right\}\right] \\
\leq\ & L_i + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^{\theta}(t), \mathcal{E}_i^{\mu}(t-1), \mathcal{E}_1^{\mu}(t-1), n_i(t-1) \geq L_i\right\}\right]}_{\omega_1} + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^{\theta}(t)}, n_i(t-1) \geq L_i\right\}\right]}_{\omega_2} \\
+\ & \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^{\mu}(t-1)}, n_i(t-1) \geq L_i\right\}\right]}_{\omega_3} + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_1^{\mu}(t-1)}, n_i(t-1) \geq L_i\right\}\right]}_{\omega_4} .
\end{aligned} \tag{38}$$

Upper bounding terms ω_2, ω_3 and ω_4 are not difficult as only concentration bounds are required.

The challenging part is to upper bound term ω_1. We decompose term ω_1 based on whether the optimal arm 1 is running Vanilla Thompson Sampling or not in round t. Define $\mathcal{T}_1(t)$ as the event that the optimal arm 1 in round t is using a fresh posterior sample in the learning. In other words, if event $\mathcal{T}_1(t)$ is true, we have $\theta_1(t) \sim \mathcal{N}\left(\hat{\mu}_{1,n_1(t-1)}, \frac{\ln^{\alpha}(T)}{n_1(t-1)}\right)$. So, $\overline{\mathcal{T}_1(t)}$

denotes the event that the optimal arm 1 is using the best posterior sample among all the previously drawn samples in the learning in round t. We have

$$\begin{aligned}\omega_1 &= \underbrace{\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\mathcal{E}_i^{\theta}(t),\mathcal{T}_1(t),\mathcal{E}_i^{\mu}(t-1),\mathcal{E}_1^{\mu}(t-1),n_i(t-1)\geq L_i\right\}\right]}_{I_1}\\ &+ \underbrace{\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\mathcal{E}_i^{\theta}(t),\overline{\mathcal{T}_1(t)},\mathcal{E}_i^{\mu}(t-1),\mathcal{E}_1^{\mu}(t-1),n_i(t-1)\geq L_i\right\}\right]}_{I_2}.\end{aligned} \tag{39}$$

Let $\mathcal{F}_{t-1}=\{h_1(\tau),h_2(\tau),\ldots,h_K(\tau),i_\tau,X_{i_\tau}(\tau),\forall\tau=1,2,\ldots,t-1\}$ collect all the history information by the end of round t. It collects the number of used posterior samples $h_i(\tau)$ for all $i\in\mathcal{A}$, the index of the pulled arm i_τ, and the observed reward $X_{i_\tau}(\tau)$ for all $\tau=1,2,\ldots,t-1$. Note that detailed information about posterior samples is not collected.

Upper bound I_1. Term I_1 in (39) will use a similar analysis to Vanilla Thompson Sampling, which links the probability of pulling a sub-optimal arm i to the probability of pulling the optimal arm 1. Note that if event $\mathcal{T}_1(t)$ is true, we know for sure the optimal arm 1 is running Vanilla Thompson Sampling in round t. This means the optimal arm 1 will have some chance to be pulled. We formalize this claim in our technical Lemma 11 below.

Lemma 11. *For any instantiation F_{t-1} of $\mathcal{F}_{t-1}$, we have*

$$\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\mathcal{T}_1(t),\mathcal{E}_i^{\theta}(t)\right\}\mid\mathcal{F}_{t-1}=F_{t-1}\right]\leq\frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)}\leq\bar{\mu}_i(t-1)|\mathcal{F}_{t-1}=F_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)}>\bar{\mu}_i(t-1)|\mathcal{F}_{t-1}=F_{t-1}\right\}}\mathbb{E}\left[\mathbf{1}\left\{i_t=1,\mathcal{T}_1(t),\mathcal{E}_i^{\theta}(t)\right\}\mid\mathcal{F}_{t-1}=F_{t-1}\right], \tag{40}$$

where $\theta_{1,n_1(t-1)}\sim\mathcal{N}\left(\hat{\mu}_{1,n_1(t-1)},\ \frac{\ln^{\alpha}(T)}{n_1(t-1)}\right)$.

With Lemma 11 in hand, we now ready to upper bound term I_1. We have

$$\begin{aligned}I_1 &= \sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\mathcal{T}_1(t),\mathcal{E}_1^{\mu}(t-1),\mathcal{E}_i^{\mu}(t-1),\mathcal{E}_i^{\theta}(t),n_i(t-1)\geq L_i\right\}\right]\\ &= \sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_i^{\mu}(t-1),\mathcal{E}_1^{\mu}(t-1),n_i(t-1)\geq L_i\right\}\cdot\underbrace{\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\mathcal{T}_1(t),\mathcal{E}_i^{\theta}(t)\right\}\mid\mathcal{F}_{t-1}\right]}_{\text{LHS of Lemma 11}}\right]\\ &\leq \sum_{t=1}^{T}\mathbb{E}\Bigg[\underbrace{\mathbf{1}\left\{\mathcal{E}_i^{\mu}(t-1),\mathcal{E}_1^{\mu}(t-1),n_i(t-1)\geq L_i\right\}\frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)}\leq\bar{\mu}_i(t-1)\mid\mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)}>\bar{\mu}_i(t-1)\mid\mathcal{F}_{t-1}\right\}}\mathbb{E}\left[\mathbf{1}\left\{i_t=1,\mathcal{T}_1(t),\mathcal{E}_i^{\theta}(t)\right\}\mid\mathcal{F}_{t-1}\right]}_{\eta}\Bigg]\\ &\leq^{(a)} \sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_1^{\mu}(t-1)\right\}\cdot\frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)}\leq\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)}>\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{t-1}\right\}}\cdot\mathbb{E}\left[\mathbf{1}\left\{i_t=1\right\}\mid\mathcal{F}_{t-1}\right]\right]\\ &= \underbrace{\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_1^{\mu}(t-1)\right\}\cdot\frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)}\leq\mu_1-\frac{\Delta_i}{2}\mid\mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)}>\mu_1-\frac{\Delta_i}{2}\mid\mathcal{F}_{t-1}\right\}}\cdot\mathbf{1}\left\{i_t=1\right\}\right]}_{\zeta},\end{aligned} \tag{41}$$

where inequality (a) uses the following arguments. For any specific F_{t-1} such that either event $\mathcal{E}_i^{\mu}(t-1)$ or $n_i(t-1)\geq L_i$ is false, we have $\eta=0$. Note that we have $0<\frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)}\leq\bar{\mu}_i(t-1)|\mathcal{F}_{t-1}=F_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)}>\bar{\mu}_i(t-1)|\mathcal{F}_{t-1}=F_{t-1}\right\}}<+\infty$. Recall $L_i=\frac{4\left(\sqrt{2}+\sqrt{5-\alpha}\right)^2\ln^{1+\alpha}(T)}{\Delta_i^2}$. For any specific F_{t-1} such that both events $\mathcal{E}_i^{\mu}(t-1)$ and $n_i(t-1)\geq L_i$ are true, we have $\bar{\mu}_i(t-1)=\hat{\mu}_{i,n_i(t-1)}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{n_i(t-1)}}\leq\mu_i+\sqrt{\frac{2\ln(T)}{n_i(t-1)}}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{n_i(t-1)}}\leq\mu_i+\sqrt{\frac{\left(\sqrt{2}+\sqrt{5-\alpha}\right)^2\ln^{1+\alpha}(T)}{n_i(t-1)}}\leq$

$\mu_i + \sqrt{\frac{\left(\sqrt{2}+\sqrt{5-\alpha}\right)^2 \ln^{1+\alpha}(T)}{L_i}} \leq \mu_i + \frac{\Delta_i}{2} = \mu_1 - \frac{\Delta_i}{2}$, which implies $\mathbb{P}\left\{\theta_{1,n_1(t-1)} > \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\} \geq \mathbb{P}\left\{\theta_{1,n_1(t-1)} > \mu_1 - \frac{\Delta_i}{2} \mid \mathcal{F}_{t-1} = F_{t-1}\right\}$.

Let $L_{1,i} := \frac{32\ln^{1+\alpha}(T)}{\Delta_i^2}$. Let $\tau_s^{(1)}$ be the round when the optimal arm 1 is pulled for the s-th time. Now, we continue upper bounding ζ by partitioning all T rounds into multiple intervals based on the pulls of the optimal arm 1. We have

$$
\begin{aligned}
\zeta &\leq \sum_{s=1}^{T} \mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}} \mathbf{1}\left\{\mathcal{E}_1^{\mu}(t-1)\right\} \cdot \frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{t-1}\right\}} \cdot \mathbf{1}\left\{i_t = 1\right\}\right] \\
&\leq \sum_{s=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{\mathcal{E}_1^{\mu}\left(\tau_{s+1}^{(1)} - 1\right)\right\} \cdot \frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_{s+1}^{(1)}-1}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_{s+1}^{(1)}-1}\right\}}\right] \\
&\leq \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] + \sum_{s=L_{1,i}+1}^{T} \mathbb{E}\left[\underbrace{\mathbf{1}\left\{\mathcal{E}_1^{\mu}\left(\tau_s^{(1)}\right)\right\} \cdot \frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} \mid \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} \mid \mathcal{F}_{\tau_s^{(1)}}\right\}}}_{\lambda}\right] \\
&\leq^{(b)} \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] + \sum_{s=L_{1,i}+1}^{T} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \leq \hat{\mu}_{1,s} - \sqrt{\frac{2\ln^{1+\alpha}(T)}{s}} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \hat{\mu}_{1,s} - \sqrt{\frac{2\ln^{1+\alpha}(T)}{s}} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] \\
&= \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] + \sum_{s=L_{1,i}+1}^{T} \mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \hat{\mu}_{1,s} - \sqrt{2\ln(T)}\sqrt{\frac{\ln^{\alpha}(T)}{s}} | \mathcal{F}_{\tau_s^{(1)}}\right\}} - 1\right] \\
&\leq^{(c)} \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] + \sum_{s=L_{1,i}+1}^{T} \mathbb{E}\left[\frac{1}{1-\frac{0.5}{T}} - 1\right] \\
&\leq \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] + 1 \quad .
\end{aligned}
\tag{42}
$$

Inequality (b) uses the following arguments. For any specific $F_{\tau_s^{(1)}}$ such that event $\mathcal{E}_1^{\mu}\left(\tau_s^{(1)}\right)$ is false, we have $\lambda = 0$. Note that we have $0 < \frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}} = F_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}} = F_{\tau_s^{(1)}}\right\}} < +\infty$. For any specific $F_{\tau_s^{(1)}}$ such that event $\mathcal{E}_1^{\mu}\left(\tau_s^{(1)}\right)$ is true, we have $\mu_1 - \frac{\Delta_i}{2} \leq \hat{\mu}_{1,s} + \sqrt{\frac{2\ln(T)}{s}} - \frac{\Delta_i}{2} \leq \hat{\mu}_{1,s} + \sqrt{\frac{2\ln^{1+\alpha}(T)}{s}} - \sqrt{\frac{8\ln^{1+\alpha}(T)}{s}} = \hat{\mu}_{1,s} - \sqrt{\frac{2\ln^{1+\alpha}(T)}{s}}$. Note that from $L_{1,i} = \frac{32\ln^{1+\alpha}(T)}{\Delta_i^2}$, we have $\frac{\Delta_i}{2} = \sqrt{\frac{32\ln^{1+\alpha}(T)}{4 \cdot L_{1,i}}} \geq \sqrt{\frac{32\ln^{1+\alpha}(T)}{4 \cdot s}} = \sqrt{\frac{8\ln^{1+\alpha}(T)}{s}}$. Inequality (c) uses Gaussian concentration bound, which gives $\mathbb{P}\left\{\theta_{1,s} > \hat{\mu}_{1,s} - \sqrt{2\ln(T)}\sqrt{\frac{\ln^{\alpha}(T)}{s}} \mid \mathcal{F}_{\tau_s^{(1)}}\right\} \geq 1 - \frac{0.5}{T}$.

To complete the proof for ζ, we claim the following result stated in Lemma 12.

Lemma 12. *Let $\tau_s^{(1)}$ be the round when the s-th pull of the optimal arm 1 occurs and $\theta_{1,s} \sim \mathcal{N}\left(\hat{\mu}_{1,s}, \frac{\ln^{\alpha}(T)}{s}\right)$. Then, for any integer $s \geq 1$, we have*

$$
\mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \leq \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] \leq 29 \quad . \tag{43}
$$

Then, we have

$$I_1 \le \zeta \le \sum_{s=1}^{L_{1,i}} \mathbb{E}\left[\frac{\mathbb{P}\left\{\theta_{1,s} \le \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} | \mathcal{F}_{\tau_s^{(1)}}\right\}}\right] + 1 \le L_{1,i} \cdot 29 + 1 \le O\left(\frac{\ln^{1+\alpha}(T)}{\Delta_i^2}\right) \quad . \tag{44}$$

Upper bound I_2. The regret decomposition for term I_2 in (39) is very similar to the one for TS-MA-α, as the optimal arm 1 is using the best posterior sample in the learning in round t. Here, for the sub-optimal arm i, we do not define an event to indicate whether the sub-optimal arm i is running Vanilla Thompson Sampling in round t or is using the best posterior sample in the learning in round t. This is because once the sub-optimal arm i has been observed enough, it is unlikely that $\theta_i(t)$ will beat $\mu_i + \frac{\Delta_i}{2}$ regardless of whether $\theta_i(t)$ is a fresh posterior sample or the best posterior sample based on the previously drawn samples. We have

$$\begin{aligned}
I_2 &= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^{\theta}(t), \overline{\mathcal{T}_1(t)}, \mathcal{E}_i^{\mu}(t-1), \mathcal{E}_1^{\mu}(t-1), n_i(t-1) \ge L_i\right\}\right] \\
&= \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^{\theta}(t), \theta_i(t) \ge \theta_1(t), \overline{\mathcal{T}_1(t)}, \mathcal{E}_i^{\mu}(t-1), \mathcal{E}_1^{\mu}(t-1), n_i(t-1) \ge L_i\right\}\right] \\
&\le^{(a)} \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{E}_i^{\theta}(t), \theta_i(t) \ge \mu_i + \Delta_i, \mathcal{E}_i^{\mu}(t-1), n_i(t-1) \ge L_i\right\}\right]}_{=0} + \underbrace{\sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) < \mu_1, \overline{\mathcal{T}_1(t)}, \mathcal{E}_1^{\mu}(t-1)\right\}\right]}_{\le 1} \\
&\le 1 \quad .
\end{aligned} \tag{45}$$

We use contradiction to prove that the first term in inequality (a) is 0. If both events $\mathcal{E}_i^{\theta}(t)$, $\mathcal{E}_i^{\mu}(t-1)$ are true and $n_i(t-1) \ge L_i$, we have $\theta_i(t) \le \hat{\mu}_{i,n_i(t-1)} + \sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{n_i(t-1)}} \le \mu_i + \sqrt{\frac{2\ln(T)}{n_i(t-1)}} + \sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{n_i(t-1)}} \le \mu_i + \sqrt{\frac{2\ln(T)}{L_i}} + \sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{L_i}} \le \mu_i + \frac{\Delta_i}{2}$, which yields a contradiction with $\theta_i(t) \ge \mu_i + \Delta_i$.

The second term in (a) reuses a similar analysis as (35). We have

$$\begin{aligned}
& \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) < \mu_1, \overline{\mathcal{T}_1(t)}, \mathcal{E}_1^{\mu}(t-1)\right\}\right] \\
\le\ & \sum_{s=1}^{T} \mathbb{E}\left[\sum_{t=\tau_s^{(1)}+1}^{\tau_{s+1}^{(1)}} \mathbf{1}\left\{\theta_1(t) < \mu_1, \overline{\mathcal{T}_1(t)}, \mathcal{E}_1^{\mu}(t-1)\right\}\right] \\
\le^{(a)}\ & \sum_{s=1}^{T} \mathbb{E}\left[\underbrace{\sum_{t=\tau_s^{(1)}+1}^{\tau_s^{(1)}+\phi} \mathbf{1}\left\{\theta_1(t) < \mu_1, \overline{\mathcal{T}_1(t)}, \mathcal{E}_1^{\mu}(t-1)\right\}}_{=0} + \sum_{t=\tau_s^{(1)}+\phi+1}^{\tau_{s+1}^{(1)}} \mathbf{1}\left\{\theta_1(t) < \mu_1, \overline{\mathcal{T}_1(t)}, \mathcal{E}_1^{\mu}(t-1)\right\}\right] \\
\le\ & \sum_{s=1}^{T} \mathbb{E}\left[\sum_{t=\tau_s^{(1)}+\phi+1}^{\tau_{s+1}^{(1)}} \mathbf{1}\left\{\max_{h\in[\phi]} \theta_{1,s}^h < \hat{\mu}_{1,s} + \sqrt{\frac{\ln(T)}{s}}\right\}\right] \\
\le\ & \sum_{s=1}^{T} \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{P}\left\{\max_{h\in[\phi]} \theta_{1,s}^h < \hat{\mu}_{1,s} + \sqrt{\frac{\ln(T)}{s}} \mid \hat{\mu}_{1,s}\right\}\right] \\
\le\ & \sum_{s=1}^{T} \mathbb{E}\left[\sum_{t=1}^{T} \prod_{h\in[\phi]} \underbrace{\mathbb{P}\left\{\theta_{1,s}^h < \hat{\mu}_{1,s} + \sqrt{\frac{\ln(T)}{s}} \mid \hat{\mu}_{1,s}\right\}}_{\eta}\right] \\
\le\ & \sum_{s=1}^{T} \sum_{t=1}^{T} \left(1 - c_0 \cdot \frac{1}{\sqrt{\ln^{(1-\alpha)}(T) \cdot T^{(1-\alpha)}}}\right)^{\phi} \\
\le\ & T^2 \cdot e^{-c_0 \cdot \frac{1}{\sqrt{\ln^{(1-\alpha)}(T) \cdot T^{(1-\alpha)}}} \cdot \phi} + 2 \\
\le\ & 1 \quad ,
\end{aligned} \tag{46}$$

where inequality (a) uses the fact for each round $t = \tau_s^{(1)} + 1, \ldots, \tau_s^{(1)} + \phi$, the optimal arm 1 is using a fresh posterior sample. So, event $\overline{\mathcal{T}_1(t)}$ cannot occur. The η term in (b) reuses the analysis of (36).

Putting together the upper bounds for I_1 and I_2 into ω_1, we have $\omega_1 \leq O\left(\frac{\ln^{1+\alpha}\ln(T)}{\Delta_i^2}\right)$. Then, we have

$$\mathbb{E}\left[n_i(T)\right] \quad \leq \quad L_i + \omega_1 + \omega_2 + \omega_3 \quad \leq \quad O\left(\frac{\ln^{1+\alpha}\ln(T)}{\Delta_i^2}\right) \quad , \tag{47}$$

which concludes the proof.

Proof of Lemma 11. For any F_{t-1}, we have

$$\begin{aligned}
& \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{T}_1(t), \mathcal{E}_i^{\theta}(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
\leq\ & \mathbb{E}\left[\mathbf{1}\left\{\theta_i(t) \leq \bar{\mu}_i(t-1), \theta_1(t) \leq \bar{\mu}_i(t-1), \theta_j(t) \leq \bar{\mu}_i(t-1), \forall j \in \mathcal{A} \setminus \{i, 1\}, \mathcal{T}_1(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
=\ & \mathbf{1}\left\{\mathcal{T}_1(t)\right\} \cdot \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) \leq \bar{\mu}_i(t-1), \theta_j(t) \leq \bar{\mu}_i(t-1), \forall j \in \mathcal{A} \setminus \{1\}\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
=\ & \mathbf{1}\left\{\mathcal{T}_1(t)\right\} \cdot \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) \leq \bar{\mu}_i(t-1)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \cdot \mathbb{E}\left[\mathbf{1}\left\{\theta_j(t) \leq \bar{\mu}_i(t-1), \forall j \in \mathcal{A} \setminus \{1\}\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right].
\end{aligned} \tag{48}$$

The first inequality uses the fact that whether event $\mathcal{T}_1(t)$ is true is determined by the history information. Note that if $h_1(t-1) = \phi - 1$, then $\mathbf{1}\left\{\mathcal{T}_1(t)\right\} = 1$. If $h_1(t-1) \in \{0, 1, \ldots, \phi - 1\}$, then $\mathbf{1}\left\{\mathcal{T}_1(t)\right\} = 0$.

For any F_{t-1}, we also have

$$\begin{aligned}
& \mathbb{E}\left[\mathbf{1}\left\{i_t = 1, \mathcal{T}_1(t), \mathcal{E}_i^{\theta}(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
\geq\ & \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) > \bar{\mu}_i(t-1) \geq \theta_j(t), \forall j \in \mathcal{A} \setminus \{1\}, \mathcal{T}_1(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
=\ & \mathbf{1}\left\{\mathcal{T}_1(t)\right\} \cdot \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) > \bar{\mu}_i(t-1), \theta_j(t) \leq \bar{\mu}_i(t-1), \forall j \in \mathcal{A} \setminus \{1\}\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
=\ & \mathbf{1}\left\{\mathcal{T}_1(t)\right\} \cdot \mathbb{E}\left[\mathbf{1}\left\{\theta_1(t) > \bar{\mu}_i(t-1)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \cdot \mathbb{E}\left[\mathbf{1}\left\{\theta_j(t) \leq \bar{\mu}_i(t-1), \forall j \in \mathcal{A} \setminus \{1\}\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right].
\end{aligned} \tag{49}$$

We categorize all the possible F_{t-1} into two groups based on whether $\mathbf{1}\left\{\mathcal{T}_1(t)\right\} = 0$ or $\mathbf{1}\left\{\mathcal{T}_1(t)\right\} = 1$.

For any F_{t-1} such that $\mathbf{1}\left\{\mathcal{T}_1(t)\right\} = 0$, combining (48) and (49) yielding

$$\begin{aligned}
& \underbrace{\mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{T}_1(t), \mathcal{E}_i^{\theta}(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right]}_{=0} \\
\leq\ & \underbrace{\frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)} \leq \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)} > \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\}}}_{>0} \cdot \underbrace{\mathbb{E}\left[\mathbf{1}\left\{i_t = 1, \mathcal{T}_1(t), \mathcal{E}_i^{\theta}(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right]}_{=0}.
\end{aligned} \tag{50}$$

Note that the coefficient $\frac{\mathbb{P}\left\{\theta_{1,n_1(t-1)} \leq \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\}}{\mathbb{P}\left\{\theta_{1,n_1(t-1)} > \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\}}$ is always positive.

For any F_{t-1} such that $\mathbf{1}\left\{\mathcal{T}_1(t)\right\} = 1$, from (48) and (49), we have

$$\begin{aligned}
& \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \mathcal{T}_1(t), \mathcal{E}_i^{\theta}(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
\leq\ & \mathbb{P}\left\{\theta_{1,n_1(t-1)} \leq \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\} \cdot \mathbb{P}\left\{\theta_j(t) \leq \bar{\mu}_i(t-1), \forall j \in \mathcal{A} \setminus \{1\} \mid \mathcal{F}_{t-1} = F_{t-1}\right\}
\end{aligned} \tag{51}$$

and

$$\begin{aligned}
& \mathbb{E}\left[\mathbf{1}\left\{i_t = 1, \mathcal{T}_1(t), \mathcal{E}_i^{\theta}(t)\right\} \mid \mathcal{F}_{t-1} = F_{t-1}\right] \\
\geq\ & \underbrace{\mathbb{P}\left\{\theta_{1,n_1(t-1)} > \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\}}_{>0} \cdot \mathbb{P}\left\{\theta_j(t) \leq \bar{\mu}_i(t-1), \forall j \in \mathcal{A} \setminus \{1\} \mid \mathcal{F}_{t-1} = F_{t-1}\right\}.
\end{aligned} \tag{52}$$

Combining (51) and (52) concludes the proof. Note that $\mathbb{P}\left\{\theta_{1,n_1(t-1)} \leq \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\}$ and $\mathbb{P}_{t-1}\left\{\theta_{1,n_1(t-1)} > \bar{\mu}_i(t-1) \mid \mathcal{F}_{t-1} = F_{t-1}\right\}$ are both positive. □

Proof of Lemma 12. The proof will be very similar to the proofs for Lemma 2. Still, we let $\mathcal{G}_{1,s}$ be a geometric random variable denoting the number of consecutive independent trials until event $\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2}$ occurs. Then, we have

$\mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s}>\mu_1-\frac{\Delta_i}{2}|\mathcal{F}_{\tau_s^{(1)}}\right\}}-1\right]=\mathbb{E}\left[\mathcal{G}_{1,s}\right]-1$. For a fixed $s\geq 1$, we use the definition of expectation and have

$$\mathbb{E}\left[\mathcal{G}_{1,s}\right] \quad = \quad \sum_{r=0}^{\infty}\mathbb{P}\left\{\mathcal{G}_{1,s}>r\right\} \quad = \quad \sum_{r=0}^{\infty}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right] \quad . \tag{53}$$

Now, we claim the following results. For any $s\geq 1$, we have

$$\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right] \quad \leq \quad \begin{cases} 1, & r\in[0,12] \quad , \\ e^{-\frac{\ln(13)}{\ln(13)+2}\sqrt{\frac{r}{\pi}}}+r^{-1}, & r\in[13,100] \quad , \\ e^{-\sqrt{\frac{r}{3\pi}}}+r^{-\frac{4}{3}}, & r\geq 101 \quad . \end{cases} \tag{54}$$

To prove the results shown in (54), we express the LHS in (54) as

$$\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right] \quad = \quad 1-\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}\leq r\mid\hat{\mu}_{1,s}\right\}\right] \quad = \quad 1-\underbrace{\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\mathcal{G}_{1,s}\leq r\right\}\mid\hat{\mu}_{1,s}\right]\right]}_{=:\gamma} \quad . \tag{55}$$

Now, we construct lower bounds for γ for three cases.

When integer $r\in[0,12]$, the proof is trivial as $\gamma\geq 0$, which implies $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right]\leq 1$.

When integer $r\in[13,100]$, we introduce $z=\sqrt{0.5\ln(r)}$ and have

$$\begin{aligned}
\gamma \quad &= \quad \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\mathcal{G}_{1,s}\leq r\right\}\mid\hat{\mu}_{1,s}\right]\right] \\
&\geq \quad \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\max_{h\in[r]}\theta_{1,s}^h>\mu_1-\frac{\Delta_i}{2}\right\}\mid\hat{\mu}_{1,s}\right]\right] \\
&\geq \quad \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\hat{\mu}_{1,s}+z\sqrt{\frac{\ln^\alpha(T)}{s}}\geq\mu_1-\frac{\Delta_i}{2}\right\}\mathbf{1}\left\{\max_{h\in[r]}\theta_{1,s}^h>\hat{\mu}_{1,s}+z\sqrt{\frac{\ln^\alpha(T)}{s}}\right\}\mid\hat{\mu}_{1,s}\right]\right] \\
&= \quad \mathbb{E}\left[\mathbf{1}\left\{\hat{\mu}_{1,s}+z\sqrt{\frac{\ln^\alpha(T)}{s}}\geq\mu_1-\frac{\Delta_i}{2}\right\}\cdot\mathbb{P}\left\{\max_{h\in[r]}\theta_{1,s}^h>\hat{\mu}_{1,s}+z\sqrt{\frac{\ln^\alpha(T)}{s}}\mid\hat{\mu}_{1,s}\right\}\right] \\
&\geq^{(a)} \quad \mathbb{E}\left[\mathbf{1}\left\{\hat{\mu}_{1,s}+\sqrt{0.5\ln(r)}\sqrt{\frac{\ln^\alpha(T)}{s}}\geq\mu_1\right\}\right]\cdot\left(1-\left(1-\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\frac{1}{2}\ln(r)}}{\frac{1}{2}\ln(r)+1}e^{-\frac{1}{4}\ln(r)}\right)^r\right) \\
&\geq^{(b)} \quad \left(1-e^{-\ln(r)}\right)\cdot\left(1-e^{-r\cdot\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\frac{1}{2}\ln(r)}}{\frac{1}{2}\ln(r)+1}\cdot r^{-\frac{1}{4}}}\right) \\
&\geq^{(c)} \quad \left(1-r^{-1}\right)\cdot\left(1-e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\frac{1}{2}\ln(r)}}{\frac{1}{2}\ln(r)+\frac{1}{\ln(13)}\ln(r)}\cdot r^{\frac{1}{4}}}\right) \\
&= \quad \left(1-r^{-1}\right)\cdot\left(1-e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}\frac{\sqrt{r^{\frac{1}{2}}}}{\sqrt{\ln(r)}}}\right) \\
&\geq^{(d)} \quad \left(1-r^{-1}\right)\cdot\left(1-e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}\right) \\
&\geq \quad 1-r^{-1}-e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}} \quad ,
\end{aligned} \tag{56}$$

which implies $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s}>r\mid\hat{\mu}_{1,s}\right\}\right]=1-\gamma\leq r^{-1}+e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}}$.

Inequality (a) uses anti-concentration bounds of Gaussian distributions and inequality (b) uses Hoeffding's inequality and $e^{-x}\geq 1-x$. Inequalities (c) and (d) use $1\leq\ln(r)/\ln(13)$ and $\sqrt{r^{0.5}/\ln(r)}\geq 1$, when $r\geq 13$.

When integer $r \geq 101$, we introduce $z = \sqrt{\frac{2}{3}\ln r}$ and have

$$\begin{aligned}
\gamma &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\mathcal{G}_{1,s} \leq r\right\} \mid \hat{\mu}_{1,s}\right]\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}\left\{\max_{h\in[r]} \theta^h_{1,s} > \mu_1 - \tfrac{\Delta_i}{2}\right\} \mid \hat{\mu}_{1,s}\right]\right] \\
&\geq \mathbb{E}\left[\mathbf{1}\left\{\hat{\mu}_{1,s} + z\sqrt{\tfrac{\ln^\alpha(T)}{s}} \geq \mu_1 - \tfrac{\Delta_i}{2}\right\} \cdot \underbrace{\mathbb{P}\left\{\max_{h\in[r]} \theta^h_{1,s} > \hat{\mu}_{1,s} + z\sqrt{\tfrac{\ln^\alpha(T)}{s}} \mid \hat{\mu}_{1,s}\right\}}_{=:\beta}\right] \quad .
\end{aligned} \tag{57}$$

We construct a lower bound for β. We have

$$\begin{aligned}
\beta &= 1 - \prod_{h\in[r]}\left(1 - \mathbb{P}\left\{\theta^h_{1,s} > \hat{\mu}_{1,s} + z\sqrt{\tfrac{\ln^\alpha(T)}{s}} \mid \hat{\mu}_{1,s}\right\}\right) \\
&= 1 - \left(1 - \mathbb{P}\left\{\theta_{1,s} > \hat{\mu}_{1,s} + z\sqrt{\tfrac{\ln^\alpha(T)}{s}} \mid \hat{\mu}_{1,s}\right\}\right)^r \\
&\geq 1 - \left(1 - \tfrac{1}{\sqrt{2\pi}}\tfrac{z}{z^2+1}e^{-0.5z^2}\right)^r \\
&\geq 1 - e^{-r\cdot\frac{1}{\sqrt{2\pi}}\frac{z}{z^2+1}e^{-0.5z^2}} \\
&= 1 - e^{-r\cdot\frac{1}{\sqrt{2\pi}}\frac{\sqrt{\frac{2}{3}\ln(r)}}{\frac{2}{3}\ln(r)+1}e^{-\frac{1}{2}\cdot\frac{2}{3}\ln(r)}} \\
&\geq 1 - e^{-r^{\frac{1}{2}}\cdot\frac{1}{\sqrt{3\pi}}\sqrt{\frac{r^{\frac{1}{3}}}{\ln r}}} \\
&\geq 1 - e^{-\sqrt{\frac{r}{3\pi}}} \quad .
\end{aligned} \tag{58}$$

Then, we have

$$\begin{aligned}
\gamma &\geq \left(1 - e^{-\sqrt{\frac{r}{3\pi}}}\right)\cdot\mathbb{P}\left\{\hat{\mu}_{1,s} + \sqrt{\tfrac{\ln r}{1.5}}\sqrt{\tfrac{\ln^\alpha(T)}{s}} \geq \mu_1\right\} \\
&\geq \left(1 - e^{-\sqrt{\frac{r}{3\pi}}}\right)\cdot\left(1 - r^{-\frac{4}{3}}\right) \\
&\geq 1 - e^{-\sqrt{\frac{r}{3\pi}}} - r^{-\frac{4}{3}} \quad ,
\end{aligned} \tag{59}$$

which implies $\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] = 1 - \gamma \leq e^{-\sqrt{\frac{r}{3\pi}}} + r^{-\frac{4}{3}}$.

Finally, we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{\mathbb{P}\left\{\theta_{1,s} > \mu_1 - \frac{\Delta_i}{2} \mid \mathcal{F}_{\tau_s(1)}\right\}} - 1\right] \\
=\ &\mathbb{E}\left[\mathcal{G}_{1,s}\right] - 1 \\
=\ &\sum_{r=0}^{\infty}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] - 1 \\
=\ &\sum_{r=0}^{12}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] + \sum_{r=13}^{100}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] + \sum_{r=101}^{\infty}\mathbb{E}\left[\mathbb{P}\left\{\mathcal{G}_{1,s} > r \mid \hat{\mu}_{1,s}\right\}\right] - 1 \\
\leq\ &13 + \sum_{r=13}^{100}\left(e^{-\sqrt{\frac{r}{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}} + \frac{1}{r}\right) + \sum_{r=101}^{\infty}\left(e^{-\sqrt{\frac{r}{3\pi}}} + \frac{1}{r^{\frac{4}{3}}}\right) - 1 \\
\leq\ &12 + \int_{12}^{100}\left(e^{-\sqrt{\frac{r}{\pi}}\cdot\frac{\ln(13)}{\ln(13)+2}} + \frac{1}{r}\right)dr + \int_{100}^{\infty}\left(e^{-\sqrt{\frac{r}{3\pi}}} + \frac{1}{r^{\frac{4}{3}}}\right)dr \\
\leq\ &12 + 10.44 + 2.13 + 3.1 + 0.65 \\
\leq\ &29 \quad ,
\end{aligned} \tag{60}$$

which concludes the proof. □

Lemma 13. *We have*

$$\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_i^\theta(t)}, n_i(t-1) \geq L_i\right\}\right] \leq \tfrac{1}{c_0} \quad . \tag{61}$$

Lemma 14. *We have*

$$\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\overline{\mathcal{E}_i^{\mu}(t-1)},n_i(t-1)\geq L_i\right\}\right] \quad \leq \quad 1 \quad . \tag{62}$$

Lemma 15. *We have*

$$\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\overline{\mathcal{E}_1^{\mu}(t-1)},n_i(t-1)\geq L_i\right\}\right] \quad \leq \quad 1 \quad . \tag{63}$$

Proof of Lemma 13. Let $\tau_s^{(i)}$ be the round when the s-th pull of the optimal arm i occurs. We have

$$\begin{aligned}
&\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\overline{\mathcal{E}_i^{\theta}(t)},n_i(t-1)\geq L_i\right\}\right]\\
\leq\ &\sum_{s=L_i}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(i)}+1}^{\tau_{s+1}^{(i)}}\mathbf{1}\left\{i_t=i,\overline{\mathcal{E}_i^{\theta}(t)}\right\}\right]\\
=\ &\sum_{s=L_i}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(i)}+1}^{\tau_{s+1}^{(i)}}\mathbf{1}\left\{i_t=i,\theta_i(t)>\hat{\mu}_{i,n_i(t-1)}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{n_i(t-1)}}\right\}\right]\\
\leq\ &\sum_{s=L_i}^{T}\mathbb{E}\left[\mathbf{1}\left\{\theta_i\left(\tau_{s+1}^{(i)}\right)>\hat{\mu}_{i,n_i\left(\tau_{s+1}^{(i)}-1\right)}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{n_i\left(\tau_{s+1}^{(i)}-1\right)}}\right\}\right]\\
=\ &\sum_{s=L_i}^{T}\mathbb{E}\left[\mathbf{1}\left\{\theta_i\left(\tau_{s+1}^{(i)}\right)>\hat{\mu}_{i,s}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{s}}\right\}\right]\\
\leq\ &\sum_{s=L_i}^{T}\mathbb{E}\left[\mathbf{1}\left\{\max_{h\in[\phi]}\theta_{i,s}^{h}>\hat{\mu}_{i,s}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{s}}\right\}\right]\\
\leq\ &\sum_{s=L_i}^{T}\sum_{h\in[\phi]}\mathbb{P}\left\{\theta_{i,s}^{h}>\hat{\mu}_{i,s}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{s}}\right\}\\
=\ &\sum_{s=L_i}^{T}\sum_{h\in[\phi]}\mathbb{E}\left[\mathbb{P}\left\{\theta_{i,s}^{h}>\hat{\mu}_{i,s}+\sqrt{\frac{(5-\alpha)\ln^{1+\alpha}(T)}{s}}\mid\hat{\mu}_{i,s}\right\}\right]\\
\leq\ &T\cdot\phi\cdot\frac{0.5}{T^{2.5-0.5\alpha}}\\
=\ &T\cdot\frac{2}{c_0}T^{0.5-0.5\alpha}\ln^{1.5-0.5\alpha}(T)\cdot\frac{0.5}{T^{2.5-0.5\alpha}}\\
\leq\ &\frac{1}{c_0}\cdot\frac{\ln^{1.5}(T)}{T}\\
\leq\ &\frac{1}{c_0}\quad,
\end{aligned} \tag{64}$$

which concludes the proof. □

Proof of Lemma 14. Recall event $\mathcal{E}_i^{\mu}(t-1)=\left\{\left|\hat{\mu}_{i,n_i(t-1)}-\mu_i\right|\leq\sqrt{\frac{2\ln(T)}{n_i(t-1)}}\right\}$. Let $\tau_s^{(i)}$ be the round when arm i is pulled for the s-th time. We have

$$\begin{aligned}
&\sum_{t=1}^{T}\mathbb{E}\left[\mathbf{1}\left\{i_t=i,\overline{\mathcal{E}_i^{\mu}(t-1)},n_i(t-1)\geq L_i\right\}\right]\\
\leq\ &\sum_{s=L_i}^{T}\mathbb{E}\left[\sum_{t=\tau_s^{(i)}+1}^{\tau_{s+1}^{(i)}}\mathbf{1}\left\{i_t=i,\overline{\mathcal{E}_i^{\mu}(t-1)}\right\}\right]\\
\leq\ &\sum_{s=L_i}^{T}\mathbb{E}\left[\mathbf{1}\left\{\overline{\mathcal{E}_i^{\mu}\left(\tau_{s+1}^{(i)}-1\right)}\right\}\right]\\
=\ &\sum_{s=L_i}^{T}\underbrace{\mathbb{P}\left\{\left|\hat{\mu}_{i,s}-\mu_i\right|>\sqrt{\frac{2\ln(T)}{s}}\right\}}_{\text{Hoeffding's inequality}}\\
\leq\ &\sum_{s=L_i}^{T}\frac{2}{T^4}\\
\leq\ &1\quad,
\end{aligned} \tag{65}$$

which concludes the proof.

□

Proof of Lemma 15. We have

$$\begin{aligned} & \sum_{t=1}^{T} \mathbb{E}\left[\mathbf{1}\left\{i_t = i, \overline{\mathcal{E}_1^{\mu}(t-1)}, n_i(t-1) \geq L_i\right\}\right] \\ \leq & \sum_{t=1}^{T}\sum_{s=1}^{T} \underbrace{\mathbb{P}\left\{|\hat{\mu}_{1,s} - \mu_1| > \sqrt{\frac{2\ln(T)}{s}}\right\}}_{\text{Hoeffding's inequality}} \\ \leq & \sum_{t=1}^{T}\sum_{s=1}^{T} e^{-4\ln(T)} \\ \leq & 1 \quad , \end{aligned} \tag{66}$$

which concludes the proof.

□

F WORST-CASE REGRET BOUNDS PROOFS

Let $\Delta_* := \sqrt{\frac{K\ln^{1+\alpha}(T)}{T}}$ be the critical gap. The total regret of pulling any sub-optimal arm i such that such that $\Delta_i < \Delta_*$ is at most $T \cdot \Delta_* = \sqrt{KT\ln^{1+\alpha}(T)}$. Now, we consider all the remaining sub-optimal arms i with $\Delta_i > \Delta_*$. For such i, the regret is upper bounded by

$$\Delta_i \mathbb{E}\left[n_i(T)\right] \leq O\left(\frac{\ln^{1+\alpha}(T)}{\Delta_i}\right)$$

which is decreasing in $\Delta_i \in (0, 1]$. Therefore, for every such i, the regret is bounded by $O\left(\sqrt{\frac{T\ln^{1+\alpha}(T)}{K}}\right)$. Taking a sum over all sub-optimal arms concludes the proofs.