Global existence and blow-up for the Euler-Poincaré equations with a class of initial data

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Abstract: In this paper we investigate the Cauchy problem of d-dimensional Euler-Poincaré equations. By choosing a class of new and special initial data, we can transform this d-dimensional Euler-Poincaré equations into the Camassa-Holm type equation in the real line. We first obtain some global existence results and then present a new blow-up result to the system under some different assumptions on this special class of initial data.

Keywords: Euler-Poincaré equations; Global existence; Blow-up.

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1 Introduction

In this paper, we consider the Cauchy problem of the following higher dimensional Euler-Poincaré equations

$$\partial_t m + u \cdot \nabla m + (\nabla u)^\top \cdot m + (\mathrm{div} u) m = 0, \quad m = (1-\Delta)u, \tag{1.1}$$

where $t \in \mathbb{R}^+$ and $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$. $u(t,x) = (u_1, u_2, \cdots, u_d)(t,x)$ denotes the velocity of the fluid and $m(t,x) = (m_1, m_2, \cdots, m_d)(t,x)$ represents the momentum. The notation $(\nabla u)^\top$ denotes the transpose of the matrix ∇u. It is useful to recast equation (1.1) in the component-wise form as

$$\partial_t m_i + \sum_{j=1}^d u_j \partial_{x_j} m_i + \sum_{j=1}^d (\partial_{x_i} u_j) m_j + m_i \sum_{j=1}^d \partial_{x_j} u_j = 0.$$

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While the Euler-Poincaré equations (1.1) is also called the higher dimensional Camassa-Holm equations. In one dimension, the system (1.1) is the classical Camassa-Holm (CH) equation

$$\partial_t m + u\partial_x m + 2m\partial_x u = 0, \quad m = (1 - \partial_{xx})u.$$

Like the KdV equation, the CH equation describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity [4, 5, 16]. It is completely integrable [4, 8], has a bi-Hamiltonian structure [7, 21], and admits exact peaked solitons of the form $ce^{-|x-ct|}(c > 0)$, which are orbitally stable [18]. It is worth mentioning that the peaked solitons present the characteristic for the travelling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [10, 14, 15, 34]. The local well-posedness and ill-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces was discussed in [11, 12, 19, 20, 22, 28, 33]. It was shown that there exist global strong solutions to the CH equation [9, 11, 12] and finite time blow-up strong solutions to the CH equation [9, 11–13]. The global conservative and dissipative solutions of CH equation were discussed in [2, 3].

The Euler-Poincaré equations arise in diverse scientific applications and enjoy several remarkable properties both in the one-dimensional and multi-dimensional cases. The Euler-Poincaré equations were first studied by Holm, Marsden, and Ratiu in 1998 as a framework for modeling and analyzing fluid dynamics [24, 25], particularly for nonlinear shallow water waves, geophysical fluids and turbulence modeling. Later, the Euler-Poincaré equations have many further interpretations beyond fluid applications. For instance, in 2-D, it is exactly the same as the averaged template matching equation for computer vision [23]. Also, the Euler-Poincaré equations have important applications in computational anatomy, it can be regarded as an evolutionary equation for a geodesic motion on a diffeomorphism group and it is associated with Euler-Poincaré reduction via symmetry (see, e.g, [26, 35]).

The rigorous analysis of the Euler-Poincaré equations with $d \geq 1$ was initiated by Chae and Liu [6] who obtained the local well-posedness in Hilbert spaces $m_0 \in H^{s+\frac{d}{2}}$, $s \geq 2$ and also gave a blow-up criterion, zero α limit and the Liouville type theorem. Li, Yu and Zhai [27] proved that the solution to higher dimensional Camassa-Holm equations with a large class of smooth initial data blows up in finite time or exists globally in time, which reveals the nonlinear depletion mechanism hidden in the Euler-Poincaré system. Luo and Yin [32] obtained a new blow-up result to the periodic Euler-Poincaré system for a special class of smooth initial data by using the rotational invariant properties of the system. By means of the Littlewood-Paley theory, Yan and Yin [36] established the local existence and uniqueness in Besov spaces $B^s_{p,r}$ with $s > \max\{\frac{3}{2}, 1 + \frac{d}{p}\}$ and $s = 1 + \frac{d}{p}$, $1 \leq p \leq 2d$, $r = 1$. For more results of the Euler-Poincaré equations, we refer the reads to see [29–31, 37]. In this paper, we consider the Euler-Poincaré equations in $\mathbb{R}^d$ with a class of special initial data be of the form $f(x_1 + x_2 + \cdots x_d)$ and study the global existence and blow-up property of the corresponding solution under certain assumptions on this initial data.

1.1 Reduction of System

Let us assume that

$$\begin{cases} \partial_t n + dv\partial_x n + 2dv_x n = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ n = (1 - d\partial_x^2)v, \\ n(0,x) = n_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Then we can find that

$$\begin{aligned} &m(t,x) = n(t, x_1 + x_2 + \cdots + x_d)\vec{e} \quad \text{with} \quad \vec{e} := (\underbrace{1, \cdots, 1}_{d}), \\ &u(t,x) = v(t, x_1 + x_2 + \cdots + x_d)\vec{e}, \end{aligned}$$

satisfy the following higher dimensional Euler-Poincaré equations

$$\begin{cases} \partial_t m + u \cdot \nabla m + (\nabla u)^\top \cdot m + (\mathrm{div} u)m = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ m = (1 - \Delta)u, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{EP}$$

Precisely speaking, the Cauchy Problem of higher dimensional Euler-Poincaré equations (EP) is transformed into that of the new Camassa-Holm type equation (1.2) in the real-line. Furthermore, we can rewrite the equation (1.2) as follows

$$\begin{cases} \partial_t v + dvv_x = -\partial_x (1 - d\partial_x^2)^{-1} \left(dv^2 + \frac{d^2}{2} (\partial_x v)^2 \right), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}. \end{cases} \tag{d-CH}$$

From now, we mainly focus on the equation (d-CH).

1.2 Main results

First, following the similar proofs for the Camassa-Holm equation in [19, 33], we can obtain the well-posedness result for (d-CH) as follows.

Theorem 1.1 (Local well-posedness). *Let $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there exists a time $T^* > 0$ and a unique strong solution $v(t,x) \in C([0,T^*); H^s(\mathbb{R})) \cap C^1\left([0,T^*); H^{s-1}(\mathbb{R})\right)$ of the Cauchy problem (d-CH).*

Remark 1.1. *Let $v_0 \in H^s(\mathbb{R})$ with $s > 2$. Assume that $u_0(x) = v_0(x_1 + x_2 + \cdots + x_d)\vec{e}$, by the embedding $H^s(\mathbb{R}) \hookrightarrow C^{s-\frac{1}{2}-\varepsilon}(\mathbb{R})$, where we take $\varepsilon = 0$ if $s - \frac{1}{2} \notin \mathbf{Z}^+$ and $\varepsilon = 0^+$ if $s - \frac{1}{2} \in \mathbf{Z}^+$, then we have $u_0(x) \in C^{s-\frac{1}{2}-\varepsilon}(\mathbb{R}^d)$. Moreover, we can obtain the unique solution $u(t,x) = v(t, x_1 + x_2 + \cdots + x_d)\vec{e} \in C([0,T^*); C^{s-\frac{1}{2}-\varepsilon}(\mathbb{R}^d))$ of the Cauchy problem (EP) with initial data $u_0(x)$. Based on the above observation, we can transform the global existence or blow-up property of solutions for the Cauchy problem (EP) into that for the Cauchy problem (d-CH).*

Finally, we state the following two global existence results and one blow-up result.

Theorem 1.2 (Global existence)**.** *Let $n_0 = v_0 - d\partial_x^2 v_0$ with $d \geq 1$. Assume that $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ satisfying one of the following*

(1) *$n_0 \geq 0$ on $(-\infty, +\infty)$;*

(2) *$n_0 \leq 0$ on $(-\infty, x_0]$ and $n_0 \geq 0$ on $[x_0, +\infty)$ for some point $x_0 \in \mathbb{R}$.*

Then the corresponding solution of the Cauchy problem (d-CH) exists globally in time.

Corollary 1.1. *Let $d \geq 1$. Assume that $u_0(x_0) = v_0(x_1^0 + x_2^0 + \cdots + x_d^0)\vec{e}$ with v_0 satisfying the assumption of Theorems 1.2. Then the corresponding solution of the Euler-Poincaré equations (EP) exists globally in time.*

Theorem 1.3 (Formation of singularities)**.** *Let $d \geq 1$. Assume that $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ satisfying that for some point $x_0 \in \mathbb{R}$*

$$v_0'(x_0) < -\frac{1}{\sqrt{d}}|v_0(x_0)|. \tag{1.3}$$

Then the corresponding solution of the Cauchy problem (d-CH) blows up in finite time.

Corollary 1.2. *Let $d \geq 1$. Assume that $u_0(x_0) = v_0(x_1^0 + x_2^0 + \cdots + x_d^0)\vec{e}$ with v_0 satisfying the assumption of Theorems 1.3. Then the corresponding solution of the Euler-Poincaré equations (EP) blows up in finite time.*

When assuming that the initial condition is stronger than (1.3), we have

Corollary 1.3. *Let $d \geq 1$. Assume that $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ satisfying $n_0 = v_0 - d\partial_x^2 v_0 \geq 0$ on $(-\infty, x_0]$ and $n_0 \leq 0$ on $[x_0, +\infty)$ for some point $x_0 \in \mathbb{R}$ and n_0 changes sign. Then the corresponding solution of the Cauchy problem (d-CH) blows up in finite time.*

2 Preliminaries

Now let us consider the associated Lagrangian flow

$$\begin{cases} \frac{\mathrm{d}q}{\mathrm{d}t} = dv(t, q(t, x)), & (t, x) \in [0, T) \times \mathbb{R}, \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.4}$$

Following the similar proofs for Camassa-Holm equation in [9], we can obtain the following results:

Lemma 2.1. *Assume that $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and T is the existence time of the corresponding solution of (d-CH). Then one has, with $n = (1 - d\partial_x^2)v$,*

$$n(t, q(t, x))q_x^2(t, x) = n_0(x), \quad (t, x) \in [0, T) \times \mathbb{R},$$

where $q(t, x) \in C^1([0, T), \mathbb{R})$ is a unique solution of the Cauchy problem (2.4) and satisfies that

$$q_x(t, x) = \exp\left(d \int_0^t v_x(\tau, q(\tau, x))\mathrm{d}\tau\right) > 0, \quad t \in [0, T).$$

Lemma 2.2. *Let $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and T is the maximal existence time of the corresponding solution of* (d-CH). *Then the corresponding solution of* (d-CH) *blows up in finite time if and only if*

$$\liminf_{t \to T} \left(\inf_{x \in \mathbb{R}} v_x(t, x) \right) = -\infty.$$

That is, singularities can arise only in the form of wave breaking.

Lemma 2.3. *Let $v_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and T is the maximal existence time of the corresponding solution of* (d-CH). *Then for any $t \in [0, T)$, we have*

$$\|v(t)\|_{L^2}^2 + d\|\partial_x v(t)\|_{L^2}^2 = \|v_0\|_{L^2}^2 + d\|\partial_x v_0\|_{L^2}^2 \le 2d\|v_0\|_{H^1}^2.$$

3 Proof of Theorems

In this section, inspired by [1, 9], we shall prove our main results. The proof of Theorem 1.1 is standard, we omit the details. Next, we begin to prove the remaining Theorems.

3.1 Proof of Theorem 1.2

Case 1: $n_0 \ge 0$ on $(-\infty, +\infty)$.
By the assumption of n_0 and Lemma 2.1, one has

$$n(t, x) \ge 0, \quad (t, x) \in [0, T) \times \mathbb{R}.$$

Since $\left(1 - d\partial_x^2\right)^{-1} f = p_d * f$ for any $f \in L^2(\mathbb{R})$ with $p_d(x) = \frac{1}{2\sqrt{d}} \exp\left(-\frac{|x|}{\sqrt{d}}\right)$, one has

$$v(t, x) = \frac{1}{2\sqrt{d}} e^{-\frac{|x|}{\sqrt{d}}} * n(t, x) = \frac{1}{2\sqrt{d}} \int_{-\infty}^{\infty} e^{-\frac{|x-\xi|}{\sqrt{d}}} n(t, \xi) \mathrm{d}\xi.$$

Thus due to Lemma 2.3, we have for $(t, x) \in [0, T) \times \mathbb{R}$

$$\begin{aligned} |\partial_x v(t, x)| &= \left| \frac{1}{2d} \int_{-\infty}^{\infty} \operatorname{sgn}(x - \xi) e^{-\frac{|x-\xi|}{\sqrt{d}}} n(t, \xi) \mathrm{d}\xi \right| \\ &\le \frac{1}{2d} \int_{-\infty}^{\infty} e^{-\frac{|x-\xi|}{\sqrt{d}}} n(t, \xi) \mathrm{d}\xi = \frac{1}{\sqrt{d}} v(t, x) \\ &\le \sqrt{\frac{\|v(t, x)\|_{L^2}^2 + d\|\partial_x v(t, x)\|_{L^2}^2}{2d}} \le \|v_0\|_{H^1}. \end{aligned}$$

In view of Lemma 2.2, this shows the existence time $T = \infty$ and completes the proof of Theorem 1.2.

Case 2: $n_0 \le 0$ on $(-\infty, x_0]$ and $n_0 \ge 0$ on $[x_0, +\infty)$ for some point $x_0 \in \mathbb{R}$.
Since $q(t, x)$ is an increasing diffeomorphism of $\mathbb{R}$ for $t \in [0, T)$, we deduce from Lemma 2.1 that

$$\begin{cases} n(t, x) \le 0, & \text{if } \ x \le q(t, x_0), \\ n(t, x) \ge 0, & \text{if } \ x \ge q(t, x_0), \end{cases} \tag{3.5}$$

and $n(t, q(t, x_0)) = 0$. Then, we have for $x \geq q(t, x_0)$

$$v_x(t, x) = -\frac{1}{\sqrt{d}} v(t, x) + \frac{1}{d} e^{\frac{x}{\sqrt{d}}} \int_x^{+\infty} e^{-\frac{\xi}{\sqrt{d}}} n(t, \xi) \mathrm{d}\xi \geq -\frac{1}{\sqrt{d}} v(t, x), \tag{3.6}$$

while $x \leq q(t, x_0)$

$$v_x(t, x) = \frac{1}{\sqrt{d}} v(t, x) - \frac{1}{d} e^{\frac{-x}{\sqrt{d}}} \int_{-\infty}^{x} e^{-\frac{\xi}{\sqrt{d}}} n(t, \xi) \mathrm{d}\xi \geq \frac{1}{\sqrt{d}} v(t, x). \tag{3.7}$$

It thus follows from the relations (3.6)-(3.7) and Lemma 2.3 that

$$v_x(t, x) \geq -\frac{1}{\sqrt{d}} |v(t, x)| \geq -\|v_0\|_{H^1}, \quad (t, x) \in [0, T) \times \mathbb{R}.$$

In view of Lemma 2.2, this shows the existence time $T = \infty$ and completes the proof of Theorem 1.3.

3.2 Proof of Theorem 1.3

We only need to show that blow-up results hold for initial data $v_0 \in H^3(\mathbb{R})$. Then the continuous dependence on initial data ensures the validity for all $H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Let $v \in C\left([0, T), H^3(\mathbb{R})\right) \cap C^1\left([0, T), H^2(\mathbb{R})\right)$ be the solution of (d-CH). Since $p_d(x) = \frac{1}{2\sqrt{d}} \exp\left(-\frac{|x|}{\sqrt{d}}\right)$, one has

$$\left(1 - d\partial_x^2\right)^{-1} f = p_d * f, \quad f \in L^2(\mathbb{R}).$$

Differentiating Eq $(\text{d-CH})_1$ with respect to x yields

$$\partial_t v_x + d v v_{xx} = -\frac{d}{2} v_x^2 + v^2 - p_d * \left(v^2 + \frac{d}{2}(\partial_x v)^2\right). \tag{3.8}$$

We introduce two new functions

$$w := \frac{1}{\sqrt{d}} v \quad \text{and} \quad V := v_x.$$

Then we can obtain from (3.8) that

$$\begin{aligned}
\partial_t V + d v V_x &= -\frac{d}{2} V^2 + d w^2 - d p_d * \left(w^2 + \frac{1}{2} V^2\right), \\
\partial_t w + d v w_x &= -d^{\frac{3}{2}} \partial_x p_d * \left(w^2 + \frac{1}{2} V^2\right).
\end{aligned}$$

From (2.4) and the above, we deduce that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(t, q(t, x)) &= \left(-\frac{d}{2} V^2 + d w^2 - d p_d * \left(w^2 + \frac{1}{2} V^2\right)\right)(t, q(t, x)), \\
\frac{\mathrm{d}}{\mathrm{d}t} w(t, q(t, x)) &= -d^{\frac{3}{2}} \left(\partial_x p_d * \left(w^2 + \frac{1}{2} V^2\right)\right)(t, q(t, x)),
\end{aligned}$$

which implies directly that

$$\frac{\mathrm{d}}{\mathrm{d}t}(w+V)(t,q(t,x)) = d\left(-\frac{1}{2}V^2 + w^2 - F(w,V)\right)(t,q(t,x)),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(w-V)(t,q(t,x)) = -d\left(-\frac{1}{2}V^2 + w^2 - G(w,V)\right)(t,q(t,x)),$$

where we set

$$F(w,V) := p_d * \left(w^2 + \frac{1}{2}V^2\right) + \sqrt{d}\partial_x p_d * \left(w^2 + \frac{1}{2}V^2\right),$$
$$G(w,V) := p_d * \left(w^2 + \frac{1}{2}V^2\right) - \sqrt{d}\partial_x p_d * \left(w^2 + \frac{1}{2}V^2\right).$$

Now we claim that

$$F(w,V)(t,x) \geq \frac{1}{2}w^2(t,x) \quad \text{and} \quad G(w,V)(t,x) \geq \frac{1}{2}w^2(t,x), \tag{3.9}$$

which in turn gives that

$$p_d * \left(w^2 + \frac{1}{2}V^2\right) = \frac{1}{2}\left(F(w,V)(t,x) + G(w,V)\right) \geq \frac{1}{2}w^2(t,x). \tag{3.10}$$

In fact, one has

$$\begin{aligned} p_d * \left(w^2 + \frac{1}{2}V^2\right) = &\frac{1}{2\sqrt{d}}e^{\frac{x}{\sqrt{d}}}\int_x^{+\infty} e^{-\frac{\xi}{\sqrt{d}}}\left(w^2 + \frac{1}{2}V^2\right)\mathrm{d}\xi \\ &+ \frac{1}{2\sqrt{d}}e^{\frac{-x}{\sqrt{d}}}\int_{-\infty}^{x} e^{\frac{\xi}{\sqrt{d}}}\left(w^2 + \frac{1}{2}V^2\right)\mathrm{d}\xi \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \sqrt{d}\partial_x p_d * \left(w^2 + \frac{1}{2}V^2\right) = &\frac{1}{2\sqrt{d}}e^{\frac{x}{\sqrt{d}}}\int_x^{+\infty} e^{-\frac{\xi}{\sqrt{d}}}\left(w^2 + \frac{1}{2}V^2\right)\mathrm{d}\xi \\ &- \frac{1}{2\sqrt{d}}e^{\frac{-x}{\sqrt{d}}}\int_{-\infty}^{x} e^{\frac{\xi}{\sqrt{d}}}\left(w^2 + \frac{1}{2}V^2\right)\mathrm{d}\xi. \end{aligned} \tag{3.12}$$

On the one hand, performing (3.11)-(3.12) yields that

$$F(w,V) = \frac{1}{\sqrt{d}}e^{\frac{x}{\sqrt{d}}}\int_x^{+\infty} e^{-\frac{\xi}{\sqrt{d}}}\left(w^2 + \frac{1}{2}V^2\right)\mathrm{d}\xi.$$

It is not difficult to verify that

$$\begin{aligned} \frac{1}{\sqrt{d}}e^{\frac{x}{\sqrt{d}}}\int_x^{+\infty} e^{-\frac{\xi}{\sqrt{d}}}\left(w^2 + V^2\right)\mathrm{d}\xi &\geq -\frac{1}{\sqrt{d}}e^{\frac{x}{\sqrt{d}}}\int_x^{+\infty} e^{-\frac{\xi}{\sqrt{d}}}(2wV)\,\mathrm{d}\xi \\ &\geq w^2 - \frac{1}{\sqrt{d}}e^{\frac{x}{\sqrt{d}}}\int_x^{+\infty} e^{-\frac{\xi}{\sqrt{d}}}w^2\mathrm{d}\xi, \end{aligned}$$

which implies

$$F(w,V) \geq \frac{1}{2}w^2. \tag{3.13}$$

On the other hand, performing (3.11)–(3.12) yields that

$$G(w,V) = \frac{1}{\sqrt{d}}e^{\frac{-x}{\sqrt{d}}}\int_{-\infty}^{x} e^{\frac{\xi}{\sqrt{d}}}\left(w^2 + \frac{1}{2}V^2\right)\mathrm{d}\xi.$$

Similarly,

$$\begin{aligned}\frac{1}{\sqrt{d}}e^{\frac{-x}{\sqrt{d}}}\int_{-\infty}^{x} e^{\frac{\xi}{\sqrt{d}}}\left(w^2 + V^2\right)\mathrm{d}\xi &\geq \frac{1}{\sqrt{d}}e^{\frac{-x}{\sqrt{d}}}\int_{-\infty}^{x} e^{\frac{\xi}{\sqrt{d}}}(2wV)\,\mathrm{d}\xi \\ &\geq w^2 - \frac{1}{\sqrt{d}}e^{-\frac{x}{\sqrt{d}}}\int_{-\infty}^{x} e^{\frac{\xi}{\sqrt{d}}}w^2\mathrm{d}\xi,\end{aligned}$$

which implies

$$\frac{1}{\sqrt{d}}e^{\frac{-x}{\sqrt{d}}}\int_{-\infty}^{x} e^{\frac{\xi}{\sqrt{d}}}\left(w^2 + \frac{1}{2}V^2\right)\mathrm{d}\xi \geq \frac{1}{2}w^2. \tag{3.14}$$

Combing (3.13) and (3.14) yields (3.9).

Using (3.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t,q(t,x)) \leq \left(-\frac{d}{2}V^2 + dw^2 - \frac{1}{2}dw^2\right)(t,q(t,x)) \leq \frac{d}{2}\left(w^2 - V^2\right)(t,q(t,x)), \tag{3.15}$$

and using (3.9)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(w + V((t,q(t,x)\right) \leq \frac{d}{2}\left(w^2 - V^2\right)(t,q(t,x)), \tag{3.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(w - V((t,q(t,x)\right) \geq -\frac{d}{2}\left(w^2 - V^2\right)(t,q(t,x)). \tag{3.17}$$

Solving the above inequalities, we obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\frac{d}{2}\int_0^t B(\tau,x)\mathrm{d}\tau}A(t,x)\Big) \geq 0,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{-\frac{d}{2}\int_0^t A(\tau,x)\mathrm{d}\tau}B(t,x)\Big) \leq 0,\end{aligned}$$

where, for simplicity we set

$$A(t,x) = (w-V)(t,q(t,x)) \quad \text{and} \quad B(t,x) = (w+V)(t,q(t,x)).$$

Since $B(0,x_0) < 0 < A(0,x_0)$, then we have $B(t,x_0) < 0 < A(t,x_0)$. Moreover, we obtain from (3.16)-(3.17) that

$$\begin{aligned}A(t,x_0) &\geq e^{\frac{d}{2}\int_0^t B(\tau,x_0)\mathrm{d}\tau}A(t,x_0) \geq A(0,x_0) > 0,\\ B(t,x_0) &\leq e^{-\frac{d}{2}\int_0^t A(\tau,x_0)\mathrm{d}\tau}B(t,x_0) \leq B(0,x_0) < 0.\end{aligned}$$

For simplicity, we denote

$$g(t) := V(t, q(t, x_0)), \quad A(t) := A(t, x_0) \quad \text{and} \quad B(t) := B(t, x_0).$$

Then, from (3.15), we have

$$g'(t) \leq \frac{d}{2} A(t)B(t) \leq \frac{d}{2} A(0)B(0). \tag{3.18}$$

Now we assume that the solution $v(t)$ of (1.2) exists globally in time $t \in [0, \infty)$, that is, $T = \infty$. We next show this leads to a contradiction.

Integrating (3.18) on $[0, t)$ yields

$$g(t) \leq g(0) + \frac{d}{2} A(0)B(0)t. \tag{3.19}$$

Since $A(0)B(0) < 0$, we have

$$\lim_{t \to \infty} g(t) = -\infty.$$

But

$$\|v(t)\|_\infty \leq \|v_0\|_{H^1} < \infty, \quad t \in [0, \infty).$$

Hence there exists some $t_0 > 0$ such that

$$g^2(t) \geq \frac{2}{d} \|v_0\|_{H^1}^2, \quad t \in [t_0, \infty)\,.$$

Combining the latter inequality with (3.18), we have derived the inequality

$$g'(t) \leq \frac{d}{2} A(t)B(t) = -\frac{d}{2} g^2(t) + \frac{1}{2} v^2 \leq -\frac{d}{4} g^2(t), \quad t \in [t_0, \infty)\,.$$

On the other hand, by the assumptions (1.3), we have

$$g(0) = v_0'(x_0) < 0.$$

It then follows from (3.19) that $g(t) < 0$ for all $t \geq 0$. The differential inequality (3.18) can be therefore solved easily for the solution $g(t), t \in [t_0, \infty)$, that is

$$\frac{1}{g\,(t_0)} - \frac{1}{g(t)} + \frac{d}{4}\,(t - t_0) \leq 0, \quad t \in [t_0, \infty)\,.$$

Since $-\frac{1}{g(t)} > 0$, we have

$$\frac{1}{g\,(t_0)} + \frac{d}{4}\,(t - t_0) < \frac{1}{g\,(t_0)} - \frac{1}{g(t)} + \frac{d}{4}\,(t - t_0) < 0, \quad t \in [t_0, \infty)\,,$$

which leads to a contradiction as $t \to \infty$. This shows that $T < \infty$ and the proof of Theorem 1.3 is complete.

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Declarations

Data Availability No data was used for the research described in the article.
Conflict of interest The authors declare that they have no conflict of interest.

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