The Correlation Energy of the Electron Gas in the Mean-Field Regime

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Abstract

We prove a rigorous lower bound on the correlation energy of interacting fermions in the mean-field regime for a wide class of singular interactions, including the Coulomb potential. Combined with the upper bound obtained in [12], our result establishes an analogue of the Gell-Mann–Brueckner formula $c_1\rho\log(\rho) + c_2\rho$ for the correlation energy of the electron gas in the high-density limit. Moreover, our analysis allows us to go beyond mean-field scaling while still covering the same class of potentials.

Contents

1 Introduction

A long-standing challenge in mathematical physics is the rigorous understanding of quantum correlations between interacting systems, based on microscopic principles. For the electron gas (e.g. jellium), this question goes back to Wigner 1934 [26] and Heisenberg 1947 [20], who recognized the difficulty of solving this task using a perturbation method. A cornerstone in the development of the correlation analysis is the *random phase approximation* (RPA) of Bohm and Pines [6, 7, 8, 23]. In this theory, electron correlation is explained by the decoupling of collective plasmon excitations and quasi-electrons interacting in the plasmon background through a screened Coulomb interaction.

The justification of the RPA has attracted notable theoretical works, including a seminal paper by Gell-Mann and Brueckner in 1957 [16], where they formally reproduced the RPA from a resummation of Feynman diagrams in a high-density electron gas. In particular, they predicted that the correlation energy of jellium, $E_{\mathrm{corr}} = E_N - E_{\mathrm{FS}}$, with density ρ is given by

$$E_{\mathrm{corr}} = c_1 \rho \log(\rho) + c_2 \rho + o(\rho), \quad \rho \to \infty, \tag{1.1}$$

with specific constants c_1, c_2. Here E_{FS} is the energy of the Fermi state given by the Slater determinant of plane waves[1]. In fact, the leading order contribution $c_1 \rho \log(\rho)$ was predicted independently by Pines [23] and Macke [22], the latter using a partial resummation of the divergent series with an effective screened Coulomb potential. The significance of the Gell-Mann-Brueckner formula (1.1) is that the second-order term $c_2 \rho$ contains the exchange contribution, which is important for a complete understanding of the electron correlation of the system.

Shortly afterwards, Sawada [24] and Sawada–Brueckner–Fukuda–Brout [25] proposed an alternative approach to the RPA which also produces correctly the leading order contribution $c_1 \rho \log(\rho)$. In this approach, the correlation energy is computed by a bosonization method where certain pairs of fermions are treated as virtual bosons, leading to a quasi-bosonic Hamiltonian which can be diagonalized explicitly by a Bogolubov transformation. The Hamiltonian approach in [24, 25] is more transparent than the resummation method in [16, 22], but unfortunately the exchange contribution of the order ρ is not taken into account in the purely bosonic picture in [24, 25].

On the mathematical side, from the techniques developed in the 1990s for large Coulomb systems by Fefferman and Seco [13], Bach [1], and Graf and Solovej [18], one can show that the correlation energy of jellium is at most of order $O(\rho^{4/3-\epsilon})$ for some small constant $\epsilon > 0$. This bound justifies that the Hartree-Fock energy, given by $c_{\mathrm{TF}} \rho^{5/3} + c_{\mathrm{D}} \rho^{4/3} + o(\rho^{4/3})$, correctly captures the full quantum mechanical energy to leading order with respect to the density (see [18, Theorem 2]). However, a rigorous justification of the Gell-Mann-Brueckner formula (1.1) is still unattainable due to various difficulties arising from both the singularity and the long-range nature of the Coulomb potential.

Recently, there has been progress in justifying a version of the Gell-Mann–Brueckner formula in the mean-field regime, where N electrons are confined within a torus of fixed volume and interact via the periodic Coulomb potential, coupled with the small factor $k_F^{-1} \sim N^{-1/3}$ as $N \to \infty$. In this setting, the long-range issue disappears but the Coulomb singularity remains a serious difficulty. In [12], we proved as a rigorous upper bound the following mean-field analogue for the correlation energy

$$E_{\mathrm{corr}} = \frac{1}{\pi} \sum_{k \in \mathbb{Z}^3_*} \int_0^\infty F\left(\frac{\hat{V}_k k_F^{-1}}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2} \right) dt + \frac{k_F^{-2}}{4\,(2\pi)^6} \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} \frac{\hat{V}_k \hat{V}_{p+q-k}}{\lambda_{k,p} + \lambda_{k,q}} + o(k_F) \tag{1.2}$$

[1] Alternatively, we could replace E_{FS} by the Hartree–Fock energy E_{HF}, but optimizing over all Slater determinants only leads to an exponentially small improvement compared to E_{FS} [17]

where $F(x) = \log(1+x) - x$ and for all $k \in \mathbb{Z}^3_* = \mathbb{Z}^3 \backslash \{0\}$ we denote

$$\lambda_{k,p} = \frac{1}{2}(|p|^2 - |p-k|^2) \text{ for } p \in L_k = \{q \in \mathbb{Z}^3 \mid |q| > k_F \geq |q-k|\}. \tag{1.3}$$

As explained in [12], a detailed evaluation of (1.2) for the Coulomb potential $\hat{V}_k \sim |k|^{-2}$ leads to an expansion of the form

$$\widetilde{c_1} k_F \log(k_F) + \widetilde{c_2} k_F + o(k_F), \tag{1.4}$$

with specific constants $\widetilde{c_1}, \widetilde{c_2}$. This expression is the analog of the Gell-Mann-Brueckner formula (1.1) for a mean-field system on a torus of fixed volume. In fact, (1.2) exactly reproduces the Gell-Mann-Brueckner formula (1.1) by formally removing the mean-field scaling and taking the thermodynamic limit. More precisely, if $k_F^{-1}\hat{V}_k$ is replaced by $4\pi e^2 |k|^{-2}$ and $(2\pi)^3$ by the volume Ω, the first term in (1.2) becomes exactly equal to the bosonic correlation contribution in [25, Eq. (34)] which is equivalent to [16, Eq. (19)] (accounting also for spin). Moreover, by applying the same procedure to the second term in (1.2) we also obtain the exchange contribution in [16, Eq. (9)], which is completely absent from the bosonic model of [25]. Roughly speaking, the rigorous proof in [12] follows the general bosonization approach in [11] which is inspired by [24, 25], but to obtain (1.2) it is necessary to refine the purely bosonic picture of [24, 25] to also capture subtle fermionic corrections.

The aim of the present work is to prove the matching lower bound for [12] and thus to fully establish (1.2) as the correlation energy of the electron gas in the mean-field regime. The main challenge in proving the lower bound compared to the analysis of the upper bound in [12] is that the a priori information available for the ground state is not sufficient to directly apply the bosonization method introduced in our previous work [11, 12]. As we will explain below, there is a big difference on the technical side for the proof of a lower bound between the treatment of smooth potentials satisfying $\sum_{k\in\mathbb{Z}^3} \hat{V}_k |k| < \infty$ and the treatment of singular potentials as we aim for in the present work.

Our proof is based on a new approach where the correlation energy is extracted directly from the Hamiltonian by completing appropriate squares containing both bosonizable and non-bosonizable terms, instead of transforming the Hamiltonian by quasi-bosonic Bogolubov transformations as in [11, 12]. This representation realizes all leading contributions of singular Coulomb-type potentials directly, but still involves several error terms that need to be controlled. To handle these we will derive several general correlation inequalities which yield the necessary error estimates. These tools should be helpful in the future not only for a better understanding of the Gell-Mann-Brueckner formula (1.1), but also for the treatment of singular interactions in other contexts.

To conclude the introduction, let us compare our work with existing results on less singular potentials. Note that if $\sum \hat{V}_k^2 |k| < \infty$, then the bosonic contribution is proportional to k_F while the exchange contribution is of negligible order $o(k_F)$. In this case, establishing a simpler form of (1.2), without the exchange contribution, has long been a very challenging problem. This problem was first resolved in [2, 3] for a smooth potential with finitely supported Fourier coefficients V_k and with sufficiently small ℓ^1-norm. Further extensions to the class of smooth potentials satisfying $\sum_{k\in\mathbb{Z}^3} \hat{V}_k |k| < \infty$, still with a lower-order exchange contribution, were obtained independently in [11, 4]. It is also worth mentioning that an earlier bound on the correlation energy that holds in the limit of small potentials was derived in [19], and that an optimal upper bound for all potentials satisfying $\sum_{k\in\mathbb{Z}^3} \hat{V}_k^2 |k| < \infty$ was discussed in [11, 4].

To see the role of the quantity $\sum_{k\in\mathbb{Z}^3} \hat{V}_k^2 |k|$, we may use the expansion

$$F(x) = \log(1+x) - x = -x^2/2 + o(x^3), \quad x \to 0, \tag{1.5}$$

to approximate the bosonic correlation contribution in (1.2) by

$$\begin{aligned}\frac{1}{\pi}\sum_{k\in\mathbb{Z}^3_*}\int_0^\infty F\left(\frac{\hat{V}_k k_F^{-1}}{(2\pi)^3}\sum_{p\in L_k}\frac{\lambda_{k,p}}{\lambda_{k,p}^2+t^2}\right)dt &\approx -\frac{1}{4(2\pi)^6}\sum_{k\in\mathbb{Z}^3_*}(\hat{V}_k k_F^{-1})^2\frac{2}{\pi}\int_0^\infty\left(\sum_{p\in L_k}\frac{\lambda_{k,p}}{\lambda_{k,p}^2+t^2}\right)^2 dt\\ &= -\frac{1}{4(2\pi)^6}\sum_{k\in\mathbb{Z}^3_*}(\hat{V}_k k_F^{-1})^2\sum_{p,q\in L_k}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}\cdot\end{aligned} \tag{1.6}$$

Since $|L_k| \sim k_F^2 \min\{|k|, k_F\}$ and $\lambda_{k,p} \sim |k| \max\{|k|, k_F\}$ in an average sense, we find that if $\sum_{k\in\mathbb{Z}^3} \hat{V}_k^2 |k| < \infty$, then the expression in (1.6) is bounded by $O(k_F)$. From this point of view, we see that the Coulomb potential is critical since the condition $\sum_{k\in\mathbb{Z}^3} \hat{V}_k^2 |k| < \infty$ barely fails for Coulomb potentials by a logarithmic divergence, which is consistent with the logarithmic term in the Gell-Mann–Brueckner formula (1.1) as well as its mean-field analogue (1.4).

Finally, let us note that the mean-field scaling $k_F^{-1} V$ is indeed special, since even for a smooth potential it does not suffice to expand $F(x)$ to a finite number of terms to obtain the full bosonic correlation contribution to leading order in k_F. This would not be the case if one considered the stronger scaling $k_F^{-1-\epsilon} V$ for a small parameter $\epsilon > 0$. On the other hand, the difficulty increases considerably when using the weaker scaling $k_F^{-1+\epsilon} V$. In the latter case, some estimates of the ground state energy for smooth potentials satisfying $\sum_{k\in\mathbb{Z}^3} \hat{V}_k |k| < \infty$ can be found in the very recent work [14]. Motivated by [14], we will also prove (1.2) as a lower bound for the weaker scaling $k_F^{-1+\epsilon} V$, where V can be as singular as the Coulomb potential. To our knowledge, this is the first time that a result on the correlation energy has been given beyond the mean-field regime.

The precise statement of our result and a brief outline of the proof will be given below.

1.1 Main Result

We consider for a given Fermi momentum $k_F > 0$ and fixed $\beta > \frac{11}{12}$ the β-scaled Hamiltonian

$$H_N = H_{\mathrm{kin}} + k_F^{-\beta} H_{\mathrm{int}} = -\sum_{i=1}^{N} \Delta_i + k_F^{-\beta} \sum_{1\leq i<j\leq N} V(x_i - x_j) \tag{1.7}$$

on $D(H_N) = D(H_{\mathrm{kin}}) = \bigwedge^N H^2(\mathbb{T}^3)$ where $\mathbb{T}^3 = [0, 2\pi]^3$ with periodic boundary conditions and

$$N = |B_F|, \quad B_F = \overline{B}(0, k_F) \cap \mathbb{Z}^3. \tag{1.8}$$

We take V to admit the Fourier decomposition

$$V(x) = (2\pi)^{-3} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k e^{ik\cdot x}, \quad \mathbb{Z}^3_* = \mathbb{Z}^3 \backslash \{0\}, \tag{1.9}$$

and make the following assumptions on the Fourier coefficients $\hat{V}_k$.

Assumption 1.1. *The Fourier coefficients $\hat{V}_k$ satisfy $\hat{V}_k = \hat{V}_{-k} \geq 0$ for all $k \in \mathbb{Z}^3_*$, are radially decreasing with respect to $k \in \mathbb{Z}^3_*$, and there exists a constant $C_V > 0$ such that*

$$\hat{V}_k \leq C_V |k|^{-2}, \quad k \in \mathbb{Z}^3_*.$$

The leading order of the ground state energy of H_N is given by the Fermi state

$$\psi_{\mathrm{FS}} = \bigwedge_{p\in B_F} u_p, \quad u_p(x) = (2\pi)^{-\frac{3}{2}} e^{ip\cdot x}, \tag{1.10}$$

with the corresponding energy (see e.g. [11, Eqs. (1.10) and (1.20)])

$$E_{\mathrm{FS}} = \langle \psi_{\mathrm{FS}}, H_N \psi_{\mathrm{FS}} \rangle = \sum_{p\in B_F} |p|^2 - \frac{1}{2(2\pi)^3} \sum_{k\in\mathbb{Z}^3_*} \hat{V}(k) \left(N - |L_k|\right) \tag{1.11}$$

where for every $k \in \mathbb{Z}^3_*$, we denoted the *lune* $L_k \subset \mathbb{Z}^3$ by

$$L_k = (B_F + k) \backslash B_F = \left\{p \in \mathbb{Z}^3 \mid |p - k| \leq k_F < |p|\right\}. \tag{1.12}$$

To describe the correlation energy, we set $\lambda_{k,p} = \frac{1}{2}(|p|^2 - |p - k|^2)$ and define the *bosonic* and *exchange contributions* by

$$E_{\mathrm{corr,bos}} = \frac{1}{\pi} \sum_{k\in\mathbb{Z}^3_*} \int_0^\infty F\left(\frac{\hat{V}_k k_F^{-\beta}}{(2\pi)^3} \sum_{p\in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right) dt, \quad F(x) = \log(1 + x) - x, \tag{1.13}$$

and

$$E_{\mathrm{corr,ex}} = \frac{k_F^{-2\beta}}{4(2\pi)^6} \sum_{k\in\mathbb{Z}^3_*} \sum_{p,q\in L_k} \frac{\hat{V}_k \hat{V}_{p+q-k}}{\lambda_{k,p}+\lambda_{k,q}}. \tag{1.14}$$

Our main result can then be stated as follows:

Theorem 1.2 (Operator and a priori estimates)**.** *Let $\frac{11}{12} < \beta \le 1$ and let V obey Assumption 1.1. Then it holds as $k_F \to \infty$ that*

$$H_N \ge E_{\mathrm{FS}} + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + \mathcal{E}$$

where $\mathcal{E}$ is an operator obeying for any $\epsilon > 0$ the lower bound

$$\mathcal{E} \ge -C_{V,\epsilon} k_F^{-\frac{1}{6}+2(1-\beta)+\epsilon} (H'_{\mathrm{kin}} + k_F) \quad \text{with} \quad H'_{\mathrm{kin}} = H_{\mathrm{kin}} - \langle \psi_{\mathrm{FS}}, H_{\mathrm{kin}} \psi_{\mathrm{FS}} \rangle .$$

Furthermore, every state $\Psi \in D(H_N)$ obeying $\langle \Psi, H_N \Psi \rangle \le \langle \psi_{\mathrm{FS}}, H_N \psi_{\mathrm{FS}} \rangle$ also satisfies

$$\left\langle \Psi, H'_{\mathrm{kin}} \Psi \right\rangle \le C_{V,\epsilon} k_F^{3-2\beta+\epsilon}.$$

Here $C_{V,\epsilon}$ denotes a general constant depending only on C_V and ϵ.

In particular, it immediately implies the ground state energy lower bound

$$\inf \sigma(H_N) \ge E_{\mathrm{FS}} + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} - O(k_F^{5/6+4(1-\beta)+\epsilon}). \tag{1.15}$$

Here are some remarks on our result.

1. In the mean-field case $\beta = 1$, the lower bound (1.15) matches the upper bound of [12], leading to a complete justification of (1.2).

Corollary 1.3 (Correlation energy in the mean-field regime)**.** *Let $\beta = 1$ and let V obey Assumption 1.1. Then it holds as $k_F \to \infty$ that, for every $\epsilon > 0$,*

$$\inf \sigma(H_N) = E_{\mathrm{FS}} + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + O(k_F^{5/6+\epsilon}).$$

Additionally, when $\beta = 1$, it always holds that $E_{\mathrm{corr,bos}} \le -C k_F$, so there is a $k_F^{-1/6+\epsilon}$ separation between the error term and $E_{\mathrm{corr,bos}}$ (the order of $E_{\mathrm{corr,ex}}$ depends on the particular potential). For regular potentials satisfying $\sum_{k\in\mathbb{Z}^3} \hat{V}_k |k| < \infty$, similar results to that of Corollary 1.3 were previously proved in [2, 3, 11, 4].

2. If there is equality in Assumption 1.1, i.e. if $\hat{V}_k \propto |k|^{-2}$ is the Coulomb potential, then

$$E_{\mathrm{corr,bos}} = O(k_F^{3-2\beta} \log(k_F)), \quad E_{\mathrm{corr,ex}} = O(k_F^{3-2\beta}). \tag{1.16}$$

Therefore, the lower bound (1.15) is a non-trivial statement for all $\frac{11}{12} < \beta \le 1$. The matching upper bound is open for $\beta < 1$ (the upper bound analysis in [12] requires mean-field scaling).

1.2 Outline of the Proof

Our method is inspired by the idea of bosonization which goes back to Sawada [24] and Sawada–Brueckner–Fukuda–Brout [25]. The key observation is that after extracting the energy of the Fermi state, the main contribution of the Hamiltonian comes from certain "bosonizable" terms, which can be written as quasi-bosonic quadratic terms in which particular pairs of fermions behave as virtual bosons. As already explained in [12], for singular potentials this bosonization method has to be implemented carefully in order to capture a subtle correction which is missed in the purely bosonic picture of [24, 25]. For regular potentials studied in [2, 3, 11, 4] the situation becomes much simpler since the purely bosonic computation is sufficient.

On the mathematical side, while we will start with the rigorous formulation of the bosonization method from [11, 12], the proof in the present paper proceeds in a very different way. Most notably, we will not use quasi-bosonic Bogolubov transformations as in [11, 12] since controlling the errors caused by these transformations would become extremely complicated due to the lack of strong a priori estimates. As a comparison, for regular potentials satisfying $\sum_{k\in\mathbb{Z}^3}\hat{V}_k|k|<\infty$ studied in [11, 4], the pointwise inequality

$$\sum_{1\leq i<j\leq N} V(x_i-x_j)-\frac{1}{2(2\pi)^3}(N^2\hat{V}_0-NV(0))=\frac{1}{2(2\pi)^3}\sum_{k\neq 0}\hat{V}_k\left|\sum_{j=1}^N e^{ik\cdot x_j}\right|^2\geq 0 \tag{1.17}$$

implies that the correlation energy in the mean-field regime is of order $O(k_F)$, leading to the a priori estimate $\langle\Psi,H'_{\text{kin}}\Psi\rangle\leq O(k_F)$ for every state satisfying $\langle\Psi,H_N\Psi\rangle\leq\langle\psi_{\text{FS}},H_N\psi_{\text{FS}}\rangle$. Unfortunately, this simple Onsager argument does not work for singular potentials. For Coulomb systems, an adaptation of the deeper techniques from [1, 18] to our mean-field situation yields an a priori bound of order $O(k_F^{3-\epsilon})$ for the correlation energy, while the stronger bound $O(k_F^{1+\epsilon})$ is typically required to apply the bosonization method from [11, 12].

To overcome this difficulty, we will derive a new representation of the Hamiltonian, wherein we extract the correlation energy directly by completing appropriate squares containing both bosonizable and non-bosonizable terms. In the bosonic picture, the realization that the ground state energy of a quadratic Hamiltonian can be extracted by completing suitable squares was first made by Bogolubov in 1947 [5]. Variations of this technique have been employed in various contexts, such as the proof of Foldy's formula for "bosonic jellium" [21], the derivation of the Lee-Huang-Yang formula for dilute Bose gases [15], and recent work [10] on the diagonalization of Bose gases beyond the Gross-Pitaevskii regime. It might therefore not seem surprising that attempting to replace quasi-bosonic Bogolubov transformations with the completion of squares should work, but the fact that the kinetic operator H'_{kin} is not expressible in terms of pairs of fermions in the same sense as the interaction term prevents a "naive" application of such an argument from working. This would also not explain why the non-bosonizable terms should be negligible.

The significance of our new formula lies not only in being the first realization of such a factorization argument for a high-density fermion system (as opposed to the low-density boson systems considered in the above works), but also in incorporating the most difficult non-bosonizable terms directly, removing the need to estimate these separately. Equipped with this representation we will then derive new correlation inequalities, which may be of independent interest, which allow us to estimate the remaining error terms as being small relative to the kinetic operator H'_{kin}. Further details of our proof are outlined as follows.

Second Quantization, Bosonizable and Non-bosonizable Terms

The starting point of the analysis is the second quantized representation of the Hamiltonian H_N, which can be decomposed as

$$H_N=E_{\text{FS}}+H_{\text{B}}+\mathcal{C}+\mathcal{Q} \tag{1.18}$$

where $E_{\text{FS}}=\langle\psi_{\text{FS}},H_N\psi_{\text{FS}}\rangle$ is the energy of the Fermi state and the *bosonizable, cubic* and *quartic* terms are given by

$$\begin{aligned}
H_{\text{B}}&=H'_{\text{kin}}+\frac{k_F^{-\beta}}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\big(2B_k^*B_k+B_kB_{-k}+B_{-k}^*B_k^*\big),\\
\mathcal{C}&=\frac{k_F^{-\beta}}{(2\pi)^3}\operatorname{Re}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\big(B_k+B_{-k}^*\big)^*D_k,\\
\mathcal{Q}&=\frac{k_F^{-\beta}}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k\left(D_k^*D_k-\sum_{p\in L_k}\big(c_p^*c_p+c_{p-k}c_{p-k}^*\big)\right),
\end{aligned} \tag{1.19}$$

respectively (see e.g. [12, eqs. 1.16 - 1.26] for the computation). Here c_p^* and c_p denote the creation and annihilation operators associated with the plane wave states with momenta $p\in\mathbb{Z}^3$, which satisfy the canonical anticommutation relations (CAR)

$$\{c_p,c_q^*\}=\delta_{p,q},\quad \{c_p,c_q\}=0=\{c_p^*,c_q^*\}. \tag{1.20}$$

Above H'_{kin} denotes the localized kinetic operator, which is

$$H'_{\text{kin}}=H_{\text{kin}}-\langle\psi_{\text{FS}},H_{\text{kin}}\psi_{\text{FS}}\rangle=\sum_{p\in B_F^c}|p|^2\,c_p^*c_p-\sum_{p\in B_F}|p|^2\,c_pc_p^*, \tag{1.21}$$

and B_k, D_k are given by

$$B_k = \sum_{p\in L_k} c^*_{p-k}c_p, \quad D_k = \sum_{p\in B_F^c\cap\left(B_F^c+k\right)} c^*_{p-k}c_p + \sum_{p\in B_F\cap(B_F+k)} c^*_{p-k}c_p. \tag{1.22}$$

Extraction of the Correlation Energy by Factorization

The correlation energy arises from the bosonizable terms H_{B}, so we start by considering these in detail. The reason for its name is the following: If we define the *excitation operators* $b_{k,p}$ and $b^*_{k,p}$ by

$$b_{k,p} = c^*_{p-k}c_p, \quad b^*_{k,p} = c^*_p c_{p-k}, \quad k\in\mathbb{Z}^3_*,\, p\in L_k, \tag{1.23}$$

then it follows immediately from the CAR that these obey commutation relations of the form

$$[b_{k,p}, b^*_{l,q}] = \delta_{k,l}\delta_{p,q} + \varepsilon_{k,l}(e_p;e_q), \quad [b_{k,p}, b_{l,q}] = 0 = [b^*_{k,p}, b^*_{l,q}], \tag{1.24}$$

which are seen to be analogous to canonical commutation relations up to a correction term $\varepsilon_{k,l}(e_p;e_q)$ (the precise form of which is not important for this outline). Furthermore, there holds the exact commutator

$$\left[H'_{\mathrm{kin}}, b^*_{k,p}\right] = 2\lambda_{k,p}b^*_{k,p} \tag{1.25}$$

which given the *quasi-bosonic* behaviour of the $b^*_{k,p}$ operators suggests an informal relation of the form

$$H'_{\mathrm{kin}} \sim \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k} 2\lambda_{k,p}b^*_{k,p}b_{k,p}, \tag{1.26}$$

and defining operators $h_k, P_k : \ell^2(L_k)\to\ell^2(L_k)$ and a vector $v_k\in\ell^2(L_k)$ by

$$\langle e_p, h_k e_q\rangle = \lambda_{k,p}\delta_{p,q}, \quad P_k = |v_k\rangle\langle v_k|, \quad \langle e_p, v_k\rangle = \sqrt{\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}}, \quad p,q\in L_k, \tag{1.27}$$

this suggests that

$$H_{\mathrm{B}} \sim \sum_{k\in\mathbb{Z}^3_*}\left(2\sum_{p,q\in L_k}\langle e_p,(h_k+P_k)e_q\rangle\, b^*_{k,p}b_{k,q} + 2\,\mathrm{Re}\sum_{p,q\in L_k}\langle e_p, P_k e_q\rangle\, b_{k,p}b_{-k,-q}\right) \tag{1.28}$$

which has the form of a quadratic Hamiltonian with respect to $b^*_{k,p}$.

Exactly Bosonic Bogolubov Factorization

Now, *if* (1.28) were a genuine identity, and *if* the operators $b^*_{k,p}$ were genuinely bosonic (i.e. if $\varepsilon_{k,l}(p,q)=0$), then this would imply that H_{B} would be *diagonalizable* by a Bogolubov transformation $e^{\mathcal{K}}$, i.e. there would exist an (explicit) Bogolubov kernel $\mathcal{K}$ such that

$$e^{\mathcal{K}}H_{\mathrm{B}}e^{-\mathcal{K}} = \sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}(E_k - h_k - P_k) + 2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, E_k e_q\rangle\, b^*_{k,p}b_{k,q} \tag{1.29}$$

where $E_k:\ell^2(L_k)\to\ell^2(L_k)$ is given in terms of h_k and P_k by $E_k = (h_k^{\frac{1}{2}}(h_k+2P_k)h_k^{\frac{1}{2}})^{\frac{1}{2}}$.
In fact (see e.g. [11, Propositions 7.1, 7.6])

$$E_{\mathrm{corr,bos}} = \sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}(E_k - h_k - P_k) \tag{1.30}$$

which explains why we refer to this as the bosonic contribution to the correlation energy.
In the exact bosonic case the transformation $e^{\mathcal{K}}$ would (for a lower bound) technically be superfluous, since "undoing" the transformation shows that

$$H_{\mathrm{B}} = E_{\mathrm{corr,bos}} + 2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, E_k e_q\rangle\, e^{-\mathcal{K}}b^*_{k,p}e^{\mathcal{K}}e^{-\mathcal{K}}b_{k,q}e^{\mathcal{K}} \tag{1.31}$$

and the transformation $e^{\mathcal{K}}$ would additionally satisfy

$$e^{-\mathcal{K}} b_{k,p} e^{\mathcal{K}} = \sum_{q \in L_k} \langle C_k e_p, e_q \rangle \, b_{k,q} + \sum_{q \in L_k} \langle e_{-q}, S_{-k} e_{-p} \rangle \, b^*_{-k,-q} \\ =: b_k(C_k e_p) + b^*_{-k}(S_{-k} e_{-p}) \tag{1.32}$$

for operators $C_k, S_k : \ell^2(L_k) \to \ell^2(L_k)$ given by

$$C_k = \frac{1}{2}(h_k^{-\frac{1}{2}} E_k^{\frac{1}{2}} + h_k^{\frac{1}{2}} E_k^{-\frac{1}{2}}), \quad S_k = \frac{1}{2}(h_k^{-\frac{1}{2}} E_k^{\frac{1}{2}} - h_k^{\frac{1}{2}} E_k^{-\frac{1}{2}}), \tag{1.33}$$

i.e.

$$H_{\mathrm{B}} = E_{\mathrm{corr,bos}} + 2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} \langle e_p, E_k e_q \rangle \left(b_k(C_k e_p) + b^*_{-k}(S_{-k} e_{-p}) \right)^* \left(b_k(C_k e_q) + b^*_{-k}(S_{-k} e_{-q}) \right) \tag{1.34}$$

which is simply an algebraic rewriting of H_{B}, as can be verified by expanding the expression and applying the definitions of E_k, C_k and S_k. Since $E_k \geq 0$, (1.34) immediately implies that $H_{\mathrm{B}} \geq E_{\mathrm{corr,bos}}$.

Quasi-Bosonic Bogolubov Factorization

Returning to the non-exact case, a result to the effect of equation (1.29) was established in [11, Theorem 1.1] (for $\beta = 1$ and potentials obeying $\sum_{k \in \mathbb{Z}^3_*} |k| \, \hat{V}_k < \infty$), in which a unitary operator $\mathcal{U}$ (a product of two quasi-bosonic Bogolubov transformations) was constructed such that

$$\mathcal{U} H_{\mathrm{B}} \mathcal{U}^* \sim E_{\mathrm{corr,bos}} + H'_{\mathrm{kin}} + 2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} \langle e_p, (E_k - h_k) e_q \rangle \, b^*_{k,p} b_{k,q}. \tag{1.35}$$

Note the difference from equation (1.29): We have the additional terms

$$H'_{\mathrm{kin}} - \sum_{k \in \mathbb{Z}^3_*} \sum_{p \in L_k} 2\lambda_{k,p} b^*_{k,p} b_{k,q}$$

which reflects the fact that the relation of equation (1.26) only holds in an indirect sense. One could hope to make this more direct, but in fact this is impossible, as it was also noted in [11, Proposition 10.1] that

$$\sum_{k \in \mathbb{Z}^3_*} \sum_{p \in L_k} 2\lambda_{k,p} b^*_{k,p} b_{k,q} = \mathcal{N}_E H'_{\mathrm{kin}} \tag{1.36}$$

so

$$H'_{\mathrm{kin}} - \sum_{k \in \mathbb{Z}^3_*} \sum_{p \in L_k} 2\lambda_{k,p} b^*_{k,p} b_{k,q} = -(\mathcal{N}_E - 1) H'_{\mathrm{kin}}$$

can not be considered small on its own. It is nonetheless the case that

$$2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} \langle e_p, (E_k - h_k) e_q \rangle \, b^*_{k,p} b_{k,q} \geq 0,$$

so this does suffice to show that $H_{\mathrm{B}} \gtrsim E_{\mathrm{corr,bos}}$, but it appears to preclude a transformation-free approach that could yield something similar to equation (1.34).

By modifying the approach this is however possible: If we similarly "undo" the transformation of equation (1.35) we see that

$$H_{\mathrm{B}} \sim E_{\mathrm{corr,bos}} + \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| |\mathcal{U}^* \tilde{c}_p \mathcal{U}|^2 + 2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} \langle e_p, (E_k - h_k) e_q \rangle \, \mathcal{U}^* b^*_{k,p} \mathcal{U} \mathcal{U}^* b_{k,q} \mathcal{U} \tag{1.37}$$

where we introduced the notation $\tilde{c}_p = \begin{cases} c_p & p \in B_F^c \\ c_p^* & p \in B_F \end{cases}$.

Now, if $\mathcal{U} = e^{\mathcal{K}}$ for a quasi-bosonic kernel $\mathcal{K}$ (defined as a "hybrid" of the kernels defining the two transformations used in [11]) one finds similarly to the exact case that

$$\mathcal{U}^* b_{k,p} \mathcal{U} \sim b_k(C_k e_p) + b^*_{-k}(S_{-k} e_{-p}) \tag{1.38}$$

while the operators $\tilde{c}_p$ obey

$$\mathcal{U}^* \tilde{c}_p \mathcal{U} \sim \tilde{c}_p + d_p^1 + d_p^2 \tag{1.39}$$

where d_p^1 and d_p^2 are given by

$$d_p^1 = \begin{cases} + \sum_{k \in \mathbb{Z}_*^3} 1_{L_k}(p) \tilde{c}_{p-k}^* b_k((C_k - 1)e_p) & p \in B_F^c \\ - \sum_{k \in \mathbb{Z}_*^3} 1_{L_k - k}(p) \tilde{c}_{p+k}^* b_k((C_k - 1)e_{p+k}) & p \in B_F \end{cases} \tag{1.40}$$

and

$$d_p^2 = \begin{cases} + \sum_{k \in \mathbb{Z}_*^3} 1_{L_k}(p) \tilde{c}_{p-k}^* b_{-k}^*(S_{-k} e_{-p}) & p \in B_F^c \\ - \sum_{k \in \mathbb{Z}_*^3} 1_{L_k - k}(p) \tilde{c}_{p+k}^* b_{-k}^*(S_{-k} e_{-p-k}) & p \in B_F \end{cases}, \tag{1.41}$$

respectively (note that these are sums of triples of fermionic creation and annihilation operators). Equation (1.37) consequently suggests an identity of the form[2]

$$\begin{aligned} H_{\mathrm{B}} &\sim E_{\mathrm{corr,bos}} + \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \left| \tilde{c}_p + d_p^1 + d_p^2 \right|^2 \\ &\quad + 2 \sum_{k \in \mathbb{Z}_*^3} \sum_{p,q \in L_k} \langle e_p, (E_k - h_k) e_q \rangle \left(b_k(C_k e_p) + b_{-k}^*(S_{-k} e_{-p}) \right)^* \left(b_k(C_k e_q) + b_{-k}^*(S_{-k} e_{-q}) \right) \end{aligned} \tag{1.42}$$

which *is* a purely algebraic statement. This is of course not exact, but the crucial point is that we can simply take the right-hand side as an *ansatz* and expand it to obtain a genuine identity for H_{B}. This is precisely what we will do in the Sections 2 and 3 (see Theorems 2.1 and 3.1) to obtain the following:

Theorem 1.4. *It holds that*

$$\begin{aligned} H_{\mathrm{B}} &= \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \left(\left| \tilde{c}_p + d_p^1 + d_p^2 \right|^2 + \left| (d_p^1 + d_p^2)^* \right|^2 \right) - 2 \sum_{k \in \mathbb{Z}_*^3} \sum_{p,q \in L_k} \langle e_p, S_k E_k S_k^* e_q \rangle \, \varepsilon_{k,k}(e_p; e_q) \\ &\quad + 2 \sum_{k \in \mathbb{Z}_*^3} \sum_{p,q \in L_k} \langle e_p, (E_k - h_k) e_q \rangle \left(b_k(C_k e_p) + b_{-k}^*(S_{-k} e_{-p}) \right)^* \left(b_k(C_k e_q) + b_{-k}^*(S_{-k} e_{-q}) \right) \\ &\quad + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + \mathcal{E}_{\mathrm{B}} \end{aligned}$$

for an operator $\mathcal{E}_{\mathrm{B}}$ which under Assumption 1.1 obeys

$$\pm \mathcal{E}_{\mathrm{B}} \leq o(1) \left(H'_{\mathrm{kin}} + k_F \right), \quad k_F \to \infty.$$

There are two things to remark about this identity: The first is that all terms on the first two lines of the right-hand side are manifestly non-negative (since $E_k - h_k, S_k E_k S_k^* \geq 0$ and $\varepsilon_{k,k}(e_p; e_q) = \delta_{p,q} \varepsilon_{k,k}(e_p; e_p) \leq 0$), and so despite their apparent complexity these terms can be ignored for a lower bound. This includes in particular all terms with 6 creation and annihilation operators - this is a consequence of the fermionic commutation relations.

The second is that although not anticipated by the motivating relation of equation (1.42), the exchange contribition $E_{\mathrm{corr,ex}}$ automatically appears during the expansion procedure. This identity thus accounts for the full correlation energy.

Handling the Cubic and Quartic Terms

The identity for the bosonizable terms essentially suffices to prove a version of Theorem 1.2 for H_{B}, but the full Hamiltonian H_N also contains the cubic and quartic terms $\mathcal{C}$ and $\mathcal{Q}$. The quartic terms are in a sense "mostly positive", but the non-definite cubic terms are difficult to estimate directly.

[2] A factorization of a similar form was recently used in [10], which inspired this approach.

Incorporation of the Small k Cubic Terms

We will deal with this issue by partially including them in the factorization identity above. To motivate this, let us note that $\mathcal{C}$ can be written as

$$\mathcal{C}=4\,\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*}\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}B_k^*D_k=4\,\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*}\left(\sum_{p\in L_k}\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}b_{k,p}^*\right)D_k \tag{1.43}$$

where the first equality follows from the observations that $D_k^*=D_{-k}$ and $[B_k,D_k^*]=0$.

If we define $w_k\in\ell^2(L_k)$ by $\langle e_p,w_k\rangle=2^{-1}(2\pi)^{-3}\hat{V}_k k_F^{-\beta}$ we can express this as

$$\mathcal{C}=4\,\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*}\left(\sum_{p\in L_k}\langle e_p,w_k\rangle\, b_{k,p}^*\right)D_k \tag{1.44}$$

which suggests how we should modify the *ansatz* we used for H_{B}: To generate expressions of the form $\sum_{p\in L_k}\langle e_p,(\cdot)\rangle\, b_{k,p}^*D_k$ we can modify the quadratic part according to

$$b_k(C_ke_p)+b_{-k}^*(S_{-k}e_{-p})\to b_k(C_ke_p)+b_{-k}^*(S_{-k}e_{-p})+\langle e_p,\eta_k\rangle D_k \tag{1.45}$$

for some $\eta_k\in\ell^2(L_k)$ (to be fixed at the end), and correspondingly include an additional term d_p^3 in the kinetic factorization, where

$$d_p^3=\begin{cases}+\sum_{k\in\mathbb{Z}^3_*}1_{L_k}(p)\,\langle e_p,\eta_k\rangle\,\tilde{c}_{p-k}^*D_k & p\in B_F^c\\ -\sum_{k\in\mathbb{Z}^3_*}1_{L_k-k}(p)\,\langle e_{p+k},\eta_k\rangle\,\tilde{c}_{p+k}^*D_k & p\in B_F\end{cases}. \tag{1.46}$$

In the Sections 4 and 5 (see Theorems 4.1 and 5.1) we show that the specific choice

$$\eta_k=\begin{cases}E_k^{-\frac{3}{2}}h_k^{\frac{1}{2}}w_k & |k|<k_F^{1/3}\\ 0 & \text{otherwise}\end{cases} \tag{1.47}$$

yields the following:

Theorem 1.5. *It holds that*

$$\begin{aligned}
&H_{\mathrm{B}}+4\,\mathrm{Re}\sum_{k\in B(0,k_F^{1/3})\cap\mathbb{Z}^3_*}\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}B_k^*D_k+\frac{k_F^{-\beta}}{2(2\pi)^3}\sum_{k\in B(0,k_F^{1/3})\cap\mathbb{Z}^3_*}\hat{V}_k\frac{2\left\langle v_k,h_k^{-1}v_k\right\rangle}{1+2\left\langle v_k,h_k^{-1}v_k\right\rangle}D_k^*D_k\\
&=\sum_{p\in\mathbb{Z}^3}\left||p|^2-k_F^2\right|\left(\left|\tilde{c}_p+d_p^1+d_p^2+d_p^3\right|^2+\left|(d_p^1+d_p^2+d_p^3)^*\right|^2\right)-2\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\varepsilon_{k,k}(e_p;S_kE_kS_k^*e_p)\\
&\quad+\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}2\left\langle e_p,(E_k-h_k)e_q\right\rangle\left(b_k(C_ke_p)+b_{-k}^*(S_{-k}e_{-p})+\langle e_p,\eta_k\rangle D_k\right)^*\\
&\qquad\qquad\qquad\qquad\cdot\left(b_k(C_ke_q)+b_{-k}^*(S_{-k}e_{-q})+\langle e_q,\eta_k\rangle D_k\right)\\
&\quad+E_{\mathrm{corr,bos}}+E_{\mathrm{corr,ex}}+\mathcal{E}_{\mathrm{B}}+\mathcal{E}_{\mathcal{C}}
\end{aligned}$$

for an operator $\mathcal{E}_{\mathcal{C}}$ which under Assumption 1.1 obeys

$$\pm\mathcal{E}_{\mathcal{C}}\leq o(1)\left(H'_{\mathrm{kin}}+k_F\right),\quad k_F\to\infty.$$

This identity only includes the "small k" part of $\mathcal{C}$, i.e. the sum over $k\in B(0,k_F^{1/3})\cap\mathbb{Z}^3_*$ (the exponent is simply the consequence of eventual optimization). This of course leaves the "large k" terms unaccounted for, but these *can* be estimated directly.

Note also the additional sum involving $D_k^*D_k$ terms, reminiscent of the quartic terms. Such expressions are unavoidable when attempting to include the cubic terms by factorization, but the crucial point here is the obvious inequality

$$\frac{2\left\langle v_k,h_k^{-1}v_k\right\rangle}{1+2\left\langle v_k,h_k^{-1}v_k\right\rangle}\leq 1. \tag{1.48}$$

That this factor is always less than 1 means that we can use the "almost positivity" of the quartic terms to partially cancel these terms.

Estimation of the Remaining Terms

The parts of H_N which remain unaccounted for are the "large k" cubic and quartic terms, which we bound in Section 6. To illustrate how to estimate these, consider for definiteness the cubic terms

$$\mathcal{E}_{\mathcal{C},\mathrm{large}} = \frac{2k_F^{-\beta}}{(2\pi)^3}\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}\hat{V}_k B_k^* D_k. \tag{1.49}$$

The key observation is that if one expands B_k^*, one can write the sum as

$$\begin{aligned}\sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}\hat{V}_k B_k^* D_k &= \sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}\sum_{p\in L_k}\hat{V}_k\tilde{c}_p^*\tilde{c}_{p-k}^* D_k \\ &= \sum_{p\in B_F^c}\tilde{c}_p^*\Bigg(\sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}1_{L_k}(p)\hat{V}_k\tilde{c}_{p-k}^* D_k\Bigg)\end{aligned} \tag{1.50}$$

and so, by the identity[3]

$$H'_{\mathrm{kin}} = \sum_{p\in B_F^c}(|p|^2-\zeta)\,c_p^*c_p + \sum_{p\in B_F}(\zeta-|p|^2)\,c_pc_p^* = \sum_{p\in\mathbb{Z}^3}\big|\,|p|^2-\zeta\big|\,\tilde{c}_p^*\tilde{c}_p \tag{1.51}$$

which is valid for any $\zeta\in[\sup_{p\in B_F}|p|^2, \inf_{p\in B_F^c}|p|^2]$, one can estimate

$$\begin{aligned}&|\langle\Psi,\mathcal{E}_{\mathcal{C},\mathrm{large}}\Psi\rangle| \\ &\leq Ck_F^{-\beta}\sqrt{\sum_{p\in B_F^c}\big|\,|p|^2-\zeta\big|\|\tilde{c}_p\Psi\|^2}\sqrt{\sum_{p\in B_F^c}\big|\,|p|^2-\zeta\big|^{-1}\Bigg\|\sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}1_{L_k}(p)\hat{V}_k\tilde{c}_{p-k}^*D_k\Psi\Bigg\|^2} \\ &\leq Ck_F^{-\beta}\sqrt{\langle\Psi,H'_{\mathrm{kin}}\Psi\rangle\sum_{p\in B_F^c}\big|\,|p|^2-\zeta\big|^{-1}\langle\Psi,A_p^*A_p\Psi\rangle}\end{aligned} \tag{1.52}$$

for

$$A_p = \sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}1_{L_k}(p)\hat{V}_k\tilde{c}_{p-k}^*D_k. \tag{1.53}$$

Clearly $A_p^*A_p\leq A_p^*A_p+A_pA_p^*=\{A_p^*,A_p\}$, and the point is that A_p is a sum of triples of fermionic creation and annihilation operators. As a consequence, the commutator consists only of sums of 4 or less creation and annihilation operators, which combined with the fact that ζ can be chosen such that

$$\sum_{p\in L_k}\big|\,|p|^2-\zeta\big|^{-1}\leq C_\epsilon k_F^{1+\epsilon} \tag{1.54}$$

(which also enters in the estimation of $\mathcal{E}_{\mathrm{B}}$ and $\mathcal{E}_{\mathcal{C}}$ from the previous steps) eventually leads to the bound

$$\pm\mathcal{E}_{\mathcal{C},\mathrm{large}}\leq C_\epsilon k_F^{1-\beta+\epsilon}\sqrt{\sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}\hat{V}_k^2}\,H'_{\mathrm{kin}}. \tag{1.55}$$

The large k quartic terms can be estimated in a similar fashion, with one exception: There remains the term

$$\frac{k_F^{-\beta}}{2(2\pi)^3}\sum_{k\in\mathbb{Z}^3_*\backslash B(0,k_F^{1/3})}\hat{V}_k\sum_{p,q\in A\cap(A+k)}c_p^*c_{q-k}^*c_qc_{p-k} \tag{1.56}$$

where $A=\mathbb{Z}^3\backslash\overline{B}(0,2k_F)$, which is to say the part of the interaction which involves momenta exclusively "far away" from the Fermi ball. This condition can however be exploited to also control this term in the same form as the other terms.

[3]This is a consequence of particle-hole symmetry, i.e. the identity $\mathcal{N}_E:=\sum_{p\in B_F^c}c_p^*c_p=\sum_{p\in B_F}c_pc_p^*$ which is valid on the N-particle space due to the fact that $N=|B_F|$.

Concluding Theorem 1.2

With all the estimates in place we thus obtain the first part of Theorem 1.2, i.e. the inequality

$$H_N \geq E_{\mathrm{FS}} + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + \mathcal{E} \tag{1.57}$$

where $\mathcal{E}$ obeys $\mathcal{E} \geq -C_{V,\epsilon} k_F^{-\frac{1}{6}+2(1-\beta)+\epsilon}(H'_{\mathrm{kin}} + k_F)$, but not the second part, i.e. the estimate

$$\left\langle \Psi, H'_{\mathrm{kin}} \Psi \right\rangle \leq C_\epsilon k_F^{3-2\beta+\epsilon} \tag{1.58}$$

for low-lying states Ψ. This however follows as a simple consequence of the first inequality, since we can write

$$2(H_N - E_{\mathrm{FS}}) = H'_{\mathrm{kin}} + (\tilde{H}_{\mathrm{B}} + \tilde{\mathcal{C}} + \tilde{\mathcal{Q}}) \tag{1.59}$$

where the tilde quantities are the same as those of equation (1.19) up to the replacement $\hat{V}_k \to 2\,\hat{V}_k$. Then by the first part (note that $\tilde{E}_{\mathrm{corr,ex}} \geq 0$)

$$\begin{aligned}\tilde{H}_{\mathrm{B}} + \tilde{\mathcal{C}} + \tilde{\mathcal{Q}} &\geq \tilde{E}_{\mathrm{corr,bos}} + \tilde{E}_{\mathrm{corr,ex}} - \tilde{C}_{V,\epsilon} k_F^{-\frac{1}{6}+2(1-\beta)+\epsilon}\big(H'_{\mathrm{kin}} + k_F\big) \\ &\geq \tilde{E}_{\mathrm{corr,bos}} - o(1)\big(H'_{\mathrm{kin}} + k_F\big), \quad k_F \to \infty,\end{aligned} \tag{1.60}$$

so

$$(1 - o(1))H'_{\mathrm{kin}} \leq 2(H_N - E_{\mathrm{FS}}) - \tilde{E}_{\mathrm{corr,bos}} + C k_F, \quad k_F \to \infty, \tag{1.61}$$

from which the second part follows by proving that $\tilde{E}_{\mathrm{corr,bos}} \leq C_\epsilon k_F^{3-2\beta+\epsilon}$.

Organization of the paper. In Section 2 we will extract the correlation energy from H_{B} by an explicit factorization. The error $\mathcal{E}_{\mathrm{B}}$ of this step is estimated in Section 3. In Section 4 we extend the exact factorization to include also the low-momentum part of the cubic terms $\mathcal{C}$. The error $\mathcal{E}_{\mathcal{C}}$ of this step is estimated in Section 5. All of the remaining terms are estimated in Section 6, leading to the conclusion of Theorem 1.2.

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2 Extraction of the Correlation Energy by Factorization

In this section we perform the computations leading to the factorized expression for H_{B}.
For convenience we recall that the operators $E_k, C_k, S_k : \ell^2(L_k) \to \ell^2(L_k)$ are defined by

$$E_k = (h_k^{\frac{1}{2}}(h_k + 2P_k)h_k^{\frac{1}{2}})^{\frac{1}{2}} \tag{2.1}$$

and

$$C_k = \frac{1}{2}(h_k^{-\frac{1}{2}} E_k^{\frac{1}{2}} + h_k^{\frac{1}{2}} E_k^{-\frac{1}{2}}), \quad S_k = \frac{1}{2}(h_k^{-\frac{1}{2}} E_k^{\frac{1}{2}} - h_k^{\frac{1}{2}} E_k^{-\frac{1}{2}}), \tag{2.2}$$

while the operators d_p^1 and d_p^2 are defined by

$$d_p^1 = \begin{cases} +\sum_{k\in\mathbb{Z}^3_*} 1_{L_k}(p)\tilde{c}^*_{p-k} b_k((C_k - 1)e_p) & p \in B_F^c \\ -\sum_{k\in\mathbb{Z}^3_*} 1_{L_k - k}(p)\tilde{c}^*_{p+k} b_k((C_k - 1)e_{p+k}) & p \in B_F \end{cases} \tag{2.3}$$

and

$$d_p^2 = \begin{cases} +\sum_{k\in\mathbb{Z}^3_*} 1_{L_k}(p)\tilde{c}^*_{p-k} b^*_{-k}(S_{-k}e_{-p}) & p \in B_F^c \\ -\sum_{k\in\mathbb{Z}^3_*} 1_{L_k - k}(p)\tilde{c}^*_{p+k} b^*_{-k}(S_{-k}e_{-p-k}) & p \in B_F \end{cases}, \tag{2.4}$$

respectively. Our goal is the following:

Theorem 2.1. *It holds that*

$$\begin{aligned} H_{\mathrm{B}} = &\sum_{p\in\mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \Big(\left|\tilde{c}_p + d_p^1 + d_p^2\right|^2 + \left|(d_p^1 + d_p^2)^*\right|^2 \Big) - 2 \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in L_k} \varepsilon_{k,k}(e_p; S_k E_k S_k^* e_p) \\ &+ 2 \sum_{k\in\mathbb{Z}^3_*} \sum_{p,q\in L_k} \langle e_p, (E_k - h_k)e_q\rangle \left(b_k(C_k e_p) + b^*_{-k}(S_{-k}e_{-p})\right)^* \left(b_k(C_k e_q) + b^*_{-k}(S_{-k}e_{-q})\right) \\ &+ E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + \mathcal{E}_{\mathrm{B}} \end{aligned}$$

for an operator $\mathcal{E}_{\mathrm{B}}$ defined below.

Quasi-Bosonic Operators

Before we start in earnest we will recall some properties of the quasi-bosonic operators we must consider. First, we define for general symmetric operators $A_k, B_k : \ell^2(L_k) \to \ell^2(L_k)$ the expressions

$$Q_1^k(A_k) = \sum_{p,q\in L_k} \langle e_p, A_k e_q\rangle\, b_{k,p}^* b_{k,q}, \quad Q_2^k(B_k) = 2\,\mathrm{Re} \sum_{p,q\in L_k} \langle e_p, B_k e_q\rangle\, b_{k,p} b_{-k,-q}, \tag{2.5}$$

in terms of which the interaction part of H_N can be written

$$\frac{k_F^{-\beta}}{2(2\pi)^3} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k\big(2B_k^* B_k + B_k B_{-k} + B_{-k}^* B_k^*\big) = \sum_{k\in\mathbb{Z}^3_*} \Big(2\,Q_1^k(P_k) + Q_2^k(P_k)\Big) \tag{2.6}$$

for $P_k = |v_k\rangle\langle v_k|$ with $v_k \in \ell^2(L_k)$ defined by $\langle e_p, v_k\rangle = \sqrt{2^{-1}(2\pi)^{-3}\hat{V}_k k_F^{-\beta}}$.
For any $\varphi \in \ell^2(L_k)$ we also define the generalized excitation operators

$$b_k(\varphi) = \sum_{p\in L_k} \langle \varphi, e_p\rangle\, b_{k,p}, \quad b_k^*(\varphi) = \sum_{p\in L_k} \langle e_p, \varphi\rangle\, b_{k,p}^*, \tag{2.7}$$

which lets us write $Q_1^k(A_k)$ and $Q_2^k(B_k)$ as

$$Q_1^k(A_k) = \sum_{p\in L_k} b_k^*(A_k e_p) b_{k,p}, \quad Q_2^k(B_k) = 2\,\mathrm{Re} \sum_{p\in L_k} b_k(B_k e_p) b_{-k,-p}. \tag{2.8}$$

The generalized excitation operators obey the quasi-bosonic commutation relations

$$\begin{aligned} [b_k(\varphi), b_l(\psi)] &= [b_k^*(\varphi), b_l^*(\psi)] = 0 \\ [b_k(\varphi), b_l^*(\psi)] &= \delta_{k,l}\,\langle \varphi, \psi\rangle + \varepsilon_{k,l}(\varphi;\psi) \end{aligned} \tag{2.9}$$

where the *exchange correction* $\varepsilon_{k,l}(\varphi;\psi)$ is given by

$$\varepsilon_{k,l}(\varphi;\psi) = -\sum_{q\in L_k\cap L_l} \langle \varphi, e_q\rangle \langle e_q, \psi\rangle\, \tilde{c}_{q-l}^* \tilde{c}_{q-k} - \sum_{q\in (L_k-k)\cap(L_l-l)} \langle \varphi, e_{q+k}\rangle \langle e_{q+l}, \psi\rangle\, \tilde{c}_{q+l}^* \tilde{c}_{q+k}. \tag{2.10}$$

Below we will often encounter expressions of the "trace form" $\sum_{i=1}^n q(Se_i, Te_i)$ for some bilinear mapping q, for example

$$Q_1^k(A_k) = \sum_{p\in L_k} b_k^*(A_k e_p) b_k(e_p) = \sum_{p\in L_k} q(A_k e_p, e_p), \quad q(\varphi,\psi) = b_k^*(\varphi) b_k(\psi). \tag{2.11}$$

For that reason we recall the following lemma which simplifies the calculations with these significantly:

Lemma 2.2. *Let V be an n-dimensional Hilbert space and let $q : V\times V \to W$ be a sesquilinear mapping into a vector space W. Then for any orthonormal basis $(e_i)_{i=1}^n$ of V and operators $S, T : V \to V$ it holds that*

$$\sum_{i=1}^n q(Se_i, Te_i) = \sum_{i=1}^n q(ST^* e_i, e_i).$$

The lemma is immediate by orthonormal expansion.
We remark that we will only consider $\ell^2(L_k)$ as a *real* vector space (so sesquilinearity is simply bilinearity).
Finally we point out that the operators E_k, C_k and S_k all obey a symmetry condition of the form

$$\langle e_p, E_k e_q\rangle = \langle e_{-p}, E_{-k} e_{-q}\rangle\,, \quad p, q \in L_k, \tag{2.12}$$

since these are directly determined by h_k and P_k which also satisfy this.

2.1 Factorization of the Interaction Terms

We begin with the terms $\sum_{k\in\mathbb{Z}^3_*}\big(2\,Q_1^k(P_k)+Q_2^k(P_k)\big)$ which come from the interaction. Since we will also need this for the kinetic terms below, we state a general identity:

Proposition 2.3. *For symmetric operators $A_k:\ell^2(L_k)\to\ell^2(L_k)$, $k\in\mathbb{Z}^3_*$, obeying*

$$\langle e_p, A_k e_q\rangle = \langle e_{-p}, A_{-k} e_{-q}\rangle\,,\quad p,q\in L_k,$$

it holds that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\,\langle e_p, A_k e_q\rangle \left(b_k(C_k e_p)+b^*_{-k}(S_{-k}e_{-p})\right)^*\left(b_k(C_k e_q)+b^*_{-k}(S_{-k}e_{-q})\right)\\
&=\sum_{k\in\mathbb{Z}^3_*}\Big(2\,Q_1^k(C_kA_kC_k^*+S_kA_kS_k^*)+Q_2^k(C_kA_kS_k^*+S_kA_kC_k^*)\Big)\\
&+\sum_{k\in\mathbb{Z}^3_*}2\,\mathrm{tr}(S_kA_kS_k^*)+2\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\varepsilon_{k,k}(e_p;S_kA_kS_k^*e_p).
\end{aligned}$$

Proof: By expanding the terms and applying Lemma 2.2 we see that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\,\langle e_p, A_k e_q\rangle \left(b_k(C_k e_p)+b^*_{-k}(S_{-k}e_{-p})\right)^*\left(b_k(C_k e_q)+b^*_{-k}(S_{-k}e_{-q})\right)\\
&=2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, A_k e_q\rangle\, b_k^*(C_ke_p)b_k(C_ke_q)+2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, A_k e_q\rangle\, b_{-k}(S_{-k}e_{-p})b^*_{-k}(S_{-k}e_{-q})\\
&+4\,\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, A_k e_q\rangle\, b_{-k}(S_{-k}e_{-p})b_k(C_ke_q) &(2.13)\\
&=2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, C_kA_kC_k^* e_q\rangle\, b^*_{k,p}b_{k,q}+2\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, S_kA_kS_k^* e_q\rangle\, b_{k,p}b^*_{k,q}\\
&+4\,\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, S_kA_kC_k^* e_q\rangle\, b_{k,p}b_{-k,-q}
\end{aligned}$$

where we also took advantage of the symmetry of A_k, C_k and S_k under $(k,p,q)\to(-k,-p,-q)$. Now

$$\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, C_kA_kC_k^* e_q\rangle\, b^*_{k,p}b_{k,q}=Q_1^k(C_kA_kC_k^*) \tag{2.14}$$

by definition, while

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, S_kA_kS_k^* e_q\rangle\, b_{k,p}b^*_{k,q}&=\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, S_kA_kS_k^* e_q\rangle\, b^*_{k,q}b_{k,p}+\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\langle e_p, S_kA_kS_k^* e_p\rangle\\
&+\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, S_kA_kS_k^* e_q\rangle\, \varepsilon_{k,k}(e_p;e_q) &(2.15)\\
&=\sum_{k\in\mathbb{Z}^3_*}Q_1^k(S_kA_kS_k^*)+\sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}(S_kA_kS_k^*)+\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\varepsilon_{k,k}(e_p;S_kA_kS_k^*e_p)
\end{aligned}$$

by symmetry of A and the fact that the matrix elements are real-valued. Similarly, renaming variables and using the symmetries involved once more

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, S_kA_kC_k^* e_q\rangle\, b_{k,p}b_{-k,-q}&=\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle C_kA_kS_k^* e_p, e_q\rangle\, b_{-k,-q}b_{k,p} &(2.16)\\
&=\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, C_kA_kS_k^* e_q\rangle\, b_{k,p}b_{-k,-q}
\end{aligned}$$

which implies

$$4\,\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, S_kA_kC_k^* e_q\rangle\, b_{k,p}b_{-k,-q}=2\,\mathrm{Re}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}\langle e_p, (C_kA_kS_k^*+S_kA_kC_k^*) e_q\rangle\, b_{k,p}b_{-k,-q}$$

$$= \sum_{k\in\mathbb{Z}^3_*} Q_2^k(C_kA_kS_k^* + S_kA_kC_k^*). \tag{2.17}$$

$\square$

This yields the following identity for the interaction terms:

Proposition 2.4. *It holds that*

$$\sum_{k\in\mathbb{Z}^3_*}\left(2\,Q_1^k(P_k) + Q_2^k(P_k)\right) = -\sum_{k\in\mathbb{Z}^3_*} 2\,Q_1^k(h_k) + E_{\mathrm{corr,bos}} - 2\sum_{k\in\mathbb{Z}^3_*}\varepsilon_{k,k}(e_p; S_kE_kS_k^*e_p)$$
$$+ \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\,\langle e_p, E_ke_q\rangle\left(b_k(C_ke_p) + b_{-k}^*(S_{-k}e_{-p})\right)^*\left(b_k(C_ke_q) + b_{-k}^*(S_{-k}e_{-q})\right).$$

Proof: From the definitions of the equations (2.1) and (2.2) it readily follows that

$$\begin{aligned} C_kE_kC_k^* + S_kE_kS_k^* &= \frac{1}{2}(h_k + 2P_k + h_k) = h_k + P_k \\ C_kE_kS_k^* + S_kE_kC_k^* &= \frac{1}{2}(h_k + 2P_k - h_k) = P_k \end{aligned} \tag{2.18}$$

and so the previous proposition tells us that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\,\langle e_p, E_ke_q\rangle\left(b_k(C_ke_p) + b_{-k}^*(S_{-k}e_{-p})\right)^*\left(b_k(C_ke_q) + b_{-k}^*(S_{-k}e_{-q})\right) \\ &= \sum_{k\in\mathbb{Z}^3_*}\left(2\,Q_1^k(h_k + P_k) + Q_2^k(P_k)\right) + \sum_{k\in\mathbb{Z}^3_*} 2\,\mathrm{tr}(S_kE_kS_k^*) + 2\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\varepsilon_{k,k}(e_p; S_kE_kS_k^*e_p) \end{aligned} \tag{2.19}$$

which can be rearranged for the claim since

$$2\,S_kE_kS_k^* = h_k + P_k - \frac{1}{2}\left(h_k^{-\frac{1}{2}}E_kh_k^{\frac{1}{2}} + h_k^{\frac{1}{2}}E_kh_k^{-\frac{1}{2}}\right) \tag{2.20}$$

whence

$$-\sum_{k\in\mathbb{Z}^3_*} 2\,\mathrm{tr}(S_kE_kS_k^*) = \sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}(E_k - h_k - P_k) = E_{\mathrm{corr,bos}} \tag{2.21}$$

as calculated in [11, Propositions 7.1, 7.6].

$\square$

2.2 Factorization of the Kinetic Terms

Clearly

$$\begin{aligned} \sum_{p\in\mathbb{Z}^3}\left|\,|p|^2 - k_F^2\right|\left|\tilde{c}_p + d_p^1 + d_p^2\right|^2 = H'_{\mathrm{kin}} &+ \sum_{p\in\mathbb{Z}^3} 2\left|\,|p|^2 - k_F^2\right|\mathrm{Re}\left(\tilde{c}_p^*d_p^1 + \tilde{c}_p^*d_p^2\right) \\ &+ \sum_{p\in\mathbb{Z}^3}\left|\,|p|^2 - k_F^2\right|\left((d_p^1)^*d_p^1 + 2\,\mathrm{Re}\left((d_p^1)^*d_p^2\right) + (d_p^2)^*d_p^2\right) \end{aligned} \tag{2.22}$$

so we consider the sums on the right-hand side in order. First the simplest:

Proposition 2.5. *It holds that*

$$\begin{aligned} \sum_{p\in\mathbb{Z}^3} 2\left|\,|p|^2 - k_F^2\right|\mathrm{Re}\left(\tilde{c}_p^*d_p^1\right) &= \sum_{k\in\mathbb{Z}^3_*} 2\,Q_1^k((C_k - 1)h_k + h_k(C_k^* - 1)) \\ \sum_{p\in\mathbb{Z}^3} 2\left|\,|p|^2 - k_F^2\right|\mathrm{Re}\left(\tilde{c}_p^*d_p^2\right) &= \sum_{k\in\mathbb{Z}^3_*} Q_2^k(S_kh_k + h_kS_k^*). \end{aligned}$$

Proof: For $p \in B_F^c$ we have from the definitions of the equations (2.3) and (2.4) that

$$\begin{aligned} \sum_{p\in B_F^c} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^1 &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* \tilde{c}_{p-k}^* b_k((C_k - 1)e_p) \\ \sum_{p\in B_F^c} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^2 &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* \tilde{c}_{p-k}^* b_{-k}^*(S_{-k}e_{-p}) \end{aligned} \tag{2.23}$$

while for $p \in B_F$ (after substituting $p \to p - k$ in the inner sums)

$$\begin{aligned} \sum_{p\in B_F} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^1 &= -\sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} \left| |p-k|^2 - k_F^2 \right| \tilde{c}_{p-k}^* \tilde{c}_p^* b_k((C_k - 1)e_p) \\ \sum_{p\in B_F} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^2 &= -\sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} \left| |p-k|^2 - k_F^2 \right| \tilde{c}_{p-k}^* \tilde{c}_p^* b_{-k}^*(S_{-k}e_{-p}). \end{aligned} \tag{2.24}$$

In the d_p^1 case this implies that together

$$\begin{aligned} \sum_{p\in\mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^1 &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} \Big(\left| |p|^2 - k_F^2 \right| + \left| |p-k|^2 - k_F^2 \right| \Big) \tilde{c}_p^* \tilde{c}_{p-k}^* b_k((C_k - 1)e_p) \\ &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\lambda_{k,p} b_{k,p}^* b_k((C_k - 1)e_p) = \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\, b_{k,p}^* b_k((C_k - 1)h_k e_p) \\ &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\, b_k^*(h_k(C_k^* - 1)e_p) b_{k,p} \end{aligned} \tag{2.25}$$

whence

$$\begin{aligned} 2\,\mathrm{Re} \sum_{p\in\mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^1 &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\big(b_k^*(h_k(C_k^* - 1)e_p) b_{k,p} + b_{k,p}^* b_k(h_k(C_k^* - 1)e_p) \big) \\ &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\, b_k^*(((C_k - 1)h_k + h_k(C_k^* - 1))e_p) b_{k,p} \\ &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\, Q_1^k((C_k - 1)h_k + h_k(C_k^* - 1)) \end{aligned} \tag{2.26}$$

and similarly, using the $(k,p,q) \to (-k,-p,-q)$ symmetry,

$$\begin{aligned} \sum_{p\in\mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^2 &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\lambda_{k,p} b_{k,p}^* b_{-k}^*(S_{-k}e_{-p}) = \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} 2\, b_{k,p}^* b_{-k}^*(S_{-k}h_{-k}e_{-p}) \\ &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} \big(b_k^*(h_k S_k^* e_p) b_{-k,-p}^* + b_{-k,-p}^* b_k^*(S_k h_k e_p) \big) \\ &= \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in L_k} b_k^*((S_k h_k + h_k S_k^*)e_p) b_{-k,-p}^* \end{aligned} \tag{2.27}$$

yielding

$$2\,\mathrm{Re} \sum_{p\in\mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^2 = \sum_{k\in\mathbb{Z}_*^3} Q_2^k(S_k h_k + h_k S_k^*). \tag{2.28}$$

□

To state the identities for terms of the form $\sum_{p\in\mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| (d_p)^* d_p$ we must define some error terms. The first is

$$\begin{aligned} \mathcal{E}_{\mathrm{B},1} = &\sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in L_k\cap L_l} \left| |p|^2 - k_F^2 \right| \tilde{c}_{p-l}^* \left[b_k^*((C_k - 1)e_p), b_l((C_l - 1)e_p) \right] \tilde{c}_{p-k} \\ &+ \sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in(L_k-k)\cap(L_l-l)} \left| |p|^2 - k_F^2 \right| \tilde{c}_{p+l}^* \left[b_k^*((C_k - 1)e_{p+k}), b_l((C_l - 1)e_{p+l}) \right] \tilde{c}_{p+k} \end{aligned} \tag{2.29}$$

and we note that the two sums are of a similar form, in that the second can be obtained from the first by the substitions $(L_k, \tilde{c}_{p-k}, e_p) \to (L_k - k, \tilde{c}_{p+k}, e_{p+k})$, which reflects the fact that the definitions of d_p^1 differ in this way depending on whether $p \in B_F^c$ or $p \in B_F$. We can thus write $\mathcal{E}_{\mathrm{B},1} = \mathcal{E}_{\mathrm{B},1}^{(1)} + \mathcal{E}_{\mathrm{B},1}^{(2)}$ for

$$\mathcal{E}_{\mathrm{B},1}^{(1)} = \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in L_k\cap L_l} ||p|^2 - k_F^2|\, \tilde{c}_{p-l}^* \left[b_k^*((C_k-1)e_p), b_l((C_l-1)e_p)\right] \tilde{c}_{p-k} \tag{2.30}$$

with $\mathcal{E}_{\mathrm{B},1}^{(2)}$ obtained from this by the above substitution. In this notation we similarly define $\mathcal{E}_{\mathrm{B},m} = \mathcal{E}_{\mathrm{B},m}^{(1)} + \mathcal{E}_{\mathrm{B},m}^{(2)}$ for $m = 2, \ldots, 5$ by

$$\begin{aligned}
\mathcal{E}_{\mathrm{B},2}^{(1)} &= \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in L_k\cap L_l} ||p|^2 - k_F^2|\, \tilde{c}_{p-l}^* b_k^*((C_k-1)e_p) \left[\tilde{c}_{p-k}, b_{-l}^*(S_{-l}e_{-p})\right] \\
\mathcal{E}_{\mathrm{B},3}^{(1)} &= \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in L_k\cap L_l} ||p|^2 - k_F^2| \tilde{c}_{p-l}^* \left[b_{-k}(S_{-k}e_{-p}), b_{-l}^*(S_{-l}e_{-p})\right] \tilde{c}_{p-k} \\
\mathcal{E}_{\mathrm{B},4}^{(1)} &= \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in L_k\cap L_l} ||p|^2 - k_F^2| \tilde{c}_{p-l}^* b_{-k}(S_{-k}e_{-p}) \left[\tilde{c}_{p-k}, b_{-l}^*(S_{-l}e_{-p})\right] \\
\mathcal{E}_{\mathrm{B},5}^{(1)} &= \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in L_k\cap L_l} ||p|^2 - k_F^2| \left[\tilde{c}_{p-l}, b_{-k}^*(S_{-k}e_{-p})\right]^* \left[\tilde{c}_{p-k}, b_{-l}^*(S_{-l}e_{-p})\right]
\end{aligned} \tag{2.31}$$

The identities then take the following forms:

Proposition 2.6. *It holds that*

$$\begin{aligned}
\sum_{p\in\mathbb{Z}^3} ||p|^2 - k_F^2|\, (d_p^1)^* d_p^1 &= \sum_{k\in\mathbb{Z}^3_*} 2\, Q_1^k((C_k-1)h_k(C_k^*-1)) - \sum_{p\in\mathbb{Z}^3} ||p|^2 - k_F^2|\, d_p^1 (d_p^1)^* - \mathcal{E}_{\mathrm{B},1} \\
2\,\mathrm{Re} \sum_{p\in\mathbb{Z}^3} ||p|^2 - k_F^2|\, (d_p^1)^* d_p^2 &= \sum_{k\in\mathbb{Z}^3_*} Q_2^k((C_k-1)h_k S_k^* + S_k h_k (C_k^*-1)) \\
&\quad - 2\,\mathrm{Re} \sum_{p\in\mathbb{Z}^3} ||p|^2 - k_F^2|\, d_p^2 (d_p^1)^* - 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},2}) \\
\sum_{p\in\mathbb{Z}^3} ||p|^2 - k_F^2|\, (d_p^2)^* d_p^2 &= \sum_{k\in\mathbb{Z}^3_*} 2\, Q_1^k(S_k h_k S_k^*) + 2 \sum_{k\in\mathbb{Z}^3_*} \mathrm{tr}(S_k h_k S_k^*) + 2 \sum_{k\in\mathbb{Z}^3_*} \varepsilon_{k,k}(e_p; S_k h_k S_k^* e_p) \\
&\quad - \sum_{p\in\mathbb{Z}^3} ||p|^2 - k_F^2|\, d_p^2 (d_p^2)^* - \mathcal{E}_{\mathrm{B},3} - 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},4}) - \mathcal{E}_{\mathrm{B},5}.
\end{aligned}$$

Proof: The first part of the derivation is similar for all three terms, so we focus on $\sum_{p\in\mathbb{Z}^3} ||p|^2 - k_F^2|\, (d_p^1)^* d_p^1$. By equation (2.3) we have for $p \in B_F^c$ that

$$\begin{aligned}
\sum_{p\in B_F^c} ||p|^2 - k_F^2|\, (d_p^1)^* d_p^1 &= \sum_{p\in B_F^c} \sum_{k,l\in\mathbb{Z}^3_*} 1_{L_k\cap L_l}(p) ||p|^2 - k_F^2|\, b_k^*((C_k-1)e_p) \tilde{c}_{p-k} \tilde{c}_{p-l}^* b_l((C_l-1)e_p) \\
&= \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in L_k} ||p|^2 - k_F^2|\, b_k^*((C_k-1)e_p) b_k((C_k-1)e_p) \\
&\quad - \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in L_k\cap L_l} ||p|^2 - k_F^2|\, b_k^*((C_k-1)e_p) \tilde{c}_{p-l}^* \tilde{c}_{p-k} b_l((C_l-1)e_p)
\end{aligned} \tag{2.32}$$

and for $p \in B_F$ that

$$\begin{aligned}
\sum_{p\in B_F} ||p|^2 - k_F^2|\, (d_p^1)^* d_p^1 &= \sum_{p\in B_F} \sum_{k,l\in\mathbb{Z}^3_*} 1_{(L_k-k)\cap(L_l-l)}(p) ||p|^2 - k_F^2|\, b_k^*((C_k-1)e_{p+k}) \tilde{c}_{p+k} \tilde{c}_{p+l}^* b_l((C_l-1)e_{p+l}) \\
&= \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in L_k} ||p-k|^2 - k_F^2|\, b_k^*((C_k-1)e_p) b_k((C_k-1)e_p) \\
&\quad - \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in(L_k-k)\cap(L_l-l)} ||p|^2 - k_F^2|\, b_k^*((C_k-1)e_{p+k}) \tilde{c}_{p+l}^* \tilde{c}_{p+k} b_l((C_l-1)e_{p+l}).
\end{aligned} \tag{2.33}$$

When summing over all $p \in \mathbb{Z}^3$ the first terms combine to form

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}\Big(\,||p|^2-k_F^2|+||p-k|^2-k_F^2|\Big)\,b_k^*((C_k-1)e_p)b_k((C_k-1)e_p) \\ &=\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}2\lambda_{k,p}b_k^*((C_k-1)e_p)b_k((C_k-1)e_p)=\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}2\,Q_1^k((C_k-1)h_k(C_k^*-1)) \end{aligned} \tag{2.34}$$

while for the second we have e.g. (using that $[b^*(\cdot),\tilde{c}^*]=0$)

$$\begin{aligned} &\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,b_k^*((C_k-1)e_p)\tilde{c}^*_{p-l}\tilde{c}_{p-k}b_l((C_l-1)e_p) \\ &=\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,\tilde{c}^*_{p-l}b_k^*((C_k-1)e_p)b_l((C_l-1)e_p)\tilde{c}_{p-k} \\ &=\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,\tilde{c}^*_{p-l}b_l((C_l-1)e_p)b_k^*((C_k-1)e_p)\tilde{c}_{p-k} \\ &+\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,\tilde{c}^*_{p-l}\,[b_k^*((C_k-1)e_p),b_l((C_l-1)e_p)]\,\tilde{c}_{p-k} \\ &=\sum_{p\in B_F^c}||p|^2-k_F^2|\,d_p^1(d_p^1)^*+\mathcal{E}^{(1)}_{\mathrm{B},1}. \end{aligned} \tag{2.35}$$

For $\sum_{p\in\mathbb{Z}^3}||p|^2-k_F^2|\,(d_p^1)^*d_p^2$ one likewise finds terms combining to form

$$\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}b^*_{-k,-p}b_k^*(((C_k-1)h_kS_k^*+S_kh_k(C_k^*-1))e_p), \tag{2.36}$$

yielding the corresponding Q_2^k terms when taking $2\,\mathrm{Re}$, and additional terms of the form

$$\begin{aligned} &\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,b_k^*((C_k-1)e_p)\tilde{c}^*_{p-l}\tilde{c}_{p-k}b^*_{-l}(S_{-l}e_{-p}) \\ &=\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,\tilde{c}^*_{p-l}b_k^*((C_k-1)e_p)b^*_{-l}(S_{-l}e_{-p})\tilde{c}_{p-k} \\ &+\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,\tilde{c}^*_{p-l}b_k^*((C_k-1)e_p)\left[\tilde{c}_{p-k},b^*_{-l}(S_{-l}e_{-p})\right] \\ &=\sum_{p\in B_F^c}||p|^2-k_F^2|\,d_p^2(d_p^1)^*+\mathcal{E}^{(1)}_{\mathrm{B},2} \end{aligned} \tag{2.37}$$

where we also used that $[b_k^*(\cdot),b_l^*(\cdot)]=0$.
Lastly one has for $\sum_{p\in\mathbb{Z}^3}||p|^2-k_F^2|\,(d_p^2)^*d_p^2$ terms combining to yield

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in L_k}2\lambda_{k,p}b_{-k}(S_{-k}e_{-p})b^*_{-k}(S_{-k}e_{-p}) \\ &=\sum_{k\in\mathbb{Z}^3_*}2\,Q_1^k(S_kh_kS_k^*)+2\sum_{k\in\mathbb{Z}^3_*}\mathrm{tr}(S_kh_kS_k^*)+2\sum_{k\in\mathbb{Z}^3_*}\varepsilon_{k,k}(e_p;S_kh_kS_k^*e_p), \end{aligned} \tag{2.38}$$

the right-hand side following as in equation (2.15), and terms of the form

$$\begin{aligned} &\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\,b_{-k}(S_{-k}e_{-p})\tilde{c}^*_{p-l}\tilde{c}_{p-k}b^*_{-l}(S_{-l}e_{-p}) \\ &=\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\tilde{c}^*_{p-l}b_{-k}(S_{-k}e_{-p})b^*_{-l}(S_{-l}e_{-p})\tilde{c}_{p-k} \\ &+2\,\mathrm{Re}\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\tilde{c}^*_{p-l}b_{-k}(S_{-k}e_{-p})\left[\tilde{c}_{p-k},b^*_{-l}(S_{-l}e_{-p})\right] \\ &+\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}||p|^2-k_F^2|\left[\tilde{c}_{p-l},b^*_{-k}(S_{-k}e_{-p})\right]^*\left[\tilde{c}_{p-k},b^*_{-l}(S_{-l}e_{-p})\right] \end{aligned} \tag{2.39}$$

$$= \sum_{p \in B_F^c} \left| |p|^2 - k_F^2 \right| d_p^2 (d_p^2)^* + \mathcal{E}_{\mathrm{B},3}^{(1)} + 2\,\mathrm{Re}\Big(\mathcal{E}_{\mathrm{B},4}^{(1)}\Big) + \mathcal{E}_{\mathrm{B},5}^{(1)}.$$

□

We can now conclude the following identity for H'_{kin}:

Proposition 2.7. *It holds that*

$$\begin{aligned} H'_{\mathrm{kin}} &= \sum_{k \in \mathbb{Z}_*^3} 2\, Q_1^k(h_k) + \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \Big(\left| \tilde{c}_p + d_p^1 + d_p^2 \right|^2 + \left| (d_p^1 + d_p^2)^* \right|^2 \Big) \\ &- \sum_{k \in \mathbb{Z}_*^3} \sum_{p,q \in L_k} 2 \left\langle e_p, h_k e_q \right\rangle \big(b_k(C_k e_p) + b_{-k}^*(C_{-k} e_{-p}) \big)^* \big(b_k(C_k e_q) + b_{-k}^*(C_{-k} e_{-q}) \big) \\ &+ \mathcal{E}_{\mathrm{B},1} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},2}) + \mathcal{E}_{\mathrm{B},3} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},4}) + \mathcal{E}_{\mathrm{B},5}. \end{aligned}$$

Proof: By rearranging the terms of equation (2.22) and inserting the identities we have derived we find

$$\begin{aligned} H'_{\mathrm{kin}} &= \sum_{k \in \mathbb{Z}_*^3} 2\, Q_1^k(h_k) + \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \Big(\left| \tilde{c}_p + d_p^1 + d_p^2 \right|^2 + \left| (d_p^1 + d_p^2)^* \right|^2 \Big) \\ &+ \mathcal{E}_{\mathrm{B},1} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},2}) + \mathcal{E}_{\mathrm{B},3} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},4}) + \mathcal{E}_{\mathrm{B},5} \\ &- \sum_{k \in \mathbb{Z}_*^3} \Big(2\, Q_1^k(C_k h_k C_k^* + S_k h_k S_k^*) + Q_2^k(C_k h_k S_k^* + S_k h_k C_k^*) \Big) \\ &- 2 \sum_{k \in \mathbb{Z}_*^3} \mathrm{tr}(S_k h_k S_k^*) - 2 \sum_{k \in \mathbb{Z}_*^3} \varepsilon_{k,k}(e_p; S_k h_k S_k^* e_p) \end{aligned} \tag{2.40}$$

and by Proposition 2.3 the terms on the two final lines combine to form

$$- \sum_{k \in \mathbb{Z}_*^3} \sum_{p,q \in L_k} 2 \left\langle e_p, h_k e_q \right\rangle \big(b_k(C_k e_p) + b_{-k}^*(C_{-k} e_{-p}) \big)^* \big(b_k(C_k e_q) + b_{-k}^*(C_{-k} e_{-q}) \big). \tag{2.41}$$

□

2.3 Extraction of $E_{\mathrm{corr,ex}}$

To conclude Theorem 2.1 it essentially only remains to identify $E_{\mathrm{corr,ex}}$. This is contained in $\mathcal{E}_{\mathrm{B},5}$: By anticommuting the commutators we can write $\mathcal{E}_{\mathrm{B},5} = -\mathcal{E}'_{\mathrm{B},5} + \mathcal{E}_{\mathrm{B},6}$ where e.g.

$$\begin{aligned} \mathcal{E}'^{(1)}_{\mathrm{B},5} &= \sum_{k,l \in \mathbb{Z}_*^3} \sum_{p \in L_k \cap L_l} \left| |p|^2 - k_F^2 \right| \left[\tilde{c}_{p-k}, b_{-l}^*(S_{-l} e_{-p}) \right] \left[\tilde{c}_{p-l}, b_{-k}^*(S_{-k} e_{-p}) \right]^* \\ \mathcal{E}^{(1)}_{\mathrm{B},6} &= \sum_{k,l \in \mathbb{Z}_*^3} \sum_{p \in L_k \cap L_l} \left| |p|^2 - k_F^2 \right| \left\{ \left[\tilde{c}_{p-l}, b_{-k}^*(S_{-k} e_{-p}) \right]^*, \left[\tilde{c}_{p-k}, b_{-l}^*(S_{-l} e_{-p}) \right] \right\}, \end{aligned} \tag{2.42}$$

and noting that

$$\begin{aligned} \left[\tilde{c}_{p-k}, b_{-l}^*(S_{-l} e_{-p}) \right] &= \sum_{q \in L_l} \left\langle e_{-q}, S_{-l} e_{-p} \right\rangle \left[\tilde{c}_{p-k}, \tilde{c}_{-q}^* \tilde{c}_{-q+l}^* \right] \\ &= - \sum_{q \in L_l} \delta_{p-k,-q+l} \left\langle e_q, S_l e_p \right\rangle \tilde{c}_{-q}^* \end{aligned} \tag{2.43}$$

for $p \in L_k$, we have

$$\begin{aligned} \mathcal{E}^{(1)}_{\mathrm{B},6} &= \sum_{k,l \in \mathbb{Z}_*^3} \sum_{p \in L_k \cap L_l} \left| |p|^2 - k_F^2 \right| \left\{ \sum_{q \in L_k} \delta_{p-l,-q+k} \left\langle S_k e_p, e_q \right\rangle \tilde{c}_{-q}, \sum_{q' \in L_l} \delta_{p-k,-q'+l} \left\langle e_{q'}, S_l e_p \right\rangle \tilde{c}_{-q'}^* \right\} \\ &= \sum_{k,l \in \mathbb{Z}_*^3} \sum_{p,q \in L_k \cap L_l} \delta_{p-k,-q+l} \left| |p|^2 - k_F^2 \right| \left\langle S_k e_p, e_q \right\rangle \left\langle e_q, S_l e_p \right\rangle \end{aligned} \tag{2.44}$$

which is simply a constant. A similar calculation shows that

$$\mathcal{E}_{\mathrm{B},6}^{(2)} = \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in(L_k-k)\cap(L_l-l)} \delta_{p+k,-q-l}\left|\,|p|^2-k_F^2\right| \langle S_k e_{p+k}, e_{q+k}\rangle \langle e_{q+l}, S_l e_{p+l}\rangle . \tag{2.45}$$

The point is that $\mathcal{E}_{\mathrm{B},6}$ is, to leading order in k_F, $E_{\mathrm{corr,ex}}$. To see this we first rewrite the expressions:

Proposition 2.8. *It holds that*

$$\mathcal{E}_{\mathrm{B},6} = \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\lambda_{k,p}\langle S_k e_p, e_q\rangle\langle e_q, S_{p+q-k}e_p\rangle .$$

Proof: We begin by noting that the Kronecker delta $\delta_{p-k,-q+l}$ implies that

$$\begin{aligned}\mathcal{E}_{\mathrm{B},6}^{(1)} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} \delta_{p-k,-q+l}\left|\,|p|^2-k_F^2\right| \langle S_k e_p, e_q\rangle\langle e_q, S_l e_p\rangle \\ &= \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} \left|\,|p|^2-k_F^2\right| \langle S_k e_p, e_q\rangle\langle e_q, S_{p+q-k} e_p\rangle\end{aligned} \tag{2.46}$$

since, as observed in [12, eq. 4.69], $p,q\in L_{p+q-k}\Leftrightarrow p,q\in L_k$. Likewise $p,q\in(L_{-p-q-k}+p+q+k)\Leftrightarrow p,q\in L_k-k$, so (using also the $(k,p,q)\to(-k,-p,-q)$ symmetry of the matrix elements)

$$\begin{aligned}\mathcal{E}_{\mathrm{B},6}^{(2)} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p,q\in(L_k-k)} \delta_{p+k,-q-l}\left|\,|p|^2-k_F^2\right| \langle S_k e_{p+k}, e_{q+k}\rangle\langle e_{q+l}, S_l e_{p+l}\rangle \\ &= \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in(L_k-k)} \left|\,|p|^2-k_F^2\right| \langle S_k e_{p+k}, e_{q+k}\rangle\left\langle e_{q-(p+q+k)}, S_{-(p+q+k)}e_{p-(p+q+k)}\right\rangle \\ &= \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in(L_k-k)} \left|\,|p|^2-k_F^2\right| \langle S_k e_{p+k}, e_{q+k}\rangle\langle e_{p+k}, S_{p+q+k} e_{q+k}\rangle \\ &= \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} \left|\,|p-k|^2-k_F^2\right| \langle S_k e_p, e_q\rangle\langle e_p, S_{p+q-k} e_q\rangle\end{aligned} \tag{2.47}$$

whence

$$\begin{aligned}\mathcal{E}_{\mathrm{B},6} &= \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} \left(\left|\,|p|^2-k_F^2\right|+\left|\,|p-k|^2-k_F^2\right|\right)\langle S_k e_p, e_q\rangle\langle e_q, S_{p+q-k} e_p\rangle \\ &= \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\lambda_{k,p}\langle S_k e_p, e_q\rangle\langle e_q, S_{p+q-k} e_p\rangle .\end{aligned} \tag{2.48}$$

□

We show in appendix section A.2 that S_k obeys $\langle e_p, S_k e_q\rangle \approx \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}$, suggesting that

$$\begin{aligned}\mathcal{E}_{\mathrm{B},6} &\approx \frac{k_F^{-2\beta}}{4(2\pi)^6}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\lambda_{k,p}\frac{\hat{V}_k}{\lambda_{k,p}+\lambda_{k,q}}\frac{\hat{V}_{p+q-k}}{\lambda_{p+q+k,p}+\lambda_{p+q+k,q}} \\ &= \frac{k_F^{-2\beta}}{4(2\pi)^6}\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} \frac{\hat{V}_k\hat{V}_{p+q-k}}{\lambda_{k,p}+\lambda_{k,q}} = E_{\mathrm{corr,ex}}\end{aligned} \tag{2.49}$$

where we used that $\lambda_{p+q+k,p}+\lambda_{p+q+k,q}=\lambda_{k,p}+\lambda_{k,q}$ and the fact that the summand on the right-hand side is symmetric in p and q. We leave the estimates to the next section, but this justifies defining $\mathcal{E}'_{\mathrm{B},6} = \mathcal{E}_{\mathrm{B},6} - E_{\mathrm{corr,ex}}$ to write

$$\mathcal{E}_{\mathrm{B},5} = E_{\mathrm{corr,ex}} - \mathcal{E}'_{\mathrm{B},5} + \mathcal{E}'_{\mathrm{B},6} \tag{2.50}$$

and Theorem 2.1 now follows from the Propositions 2.4 and 2.7 with

$$\mathcal{E}_{\mathrm{B}} = \mathcal{E}_{\mathrm{B},1} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},2}) + \mathcal{E}_{\mathrm{B},3} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},4}) - \mathcal{E}'_{\mathrm{B},5} + \mathcal{E}'_{\mathrm{B},6}. \tag{2.51}$$

3 Estimation of $\mathcal{E}_{\mathrm{B}}$

In this section we bound the error term $\mathcal{E}_{\mathrm{B}}$ appearing in Theorem 2.1, obtaining the following estimate:

Theorem 3.1. *For any symmetric set $S \subset \mathbb{Z}^3_*$ and $\epsilon > 0$ it holds as $k_F \to \infty$ that*

$$\pm\mathcal{E}_{\mathrm{B}} \le C_\epsilon k_F^{2(1-\beta)+\epsilon}\left(\sqrt{\sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k^2} + k_F^{-\frac{1}{2}}\sum_{k\in S}\hat{V}_k\right)\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}\left(H'_{\mathrm{kin}}+k_F\right) + Ck_F^{3(1-\beta)}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^3$$

for constants $C, C_\epsilon > 0$ with C independent of all quantities and C_ϵ depending only on ϵ.

Reduction to Schematic Forms

Recall that $\mathcal{E}_{\mathrm{B}}$ was defined to be

$$\mathcal{E}_{\mathrm{B}} = \mathcal{E}_{\mathrm{B},1} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},2}) + \mathcal{E}_{\mathrm{B},3} + 2\,\mathrm{Re}(\mathcal{E}_{\mathrm{B},4}) - \mathcal{E}'_{\mathrm{B},5} + \mathcal{E}'_{\mathrm{B},6}, \tag{3.1}$$

the sub-terms $\mathcal{E}_{\mathrm{B},1},\ldots,\mathcal{E}'_{\mathrm{B},5}$ being defined in the equations (2.30), (2.31), (2.42) and $\mathcal{E}'_{\mathrm{B},6}$ being

$$\mathcal{E}'_{\mathrm{B},6} = \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\lambda_{k,p}\left\langle S_k e_p, e_q\right\rangle\left\langle e_q, S_{p+q-k}e_p\right\rangle - E_{\mathrm{corr,ex}}. \tag{3.2}$$

Consider $\mathcal{E}_{\mathrm{B},1}$, which is the sum of the two terms

$$\begin{aligned}\mathcal{E}^{(1)}_{\mathrm{B},1} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\left||p|^2-k_F^2\right|\tilde{c}^*_{p-l}\left[b_k^*((C_k-1)e_p), b_l((C_l-1)e_p)\right]\tilde{c}_{p-k}\\ \mathcal{E}^{(2)}_{\mathrm{B},1} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in(L_k-k)\cap(L_l-l)}\left||p|^2-k_F^2\right|\tilde{c}^*_{p+l}\left[b_k^*((C_k-1)e_{p+k}), b_l((C_l-1)e_{p+l})\right]\tilde{c}_{p+k}.\end{aligned} \tag{3.3}$$

As already noted, these terms are clearly similar. Indeed, they are both of the schematic form

$$\tilde{\mathcal{E}}_{\mathrm{B},1} = \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\tilde{c}^*_{p\mp l}\left[b_k^*(\varphi_{k,p}), b_l(\varphi_{l,p})\right]\tilde{c}_{p\mp k} \tag{3.4}$$

where the sets M_k, the signs $p\mp k$ and $\varphi_{k,p}\in\ell^2(L_k)$ are given by

$$(M_k, p\mp k, \varphi_{k,p}) = \begin{cases}\left(L_k, p-k, \sqrt{\left||p|^2-k_F^2\right|}(C_k-1)e_p\right) & \text{for } \mathcal{E}^{(1)}_{\mathrm{B},1}\\ \left(L_k-k, p+k, \sqrt{\left||p|^2-k_F^2\right|}(C_k-1)e_{p+k}\right) & \text{for } \mathcal{E}^{(2)}_{\mathrm{B},1}\end{cases}. \tag{3.5}$$

It thus suffices to obtain estimates for the schematic form of equation (3.4) rather than the specific terms $\mathcal{E}^{(1)}_{\mathrm{B},1}$ and $\mathcal{E}^{(2)}_{\mathrm{B},1}$. The same is true of the other error terms: $\mathcal{E}_{\mathrm{B},2}$, for instance, consists of the terms

$$\begin{aligned}\mathcal{E}^{(1)}_{\mathrm{B},2} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in L_k\cap L_l}\left||p|^2-k_F^2\right|\tilde{c}^*_{p-l}b_k^*((C_k-1)e_p)\left[\tilde{c}_{p-k}, b^*_{-l}(S_{-l}e_{-p})\right]\\ \mathcal{E}^{(2)}_{\mathrm{B},2} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in(L_k-k)\cap(L_l-l)}\left||p|^2-k_F^2\right|\tilde{c}^*_{p+l}b_k^*((C_k-1)e_{p+k})\left[\tilde{c}_{p+k}, b^*_{-l}(S_{-l}e_{-p-l})\right]\end{aligned} \tag{3.6}$$

which we can likewise summarize in the schematic form

$$\tilde{\mathcal{E}}_{\mathrm{B},2} = \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\tilde{c}^*_{p\mp l}b_k^*(\varphi_{k,p})\left[\tilde{c}_{p\mp k}, b^*_{-l}(\psi_{-l,-p})\right] \tag{3.7}$$

provided we also define $\psi_{l,p}\in\ell^2(L_l)$ by

$$\psi_{l,p} = \begin{cases}\sqrt{\left||p|^2-k_F^2\right|}S_l e_p & \text{for } \mathcal{E}^{(1)}_{\mathrm{B},2}\\ \sqrt{\left||p|^2-k_F^2\right|}S_l e_{p+l} & \text{for } \mathcal{E}^{(2)}_{\mathrm{B},2}\end{cases}. \tag{3.8}$$

The quantities of the equations (3.5) and (3.8) suffice to write all error terms schematically as

$$\begin{aligned}
\tilde{\mathcal{E}}_{\mathrm{B},1} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l} \tilde{c}^*_{p\mp l}\left[b^*_k(\varphi_{k,p}), b_l(\varphi_{l,p})\right]\tilde{c}_{p\mp k} \\
\tilde{\mathcal{E}}_{\mathrm{B},2} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l} \tilde{c}^*_{p\mp l} b^*_k(\varphi_{k,p})\left[\tilde{c}_{p\mp k}, b^*_{-l}(\psi_{-l,-p})\right] \\
\tilde{\mathcal{E}}_{\mathrm{B},3} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l} \tilde{c}^*_{p\mp l}\left[b_{-k}(\psi_{-k,-p}), b^*_{-l}(\psi_{-l,-p})\right]\tilde{c}_{p\mp k} \\
\tilde{\mathcal{E}}_{\mathrm{B},4} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l} \tilde{c}^*_{p\mp l} b_{-k}(\psi_{-k,-p})\left[\tilde{c}_{p\mp k}, b^*_{-l}(\psi_{-l,-p})\right] \\
\tilde{\mathcal{E}}'_{\mathrm{B},5} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l} \left[\tilde{c}_{p\mp k}, b^*_{-l}(\psi_{-l,-p})\right]\left[\tilde{c}_{p\mp l}, b^*_{-k}(\psi_{-k,-p})\right]^*
\end{aligned} \tag{3.9}$$

and it is these general forms which we will estimate. We will then insert the particular expressions for $\varphi_{k,p}$ and $\psi_{l,p}$ at the end to obtain Theorem 3.1.

3.1 Estimation of $\tilde{\mathcal{E}}_{\mathrm{B},1}$ and $\tilde{\mathcal{E}}_{\mathrm{B},2}$

The schematic forms of $\tilde{\mathcal{E}}_{\mathrm{B},1}$ and $\tilde{\mathcal{E}}_{\mathrm{B},2}$ display the typical structure we will need to consider, so we first consider these in detail.

We begin with $\tilde{\mathcal{E}}_{\mathrm{B},1}$, which since $[b^*_k(\varphi_{k,p}), b_l(\varphi_{l,p})] = -\delta_{k,l}\|\varphi_{k,p}\|^2 - \varepsilon_{l,k}(\varphi_{l,p};\varphi_{k,p})$ can be further decomposed as

$$\begin{aligned}
\tilde{\mathcal{E}}_{\mathrm{B},1} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l} \tilde{c}^*_{p\mp l}\left[b^*_k(\varphi_{k,p}), b_l(\varphi_{l,p})\right]\tilde{c}_{p\mp k} \\
&= -\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\|\varphi_{k,p}\|^2\,\tilde{c}^*_{p\mp k}\tilde{c}_{p\mp k} + \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\sum_{q\in L_k\cap L_l}\langle\varphi_{l,p}, e_q\rangle\langle e_q,\varphi_{k,p}\rangle\,\tilde{c}^*_{p\mp l}\tilde{c}^*_{q-k}\tilde{c}_{q-l}\tilde{c}_{p\mp k} \\
&\quad+ \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\sum_{q\in(L_k-k)\cap(L_l-l)}\langle\varphi_{l,p}, e_{q+l}\rangle\langle e_{q+k},\varphi_{k,p}\rangle\,\tilde{c}^*_{p\mp l}\tilde{c}^*_{q+k}\tilde{c}_{q+l}\tilde{c}_{p\mp k} \\
&=: -\tilde{\mathcal{E}}_{\mathrm{B},1,1} + \tilde{\mathcal{E}}_{\mathrm{B},1,2} + \tilde{\mathcal{E}}_{\mathrm{B},1,3}.
\end{aligned} \tag{3.10}$$

To control this we will use the following:

Lemma 3.2. *For any $A\subset\mathbb{Z}^3$ with $|A|\leq\left|\overline{B}(0,2k_F)\cap\mathbb{Z}^3\right|$ and any $\epsilon>0$ it holds that*

$$\sum_{p\in A}\frac{1}{\left||p|^2-\zeta\right|}\leq C_\epsilon k_F^{1+\epsilon}$$

for a constant $C_\epsilon>0$ depending only on ϵ.

For the proof see appendix section A.1.

We can now prove the following:

Proposition 3.3. *For any symmetric set $S\subset\mathbb{Z}^3_*$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$\pm\tilde{\mathcal{E}}_{\mathrm{B},1,1}\leq\sum_{k\in\mathbb{Z}^3_*}\max_{p\in M_k}\|\varphi_{k,p}\|^2 H'_{\mathrm{kin}}$$

$$\pm\tilde{\mathcal{E}}_{\mathrm{B},1,2},\pm\tilde{\mathcal{E}}_{\mathrm{B},1,3}\leq C_\epsilon\left(k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}^3_*\setminus S}\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2+\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2}\right)^2\right)H'_{\mathrm{kin}}$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Note that $\max_{p\in M_k}\|\varphi_{k,p}\|^2\leq Ck_F^{1-2\beta}\hat{V}_k^2$ and $\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2\leq Ck_F^{1-2\beta}\hat{V}_k^2$, see (3.42) below.

Proof: The estimate for $\tilde{\mathcal{E}}_{\mathrm{B},1,1}$ is immediate, since

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathrm{B},1,1}&=\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_k}\|\varphi_{k,p}\|^2\,\tilde{c}_{p\mp k}^*\tilde{c}_{p\mp k}\leq\sum_{k\in\mathbb{Z}_*^3}\max_{p\in M_k}\|\varphi_{k,p}\|^2\sum_{p\in M_k}\tilde{c}_{p\mp k}^*\tilde{c}_{p\mp k}\\&\leq\left(\sum_{k\in\mathbb{Z}_*^3}\max_{p\in M_k}\|\varphi_{k,p}\|^2\right)\mathcal{N}_E\leq\left(\sum_{k\in\mathbb{Z}_*^3}\max_{p\in M_k}\|\varphi_{k,p}\|^2\right)H'_{\mathrm{kin}},\end{aligned}\tag{3.11}$$

where we used that $\mathcal{N}_E\leq H'_{\mathrm{kin}}$ at the end (a consequence of the representation $H'_{\mathrm{kin}}=\sum_{p\in\mathbb{Z}^3}\left||p|^2-\zeta\right|\tilde{c}_p^*\tilde{c}_p$). The terms $\tilde{\mathcal{E}}_{\mathrm{B},1,2}$ and $\tilde{\mathcal{E}}_{\mathrm{B},1,3}$ are similar, so we focus on $\tilde{\mathcal{E}}_{\mathrm{B},1,2}$. For this we note that for any $\Psi\in D(H'_{\mathrm{kin}})$

$$\begin{aligned}\left|\left\langle\Psi,\tilde{\mathcal{E}}_{\mathrm{B},1,2}\Psi\right\rangle\right|&\leq\sum_{k,l\in\mathbb{Z}_*^3}\sum_{p\in M_k\cap M_l}\sum_{q\in L_k\cap L_l}|\langle\varphi_{l,p},e_q\rangle|\,|\langle e_q,\varphi_{k,p}\rangle|\,\|\tilde{c}_{p\mp l}\tilde{c}_{q-k}\Psi\|\,\|\tilde{c}_{p\mp k}\tilde{c}_{q-l}\Psi\|\\&\leq\sum_{p,q\in\mathbb{Z}^3}\left(\sum_{k\in\mathbb{Z}_*^3}1_{M_k}(p)1_{L_k}(q)\,|\langle e_q,\varphi_{k,p}\rangle|\,\|\tilde{c}_{q-k}\Psi\|\right)^2\\&\leq 2\sum_{p,q\in\mathbb{Z}^3}\left(\sum_{k\in\mathbb{Z}_*^3\backslash S}1_{M_k}(p)1_{L_k}(q)\,|\langle e_q,\varphi_{k,p}\rangle|\,\|\tilde{c}_{q-k}\Psi\|\right)^2\\&\quad+2\sum_{p,q\in\mathbb{Z}^3}\left(\sum_{k\in S}1_{M_k}(p)1_{L_k}(q)\,|\langle e_q,\varphi_{k,p}\rangle|\,\|\tilde{c}_{q-k}\Psi\|\right)^2\end{aligned}\tag{3.12}$$

where we used the triangle and Cauchy-Schwarz inequalities and that e.g. $\|\tilde{c}_{p\pm l}\|_{\mathrm{op}}=1$. The first term on the right-hand side can be further estimated as

$$\begin{aligned}&\sum_{p,q\in\mathbb{Z}^3}\left(\sum_{k\in\mathbb{Z}_*^3\backslash S}1_{M_k}(p)1_{L_k}(q)\,|\langle e_q,\varphi_{k,p}\rangle|\,\|\tilde{c}_{q-k}\Psi\|\right)^2\\&\leq\sum_{p,q\in\mathbb{Z}^3}\left(\sum_{k\in\mathbb{Z}_*^3\backslash S}1_{M_k}(p)\frac{1_{L_k}(q)}{\left||q-k|^2-\zeta\right|}\,|\langle e_q,\varphi_{k,p}\rangle|^2\right)\left(\sum_{k\in\mathbb{Z}_*^3\backslash S}1_{L_k}(q)\left||q-k|^2-\zeta\right|\,\|\tilde{c}_{q-k}\Psi\|^2\right)\\&\leq\sum_{k\in\mathbb{Z}_*^3\backslash S}\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2\sum_{q\in L_k}\frac{1_{L_k}(q)}{\left||q-k|^2-\zeta\right|}\left\langle\Psi,H'_{\mathrm{kin}}\Psi\right\rangle\\&\leq C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}_*^3\backslash S}\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2\left\langle\Psi,H'_{\mathrm{kin}}\Psi\right\rangle\end{aligned}\tag{3.13}$$

where we could apply Lemma 3.2 since $|L_k|\leq|B_F|\leq\left|\overline{B}(0,2k_F)\cap\mathbb{Z}^3\right|$.

For the second we instead expand and bound

$$\begin{aligned}&\sum_{p,q\in\mathbb{Z}^3}\left(\sum_{k\in S}1_{M_k}(p)1_{L_k}(q)\,|\langle e_q,\varphi_{k,p}\rangle|\,\|\tilde{c}_{q-k}\Psi\|\right)^2\\&=\sum_{k,l\in S}\sum_{p\in M_k\cap M_l}\sum_{q\in L_k\cap L_l}|\langle e_q,\varphi_{k,p}\rangle|\,|\langle e_q,\varphi_{l,p}\rangle|\,\|\tilde{c}_{q-k}\Psi\|\,\|\tilde{c}_{q-l}\Psi\|\\&\leq\sum_{k,l\in S}\sum_{p\in M_k\cap M_l}\left(\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|\right)\left(\max_{q\in L_l}|\langle e_q,\varphi_{l,p}\rangle|\right)\sqrt{\sum_{q\in L_k\cap L_l}\|\tilde{c}_{q-k}\Psi\|^2}\sqrt{\sum_{q\in L_k\cap L_l}\|\tilde{c}_{q-l}\Psi\|^2}\\&\leq\sum_{k,l\in S}\sqrt{\sum_{p\in M_k\cap M_l}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2}\sqrt{\sum_{p\in M_k\cap M_l}\max_{q\in L_l}|\langle e_q,\varphi_{l,p}\rangle|^2}\,\langle\Psi,\mathcal{N}_E\Psi\rangle\\&\leq\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2}\right)^2\left\langle\Psi,H'_{\mathrm{kin}}\Psi\right\rangle.\end{aligned}\tag{3.14}$$

$\square$

For $\tilde{\mathcal{E}}_{\mathrm{B},2}$ we compute that when $p \in M_k \cap M_l$ with $M_k = L_k$

$$\begin{aligned}\left[\tilde{c}_{p\mp k}, b_{-l}^*(\psi_{-l,-p})\right] &= \sum_{q\in L_{-l}} \langle e_q, \psi_{-l,-p}\rangle \left[c_{p-k}^*, c_q^* c_{q+l}\right] = -\sum_{q\in L_{-l}} \delta_{p-k,q+l} \langle e_q, \psi_{-l,-p}\rangle c_q^* \\ &= -1_{L_{-l}}(p-k-l) \langle e_{p-k-l}, \psi_{-l,-p}\rangle \tilde{c}_{p-k-l}^*\end{aligned} \tag{3.15}$$

and likewise when $p \in M_k \cap M_l$ with $M_k = (L_k - k)$

$$\begin{aligned}\left[\tilde{c}_{p\mp k}, b_{-l}^*(\psi_{-l,-p})\right] &= \sum_{q\in L_{-l}} \langle e_q, \psi_{-l,-p}\rangle \left[c_{p+k}, c_q^* c_{q+l}\right] = \sum_{q\in L_{-l}} \delta_{p+k,q} \langle e_q, \psi_{-l,-p}\rangle c_{q+l} \\ &= 1_{L_{-l}}(p+k) \langle e_{p+k}, \psi_{-l,-p}\rangle \tilde{c}_{p+k+l}^* = 1_{L_{-l}+l}(p+k+l) \langle e_{p+k}, \psi_{-l,-p}\rangle \tilde{c}_{p+k+l}^*.\end{aligned} \tag{3.16}$$

We can summarize this as

$$\left[\tilde{c}_{p\mp k}, b_{-l}^*(\psi_{-l,-p})\right] = \mp 1_{M_{-l}}(p\mp k\mp l) \left\langle e_{p\mp k\mp l'}, \psi_{-l,-p}\right\rangle \tilde{c}_{p\mp k\mp l}^* \tag{3.17}$$

where $l' = l$ when $M_k = L_k$ and $l' = 0$ when $M_k = L_k - k$ (the presence or absence of this will not make a difference to the estimation below, so this definition is convenient).
Using also that $[\tilde{c}^*, b^*(\cdot)] = 0$ we can then write $\tilde{\mathcal{E}}_{\mathrm{B},2}$ as

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathrm{B},2} &= \mp \sum_{k,l\in\mathbb{Z}_*^3} \sum_{p\in M_k\cap M_l} 1_{M_{-l}}(p\mp k\mp l) \left\langle e_{p\mp k\mp l'}, \psi_{-l,-p}\right\rangle \tilde{c}_{p\mp l}^* b_k^*(\varphi_{k,p}) \tilde{c}_{p\mp k\mp l}^* \\ &= \pm \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in M_k} \left(\sum_{l\in\mathbb{Z}_*^3} 1_{M_l}(p) 1_{M_{-l}}(p\mp k\mp l) \left\langle \psi_{-l,-p}, e_{p\mp k\mp l'}\right\rangle \tilde{c}_{p\mp l} \tilde{c}_{p\mp k\mp l}\right)^* b_k^*(\varphi_{k,p}).\end{aligned} \tag{3.18}$$

To control this we note the following bounds from [11, Propositions 4.4, A.1, A.2]:

Proposition 3.4. *For any $k \in \mathbb{Z}_*^3$ and $\varphi \in \ell^2(L_k)$ it holds that*

$$b_k^*(\varphi) b_k(\varphi) \le \left\langle \varphi, h_k^{-1}\varphi\right\rangle H'_{\mathrm{kin}}, \quad b_k(\varphi) b_k^*(\varphi) \le \left\langle \varphi, h_k^{-1}\varphi\right\rangle H'_{\mathrm{kin}} + \|\varphi\|^2.$$

Proposition 3.5. *For any $k \in \mathbb{Z}_*^3$ it holds as $k_F \to \infty$ that*

$$\sum_{p\in L_k} \lambda_{k,p}^{-1} \le C k_F, \quad |L_k| \le C k_F^2 \min\{|k|, k_F\},$$

for a constant $C > 0$ independent of all quantities.

With this we can prove the following:

Proposition 3.6. *For any symmetric set $S \subset \mathbb{Z}_*^3$ and $\epsilon > 0$ it holds as $k_F \to \infty$ that*

$$\begin{aligned}\pm\tilde{\mathcal{E}}_{\mathrm{B},2} \le C_\epsilon &\left(\sqrt{k_F^{1+\epsilon} \sum_{k\in\mathbb{Z}_*^3\setminus S} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2} + \sum_{k\in S} \sqrt{\sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2}\right) \\ &\cdot \sqrt{k_F \sum_{k\in\mathbb{Z}_*^3} \min\{|k|, k_F\} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \varphi_{k,p}\rangle|^2} \left(H'_{\mathrm{kin}} + k_F\right)\end{aligned}$$

for a constant $C_\epsilon > 0$ depending only on ϵ.

Note that $\max_{p\in M_k} \|\psi_{k,p}\|^2 \le C k_F^{1-2\beta} \hat{V}_k^2$ and $\sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2 \le C k_F^{1-2\beta} \hat{V}_k^2$, namely $\psi_{k,p}$'s satisfy the same bounds as $\varphi_{k,p}$'s, see (3.42) below.

Proof: By the computation above we can for any $\Psi \in D(H'_{\mathrm{kin}})$ estimate

$$\left\langle \Psi, \tilde{\mathcal{E}}_{\mathrm{B},2}\Psi\right\rangle \le \sum_{k\in\mathbb{Z}_*^3} \sum_{p\in M_k} \left\| \sum_{l\in\mathbb{Z}_*^3\setminus S} 1_{M_l}(p) 1_{M_{-l}}(p\mp k\mp l) \left\langle \psi_{-l,-p}, e_{p\mp k\mp l'}\right\rangle \tilde{c}_{p\mp l} \tilde{c}_{p\mp k\mp l} \Psi \right\| \left\| b_k^*(\varphi_{k,p})\Psi\right\| \tag{3.19}$$

$$+\sum_{l\in S}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}1_{M_l}(p)1_{M_{-l}}(p\mp k\mp l)\left|\left\langle \psi_{-l,-p},e_{p\mp k\mp l'}\right\rangle\right|\left\Vert \tilde{c}_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\right\Vert \left\Vert b_k^*(\varphi_{k,p})\Psi\right\Vert$$

and by Cauchy-Schwarz the terms on the right-hand side can be bounded by the terms

$$\begin{gathered}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left\Vert\sum_{l\in\mathbb{Z}^3_*\backslash S}1_{M_l}(p)1_{M_{-l}}(p\mp k\mp l)\left\langle \psi_{-l,-p},e_{p\mp k\mp l'}\right\rangle\tilde{c}_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\right\Vert^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left\Vert b_k^*(\varphi_{k,p})\Psi\right\Vert^2},\\ \sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}1_{M_l}(p)1_{M_{-l}}(p\mp k\mp l)\left|\left\langle \psi_{-l,-p},e_{p\mp k\mp l'}\right\rangle\right|^2\left\Vert \tilde{c}_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\right\Vert^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left\Vert b_k^*(\varphi_{k,p})\Psi\right\Vert^2}.\end{gathered}\tag{3.20}$$

Beginning with the common factor $\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\Vert b_k^*(\varphi_{k,p})\Psi\Vert^2$ we can apply the Propositions 3.4 and 3.5 to see that

$$\begin{aligned}\Vert b_k^*(\varphi_{k,p})\Psi\Vert^2&\leq\left\langle\varphi_{k,p},h_k^{-1}\varphi_{k,p}\right\rangle\left\langle\Psi,H'_{\text{kin}}\Psi\right\rangle+\Vert\varphi_{k,p}\Vert^2\Vert\Psi\Vert^2\\&=\sum_{q\in L_k}\frac{1}{\lambda_{k,q}}\left|\langle e_q,\varphi_{k,p}\rangle\right|^2\left\langle\Psi,H'_{\text{kin}}\Psi\right\rangle+\sum_{q\in L_k}\left|\langle e_q,\varphi_{k,p}\rangle\right|^2\Vert\Psi\Vert^2\\&\leq Ck_F\max_{q\in L_k}\left|\langle e_q,\varphi_{k,p}\rangle\right|^2\left\langle\Psi,H'_{\text{kin}}\Psi\right\rangle+Ck_F^2\min\{|k|,k_F\}\max_{q\in L_k}\left|\langle e_q,\varphi_{k,p}\rangle\right|^2\Vert\Psi\Vert^2\\&\leq Ck_F\min\{|k|,k_F\}\max_{q\in L_k}\left|\langle e_q,\varphi_{k,p}\rangle\right|^2\left\langle\Psi,\left(H'_{\text{kin}}+k_F\right)\Psi\right\rangle\end{aligned}\tag{3.21}$$

for any $k\in\mathbb{Z}^3_*$ and $p\in M_k$, whence

$$\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\Vert b_k^*(\varphi_{k,p})\Psi\Vert^2\leq Ck_F\sum_{k\in\mathbb{Z}^3_*}\min\{|k|,k_F\}\sum_{p\in M_k}\max_{q\in L_k}\left|\langle e_q,\varphi_{k,p}\rangle\right|^2\left\langle\Psi,\left(H'_{\text{kin}}+k_F\right)\Psi\right\rangle.\tag{3.22}$$

For the remaining factors of equation (3.20) we begin with

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left\Vert\sum_{l\in\mathbb{Z}^3_*\backslash S}1_{M_l}(p)1_{M_{-l}}(p\mp k\mp l)\left\langle \psi_{-l,-p},e_{p\mp k\mp l'}\right\rangle\tilde{c}_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\right\Vert^2\\&\leq\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left(\sum_{l\in\mathbb{Z}^3_*\backslash S}1_{M_l}(p)\frac{1_{M_{-l}}(p\mp k\mp l)}{\left||p\mp k\mp l|^2-\zeta\right|}\left|\left\langle e_{p\mp k\mp l'},\psi_{-l,-p}\right\rangle\right|^2\right)\\&\qquad\cdot\left(\sum_{l\in\mathbb{Z}^3_*\backslash S}1_{M_{-l}}(p\mp k\mp l)\left||p\mp k\mp l|^2-\zeta\right|\Vert\tilde{c}_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\Vert^2\right)\\&\leq\sum_{l\in\mathbb{Z}^3_*\backslash S}\sum_{p\in M_l}\max_{q\in L_l}\left|\langle e_{-q},\psi_{-l,-p}\rangle\right|^2\sum_{k\in\mathbb{Z}^3_*}\frac{1_{M_{-l}}(p\mp k\mp l)}{\left||p\mp k\mp l|^2-\zeta\right|}\left\langle\Psi,H'_{\text{kin}}\Psi\right\rangle\\&\leq C_\epsilon k_F^{1+\epsilon}\sum_{l\in\mathbb{Z}^3_*\backslash S}\sum_{p\in M_l}\max_{q\in L_l}\left|\langle e_q,\psi_{l,p}\rangle\right|^2\left\langle\Psi,H'_{\text{kin}}\Psi\right\rangle\end{aligned}\tag{3.23}$$

where we used that the presence of the indicator function $1_{M_{-l}}(p\mp k\mp l)$ restricts the k summation to a set of cardinality at most $|M_{-l}|=|L_l|\leq|B_F|\leq\left|\overline{B}(0,2k_F)\cap\mathbb{Z}^3\right|$, so Lemma 3.2 applies.
For the last factor of equation (3.20) we simply note that

$$\begin{aligned}&\sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}1_{M_l}(p)1_{M_{-l}}(p\mp k\mp l)\left|\left\langle \psi_{-l,-p},e_{p\mp k\mp l'}\right\rangle\right|^2\left\Vert \tilde{c}_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\right\Vert^2}\\&\leq\sum_{l\in S}\sqrt{\sum_{p\in M_l}\max_{q\in L_l}\left|\langle e_{-q},\psi_{-l,-p}\rangle\right|^2\sum_{k\in\mathbb{Z}^3_*}1_{M_{-l}}(p\mp k\mp l)\Vert\tilde{c}_{p\mp k\mp l}\Psi\Vert^2}\\&\leq\sum_{l\in S}\sqrt{\sum_{p\in M_l}\max_{q\in L_l}\left|\langle e_{-q},\psi_{-l,-p}\rangle\right|^2}\sqrt{\langle\Psi,\mathcal{N}_E\Psi\rangle}\leq\sum_{l\in S}\sqrt{\sum_{p\in M_l}\max_{q\in L_l}\left|\langle e_q,\psi_{l,p}\rangle\right|^2}\sqrt{\langle\Psi,H'_{\text{kin}}\Psi\rangle}.\end{aligned}\tag{3.24}$$

The proposition now follows by combining the estimates.

□

3.2 Estimation of $\tilde{\mathcal{E}}_{\mathrm{B},3}$, $\tilde{\mathcal{E}}_{\mathrm{B},4}$ and $\mathcal{E}'_{\mathrm{B},5}$

We now bound the remaining forms of equation (3.9). First is $\tilde{\mathcal{E}}_{\mathrm{B},3}$, which is an analog of $\tilde{\mathcal{E}}_{\mathrm{B},1}$: We can write it as (substituting $(k,l,p)\to(-k,-l,-p)$ first)

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathrm{B},3} &= \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\tilde{c}^*_{-p\pm l}\left[b_k(\psi_{k,p}),b_l^*(\psi_{l,p})\right]\tilde{c}_{-p\pm k}\\ &= \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\|\psi_{k,p}\|^2\,\tilde{c}^*_{-p\pm k}\tilde{c}_{-p\pm k} - \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\sum_{q\in L_k\cap L_l}\langle\psi_{k,p},e_q\rangle\langle e_q,\psi_{l,p}\rangle\,\tilde{c}^*_{-p\pm l}\tilde{c}^*_{q-l}\tilde{c}_{q-k}\tilde{c}_{-p\pm k}\\ &\quad- \sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\sum_{q\in(L_k-k)\cap(L_l-l)}\langle\psi_{k,p},e_{q+k}\rangle\langle e_{q+l},\psi_{l,p}\rangle\,\tilde{c}^*_{-p\pm l}\tilde{c}^*_{q+l}\tilde{c}_{q+k}\tilde{c}_{-p\pm k} \\ &:= \tilde{\mathcal{E}}_{\mathrm{B},3,1}-\tilde{\mathcal{E}}_{\mathrm{B},3,2}-\tilde{\mathcal{E}}_{\mathrm{B},3,3}.\end{aligned}\tag{3.25}$$

The following can now be concluded exactly as we did Proposition 3.3:

Proposition 3.7. *For any symmetric set $S\subset\mathbb{Z}^3_*$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$\begin{aligned}\pm\tilde{\mathcal{E}}_{\mathrm{B},3,1} &\leq \sum_{k\in\mathbb{Z}^3_*}\max_{p\in M_k}\|\psi_{k,p}\|^2\,H'_{\mathrm{kin}}\\ \pm\tilde{\mathcal{E}}_{\mathrm{B},3,2},\ \pm\tilde{\mathcal{E}}_{\mathrm{B},3,3} &\leq C_\epsilon\left(k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}^3_*\setminus S}\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\psi_{k,p}\rangle|^2+\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\psi_{k,p}\rangle|^2}\right)^2\right)H'_{\mathrm{kin}}\end{aligned}$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Proof: $\tilde{\mathcal{E}}_{\mathrm{B},3,1}$ is of the exact same form as $\tilde{\mathcal{E}}_{\mathrm{B},1,1}$, the only difference being the substitution $(\varphi_{k,p},p\mp k)\to(\psi_{k,p},-p\pm k)$, so the first estimate follows exactly as in Proposition 3.3.
$\tilde{\mathcal{E}}_{\mathrm{B},3,2}$ and $\tilde{\mathcal{E}}_{\mathrm{B},3,3}$ are similar, so we consider $\tilde{\mathcal{E}}_{\mathrm{B},3,2}$. This immediately factorizes as

$$\tilde{\mathcal{E}}_{\mathrm{B},3,2}=\sum_{p,q\in\mathbb{Z}^3}\left|\sum_{k\in\mathbb{Z}^3_*}1_{M_k}(p)1_{L_k}(q)\,\langle\psi_{k,p},e_q\rangle\,\tilde{c}_{-p\pm k}\tilde{c}_{q-k}\right|^2\tag{3.26}$$

so for any $\Psi\in D(H'_{\mathrm{kin}})$

$$\left|\left\langle\Psi,\tilde{\mathcal{E}}_{\mathrm{B},3,2}\Psi\right\rangle\right|\leq\sum_{p,q\in\mathbb{Z}^3}\left(\sum_{k\in\mathbb{Z}^3_*}1_{M_k}(p)1_{L_k}(q)\,|\langle\psi_{k,p},e_q\rangle|\,\|\tilde{c}_{q-k}\Psi\|\right)^2\tag{3.27}$$

which subject to the substitution $\varphi_{k,p}\to\psi_{k,p}$ is the same as that of equation (3.12), whence the second estimate follows.

$\square$

Using equation (3.17) we can write $\tilde{\mathcal{E}}_{\mathrm{B},4}$ as

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathrm{B},4} &= \mp\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}1_{M_{-l}}(p\mp k\mp l)\left\langle e_{p\mp k\mp l'},\psi_{-l,-p}\right\rangle\tilde{c}^*_{p\mp l}b_{-k}(\psi_{-k,-p})\tilde{c}^*_{p\mp k\mp l}\\ &= \pm\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left(\sum_{l\in\mathbb{Z}^3_*}1_{M_l}(p)1_{M_{-l}}(p\mp k\mp l)\left\langle\psi_{-l,-p},e_{p\mp k\mp l'}\right\rangle\tilde{c}_{p\mp l}\tilde{c}_{p\mp k\mp l}\right)^*b_{-k}(\psi_{-k,-p})\\ &\mp\sum_{k,l\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}1_{M_{-l}}(p\mp k\mp l)\left\langle e_{p\mp k\mp l'},\psi_{-l,-p}\right\rangle\tilde{c}^*_{p\mp l}\left[b_{-k}(\psi_{-k,-p}),\tilde{c}^*_{p\mp k\mp l}\right]\\ &=: \tilde{\mathcal{E}}_{\mathrm{B},4,1}+\tilde{\mathcal{E}}_{\mathrm{B},4,2}\end{aligned}\tag{3.28}$$

and these terms can be bounded in the following manner:

Proposition 3.8. *For any symmetric set $S \subset \mathbb{Z}^3_*$ and $\epsilon > 0$ it holds as $k_F \to \infty$ that*

$$\pm\tilde{\mathcal{E}}_{\mathrm{B},4,1} \leq C_\epsilon \left(\sqrt{k_F^{1+\epsilon} \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2} + \sum_{k\in S} \sqrt{\sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2} \right) \cdot \sqrt{k_F \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2 H'_{\mathrm{kin}}}$$

$$\pm\tilde{\mathcal{E}}_{\mathrm{B},4,2} \leq \sqrt{\sum_{k\in\mathbb{Z}^3_*} \max_{p\in M_k} \|\psi_{k,p}\|^2} \sqrt{\sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2 H'_{\mathrm{kin}}}$$

for a constant $C_\epsilon > 0$ depending only on ϵ.

Proof: $\tilde{\mathcal{E}}_{\mathrm{B},4,1}$ is of the same form as $\tilde{\mathcal{E}}_{\mathrm{B},2}$ up to the substitution $b^*(\varphi_{k,p}) \to b_{-k}(\psi_{-k,-p})$, so the first estimate follows as in Proposition 3.6 after noting that we now simply have

$$\begin{aligned} \|b_{-k}(\psi_{-k,-p})\Psi\| &\leq \left\langle \psi_{-k,-p}, h_{-k}^{-1}\psi_{-k,-p}\right\rangle \left\langle \Psi, H'_{\mathrm{kin}}\Psi\right\rangle \\ &\leq C k_F \max_{q\in L_k} |\langle e_{-q}, \psi_{-k,-p}\rangle|^2 \left\langle \Psi, H'_{\mathrm{kin}}\Psi\right\rangle \end{aligned} \tag{3.29}$$

by Proposition 3.4, rather than the more complicated bound of equation (3.21) which was needed for $\tilde{\mathcal{E}}_{\mathrm{B},2}$. For $\tilde{\mathcal{E}}_{\mathrm{B},4,2}$ we compute that when $p \in M_k = L_k$

$$\begin{aligned} \left[b_{-k}(\psi_{-k,-p}), \tilde{c}^*_{p\mp k\mp l}\right] &= \sum_{q\in L_{-k}} \langle \psi_{-k,-p}, e_q\rangle \left[c^*_{q+k}c_q, c^*_{p-k-l}\right] = \sum_{q\in L_{-k}} \delta_{q,p-k-l} \langle \psi_{-k,-p}, e_q\rangle c^*_{q+k} \\ &= 1_{L_{-k}}(p-k-l) \langle \psi_{-k,-p}, e_{p-k-l}\rangle \tilde{c}_{p-l} \end{aligned} \tag{3.30}$$

while when $p \in M_k = (L_k - k)$

$$\begin{aligned} \left[b_{-k}(\psi_{-k,-p}), \tilde{c}^*_{p\mp k\mp l}\right] &= \sum_{q\in L_{-k}} \langle \psi_{-k,-p}, e_q\rangle \left[c^*_{q+k}c_q, c_{p+k+l}\right] = -\sum_{q\in L_{-k}} \delta_{q,p+l} \langle \psi_{-k,-p}, e_q\rangle c_q \\ &= -1_{L_{-k}}(p+l) \langle \psi_{-k,-p}, e_{p+l}\rangle \tilde{c}_{p+l} = -1_{L_{-k}+k}(p+k+l) \langle \psi_{-k,-p}, e_{p+l}\rangle \tilde{c}_{p+l} \end{aligned} \tag{3.31}$$

which we can summarize as

$$\left[b_{-k}(\psi_{-k,-p}), \tilde{c}^*_{p\mp k\mp l}\right] = \pm 1_{M_{-k}}(p\mp k\mp l) \left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle \tilde{c}_{p\mp l}. \tag{3.32}$$

$\tilde{\mathcal{E}}_{\mathrm{B},4,2}$ thus takes the form

$$\tilde{\mathcal{E}}_{\mathrm{B},4,2} = -\sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in M_k\cap M_l} 1_{M_{-k}}(p\mp k\mp l) 1_{M_{-l}}(p\mp k\mp l) \left\langle e_{p\mp k\mp l'}, \psi_{-l,-p}\right\rangle \left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle \tilde{c}^*_{p\mp l}\tilde{c}_{p\mp l} \tag{3.33}$$

so by Cauchy-Schwarz we may for any $\Psi \in D(H'_{\mathrm{kin}})$ estimate that

$$\begin{aligned} \left|\left\langle \Psi, \tilde{\mathcal{E}}_{\mathrm{B},4,2}\Psi\right\rangle\right| &\leq \sqrt{\sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in M_k\cap M_l} 1_{M_{-k}}(p\mp k\mp l) 1_{M_{-l}}(p\mp k\mp l) \left|\left\langle e_{p\mp k\mp l'}, \psi_{-l,-p}\right\rangle\right|^2 \|\tilde{c}_{p\mp l}\Psi\|^2} \\ &\cdot \sqrt{\sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in M_k\cap M_l} 1_{M_{-k}}(p\mp k\mp l) 1_{M_{-l}}(p\mp k\mp l) \left|\left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle\right|^2 \|\tilde{c}_{p\mp l}\Psi\|^2}. \end{aligned} \tag{3.34}$$

The first quantity can be controlled by writing it as

$$\begin{aligned} &\sum_{l\in\mathbb{Z}^3_*} \sum_{p\in M_l} \left(\sum_{k\in\mathbb{Z}^3_*} 1_{M_k}(p) 1_{M_{-k}}(p\mp k\mp l) 1_{M_{-l}}(p\mp k\mp l) \left|\left\langle e_{p\mp k\mp l'}, \psi_{-l,-p}\right\rangle\right|^2 \right) \|\tilde{c}_{p\mp l}\Psi\|^2 \\ &\leq \sum_{l\in\mathbb{Z}^3_*} \sum_{p\in M_l} \|\psi_{-l,-p}\|^2 \|\tilde{c}_{p\mp l}\Psi\|^2 \leq \sum_{l\in\mathbb{Z}^3_*} \max_{p\in M_l} \|\psi_{-l,-p}\|^2 \sum_{p\in M_l} \|\tilde{c}_{p\mp l}\Psi\|^2 \end{aligned} \tag{3.35}$$

$$\leq \sum_{l\in\mathbb{Z}^3_*} \max_{p\in M_l} \|\psi_{-l,-p}\|^2 \langle \Psi, \mathcal{N}_E \Psi\rangle \leq \sum_{l\in\mathbb{Z}^3_*} \max_{p\in M_l} \|\psi_{l,p}\|^2 \left\langle \Psi, H'_{\mathrm{kin}} \Psi \right\rangle.$$

For the second we instead estimate

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \sum_{l\in\mathbb{Z}^3_*} M_l(p) 1_{M_{-k}}(p\mp k\mp l) 1_{M_{-l}}(p\mp k\mp l) \left|\left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle\right|^2 \|\tilde{c}_{p\mp l}\Psi\|^2 \\
&\leq \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_{-q}, \psi_{-k,-p}\rangle|^2 \left(\sum_{l\in\mathbb{Z}^3_*} M_l(p) 1_{M_{-k}}(p\mp k\mp l) 1_{M_{-l}}(p\mp k\mp l) \|\tilde{c}_{p\mp l}\Psi\|^2\right) \qquad (3.36) \\
&\leq \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_{-q}, \psi_{-k,-p}\rangle|^2 \langle \Psi, \mathcal{N}_E \Psi\rangle \leq \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2 \left\langle \Psi, H'_{\mathrm{kin}} \Psi\right\rangle
\end{aligned}$$

and the claim follows.

□

Finally we have $\mathcal{E}'_{\mathrm{B},5}$:

Proposition 3.9. *It holds as $k_F\to\infty$ that*

$$\pm\tilde{\mathcal{E}}'_{\mathrm{B},5} \leq \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2 H'_{\mathrm{kin}}.$$

Proof: From equation (3.17) we have

$$\tilde{\mathcal{E}}'_{\mathrm{B},5} = \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in M_k\cap M_l} 1_{M_{-l}}(p\mp k\mp l) 1_{M_{-k}}(p\mp k\mp l) \left\langle e_{p\mp k\mp l'}, \psi_{-l,-p}\right\rangle \left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle \tilde{c}^*_{p\mp k\mp l}\tilde{c}_{p\mp k\mp l}$$

and since the summand is symmetric in k and l we can for any $\Psi\in D(H'_{\mathrm{kin}})$ estimate using Cauchy-Schwarz

$$\begin{aligned}
\left|\left\langle \Psi, \tilde{\mathcal{E}}'_{\mathrm{B},5}\Psi\right\rangle\right| &\leq \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p\in M_k\cap M_l} 1_{M_{-l}}(p\mp k\mp l) 1_{M_{-k}}(p\mp k\mp l) \left|\left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle\right|^2 \|\tilde{c}_{p\mp k\mp l}\Psi\|^2 \qquad (3.37) \\
&\leq \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_{-q}, \psi_{-k,-p}\rangle|^2 \left(\sum_{l\in\mathbb{Z}^3_*} 1_{M_l}(p) 1_{M_{-l}}(p\mp k\mp l) 1_{M_{-k}}(p\mp k\mp l) \|\tilde{c}_{p\mp k\mp l}\Psi\|^2\right) \\
&\leq \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_{-q}, \psi_{-k,-p}\rangle|^2 \langle \Psi, \mathcal{N}_E \Psi\rangle \leq \sum_{k\in\mathbb{Z}^3_*} \sum_{p\in M_k} \max_{q\in L_k} |\langle e_q, \psi_{k,p}\rangle|^2 \left\langle \Psi, H'_{\mathrm{kin}} \Psi\right\rangle.
\end{aligned}$$

□

3.3 Proof of Theorem 3.1

We are now ready to insert the particular $\varphi_{k,p}$'s and $\psi_{k,p}$'s of our problem to conclude Theorem 3.1. To estimate the relevant quantities we will need the following matrix element estimates on the one-body operators C_k and S_k:

Proposition 3.10. *For any $k\in\mathbb{Z}^3_*$ and $p,q\in L_k$ it holds that*

$$\begin{aligned}
|\langle e_p, (C_k-1)e_q\rangle|, |\langle e_p, S_k e_q\rangle| &\leq C\frac{\hat{V}_k k_F^{-\beta}}{\lambda_{k,p}+\lambda_{k,q}} \\
\left|\langle e_p, S_k e_q\rangle - \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}\right| &\leq C\frac{\hat{V}_k^2 k_F^{1-2\beta}}{\lambda_{k,p}+\lambda_{k,q}}
\end{aligned}$$

for a constant $C>0$ independent of all quantities.

The proof of these estimates is similar to that of the one-body estimates of [11, Section 7] so we leave this to appendix section A.2.

With these estimates we can also bound $\mathcal{E}'_{\mathrm{B},6}$:

Proposition 3.11. *It holds as $k_F \to \infty$ that*

$$\pm\mathcal{E}'_{\mathrm{B},6} \leq C k_F^{3(1-\beta)} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^3$$

for a constant $C > 0$ independent of all quantities.

Proof: As in equation (2.49), the fact that $\lambda_{k,p} + \lambda_{k,q} = \lambda_{l,p} + \lambda_{l,q}$ when there is a Kronecker delta $\delta_{p+q,k+l}$ means that we can write

$$\mathcal{E}'_{\mathrm{B},6} = \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l}(\lambda_{k,p}+\lambda_{k,q}) \langle S_k e_p, e_q\rangle \langle e_q, S_l e_p\rangle - E_{\mathrm{corr,ex}} \tag{3.38}$$

$$E_{\mathrm{corr,ex}} = \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l}(\lambda_{k,p}+\lambda_{k,q}) \left(\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}\right)\left(\frac{\hat{V}_l k_F^{-\beta}}{2(2\pi)^3}\frac{1}{\lambda_{l,p}+\lambda_{l,q}}\right)$$

so $\mathcal{E}'_{\mathrm{B},6}$ can be written as the sum of two terms

$$\begin{aligned}\mathcal{E}'_{\mathrm{B},6} &= \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l}(\lambda_{l,p}+\lambda_{l,q}) \left(\langle S_k e_p, e_q\rangle - \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}\right)\langle e_q, S_l e_p\rangle \\ &+ \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l}(\lambda_{k,p}+\lambda_{k,q}) \left(\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\frac{1}{\lambda_{k,p}+\lambda_{k,q}}\right)\left(\langle e_q, S_l e_p\rangle - \frac{\hat{V}_l k_F^{-\beta}}{2(2\pi)^3}\frac{1}{\lambda_{l,p}+\lambda_{l,q}}\right).\end{aligned} \tag{3.39}$$

By the estimates of Proposition 3.10 these terms can be estimated in a similar form for

$$\begin{aligned}\left|\mathcal{E}'_{\mathrm{B},6}\right| &\leq C \sum_{k,l\in\mathbb{Z}^3_*} \sum_{p,q\in L_k\cap L_l} \delta_{p+q,k+l}(\lambda_{l,p}+\lambda_{l,q}) \frac{\hat{V}_k^2 k_F^{1-2\beta}}{(\lambda_{k,p}+\lambda_{k,q})^2}\frac{\hat{V}_l k_F^{-\beta}}{\lambda_{l,p}+\lambda_{l,q}} \\ &= C k_F^{1-3\beta} \sum_{k,l\in\mathbb{Z}^3_*} \hat{V}_k^2 \hat{V}_l \sum_{p,q\in L_k\cap L_l} \frac{\delta_{p+q,k+l}}{(\lambda_{k,p}+\lambda_{k,q})^2}.\end{aligned} \tag{3.40}$$

Using that $\delta_{p+q,k+l}(\lambda_{k,p}+\lambda_{k,q}) = \delta_{p+q,k+l}(\lambda_{l,p}+\lambda_{l,q})$ again, Hölder's inequality now implies

$$\left|\mathcal{E}'_{\mathrm{B},6}\right| \leq C k_F^{1-3\beta} \sum_{k,l\in\mathbb{Z}^3_*} \hat{V}_k^3 \sum_{p,q\in L_k} \frac{\delta_{p+q,k+l}}{(\lambda_{k,p}+\lambda_{k,q})^2} \leq C k_F^{1-3\beta} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^3 \left(\sum_{p\in L_k}\frac{1}{\lambda_{k,p}}\right)^2 \leq C k_F^{3(1-\beta)} \sum_{k\in\mathbb{Z}^3_*} \hat{V}_k^3 \tag{3.41}$$

where we also used the bound of Proposition 3.5.

□

We now conclude the main result of this section:

Theorem (3.1). *For any symmetric set $S \subset \mathbb{Z}^3_*$ and $\epsilon > 0$ it holds as $k_F \to \infty$ that*

$$\pm\mathcal{E}_{\mathrm{B}} \leq C_\epsilon k_F^{2(1-\beta)+\epsilon}\left(\sqrt{\sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k^2} + k_F^{-\frac{1}{2}}\sum_{k\in S}\hat{V}_k\right)\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2 \min\{|k|, k_F\}}\,(H'_{\mathrm{kin}} + k_F) + C k_F^{3(1-\beta)}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^3$$

for constants $C, C_\epsilon > 0$ with C independent of all quantities and C_ϵ depending only on ϵ.

Proof: Recalling the definition of equation (3.5), we can use Proposition 3.10 to estimate that

$$\begin{aligned}\max_{p\in M_k}\|\varphi_{k,p}\|^2 &= \max_{p\in L_k}\sum_{q\in L_k}\left||p|^2-k_F^2\right|\left|\langle e_q,(C_k-1)e_p\rangle\right|^2 \leq C k_F^{-2\beta}\hat{V}_k^2 \max_{p\in L_k}\sum_{q\in L_k}\frac{\left||p|^2-k_F^2\right|}{(\lambda_{k,p}+\lambda_{k,q})^2} \\ &\leq C k_F^{-2\beta}\hat{V}_k^2\sum_{q\in L_k}\lambda_{k,q}^{-1} \leq C k_F^{1-2\beta}\hat{V}_k^2\end{aligned} \tag{3.42}$$

$$\sum_{p\in M_k}\max_{q\in L_k}\left|\langle e_q,\varphi_{k,p}\rangle\right|^2 = \sum_{p\in L_k}\max_{q\in L_k}\left||p|^2-k_F^2\right|\left|\langle e_q,(C_k-1)e_p\rangle\right|^2 \leq C k_F^{-2\beta}\hat{V}_k^2\sum_{p\in L_k}\max_{q\in L_k}\frac{\left||p|^2-k_F^2\right|}{(\lambda_{k,p}+\lambda_{k,q})^2}$$

$$\leq C k_F^{-2\beta} \hat{V}_k^2 \sum_{p \in L_k} \lambda_{k,p}^{-1} \leq C k_F^{1-2\beta} \hat{V}_k^2$$

when $p \in M_k = L_k$, where we used that $\left| |p|^2 - k_F^2 \right| \leq \left| |p|^2 - k_F^2 \right| + \left| |p-k|^2 - k_F^2 \right| = 2\lambda_{k,p}$. This is also true when $p \in M_k = (L_k - k)$ (the only difference being the substitition $\left| |p|^2 - k_F^2 \right| \to \left| |p-k|^2 - k_F^2 \right|$ in the formulas above) and, since the estimate for $(C_k - 1)$ is also valid for S_k, the same estimates hold when $\psi_{k,p}$ is substituted for $\varphi_{k,p}$.

Consequently all the estimates for $\tilde{\mathcal{E}}_{\mathrm{B},1}, \ldots, \tilde{\mathcal{E}}'_{\mathrm{B},5}$ of the Propositions 3.3, 3.6, 3.7, 3.8 and 3.9 can be dominated by

$$C_\epsilon k_F^{2(1-\beta)+\epsilon} \left(\sqrt{\sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k^2} + k_F^{-\frac{1}{2}} \sum_{k \in S} \hat{V}_k \right) \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \left(H'_{\mathrm{kin}} + k_F \right) \tag{3.43}$$

and including also our estimate of $\mathcal{E}'_{\mathrm{B},6}$ yields the claim.

□

4 Inclusion of the "Small k" Cubic Terms

In this section we perform the computations leading to the incorporation of the "small k" cubic terms into the factorization of H_{B}.

For convenience we recall that the (full) cubic terms can be written

$$\mathcal{C} = 4 \operatorname{Re} \sum_{k \in \mathbb{Z}^3_*} b_k^*(w_k) D_k \tag{4.1}$$

where

$$w_k = \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3} \sum_{p \in L_k} e_p = \sqrt{\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}} v_k. \tag{4.2}$$

We furthermore define

$$\eta_k = \begin{cases} E_k^{-\frac{3}{2}} h_k^{\frac{1}{2}} w_k & k \in S \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

for a fixed symmetric subset $S \subset \mathbb{Z}^3_*$ (to be optimized over at the end) and

$$d_p^3 = \begin{cases} +\sum_{k \in S} 1_{L_k}(p) \langle e_p, \eta_k \rangle \tilde{c}^*_{p-k} D_k & p \in B_F^c \\ -\sum_{k \in S} 1_{L_k - k}(p) \langle e_{p+k}, \eta_k \rangle \tilde{c}^*_{p+k} D_k & p \in B_F \end{cases}. \tag{4.4}$$

We will prove the following:

Theorem 4.1. *For any symmetric set $S \subset \mathbb{Z}^3_*$ it holds that*

$$\begin{aligned}
& H_{\mathrm{B}} + 4 \operatorname{Re} \sum_{k \in S} b_k^*(w_k) D_k + \frac{k_F^{-\beta}}{2(2\pi)^3} \sum_{k \in S} \hat{V}_k \frac{2 \langle v_k, h_k^{-1} v_k \rangle}{1 + 2 \langle v_k, h_k^{-1} v_k \rangle} D_k^* D_k \\
= & \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \left(\left| \tilde{c}_p + d_p^1 + d_p^2 + d_p^3 \right|^2 + \left| (d_p^1 + d_p^2 + d_p^3)^* \right|^2 \right) - 2 \sum_{k \in \mathbb{Z}^3_*} \sum_{p \in L_k} \varepsilon_{k,k}(e_p; S_k E_k S_k^* e_p) \\
& + \sum_{k \in \mathbb{Z}^3_*} \sum_{p,q \in L_k} 2 \langle e_p, (E_k - h_k) e_q \rangle \left(b_k(C_k e_p) + b^*_{-k}(S_{-k} e_{-p}) + \langle e_p, \eta_k \rangle D_k \right)^* \left(b_k(C_k e_q) + b^*_{-k}(S_{-k} e_{-q}) + \langle e_q, \eta_k \rangle D_k \right) \\
& + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + \mathcal{E}_{\mathrm{B}} + \mathcal{E}_{\mathcal{C}}
\end{aligned}$$

for an operator $\mathcal{E}_{\mathcal{C}}$ defined below.

4.1 Expansion of the Potential Terms

As in Section 2 we first consider the potential part of the factorization. For that we first have the following:

Proposition 4.2. *For any symmetric operators $A_k : \ell^2(L_k) \to \ell^2(L_k)$, $k \in \mathbb{Z}^3_*$, obeying*

$$\langle e_p, A_k e_q \rangle = \langle e_{-p}, A_{-k} e_{-q} \rangle, \quad p, q \in L_k,$$

it holds that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\langle e_p, A_k e_q\rangle \left(b_k(C_k e_p) + b^*_{-k}(S_{-k}e_{-p}) + \langle e_p, \eta_k\rangle D_k\right)^* \left(b_k(C_k e_q) + b^*_{-k}(S_{-k}e_{-q}) + \langle e_q, \eta_k\rangle D_k\right) \\
&= \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\langle e_p, A_k e_q\rangle \left(b_k(C_k e_p) + b^*_{-k}(S_{-k}e_{-p})\right)^* \left(b_k(C_k e_q) + b^*_{-k}(S_{-k}e_{-q})\right) \\
&+ 4\,\mathrm{Re}\sum_{k\in S} b^*_k((C_k + S_k)A_k\eta_k)D_k + \sum_{k\in S}\sum_{p,q\in L_k} 2\langle \eta_k, A_k\eta_k\rangle D^*_k D_k
\end{aligned}$$

Proof: This is immediate by expansion upon noting that

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\langle e_p, A_k e_q\rangle \langle e_q, \eta_k\rangle b^*_k(C_k e_p)D_k &= 2\sum_{k\in S}\sum_{p,q\in L_k} b^*_k(C_k A_k\eta_k)D_k \\
\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\langle e_p, A_k e_q\rangle \langle \eta_k, e_p\rangle \langle e_q, \eta_k\rangle D^*_k D_k &= \sum_{k\in S} 2\langle \eta_k, A_k\eta_k\rangle D^*_k D_k
\end{aligned} \tag{4.5}$$

and (using also that the quantities $\langle e_q, \eta_k\rangle$ are real and obey $\langle e_{-q}, \eta_{-k}\rangle = \langle e_q, \eta_k\rangle$)

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\langle e_p, A_k e_q\rangle \langle e_q, \eta_k\rangle b_{-k}(S_{-k}e_{-p})D_k &= 2\sum_{k\in S} b_{-k}(S_{-k}h_{-k}\eta_{-k})D^*_{-k} \\
&= 2\sum_{k\in S} D^*_k b_k(S_k h_k\eta_k)
\end{aligned} \tag{4.6}$$

as it holds in general that $[b_k(\cdot), D^*_k] = 0$.

$\square$

This allows us to conclude a generalization of Proposition 2.4:

Proposition 4.3. *It holds that*

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^3_*}\left(2\,Q^k_1(P_k) + Q^k_2(P_k)\right) + 4\,\mathrm{Re}\sum_{k\in S} b^*_k(w_k)D_k + \frac{k_F^{-\beta}}{2(2\pi)^3}\sum_{k\in S}\hat{V}_k \frac{2\left\langle v_k, h_k^{-1}v_k\right\rangle}{1 + 2\left\langle v_k, h_k^{-1}v_k\right\rangle} D^*_k D_k \\
&= -\sum_{k\in\mathbb{Z}^3_*} 2\,Q^k_1(h_k) + E_{\mathrm{corr,bos}} - \sum_{k\in\mathbb{Z}^3_*}\varepsilon_{k,k}(e_p; S_k E_k S^*_k e_p) \\
&+ \sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k} 2\langle e_p, E_k e_q\rangle \left(b_k(C_k e_p) + b^*_{-k}(S_{-k}e_{-p}) + \langle e_p, \eta_k\rangle D_k\right)^* \left(b_k(C_k e_q) + b^*_{-k}(S_{-k}e_{-q}) + \langle e_q, \eta_k\rangle D_k\right).
\end{aligned}$$

Proof: The only terms above which are not accounted for by Proposition 2.4 after applying the previous proposition are the final two terms on the left-hand side. These arise since η_k obeys (for $k \in S$)

$$(C_k + S_k)E_k\eta_k = h_k^{-\frac{1}{2}} E_k^{\frac{1}{2}} E_k E_k^{-\frac{3}{2}} h_k^{\frac{1}{2}} w_k = w_k \tag{4.7}$$

whence

$$4\,\mathrm{Re}\sum_{k\in S} b^*_k((C_k + S_k)E_k\eta_k)D_k = 4\,\mathrm{Re}\sum_{k\in S} b^*_k(w_k)D_k, \tag{4.8}$$

while by the definition of w_k and η_k

$$\langle \eta_k, E_k\eta_k\rangle = \left\langle w_k, h_k^{\frac{1}{2}} E_k^{-2} h_k^{\frac{1}{2}} w_k\right\rangle = \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\left\langle v_k, (h_k + 2P_k)^{-1} v_k\right\rangle \tag{4.9}$$

and by the Sherman-Morrison formula

$$(h_k + 2P_k)^{-1} = h_k^{-1} - \frac{2}{1 + 2\left\langle v_k, h_k^{-1} v_k \right\rangle} h_k^{-1} P_k h_k^{-1} \tag{4.10}$$

so

$$\langle \eta_k, E_k \eta_k \rangle = \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3} \left(\left\langle v_k, h_k^{-1} v_k \right\rangle - \frac{2 \left\langle v_k, h_k^{-1} v_k \right\rangle^2}{1 + 2\left\langle v_k, h_k^{-1} v_k \right\rangle} \right) = \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3} \frac{\left\langle v_k, h_k^{-1} v_k \right\rangle}{1 + 2\left\langle v_k, h_k^{-1} v_k \right\rangle} \tag{4.11}$$

i.e.

$$\sum_{k \in S} 2 \langle \eta_k, E_k \eta_k \rangle D_k^* D_k = \sum_{k \in S} \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3} \frac{2 \left\langle v_k, h_k^{-1} v_k \right\rangle}{1 + 2\left\langle v_k, h_k^{-1} v_k \right\rangle} D_k^* D_k. \tag{4.12}$$

□

4.2 Expansion of The Kinetic Terms

Obviously

$$\begin{aligned} &\sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \left| \tilde{c}_p + d_p^1 + d_p^2 + d_p^3 \right|^2 \\ &= \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \left| \tilde{c}_p + d_p^1 + d_p^2 \right|^2 + 2 \operatorname{Re} \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^3 \\ &+ \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| (d_p^3)^* d_p^3 + 2 \operatorname{Re} \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| (d_p^1)^* d_p^3 + 2 \operatorname{Re} \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| (d_p^2)^* d_p^3 \end{aligned} \tag{4.13}$$

and the term $\sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \left| \tilde{c}_p + d_p^1 + d_p^2 \right|^2$ is what we considered in Section 2, so we examine the remaining expressions. First the simplest:

Proposition 4.4. *It holds that*

$$2 \operatorname{Re} \sum_{p \in \mathbb{Z}^3} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^3 = 4 \operatorname{Re} \sum_{k \in S} b_k^* (h_k \eta_k) D_k.$$

Proof: It follows directly from equation (4.4) that

$$\begin{aligned} \sum_{p \in B_F^c} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^3 &= + \sum_{k \in S} \sum_{p \in L_k} \left| |p|^2 - k_F^2 \right| \langle e_p, \eta_k \rangle \tilde{c}_p^* \tilde{c}_{p-k}^* D_k \\ \sum_{p \in B_F} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^3 &= - \sum_{k \in S} \sum_{p \in L_k} \left| |p-k|^2 - k_F^2 \right| \langle e_p, \eta_k \rangle \tilde{c}_{p-k}^* \tilde{c}_p^* D_k \end{aligned} \tag{4.14}$$

so

$$\begin{aligned} \sum_{p \in \mathbb{Z}^3_*} \left| |p|^2 - k_F^2 \right| \tilde{c}_p^* d_p^3 &= \sum_{k \in S} \sum_{p \in L_k} \left(\left| |p|^2 - k_F^2 \right| + \left| |p-k|^2 - k_F^2 \right| \right) \langle e_p, \eta_k \rangle \tilde{c}_p^* \tilde{c}_{p-k}^* D_k \\ &= \sum_{k \in S} \sum_{p \in L_k} 2 \lambda_{k,p} \langle e_p, \eta_k \rangle b_{k,p}^* D_k = 2 \sum_{k \in \mathbb{Z}^3_*} b_k^* (h_k \eta_k) D_k \end{aligned} \tag{4.15}$$

which implies the claim.

□

For the remaining terms of equation (4.13) we must again define a number of error terms. In the notation of Section 2 the first of these are $\mathcal{E}_{\mathcal{C},m} = \mathcal{E}_{\mathcal{C},m}^{(1)} + \mathcal{E}_{\mathcal{C},m}^{(2)}$ where

$$\begin{aligned} \mathcal{E}_{\mathcal{C},1}^{(1)} &= \sum_{k,l \in S} \sum_{p \in L_k \cap L_l} \left| |p|^2 - k_F^2 \right| \langle \eta_k, e_p \rangle \langle e_p, \eta_l \rangle \tilde{c}_{p-l}^* [D_k^*, D_l] \tilde{c}_{p-k} \\ \mathcal{E}_{\mathcal{C},2}^{(1)} &= \sum_{k,l \in S} \sum_{p \in L_k \cap L_l} \left| |p|^2 - k_F^2 \right| \langle \eta_k, e_p \rangle \langle e_p, \eta_l \rangle \tilde{c}_{p-l}^* D_k^* [\tilde{c}_{p-k}, D_l] \end{aligned} \tag{4.16}$$

$$\mathcal{E}_{\mathcal{C},3}^{(1)} = \sum_{k,l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle \eta_k, e_p\rangle \langle e_p, \eta_l\rangle [\tilde{c}_{p-l}, D_k]^* [\tilde{c}_{p-k}, D_l]$$

and the substitutions in going from $\mathcal{E}_{\mathcal{C},m}^{(1)}$ to $\mathcal{E}_{\mathcal{C},m}^{(2)}$ now also includes $\langle \eta_k, e_p\rangle \to \langle \eta_k, e_{p+k}\rangle$. We can then state

Proposition 4.5. *It holds that*

$$\sum_{p\in\mathbb{Z}^3} \left||p|^2 - k_F^2\right| (d_p^3)^* d_p^3 = 2\sum_{k\in S} \langle \eta_k, h_k\eta_k\rangle D_k^* D_k + \sum_{p\in\mathbb{Z}^3} \left||p|^2 - k_F^2\right| d_p^3 (d_p^3)^* - \mathcal{E}_{\mathcal{C},1} - 2\operatorname{Re}(\mathcal{E}_{\mathcal{C},2}) - \mathcal{E}_{\mathcal{C},3}.$$

Proof: By the definition of equation (4.4) we have that for $p \in B_F^c$

$$\begin{aligned}\sum_{p\in B_F^c} \left||p|^2 - k_F^2\right| (d_p^3)^* d_p^3 &= \sum_{k,l\in S}\sum_{p\in B_F^c} 1_{L_k\cap L_l}(p) \left||p|^2 - k_F^2\right| \langle \eta_k, e_p\rangle \langle e_p, \eta_l\rangle D_k^* \tilde{c}_{p-k} \tilde{c}_{p-l}^* D_l \\ &= \sum_{k\in S}\sum_{p\in L_k} \left||p|^2 - k_F^2\right| |\langle e_p, \eta_k\rangle|^2 D_k^* D_k \\ &\quad - \sum_{k,l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle \eta_k, e_p\rangle \langle e_p, \eta_l\rangle D_k^* \tilde{c}_{p-l}^* \tilde{c}_{p-k} D_l\end{aligned} \tag{4.17}$$

and similarly, when $p \in B_F$,

$$\begin{aligned}\sum_{p\in B_F} \left||p|^2 - k_F^2\right| (d_p^3)^* d_p^3 &= \sum_{k\in S}\sum_{p\in L_k} \left||p-k|^2 - k_F^2\right| |\langle e_p, \eta_k\rangle|^2 D_k^* D_k \\ &\quad - \sum_{k,l\in S}\sum_{p\in (L_k-k)\cap(L_l-l)} \left||p|^2 - k_F^2\right| \langle \eta_k, e_{p+k}\rangle \langle e_{p+l}, \eta_l\rangle D_k^* \tilde{c}_{p+l}^* \tilde{c}_{p+k} D_l.\end{aligned} \tag{4.18}$$

The leading terms combine to form

$$\begin{aligned}&\sum_{k\in S}\sum_{p\in L_k} \left(\left||p|^2 - k_F^2\right| + \left||p-k|^2 - k_F^2\right|\right) |\langle e_p, \eta_k\rangle|^2 D_k^* D_k \\ &= \sum_{k\in S}\left(\sum_{p\in L_k} 2\lambda_{k,p} |\langle e_p, \eta_k\rangle|^2\right) D_k^* D_k = 2\sum_{k\in S} \langle \eta_k, h_k\eta_k\rangle D_k^* D_k\end{aligned} \tag{4.19}$$

while the remaining terms obey e.g.

$$\begin{aligned}&\sum_{k,l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle \eta_k, e_p\rangle \langle e_p, \eta_l\rangle D_k^* \tilde{c}_{p-l}^* \tilde{c}_{p-k} D_l \\ &= \sum_{k,l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle \eta_k, e_p\rangle \langle e_p, \eta_l\rangle \tilde{c}_{p-l}^* D_k^* D_l \tilde{c}_{p-k} \\ &+ 2\operatorname{Re}\sum_{k,l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle \eta_k, e_p\rangle \langle e_p, \eta_l\rangle \tilde{c}_{p-l}^* D_k^* [\tilde{c}_{p-k}, D_l] \\ &+ \sum_{k,l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle \eta_k, e_p\rangle \langle e_p, \eta_l\rangle [\tilde{c}_{p-l}, D_k]^* [\tilde{c}_{p-k}, D_l] \\ &= \sum_{p\in B_F^c} \left||p|^2 - k_F^2\right| d_p^3 (d_p^3)^* + \mathcal{E}_{\mathcal{C},1}^{(1)} + 2\operatorname{Re}\left(\mathcal{E}_{\mathcal{C},2}^{(1)}\right) + \mathcal{E}_{\mathcal{C},3}^{(1)}.\end{aligned} \tag{4.20}$$

$\square$

For the last terms of equation (4.13) we define the final error terms by

$$\begin{aligned}\mathcal{E}_{\mathcal{C},4}^{(1)} &= \sum_{k\in\mathbb{Z}_*^3}\sum_{l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle e_p, \eta_l\rangle \tilde{c}_{p-l}^* [b_k^*((C_k - 1)e_p)\tilde{c}_{p-k}, D_l] \\ \mathcal{E}_{\mathcal{C},5}^{(1)} &= \sum_{k\in\mathbb{Z}_*^3}\sum_{l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle e_p, \eta_l\rangle \tilde{c}_{p-l}^* [b_{-k}(S_{-k}e_{-p})\tilde{c}_{p-k}, D_l] \\ \mathcal{E}_{\mathcal{C},6}^{(1)} &= \sum_{k\in\mathbb{Z}_*^3}\sum_{l\in S}\sum_{p\in L_k\cap L_l} \left||p|^2 - k_F^2\right| \langle e_p, \eta_l\rangle \left[b_{-k}(S_{-k}e_{-p}), \tilde{c}_{p-l}^*\right] \tilde{c}_{p-k} D_l\end{aligned} \tag{4.21}$$

and compute the following:

Proposition 4.6. *It holds that*

$$2\,\mathrm{Re}\sum_{p\in\mathbb{Z}^3}\big|\,|p|^2-k_F^2\big|\,(d_p^1)^*d_p^3=4\,\mathrm{Re}\sum_{k\in S}b_k^*((C_k-1)h_k\eta_k)D_k+2\,\mathrm{Re}\sum_{p\in\mathbb{Z}^3}\big|\,|p|^2-k_F^2\big|\,d_p^3(d_p^1)^*-2\,\mathrm{Re}(\mathcal{E}_{\mathcal{C},4})$$
$$2\,\mathrm{Re}\sum_{p\in\mathbb{Z}^3}\big|\,|p|^2-k_F^2\big|\,(d_p^2)^*d_p^3=4\,\mathrm{Re}\sum_{k\in S}b_k^*(S_kh_k\eta_k)D_k+2\,\mathrm{Re}\sum_{p\in\mathbb{Z}^3}\big|\,|p|^2-k_F^2\big|\,d_p^3(d_p^2)^*-2\,\mathrm{Re}(\mathcal{E}_{\mathcal{C},5}+\mathcal{E}_{\mathcal{C},6}).$$

Proof: As in the previous proposition it is easily verified that

$$\begin{aligned}\sum_{p\in B_F^c}\big|\,|p|^2-k_F^2\big|\,(d_p^1)^*d_p^3&=\sum_{k\in S}\sum_{p\in L_k}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_k\rangle\,b_k^*((C_k-1)e_p)D_k\\&\quad-\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_l\rangle\,b_k^*((C_k-1)e_p)\tilde{c}_{p-l}^*\tilde{c}_{p-k}D_l\qquad(4.22)\\\sum_{p\in B_F}\big|\,|p|^2-k_F^2\big|\,(d_p^1)^*d_p^3&=\sum_{k\in S}\sum_{p\in L_k}\big|\,|p-k|^2-k_F^2\big|\,\langle e_p,\eta_k\rangle\,b_k^*((C_k-1)e_p)D_k\\&\quad-\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in(L_k-k)\cap(L_l-l)}\big|\,|p|^2-k_F^2\big|\,\langle e_{p+l},\eta_l\rangle\,b_k^*((C_k-1)e_{p+k})\tilde{c}_{p+l}^*\tilde{c}_{p+k}D_l\end{aligned}$$

and the first terms form

$$\begin{aligned}&\sum_{k\in S}\left(\sum_{p\in L_k}\Big(\big|\,|p|^2-k_F^2\big|+\big|\,|p-k|^2-k_F^2\big|\Big)\,\langle e_p,\eta_k\rangle\,b_k^*((C_k-1)e_p)\right)D_k\qquad(4.23)\\&=2\sum_{k\in S}\left(\sum_{p\in L_k}\langle e_p,h_k\eta_k\rangle\,b_k^*((C_k-1)e_p)\right)D_k=2\sum_{k\in S}b_k^*((C_k-1)h_k\eta_k)D_k\end{aligned}$$

whereas the second terms obey

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_l\rangle\,b_k^*((C_k-1)e_p)\tilde{c}_{p-l}^*\tilde{c}_{p-k}D_l\\&=\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_l\rangle\,\tilde{c}_{p-l}^*D_lb_k^*((C_k-1)e_p)\tilde{c}_{p-k}\qquad(4.24)\\&+\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_l\rangle\,\tilde{c}_{p-l}^*\,[b_k^*((C_k-1)e_p)\tilde{c}_{p-k},D_l]\\&=\sum_{p\in B_F^c}\big|\,|p|^2-k_F^2\big|\,d_p^3(d_p^1)^*+\mathcal{E}_{\mathcal{C},4}^{(1)}\end{aligned}$$

where we also used that $[b_k^*(\cdot),\tilde{c}^*]=0$.
For the $(d_p^2)^*d_p^3$ sum one similarly finds terms combining to yield

$$\begin{aligned}&\sum_{k\in S}\sum_{p\in L_k}\Big(\big|\,|p|^2-k_F^2\big|+\big|\,|p-k|^2-k_F^2\big|\Big)\,\langle e_p,\eta_k\rangle\,b_{-k}(S_{-k}e_{-p})D_k\qquad(4.25)\\&=2\sum_{k\in S}b_{-k}(S_{-k}h_{-k}\eta_{-k})D_{-k}^*=2\sum_{k\in S}D_k^*b_k(S_kh_k\eta_k)\end{aligned}$$

as in equation (4.6), and additional terms of the form

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_l\rangle\,b_{-k}(S_{-k}e_{-p})\tilde{c}_{p-l}^*\tilde{c}_{p-k}D_l\\&=\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_l\rangle\,\tilde{c}_{p-l}^*D_lb_{-k}(S_{-k}e_{-p})\tilde{c}_{p-k}\\&+\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\big|\,|p|^2-k_F^2\big|\,\langle e_p,\eta_l\rangle\,\tilde{c}_{p-l}^*\,[b_{-k}(S_{-k}e_{-p})\tilde{c}_{p-k},D_l]\qquad(4.26)\end{aligned}$$

$$+\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in L_k\cap L_l}\left||p|^2-k_F^2\right|\langle e_p,\eta_l\rangle\left[b_{-k}(S_{-k}e_{-p}),\tilde{c}^*_{p-l}\right]\tilde{c}_{p-k}D_l$$
$$=\sum_{p\in B_F^c}\left||p|^2-k_F^2\right|d_p^3(d_p^2)^*+\mathcal{E}^{(1)}_{\mathcal{C},5}+\mathcal{E}^{(1)}_{\mathcal{C},6}.$$

□

We can now conclude the generalization of Proposition 2.7:

Proposition 4.7. *It holds that*

$$H'_{\text{kin}}=\sum_{k\in\mathbb{Z}^3_*}2\,Q_1^k(h_k)+\sum_{p\in\mathbb{Z}^3}\left||p|^2-k_F^2\right|\left(\left|\tilde{c}_p+d_p^1+d_p^2+d_p^3\right|^2+\left|(d_p^1+d_p^2+d_p^3)^*\right|^2\right)$$
$$-\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}2\langle e_p,h_ke_q\rangle\left(b_k(C_ke_p)+b^*_{-k}(S_{-k}e_{-p})+\langle e_p,\eta_k\rangle D_k\right)^*\left(b_k(C_ke_q)+b^*_{-k}(S_{-k}e_{-q})+\langle e_q,\eta_k\rangle D_k\right)$$
$$+E_{\text{corr,ex}}+\mathcal{E}_{\text{B}}+\mathcal{E}_{\mathcal{C}}$$

for

$$\mathcal{E}_{\mathcal{C}}=\mathcal{E}_{\mathcal{C},1}+2\,\text{Re}(\mathcal{E}_{\mathcal{C},2})+\mathcal{E}_{\mathcal{C},3}+2\,\text{Re}(\mathcal{E}_{\mathcal{C},4}+\mathcal{E}_{\mathcal{C},5}+\mathcal{E}_{\mathcal{C},6}).$$

Proof: From equation (4.13), the propositions above and the computation of Section 2 we have

$$\begin{aligned}&\sum_{p\in\mathbb{Z}^3}\left||p|^2-k_F^2\right|\left|\tilde{c}_p+d_p^1+d_p^2+d_p^3\right|^2\\
&=H'_{\text{kin}}-\sum_{k\in\mathbb{Z}^3_*}2\,Q_1^k(h_k)-E_{\text{corr,ex}}-\mathcal{E}_{\text{B}}+\sum_{p\in\mathbb{Z}^3}\left||p|^2-k_F^2\right|\left|(d_p^1+d_p^2)^*\right|^2\\
&+2\,\text{Re}\sum_{p\in\mathbb{Z}^3}\left||p|^2-k_F^2\right|d_p^3(d_p^1+d_p^2)^*+\sum_{p\in\mathbb{Z}^3}\left||p|^2-k_F^2\right|d_p^3(d_p^3)^*\\
&-\mathcal{E}_{\mathcal{C},1}-2\,\text{Re}(\mathcal{E}_{\mathcal{C},2})-\mathcal{E}_{\mathcal{C},3}-2\,\text{Re}(\mathcal{E}_{\mathcal{C},4}+\mathcal{E}_{\mathcal{C},5}+\mathcal{E}_{\mathcal{C},6})\\
&+\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}2\langle e_p,h_ke_q\rangle\left(b_k(C_ke_p)+b^*_{-k}(C_{-k}e_{-p})\right)^*\left(b_k(C_ke_q)+b^*_{-k}(C_{-k}e_{-q})\right)\\
&+4\,\text{Re}\sum_{k\in S}b_k^*((C_k+S_k)h_k\eta_k)D_k+2\sum_{k\in S}\langle\eta_k,h_k\eta_k\rangle D_k^*D_k.\end{aligned}\tag{4.27}$$

By Proposition 4.2 the terms on the two final lines combine to form

$$\sum_{k\in\mathbb{Z}^3_*}\sum_{p,q\in L_k}2\langle e_p,h_ke_q\rangle\left(b_k(C_ke_p)+b^*_{-k}(S_{-k}e_{-p})+\langle e_p,\eta_k\rangle D_k\right)^*\left(b_k(C_ke_q)+b^*_{-k}(S_{-k}e_{-q})+\langle e_q,\eta_k\rangle D_k\right)$$

whereupon the claim follows by rearranging the equation.

□

Theorem 4.1 now follows by combining Proposition 4.3 and Proposition 4.7.

5 Estimation of $\mathcal{E}_{\mathcal{C}}$

In this section we bound the new error term $\mathcal{E}_{\mathcal{C}}$ of Theorem 4.1, which consists of six sub-terms

$$\mathcal{E}_{\mathcal{C}}=\mathcal{E}_{\mathcal{C},1}+2\,\text{Re}(\mathcal{E}_{\mathcal{C},2})+\mathcal{E}_{\mathcal{C},3}+2\,\text{Re}(\mathcal{E}_{\mathcal{C},4}+\mathcal{E}_{\mathcal{C},5}+\mathcal{E}_{\mathcal{C},6})\tag{5.1}$$

which are given by the equations (4.16) and (4.21).
We will prove the following:

Theorem 5.1. *For any symmetric set $S\subset\mathbb{Z}^3_*$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$\pm\mathcal{E}_{\mathcal{C}}\leq C_\epsilon k_F^{2(1-\beta)+\epsilon}\left(k_F^{-\frac{1}{2}}\sum_{k\in S}\hat{V}_k\right)\left(\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}}+k_F^{-\frac{1}{2}}\sum_{k\in S}\hat{V}_k\right)(H'_{\text{kin}}+k_F)$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Before we begin the estimation, we write the terms of $\mathcal{E}_{\mathcal{C}}$ more conveniently by introducing the quantity

$$\tilde{\eta}_{k,p} = \begin{cases} \sqrt{\left| |p|^2 - k_F^2 \right|} \langle e_p, \eta_k \rangle & p \in B_F^c \\ \sqrt{\left| |p|^2 - k_F^2 \right|} \langle e_{p+k}, \eta_k \rangle & p \in B_F \end{cases}, \tag{5.2}$$

which recalling also the definitions of Section 3 lets us represent the different expressions defining $\mathcal{E}_{\mathcal{C}}$ by the schematic forms

$$\begin{aligned} \tilde{\mathcal{E}}_{\mathcal{C},1} &= \sum_{k,l \in S} \sum_{p \in M_k \cap M_l} \tilde{\eta}_{k,p} \tilde{\eta}_{l,p} \tilde{c}_{p \mp l}^* \left[D_k^*, D_l \right] \tilde{c}_{p \mp k} \\ \tilde{\mathcal{E}}_{\mathcal{C},2} &= \sum_{k,l \in S} \sum_{p \in M_k \cap M_l} \tilde{\eta}_{k,p} \tilde{\eta}_{l,p} \tilde{c}_{p \mp l}^* D_k^* \left[\tilde{c}_{p \mp k}, D_l \right] \\ \tilde{\mathcal{E}}_{\mathcal{C},3} &= \sum_{k,l \in S} \sum_{p \in M_k \cap M_l} \tilde{\eta}_{k,p} \tilde{\eta}_{l,p} \left[\tilde{c}_{p \mp l}, D_k \right]^* \left[\tilde{c}_{p \mp k}, D_l \right] \\ \tilde{\mathcal{E}}_{\mathcal{C},4} &= \sum_{k \in \mathbb{Z}_*^3} \sum_{l \in S} \sum_{p \in M_k \cap M_l} \tilde{\eta}_{l,p} \tilde{c}_{p \mp l}^* \left[b_k^*(\varphi_{k,p}) \tilde{c}_{p \mp k}, D_l \right] \\ \tilde{\mathcal{E}}_{\mathcal{C},5} &= \sum_{k \in \mathbb{Z}_*^3} \sum_{l \in S} \sum_{p \in M_k \cap M_l} \tilde{\eta}_{l,p} \tilde{c}_{p \mp l}^* \left[b_{-k}(\psi_{-k,-p}) \tilde{c}_{p \mp k}, D_l \right] \\ \tilde{\mathcal{E}}_{\mathcal{C},6} &= \sum_{k \in \mathbb{Z}_*^3} \sum_{l \in S} \sum_{p \in M_k \cap M_l} \tilde{\eta}_{l,p} \left[b_{-k}(\psi_{-k,-p}), \tilde{c}_{p \mp l}^* \right] \tilde{c}_{p \mp k} D_l. \end{aligned} \tag{5.3}$$

We also recall that $D_k = D_{1,k} + D_{2,k}$ where

$$\begin{aligned} D_{1,k} &= \sum_{p \in B_F^c \cap \left(B_F^c + k \right)} c_{p-k}^* c_p = + \sum_{p \in B_F^c \cap \left(B_F^c - k \right)} \tilde{c}_p^* \tilde{c}_{p+k} \\ D_{2,k} &= \sum_{p \in B_F \cap (B_F + k)} c_{p-k}^* c_p = - \sum_{p \in B_F \cap (B_F + k)} \tilde{c}_p^* \tilde{c}_{p-k} \end{aligned} \tag{5.4}$$

which we can abbreviate as

$$D_{j,k} = \pm \sum_{p \in B_F^\circ \cap \left(B_F^\circ \mp k \right)} \tilde{c}_p^* \tilde{c}_{p \pm k}, \quad (B_F^\circ, \pm) = \begin{cases} (B_F^c, +) & j = 1 \\ (B_F, -) & j = 2 \end{cases}. \tag{5.5}$$

5.1 Estimation of $\tilde{\mathcal{E}}_{\mathcal{C},1}$, $\tilde{\mathcal{E}}_{\mathcal{C},2}$ and $\tilde{\mathcal{E}}_{\mathcal{C},3}$

We begin with the error terms arising from the $(d_p^3)^* d_p^3$ part of the factorization. For $\tilde{\mathcal{E}}_{\mathcal{C},1}$ we need to calculate the commutator $[D_k^*, D_l]$. Since $[D_{1,k}, D_{2,l}^*] = 0$ we need only consider the commutator $[D_{j,k}^*, D_{j,l}]$. This we compute to be

$$\begin{aligned} \left[D_{j,k}^*, D_{j,l} \right] &= \sum_{p \in B_F^\circ \cap \left(B_F^\circ \mp k \right)} \sum_{q \in B_F^\circ \cap \left(B_F^\circ \mp l \right)} \left[\tilde{c}_{p \pm k}^* \tilde{c}_p, \tilde{c}_q^* \tilde{c}_{q \pm l} \right] \\ &= \sum_{p \in B_F^\circ \cap \left(B_F^\circ \mp k \right)} \sum_{q \in B_F^\circ \cap \left(B_F^\circ \mp l \right)} \tilde{c}_{p \pm k}^* \left\{ \tilde{c}_p, \tilde{c}_q^* \right\} \tilde{c}_{q \pm l} - \sum_{p \in B_F^\circ \cap \left(B_F^\circ \mp k \right)} \sum_{q \in B_F^\circ \cap \left(B_F^\circ \mp l \right)} \tilde{c}_q^* \left\{ \tilde{c}_{p \pm k}^*, \tilde{c}_{q \pm l} \right\} \tilde{c}_p \\ &= \sum_{q \in B_F^\circ \cap \left(B_F^\circ \mp k \right) \cap \left(B_F^\circ \mp l \right)} \tilde{c}_{q \pm k}^* \tilde{c}_{q \pm l} - \sum_{q \in B_F^\circ \cap \left(B_F^\circ \pm k \right) \cap \left(B_F^\circ \pm l \right)} \tilde{c}_{q \mp l}^* \tilde{c}_{q \mp k}. \end{aligned} \tag{5.6}$$

We can now estimate $\tilde{\mathcal{E}}_{\mathcal{C},1}$ as follows:

Proposition 5.2. *For any symmetric set $S \subset \mathbb{Z}_*^3$ it holds as $k_F \to \infty$ that*

$$\pm \tilde{\mathcal{E}}_{\mathcal{C},1} \leq 2 \left(\sum_{k \in S} \sqrt{\sum_{p \in M_k} \tilde{\eta}_{k,p}^2} \right)^2 H_{\text{kin}}'.$$

Proof: For any $\Psi \in D(H'_{\text{kin}})$ we can estimate

$$\left|\left\langle \Psi, \tilde{\mathcal{E}}_{\mathcal{C},1}\Psi \right\rangle\right| \leq \sum_{k,l\in S}\sum_{p\in M_k\cap M_l}\sum_{q\in B_F^\circ\cap\left(B_F^\circ\mp k\right)\cap\left(B_F^\circ\mp l\right)} |\tilde{\eta}_{k,p}\tilde{\eta}_{l,p}| \, \|\tilde{c}_{q\pm k}\tilde{c}_{p\mp l}\Psi\| \, \|\tilde{c}_{q\pm l}\tilde{c}_{p\mp k}\Psi\| \\ + \sum_{k,l\in S}\sum_{p\in M_k\cap M_l}\sum_{q\in B_F^\circ\cap\left(B_F^\circ\pm k\right)\cap\left(B_F^\circ\pm l\right)} |\tilde{\eta}_{k,p}\tilde{\eta}_{l,p}| \, \|\tilde{c}_{q\mp l}\tilde{c}_{p\mp l}\Psi\| \, \|\tilde{c}_{q\mp k}\tilde{c}_{p\mp k}\Psi\| \tag{5.7}$$

and we focus on the first sum. Using that $\|c_{p\mp k}\|_{\text{op}} \leq 1$ we can bound this by

$$\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\sum_{q\in\left(B_F^\circ\mp k\right)}\tilde{\eta}_{k,p}^2\,\|\tilde{c}_{q+k}\Psi\|^2}\right)^2 \leq \left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\tilde{\eta}_{k,p}^2}\right)^2 \langle\Psi,\mathcal{N}_E\Psi\rangle \tag{5.8}$$

whence the claim follows since $\mathcal{N}_E \leq H'_{\text{kin}}$.

$\square$

For $\tilde{\mathcal{E}}_{\mathcal{C},2}$ and $\tilde{\mathcal{E}}_{\mathcal{C},3}$ we need the commutator $[\tilde{c}_{p\mp k}, D_l]$. When $M_k = L_k$ (so $p \mp k = p - k \in B_F$) this is

$$[\tilde{c}_{p\mp k}, D_l] = -\sum_{q\in B_F\cap(B_F+l)}\left[\tilde{c}_{p-k}, \tilde{c}_q^*\tilde{c}_{q-l}\right] = -\sum_{q\in B_F\cap(B_F+l)}\delta_{p-k,q}\tilde{c}_{q-l} \\ = -1_{B_F}(p-k-l)\tilde{c}_{p-k-l} \tag{5.9}$$

and likewise when $M_k = L_k - k$ (so $p\mp k = p+k \in B_F^c$)

$$[\tilde{c}_{p\mp k}, D_l] = \sum_{q\in B_F^c\cap\left(B_F^c-l\right)}\left[\tilde{c}_{p+k}, \tilde{c}_q^*\tilde{c}_{q+l}\right] = \sum_{q\in B_F^c\cap\left(B_F^c-l\right)}\delta_{p+k,q}\tilde{c}_{q+l} \\ = 1_{B_F^c}(p+k+l)\tilde{c}_{p+k+l}. \tag{5.10}$$

We can summarize these in the common expression

$$[\tilde{c}_{p\mp k}, D_l] = \mp 1_{B_F^\circ}(p\mp k\mp l)\tilde{c}_{p\mp k\mp l}, \quad B_F^\circ = \begin{cases} B_F & M_k = L_k \\ B_F^c & M_k = L_k - k\end{cases}, \tag{5.11}$$

and write

$$\tilde{\mathcal{E}}_{\mathcal{C},2} = \sum_{k,l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{k,p}\tilde{\eta}_{l,p}\tilde{c}_{p\mp l}^*D_k^*\left[\tilde{c}_{p\mp k}, D_l\right] \\ = \sum_{k,l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{k,p}\tilde{\eta}_{l,p}D_k^*\tilde{c}_{p\mp l}^*\left[\tilde{c}_{p\mp k}, D_l\right] + \tilde{\mathcal{E}}_{\mathcal{C},3} = \tilde{\mathcal{E}}_{\mathcal{C},2,2} - \tilde{\mathcal{E}}_{\mathcal{C},3} \tag{5.12}$$

where $\tilde{\mathcal{E}}_{\mathcal{C},2,2}$ is then

$$\tilde{\mathcal{E}}_{\mathcal{C},2,2} = \sum_{k,l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{k,p}\tilde{\eta}_{l,p}D_k^*\tilde{c}_{p\mp l}^*\left[\tilde{c}_{p\mp k}, D_l\right] \\ = \mp\sum_{k\in S}\sum_{p\in M_k}\tilde{\eta}_{k,p}D_k^*\left(\sum_{l\in S}1_{M_l}(p)1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}_{l,p}\tilde{c}_{p\mp l}^*\tilde{c}_{p\mp k\mp l}\right). \tag{5.13}$$

To handle the presence of the D_k^* factor we need the following:

Proposition 5.3. *For any $k\in\mathbb{Z}_*^3$ and $\epsilon > 0$ it holds that*

$$D_{1,k}^*D_{1,k},\, D_{2,k}^*D_{2,k} \leq C_\epsilon k_F^{1+\epsilon}H'_{\text{kin}}$$

for a constant $C_\epsilon > 0$ depending only on ϵ.

Proof: The bound for $D_{2,k}$ follows immediately from Lemma 3.2 as

$$\|D_{2,k}\Psi\| \leq \sum_{p\in B_F\cap(B_F+k)} \left\|\tilde{c}_p^*\tilde{c}_{p-k}\Psi\right\| \leq \sqrt{\sum_{p\in B_F\cap(B_F+k)} \frac{1}{\left|\,|p-k|^2-\zeta\right|}}\sqrt{\sum_{p\in B_F\cap(B_F+k)} \left|\,|p-k|^2-\zeta\right| \|\tilde{c}_{p-k}\Psi\|^2}$$
$$\leq \sqrt{C_\epsilon k_F^{1+\epsilon}\left\langle \Psi, H'_{\text{kin}}\Psi\right\rangle}. \tag{5.14}$$

For $D_{1,k}$ we define the sets

$$A_1 = \{p\in B_F^c \mid |p|\leq 2k_F\}, \quad A_2 = \{p\in B_F^c \mid |p|>2k_F\}, \tag{5.15}$$

and use the triangle inequality to see that

$$\|D_{1,k}\Psi\| \leq \left(\sum_{p\in A_1\cap(A_1-k)} + \sum_{p\in A_1\cap(A_2-k)} + \sum_{p\in A_2\cap(A_1-k)}\right)\left\|\tilde{c}_p^*\tilde{c}_{p+k}\Psi\right\| + \|D_{0,k}\Psi\| \tag{5.16}$$

where

$$D_{0,k} = \sum_{p\in A_2\cap(A_2-k)} \tilde{c}_p^*\tilde{c}_{p+k}. \tag{5.17}$$

The first three sums can be estimated in the same manner as we did $D_{2,k}^*D_{2,k}$, so we need only consider $D_{0,k}$ further. For this we note that

$$\begin{aligned} D_{0,k}^*D_{0,k} &= \sum_{p,q\in A_2\cap(A_2-k)} \tilde{c}_{p+k}^*\tilde{c}_p\tilde{c}_q^*\tilde{c}_{q+k} \\ &= \sum_{p,q\in A_2\cap(A_2-k)} \tilde{c}_q^*\tilde{c}_{p+k}^*\tilde{c}_p\tilde{c}_{q+k} + \sum_{p\in A_2\cap(A_2-k)} \tilde{c}_{p+k}^*\tilde{c}_{p+k} \leq (\mathcal{N}_E')^2 \end{aligned} \tag{5.18}$$

where $\mathcal{N}_E' = \sum_{p\in A_2}\tilde{c}_p^*\tilde{c}_p$, since

$$\begin{aligned} \sum_{p,q\in A_2\cap(A_2-k)} \left\langle \Psi, \tilde{c}_q^*\tilde{c}_{p+k}^*\tilde{c}_p\tilde{c}_{q+k}\Psi\right\rangle &\leq \sqrt{\sum_{p,q\in A_2\cap(A_2-k)} \|\tilde{c}_{p+k}\tilde{c}_q\Psi\|^2}\sqrt{\sum_{p,q\in A_2\cap(A_2-k)} \|\tilde{c}_p\tilde{c}_{q+k}\Psi\|^2} \\ &\leq \sum_{p,q\in A_2} \|\tilde{c}_p\tilde{c}_q\Psi\|^2 = \left\langle \Psi, \mathcal{N}_E'(\mathcal{N}_E'-1)\Psi\right\rangle. \end{aligned} \tag{5.19}$$

Now, $\mathcal{N}_E'$ can be estimated in two different ways. First we clearly have that

$$\mathcal{N}_E' \leq \mathcal{N}_E \leq |B_F| \leq Ck_F^3, \tag{5.20}$$

but the condition $p\in A_2$ also lets us estimate

$$\mathcal{N}_E' = \sum_{p\in A_2} \frac{|p|^2-k_F^2}{|p|^2-k_F^2}\tilde{c}_p^*\tilde{c}_p \leq \frac{1}{3k_F^2}\sum_{p\in A_2}\left(|p|^2-k_F^2\right)\tilde{c}_p^*\tilde{c}_p \leq \frac{1}{3}k_F^{-2}H'_{\text{kin}} \tag{5.21}$$

and combining the two we conclude that

$$D_{0,k}^*D_{0,k} \leq (\mathcal{N}_E')^2 \leq Ck_F H'_{\text{kin}}. \tag{5.22}$$

□

We can now estimate $\tilde{\mathcal{E}}_{\mathcal{C},2,2}$ and $\tilde{\mathcal{E}}_{\mathcal{C},3}$:

Proposition 5.4. *For any symmetric set $S\subset\mathbb{Z}_*^3$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$\begin{aligned} \pm\tilde{\mathcal{E}}_{\mathcal{C},2,2} &\leq C_\epsilon\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\tilde{\eta}_{k,p}^2}\right)\sqrt{k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_k}\tilde{\eta}_{k,p}^2 H'_{\text{kin}}} \\ \pm\tilde{\mathcal{E}}_{\mathcal{C},3} &\leq \sum_{k\in S}\sum_{p\in M_k}\tilde{\eta}_{k,p}^2 H'_{\text{kin}} \end{aligned}$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Proof: From equation (5.13) and Cauchy-Schwarz we see that we can for any $\Psi \in D(H'_{\text{kin}})$ estimate

$$\left|\left\langle \Psi, \tilde{\mathcal{E}}_{\mathcal{C},2,2}\Psi\right\rangle\right| \leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\tilde{\eta}^2_{k,p}\left\|D_k\Psi\right\|^2}\sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}1_{M_l}(p)1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}^2_{l,p}\|\tilde{c}^*_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\|^2}. \tag{5.23}$$

It is immediate from Proposition 5.3 that the first factor can be bounded as

$$\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\tilde{\eta}^2_{k,p}\left\|D_k\Psi\right\|^2 \leq C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\tilde{\eta}^2_{k,p}\left\langle\Psi, H'_{\text{kin}}\Psi\right\rangle \tag{5.24}$$

so we turn to the latter. For this we simply bound

$$\begin{aligned}&\sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}1_{M_l}(p)1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}^2_{l,p}\|\tilde{c}^*_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\|^2} \\ &\leq \sum_{l\in S}\sqrt{\sum_{p\in M_l}\tilde{\eta}^2_{l,p}\sum_{k\in\mathbb{Z}^3_*}1_{B_F^\circ}(p\mp k\mp l)\left\|\tilde{c}_{p\mp k\mp l}\Psi\right\|^2} \leq \sum_{l\in S}\sqrt{\sum_{p\in M_l}\tilde{\eta}^2_{l,p}}\sqrt{\langle\Psi,\mathcal{N}_E\Psi\rangle}\end{aligned} \tag{5.25}$$

and use that $\mathcal{N}_E \leq H'_{\text{kin}}$.
For $\tilde{\mathcal{E}}_{\mathcal{C}}$ we note that by equation (5.11) this is

$$\tilde{\mathcal{E}}_{\mathcal{C},3} = \sum_{k,l\in S}\sum_{p\in M_k\cap M_l}1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}_{k,p}\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp k\mp l}\tilde{c}_{p\mp k\mp l} \tag{5.26}$$

whence

$$\begin{aligned}\left|\left\langle\Psi,\tilde{\mathcal{E}}_{\mathcal{C},3}\Psi\right\rangle\right| &\leq \sum_{k,l\in S}\sum_{p\in M_k\cap M_l}1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}^2_{k,p}\left\|\tilde{c}_{p\mp k\mp l}\Psi\right\|^2 \\ &\leq \sum_{k\in S}\sum_{p\in M_k}\tilde{\eta}^2_{k,p}\sum_{l\in S}1_{B_F^\circ}(p\mp k\mp l)\left\|\tilde{c}_{p\mp k\mp l}\Psi\right\|^2 \\ &\leq \sum_{k\in S}\sum_{p\in M_k}\tilde{\eta}^2_{k,p}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle \leq \sum_{k\in S}\sum_{p\in M_k}\tilde{\eta}^2_{k,p}\left\langle\Psi, H'_{\text{kin}}\Psi\right\rangle.\end{aligned} \tag{5.27}$$

□

5.2 Estimation of $\tilde{\mathcal{E}}_{\mathcal{C},4}$, $\tilde{\mathcal{E}}_{\mathcal{C},5}$ and $\tilde{\mathcal{E}}_{\mathcal{C},6}$

Now we come to the "mixed" terms of $\mathcal{E}_{\mathcal{C}}$, which include also $b_k(\cdot)$ expressions. The first of these, $\tilde{\mathcal{E}}_{\mathcal{C},4}$, is

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathcal{C},4} &= \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp l}\left[b^*_k(\varphi_{k,p})\tilde{c}_{p\mp k}, D_l\right] \\ &= \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp l}\left[b^*_k(\varphi_{k,p}), D_l\right]\tilde{c}_{p\mp k} \\ &+ \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp l}b^*_k(\varphi_{k,p})\left[\tilde{c}_{p\mp k}, D_l\right] =: \tilde{\mathcal{E}}_{\mathcal{C},4,1} + \tilde{\mathcal{E}}_{\mathcal{C},4,2}\end{aligned} \tag{5.28}$$

and we can write the second, $\tilde{\mathcal{E}}_{\mathcal{C},5}$, in the similar form

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathcal{C},5} &= \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp l}\left[b_{-k}(\psi_{-k,-p}), D_l\right]\tilde{c}_{p\mp k} \\ &+ \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp l}b_{-k}(\psi_{-k,-p})\left[\tilde{c}_{p\mp k}, D_l\right] =: \tilde{\mathcal{E}}_{\mathcal{C},5,1} + \tilde{\mathcal{E}}_{\mathcal{C},5,2}.\end{aligned} \tag{5.29}$$

To bound the commutators of the form $[b^*_k(\cdot), D_l]$ we prove the following:

Proposition 5.5. *For any $k,l\in\mathbb{Z}^3_*$, $p\in M_k$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$\left|[b_k^*(\varphi_{k,p}),D_l]^*\,\tilde{c}_{p\mp l}\right|^2\leq C_\epsilon k_F^{1+\epsilon}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2\,H'_{\text{kin}}$$
$$\left|[b_{-k}(\psi_{-k,-p}),D_l]^*\,\tilde{c}_{p\mp l}\right|^2\leq C_\epsilon k_F^{1+\epsilon}\max_{q\in L_k}|\langle e_{-q},\psi_{-k,-p}\rangle|^2\,H'_{\text{kin}}+2\left\|\psi_{-k,-p}\right\|^2\tilde{c}^*_{p\mp l}\tilde{c}_{p\mp l}$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Proof: Computing that

$$\begin{aligned}\left[b^*_{k,q},D_l\right]&=\sum_{p\in B_F^c\cap\left(B_F^c-l\right)}\left[\tilde{c}_q^*\tilde{c}_{q-k}^*,\tilde{c}_p^*\tilde{c}_{p+l}\right]-\sum_{p\in B_F\cap(B_F+l)}\left[\tilde{c}_q^*\tilde{c}_{q-k}^*,\tilde{c}_p^*\tilde{c}_{p-l}\right]\\&=-\sum_{p\in B_F^c\cap\left(B_F^c-l\right)}\tilde{c}_p^*\left\{\tilde{c}_q^*,\tilde{c}_{p+l}\right\}\tilde{c}_{q-k}^*+\sum_{p\in B_F\cap(B_F+l)}\tilde{c}_q^*\tilde{c}_p^*\left\{\tilde{c}_{q-k}^*,\tilde{c}_{p-l}\right\}\\&=-1_{B_F^c}(q-l)\tilde{c}_{q-l}^*\tilde{c}_{q-k}^*+1_{B_F}(q-k+l)\tilde{c}_q^*\tilde{c}_{q-k+l}^*\end{aligned}\tag{5.30}$$

we see that the commutator $[b_k^*(\varphi_{k,p}),D_l]$ can be written as

$$\begin{aligned}&[b_k^*(\varphi_{k,p}),D_l]=\sum_{q\in L_k}\langle e_q,\varphi_{k,p}\rangle\left[b^*_{k,q},D_l\right]\\&=-\sum_{q\in L_k}1_{B_F^c}(q-l)\,\langle e_q,\varphi_{k,p}\rangle\,\tilde{c}_{q-l}^*\tilde{c}_{q-k}^*+\sum_{q\in L_k}1_{B_F}(q-k+l)\,\langle e_q,\varphi_{k,p}\rangle\,\tilde{c}_q^*\tilde{c}_{q-k+l}^*\\&=\sum_{q\in L_k}1_{B_F^c}(q-l)\,\langle e_q,\varphi_{k,p}\rangle\,\tilde{c}_{q-k}^*\tilde{c}_{q-l}^*+\sum_{q\in(L_k-k)}1_{B_F}(q+l)\,\langle e_{q+k},\varphi_{k,p}\rangle\,\tilde{c}_{q+k}^*\tilde{c}_{q+l}^*.\end{aligned}\tag{5.31}$$

Consequently, for any $\Psi\in D(H'_{\text{kin}})$,

$$\begin{aligned}\left\|[b_k^*(\varphi_{k,p}),D_l]^*\,\tilde{c}_{p\mp l}\Psi\right\|&\leq\sum_{q\in L_k}1_{B_F^c}(q-l)\,|\langle e_q,\varphi_{k,p}\rangle|\,\|\tilde{c}_{q-l}\tilde{c}_{q-k}\tilde{c}_{p\mp l}\Psi\|\\&\quad+\sum_{q\in(L_k-k)}1_{B_F}(q+l)\,|\langle e_{q+k},\varphi_{k,p}\rangle|\,\|\tilde{c}_{q+l}\tilde{c}_{q+k}\tilde{c}_{p\mp l}\Psi\|\\&\leq\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|\sqrt{\sum_{q\in L_k}\frac{1}{\left||q-k|^2-\zeta\right|}}\sqrt{\sum_{q\in L_k}\left||q-k|^2-\zeta\right|\|\tilde{c}_{q-k}\Psi\|^2}\\&\quad+\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|\sqrt{\sum_{q\in(L_k-k)}\frac{1}{\left||q+k|^2-\zeta\right|}}\sqrt{\sum_{q\in(L_k-k)}\left||q+k|^2-\zeta\right|\|\tilde{c}_{q+k}\Psi\|^2}\\&\leq\sqrt{C_\epsilon k_F^{1+\epsilon}\max_{q\in L_k}|\langle e_q,\varphi_{k,p}\rangle|^2\left\langle\Psi,H'_{\text{kin}}\Psi\right\rangle}.\end{aligned}\tag{5.32}$$

For $[b_{-k}(\psi_{-k,-p}),D_l]^*\,\tilde{c}_{p\mp l}$ we note that from the calculation of equation (5.31)

$$\begin{aligned}[b_{-k}(\psi_{-k,-p}),D_l]^*&=-\left[b^*_{-k}(\psi_{-k,-p}),D_l^*\right]=-\left[b^*_{-k}(\psi_{-k,-p}),D_{-l}\right]\\&=\sum_{q\in L_k}1_{B_F^c}(q-l)\,\langle e_{-q},\psi_{-k,-p}\rangle\,\tilde{c}_{-q+k}^*\tilde{c}_{-q+l}^*\\&\quad+\sum_{q\in(L_k-k)}1_{B_F}(q+l)\,\langle e_{-q-k},\psi_{-k,-p}\rangle\,\tilde{c}_{-q-k}^*\tilde{c}_{-q-l}^*\end{aligned}\tag{5.33}$$

as $D_l^*=D_{-l}$. Now, note that either of these sums in fact of the $b^*(\cdot)$ form, since the q summation ranges and indicator functions force one momenta to lie inside B_F and the other outside. We can take advantage of this to estimate $\left|[b_{-k}(\psi_{-k,-p}),D_l]^*\,\tilde{c}_{p\mp l}\right|^2$ by commutation, as (considering the first term for definiteness)

$$\left|\sum_{q\in L_k}1_{B_F^c}(q-l)\,\langle e_{-q},\psi_{-k,-p}\rangle\,\tilde{c}_{-q+k}^*\tilde{c}_{-q+l}^*\tilde{c}_{p\mp l}\right|^2=\left|\sum_{q\in L_k}1_{B_F^c}(q-l)\,\langle\psi_{-k,-p},e_{-q}\rangle\,\tilde{c}_{-q+l}\tilde{c}_{-q+k}\tilde{c}_{p\mp l}\right|^2$$

$$
\begin{aligned}
&+\sum_{q,q'\in L_k} 1_{B_F^c}(q-l)1_{B_F^c}(q'-l)\left\langle e_{-q},\psi_{-k,-p}\right\rangle\left\langle \psi_{-k,-p},e_{-q'}\right\rangle \tilde{c}_{p\mp l}^*\left[\tilde{c}_{-q+l}\tilde{c}_{-q+k},\tilde{c}_{-q'+k}^*\tilde{c}_{-q'+l}^*\right]\tilde{c}_{p\mp l}\\
&\leq C_\epsilon k_F^{1+\epsilon}\max_{q\in L_k}\left|\left\langle e_{-q},\psi_{-k,-p}\right\rangle\right|^2 H'_{\mathrm{kin}}+\sum_{q\in L_k}1_{B_F^c}(q-l)\left|\left\langle e_{-q},\psi_{-k,-p}\right\rangle\right|^2\tilde{c}_{p\mp l}^*\tilde{c}_{p\mp l} \qquad (5.34)\\
&\leq C_\epsilon k_F^{1+\epsilon}\max_{q\in L_k}\left|\left\langle e_{-q},\psi_{-k,-p}\right\rangle\right|^2 H'_{\mathrm{kin}}+\left\|\psi_{-k,-p}\right\|^2\tilde{c}_{p\mp l}^*\tilde{c}_{p\mp l}
\end{aligned}
$$

where the first bound follows as in equation (5.32) whereas the second follows from the $b^*(\cdot)$ form since

$$
\begin{aligned}
\left[\tilde{c}_{-q+l}\tilde{c}_{-q+k},\tilde{c}_{-q'+k}^*\tilde{c}_{-q'+l}^*\right]&=\tilde{c}_{-q+l}\left\{\tilde{c}_{-q+k},\tilde{c}_{-q'+k}^*\right\}\tilde{c}_{-q'+l}^*-\tilde{c}_{-q'+k}^*\left\{\tilde{c}_{-q+l},\tilde{c}_{-q'+l}^*\right\}\tilde{c}_{-q+k} \qquad (5.35)\\
&=\delta_{q,q'}\left(1-\tilde{c}_{-q+k}^*\tilde{c}_{-q+k}-\tilde{c}_{-q+l}^*\tilde{c}_{-q+l}\right).
\end{aligned}
$$

The same argument applies to the second term of equation (5.33) and the proposition follows.

□

The first error terms can then be estimated:

Proposition 5.6. *For any symmetric set $S\subset\mathbb{Z}_*^3$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$
\pm\tilde{\mathcal{E}}_{\mathcal{C},4,1},\pm\tilde{\mathcal{E}}_{\mathcal{C},5,1}\leq C_\epsilon k_F^{1-\beta+\epsilon}\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\tilde{\eta}_{k,p}^2}\right)\sqrt{\sum_{k\in S}\hat{V}_k^2}H'_{\mathrm{kin}}
$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Proof: By the first bound above we can for any $\Psi\in D(H'_{\mathrm{kin}})$ estimate

$$
\begin{aligned}
\left|\left\langle\Psi,\tilde{\mathcal{E}}_{\mathcal{C},4,1}\Psi\right\rangle\right|&\leq\sum_{k\in\mathbb{Z}_*^3}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\left\|\left[b_k^*(\varphi_{k,p}),D_l\right]^*\tilde{c}_{p\mp l}\Psi\right\|\left\|\tilde{c}_{p\mp k}\Psi\right\|\\
&\leq\sqrt{C_\epsilon k_F^{1+\epsilon}\left\langle\Psi,H'_{\mathrm{kin}}\Psi\right\rangle}\sum_{l\in S}\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\max_{q\in L_k}\left|\left\langle e_q,\varphi_{k,p}\right\rangle\right|\left\|\tilde{c}_{p\mp k}\Psi\right\| \qquad (5.36)\\
&\leq\sqrt{C_\epsilon k_F^{1+\epsilon}\left\langle\Psi,H'_{\mathrm{kin}}\Psi\right\rangle}\sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}^2\left\|\tilde{c}_{p\mp k}\Psi\right\|^2}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_k}\max_{q\in L_k}\left|\left\langle e_q,\varphi_{k,p}\right\rangle\right|^2}\\
&\leq\left(\sum_{l\in S}\sqrt{\sum_{p\in M_l}\tilde{\eta}_{l,p}^2}\right)\sqrt{C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_k}\max_{q\in L_k}\left|\left\langle e_q,\varphi_{k,p}\right\rangle\right|^2}\sqrt{\left\langle\Psi,H'_{\mathrm{kin}}\Psi\right\rangle\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle}
\end{aligned}
$$

which upon using that $\mathcal{N}_E\leq H'_{\mathrm{kin}}$ and recalling $\sum_{p\in M_k}\max_{q\in L_k}\left|\left\langle e_q,\varphi_{k,p}\right\rangle\right|^2\leq Ck_F^{1-2\beta}\hat{V}_k^2$ gives the first bound. For the second we likewise have

$$
\begin{aligned}
\left|\left\langle\Psi,\tilde{\mathcal{E}}_{\mathcal{C},5,1}\Psi\right\rangle\right|&\leq\sqrt{C_\epsilon k_F^{1+\epsilon}\left\langle\Psi,H'_{\mathrm{kin}}\Psi\right\rangle}\sum_{k\in\mathbb{Z}_*^3}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\max_{q\in L_k}\left|\left\langle e_{-q},\psi_{-k,-p}\right\rangle\right|\left\|\tilde{c}_{p\mp k}\Psi\right\|\\
&\quad+\sqrt{2}\sum_{k\in\mathbb{Z}_*^3}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\left\|\psi_{-k,-p}\right\|\left\|\tilde{c}_{p\mp l}\Psi\right\|\left\|\tilde{c}_{p\mp k}\Psi\right\| \qquad (5.37)
\end{aligned}
$$

and the first can be estimated as we did the previous one, whereas the second obeys

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}_*^3}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\left\|\psi_{-k,-p}\right\|\left\|\tilde{c}_{p\mp l}\Psi\right\|\left\|\tilde{c}_{p\mp k}\Psi\right\|\\
&\leq\sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_l}\tilde{\eta}_{l,p}^2\left\|\tilde{c}_{p\mp k}\Psi\right\|^2}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\sum_{p\in M_l}\left\|\psi_{-k,-p}\right\|^2\left\|\tilde{c}_{p\mp l}\Psi\right\|^2} \qquad (5.38)\\
&\leq\sum_{l\in S}\sqrt{\sum_{p\in M_l}\tilde{\eta}_{l,p}^2}\sqrt{\sum_{k\in\mathbb{Z}_*^3}\max_{p\in M_k}\left\|\psi_{k,p}\right\|^2}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle
\end{aligned}
$$

and we recall that $\max_{p\in M_k}\left\|\psi_{k,p}\right\|^2\leq Ck_F^{1-2\beta}\hat{V}_k^2$.

$\square$

Recalling equation (5.11), we see that $\tilde{\mathcal{E}}_{\mathcal{C},4,2}$ can be written as

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathcal{C},4,2} &= \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp l}b^*_k(\varphi_{k,p})\left[\tilde{c}_{p\mp k},D_l\right] \\ &= \mp\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}_{l,p}b^*_k(\varphi_{k,p})\tilde{c}^*_{p\mp l}\tilde{c}_{p\mp k\mp l}\end{aligned} \tag{5.39}$$

since $[\tilde{c}^*,b^*(\cdot)]=0$. Now, $[\tilde{c}^*,b(\cdot)]\neq 0$, but we can nonetheless write $\tilde{\mathcal{E}}_{\mathcal{C},5,2}$ in the similar form

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathcal{C},5,2} &= \mp\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in\mathbb{Z}^3_*\setminus S}\sum_{p\in M_k\cap M_l}1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}_{l,p}\tilde{c}^*_{p\mp l}b_{-k}(\psi_{-k,-p})\tilde{c}_{p\mp k\mp l} \\ &= \mp\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in\mathbb{Z}^3_*\setminus S}\sum_{p\in M_k\cap M_l}1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}_{l,p}b_{-k}(\psi_{-k,-p})\tilde{c}^*_{p\mp l}\tilde{c}_{p\mp k\mp l}\end{aligned} \tag{5.40}$$

as equation (3.17) implies that

$$\left[\tilde{c}^*_{p\mp l},b_{-k}(\psi_{-k,-p})\right]=\pm 1_{M_{-k}}(p\mp k\mp l)\left\langle\psi_{-k,-p},e_{p\mp k'\mp l}\right\rangle\tilde{c}_{p\mp k\mp l} \tag{5.41}$$

and the two indicator functions for $p\mp k\mp l$ have disjoint support.

We now bound these terms:

Proposition 5.7. *For any symmetric set $S\subset\mathbb{Z}^3_*$ it holds as $k_F\to\infty$ that*

$$\begin{aligned}\pm\tilde{\mathcal{E}}_{\mathcal{C},4,2} &\leq Ck_F^{1-\beta}\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\tilde{\eta}^2_{k,p}}\right)\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2 H'_{\mathrm{kin}}} \\ \pm\tilde{\mathcal{E}}_{\mathcal{C},5,2} &\leq Ck_F^{1-\beta}\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\tilde{\eta}^2_{k,p}}\right)\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}(H'_{\mathrm{kin}}+k_F)}\end{aligned}$$

for a constant $C>0$ independent of all quantities.

Proof: For any $\Psi\in D(H'_{\mathrm{kin}})$ we can estimate

$$\begin{aligned}\left|\left\langle\Psi,\tilde{\mathcal{E}}_{\mathcal{C},4,2}\Psi\right\rangle\right| &\leq \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l}1_{B_F^\circ}(p\mp k\mp l)\left|\tilde{\eta}_{l,p}\right|\left\|b_k(\varphi_{k,p})\Psi\right\|\left\|\tilde{c}^*_{p\mp l}\tilde{c}_{p\mp k\mp l}\Psi\right\| \\ &\leq \sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left\|b_k(\varphi_{k,p})\Psi\right\|^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}1_{B_F^\circ}(p\mp k\mp l)\tilde{\eta}^2_{l,p}\left\|\tilde{c}_{p\mp k\mp l}\Psi\right\|^2} \\ &\leq \sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left\|b_k(\varphi_{k,p})\Psi\right\|^2}\sum_{l\in S}\sqrt{\sum_{p\in M_l}\tilde{\eta}^2_{l,p}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle}\end{aligned} \tag{5.42}$$

and similarly

$$\left|\left\langle\Psi,\tilde{\mathcal{E}}_{\mathcal{C},5,2}\Psi\right\rangle\right|\leq\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left\|b^*_k(\psi_{-k,-p})\Psi\right\|^2}\sum_{l\in S}\sqrt{\sum_{p\in M_l}\tilde{\eta}^2_{l,p}\left\langle\Psi,\mathcal{N}_E\Psi\right\rangle}. \tag{5.43}$$

Now, as in the Propositions 3.6 and 3.8 it holds that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left|b_k(\varphi_{k,p})\right|^2 &\leq Ck_F^{2(1-\beta)}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2 H'_{\mathrm{kin}} \\ \sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\left|b^*_k(\psi_{-k,-p})\right|^2 &\leq Ck_F^{2(1-\beta)}\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}\left(H'_{\mathrm{kin}}+k_F\right)\end{aligned} \tag{5.44}$$

from which the claim follows.

$\square$

By equation (5.41), the final error term is

$$\begin{aligned}\tilde{\mathcal{E}}_{\mathcal{C},6} &= \pm\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l} 1_{M_{-k}}(p\mp k\mp l)\left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle \tilde{\eta}_{l,p}\tilde{c}_{p\mp k\mp l}\tilde{c}_{p\mp k}D_l \\ &= \pm\sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l} 1_{M_{-k}}(p\mp k\mp l)\left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle \tilde{\eta}_{l,p}D_l\tilde{c}_{p\mp k\mp l}\tilde{c}_{p\mp k}\end{aligned} \tag{5.45}$$

where we could commute D_l to the left due to the indicator function of the commutator

$$[\tilde{c}_{p\mp k}, D_l] = \mp 1_{B_F^\circ}(p\mp k\mp l)\tilde{c}_{p\mp k\mp l} \tag{5.46}$$

and, as is readily computed,

$$[\tilde{c}_{p\mp k\mp l}, D_l] = \begin{cases} 1_{B_F^c}(p-k)\tilde{c}_{p-k} & M_k = L_k \\ 1_{B_F}(p+k)\tilde{c}_{p+k} & M_k = L_k - k\end{cases} \tag{5.47}$$

which vanishes for $p\in M_k$. $\tilde{\mathcal{E}}_{\mathcal{C},6}$ can be controlled as follows:

Proposition 5.8. *For any symmetric set $S\subset\mathbb{Z}^3_*$ it holds as $k_F\to\infty$ that*

$$\pm\tilde{\mathcal{E}}_{\mathcal{C},6} \leq C_\epsilon k_F^{1-\beta+\epsilon}\left(\sum_{k\in S}\sqrt{\sum_{p\in M_k}\tilde{\eta}_{k,p}^2}\right)\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2}H'_{\mathrm{kin}}$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Proof: For any $\Psi\in D(H'_{\mathrm{kin}})$ we have

$$\begin{aligned}\left|\left\langle \Psi, \tilde{\mathcal{E}}_{\mathcal{C},6}\Psi\right\rangle\right| &\leq \sum_{k\in\mathbb{Z}^3_*}\sum_{l\in S}\sum_{p\in M_k\cap M_l} 1_{M_{-k}}(p\mp k\mp l)\left|\left\langle \psi_{-k,-p}, e_{p\mp k'\mp l}\right\rangle\right| |\tilde{\eta}_{l,p}| \, \|D_l^*\Psi\| \, \|\tilde{c}_{p\mp k\mp l}\tilde{c}_{p\mp k}\Psi\| \\ &\leq \sqrt{C_\epsilon k_F^{1+\epsilon}\left\langle \Psi, H'_{\mathrm{kin}}\Psi\right\rangle}\sum_{l\in S}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\max_{q\in L_k}|\langle e_{-q},\psi_{-k,-p}\rangle| \, |\tilde{\eta}_{l,p}| \, \|\tilde{c}_{p\mp k}\Psi\| \\ &\leq \sqrt{C_\epsilon k_F^{1+\epsilon}\left\langle \Psi, H'_{\mathrm{kin}}\Psi\right\rangle}\sum_{l\in S}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\max_{q\in L_k}|\langle e_{-q},\psi_{-k,-p}\rangle|^2}\sqrt{\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k\cap M_l}\tilde{\eta}_{l,p}^2\|\tilde{c}_{p\mp k}\Psi\|^2} \\ &\leq \sqrt{C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}^3_*}\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\psi_{k,p}\rangle|^2}\sum_{l\in S}\sqrt{\sum_{p\in M_l}\tilde{\eta}_{l,p}^2}\sqrt{\left\langle \Psi, H'_{\mathrm{kin}}\Psi\right\rangle\langle\Psi,\mathcal{N}_E\Psi\rangle}.\end{aligned} \tag{5.48}$$

Recalling $\sum_{p\in M_k}\max_{q\in L_k}|\langle e_q,\psi_{k,p}\rangle|^2 \leq Ck_F^{1-2\beta}\hat{V}_k^2$ and using $\mathcal{N}_E\leq H'_{\mathrm{kin}}$ we have the claim.

□

We can now conclude the main result of this section:

Theorem (5.1)**.** *For any symmetric set $S\subset\mathbb{Z}^3_*$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$\pm\mathcal{E}_{\mathcal{C}} \leq C_\epsilon k_F^{2(1-\beta)+\epsilon}\left(k_F^{-\frac{1}{2}}\sum_{k\in S}\hat{V}_k\right)\left(\sqrt{\sum_{k\in\mathbb{Z}^3_*}\hat{V}_k^2\min\{|k|,k_F\}} + k_F^{-\frac{1}{2}}\sum_{k\in S}\hat{V}_k\right)(H'_{\mathrm{kin}}+k_F)$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Proof: By definition of $\tilde{\eta}_{k,p}$, the sum $\sum_{p\in M_k}\tilde{\eta}_{k,p}^2$ is (as $h_k\leq E_k$)

$$\begin{aligned}\sum_{p\in M_k}\tilde{\eta}_{k,p}^2 &= \sum_{p\in L_k}\left||p|^2-k_F^2\right||\langle e_p,\eta_k\rangle|^2 \leq 2\sum_{p\in L_k}\lambda_{k,p}|\langle e_p,\eta_k\rangle|^2 \\ &= 2\langle\eta_k,h_k\eta_k\rangle \leq 2\langle\eta_k,E_k\eta_k\rangle\end{aligned} \tag{5.49}$$

for both $M_k=L_k$ and $M_k=L_k-k$ (the only difference being $\left||p|^2-k_F^2\right|\to\left||p-k|^2-k_F^2\right|$ in the first line). We calculated the inner product in equation (4.11), with the result

$$\langle\eta_k,E_k\eta_k\rangle = \frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\frac{\left\langle v_k,h_k^{-1}v_k\right\rangle}{1+2\left\langle v_k,h_k^{-1}v_k\right\rangle} \leq Ck_F^{-\beta}\hat{V}_k\left\langle v_k,h_k^{-1}v_k\right\rangle \leq Ck_F^{1-2\beta}\hat{V}_k^2 \tag{5.50}$$

from which it is seen that all the bounds of the Propositions 5.2, 5.4, 5.6, 5.7 and 5.8 can be controlled in the claimed manner.

□

6 Estimation of the Remaining Terms

The results of the previous sections can be summarized in the inequality

$$H_N \geq E_{\mathrm{FS}} + E_{\mathrm{corr,bos}} + E_{\mathrm{corr,ex}} + \mathcal{E}_{\mathrm{B}} + \mathcal{E}_{\mathcal{C}} + 2^{-1}(2\pi)^{-3}k_F^{-\beta}\left(\mathcal{E}_S + \mathcal{E}_{\mathbb{Z}^3_*\backslash S}\right) \tag{6.1}$$

where $\mathcal{E}_S$ and $\mathcal{E}_{\mathbb{Z}^3_*\backslash S}$ are given by

$$\begin{aligned}\mathcal{E}_S &= \sum_{k\in S}\hat{V}_k\left(\left(1 - \frac{2\left\langle v_k, h_k^{-1}v_k\right\rangle}{1+2\left\langle v_k, h_k^{-1}v_k\right\rangle}\right)D_k^* D_k - \sum_{p\in L_k}\left(\tilde{c}_p^*\tilde{c}_p + \tilde{c}_{p-k}^*\tilde{c}_{p-k}\right)\right) \\ \mathcal{E}_{\mathbb{Z}^3_*\backslash S} &= \sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k\left(4\,\mathrm{Re}(B_k^* D_k) + D_k^* D_k - \sum_{p\in L_k}\left(\tilde{c}_p^*\tilde{c}_p + \tilde{c}_{p-k}^*\tilde{c}_{p-k}\right)\right)\end{aligned} \tag{6.2}$$

with $\mathcal{E}_{\mathrm{B}}$ and $\mathcal{E}_{\mathcal{C}}$ obeying the estimates of the Theorems 3.1 and 5.1.

In this section we conclude the proof of Theorem 1.2 by estimating these final error terms. To state the main results of this section we define

$$\begin{aligned}\mathcal{E}_{\mathcal{Q},4} &= 2\sum_{p\in B_F}\left(\sum_{k\in\mathbb{Z}^3_*\backslash S}1_{B_F}(p+k)\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p - 2\sum_{p\in B_F^c}\left(\sum_{k\in\mathbb{Z}^3_*\backslash S}1_{B_F}(p-k)\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p \\ \mathcal{E}_{\mathcal{Q},5} &= \sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k\sum_{p,q\in A\cap(A+k)}\tilde{c}_p^*\tilde{c}_{q-k}^*\tilde{c}_q\tilde{c}_{p-k}\end{aligned} \tag{6.3}$$

where $A = \{p \in B_F^c \mid |p| > 2k_F\}$. The estimates are as follows:

Theorem 6.1. *Let $\hat{V}_k$ be radially decreasing with respect to $k \in \mathbb{Z}^3_*$. Then for any symmetric set $S \subset \mathbb{Z}^3_*$ and $\epsilon > 0$ it holds that*

$$\begin{aligned}\mathcal{E}_S &\geq -2\sum_{k\in S}\hat{V}_k H'_{\mathrm{kin}} \\ \mathcal{E}_{\mathbb{Z}^3_*\backslash S} - \mathcal{E}_{\mathcal{Q},4} - \mathcal{E}_{\mathcal{Q},5} &\geq -C_\epsilon k_F^{1+\epsilon}\sqrt{\sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k^2}\,H'_{\mathrm{kin}} \\ \mathcal{E}_{\mathcal{Q},4} &\geq -2\left(\sum_{k\in S}\hat{V}_k + \sup_{p\in B_F^c}\hat{V}_p\right)H'_{\mathrm{kin}}\end{aligned}$$

for a constant $C_\epsilon > 0$ depending only on ϵ. Furthermore, for any $S' \subset \mathbb{Z}^3_*$ containing S and $\overline{B}(0,3k_F)\cap\mathbb{Z}^3$,

$$\mathcal{E}_{\mathcal{Q},5} \geq -C'_\epsilon\left(k_F^{-2}\sum_{k\in S'}\hat{V}_k + k_F^3\left(\sum_{k\in\mathbb{Z}^3_*\backslash S'}\hat{V}_k\,|k|^{-2} + \sqrt{\sum_{k\in\mathbb{Z}^3_*\backslash S'}\hat{V}_k^2\,|k|^{-(1-\epsilon)}}\right)\right)H'_{\mathrm{kin}}$$

for a constant $C'_\epsilon > 0$ depending only on ϵ.

Estimation of $\mathcal{E}_S$

The bound for $\mathcal{E}_S$ is almost immediate from the observation that

$$\begin{aligned}\mathcal{E}_S &= \sum_{k\in S}\hat{V}_k\left(\left(1 - \frac{2\left\langle v_k, h_k^{-1}v_k\right\rangle}{1+2\left\langle v_k, h_k^{-1}v_k\right\rangle}\right)D_k^* D_k - \sum_{p\in L_k}\left(\tilde{c}_p^*\tilde{c}_p + \tilde{c}_{p-k}^*\tilde{c}_{p-k}\right)\right) \\ &\geq -\sum_{k\in S}\hat{V}_k\sum_{p\in L_k}\left(\tilde{c}_p^*\tilde{c}_p + \tilde{c}_{p-k}^*\tilde{c}_{p-k}\right)\end{aligned} \tag{6.4}$$

which leads to the following:

Proposition 6.2. *For any symmetric set $S \subset \mathbb{Z}^3_*$ it holds that*

$$\mathcal{E}_S \geq -2 \sum_{k \in S} \hat{V}_k H'_{\text{kin}}.$$

Proof: From the above inequality we rearrange the sums to see that

$$\begin{aligned} \sum_{k \in S} \hat{V}_k \sum_{p \in L_k} \left(\tilde{c}^*_p \tilde{c}_p + \tilde{c}^*_{p-k} \tilde{c}_{p-k} \right) &= \sum_{p \in B_F^c} \left(\sum_{k \in S} 1_{L_k}(p) \hat{V}_k \right) \tilde{c}^*_p \tilde{c}_p + \sum_{p \in B_F} \left(\sum_{k \in S} 1_{L_k - k}(p) \hat{V}_k \right) \tilde{c}^*_p \tilde{c}_p \\ &\leq \sum_{k \in S} \hat{V}_k \left(\sum_{p \in B_F^c} \tilde{c}^*_p \tilde{c}_p + \sum_{p \in B_F} \tilde{c}^*_p \tilde{c}_p \right) \leq 2 \sum_{k \in S} \hat{V}_k \mathcal{N}_E \end{aligned} \tag{6.5}$$

and the claim now follows from the fact that $\mathcal{N}_E \leq H'_{\text{kin}}$.

□

6.1 Preliminary Analysis of Large k Terms

We split the large k terms into a cubic part and a quartic part as $\mathcal{E}_{\mathbb{Z}^3_* \setminus S} = 4 \operatorname{Re}\left(\mathcal{E}_{\mathcal{C}, \mathbb{Z}^3_* \setminus S}\right) + \mathcal{E}_{\mathcal{Q}}$ where

$$\mathcal{E}_{\mathcal{C}, \mathbb{Z}^3_* \setminus S} = \sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k B^*_k D_k, \quad \mathcal{E}_{\mathcal{Q}} = \sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k \left(D^*_k D_k - \sum_{p \in L_k} \left(\tilde{c}^*_p \tilde{c}_p + \tilde{c}^*_{p-k} \tilde{c}_{p-k} \right) \right), \tag{6.6}$$

and we recall that $B_k = \sum_{p \in L_k} \tilde{c}_{p-k} \tilde{c}_p$ and $D_k = D_{1,k} + D_{2,k}$ for

$$D_{1,k} = + \sum_{p \in B_F^c \cap \left(B_F^c - k \right)} \tilde{c}^*_p \tilde{c}_{p+k}, \quad D_{2,k} = - \sum_{p \in B_F \cap (B_F + k)} \tilde{c}^*_p \tilde{c}_{p-k}. \tag{6.7}$$

We split the cubic terms $\mathcal{E}_{\mathcal{C}, \mathbb{Z}^3_* \setminus S}$ further into a $D_{1,k}$ part and a $D_{2,k}$ part as

$$\begin{aligned} \mathcal{E}_{\mathcal{C}, \mathbb{Z}^3_* \setminus S} &= \sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k \sum_{p \in L_k} \tilde{c}^*_p \tilde{c}^*_{p-k} D_{1,k} - \sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k \sum_{p \in (L_k - k)} \tilde{c}^*_p \tilde{c}^*_{p+k} D_{2,k} \\ &=: \mathcal{E}^{(1)}_{\mathcal{C}, \mathbb{Z}^3_* \setminus S} - \mathcal{E}^{(2)}_{\mathcal{C}, \mathbb{Z}^3_* \setminus S} \end{aligned} \tag{6.8}$$

and for the quartic terms we note that

$$\begin{aligned} D^*_{1,k} D_{1,k} &= + \sum_{p \in B_F^c \cap \left(B_F^c + k \right)} \tilde{c}^*_p D_{1,k} \tilde{c}_{p-k} + \sum_{p \in B_F^c \cap \left(B_F^c + k \right)} \tilde{c}^*_p \tilde{c}_p \\ D^*_{2,k} D_{2,k} &= - \sum_{p \in B_F \cap (B_F - k)} \tilde{c}^*_p D_{2,k} \tilde{c}_{p+k} + \sum_{p \in B_F \cap (B_F - k)} \tilde{c}^*_p \tilde{c}_p \end{aligned} \tag{6.9}$$

and (since e.g. $D^*_{1,k} = D_{1,-k}$ and $[D_{1,k}, D_{2,l}] = 0$)

$$\sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k \left(D^*_{1,k} D_{2,k} + D^*_{2,k} D_{1,k} \right) = -2 \sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k \sum_{p \in B_F \cap (B_F - k)} \tilde{c}^*_p D_{1,k} \tilde{c}_{p+k} \tag{6.10}$$

whence $\mathcal{E}_{\mathcal{Q}}$ can be decomposed as

$$\mathcal{E}_{\mathcal{Q}} = \mathcal{E}_{\mathcal{Q},1} - \mathcal{E}_{\mathcal{Q},2} - 2\mathcal{E}_{\mathcal{Q},3} + \mathcal{E}_{\mathcal{Q},4} \tag{6.11}$$

where

$$\begin{aligned} \mathcal{E}_{\mathcal{Q},1} &= \sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k \sum_{p \in B_F^c \cap \left(B_F^c + k \right)} \tilde{c}^*_p D_{1,k} \tilde{c}_{p-k} \\ \mathcal{E}_{\mathcal{Q},2} &= \sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k \sum_{p \in B_F \cap (B_F - k)} \tilde{c}^*_p D_{2,k} \tilde{c}_{p+k} \end{aligned} \tag{6.12}$$

$$\mathcal{E}_{\mathcal{Q},3} = \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in B_F\cap(B_F-k)} \tilde{c}^*_p D_{1,k}\tilde{c}_{p+k}$$

and we noted that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \left(\sum_{p\in B_F^c\cap\left(B_F^c+k\right)} \tilde{c}^*_p\tilde{c}_p + \sum_{p\in B_F\cap(B_F-k)} \tilde{c}^*_p\tilde{c}_p - \sum_{p\in L_k} \left(\tilde{c}^*_p\tilde{c}_p + \tilde{c}^*_{p-k}\tilde{c}_{p-k}\right) \right) \\ &= \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \left(\sum_{p\in B_F^c} \left(1_{B_F^c}(p-k) - 1_{B_F}(p-k)\right)\tilde{c}^*_p\tilde{c}_p + \sum_{p\in B_F} \left(1_{B_F}(p+k) - 1_{B_F^c}(p+k)\right)\tilde{c}^*_p\tilde{c}_p \right) \\ &= 2\sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \left(-\sum_{p\in B_F^c} 1_{B_F}(p-k)\tilde{c}^*_p\tilde{c}_p + \sum_{p\in B_F} 1_{B_F}(p+k)\tilde{c}^*_p\tilde{c}_p \right) = \mathcal{E}_{\mathcal{Q},4} \end{aligned} \tag{6.13}$$

since $\sum_{p\in B_F^c} \tilde{c}^*_p\tilde{c}_p = \mathcal{N}_E = \sum_{p\in B_F} \tilde{c}^*_p\tilde{c}_p$.

Now we decompose $\mathcal{E}_{\mathcal{Q},1}$ further: Defining (as in Proposition 5.3)

$$A_1 = \{p\in B_F^c \mid |p| \le 2k_F\}, \quad A_2 = \{p\in B_F^c \mid |p| > 2k_F\}, \tag{6.14}$$

and

$$D_{0,k} = \sum_{p\in A_2\cap(A_2-k)} \tilde{c}^*_p\tilde{c}_{p+k} \tag{6.15}$$

we split the sum of $\mathcal{E}_{\mathcal{Q},1}$ into 4 parts depending on whether p and $p-k$ are in A_1 or A_2:

$$\begin{aligned} \sum_{p\in B_F^c\cap\left(B_F^c+k\right)} \tilde{c}^*_p D_{1,k}\tilde{c}_{p-k} &= \sum_{p\in A_1\cap(A_1+k)} \tilde{c}^*_p D_{1,k}\tilde{c}_{p-k} + \sum_{p\in A_2\cap(A_2+k)} \tilde{c}^*_p D_{1,k}\tilde{c}_{p-k} \\ &\quad + \sum_{p\in A_2\cap(A_1+k)} \tilde{c}^*_p D_{1,k}\tilde{c}_{p-k} + \left(\sum_{p\in(A_1-k)\cap A_2} \tilde{c}^*_p D_{1,-k}\tilde{c}_{p+k} \right)^*. \end{aligned} \tag{6.16}$$

The second sum of this equation can be written in terms of $D_{0,k}$ as

$$\begin{aligned} \sum_{p\in A_2\cap(A_2+k)} \tilde{c}^*_p D_{1,k}\tilde{c}_{p-k} &= \sum_{q\in B_F^c\cap\left(B_F^c-k\right)} \tilde{c}^*_q D_{0,-k}\tilde{c}_{q+k} \\ &= \sum_{q\in A_1\cap(A_1-k)} \tilde{c}^*_q D_{0,-k}\tilde{c}_{q+k} + \sum_{q\in A_2\cap(A_2-k)} \tilde{c}^*_q D_{0,-k}\tilde{c}_{q+k} \\ &\quad + \sum_{q\in A_2\cap(A_1-k)} \tilde{c}^*_q D_{0,-k}\tilde{c}_{q+k} + \left(\sum_{q\in(A_1+k)\cap A_2} \tilde{c}^*_q D_{0,k}\tilde{c}_{q-k} \right)^* \end{aligned} \tag{6.17}$$

so all in all (substituting also $k\to -k$ in some sums to group terms together)

$$\mathcal{E}_{\mathcal{Q},1} = \mathcal{E}^1_{\mathcal{Q},1} + 2\,\mathrm{Re}\left(\mathcal{E}^2_{\mathcal{Q},1}\right) + \mathcal{E}^1_{\mathcal{Q},0} + 2\,\mathrm{Re}\left(\mathcal{E}^2_{\mathcal{Q},0}\right) + \mathcal{E}_{\mathcal{Q},5} \tag{6.18}$$

where for $a = 1, 2$

$$\mathcal{E}^a_{\mathcal{Q},j} = \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in A_a\cap(A_1+k)} \tilde{c}^*_p D_{j,k}\tilde{c}_{p-k}, \quad j = 0, 1, \tag{6.19}$$

and we recognized that

$$\sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{q\in A_2\cap(A_2-k)} \tilde{c}^*_q D_{0,-k}\tilde{c}_{q+k} = \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p,q\in A_2\cap(A_2+k)} \tilde{c}^*_q\tilde{c}^*_{p-k}\tilde{c}_p\tilde{c}_{q-k} = \mathcal{E}_{\mathcal{Q},5}. \tag{6.20}$$

Schematic Forms

By the decompositions above we see that to obtain the second estimate of Theorem 6.1, i.e. that on $\mathcal{E}_{\mathbb{Z}^3_*\setminus S} - \mathcal{E}_{\mathcal{Q},4} - \mathcal{E}_{\mathcal{Q},5}$, it suffices to estimate the sums

$$\mathcal{E}^1_{\mathcal{C},\mathbb{Z}^3_*\setminus S} = \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in L_k} \tilde{c}^*_p \tilde{c}^*_{p-k} D_{1,k}, \quad \mathcal{E}^2_{\mathcal{C},\mathbb{Z}^3_*\setminus S} = \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in(L_k-k)} \tilde{c}^*_p \tilde{c}^*_{p+k} D_{2,k}, \tag{6.21}$$

and

$$\begin{aligned}\mathcal{E}^a_{\mathcal{Q},j} &= \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in A_a\cap(A_1+k)} \tilde{c}^*_p D_{j,k} \tilde{c}_{p-k} \\ \mathcal{E}_{\mathcal{Q},2} &= \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in B_F\cap(B_F-k)} \tilde{c}^*_p D_{2,k} \tilde{c}_{p+k} \\ \mathcal{E}_{\mathcal{Q},3} &= \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in B_F\cap(B_F-k)} \tilde{c}^*_p D_{1,k} \tilde{c}_{p+k}\end{aligned} \tag{6.22}$$

for $a = 1, 2$.
We can summarize these in two schematic forms: $\mathcal{E}^1_{\mathcal{C},\mathbb{Z}^3_*\setminus S}$ and $\mathcal{E}^2_{\mathcal{C},\mathbb{Z}^3_*\setminus S}$ are both of the form

$$\tilde{\mathcal{E}}_{\mathcal{C},\mathbb{Z}^3_*\setminus S} = \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in\mathbb{Z}^3} 1_{S_k}(p) \tilde{c}^*_p \tilde{c}^*_{p\mp k} D_{j,k} \tag{6.23}$$

where

$$\left(S_k, \tilde{c}^*_{p\mp k}, D_{j,k}\right) = \begin{cases} \left(L_k, \tilde{c}^*_{p-k}, D_{1,k}\right) & \text{for } \mathcal{E}^1_{\mathcal{C},\mathbb{Z}^3_*\setminus S} \\ \left(L_k - k, \tilde{c}^*_{p+k}, D_{2,k}\right) & \text{for } \mathcal{E}^2_{\mathcal{C},\mathbb{Z}^3_*\setminus S} \end{cases}, \tag{6.24}$$

while $\mathcal{E}^a_{\mathcal{Q},j}$, $\mathcal{E}_{\mathcal{Q},2}$ and $\mathcal{E}_{\mathcal{Q},3}$ are all of the form

$$\tilde{\mathcal{E}}_{\mathcal{Q}} = \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k \sum_{p\in\mathbb{Z}^3} 1_{S_k}(p) \tilde{c}^*_p D_{j,k} \tilde{c}_{p\mp k} \tag{6.25}$$

where

$$(S_k, \tilde{c}_{p\mp k}, D_{j,k}) = \begin{cases} (A_a \cap (A_1 + k), \tilde{c}_{p-k}, D_{j,k}) & \text{for } \mathcal{E}^a_{\mathcal{Q},j} \\ (B_F \cap (B_F - k), \tilde{c}_{p+k}, D_{2,k}) & \text{for } \mathcal{E}_{\mathcal{Q},2} \\ (B_F \cap (B_F - k), \tilde{c}_{p+k}, D_{1,k}) & \text{for } \mathcal{E}_{\mathcal{Q},3} \end{cases}. \tag{6.26}$$

It consequently suffices to estimate these schematic forms. Noting that $\tilde{\mathcal{E}}_{\mathcal{C},\mathbb{Z}^3_*\setminus S}$ can be written as

$$\tilde{\mathcal{E}}_{\mathcal{C},\mathbb{Z}^3_*\setminus S} = \sum_{p\in\mathbb{Z}^3} \tilde{c}^*_p \left(\sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) \hat{V}_k \tilde{c}^*_{p\mp k} D_{j,k} \right) \tag{6.27}$$

we can for any $\Psi \in D(H'_{\text{kin}})$ estimate

$$\begin{aligned}\left|\left\langle \Psi, \tilde{\mathcal{E}}_{\mathcal{C},\mathbb{Z}^3_*\setminus S}\Psi \right\rangle\right| &\leq \sqrt{\sum_{p\in\mathbb{Z}^3} \left| |p|^2 - \zeta \right| \|\tilde{c}_p \Psi\|^2} \sqrt{\sum_{p\in\mathbb{Z}^3} \frac{1}{\left| |p|^2 - \zeta \right|} \left\| \sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) \hat{V}_k \tilde{c}^*_{p\mp k} D_{j,k} \Psi \right\|^2} \\ &\leq \sqrt{\left\langle \Psi, H'_{\text{kin}} \Psi \right\rangle \sum_{p\in\mathbb{Z}^3} \left| |p|^2 - \zeta \right|^{-1} \left\langle \Psi, T^{\mathcal{C}}_p \Psi \right\rangle}\end{aligned} \tag{6.28}$$

where $T^{\mathcal{C}}_p = \left| \sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) \hat{V}_k \tilde{c}^*_{p\mp k} D_{j,k} \right|^2$, and similarly

$$\left|\left\langle \Psi, \tilde{\mathcal{E}}_{\mathcal{Q}}\Psi \right\rangle\right| \leq \sqrt{\left\langle \Psi, H'_{\text{kin}} \Psi \right\rangle \sum_{p\in\mathbb{Z}^3} \left| |p|^2 - \zeta \right|^{-1} \left\langle \Psi, T^{\mathcal{Q}}_p \Psi \right\rangle} \tag{6.29}$$

for $T^{\mathcal{Q}}_p = \left| \sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) \hat{V}_k D_{j,k} \tilde{c}_{p\mp k} \right|^2$.

6.2 Estimation of $T_p^{\mathcal{C}}$ and $T_p^{\mathcal{Q}}$

By expansion and (anti-)commutation we can write $T_p^{\mathcal{C}}$ as

$$\begin{aligned} T_p^{\mathcal{C}} &= \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l D_{j,k}^*\tilde{c}_{p\mp k}\tilde{c}_{p\mp l}^* D_{j,l} \\ &= -\sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l D_{j,k}^*\tilde{c}_{p\mp l}^*\tilde{c}_{p\mp k} D_{j,l} + \sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2 D_{j,k}^* D_{j,k} \\ &= -\left|\sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k D_{j,k}^*\tilde{c}_{p\mp k}\right|^2 + \sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2 D_{j,k}^* D_{j,k} \\ &\quad - \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\tilde{c}_{p\mp l}^*\left[D_{j,k}^*, D_{j,l}\right]\tilde{c}_{p\mp k} \end{aligned} \tag{6.30}$$

where we also used that $[\tilde{c}_{p\pm k}, D_{j,l}] = 0$ in this particular case, since for $\mathcal{E}^1_{\mathcal{C},\mathbb{Z}^3_*\backslash S}$ the momenta $p\mp k = p-k \in B_F$ but $D_{j,l} = D_{1,l}$ only involves momenta in B_F^c, and vice versa for $\mathcal{E}^2_{\mathcal{C},\mathbb{Z}^3_*\backslash S}$.
Now, the first term of the right-hand side of the equation above is manifestly negative, so we have the bound

$$\begin{aligned} T_p^{\mathcal{C}} &\le \sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2 D_{j,k}^* D_{j,k} - \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\tilde{c}_{p\mp l}^*\left[D_{j,k}^*, D_{j,l}\right]\tilde{c}_{p\mp k} \\ &=: T_p^{\mathcal{C},1} - T_p^{\mathcal{C},2}. \end{aligned} \tag{6.31}$$

We computed the commutator $[D_{j,k}^*, D_{j,l}]$ in equation (5.6), with the result

$$\left[D_{j,k}^*, D_{j,l}\right] = \sum_{q\in B_F^\circ\cap\left(B_F^\circ\mp k\right)\cap\left(B_F^\circ\mp l\right)} \tilde{c}_{q\pm k}^*\tilde{c}_{q\pm l} - \sum_{q\in B_F^\circ\cap\left(B_F^\circ\pm k\right)\cap\left(B_F^\circ\pm l\right)} \tilde{c}_{q\mp l}^*\tilde{c}_{q\mp k}. \tag{6.32}$$

Performing the substitution $q\mapsto q+k+l$ in the second sum, we can also write this as

$$\begin{aligned} \left[D_{j,k}^*, D_{j,l}\right] &= \sum_{q\in\left(B_F^\circ\mp k\right)\cap\left(B_F^\circ\mp l\right)} \left(1_{B_F^\circ}(q) - 1_{B_F^\circ}(q\pm k\pm l)\right)\tilde{c}_{q\pm k}^*\tilde{c}_{q\pm l} \\ &= \sum_{q\in\left(B_F^\circ\mp k\right)\cap\left(B_F^\circ\mp l\right)} \left(1_{B_F^\circ}(q)1_{(B_F^\circ)^c}(q\pm k\pm l) - 1_{(B_F^\circ)^c}(q)1_{B_F^\circ}(q\pm k\pm l)\right)\tilde{c}_{q\pm k}^*\tilde{c}_{q\pm l} \end{aligned} \tag{6.33}$$

where we used the indicator function identity $1_A(x) - 1_A(y) = 1_A(x)1_{A^c}(y) - 1_{A^c}(x)1_A(y)$. Writing out the possible choices of $B_F^\circ = B_F, B_F^c$ it is straightforward to see that there holds the alternative identity

$$\begin{aligned} \mp\left[D_{j,k}^*, D_{j,l}\right] = &\sum_{q\in B_F\cap\left(B_F^\circ\mp k\right)\cap\left(B_F^\circ\mp l\right)} 1_{B_F^c}(q\pm k\pm l)\tilde{c}_{q\pm k}^*\tilde{c}_{q\pm l} \\ &- \sum_{q\in B_F\cap\left(B_F^\circ\pm k\right)\cap\left(B_F^\circ\pm l\right)} 1_{B_F^c}(q\mp k\mp l)\tilde{c}_{q\mp l}^*\tilde{c}_{q\mp k}. \end{aligned} \tag{6.34}$$

Using this identity we can now estimate $\sum_{p\in\mathbb{Z}^3}\left||p|^2-\zeta\right|^{-1}T_p^{\mathcal{C}}$ as follows:

Proposition 6.3. *For any symmetric set $S\subset\mathbb{Z}^3_*$ and $\epsilon>0$ it holds as $k_F\to\infty$ that*

$$\sum_{p\in\mathbb{Z}^3}\frac{1}{\left||p|^2-\zeta\right|}T_p^{\mathcal{C}} \le C_\epsilon k_F^{2+\epsilon}\sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k^2 H'_{\mathrm{kin}}$$

for a constant $C_\epsilon>0$ depending only on ϵ.

Proof: $T_p^{\mathcal{C},1}$ can immediately be bounded as

$$T_p^{\mathcal{C},1} = \sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2 D_{j,k}^* D_{j,k} \le C_\epsilon k_F^{1+\epsilon}\left(\sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2\right)H'_{\mathrm{kin}} \tag{6.35}$$

by Proposition 5.3, and by equation (6.34) we can for any $\Psi \in D(H'_{\text{kin}})$ estimate $\left\langle \Psi, T_p^{\mathcal{C},2}\Psi \right\rangle$ as

$$\begin{aligned}
\left|\left\langle \Psi, T_p^{\mathcal{C},2}\Psi \right\rangle\right| &\leq 2\sum_{q\in B_F}\left(\sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{B_F^\circ\mp k}(q)\hat{V}_k\,\|\tilde{c}_{q\pm k}\Psi\|\right)^2 \\
&\leq \sum_{q\in B_F}\left(\sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2\frac{1_{B_F^\circ\mp k}(q)}{\left||q\mp k|^2-\zeta\right|}\right)\left(\sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{B_F^\circ\mp k}(q)\left||q\mp k|^2-\zeta\right|\|\tilde{c}_{q\mp k}\Psi\|^2\right) \qquad (6.36)\\
&\leq \sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2\sum_{q\in B_F}\frac{1_{B_F^\circ\mp k}(q)}{\left||q\mp k|^2-\zeta\right|}\left\langle \Psi, H'_{\text{kin}}\Psi\right\rangle \leq C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2\left\langle \Psi, H'_{\text{kin}}\Psi\right\rangle
\end{aligned}$$

where Proposition 3.2 could be applied due to the condition $q \in B_F$ in the sum. Combining these estimates and applying Proposition 3.2 once more, we conclude that

$$\begin{aligned}
\sum_{p\in\mathbb{Z}^3}\frac{1}{\left||p|^2-\zeta\right|}T_p^{\mathcal{C}} &\leq C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k^2\sum_{p\in\mathbb{Z}^3}\frac{1_{S_k}(p)}{\left||p|^2-\zeta\right|}H'_{\text{kin}} \qquad (6.37)\\
&\leq C'_\epsilon k_F^{2+\epsilon'}\sum_{k\in\mathbb{Z}^3_*\backslash S}\hat{V}_k^2 H'_{\text{kin}}.
\end{aligned}$$

□

As we did for $T_p^{\mathcal{C}}$, we expand $T_p^{\mathcal{Q}}$ and commute for the identity

$$\begin{aligned}
T_p^{\mathcal{Q}} &= \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\tilde{c}^*_{p\mp k}D^*_{j,k}D_{j,l}\tilde{c}_{p\mp l} \\
&= \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\left(D_{j,l}\tilde{c}^*_{p\mp k}+\left[\tilde{c}^*_{p\mp k},D_{j,l}\right]\right)\left(\tilde{c}_{p\mp l}D^*_{j,k}+\left[D^*_{j,k},\tilde{c}_{p\mp l}\right]\right) \\
&+ \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\tilde{c}^*_{p\mp k}\left[D^*_{j,k},D_{j,l}\right]\tilde{c}_{p\mp l} \\
&= -\left|\sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k\tilde{c}^*_{p\mp k}D^*_{j,k}\right|^2 + \sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2 D_{j,k}D^*_{j,k} \qquad (6.38)\\
&+ \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\tilde{c}^*_{p\mp k}\left[D^*_{j,k},D_{j,l}\right]\tilde{c}_{p\mp l} \\
&+ 2\,\mathrm{Re}\sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l D_{j,l}\tilde{c}^*_{p\mp k}\left[D^*_{j,k},\tilde{c}_{p\mp l}\right] \\
&+ \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\left[D^*_{j,l},\tilde{c}_{p\mp k}\right]^*\left[D^*_{j,k},\tilde{c}_{p\mp l}\right],
\end{aligned}$$

which yields the inequality

$$T_p^{\mathcal{Q}} \leq T_p^{\mathcal{Q},1} + T_p^{\mathcal{Q},2} + 2\,\mathrm{Re}\left(T_p^{\mathcal{Q},3}\right) + T_p^{\mathcal{Q},4} \qquad (6.39)$$

where

$$\begin{aligned}
T_p^{\mathcal{Q},1} &= \sum_{k\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)\hat{V}_k^2 D_{j,k}D^*_{j,k} \\
T_p^{\mathcal{Q},2} &= \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\tilde{c}^*_{p\mp k}\left[D^*_{j,k},D_{j,l}\right]\tilde{c}_{p\mp l} \qquad (6.40)\\
T_p^{\mathcal{Q},3} &= \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l D_{j,l}\tilde{c}^*_{p\mp k}\left[D^*_{j,k},\tilde{c}_{p\mp l}\right] \\
T_p^{\mathcal{Q},4} &= \sum_{k,l\in\mathbb{Z}^3_*\backslash S} 1_{S_k}(p)1_{S_l}(p)\hat{V}_k\hat{V}_l\left[D^*_{j,l},\tilde{c}_{p\mp k}\right]^*\left[D^*_{j,k},\tilde{c}_{p\mp l}\right].
\end{aligned}$$

We can then estimate $\sum_{p\in\mathbb{Z}^3}\left||p|^2-\zeta\right|^{-1}T_p^{\mathcal{Q}}$ in a similar fashion:

Proposition 6.4. *For any symmetric set $S \subset \mathbb{Z}^3_*$ and $\epsilon > 0$ it holds as $k_F \to \infty$ that*

$$\sum_{p\in\mathbb{Z}^3} \frac{1}{\left| |p|^2 - \zeta \right|} T_p^{\mathcal{Q}} \le C_\epsilon k_F^{2+\epsilon} \sum_{k\in\mathbb{Z}^3_*\setminus S} \hat{V}_k^2 H'_{\mathrm{kin}}$$

for a constant $C_\epsilon > 0$ depending only on ϵ.

Proof: Exactly as in the previous proposition we see that the bound

$$T_p^{\mathcal{Q},1},\, T_p^{\mathcal{Q},2} \le C_\epsilon k_F^{1+\epsilon} \left(\sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) \hat{V}_k^2 \right) H'_{\mathrm{kin}} \tag{6.41}$$

holds: The $T_p^{\mathcal{Q},1}$ bound follows since $D_{j,k}D_{j,k}^* = D_{j,-k}^* D_{j,-k}$ and Proposition 5.3 is also valid for $D_{0,k}^* D_{0,k}$ (indeed, this is the final equality of the proposition), and the $T_p^{\mathcal{Q},2}$ bound follows since it is readily computed that $[D_{0,k}^*, D_{0,l}]$ can also be written in the form

$$\begin{aligned} -\left[D_{0,k}^*, D_{0,l}\right] = &\sum_{q\in A_2^c\cap(A_2-k)\cap(A_2-l)} 1_{A_2}(q+k+l) \tilde{c}_{q+k}^* \tilde{c}_{q+l} \\ &- \sum_{q\in A_2^c\cap(A_2+k)\cap(A_2+l)} 1_{A_2}(q-k-l) \tilde{c}_{q-l}^* \tilde{c}_{q-k} \end{aligned} \tag{6.42}$$

and the fact that $A_2^c = \overline{B}(0, 2k_F) \cap \mathbb{Z}^3$ ensures that Proposition 3.2 still applies.
For $T_p^{\mathcal{Q},3}$ (and $T_p^{\mathcal{Q},4}$) we calculate

$$\begin{aligned} \left[D_{j,k}^*, \tilde{c}_{p\mp l}\right] &= \pm \sum_{q\in B_F^\circ\cap(B_F^\circ\mp k)} \left[\tilde{c}_{q\pm k}^* \tilde{c}_q, \tilde{c}_{p\mp l}\right] = \mp \sum_{q\in B_F^\circ\cap(B_F^\circ\mp k)} \delta_{q\pm k, p\mp l} \tilde{c}_q \\ &= \mp 1_{B_F^\circ}(p\mp l) 1_{B_F^\circ}(p\mp k\mp l) \tilde{c}_{p\mp k\mp l} \end{aligned} \tag{6.43}$$

so $T_p^{\mathcal{Q},3}$ can be written as

$$\begin{aligned} T_p^{\mathcal{Q},3} &= \mp \sum_{k,l\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) 1_{S_l}(p) 1_{B_F^\circ}(p\mp l) 1_{B_F^\circ}(p\mp k\mp l) \hat{V}_k \hat{V}_l D_{j,l} \tilde{c}_{p\mp k}^* \tilde{c}_{p\mp k\mp l} \\ &= \mp \sum_{l\in\mathbb{Z}^3_*\setminus S} 1_{S_l}(p) 1_{B_F^\circ}(p\mp l) \hat{V}_l D_{j,l} \left(\sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) 1_{B_F^\circ}(p\mp k\mp l) \hat{V}_k \tilde{c}_{p\mp k}^* \tilde{c}_{p\mp k\mp l} \right) \end{aligned}$$

which implies the estimate

$$\left| \left\langle \Psi, T_p^{\mathcal{Q},3} \Psi \right\rangle \right| \le \sqrt{\sum_{l\in\mathbb{Z}^3_*\setminus S} 1_{S_l}(p) \hat{V}_l^2 \left\| D_{j,l}\Psi \right\|^2} \sqrt{\sum_{l\in\mathbb{Z}^3_*\setminus S} 1_{S_l}(p) \left\| \sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) 1_{B_F^\circ}(p\mp k\mp l) \hat{V}_k \tilde{c}_{p\mp k}^* \tilde{c}_{p\mp k\mp l} \Psi \right\|^2}.$$

Again

$$\sum_{l\in\mathbb{Z}^3_*\setminus S} 1_{S_l}(p) \hat{V}_l^2 \left\| D_{j,l}\Psi \right\|^2 \le C_\epsilon k_F^{1+\epsilon} \sum_{l\in\mathbb{Z}^3_*\setminus S} 1_{S_l}(p) \hat{V}_l^2 \left\langle \Psi, H'_{\mathrm{kin}} \Psi \right\rangle \tag{6.44}$$

while the second factor obeys

$$\begin{aligned} &\sum_{l\in\mathbb{Z}^3_*\setminus S} 1_{S_l}(p) \left\| \sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) 1_{B_F^\circ}(p\mp k\mp l) \hat{V}_k \tilde{c}_{p\mp k}^* \tilde{c}_{p\mp k\mp l} \Psi \right\|^2 \\ &\le \sum_{l\in\mathbb{Z}^3_*\setminus S} 1_{S_l}(p) \left(\sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{S_k}(p) \frac{1_{B_F^\circ}(p\mp k\mp l)}{\left| |p\mp k\mp l|^2 - \zeta \right|} \hat{V}_k^2 \right) \\ &\qquad \cdot \left(\sum_{k\in\mathbb{Z}^3_*\setminus S} 1_{B_F^\circ}(p\mp k\mp l) \left| |p\mp k\mp l|^2 - \zeta \right| \left\| \tilde{c}_{p\mp k\mp l} \Psi \right\|^2 \right) \end{aligned} \tag{6.45}$$

$$\leq \sum_{k\in\mathbb{Z}_*^3\setminus S} 1_{S_k}(p)\hat{V}_k^2 \sum_{l\in\mathbb{Z}_*^3\setminus S} 1_{S_l}(p)\frac{1_{B_F^\circ}(p\mp k\mp l)}{\left||p\mp k\mp l|^2-\zeta\right|}\langle\Psi, H'_{\mathrm{kin}}\Psi\rangle$$
$$\leq C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}_*^3\setminus S} 1_{S_k}(p)\hat{V}_k^2\langle\Psi, H'_{\mathrm{kin}}\Psi\rangle$$

where we could use Proposition 3.2 once more since the summation over l is restricted by the indicator function $1_{S_l}(p)$, with S_l being either $B_F\cap(B_F-l)\subset(B_F-l)$ or $A_a\cap(A_1+l)\subset(A_1+l)$, since $|B_F|,|A_1|\leq\left|\overline{B}(0,2k_F)\cap\mathbb{Z}^3\right|$.

Combining the two estimates we get

$$\pm T_p^{\mathcal{Q},3}\leq C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}_*^3\setminus S} 1_{S_k}(p)\hat{V}_k^2 H'_{\mathrm{kin}} \tag{6.46}$$

and equation (6.43) also yields

$$T_p^{\mathcal{Q},4}=\sum_{k,l\in\mathbb{Z}_*^3\setminus S} 1_{S_k}(p)1_{S_l}(p)1_{B_F^\circ}(p\mp k)1_{B_F^\circ}(p\mp l)1_{B_F^\circ}(p\mp k\mp l)\hat{V}_k\hat{V}_l\tilde{c}^*_{p\mp k\mp l}\tilde{c}_{p\mp k\mp l} \tag{6.47}$$

which as the summand is symmetric in k and l can be estimated by

$$\begin{aligned} T_p^{\mathcal{Q},4} &\leq \sum_{k,l\in\mathbb{Z}_*^3\setminus S} 1_{S_k}(p)1_{S_l}(p)1_{B_F^\circ}(p\mp k)1_{B_F^\circ}(p\mp l)1_{B_F^\circ}(p\mp k\mp l)\hat{V}_k^2\tilde{c}^*_{p\mp k\mp l}\tilde{c}_{p\mp k\mp l} \\ &\leq \sum_{k\in\mathbb{Z}_*^3\setminus S} 1_{S_k}(p)\hat{V}_k^2\mathcal{N}_E\leq\sum_{k\in\mathbb{Z}_*^3\setminus S} 1_{S_k}(p)\hat{V}_k^2 H'_{\mathrm{kin}}. \end{aligned} \tag{6.48}$$

All in all this shows that $T_p^{\mathcal{Q}}\leq C_\epsilon k_F^{1+\epsilon}\sum_{k\in\mathbb{Z}_*^3\setminus S}1_{S_k}(p)\hat{V}_k^2 H'_{\mathrm{kin}}$ and the claim now follows as in Proposition 6.4.

□

By the equations (6.28) and (6.29) combined with these propositions we see that

$$\pm\mathcal{E}_{\mathcal{C},\mathbb{Z}_*^3\setminus S},\ \pm(\mathcal{E}_{\mathcal{Q}}-\mathcal{E}_{\mathcal{Q},4}-\mathcal{E}_{\mathcal{Q},5})\leq C_\epsilon k_F^{1+\epsilon}\sqrt{\sum_{k\in\mathbb{Z}_*^3\setminus S}\hat{V}_k^2}H'_{\mathrm{kin}} \tag{6.49}$$

which implies the second estimate of Theorem 6.1.

6.3 Estimation of $\mathcal{E}_{\mathcal{Q},4}$ and $\mathcal{E}_{\mathcal{Q},5}$

For $\mathcal{E}_{\mathcal{Q},4}$ we can make the following estimate:

Proposition 6.5. *Let $\hat{V}_k$ be radially decreasing with respect to $k\in\mathbb{Z}_*^3$. Then for any symmetric $S\subset\mathbb{Z}_*^3$ it holds that*

$$\mathcal{E}_{\mathcal{Q},4}\geq-2\left(\sum_{k\in S}\hat{V}_k+\sup_{p\in B_F^c}\hat{V}_p\right)H'_{\mathrm{kin}}.$$

Proof: We can write $\mathcal{E}_{\mathcal{Q},4}$ in the form

$$\begin{aligned} \mathcal{E}_{\mathcal{Q},4} &= 2\sum_{p\in B_F}\left(\sum_{k\in\mathbb{Z}_*^3}1_{B_F}(p+k)\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p-2\sum_{p\in B_F^c}\left(\sum_{k\in\mathbb{Z}_*^3}1_{B_F}(p-k)\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p \\ &\quad -2\sum_{p\in B_F}\left(\sum_{k\in S}1_{B_F}(p+k)\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p+2\sum_{p\in B_F^c}\left(\sum_{k\in S}1_{B_F}(p-k)\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p \end{aligned} \tag{6.50}$$

which, estimating the final line as in Proposition 6.2, implies that

$$\mathcal{E}_{\mathcal{Q},4}\geq 2\sum_{p\in B_F}\left(\sum_{k\in(B_F+p)\setminus\{0\}}\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p-2\sum_{p\in B_F^c}\left(\sum_{k\in(B_F+p)}\hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p-2\sum_{k\in S}\hat{V}_k H'_{\mathrm{kin}} \tag{6.51}$$

where we also absorbed the indicator functions into the summation range (and substituted $k \to -k$ in the first sum).

Now we note that there must exist a $\nu \in \mathbb{R}$ such that

$$\sup_{p\in B_F^c} \sum_{k\in(B_F+p)\backslash\{p\}} \hat{V}_k \leq \nu \leq \inf_{p\in B_F} \sum_{k\in(B_F+p)\backslash\{0\}} \hat{V}_k \tag{6.52}$$

by the assumption that $\hat{V}_k$ is radially decreasing, since each sum is over $|B_F|-1$ points. Then by particle-hole symmetry we have

$$\begin{aligned}
&\sum_{p\in B_F}\left(\sum_{k\in(B_F+p)\backslash\{0\}} \hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p - \sum_{p\in B_F^c}\left(\sum_{k\in(B_F+p)} \hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p \\
&= \sum_{p\in B_F}\left(\sum_{k\in(B_F+p)\backslash\{0\}} \hat{V}_k - \nu\right)\tilde{c}_p^*\tilde{c}_p + \sum_{p\in B_F^c}\left(\nu - \sum_{k\in(B_F+p)\backslash\{p\}} \hat{V}_k\right)\tilde{c}_p^*\tilde{c}_p - \sum_{p\in B_F^c} \hat{V}_p\tilde{c}_p^*\tilde{c}_p \\
&\geq -\sum_{p\in B_F^c} \hat{V}_p\tilde{c}_p^*\tilde{c}_p \geq -\sup_{p\in B_F^c} \hat{V}_p\,\mathcal{N}_E \geq -\sup_{p\in B_F^c} \hat{V}_p\,H'_{\mathrm{kin}}
\end{aligned} \tag{6.53}$$

for the claim.

$\square$

Estimation of $\mathcal{E}_{\mathcal{Q},5}$

Finally we come to $\mathcal{E}_{\mathcal{Q},5}$, which we recall is

$$\mathcal{E}_{\mathcal{Q},5} = \sum_{k\in\mathbb{Z}_*^3\backslash S} \hat{V}_k \sum_{p,q\in A_2\cap(A_2+k)} \tilde{c}_p^*\tilde{c}_{q-k}^*\tilde{c}_q\tilde{c}_{p-k} \tag{6.54}$$

where $A_2 = \mathbb{Z}^3\backslash\overline{B}(0,2k_F)$. Noting as in Proposition 5.3 that

$$\sum_{p,q\in A_2\cap(A_2+k)} \tilde{c}_p^*\tilde{c}_{q-k}^*\tilde{c}_q\tilde{c}_{p-k} = D_{0,k}^*D_{0,k} - \sum_{p\in A_2\cap(A_2+k)} \tilde{c}_p^*\tilde{c}_p \geq -3^{-1}k_F^{-2}H'_{\mathrm{kin}} \tag{6.55}$$

we can for any $S' \subset \mathbb{Z}_*^3$ containing S estimate

$$\begin{aligned}
\mathcal{E}_{\mathcal{Q},5} &= \sum_{k\in S'\backslash S} \hat{V}_k \sum_{p,q\in A_2\cap(A_2+k)} \tilde{c}_p^*\tilde{c}_{q-k}^*\tilde{c}_q\tilde{c}_{p-k} + \sum_{k\in\mathbb{Z}_*^3\backslash S'} \hat{V}_k \sum_{p,q\in A_2\cap(A_2+k)} \tilde{c}_p^*\tilde{c}_{q-k}^*\tilde{c}_q\tilde{c}_{p-k} \\
&\geq -\frac{1}{3}k_F^{-2}\sum_{k\in S'} \hat{V}_k H'_{\mathrm{kin}} + \sum_{k\in\mathbb{Z}_*^3\backslash S'} \hat{V}_k \sum_{p,q\in A_2\cap(A_2+k)} c_p^*c_{q-k}^*c_qc_{p-k}
\end{aligned} \tag{6.56}$$

and the k_F^{-2} factor ensures that we can take S' to be considerably larger than S without worsening the overall estimate. The remaining sum then not only involves exclusively momenta which are large, but we can also assume k to be large. In that case we can make the following estimate:

Proposition 6.6. *For any $\epsilon > 0$ and S' containing $\overline{B}(0,3k_F)\cap\mathbb{Z}^3$ it holds that*

$$\pm\sum_{k\in\mathbb{Z}_*^3\backslash S'} \hat{V}_k \sum_{p,q\in A_2\cap(A_2+k)} c_p^*c_{q-k}^*c_qc_{p-k} \leq C_\epsilon\left(\sum_{k\in\mathbb{Z}_*^3\backslash S'} \frac{\hat{V}_k}{|k|^2} + \sqrt{\sum_{k\in\mathbb{Z}_*^3\backslash S'} \hat{V}_k^2\,|k|^{-(1-\epsilon)}}\right)\mathcal{N}_E H'_{\mathrm{kin}}$$

for a constant $C_\epsilon > 0$ depending only on ϵ.

Proof: By the triangle inequality we have for all $k \in \mathbb{Z}_*^3\backslash S' \subset \mathbb{Z}_*^3\backslash\overline{B}(0,3k_F)$ that

$$3^{-1}|k| \leq |k| - 2k_F \leq |p| - k_F + |p-k| - k_F \leq \sqrt{|p|^2 - k_F^2} + \sqrt{|p-k|^2 - k_F^2} \tag{6.57}$$

when $p \in A_2 \cap (A_2 + k)$, so for any $\Psi \in D(H'_{\mathrm{kin}})$ we can estimate

$$\begin{aligned}
&\left|\sum_{k \in \mathbb{Z}^3_* \backslash S'} \hat{V}_k \sum_{p,q \in A_2 \cap (A_2+k)} \left\langle \Psi, c_p^* c_{q-k}^* c_q c_{p-k} \Psi \right\rangle \right| \\
&\leq \sum_{k \in \mathbb{Z}^3_* \backslash S'} \hat{V}_k \sum_{p,q \in A_2 \cap (A_2+k)} \| c_{q-k} c_p \Psi \| \, \| c_q c_{p-k} \Psi \| \\
&\leq \sum_{k \in \mathbb{Z}^3_* \backslash S'} \hat{V}_k \sum_{p,q \in A_2 \cap (A_2+k)} \frac{\sqrt{|p|^2 - k_F^2} + \sqrt{|p-k|^2 - k_F^2}}{3^{-1} |k|} \| c_{q-k} c_p \Psi \| \, \| c_q c_{p-k} \Psi \| \\
&= 6 \sum_{k \in \mathbb{Z}^3_* \backslash S'} \frac{\hat{V}_k}{|k|} \sum_{p,q \in A_2 \cap (A_2+k)} \sqrt{|p|^2 - k_F^2} \, \| c_{q-k} c_p \Psi \| \, \| c_q c_{p-k} \Psi \|
\end{aligned} \tag{6.58}$$

where we also made the substitutions $p \to p + k$ and $k \to -k$ in one sum to reduce to the same expression. Now we split the q summation into a $|q| \geq |k|$ and $|q| < |k|$ part. In the first case we can estimate

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^3_* \backslash S'} \frac{\hat{V}_k}{|k|} \sum_{p,q \in A_2 \cap (A_2+k)} 1_{B(0,|k|)^c}(q) \sqrt{|p|^2 - k_F^2} \, \| c_{q-k} c_p \Psi \| \, \| c_q c_{p-k} \Psi \| \\
&\leq \sum_{k \in \mathbb{Z}^3_* \backslash S'} \frac{\hat{V}_k}{|k|} \sum_{p,q \in A_2 \cap (A_2+k)} \frac{|q| - k_F}{|k| - k_F} \sqrt{|p|^2 - k_F^2} \, \| c_{q-k} c_p \Psi \| \, \| c_q c_{p-k} \Psi \| \\
&\leq \frac{3}{2} \sum_{k \in \mathbb{Z}^3_* \backslash S'} \frac{\hat{V}_k}{|k|^2} \sum_{p,q \in A_2 \cap (A_2+k)} \sqrt{|p|^2 - k_F^2} \sqrt{|q|^2 - k_F^2} \, \| c_{q-k} c_p \Psi \| \, \| c_q c_{p-k} \Psi \| \\
&\leq \frac{3}{2} \sum_{k \in \mathbb{Z}^3_* \backslash S'} \frac{\hat{V}_k}{|k|^2} \sum_{p,q \in A_2 \cap (A_2+k)} \left(|q|^2 - k_F^2 \right) \| c_q c_{p-k} \Psi \|^2 \\
&\leq \frac{3}{2} \sum_{k \in \mathbb{Z}^3_* \backslash S'} \frac{\hat{V}_k}{|k|^2} \left\langle \Psi, \mathcal{N}_E H'_{\mathrm{kin}} \Psi \right\rangle
\end{aligned} \tag{6.59}$$

as also $|k| - k_F > \frac{2}{3} |k|$ for $k \in \mathbb{Z}^3_* \backslash S'$. Meanwhile, in the second case,

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^3_* \backslash S'} \frac{\hat{V}_k}{|k|} \sum_{p,q \in A_2 \cap (A_2+k)} 1_{B(0,|k|)}(q) \sqrt{|p|^2 - k_F^2} \, \| c_{q-k} c_p \Psi \| \, \| c_q c_{p-k} \Psi \| \\
&\leq \sqrt{\sum_{k \in \mathbb{Z}^3_* \backslash S'} \hat{V}_k^2 |k|^{-(1-\epsilon)} \sum_{p,q \in A_2 \cap (A_2+k)} \left(|q|^2 - k_F^2 \right) \| c_q c_{p-k} \Psi \|^2} \\
&\quad \cdot \sqrt{\sum_{k \in \mathbb{Z}^3_* \backslash S'} \sum_{p,q \in A_2 \cap (A_2+k)} \frac{1_{B(0,|k|)}(q)}{|k|^{1+\epsilon}} \frac{|p|^2 - k_F^2}{|q|^2 - k_F^2} \| c_{q-k} c_p \Psi \|^2} \\
&\leq \sqrt{\sum_{k \in \mathbb{Z}^3_* \backslash S'} \hat{V}_k^2 |k|^{-(1-\epsilon)} \left\langle \Psi, \mathcal{N}_E H'_{\mathrm{kin}} \Psi \right\rangle} \\
&\quad \cdot \sqrt{\sum_{q \in A_2} \frac{1}{|q|^{1+\epsilon} \left(|q|^2 - k_F^2 \right)} \sum_{k \in \mathbb{Z}^3_* \backslash S'} \sum_{p \in A_2 \cap (A_2+k)} \left(|p|^2 - k_F^2 \right) \| c_{q-k} c_p \Psi \|^2} \\
&\leq \sqrt{\sum_{k \in \mathbb{Z}^3_* \backslash S'} \hat{V}_k^2 |k|^{-(1-\epsilon)}} \sqrt{\sum_{q \in A_2} \frac{1}{|q|^{1+\epsilon} \left(|q|^2 - k_F^2 \right)}} \left\langle \Psi, \mathcal{N}_E H'_{\mathrm{kin}} \Psi \right\rangle
\end{aligned} \tag{6.60}$$

and for $q \in A_2$, $k_F^2 \leq \frac{1}{4} |q|^2$, so

$$\sum_{q \in A_2} \frac{1}{|q|^{1+\epsilon} \left(|q|^2 - k_F^2 \right)} \leq \frac{4}{3} \sum_{q \in A_2} \frac{1}{|q|^{3+\epsilon}} \leq C_\epsilon. \tag{6.61}$$

$\square$

By inserting this bound in equation (6.56) and using the trivial bound $\mathcal{N}_E H'_{\text{kin}} \leq |B_F| H'_{\text{kin}} \leq C k_F^3 H'_{\text{kin}}$ we arrive at the final estimate of Theorem 6.1.

6.4 Proof of Theorem 1.2

We can now prove the first part of Theorem 1.2:

Proposition 6.7. *Let $\frac{1}{6} \leq \beta \leq 1$ and let V obey Assumption 1.1. Then it holds as $k_F \to \infty$ that*

$$H_N \geq E_{\text{FS}} + E_{\text{corr,bos}} + E_{\text{corr,ex}} + \mathcal{E}$$

for an operator $\mathcal{E}$ obeying

$$\mathcal{E} \geq -C_{V,\epsilon} k_F^{-\frac{1}{6}+2(1-\beta)+\epsilon} \left(H'_{\text{kin}} + k_F\right)$$

for any $\epsilon > 0$ where $C_{V,\epsilon} > 0$ is a constant depending only on C_V and ϵ.

Proof: As remarked in the beginning of the section we have by Theorem 4.1 that

$$H_N \geq E_{\text{FS}} + E_{\text{corr,bos}} + E_{\text{corr,ex}} + \mathcal{E}$$

for $\mathcal{E} = \mathcal{E}_{\text{B}} + \mathcal{E}_{\mathcal{C}} + 2^{-1}(2\pi)^{-3} k_F^{-\beta} \left(\mathcal{E}_S + \mathcal{E}_{\mathbb{Z}^3_* \setminus S}\right)$. By the Theorems 3.1 and 5.1, $\mathcal{E}_{\text{B}}$ and $\mathcal{E}_{\mathcal{C}}$ obey the bounds

$$\pm\mathcal{E}_{\text{B}} \leq C_\epsilon k_F^{2(1-\beta)+\epsilon} \left(\sqrt{\sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k^2} + k_F^{-\frac{1}{2}} \sum_{k \in S} \hat{V}_k\right) \sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} \left(H'_{\text{kin}} + k_F\right) + C k_F^{3(1-\beta)} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^3$$

$$\pm\mathcal{E}_{\mathcal{C}} \leq C_\epsilon k_F^{2(1-\beta)+\epsilon} \left(k_F^{-\frac{1}{2}} \sum_{k \in S} \hat{V}_k\right) \left(\sqrt{\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\}} + k_F^{-\frac{1}{2}} \sum_{k \in S} \hat{V}_k\right) \left(H'_{\text{kin}} + k_F\right) \tag{6.62}$$

and under Assumption 1.1 it holds that with $S = \overline{B}(0, k_F^{1/3}) \cap \mathbb{Z}^3_*$

$$\sqrt{\sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k^2},\ k_F^{-\frac{1}{2}} \sum_{k \in S} \hat{V}_k \leq C'_V k_F^{-\frac{1}{6}}, \quad \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^3 \leq C'_V, \tag{6.63}$$

and

$$\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\} \leq C'_V \log(k_F) \leq C_{V,\epsilon} k_F^\epsilon \tag{6.64}$$

which together imply a bound of the form

$$\mathcal{E}_{\text{B}} + \mathcal{E}_{\mathcal{C}} \geq -C_{V,\epsilon} k_F^{-\frac{1}{6}+2(1-\beta)+\epsilon} \left(H'_{\text{kin}} + k_F\right) \tag{6.65}$$

where we used the assumption $\beta \geq \frac{1}{6}$ to absorb the $k_F^{3(1-\beta)}$ term into the rest.

By Theorem 6.1 it follows that with $S' = \overline{B}(0, k_F^{5/2}) \cap \mathbb{Z}^3_*$

$$\begin{aligned} \mathcal{E}_S + \mathcal{E}_{\mathbb{Z}^3_* \setminus S} &\geq -C_\epsilon \left(k_F^{1+\epsilon} \sqrt{\sum_{k \in \mathbb{Z}^3_* \setminus S} \hat{V}_k^2} + \sum_{k \in S} \hat{V}_k + \sup_{p \in B_F^c} \hat{V}_p\right) H'_{\text{kin}} \\ &\quad - C'_\epsilon \left(k_F^{-2} \sum_{k \in S'} \hat{V}_k + k_F^3 \left(\sum_{k \in \mathbb{Z}^3_* \setminus S'} \hat{V}_k |k|^{-2} + \sqrt{\sum_{k \in \mathbb{Z}^3_* \setminus S'} \hat{V}_k^2 |k|^{-(1-\epsilon)}}\right)\right) H'_{\text{kin}} \\ &\geq -C_{V,\epsilon} k_F^{-\frac{1}{6}+1+\epsilon} H'_{\text{kin}} \end{aligned} \tag{6.66}$$

so all in all

$$\mathcal{E} = \mathcal{E}_{\text{B}} + \mathcal{E}_{\mathcal{C}} + \frac{k_F^{-\beta}}{2(2\pi)^3} \left(\mathcal{E}_S + \mathcal{E}_{\mathbb{Z}^3_* \setminus S}\right) \geq -C_{V,\epsilon} k_F^{-\frac{1}{6}+2(1-\beta)+\epsilon} \left(H'_{\text{kin}} + k_F\right) \tag{6.67}$$

since $\beta \leq 1$.

$\square$

As remarked in the introduction, we can by this result conclude the inequality

$$(1-o(1))H'_{\text{kin}} \leq 2(H_N - E_{\text{FS}}) - \tilde{E}_{\text{corr,bos}} + Ck_F, \quad k_F \to \infty, \tag{6.68}$$

when $\beta > \frac{11}{12}$ (to ensure $-\frac{1}{6} + 2(1-\beta) + \epsilon < 0$ for some ϵ), where

$$\tilde{E}_{\text{corr,bos}} = \frac{1}{\pi} \sum_{k \in \mathbb{Z}^3_*} \int_0^\infty F\left(\frac{2\hat{V}_k k_F^{-\beta}}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right) dt, \quad F(x) = \log(1+x) - x. \tag{6.69}$$

The second part of Theorem 1.2 is now an immediate consequence of the following:

Proposition 6.8. *Let V obey Assumption 1.1. Then for any $\epsilon > 0$ it holds that*

$$-\tilde{E}_{\text{corr,bos}} \leq C_{V,\epsilon} k_F^{3-2\beta+\epsilon}, \quad k_F \to \infty,$$

for a constant $C_{V,\epsilon} > 0$ depending only on C_V and ϵ.

Proof: By the inequality $\log(1+x) \geq x - \frac{1}{2}x^2$, valid for all $x \geq 0$, we see that

$$\begin{aligned}
-\tilde{E}_{\text{corr,bos}} &\leq \frac{1}{\pi} \sum_{k \in \mathbb{Z}^3_*} \int_0^\infty \frac{1}{2}\left(\frac{2\hat{V}_k k_F^{-\beta}}{(2\pi)^3} \sum_{p \in L_k} \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2}\right)^2 dt \\
&= \frac{4k_F^{-2\beta}}{(2\pi)^7} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \sum_{p,q \in L_k} \int_0^\infty \frac{\lambda_{k,p}}{\lambda_{k,p}^2 + t^2} \frac{\lambda_{k,q}}{\lambda_{k,q}^2 + t^2} dt \\
&= \frac{k_F^{-2\beta}}{(2\pi)^6} \sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \sum_{p,q \in L_k} \frac{1}{\lambda_{k,p} + \lambda_{k,q}}
\end{aligned} \tag{6.70}$$

where we applied the integral identity $\int_0^\infty \frac{a}{a^2+t^2} \frac{b}{b^2+t^2} dt = \frac{\pi}{2}(a+b)^{-1}$, valid for all $a, b > 0$. Now

$$\sum_{p,q \in L_k} \frac{1}{\lambda_{k,p} + \lambda_{k,q}} \leq |L_k| \sum_{p \in L_k} \frac{1}{\lambda_{k,p}} \leq Ck_F^3 \min\{|k|, k_F\} \tag{6.71}$$

and as noted above, $\sum_{k \in \mathbb{Z}^3_*} \hat{V}_k^2 \min\{|k|, k_F\} \leq C_{V,\epsilon} k_F^\epsilon$ under Assumption 1.1, from which the claim follows.

$\square$

A Appendix

A.1 Kinetic Sum Estimates

We will use the following estimate for the number of lattice points on a sphere (see for instance [9, Section 2]):

Proposition A.1. *For any $n \in \mathbb{N}$ and $\epsilon > 0$ it holds that*

$$r_3(n) := |\{p \in \mathbb{Z}^3 \mid |p|^2 = n\}| \leq C_\epsilon n^{\frac{1}{2}+\epsilon}$$

for a constant $C_\epsilon > 0$ depending only on ϵ.

We note that

$$\zeta = \frac{1}{2}\left(\inf_{p \in B_F^c} |p|^2 + \sup_{q \in B_F} |q|^2\right) \tag{A.1}$$

obeys $|\zeta - k_F^2| \leq k_F + 1$ ([11, eq. A.90]) and crucially

$$||p|^2 - \zeta| \geq \frac{1}{2}, \quad \forall p \in \mathbb{Z}^3. \tag{A.2}$$

Lemma (3.2)**.** *For any $A \subset \mathbb{Z}^3$ with $|A| \leq \left|\overline{B}(0,2k_F) \cap \mathbb{Z}^3\right|$ and any $\epsilon > 0$ it holds that*

$$\sum_{p \in A} \frac{1}{\left| |p|^2 - \zeta \right|} \leq C_\epsilon k_F^{1+\epsilon}$$

for a constant $C_\epsilon > 0$ depending only on ϵ.

Proof: By rearrangement it holds for sufficiently large k_F that

$$\sum_{p \in A} \frac{1}{\left| |p|^2 - \zeta \right|} \leq \sum_{p \in \overline{B}(0,2k_F) \cap \mathbb{Z}^3_*} \frac{1}{\left| |p|^2 - \zeta \right|} + \frac{1}{2} k_F^{-2} \tag{A.3}$$

since $\left| |p|^2 - \zeta \right|^{-1}$ is radially increasing for $p \in B_F$, radially decreasing for $p \in B_F^c$, and

$$\left| |p|^2 - \zeta \right|^{-1} \leq \left(4k_F^2 - \zeta\right)^{-1} \leq \left(3k_F^2 - k_F - 1\right)^{-1} \leq \frac{1}{2} k_F^{-2}, \quad k_F \to \infty, \tag{A.4}$$

for $p \notin \overline{B}(0,2k_F)$ while

$$\left| |p|^2 - \zeta \right|^{-1} \geq \zeta^{-1} \geq \left(k_F^2 + k_F + 1\right)^{-1} \geq \frac{1}{2} k_F^{-2}, \quad k_F \to \infty, \tag{A.5}$$

for $p \in B_F$. The sum can now be written as

$$\sum_{p \in \overline{B}(0,2k_F) \cap \mathbb{Z}^3_*} \frac{1}{\left| |p|^2 - \zeta \right|} = \sum_{n=1}^{\lfloor 4k_F^2 \rfloor} \frac{r_3(n)}{|n - \zeta|} = \sum_{n=1}^{m} \frac{r_3(n)}{\zeta - n} + \sum_{n=m'}^{\lfloor 4k_F^2 \rfloor} \frac{r_3(n)}{n - \zeta} \tag{A.6}$$

where

$$m = \sup_{q \in B_F} |q|^2, \quad m' = \inf_{p \in B_F^c} |p|^2. \tag{A.7}$$

We can use Proposition A.1 and the fact that $t \mapsto \sqrt{t}(\zeta - t)^{-1}$ is increasing for $t \in (0, \zeta)$ to estimate the first sum as

$$\sum_{n=1}^{m} \frac{r_3(n)}{\zeta - n} \leq C_\epsilon \sum_{n=1}^{m} \frac{n^{\frac{1}{2}+\frac{1}{2}\epsilon}}{\zeta - n} \leq C_\epsilon k_F^\epsilon \left(\frac{\sqrt{m}}{\zeta - m} + \sum_{n=1}^{m-1} \frac{\sqrt{n}}{\zeta - n} \right) \leq C_\epsilon k_F^\epsilon \left(2\sqrt{m} + \int_1^{m-\frac{1}{2}} \frac{\sqrt{t}}{\zeta - t} \, dt \right) \tag{A.8}$$

where we also used that equation (A.2) implies that $(\zeta - m)^{-1} \leq 2$. The integral obeys

$$\begin{aligned} \int_1^{m-\frac{1}{2}} \frac{\sqrt{t}}{\zeta - t} \, dt &\leq \int_0^m \frac{2(\sqrt{t})^2}{\zeta - (\sqrt{t})^2} (\sqrt{t})' dt = \int_0^{\sqrt{m}} \frac{2t^2}{\zeta - t^2} dt = \sqrt{\zeta} \log\left(\frac{\sqrt{\zeta} + \sqrt{m}}{\sqrt{\zeta} - \sqrt{m}} \right) - 2\sqrt{m} \\ &= \sqrt{\zeta} \log\left(\frac{\left(\sqrt{\zeta} + \sqrt{m}\right)^2}{\zeta - m} \right) - 2\sqrt{m} \leq \sqrt{\zeta} \log\left(8\sqrt{\zeta}\right) - 2\sqrt{m} \end{aligned} \tag{A.9}$$

whence

$$\sum_{n=1}^{m} \frac{r_3(n)}{\zeta - n} \leq C_\epsilon k_F^\epsilon \sqrt{\zeta} \log\left(8\sqrt{\zeta}\right) \leq C'_\epsilon k_F^{1+\epsilon'}. \tag{A.10}$$

For the other sum we can similarly estimate

$$\sum_{n=m'}^{\lfloor 4k_F^2 \rfloor} \frac{r_3(n)}{n - \zeta} \leq C_\epsilon k_F^\epsilon \left(\frac{\sqrt{m'}}{m' - \zeta} + \sum_{n=m'+1}^{\lfloor 4k_F^2 \rfloor} \frac{\sqrt{n}}{n - \zeta} \right) \leq C_\epsilon k_F^\epsilon \left(2\sqrt{m'} + \int_{m'+\frac{1}{2}}^{\lfloor 4k_F^2 \rfloor} \frac{\sqrt{t}}{t - \zeta} \, dt \right) \tag{A.11}$$

as $t \mapsto \sqrt{t}(t - \zeta)^{-1}$ is decreasing on (ζ, ∞). This integral can be bounded as

$$\int_{m'+\frac{1}{2}}^{\lfloor 4k_F^2 \rfloor} \frac{\sqrt{t}}{t - \zeta} \, dt \leq \int_{\sqrt{m'}}^{2k_F} \frac{2t^2}{t^2 - \zeta} \, dt = 4k_F - 2\sqrt{m'} + \int_{\sqrt{m'}}^{2k_F} \frac{2\zeta}{t^2 - \zeta} \, dt$$

$$
\begin{aligned}
&= 4k_F - 2\sqrt{m'} + \sqrt{\zeta}\left(\log\left(\frac{2k_F-\sqrt{\zeta}}{2k_F+\sqrt{\zeta}}\right) - \log\left(\frac{\sqrt{m'}-\sqrt{\zeta}}{\sqrt{m'}+\sqrt{\zeta}}\right)\right) \\
&\leq 4k_F - 2\sqrt{m'} + \sqrt{\zeta}\log(8m')
\end{aligned} \tag{A.12}
$$

whence

$$
\sum_{n=m'}^{\lfloor 4k_F^2 \rfloor} \frac{r_3(n)}{n-\zeta} \leq C_\epsilon k_F^\epsilon \left(4k_F + \sqrt{\zeta}\log(8m')\right) \leq C'_\epsilon k_F^{1+\epsilon'}. \tag{A.13}
$$

Combining the estimates yields the claim.

□

A.2 One-Body Operator Estimates

Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional Hilbert space, $h : V \to V$ be a positive self-adjoint operator with eigenbasis $(x_i)_{i=1}^n$ and eigenvalues $(\lambda_i)_{i=1}^n$, and let $v \in V$ be a vector with $\langle x_i, v \rangle \geq 0$ for all $1 \leq i \leq n$.
We define

$$
E = (h^{\frac{1}{2}}(h + 2P_v)h^{\frac{1}{2}})^{\frac{1}{2}} = (h^2 + 2P_{h^{\frac{1}{2}}v})^{\frac{1}{2}} \tag{A.14}
$$

where $P_w = |w\rangle\langle w|$ for any $w \in V$, and in terms of this further define

$$
C = \frac{1}{2}\left(h^{-\frac{1}{2}}E^{\frac{1}{2}} + h^{\frac{1}{2}}E^{-\frac{1}{2}}\right), \quad S = \frac{1}{2}\left(h^{-\frac{1}{2}}E^{\frac{1}{2}} - h^{\frac{1}{2}}E^{-\frac{1}{2}}\right). \tag{A.15}
$$

Note that $E^{\frac{1}{2}}$ is the fourth root of a rank one perturbation. As in [11, Proposition 9.9], the Sherman-Morrison formula

$$
(A + gP_w)^{-1} = A^{-1} - \frac{g}{1 + g\langle w, A^{-1}w\rangle} P_{A^{-1}w} \tag{A.16}
$$

and the integral identity $a^{\frac{1}{4}} = \frac{2\sqrt{2}}{\pi}\int_0^\infty \left(1 - t^4(a+t^4)^{-1}\right)dt$, $a \geq 0$, yields the following characterization of such operator roots:

Proposition A.2. *Let $A : V \to V$ be a positive self-adjoint operator. Then for any $w \in V$ it holds that*

$$
\begin{aligned}
(A + P_w)^{\frac{1}{4}} &= A^{\frac{1}{4}} + \frac{2\sqrt{2}}{\pi}\int_0^\infty \frac{t^4}{1 + \left\langle w, (A+t^4)^{-1}w \right\rangle} P_{(A+t^4)^{-1}w}\, dt \\
(A + P_w)^{-\frac{1}{4}} &= A^{-\frac{1}{4}} - \frac{2\sqrt{2}}{\pi}\int_0^\infty \frac{t^4}{1 + \left\langle w, A^{-1}(A^{-1}+t^4)^{-1}w \right\rangle t^4} P_{A^{-1}(A^{-1}+t^4)^{-1}w}\, dt.
\end{aligned}
$$

This implies the following:

Proposition A.3. *For all $1 \leq i, j \leq n$ it holds that*

$$
\begin{aligned}
\frac{1}{1 + 2\langle v, h^{-1}v\rangle}\frac{2\sqrt{\lambda_i\lambda_j}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j} &\leq \left\langle x_i, \left(E^{\frac{1}{2}} - h^{\frac{1}{2}}\right)x_j\right\rangle \leq \frac{2\sqrt{\lambda_i\lambda_j}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j} \\
\frac{1}{1 + 2\langle v, h^{-1}v\rangle}\frac{2}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j} &\leq \left\langle x_i, \left(h^{-\frac{1}{2}} - E^{-\frac{1}{2}}\right)x_j\right\rangle \leq \frac{2}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\langle x_i, v\rangle\langle v, x_j\rangle}{\lambda_i+\lambda_j}.
\end{aligned}
$$

Proof: Taking $A = h^2$ and $w = \sqrt{2}h^{\frac{1}{2}}v$ in Proposition A.2, we have

$$
\begin{aligned}
E^{\frac{1}{2}} &= h^{\frac{1}{2}} + \frac{4\sqrt{2}}{\pi}\int_0^\infty \frac{t^4}{1 + 2\left\langle v, h(h^2+t^4)^{-1}v\right\rangle} P_{h^{\frac{1}{2}}(h^2+t^4)^{-1}v}\, dt \\
E^{-\frac{1}{2}} &= h^{-\frac{1}{2}} - \frac{4\sqrt{2}}{\pi}\int_0^\infty \frac{t^4}{1 + 2\left\langle v, h^{-1}(h^{-2}+t^4)^{-1}v\right\rangle t^4} P_{h^{-\frac{3}{2}}(h^{-2}+t^4)^{-1}v}\, dt
\end{aligned} \tag{A.17}
$$

so

$$\left\langle x_i,\left(E^{\frac{1}{2}}-h^{\frac{1}{2}}\right)x_j\right\rangle=\frac{4\sqrt{2}}{\pi}\int_0^\infty\frac{t^4}{1+2\left\langle v,h(h^2+t^4)^{-1}v\right\rangle}\frac{\sqrt{\lambda_i}\left\langle x_i,v\right\rangle}{\lambda_i^2+t^4}\frac{\sqrt{\lambda_j}\left\langle v,x_j\right\rangle}{\lambda_j^2+t^4}dt \tag{A.18}$$

$$\left\langle x_i,\left(h^{\frac{1}{2}}-E^{-\frac{1}{2}}\right)x_j\right\rangle=\frac{4\sqrt{2}}{\pi}\int_0^\infty\frac{t^4}{1+2\left\langle v,h^{-1}(h^{-2}+t^4)^{-1}v\right\rangle t^4}\frac{\lambda_i^{-\frac{3}{2}}\left\langle x_i,v\right\rangle}{\lambda_i^{-2}+t^4}\frac{\lambda_j^{-\frac{3}{2}}\left\langle v,x_j\right\rangle}{\lambda_j^{-2}+t^4}dt$$

and the estimates now follow from the fact that

$$0\leq\left\langle v,h\left(h^2+t^4\right)^{-1}v\right\rangle,\left\langle v,h^{-1}\left(h^{-2}+t^4\right)^{-1}v\right\rangle t^4\leq\left\langle v,h^{-1}v\right\rangle \tag{A.19}$$

for all $t\geq 0$, as well as the integral identities (for $a,b>0$)

$$\int_0^\infty\frac{a^{-\frac{3}{2}}}{a^{-2}+t^4}\frac{b^{-\frac{3}{2}}}{b^{-2}+t^4}t^4dt=\int_0^\infty\frac{1}{a^2+t^4}\frac{1}{b^2+t^4}t^4dt=\frac{\pi}{2\sqrt{2}}\frac{1}{\sqrt{a}+\sqrt{b}}\frac{1}{a+b}. \tag{A.20}$$

□

This leads to the following bounds for C and S:

Proposition A.4. *For all $1\leq i,j\leq n$ it holds that*

$$\left|\left\langle x_i,(C-1)x_j\right\rangle\right|,\left|\left\langle x_i,Sx_j\right\rangle\right|\leq\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}$$

$$\left|\left\langle x_i,Sx_j\right\rangle-\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\right|\leq 2\left\langle v,h^{-1}v\right\rangle\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}.$$

Proof: From the definition of C we have

$$\left\langle x_i,(C-1)x_j\right\rangle=\frac{1}{2}\left(\lambda_i^{-\frac{1}{2}}\left\langle x_i,\left(E^{\frac{1}{2}}-h^{\frac{1}{2}}\right)x_j\right\rangle-\lambda_i^{\frac{1}{2}}\left\langle x_i,\left(h^{-\frac{1}{2}}-E^{-\frac{1}{2}}\right)x_i\right\rangle\right) \tag{A.21}$$

and by the proposition

$$0\leq\lambda_i^{-\frac{1}{2}}\left\langle x_i,\left(E^{\frac{1}{2}}-h^{\frac{1}{2}}\right)x_j\right\rangle\leq\frac{2\sqrt{\lambda_j}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\leq 2\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j} \tag{A.22}$$

$$0\leq\lambda_i^{\frac{1}{2}}\left\langle x_i,\left(h^{-\frac{1}{2}}-E^{-\frac{1}{2}}\right)x_i\right\rangle\leq\frac{2\sqrt{\lambda_i}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\leq 2\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}$$

whence the claim for $C-1$. For S we have

$$\begin{aligned}\left\langle x_i,Sx_j\right\rangle&=\frac{1}{2}\left(\lambda_i^{-\frac{1}{2}}\left\langle x_i,\left(E^{\frac{1}{2}}-h^{\frac{1}{2}}\right)x_j\right\rangle+\lambda_i^{\frac{1}{2}}\left\langle x_i,\left(h^{-\frac{1}{2}}-E^{-\frac{1}{2}}\right)x_i\right\rangle\right)\\&\leq\frac{1}{2}\left(\frac{2\sqrt{\lambda_j}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}+\frac{2\sqrt{\lambda_i}}{\sqrt{\lambda_i}+\sqrt{\lambda_j}}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\right)=\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\end{aligned} \tag{A.23}$$

hence the general bound for S, and also

$$\left\langle x_i,Sx_j\right\rangle\geq\frac{1}{1+2\left\langle v,h^{-1}v\right\rangle}\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j} \tag{A.24}$$

whence

$$\left|\left\langle x_i,Sx_j\right\rangle-\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\right|\leq\left(1-\frac{1}{1+2\left\langle v,h^{-1}v\right\rangle}\right)\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}\leq 2\left\langle v,h^{-1}v\right\rangle\frac{\left\langle x_i,v\right\rangle\left\langle v,x_j\right\rangle}{\lambda_i+\lambda_j}. \tag{A.25}$$

□

Proposition 3.10 now follows by the substitutions $\lambda_i\to\lambda_{k,p}$, $\left\langle x_i,v\right\rangle\to\sqrt{\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}}$ and using that

$$\left\langle v_k,h_k^{-1}v_k\right\rangle=\frac{\hat{V}_k k_F^{-\beta}}{2(2\pi)^3}\sum_{p\in L_k}\lambda_{k,p}^{-1}\leq C\hat{V}_k k_F^{1-\beta}. \tag{A.26}$$

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