DIMENSION-FREE ESTIMATES FOR POSITIVITY-PRESERVING RIESZ TRANSFORMS RELATED TO SCHRÖDINGER OPERATORS WITH CERTAIN POTENTIALS

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Abstract. We study the $L^\infty(\mathbb{R}^d)$ boundedness for Riesz transforms of the form $V^a(-\frac{1}{2}\Delta+V)^{-a}$, where $a>0$ and V is a non-negative potential with power growth acting independently on each coordinate. We factorize the semigroup e^{-tL} into one-dimensional factors, estimate them separately and combine the results to estimate the original semigroup. Similar results with additional assumption $a\leqslant 1$ are obtained on $L^1(\mathbb{R}^d)$.

1. Introduction

In this paper we consider the Schrödinger operator L on $\mathbb{R}^d$ given by

$$L=-\frac{1}{2}\Delta+V,$$

with V being a non-negative potential, and the associated Riesz transform

$$R_V^a f(x)=V(x)^a L^{-a} f(x)=\frac{V(x)^a}{\Gamma(a)}\int_0^\infty e^{-tL}f(x)\,t^{a-1}\,dt,\quad a>0. \tag{1.1}$$

Such Riesz transforms related to Schrödinger operators have been studied by numerous authors, see [1, 2, 3, 4, 13, 14]. For general $V\in L^2_{\mathrm{loc}}$ Sikora proved in [13, Theorem 1] that $R_V^{1/2}$ is bounded on L^p for $1<p\leqslant 2$ (in fact the result applies not only to Riesz transforms on $\mathbb{R}^d$ but also on more general doubling spaces), it is also well known that R_V^1 is bounded on L^1 with norm estimated by 1, see for example [5], [8, Lemma 6] and [2, Theorem 4.3]. When the potential V belongs to the reverse Hölder class B_q for some $q\geqslant\frac{d}{2}$, then it is known, see [12, Theorem 5.10], that R_V^1 is bounded on L^1. There are also two results regarding polynomial potentials, namely Dziubański [3, Theorem 4.5] proved that R_V^a, $a>0$, is bounded on L^1 and L^∞ if V is a polynomial and then Urban and Zienkiewicz proved in [14, Theorem 1.1] that R_V^1 is bounded on L^∞ independently of the dimension for V being a polynomial satisfying a certain condition of C. Fefferman. Recently it has been proved in [10] that R_V^a is bounded on L^p with $0\leqslant a\leqslant 1/p$ and $1<p\leqslant 2$ for general $V\in L^1_{\mathrm{loc}}$ and that R_V^a, $a>0$, is bounded on L^1 and L^∞ if the potential V has polynomial or exponential growth.

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Obtaining dimension-free bounds for the Riesz transforms related to Schrödinger operators seems to be a significantly harder task. The only available results are the aforementioned paper by Urban and Zienkiewicz [14], the well-known bound for R_V^1 for general potentials and a result regarding a particular case of $R_V^{1/2}$ with $V(x) = |x|^2$, see [6, 11, 9]. Our goal is to extend these dimension-free results and get L^∞ bounds for R_V^a with $a > 0$ and L^1 bounds for R_V^a with $a \leqslant 1$ when the potential V is of the form

$$V(x) = V_1(x) + \cdots + V_d(x), \tag{1.2}$$

where each V_i acts only on the i-th coordinate of the argument x and has polynomial growth with the exponent not greater than 2, i.e. there are absolute constants m and M such that

$$m|x_i|^\alpha \leqslant V_i(x) \leqslant M|x_i|^\alpha \tag{1.3}$$

for some $0 < \alpha \leqslant 2$. This holds for example if $V_i(x) = x_i^2$ and $V(x) = |x|^2$, which results in the operator $L = -\frac{1}{2}\Delta + |x|^2$ called the harmonic oscillator. The reason why we can only handle $\alpha \leqslant 2$ is related to the distribution of the Brownian motion, which arises in the Feynman–Kac formula (2.3), and is visible in (3.10).

By the definition (1.1) of R_V^a and the positivity-preserving property of the semigroup e^{-tL} obtaining the L^∞ bounds for R_V^a amounts to estimating the value of $R_V^a(\mathbb{1})(x)$ independently of x and d, which in turn hints that the main part of the proof is estimating the semigroup applied to the constant function 1, i.e. $e^{-tL}(\mathbb{1})$. The particular structure of V (1.2) lets us write

$$L = \sum_{i=1}^d L_i, \quad \text{where } L_i = -\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + V_i, \tag{1.4}$$

and, as a consequence, factorize the semigroup e^{-tL} in the following way

$$e^{-tL} = \prod_{i=1}^d e^{-tL_i} \quad \text{and hence} \quad e^{-tL}(\mathbb{1})(x) = \prod_{i=1}^d e^{-tL_i}(\mathbb{1})(x). \tag{1.5}$$

This is the key property allowing us to get estimates that does not depend on the dimension d.

The main result of the paper is the following theorem.

Theorem 1.1. *Fix $0 < \alpha \leqslant 2$ and let V given by (1.2) satisfy (1.3). For $a > 0$ let the Riesz transform R_V^a be defined as in (1.1). Then there is a constant $C > 0$ depending on m, M, and α and independent of the dimension d such that*

$$\|R_V^a f\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|f\|_{L^\infty(\mathbb{R}^d)}, \quad f \in L^\infty(\mathbb{R}^d).$$

As a by-product of our considerations we also obtain L^1 estimates for R_V^a, but only for a limited range of a. The reason for this is that we need to use concavity of the function x^a.

Theorem 1.2. *Fix $0 < \alpha \leqslant 2$ and let V given by (1.2) satisfy (1.3). For $0 < a \leqslant 1$ let the Riesz transform R_V^a be defined as in (1.1). Then there is a constant $C > 0$ depending on m, M, and α and independent of the dimension d such that*

$$\|R_V^a f\|_{L^1(\mathbb{R}^d)} \leqslant C\|f\|_{L^1(\mathbb{R}^d)}, \quad f \in L^1(\mathbb{R}^d).$$

Remark. For technical reasons we will assume that $d \geqslant 3$. The case of $d = 1, 2$ follows from previous results, e.g. [10].

1.1. **Structure and methods.** The main part of the proof is contained in Section 3 where we prove that the one-dimensional semigroups e^{-tL_i} decay exponentially in t and $V(x)$ for small values of t, i.e. we have

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c_N tV_i(x)} \quad \text{for } t \leqslant N.$$

It is noteworthy that the constant in front of the exponential in the above estimate is 1, which means that we can multiply one-dimensional bounds to estimate the full semigroup e^{-tL} without constants growing with the dimension. The proof is divided into three cases depending on the value of $|x_i|$ and $tV_i(x)$ but in all of them the main ingredient is the Feynman–Kac formula (2.3).

In Section 4 we use results from Section 3 and a similar result [10, Lemma 4.1] giving an exponential decay of the semigroup for large values of t, namely

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-ct} \quad \text{for } t \geqslant N,$$

to estimate the L^∞ norm of R^a_V.

Finally in Section 5 we estimate the L^1 norm of the Riesz transform. We use duality between L^∞ and L^1 which reduces estimating the L^1 norm of the operator $R^a_V = V^a L^{-a}$ to estimating the L^∞ norm of the adjoint operator

$$(L^{-a}V^a)f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a f)(x)\, t^{a-1}\, dt.$$

Again, using the positivity-preserving property of the semigroup e^{-tL} reduces the task to estimating $e^{-tL}(V^a)$. In this case, although the factorization (1.5) of the semigroup as an operator still applies, it does not behave well when the semigroup is applied to V^a instead of the constant function, so we use the following formula

$$e^{-tL}(V) = \sum_{i=1}^d e^{-tL}(V_i) = \sum_{i=1}^d e^{-tL^i}(\mathbb{1})\, e^{-tL_i}(V_i), \quad \text{where } L^i = L - L_i.$$

1.2. **Notation.** We conclude the introduction by establishing some useful notation used throughout the paper.

(1) We abbreviate $L^p(\mathbb{R}^d)$ to L^p and $\|\cdot\|_{L^p}$ to $\|\cdot\|_p$. For a linear operator T acting on L^p we denote its norm by $\|T\|_{p\to p}$.
(2) By $\mathbb{1}$ we denote the constant function 1 and by $\mathbb{1}_X$ we denote the characteristic function of the set X.
(3) The space of smooth compactly supported functions on $\mathbb{R}^d$ is denoted by C_c^∞.
(4) For two quantities A and B we write $A \lesssim B$ if $A \leqslant CB$ for some constant $C > 0$ which may depend on m, M and α and is independent of the dimension d. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.
(5) For $x \in \mathbb{R}^d$ we denote its components by $x_1, \dots, x_d$, i.e. $x = (x_1, \dots, x_d)$.
(6) For a random variable X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $A \subseteq \mathbb{R}$ we denote $\mathbb{P}(X \in A) \coloneqq \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\})$.

2. Definitions

We begin by defining the semigroup e^{-tL} and then we proceed to defining the Riesz transform R_V^a. By the result of Kato [7, p. 137] the operator $L = -\frac{1}{2}\Delta + V$ is essentially self-adjoint on C_c^∞ and hence it has a non-negative self-adjoint extension. This in turn means that L generates a strongly continuous semigroup of contractions on L^2 which can be expressed using the Feynman–Kac formula

$$e^{-tL}f(x) = \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds} f(X_t)\right], \quad f \in L^2, \tag{2.1}$$

where the expectation $\mathbb{E}_x$ is taken with respect to the Wiener measure of the standard d-dimensional Brownian motion $\{X_s\}_{s>0}$ starting at $x \in \mathbb{R}^d$; here $X_s = (X_s^1, \ldots, X_s^d)$. Since the right-hand makes sense also for $f \in L^\infty$, we use the Feynman–Kac formula to define e^{-tL} acting on L^∞ as

$$e^{-tL}f(x) \coloneqq \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds} f(X_t)\right], \quad f \in L^\infty. \tag{2.2}$$

Similarly, using the fact that V, and hence L, act on each coordinate separately, see (1.2) and (1.4), we define one-dimensional semigroups e^{-tL_i}, $i = 1, \ldots, d$, as follows

$$e^{-tL_i}f(x) \coloneqq \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds} f_{x^i}(X_t^i)\right], \quad f \in L^\infty, \tag{2.3}$$

where

$$f_{x^i}(y) = f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_d).$$

Here the expectation $\mathbb{E}_{x_i}$ is taken with regards to the Wiener measure of the standard one-dimensional Brownian motion $\{X_s^i\}_{s>0}$ starting at $x_i \in \mathbb{R}$.

As the next lemma shows, this is the definition that suits best our purpose of factorizing the semigroup e^{-tL} into one-dimensional factors e^{-tL_i}.

Lemma 2.1. *Fix d and let the d-dimensional semigroup e^{-tL} be given by* (2.2) *and the one-dimensional semigroup e^{-tL_i} by* (2.3)*. Then for $f \in L^\infty$ we have*

$$e^{-tL}f(x) = \left(\left(\prod_{i=1}^d e^{-tL_i}\right) f\right)(x) \quad \text{and} \quad e^{-tL}(\mathbb{1})(x) = \prod_{i=1}^d \left(e^{-tL_i}(\mathbb{1})(x)\right). \tag{2.4}$$

Proof. We will prove by induction that for $k = 1, \ldots, d$ we have

$$\left(\left(\prod_{i=1}^k e^{-tL_i}\right) f\right)(x) = \mathbb{E}_{(x_1,\ldots,x_k)}\left[e^{-\int_0^t \sum_{i=1}^k V_i(X_s)\,ds} f(X_t^1, \ldots, X_t^k, x_{k+1}, \ldots, x_d)\right], \tag{2.5}$$

which justifies the first formula in (2.4) if we take $k = d$.

The case $k = 1$ is clear from the definition (2.3) of e^{-tL_1}. Now suppose that (2.5) holds. Then

$$\begin{aligned}
\left(\left(\prod_{i=1}^{k+1} e^{-tL_i}\right) f\right)(x) &= \mathbb{E}_{x_{k+1}}\left[e^{-\int_0^t V_{k+1}(X_s)\,ds} \left(\left(\prod_{i=1}^k e^{-tL_i}\right) f\right)_{x^{k+1}} (X_t^{k+1})\right] \\
&= \mathbb{E}_{x_{k+1}}\left[e^{-\int_0^t V_{k+1}(X_s)\,ds}\, \mathbb{E}_{(x_1,\ldots,x_k)}\left[e^{-\int_0^t \sum_{i=1}^k V_i(X_s)\,ds} f(X_t^1, \ldots, X_t^k, X_t^{k+1}, x_{k+2}, \ldots, x_d)\right]\right] \\
&= \mathbb{E}_{(x_1,\ldots,x_{k+1})}\left[e^{-\int_0^t \sum_{i=1}^{k+1} V_i(X_s)\,ds} f(X_t^1, \ldots, X_t^{k+1}, x_{k+2}, \ldots, x_d)\right].
\end{aligned}$$

Note that we can use the same Brownian motion in the inner and in the outer expected value since its coordinates are independent of each other and $V_i(X_s)$ depends only on X_s^i.

The second formula in (2.4) follows from the definitions of e^{-tL} and e^{-tL_i} and the fact that the coordinates of d-dimensional Brownian motion are independent. □

Now we take $a>0$ and a non-negative function $f\in L^\infty$ and define the Riesz transform

$$R_V^a f(x) = \frac{V(x)^a}{\Gamma(a)}\int_0^\infty e^{-tL} f(x)\, t^{a-1}\, dt, \tag{2.6}$$

where $e^{-tL}f(x)$ is defined as in (2.2). Lastly, we use the positivity-preserving property of the semigroup e^{-tL}, which means that $e^{-tL}f\geqslant 0$ whenever $f\geqslant 0$, to rewrite the main theorem in a simpler form. Namely, we have

$$\left|e^{-tL}f(x)\right| \leqslant e^{-tL}\left(\|f\|_\infty \mathbb{1}\right)(x) = \|f\|_\infty\, e^{-tL}(\mathbb{1})(x), \quad f\in L^\infty,$$

which means that the Riesz transform R_V^a is bounded on L^∞ if

$$\|R_V^a(\mathbb{1})\| < \infty$$

with its norm being

$$\|R_V^a\|_{\infty\to\infty} = \|R_V^a(\mathbb{1})\|_\infty.$$

Thus, Theorem 1.1 can be rewritten as

Theorem 2.2. *Fix $0<\alpha\leqslant 2$ and let V given by (1.2) satisfy (1.3). For $a>0$ let the Riesz transform R_V^a be defined as in (2.6). Then there is a constant $C>0$ depending on m, M, and α and independent of the dimension d such that*

$$\|R_V^a(\mathbb{1})\|_\infty \leqslant C.$$

3. One-dimensional estimates

In this section we prove the aforementioned exponential decay of the one-dimensional semigroup which we will then combine to estimate the semigroup e^{-tL}.

Lemma 3.1. *For every $N>0$ there is a constant $c_N>0$ such that*

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c_N t V_i(x)} \tag{3.1}$$

for all $x\in\mathbb{R}^d$ and $0\leqslant t\leqslant N$. Moreover, if $|x_i|\leqslant 4$, then

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c_N\left(t^{\frac{\alpha}{2}+1}+tV_i(x)\right)}, \quad t\leqslant N. \tag{3.2}$$

Proof. First we will show that (3.1) is satisfied for $0\leqslant t\leqslant t_0$ for some t_0 and then we will extend the estimate to all $0\leqslant t\leqslant N$.

We begin with the case $|x_i|\leqslant 4$. We will make use of the inequality

$$e^{-x}\leqslant 1-x+\frac{x^2}{2}, \qquad x\geqslant 0. \tag{3.3}$$

The Feynman–Kac formula (2.3) together with (3.3) give

$$e^{-tL_i}(\mathbb{1})(x) \leqslant 1-\mathbb{E}_{x_i}\left[\int_0^t V_i(X_s)\,ds\right]+\frac{1}{2}\mathbb{E}_{x_i}\left[\left(\int_0^t V_i(X_s)\,ds\right)^2\right]. \tag{3.4}$$

We need to estimate the first and the second expected value in the expression above. In order to do this we will use the fact that for any $a, b \geqslant 0$ and $\alpha > 0$ we have

$$(a+b)^\alpha \approx a^\alpha + b^\alpha, \tag{3.5}$$

and an estimate for the moments of normal distribution

$$\mathbb{E}\left|X_s^i\right|^\alpha \approx s^{\alpha/2}. \tag{3.6}$$

Let us begin by estimating $\mathbb{E}_{x_i} V_i(X_s)$ from below and assume without loss of generality that $x_i \geqslant 0$.

$$\mathbb{E}_{x_i} V_i(X_s) \gtrsim \mathbb{E}_0\left|X_s^i + x_i\right|^\alpha \geqslant \mathbb{E}_0\left[\mathbb{1}_{\{X_s^i \geqslant 0\}}(X_s^i + x_i)^\alpha\right] \approx s^{\alpha/2} + x_i^\alpha$$

Integrating this gives

$$\mathbb{E}_{x_i}\left[\int_0^t V_i(X_s)\, ds\right] \gtrsim t^{\frac{\alpha}{2}+1} + t x_i^\alpha.$$

Now we estimate the last term in (3.4) using Cauchy–Schwarz inequality.

$$\begin{aligned}\mathbb{E}_{x_i}\left[\left(\int_0^t V_i(X_s)\, ds\right)^2\right] &\lesssim t\int_0^t \mathbb{E}_0\left[\left|X_s^i + x_i\right|^{2\alpha}\right] ds \lesssim t\int_0^t \mathbb{E}_0\left[\left|X_s^i\right|^{2\alpha} + x_i^{2\alpha}\right] ds \\ &\approx t\int_0^t s^\alpha + x_i^{2\alpha} ds \approx t^{\alpha+2} + t^2 x_i^{2\alpha} \lesssim \left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right)^2.\end{aligned}$$

Plugging this into (3.4), recalling that $|x_i| \leqslant 4$, and choosing t_0 sufficiently small yields

$$\begin{aligned} e^{-tL_i}(\mathbb{1})(x) &\leqslant 1 - c_1\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right) + c_2\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right)^2 \\ &\leqslant 1 - c\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right) \leqslant e^{-c\left(t^{\frac{\alpha}{2}+1} + t x_i^\alpha\right)}\end{aligned}$$

which implies (3.1) and (3.2) for $t \leqslant t_0$.

The second case is when $|x_i| > 4$ and $tV_i(x) \leqslant 2A\log 5$, where $A = \frac{2^\alpha M}{m}$ with m and M as in (1.3). We will roughly show that then we have

$$\frac{d}{dt} e^{-tL_i}(\mathbb{1})(x) = -e^{-tL_i}(V_i)(x) \leqslant -cV_i(x). \tag{3.7}$$

However since the equality may not hold, we replace V_i with $V_i^n(x) = \min(V_i(x), n)$ for any $n > 0$, establish (3.7) for V_i^n, then we prove (3.1) for V_i^n and finally we deduce (3.1) for V_i.

Recall that V_i satisfies $m|x_i|^\alpha \leqslant V_i(x) \leqslant M|x_i|^\alpha$ and take $x_i, y_i \in \mathbb{R}$ such that $|x_i - y_i| \leqslant \frac{|x_i|}{2}$. Then $\frac{|x_i|}{2} \leqslant |y_i| \leqslant 2|x_i|$ so that we have

$$V_i(y) \leqslant M|y_i|^\alpha \leqslant 2^\alpha M|x_i|^\alpha \leqslant \frac{2^\alpha M}{m} V_i(x) = AV_i(x)$$

and

$$V_i(y) \geqslant m|y_i|^\alpha \geqslant m\frac{|x_i|^\alpha}{2^\alpha} \geqslant \frac{m}{2^\alpha M} V_i(x) = \frac{1}{A} V_i(x)$$

We also calculate the probability that $\sup_{0 \leqslant s \leqslant t}\left|X_s^i - x_i\right| \geqslant \frac{|x_i|}{2}$ using the reflection principle to get

$$\mathbb{P}\left(\sup_{0 \leqslant s \leqslant t}\left|X_s^i - x_i\right| \geqslant \frac{|x_i|}{2}\right) \leqslant 4e^{-\frac{|x_i|^2}{8t}}. \tag{3.8}$$

Now, for $n > 0$, we define $V_i^n(x) = \min(V_i(x), n)$ and $L_i^n = -\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + V_i^n$ and use the Feynman–Kac formula and (3.8) to get

$$\begin{aligned}
e^{-tL_i^n}(V_i^n)(x) &= \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i^n(X_s)\,ds} V_i^n(X_t)\right] \geqslant \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds} V_i^n(X_t)\right] \\
&\geqslant \mathbb{P}\left(\forall_{0\leqslant s\leqslant t}\ \tfrac{V_i^n(x)}{A} \leqslant V_i^n(X_s) \text{ and } V_i(X_s) \leqslant AV_i(x)\right) \frac{V_i^n(x)}{A} e^{-AtV_i(x)} \\
&\geqslant \mathbb{P}\left(\forall_{0\leqslant s\leqslant t}\ \tfrac{V_i(x)}{A} \leqslant V_i(X_s) \leqslant AV_i(x)\right) \frac{V_i^n(x)}{A} e^{-AtV_i(x)} \\
&\geqslant \frac{V_i^n(x)}{A}\left(1 - 8e^{-\frac{|x_i|^2}{8t}}\right) e^{-2A^2\log 5} \\
&\geqslant \frac{V_i^n(x)}{A}\left(1 - 8e^{-\frac{4^2}{8t_0}}\right) e^{-2A^2\log 5} \geqslant cV_i^n(x)
\end{aligned}$$

if t_0 is sufficiently small, which proves that

$$\frac{d}{dt} e^{-tL_i^n}(\mathbb{1})(x) = -e^{-tL_i^n}(V_i^n)(x) \leqslant -cV_i^n(x). \tag{3.9}$$

Differentiation is allowed here by the Leibniz integral rule. Now we show that this implies a version of (3.1) with V_i^n. Consider the function

$$f(t) = e^{-tL_i^n}(\mathbb{1})(x)\, e^{ctV_i^n(x)}.$$

If we differentiate it and use (3.9), we get

$$\begin{aligned}
f'(t) &= \frac{d}{dt} e^{-tL_i^n}(\mathbb{1})(x)\, e^{ctV_i^n(x)} + cV_i^n(x)\, e^{-tL_i^n}(\mathbb{1})(x)\, e^{ctV_i^n(x)} \\
&\leqslant -cV_i^n(x)\, e^{ctV_i^n(x)} + cV_i^n(x)\, e^{ctV_i^n(x)} = 0.
\end{aligned}$$

Since $f(0) = 1$, we conclude that

$$e^{-tL_i^n}(\mathbb{1})(x) \leqslant e^{-ctV_i^n(x)}.$$

Now we take the limit as n goes to infinity on both sides of the inequality. The left-hand side becomes

$$\lim_{n\to\infty} e^{-tL_i^n}(\mathbb{1})(x) = \lim_{n\to\infty} \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i^n(X_s)\,ds}\right] = \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds}\right] = e^{-tL_i}(\mathbb{1})(x).$$

Passing with the limit under the integral sign is allowed since the integrand is dominated by the constant function $\mathbb{1}$ which is integrable. On the right-hand side we get

$$\lim_{n\to\infty} e^{-ctV_i^n(x)} = e^{-ctV_i(x)},$$

so altogether we get (3.1).

The last case to consider is $|x_i| > 4$ and $tV_i(x) > 2A\log 5$. We choose sufficiently small t_0 and use (3.8) to obtain

$$\begin{aligned}
e^{-tL_i}(\mathbb{1})(x) &\leqslant e^{-\frac{tV_i(x)}{A}} \mathbb{P}\left(\forall_{0\leqslant s\leqslant t}\ \tfrac{V_i(x)}{A} \leqslant V_i(X_s)\right) + 1\cdot \mathbb{P}\left(\exists_{0\leqslant s\leqslant t}\ \tfrac{V_i(x)}{A} > V_i(X_s)\right) \\
&\leqslant e^{-\frac{tV_i(x)}{A}} + 4e^{-\frac{|x_i|^2}{8t}} \leqslant 5e^{-\frac{tV_i(x)}{A}} \leqslant e^{-\frac{tV_i(x)}{2A}},
\end{aligned} \tag{3.10}$$

which is (3.1). In the second-to-last inequality we used the assumption $\alpha \leqslant 2$.

Recall that we have just proved that

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-ctV_i(x)}$$

is satisfied for $t \leqslant t_0$ and $x \in \mathbb{R}^d$. If $N \leqslant t_0$, then the proof is finished, so suppose that $N > t_0$ and take $t \in [t_0, N]$. Then we have

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-t_0L_i}(\mathbb{1})(x) \leqslant e^{-ct_0V_i(x)} = e^{-c\frac{t_0}{t}tV_i(x)} \leqslant e^{-c\frac{t_0}{N}tV_i(x)} = e^{-c_NtV_i(x)}.$$

The inequality (3.2) can be extended to $t \in [0, N]$ in a very similar way. Suppose that $N > t$ and take $t \in [t_0, N]$. Then

$$e^{-tL_i}(\mathbb{1})(x) \leqslant e^{-c\left(t_0^{\frac{\alpha}{2}+1}+t_0V_i(x)\right)} \leqslant e^{-c\left(\left(\frac{t_0}{N}\right)^{\frac{\alpha}{2}+1}t^{\frac{\alpha}{2}+1}+\frac{t_0}{N}tV_i(x)\right)} = e^{-c_N\left(t^{\frac{\alpha}{2}+1}+tV_i(x)\right)}.$$

This finishes the proof. □

4. L^∞ dimension-free estimates — proof of Theorem 2.2

In this section we prove Theorem 2.2 using one-dimensional estimates from Lemma 3.1. The other relevant result is [10, Lemma 4.1], which guarantees that there exist universal constants $C > 0$ and $\delta' > 0$ such that

$$e^{-tL_i}(\mathbb{1})(x) \leqslant Ce^{-\delta' t}$$

for all $i = 1, \ldots, d$ and $x \in \mathbb{R}^d$. This in turn means, thanks to (2.4), that we have

$$e^{-tL}(\mathbb{1})(x) \leqslant e^{-d\delta t} \tag{4.1}$$

for $x \in \mathbb{R}^d$ and $t \geqslant N$, where $N > 0$ and $\delta > 0$ are universal constants.

First we estimate the upper part of the integral in (2.6), i.e. the integral from N to ∞, dividing the calculations into two cases depending on the value of a. If $a < 1$, then

$$\begin{aligned} V(x)^a \int_N^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt &\leqslant V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \int_N^\infty e^{-\frac{t}{2}\delta d}\, t^{a-1}\, dt \\ &\lesssim \frac{N^{a-1}}{\delta d} V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \lesssim \frac{1}{d}\sum_{i=1}^d V_i(x)^a\, e^{-\frac{N}{2}L_i}(\mathbb{1})(x). \end{aligned}$$

In the last inequality we used the fact that

$$(x_1 + \cdots + x_d)^a \leqslant x_1^a + \ldots x_d^a$$

for $a \leqslant 1$ and $x_i \geqslant 0$.

If, on the other hand, $a \geqslant 1$, then

$$\begin{aligned} V(x)^a \int_N^\infty e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt &\leqslant V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \int_N^\infty e^{-\frac{t}{2}\delta d}\, t^{a-1}\, dt \\ &\lesssim \frac{1}{(\delta d)^a} V(x)^a\, e^{-\frac{N}{2}L}(\mathbb{1})(x) \lesssim \frac{1}{d}\sum_{i=1}^d V_i(x)^a\, e^{-\frac{N}{2}L_i}(\mathbb{1})(x). \end{aligned}$$

Here in the last inequality we used that fact that

$$(x_1 + \cdots + x_d)^a \leqslant d^{a-1}\,(x_1^a + \cdots + x_d^a)$$

for $a \geqslant 1$ and $x_i \geqslant 0$, which follows from Jensen's inequality or Hölder's inequality. Thus, we have reduced our problem to the one-dimensional case of estimating $V_i^a e^{-\frac{N}{2}L_i}(\mathbb{1})$ which can be done by invoking (3.1), namely

$$V_i(x)^a\, e^{-\frac{N}{2}L_i}(\mathbb{1})(x) \leqslant V_i(x)^a\, e^{-\frac{N}{2}c_N V_i(x)} \leqslant \left(\frac{2a}{Nc_N e}\right)^a \tag{4.2}$$

Then we handle the lower part of the integral in (2.6). We estimate $e^{-tL}(\mathbb{1})(x)$ for $t \leqslant N$ independently of x and d by using (3.1) and the factorization property (2.4), which gives

$$e^{-tL}(\mathbb{1})(x) \leqslant e^{-c_N tV(x)},$$

and then integrate

$$V(x)^a \int_0^N e^{-tL}(\mathbb{1})(x)\, t^{a-1}\, dt \leqslant V(x)^a \int_0^\infty e^{-c_N tV(x)}\, t^{a-1}\, dt \lesssim c_N^{-a}.$$

This completes the proof of Theorem 2.2.

5. L^1 DIMENSION-FREE ESTIMATES

In this section we will again use the one-dimensional estimates for the semigroups e^{-tL_i} to prove dimension-free estimates of the L^1 norm of R_V^a for $0 < a \leqslant 1$. The idea is to estimate the L^∞ norm of the adjoint operator formally given by

$$(L^{-a}V^a)f(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a f)(x)\, t^{a-1}\, dt.$$

As before, the positivity-preserving property of e^{-tL} lets us reduce the task to estimating the L^∞ norm of

$$L^{-a}(V^a)(x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt. \tag{5.1}$$

However, since V^a may be unbounded, it is not clear if the integral above is a measurable function of x. The issue was addressed in detail in [10, Section 5]. Briefly, we define

$$L^{-a}(V^a)(x) := \lim_{N\to\infty} \frac{1}{\Gamma(a)} \int_0^\infty e^{-tL}(V^a \mathbb{1}_{|V|<N})(x)\, t^{a-1}\, e^{-t/N}\, dt \tag{5.2}$$

and we see that each integral is finite and measurable by [10, Lemma 3.1], hence the limit is also measurable. Later it will turn out that the integral in (5.1) is finite, which lets us handle (5.1) instead of using (5.2). As in the L^∞ case this lets us reformulate Theorem 1.2 in the following way

Theorem 5.1. *Fix $0 < \alpha \leqslant 2$ and let V given by (1.2) satisfy (1.3). For $0 < a \leqslant 1$ let the Riesz transform R_V^a be defined as in (2.6). Then there is a constant $C > 0$ depending on m, M, and α and independent of the dimension d such that*

$$\left\|L^{-a}(V^a)\right\|_\infty \leqslant C.$$

Before we move to the proof, we need two general results regarding the semigroup e^{-tL}. The first one is a factorization property for $e^{-tL}(V)$

$$e^{-tL}(V) = \sum_{i=1}^d e^{-tL}(V_i) = \sum_{i=1}^d e^{-tL^i}(\mathbb{1})\, e^{-tL_i}(V_i), \quad \text{where } L^i = L - L_i. \tag{5.3}$$

The second one is an estimate for $e^{-tL_i}(V_i^a)$

$$\begin{aligned}
e^{-tL_i}(V_i^a)(x) &= \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds}\, V_i(X_t)^a\right] \\
&\lesssim \mathbb{E}_0\left[e^{-\int_0^t V_i(X_s+x)\,ds}\, V_i(X_t)^a\right] + \mathbb{E}_0\left[e^{-\int_0^t V_i(X_s+x)\,ds}\, V_i(x)^a\right] \\
&\lesssim \mathbb{E}_0\left[V_i(X_t)^a\right] + V_i(x)^a\, \mathbb{E}_{x_i}\left[e^{-\int_0^t V_i(X_s)\,ds}\right] \\
&\lesssim t^{\frac{a\alpha}{2}} + V_i(x)^a\, e^{-tL_i}(\mathbb{1})(x), \qquad (5.4)
\end{aligned}$$

valid for $t>0$ and $x \in \mathbb{R}^d$. Here we used estimate (1.3) for V and (3.5). Now we are in position to prove Theorem 5.1

Proof of Theorem 5.1. We begin with the upper part of the integral in (5.1), i.e. the integral from N to ∞. Using subadditivity of the function x^a for $a \leqslant 1$, factorization (5.3) and (4.1) we obtain

$$\begin{aligned}
\int_N^\infty e^{-tL}(V^a)(x)\, t^{a-1}\, dt &\leqslant \int_N^\infty \sum_{i=1}^d e^{-tL}(V_i^a)(x)\, t^{a-1}\, dt \\
&\leqslant \int_N^\infty \sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\, e^{-tL_i}(V_i^a)(x)\, t^{a-1}\, dt \\
&\leqslant \int_N^\infty \sum_{i=1}^d e^{-t\delta(d-1)} e^{-tL_i}(V_i^a)(x)\, t^{a-1}\, dt \\
&\leqslant N^{a-1} \sum_{i=1}^d \int_N^\infty \left\|e^{-tL_i}(V_i^a)\right\|_\infty e^{-t\delta(d-1)}\, dt \\
&\lesssim \sum_{i=1}^d \left\|e^{-NL_i}(V_i^a)\right\|_\infty \int_N^\infty e^{-t\delta(d-1)}\, dt \\
&\lesssim \frac{1}{d-1} \sum_{i=1}^d \left\|e^{-NL_i}(V_i^a)\right\|_\infty.
\end{aligned}$$

Then we use (5.4) and (3.1) and we estimate the resulting function similarly to (4.2).

To deal with the lower part we use the inequality

$$e^{-tL}(V^a) \leqslant e^{-tL}(V)^a, \quad a \leqslant 1,$$

which follows from Hölder's inequality. We use this and (5.3) to get

$$\int_0^N e^{-tL}(V^a)(x)\, t^{a-1}\, dt \leqslant \int_0^N \left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\, e^{-tL_i}(V_i)(x)\right)^a t^{a-1}\, dt.$$

Then we use (5.4) and obtain

$$\begin{aligned}\int_0^N e^{-tL}(V^a)(x)\,t^{a-1}\,dt &\lesssim \int_0^N \left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\,\left(t^{\frac{\alpha}{2}}+V_i(x)\,e^{-tL_i}(\mathbb{1})(x)\right)\right)^a t^{a-1}\,dt\\ &= \int_0^N \left(V(x)e^{-tL}(\mathbb{1})(x)+t^{\frac{\alpha}{2}}\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt\\ &\leqslant \int_0^N V(x)^a e^{-tL}(\mathbb{1})(x)^a\,t^{a-1}\,dt+\int_0^N t^{\frac{a\alpha}{2}}\left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt.\end{aligned}$$

To the first integral we apply (3.1) and factorization (2.4), which lets us estimate the first integral by a constant independent of x and the dimension d. To estimate the second integral we fix $x=(x_1,\dots,x_d)$ and divide its coordinates x_j into those whose absolute value is greater than 4 and all others. Say that there are k coordinates greater than 4 and $d-k$ not greater than 4. Then we consider three cases.

First we assume that $k=0$ and apply (3.2) and (2.4) to get

$$\int_0^N t^{\frac{a\alpha}{2}}\left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt \leqslant \int_0^N d^a e^{-a(d-1)c_N t^{\frac{\alpha}{2}+1}}\,t^{\frac{a\alpha}{2}}\,t^{a-1}\,dt \lesssim \frac{d^a}{d^a}=1.$$

In the last inequality we used

$$\int_0^\infty e^{-At^\beta}\,t^\gamma\,dt=\frac{\Gamma\left(\frac{\gamma+1}{\beta}\right)}{\beta A^{\frac{\gamma+1}{\beta}}}, \tag{5.5}$$

with $A=a(d-1)c_N$, $\beta=\frac{\alpha}{2}+1$ and $\gamma=\frac{a\alpha}{2}+a-1$.

Then if $k=d$, we apply (3.1) and (2.4) and use the fact $V_i(x)\geqslant m\cdot 4^\alpha$ which gives

$$\begin{aligned}\int_0^N t^{\frac{a\alpha}{2}}\left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt &\leqslant \int_0^N d^a e^{-4^\alpha m a c_N t(d-1)}\,t^{\frac{a\alpha}{2}}\,t^{a-1}\,dt\\ &\lesssim \int_0^N d^a e^{-4^\alpha m a c_N td}\,t^{a-1}\,dt \lesssim 1.\end{aligned}$$

The third case is when $0<k<d$ in which the estimate is a mixture of the estimates for $k=0$ and $k=d$. Observe that each $(d-1)$-element subsequence of $(x_1,\dots,x_d)$ has at least $k-1$ elements greater than 4 and at least $d-k-1$ elements not greater than 4. By (3.1) and (3.2) this means that

$$\int_0^N t^{\frac{a\alpha}{2}}\left(\sum_{i=1}^d e^{-tL^i}(\mathbb{1})(x)\right)^a t^{a-1}\,dt \leqslant \int_0^N d^a e^{-4^\alpha m a c_N t(k-1)}\,e^{-ac_N(d-k-1)t^{\frac{\alpha}{2}+1}}t^{\frac{a\alpha}{2}}\,t^{a-1}\,dt.$$

Then we use Hölder's inequality with $p=\frac{d-2}{k-1}$ ($p=\infty$ if $k=1$) and $q=\frac{d-2}{d-k-1}$ ($q=\infty$ if $k=d-1$) to the functions $e^{-4^\alpha m a c_N t(k-1)}$ and $e^{-ac_N(d-k-1)t^{\frac{\alpha}{2}+1}}t^{\frac{a\alpha}{2}}$ with respect to the measure $t^{a-1}\,dt$ which yields

$$\begin{aligned}
&d^a \int_0^N e^{-4^\alpha mac_N t(k-1)} \cdot e^{-ac_N(d-k-1)t^{\frac{\alpha}{2}+1}} t^{\frac{a\alpha}{2}} \cdot t^{a-1}\,dt \\
&\quad\lesssim d^a \left(\int_0^N e^{-4^\alpha mac_N t(d-2)}\, t^{a-1}\,dt\right)^{1/p} \left(\int_0^N e^{-ac_N(d-2)t^{\frac{\alpha}{2}+1}} t^{\frac{a\alpha}{2}}\, t^{a-1}\,dt\right)^{1/q} \\
&\quad\lesssim d^a \left(\frac{1}{d^a}\right)^{1/p} \left(\frac{1}{d^a}\right)^{1/q} = 1.
\end{aligned}$$

Again, in the last inequality we used (5.5). □

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