ON THE EXISTENCE OF APPROXIMATE PROBLEMS THAT PRESERVE THE TYPE OF A BIFURCATION POINT OF A NONLINEAR PROBLEM. APPLICATION TO THE STATIONARY NAVIER-STOKES EQUATIONS

CĂTĂLIN - LIVIU BICHIR

ABSTRACT. We consider a nonlinear problem $F(\lambda, u) = 0$ on infinite-dimensional Banach spaces that correspond to the steady-state bifurcation case. In the literature, it is found again a bifurcation point of the approximate problem $F_h(\lambda_h, u_h) = 0$ only in some cases. We prove that, in every situation, given F_h that approximates F, there exists an approximate problem $F_h(\lambda_h, u_h) - \varrho_h = 0$ that has a bifurcation point with the same properties as the bifurcation point of $F(\lambda, u) = 0$. First, we formulate, for a function $\widehat{F}$ defined on general Banach spaces, some sufficient conditions for the existence of an equation that has a bifurcation point of certain type. For the proof of this result, we use some methods from variational analysis, Graves' theorem, one of its consequences and the contraction mapping principle for set-valued mappings. These techniques allow us to prove the existence of a solution with some desired components that equal zero of an overdetermined extended system. We then obtain the existence of a constant (or a function) $\widehat{\varrho}$ so that the equation $\widehat{F}(\lambda, u) - \widehat{\varrho} = 0$ has a bifurcation point of certain type. This equation has $\widehat{F}(\lambda, u) = 0$ as a perturbation. It is also made evident a class of maps C^p - equivalent (right equivalent) at the bifurcation point to $\widehat{F}(\lambda, u) - \widehat{\varrho}$ at the bifurcation point. Then, for the study of the approximation of $F(\lambda, u) = 0$, we give conditions that relate the exact and the approximate functions. As an application of the theorem on general Banach spaces, we formulate conditions in order to obtain the existence of the approximate equation $F_h(\lambda_h, u_h) - \varrho_h = 0$. For example, we consider the finite element approximation of stationary Navier-Stokes equations.

1. INTRODUCTION

For a steady-state bifurcation problem on infinite-dimensional Banach spaces, we study the existence of an approximate problem that has a bifurcation point with the same properties as the bifurcation point of the given nonlinear problem. The problem of bifurcation is present in the analysis of many mathematical models of phenomena from the physical world. Generally, these models are formulated with an equation on Banach spaces,

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infinite or finite-dimensional. Examples of infinite-dimensional problems are from fluid mechanics, solid mechanics, elasticity, nonlinear vibrations, structural analysis, ocean, atmosphere and climate models and so on. As far as the finite-dimensional problems are concerned, examples are from medicine (cardiology, neuroscience), biology, chemistry, economy, etc. In both cases, infinite-dimensional and finite-dimensional, practical computations are necessary. For this purpose, the infinite-dimensional problem must be approximated by a finite-dimensional problem using methods such as finite element method, finite differences method, finite volume method, spectral methods or wavelets. The first problem is named "exact", defined on exact spaces, by an exact equation and it has exact solutions. The second equation is the approximate equation and the related entities are called "approximate".

We consider equations that correspond to the steady-state bifurcation case. We retain the exact equation (1.1) and the hypothesis (1.2) on the bifurcation point from [18], where Crouzeix and Rappaz study the bifurcation problems and their approximations. Usually, the Liapunov - Schmidt method is applied and it is obtained that, locally, around the bifurcation point, the solution set of the equation on Banach spaces is in one-to-one correspondence with the solution set of the classical bifurcation equation. The study of the solutions of the classical bifurcation equation, using singularity theory, is performed in [31, 32]. As the authors of [18] specify, their method, for the exact equation, is equivalent to the Lyapunov-Schmidt method.

Let W and Z be real Banach spaces. Let $m \geq 1$, $p \geq 2$. Let $F : \mathbb{R}^m \times W \to Z$ be a nonlinear function of class C^p. Consider the equation in $(\lambda, u) \in \mathbb{R}^m \times W$

$$F(\lambda, u) = 0\,. \tag{1.1}$$

Assume that (λ_0, u_0) is a solution of (1.1) that satisfies the hypothesis ([18]):

$$\begin{aligned} &D_u F(\lambda_0, u_0) \text{ is a Fredholm operator of } W \text{ onto } Z \text{ with index zero}\,,\\ &\qquad n \geq 1 \text{ and } q \geq 1\,, \end{aligned} \tag{1.2}$$

where $n = dim\ Ker(D_u F(\lambda_0, u_0))$ and $q = codim\ Range(DF(\lambda_0, u_0))$. The solution (λ_0, u_0) is called *a bifurcation point of problem (1.1)*. If (λ_1, u_1) is a solution of (1.1) and $D_u F(\lambda_1, u_1)$ is an isomorphism of W onto Z, then (λ_1, u_1) is *a regular solution (a regular point) of problem (1.1)*.

Let $\breve{W}$, $\breve{Z}$ be some real Banach spaces and let $\breve{F} : \mathbb{R}^m \times \breve{W} \to \breve{Z}$. If a solution $(\breve{\lambda}_0, \breve{u}_0)$ of the equation $\breve{F}(\breve{\lambda}, \breve{u}) = 0$ satisfies hypothesis (1.2), with the same n and q, we say that $(\breve{\lambda}_0, \breve{u}_0)$ is *a bifurcation point of the same type as* (λ_0, u_0).

The above spaces W and Z are both infinite-dimensional or they are both finite-dimensional. If they are infinite-dimensional, then equation (1.1) is approximated by an equation

$$F_h(\lambda_h, u_h) = 0\,, \tag{1.3}$$

where $F_h : \mathbb{R}^m \times W_h \to Z_h$, W_h is a closed subspace of W and Z_h is a closed subspace of Z. W_h and Z_h are both infinite-dimensional spaces or they are both finite-dimensional spaces with $dim\ W_h = dim\ Z_h$. Usually, (1.3) is obtained by finite element method or by the other methods mentioned above [6, 7, 8, 9, 11, 12, 13, 17, 18, 23, 29, 39, 43, 55, 62, 67, 68]. The theoretical Galerkin method is also taken into account.

A question arises with regard to an approximate equation (1.3):

(Q_1) Has the approximate equation (1.3) also a bifurcation point ? In case that it exists, is this point of the same type as (λ_0, u_0)*?*

In case of the simple limit point, the point is generic ([31]) and the approximate problem has a simple limit point (λ_{0h}, u_{0h}) ([12, 18]).

In case of the simple bifurcation point of the problems on Banach spaces, it is found again a bifurcation point of the approximate problem only in some situations. In two particular cases ([13, 14, 18]), which are generic, bifurcation from the trivial branch and symmetry-breaking bifurcation, a simple bifurcation point (λ_{0h}, u_{0h}) of the approximate problem exists (numerical bifurcation) and there is a diffeomorphism between the solution set of the approximate bifurcation equation and a degenerate hyperbola. In the general case ([3, 13, 14, 18, 52]), in the hyperbolic case, the solution set of the approximate equation is composed of two branches that do not intersect. This solution set and the solution set of the approximate bifurcation equation are diffeomorphic to a part of a nondegenerate hyperbola (imperfect numerical bifurcation). In the course of their study, Brezzi, Rappaz and Raviart made evident a perturbed approximate bifurcation equation (equation (3.21), page 11, [13]) of the approximate bifurcation equation. The branches of this perturbed equation intersect transversally in a point. In this general case, Weber [65] propose to calculate an approximation of the exact solution as a component of the solution of an approximate adequate extended system and then, compute two approximate branches that intersect in this point. In this way, bifurcation is not destroyed by approximation.

We mention that the terminology, in [18], is the following: simple bifurcation points are simple bifurcation points (fold bifurcation and cusp bifurcation in [13], transcritical in [17], transcritical and pitchfork (subcritical and supercritical) in [22] and [45]) or double limit points.

We cite, among other references, [17, 25, 28, 31, 33, 45, 63, 66, 69, 70] for a discussion about the genericity of bifurcation points. For the exact stationary Navier - Stokes equations, there exist bifurcation points that are not generic [25, 63]. For the bifurcation of the solutions of the stationary Navier - Stokes equations, we mention [7, 9, 12, 17, 23, 25, 26, 28, 45, 62, 63, 64, 66, 70]. For the approximate case of these equations, studies are performed in, e.g., [12, 17].

Referring to the results mentioned above, from [18], concerning imperfect numerical bifurcation, and to the theory from [31, 32], Georgescu [27] interpreted the approximate equation (1.3) as a perturbation of the exact equation (1.1) on different spaces. She also suggested us [27] that there probably exists a perturbation of (1.3) which has the approximate bifurcation point that we sought at (1.3) in all the situations given by the hypothesis (1.2).

As we have seen, in literature, the above question *(Q_1)* has a positive answer only in some situations. Moreover, (1.3) cannot be used to study the qualitative aspects of (1.1). When we use an approximation method, we expect not only to find some branches of approximate solutions but also information about the qualitative aspects of the exact equation. On the other hand, we expect that (1.3) is the perturbation of an approximate equation that has a bifurcation point, not only of (1.1) (this follows e.g. from (the interpretation of) [31, 32], from the discussion [27] described above and from the interpretation of the approximate bifurcation equation (3.21), page 11, [13]). In order to obtain a positive answer in all the situations, let us replace the question *(Q_1)* by the following question

(Q_2) Does an approximate problem that preserves the type of (λ_0, u_0) *exist in all the situations given by the hypothesis (1.2)? If this approximate problem exists, is the given problem (1.3) a perturbation of this one?*

We prove an affirmative answer to *(Q_2)*. To the best of our knowledge, this approach and the results we prove are new. We do not discuss if the exact bifurcation point is generic or not. We prove that if an exact bifurcation point exists, satisfying the hypothesis (1.2), then, for an approximation method, there exists an approximate equation that has a bifurcation point with the same properties (hypotheses).

To be specific, given a function F_h that approximates F, we prove that there exists ϱ_h such that the equation

$$F_h(\lambda_h, u_h) - \varrho_h = 0\,, \tag{1.4}$$

has a bifurcation point (λ_{0h}, u_{0h}) of the same type as the bifurcation point (λ_0, u_0) of (1.1). ϱ_h is a constant. The usual approximate equation (1.3) is a perturbation of the new approximate equation (1.4). The result for (1.4) is local. Equation (1.4) can be used in order to study the qualitative aspects of (1.1). Moreover, there exists a class of maps C^p - equivalent (right equivalent) at (λ_{0h}, u_{0h}) to $F_h(\lambda_h, u_h) - \varrho_h$ at (λ_{0h}, u_{0h}) and that satisfies the hypothesis (1.2) in (λ_{0h}, u_{0h}). The problem can be formulated as an inverse problem: given F_h, there exists ϱ_h and it must be determined such that (1.4) has a bifurcation point of the same type as the exact equation (1.1). (1.4) approximates (1.1). Not every approximate equation of (1.1) has a bifurcation point. Equation (1.4) is a particular form of (1.3), obtained by replacing $F_h(\lambda_h, u_h)$ with $F_h(\lambda_h, u_h) - \varrho_h$. If (1.1) has two bifurcation points satisfying hypothesis (1.2), it is possible that the corresponding two ϱ_h are not equal. These results do not contradict the present literature results.

The equation (1.4) and the conclusion for this are the consequences of the formulation of two main results that we introduce: (i) Theorem 3.5 about the equivalence between the properties of a bifurcation point that satisfies (1.2) and the existence of the solution of an overdetermined extended system; (ii) Theorem 5.4 where we formulate some sufficient conditions and we establish, on general Banach spaces, the existence of an equation that has a bifurcation point of certain type. The reasoning we use is the following: if the exact problem (1.1) has a bifurcation point, we construct an adequate extended system applying the direct implication of the first theorem. This system is approximated and the proof of the second theorem furnishes an extended system that satisfies the hypotheses of the converse implication of the first theorem. In this way, we obtain (1.4) and the fact that (1.4) has a bifurcation point of the same type as the bifurcation point of (1.1). Practically, we use Theorem 5.4. This second theorem is generally valid and regards not only the approximate equations, but also the exact equations. Theorem 5.4 is a result in its own right. Theorem 5.4 allows us to obtain the existence of a bifurcation problem for which a given problem is a perturbation. Theorem 7.6 and Theorem 7.7 give the affirmative answer to the question *(Q_2)*. We also give some conditions that relate the exact and the approximate functions in Theorem 7.4. In Corollary 9.1, we obtain ϱ in the form of a function of (λ, u), where ϱ is the corresponding form of ϱ_h, from (1.4), in the infinite-dimensional case.

In our approach, the numerical analysis and the numerical experiments must be performed using the inverse problem attached to (1.4) and not, as usual, using (1.3).

Our results can be applied to the particular case of the exact simple bifurcation point of (1.1), in the general case, in the hyperbolic case, studied in [13, 14, 18], mentioned above. Let us consider (1.1), (1.3) and (1.4) in this case. We obtain the approximate equation (1.4) that has a (an approximate) simple bifurcation point in this case. This (1.4) approximates (1.1). The equation (1.3) used in [13, 14, 18] is impractical in order to regain the qualitative aspects of (1.1), as we saw above; the equation (1.4) maintains the qualitative aspects of (1.1). The Liapunov - Schmidt method or the alternate equivalent method of Crouzeix and Rappaz can be applied to (1.4) as to (1.1). There results a (classical) (approximate) bifurcation equation and there is a diffeomorphism between the solution set of this one and a degenerate hyperbola. The solution set of (1.4) is composed of two branches that intersect in the simple bifurcation point. The approximate bifurcation equation from [13, 14, 18] is obtained using mathematical entities related to (1.1); the (approximate) bifurcation equation for (1.4) can be constructed using only mathematical entities related to (1.4). We have three other remarks related to the results from the literature: 1. Recall the perturbed approximate bifurcation equation whose branches intersect transversally in a point (equation (3.21), page 11, [13]), mentioned above, made evident by Brezzi, Rappaz and Raviart. This approximate bifurcation equation is not related to any approximate equation in [13]. We do not perform a study to answer if the approximate equation (1.4) corresponds to this perturbed approximate bifurcation equation, but it seems that this is the case. 2. We can interpret that the approximation of the exact solution calculated by Weber [65], cited above, is the solution of an approximate equation of the form (1.4). 3. In each of the generic cases of simple limit point, bifurcation from the trivial branch and symmetry-breaking bifurcation, in [12, 13, 14, 18], an estimate $|\lambda_{0h} - \lambda_0|$ is given and it is not proven that λ_{0h} equals λ_0. The bifurcation point (λ_{0h}, u_{0h}) of (1.4) has also this limitation.

The results can be applied in nonlinear functional analysis (bifurcation theory, nonlinear Fredholm operators), singularity theory, analysis on manifolds, modelling, hydrodynamic stability and bifurcation, solid mechanics, PDEs, other mathematical models where bifurcation is present, infinite-dimensional and finite-dimensional dynamical systems, numerical methods.

Let us observe that if some numerical algorithms are implemented in order to determine the bifurcation point of (1.4), then, on a computer, it is obtained an approximation $(\lambda_{0\epsilon}, u_{0\epsilon})$ of (λ_{0h}, u_{0h}) which is a solution of an equation of the form of (1.4),

$$F_\epsilon(\lambda_\epsilon, u_\epsilon) - \varrho_\epsilon = 0\,, \tag{1.5}$$

where F_ϵ is an approximation of F_h in the computer's arithmetics.

Under the conditions of the above discussion, the equilibria (stationary) solutions, at least locally, for an approximate study of an evolution equation, are given by (1.4) and not by (1.3). In other words, at least locally (related to equilibria), the approximation of

$$\frac{\partial u}{\partial t} - F(\lambda, u) = 0 \tag{1.6}$$

is

$$\frac{\partial u_h}{\partial t} - F_h(\lambda_h, u_h) + \varrho_h = 0\,. \tag{1.7}$$

The text is organized as follows.

In our work, we use the method of Crouzeix and Rappaz [18], so we remind it briefly in Section 2.1.

In literature [7, 8, 16, 17, 18, 30, 33, 34, 35, 37, 38, 40, 41, 42, 44, 47, 48, 50, 52, 53, 56, 57, 58, 65], an extended system is used in order to reduce a problem that presents a bifurcation to a problem without a bifurcation. We develop the work on the basis of a connection between the properties of a bifurcation point of a nonlinear equation on Banach spaces and a somewhat new extended system (this one is constructed based on a local C^p - diffeomorphism related to the bifurcation point). Sections 3 and 4 are devoted to this subject.

In Section 5, we formulate and we prove the main result about the existence of an equation of form (1.4), on infinite-dimensional Banach spaces, which has a solution (λ_0, u_0) that satisfies the hypothesis (1.2). These developments are based on the methods presented in the monograph [21] of Dontchev and Rockafellar. Between them, there are the Graves' theorem, one of its consequences and the contraction mapping principle for set-valued mappings. These are reminded in Section 2.2. The results that we obtain in Section 5 are placed in the formalism of [11, 12, 13, 14, 18, 21, 29] and of Graves' theorem [21].

In Section 6, the existence of a class of maps equivalent to $F_h(\lambda_h, u_h) - \varrho_h$ is made evident.

In Section 7, the case of the approximate equation is studied. If the exact problem has a bifurcation point, a theorem that connects the exact and the approximate problems is formulated. Then, the main result from Section 5 is formulated for the approximate case.

In Section 8, we relate the exact and the finite element formulations from [29], for the Dirichlet problem for the stationary Navier-Stokes equations, to the framework of Section 7.

In Section 9, a complement to Theorem 5.4 is formulated. Instead of a constant ϱ in the equation (5.28), we obtain ϱ in the form of a function of (λ, u).

In Section 10, an intended further research is presented.

2. Preliminaries

2.1. **The setting of Crouzeix and Rappaz** [18]**.** Let us retain the equation (1.1) and the hypothesis (1.2) considered above following the work of Crouzeix and Rappaz [18]. First, Crouzeix and Rappaz treat the case $q = 0$ where they reduce a problem that has bifurcation to a problem without bifurcation (in Chapter 4, [18]). Second, for the case $q \geq 1$, they reduce the study to the case $q = 0$ (in Chapter 6, [18]). In this subsection, for the sake of brevity, we remind these in the inverse order, by modifying the presentation of Crouzeix and Rappaz [18]. The problem without bifurcation is obtained directly for the reduced problem.

Under the hypothesis that (λ_0, u_0) is a bifurcation solution satisfying hypothesis (1.2) (recall that $q \geq 1$), $Z_2 = Range(DF(\lambda_0, u_0))$ is closed in Z, $DF(\lambda_0, u_0)$ is a Fredholm operator of $\mathbb{R}^m \times W$ onto Z with index m and $Z = Z_1 \oplus Z_2$, where $Z_1 = sp\ \{\bar{a}_1, \ldots, \bar{a}_q\}$ and $\bar{a}_1, \ldots, \bar{a}_q$ are some linearly independent some linearly independent elements from Z. Let $X = \mathbb{R}^{q+m} \times W$, $Y = \mathbb{R}^{q+m} \times Z$, $f = (f^1, \ldots, f^q) \in \mathbb{R}^q$, $f_0 = 0 \in \mathbb{R}^q$, $x = (f, \lambda, u)$, $x_0 = (f_0, \lambda_0, u_0) \in X$. Crouzeix and Rappaz define the function

$$G : X \to Z,\ G(x) = F(\lambda, u) - \sum_{i=1}^{q} f^i \bar{a}_i\,, \tag{2.1}$$

and they replace the study of (1.1) around (λ_0, u_0) by the study of the case $Range(DG(x_0)) = Z$ (corresponding to the case $q = 0$ in Chapter 4, [18]) for the problem

$$G(x) = 0\,, \tag{2.2}$$

around the solution x_0 of it. For this, they use the remark that (λ, u) is a solution of (1.1) is equivalent to the fact that x is a solution of (2.2) satisfying $f = 0$.

$G(x_0) = 0$ and $G(x) = F(\lambda, u)$ if $f = 0$. $DG(x)y = DG(f, \lambda, u)(g, \mu, w) = DF(\lambda, u)(\mu, w) - \sum_{i=1}^{q} g^i \bar{a}_i$ for $y = (g, \mu, w)$. $dim\ Ker(DG(x_0)) = q+m$,

$$Ker(DG(x_0)) = \{y = (g, \mu, w) \in X; g = 0, (\mu, w) \in Ker(DF(\lambda_0, u_0))\}\,. \tag{2.3}$$

$Range(DG(x_0)) = Z$ and $DG(x_0) \in L(X, Z)$ is a Fredholm operator with index $q+m$. This is equivalent to $D_u G(0, \lambda_0, u_0)$ is a Fredholm operator of W onto Z with index zero and $Range(DG(x_0)) = Z$ (according to Chapter 4, [18]).

In order to reduce the problem (2.2) to a problem without bifurcation, Crouzeix and Rappaz justify the introduction of an operator B and the reduction of the problem of the study of the solutions of (2.2), in a neighborhood of x_0, to the study of the solutions of the extended system in $(\theta, x) \in \mathbb{R}^{q+m} \times X$

$$\mathcal{F}(\theta, x) = 0\,, \tag{2.4}$$

in a neighborhood of its regular solution $(0, x_0)$, where

$$\mathcal{F} : \mathbb{R}^{q+m} \times X \to Y\,,\ \mathcal{F}(\theta, x) = \begin{bmatrix} B(x) - B(x_0) - \theta \\ G(x) \end{bmatrix}. \tag{2.5}$$

The continuous linear operator $B \in L(X, \mathbb{R}^{q+m})$ is introduced such that it satisfies $Ker(DG(x_0)) \cap Ker(B) = \{0\}$. B is an isomorphism of $Ker(DG(x_0))$ onto $\mathbb{R}^{q+m}$. To choose B is equivalent to choose $q + m$ linear forms χ_i, $i = 1, \ldots, q + m$, on X that are linearly independent on $Ker(DG(x_0))$. By identifying $Ker(DG(x_0))$ to $\mathbb{R}^{q+m}$, $B(x)$ can be considered the component of $x \in X$ on $Ker(DG(x_0))$ with respect to the decomposition $X = Ker(DG(x_0)) \oplus Ker(B)$. The function $\mathcal{F}$ is of class C^p.

$$D_x\mathcal{F}(0, x_0)y = \begin{bmatrix} B(y) \\ DG(x_0)y \end{bmatrix}. \tag{2.6}$$

Since $Ker(D_x\mathcal{F}(0, x_0)) = \{0\}$ and $Range(D_x\mathcal{F}(0, x_0)) = Y$, there results that $D_x\mathcal{F}(0, x_0)$ is an isomorphism of X onto Y. Implicit functions theorem leads to

Lemma 2.1. *(Lemma 4.1 and Lemma 6.1,* [18]*). Under the above hypotheses, there exists a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^{q+m}$, a neighborhood $\mathcal{U}$ of x_0 in X and a unique C^p - mapping $x : \theta \in \mathcal{V} \to x(\theta) = (f(\theta), \lambda(\theta), u(\theta)) \in \mathcal{U}$ satisfying: a) $G(x(\theta)) = 0$, $\theta = B(f(\theta) - f_0, \lambda(\theta) - \lambda_0, u(\theta) - u_0)$, $\forall \theta \in \mathcal{V}$; b) $x(0) = x_0$; c) $(\theta, x(\theta))$ is a regular point of (2.4), $\forall \theta \in \mathcal{V}$; d) if $y = (g, \mu, v) \in \mathcal{U}$ is such that $G(y) = 0$, then $g = f(\theta)$, $\mu = \lambda(\theta)$, $v = u(\theta)$, with $\theta = B(g - f_0, \mu - \lambda_0, v - u_0)$.*

Equation $f(\theta) = 0$ is the bifurcation equation of problem (1.1) ([18]). Crouzeix and Rappaz specify that their method is equivalent to the Lyapunov-Schmidt method. They call "classical" the bifurcation equation related to the Lyapunov-Schmidt method.

For the case $q \geq 1$, Crouzeix and Rappaz use B reduced to $\mathbb{R}^m \times W$, $B \in L(\mathbb{R}^m \times W, \mathbb{R}^{q+m})$. In the above presentation and in Lemma 2.1, we maintain the operator $B \in L(X, \mathbb{R}^{q+m})$ corresponding to the case in Chapter 4, [18].

The problem (2.4), in the case $q = 0$, is also formulated in a different context in [21]. Related to the study of a linear Fredholm operator and of the system $D_x\mathcal{F}(0, x)y - [\theta', 0]^T = 0$, where x and $\theta' \in \mathbb{R}^{q+m}$ are fixed, the use of an operator similar to B (from [18]) and of some elements similar to $\bar{a}_1, \ldots, \bar{a}_q$ (from [18]) is found in [6, 7, 8, 21, 34, 35, 41, 44].

Given two normed (linear) spaces E and F, $L(E, F)$ is the space of all continuous linear mappings (operators) $K : E \to F$. An isomorphism of E onto F is a linear, continuous and bijective mapping $K : E \to F$ whose inverse K^{-1} is continuous [6, 18, 21, 24, 29, 36, 59, 60, 61].

Lemma 2.2. *(The hypotheses of a formulation of the inverse function theorem, Theorem I.2.2,* [14] *and a partial result from the proof of this one). For $v \in X$ and the function $\widehat{G} : X \to Z$ of class C^p, $p \geq 1$, we assume that $D\widehat{G}(v) \in L(X, Z)$ is an isomorphism and that β satisfies $2\gamma L_{\widehat{G}}(\beta) \leq 1$, with $\gamma = \widetilde{\gamma}(\widehat{G}, v, X, Z)$ and $L_{\widehat{G}}(\beta) = \widetilde{L}(\widehat{G}, v, x, \beta, X, Z)$, where we use the notations (2.11) and (2.12) below. Then, for any $z \in int\mathbb{B}_{\frac{\beta}{2\gamma}}(\widehat{G}(v))$,* *the equation* $\widehat{G}(x) = z$ *has a unique solution* x *in* $\mathbb{B}_{\beta_1}(v)$*, where* $\beta_1 = 2\gamma\|\widehat{G}(v) - z\|_Z \leq \beta$.

2.2. **Contraction mapping principle for set-valued mappings and Graves' theorem** [21]**.** Let us present the following mathematical entities as they are introduced in the monograph [21] of Dontchev and Rockafellar.

Let (X, ρ) be a metric space. For the sets C and D in X and $x \in C$,

$d(x, C) = \inf_{x' \in C} \rho(x, x')$, $e(C, D) = \sup_{x \in C} d(x, D)$,

with the convention $e(\emptyset, D) = 0$ when $D \neq \emptyset$ and $e(\emptyset, D) = \infty$ otherwise.

$\rho(x, y) = \|x - y\|$ if $(X, \|\cdot\|)$ is a normed space.

Let X, Y be Banach spaces. Let $F : X \rightrightarrows Y$ be a set-valued mapping, that is, for $x \in X$, F assigns a set $F(x)$ that contains one or more elements of Y or it is empty. The graph of F is the set *gph* $F = \{(x, y) \in X \times Y | y \in F(x)\}$. The domain of F is the set *dom* $F = \{x | F(x) \neq \emptyset\}$. The range of F is the set *rge* $F = \{y | y \in F(x)$ for some $x\}$. A set-valued mapping $F : X \rightrightarrows Y$ has an inverse $F^{-1} : Y \rightrightarrows X$, $F^{-1}(y) = \{\ x \mid y \in F(x)\ \}$. Let us retain that $x \in F^{-1}(y) \Leftrightarrow y \in F(x)$.

Let $A \in L(X, Y)$ be a surjective mapping. A consequence of Theorem 5A.1 (Banach open mapping theorem), page 253, [21], is the existence of a $\kappa > 0$ such that $d(0, A^{-1}(y)) \leq \kappa\|y\|$, for all y. The regularity modulus is defined by

$$reg\, A = \sup_{\|y\| \leq 1} d(0, A^{-1}(y))\,, \tag{2.7}$$

$reg\, A < \infty$. For a Banach space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}$, let $\mathbb{B}_a(\tilde{s}) = \{s \in \mathcal{X}; \|\tilde{s} - s\|_{\mathcal{X}} \leq a\}$ be the closed ball with center $\tilde{s}$ and radius a.

Theorem 2.3. *(Contraction mapping principle for set-valued mappings, Theorem 5E.2, page 284,* [21]*) Let* (X, ρ) *be a complete metric space, and consider a set-valued mapping* $T : X \rightrightarrows X$ *and a point* $\bar{x} \in X$*. Suppose that there exist scalars* $a > 0$ *and* $\lambda \in (0, 1)$ *such that the set* $gphT \cap (\mathbb{B}_a(\bar{x}) \times \mathbb{B}_a(\bar{x}))$ *is closed and*

I. $d(\bar{x}, T(\bar{x})) < a(1 - \lambda)$*;*

II. $e(T(u) \cap \mathbb{B}_a(\bar{x}), T(v)) \leq \lambda\rho(u, v)$ *for all* u*,* $v \in \mathbb{B}_a(\bar{x})$*.*

Then, T *has a fixed point in* $\mathbb{B}_a(\bar{x})$*; that is, there exists* $x \in \mathbb{B}_a(\bar{x})$ *such that* $x \in T(x)$*.*

Theorem 2.4. *(Graves, Theorem 5D.2, page 276,* [21]*) Let* X *and* Y *be Banach spaces. Consider a function* $f : X \to Y$ *and a point* $\bar{x} \in int\, dom\, f$ *and let* f *be continuous in* $\mathbb{B}_\varepsilon(\bar{x})$ *for some* $\varepsilon > 0$*. Let* $A \in L(X, Y)$ *be surjective and let* $\kappa \geq reg\, A$*. Suppose there is a nonnegative* μ *such that* $\kappa\mu < 1$ *and*

$$\|f(x) - f(x') - A(x - x')\| \leq \mu\|x - x'\| \text{ whenever } x, x' \in \mathbb{B}_\varepsilon(\bar{x})\,. \tag{2.8}$$

Then, in terms of $\bar{y} = f(\bar{x})$ *and* $c = \kappa^{-1} - \mu$*, if* y *is such that* $\|y - \bar{y}\| \leq c\varepsilon$*, then the equation* $y = f(x)$ *has a solution* $x \in \mathbb{B}_\varepsilon(\bar{x})$*.*

Corollary 2.5. *(A consequence of Graves' theorem, pages 277-278,* [21]*) Let (2.8) hold for* x*,* $x' \in \mathbb{B}_\varepsilon(\bar{x})$ *and choose a positive* $\tau < \varepsilon$*. Then, there is a neighborhood* U *of* $\bar{x}$ *such that* $\mathbb{B}_\tau(x) \subset \mathbb{B}_\varepsilon(\bar{x})$ *for all* $x \in U$*. Make* U

smaller if necessary so that $\|f(x)-f(\bar{x})\| < c\tau$ *for* $x \in U$*. Pick* $x \in U$ *and a neighborhood* V *of* $\bar{y}$ *such that* $\|y - f(x)\| \le c\tau$ *for* $y \in V$*. Then,*

$$d(x, f^{-1}(y)) \le \frac{\kappa}{1-\kappa\mu}\|y - f(x)\| \text{ for } (x,y) \in U \times V\,. \tag{2.9}$$

This is the metric regularity property of the function f *at* $\bar{x}$ *for* $\bar{y}$*.*

If for every $\mu > 0$ there exists $\varepsilon > 0$ such that (2.8) holds for every x, x' $\in \mathbb{B}_\varepsilon(\bar{x})$, then A is, by definition ([20], [21]), the strict derivative of f at $\bar{x}$, $A = Df(\bar{x})$.

Theorem 2.6. *(Theorem 1.3,* [20]*) Let* X *and* Y *be Banach spaces, let* $\bar{x}$ $\in X$ *and let* $f : X \to Y$ *be a function which is strictly differentiable at* $\bar{x}$ *(see the definition at page 31,* [21]*). Suppose that* $Df(\bar{x})$ *is onto. Then, there exist a neighborhood* W *of* $\bar{x}$ *and a constant* $c > 0$ *such that for every* $x \in W$ *and* $\tau > 0$ *with* $\mathbb{B}_\tau(x) \subset W$*,*

$$\mathbb{B}_{c\tau}(f(x)) \subset f(\mathbb{B}_\tau(x))\,. \tag{2.10}$$

Let us consider the notations γ and $L(\varepsilon)$ from [14, 18, 29] in the following way:

$$\widetilde{\gamma}(f, \bar{x}, X, Y) = \|Df(\bar{x})^{-1}\|_{L(Y,X)}\,, \tag{2.11}$$

$$\widetilde{L}(f, \bar{x}, x, \varepsilon, X, Y) = \sup_{x \in \mathbb{B}_\varepsilon(\bar{x})} \|Df(\bar{x}) - Df(x)\|_{L(X,Y)}\,. \tag{2.12}$$

3. The main theorem on the extended systems considered in the work

We formulate a theorem for the equivalence between the properties of the bifurcation point (λ_0, u_0) of (1.1), satisfying hypothesis (1.2), and the existence of the solution of an (overdetermined) extended system. This Section and Section 4 are devoted to this subject.

Let us consider $\theta = 0$ in equation (2.4) and conclude that we can introduce an equation based on the value $B(x_0)$. Let us denote

$$\theta_* = B(x_0)\,. \tag{3.1}$$

It follows that the equation

$$\begin{aligned} B(x) - \theta_* &= 0\,, \\ G(x) &= 0 \end{aligned} \tag{3.2}$$

has an unique solution $x = x_0 = (f_0, \lambda_0, u_0)$.

There results the existence of a $\theta_* \in \mathbb{R}^{q+m}$ such that we can consider equation (3.2) on its own and we can formulate questions about the existence, the uniqueness and the properties of the solution (λ_0, u_0) of (1.1). Since our results are results of existence, it is not necessary to know (to have explicitely) the value of θ_*. In the sequel, we use (3.1) in two respects: 1) Given B and x_0, θ_* results from (3.1). 2) Given B and θ_*, x_0 is a solution of (3.1).

Take some fixed $\theta_0 \in \mathbb{R}^{q+m}$. Consider $p \ge 2$. Let us introduce the function

$$\Psi : X \to Y,\ \Psi(x) = \begin{bmatrix} B(x) - \theta_0 \\ G(x) \end{bmatrix}, \tag{3.3}$$

and the equation in $x \in X$

$$\Psi(x) = 0\,. \tag{3.4}$$

We have $\Psi(x_0) = \mathcal{F}(0, x_0)$ and $D\Psi(x_0) = D_x\mathcal{F}(0, x_0)$, where $D\Psi(x)\overline{x} = [B(\overline{x}), DG(x)\overline{x}]^T$ and $[\cdot]^T$ denotes the transpose of the row vector $[\cdot]$.

As $DG(x_0)$ is associated to the Fredholm operator $DF(\lambda_0, u_0)$, let us introduce a function H, of class C^{p-1}, associated to $D_uF(\lambda_0, u_0)$. The space $Z_4 = Range(D_uF(\lambda_0, u_0))$ is closed in Z, $Z = Z_3 \oplus Z_4$, where $Z_3 = sp\ \{\bar{b}_1, \dots, \bar{b}_n\}$ and $\bar{b}_1, \dots, \bar{b}_n$ are some linearly independent elements from Z. Let $\Delta = \mathbb{R}^n \times W$, $\Sigma = \mathbb{R}^n \times Z$, $e = (e^1, \dots, e^n) \in \mathbb{R}^n$, $\mathrm{z} = (e, v) \in \Delta$. Define the function H by

$$H : \mathbb{R}^m \times W \times \Delta \to Z,\ H(\lambda, u, \mathrm{z}) = D_uF(\lambda, u)v - \sum_{k=1}^{n} e^k \bar{b}_k\,. \tag{3.5}$$

$$Ker(H(\lambda_0, u_0, \cdot)) = \{\mathrm{z} = (e, v) \in \Delta; e = 0, v \in Ker(D_uF(\lambda_0, u_0))\}\,. \tag{3.6}$$

It follows that $dim\ Ker(H(\lambda_0, u_0, \cdot)) = n$, $Range(H(\lambda_0, u_0, \cdot)) = Z$ and $H(\lambda_0, u_0, \cdot) \in L(\Delta, Z)$ is a Fredholm operator with index n. We have $DH(\lambda, u, \mathrm{z})(\overline{\lambda}, \overline{u}, \overline{\mathrm{z}}) = D_{(\lambda,u)}(D_uF(\lambda, u)v)(\overline{\lambda}, \overline{u}) + D_uF(\lambda, u)\overline{v} - \sum_{k=1}^{n} \overline{e}^{\,k}\bar{b}_k$.

Consider an operator $\bar{\mathcal{B}}$, similar to B, $\bar{\mathcal{B}} \in L(\Delta, \mathbb{R}^n)$ such that $Ker(H(\lambda_0, u_0, \cdot)) \cap Ker(\bar{\mathcal{B}}) = \{0\}$. $\bar{\mathcal{B}}$ is an isomorphism of $Ker(H(\lambda_0, u_0, \cdot))$ onto $\mathbb{R}^n$ and the decomposition $\Delta = Ker(H(\lambda_0, u_0, \cdot)) \oplus Ker(\mathcal{B}_n)$ holds. To choose $\bar{\mathcal{B}}$ is equivalent to choose n linear forms $\bar{\chi}_k$, $k = 1, \dots, n$, on Δ that are linearly independent on $Ker(H(\lambda_0, u_0, \cdot))$.

Define the functions $\Phi_G(x, \cdot) : X \to Y$ and $\Phi_H(x, \cdot) : \Delta \to \Sigma$ by

$$\Phi_G(x, y) = \begin{bmatrix} B(y) \\ DG(x)y \end{bmatrix} \text{ and } \Phi_H(x, \mathrm{z}) = \begin{bmatrix} \bar{\mathcal{B}}(\mathrm{z}) \\ H(\lambda, u, \mathrm{z}) \end{bmatrix}.$$

We have $\Phi_G(x, y) = D\Psi(x)y$ and

$$D\Phi_G(x, y)(\overline{x}, \overline{y}) = \begin{bmatrix} B(\overline{y}) \\ D^2F(\lambda, u)((\mu, w), (\overline{\lambda}, \overline{u})) + DG(x)\overline{y} \end{bmatrix},$$

$$D\Phi_H(x, \mathrm{z})(\overline{x}, \overline{\mathrm{z}}) = \begin{bmatrix} \bar{\mathcal{B}}(\overline{\mathrm{z}}) \\ D_{(\lambda,u)}(D_uF(\lambda, u)v)(\overline{\lambda}, \overline{u}) + H(\lambda, u, \overline{\mathrm{z}}) \end{bmatrix}.$$

We remember that $X = \mathbb{R}^{q+m} \times W$, $Y = \mathbb{R}^{q+m} \times Z$, $f = (f^1, \dots, f^q) \in \mathbb{R}^q$, $f_0 = 0 \in \mathbb{R}^q$, $x = (f, \lambda, u)$, $x_0 = (f_0, \lambda_0, u_0) \in X$.

Related to $DF(\lambda_0, u_0)$, we denote $g = (g^1, \dots, g^q)$, $g_i = (g_i^1, \dots, g_i^q) \in \mathbb{R}^q$, $g_0 = 0$, $g_{i,0} = 0 \in \mathbb{R}^q$ and $y = (g, \mu, w)$, $y_0 = (g_0, \mu_0, w_0)$, $y_i = (g_i, \mu_i, w_i)$, $y_{i,0} = (g_{i,0}, \mu_{i,0}, w_{i,0}) \in X$, $i = 1, \dots, q + m$.

Related to $D_uF(\lambda_0, u_0)$, we denote $e_k = (e_k^1, \dots, e_k^n) \in \mathbb{R}^n$, $e_{k,0} = 0 \in \mathbb{R}^n$ and $\mathrm{z}_k = (e_k, v_k)$, $\mathrm{z}_{k,0} = (e_{k,0}, v_{k,0}) \in \Delta$, $k = 1, \dots, n$.

We also denote $\Gamma = X^{1+q+m} \times \Delta^n$, $\Sigma = (\mathbb{R}^{q+m} \times Z)^{1+q+m} \times (\mathbb{R}^n \times Z)^n$ and $s = (x, y_1, \dots, y_{q+m}, \mathrm{z}_1, \dots, \mathrm{z}_n)$, $s_0 = (x_0, y_{1,0}, \dots, y_{q+m,0}, \mathrm{z}_{1,0}, \dots, \mathrm{z}_{n,0}) \in \Gamma$.

On the product space $Y_0 = \prod_{\jmath=1}^{N} Y_\jmath$, we use the norm $\|\cdot\|_{(1)}$, $\|\kappa_0\|_{(1)} = \|\kappa_0\|_Y = \sum_{\jmath=1}^{N} \|\kappa_\jmath\|_{Y_\jmath}$, where $Y_\jmath$ is a normed space with norm $\|\cdot\|_{Y_\jmath}$, for $\jmath = 1, \dots, N$, and $\kappa_0 = (\kappa_1, \dots, \kappa_N) \in Y$. We write $\|\cdot\|$ instead of $\|\cdot\|_E$

or $\|\cdot\|_{L(E,F)}$ when the spaces E and F are clear. Let I_N be the identity operator on $\mathbb{R}^N$, $N \geq 1$. Let I_W be the identity operator on W.

δ_{ij} is the Kronecker delta and $\delta_k^N = \{\delta_{k1}, \ldots, \delta_{kk}, \ldots, \delta_{kN}\}$, for $1 \leq k \leq N$, but $\{\delta_1^N, \ldots, \delta_N^N\}$ can be any fixed basis of $\mathbb{R}^N$.

For the solution x_0 of (3.2), a basis of $Ker(DF(\lambda_0, u_0))$ and a basis of $Ker(D_uF(\lambda_0, u_0))$ can be determined by the solution s_0 of (3.8) (where $\theta_0 = \theta_*$) below.

Together with θ_0, let us consider some elements $\bar{a}_1, \ldots, \bar{a}_q$, $\bar{b}_1, \ldots, \bar{b}_n \in Z$. Having in mind the local C^p - diffeomorphism Ψ at x_0, let us introduce the function $S : \Gamma \to \Sigma$ ([5]),

$$(3.7) \quad S(s) = \begin{bmatrix} \Psi(x) \\ B(y_i) - \delta_i^{q+m} \\ DG(x)y_i \\ \bar{\mathcal{B}}(\mathrm{z}_k) - \delta_k^n \\ H(\lambda, u, \mathrm{z}_k) \end{bmatrix} \text{ or } S(s) = \begin{bmatrix} \Psi(x) \\ \Phi_G(x, y_i) - \begin{bmatrix} \delta_i^{q+m} \\ 0 \end{bmatrix} \\ \Phi_H(x, \mathrm{z}_k) - \begin{bmatrix} \delta_k^n \\ 0 \end{bmatrix} \end{bmatrix},$$

for all $i = 1, \ldots, q+m$, $k = 1, \ldots, n$, with the following convention (in the left formulation of $S(s)$) that we use throughout the paper: the second and the third rows (components) are taken $q+m$ times, for all the values of i, and the fourth and the fifth rows (components) are taken n times, for all the values of k. Consider the extended system in s

$$(3.8) \qquad S(s) = 0 \ .$$

In (3.8), x is determined by equation $\Psi(x) = 0$ and the rest of components of s are determined by the rest of the equations.

Remark 3.1. Let us observe that $DG(f, \lambda, u)(g, \mu, w) = DG(f', \lambda, u)(g, \mu, w)$ for $f \neq f'$.

We have

$$(3.9) \qquad DS(s)\overline{s} = \begin{bmatrix} B(\overline{x}) \\ DG(x)\overline{x} \\ B(\overline{y}_i) \\ D^2F(\lambda, u)((\mu_i, w_i), (\overline{\lambda}, \overline{u})) + DG(x)\overline{y}_i \\ \bar{\mathcal{B}}(\overline{\mathrm{z}}_k) \\ D_{(\lambda,u)}(D_uF(\lambda, u)v_k)(\overline{\lambda}, \overline{u}) + H(\lambda, u, \overline{\mathrm{z}}_k) \end{bmatrix},$$

$i = 1, \ldots, q+m$, $k = 1, \ldots, n$.

Let us define the function $\Phi : X \times \Gamma \to \Sigma$,

$$(3.10) \qquad \Phi(x, \phi') = \begin{bmatrix} \Phi_G(x, y') \\ \Phi_G(x, y_i') \\ \Phi_H(x, \mathrm{z}_k') \end{bmatrix},$$

for all $i = 1, \ldots, q+m$, $k = 1, \ldots, n$, where $\phi' = (y', y_1', \ldots, y_{q+m}', \mathrm{z}_1', \ldots, \mathrm{z}_n')$.

We have

$$(3.11) \quad D\Phi(x,\phi')(\overline{x},\overline{\phi}') = \begin{bmatrix} B(\overline{y}') \\ D^2F(\lambda,u)((\mu',w'),(\overline{\lambda},\overline{u})) + DG(x)\overline{y}' \\ B(\overline{y}'_i) \\ D^2F(\lambda,u)((\mu'_i,w'_i),(\overline{\lambda},\overline{u})) + DG(x)\overline{y}'_i \\ \bar{\mathcal{B}}(\overline{z}'_k) \\ D_{(\lambda,u)}(D_uF(\lambda,u)v'_k)(\overline{\lambda},\overline{u}) + H(\lambda,u,\overline{z}'_k) \end{bmatrix},$$

Remark 3.2. Related to the solution (λ_0,u_0) of (1.1), to a basis of $Ker(DF(\lambda_0,u_0))$ and to a basis of $Ker(D_uF(\lambda_0,u_0))$, let us observe that the following statements i) and ii) must be equivalent:

i) $\theta_0 = \theta_*$ from (3.1), $g_i = 0$, $e_k = 0$, $i = 1,\ldots,q+m$, $k = 1,\ldots,n$.

ii) There exists $\theta_0 \in \mathbb{R}^{q+m}$ such that $f = 0$, $g_i = 0$, $e_k = 0$, $i = 1,\ldots,q+m$, $k = 1,\ldots,n$.

Lemma 3.3. *Let $s \in \Gamma$. The following statements i) and ii) are equivalent:*

i) $D\Psi(x)$ is an isomorphism of X onto Y, $DS(s)$ is an isomorphism of Γ onto Σ

ii) $\Phi_G(x,\cdot)$ is an isomorphism of X onto Y, $\Phi_H(x,\cdot)$ is an isomorphism of Δ onto Σ.

Proof. For the implication (i) $\Rightarrow$ (ii), we mentions the followings remarks: $\Phi_G(x,\cdot) = D\Psi(x)$. Let us take the last component of $DS(s)$ and obtain that $\Phi_H(x,\cdot)$ is bijective from the study of the equation in $\overline{z}$

$$\Phi_H(x,\overline{z}) = [\varsigma',\varsigma'']^T - [0, D_{(\lambda,u)}(D_uF(\lambda,u)v_n)(\overline{\lambda},\overline{u})]^T,$$

for $(\varsigma',\varsigma'') \in Y$ and $(\overline{\lambda},\overline{u})$ fixed in $\mathbb{R}^m \times W$.

□

Lemma 3.4. *Let the assumptions (i) or (ii) of Lemma 3.3 hold. Then, $\Phi(x,\cdot)$ is an isomorphism of Γ onto Σ.*

We study the case $q \geq 1$. The results can be transferred, easily, to the case $q = 0$. We formulate, now, the main theorem on the extended systems considered above.

Theorem 3.5. *Let F, $n \geq 1$, $q \geq 1$, $\bar{a}_1,\ldots,\bar{a}_q$, $\bar{b}_1,\ldots,\bar{b}_n \in Z\backslash\{0\}$ and G, H defined above. Let $B = (\chi_1,\ldots,\chi_{q+m}) \in L(X,\mathbb{R}^{q+m})$ and $\bar{\mathcal{B}} = (\bar{\chi}_1,\ldots,\bar{\chi}_n) \in L(\Delta,\mathbb{R}^n)$, where χ_1, …, χ_{q+m} are $(q+m)$ nonzero linear forms on X and $\bar{\chi}_1$, …, $\bar{\chi}_n$ are n nonzero linear forms on Δ. The following statements (i) and (ii) are equivalent*

(i) Let (1.1). (λ_0,u_0) is a solution of (1.1). Hypothesis (1.2) holds for (λ_0,u_0). $Z = Z_1 \oplus Z_2$ with $\bar{a}_1,\ldots,\bar{a}_q$ linearly independent. $Z = Z_3 \oplus Z_4$ with $\bar{b}_1,\ldots,\bar{b}_n$ linearly independent. x_0 is the unique solution of (2.2). B and $\bar{\mathcal{B}}$ are such that

$$(3.12) \qquad Ker(DG(x_0)) \cap Ker(B) = \{0\},$$

$$(3.13) \qquad Ker(H(\lambda_0,u_0,\cdot)) \cap Ker(\bar{\mathcal{B}}) = \{0\}.$$

where these kernels are given by (2.3) and (3.6). We have

$$(3.14) \qquad \theta_0 = B(x_0) \in \mathbb{R}^{q+m}.$$

(ii) Fix $\theta_0 \in \mathbb{R}^{q+m}$*, a basis* $\{\delta_i^{q+m}\}_{i=1,\dots,q+m}$ *of* $\mathbb{R}^{q+m}$ *and a basis* $\{\delta_k^n\}_{k=1,\dots,n}$ *of* $\mathbb{R}^n$*. Consider* Ψ *and (3.4) together with* S *and (3.8). We have:*

(a) The system (3.8) has the solution $s_0 = (x_0,\, y_{1,0},\, \dots,\, y_{q+m,0},\, \mathrm{z}_{1,0},\, \dots,\, \mathrm{z}_{n,0})$*, where* $x_0 = (f_0, \lambda_0, u_0)$*,* $y_{i,0} = (g_{i,0}, \mu_{i,0}, w_{i,0})$*,* $\mathrm{z}_{k,0} = (e_{k,0}, v_{k,0})$*,* $f_0 = 0$*,* $g_{i,0} = 0$*,* $e_{k,0} = 0$*,* $i = 1, \dots, q+m$*,* $k = 1, \dots, n$ *and the component* x_0 *of* s_0 *is the solution of (3.4).*

(b) $D\Psi(x_0)$ *is an isomorphism of* X *onto* Y*;* $DS(s_0)$ *is an isomorphism of* Γ *onto* Σ*.*

Corollary 3.6. *Theorem 3.5 remains true if we replace* B *by a nonlinear* B_N *in the definition (3.3) of* Ψ*.*

Proof. We first use Lemma 4.1 (i) and (ii) below. Then, we observe that, under the assumptions of the statement (ii) (b), we obtain that $\Phi_G(x,\cdot)$ is an isomorphism of X onto Y as in the proof of Lemma 3.3.

□

Remark 3.7. Since we have $f_0 = 0$, $g_{i,0} = 0$ $(i = 1, \dots, q+m)$ and $e_{k,0} = 0$ $(k = 1, \dots, n)$, it follows that the results of Theorem 3.5 do not depend on the choice of $\bar{a}_1, \dots, \bar{a}_q$, $\bar{b}_1, \dots, \bar{b}_n$.

Remark 3.8. The results of Theorem 3.5 do not depend on the choice of $B = (\chi_1, \dots, \chi_{q+m})$ and $\bar{\mathcal{B}} = (\bar{\chi}_1, \dots, \bar{\chi}_n)$.

Corollary 3.9. *Theorem 3.5 ii) a) is equivalent to the following formulation:* s_0 *is a solution of the overdetermined system*

$$S(s) = 0\,, f = 0\,,\ g_i = 0\,,\ e_k = 0\,,\ i = 1, \dots, q+m\,,\ k = 1, \dots, n\,. \tag{3.15}$$

Corollary 3.10. *In the case* $q = 0$*, we replace* G *by* F *in the definition (3.7) of* S *and (3.15) is reduced to*

$$\begin{aligned}
&B(\lambda, u) - \theta_0 = 0\,,\\
&F(\lambda, u) = 0\,,\\
&B(\mu_i, w_i) - \delta_i^m = 0\,,\\
&DF(\lambda, u)(\mu_i, w_i) = 0\,,\\
&\bar{\mathcal{B}}(\mathrm{z}_k) - \delta_k^n = 0\,,\\
&H(\lambda, u, \mathrm{z}_k) = 0\,,\\
&e_k = 0\,,\ k = 1, \dots, n\,.
\end{aligned} \tag{3.16}$$

Let us remark that the study of a nonlinear Fredholm operator $\widehat{F}$ can be reduced to the study of the equation (1.1) satisfying hypothesis (1.2). Let $\widehat{U}$ be an open set, $\widehat{U} \subseteq W$, and $\widehat{F} : \widehat{U} \to Z$. Let us fix $\widehat{u}_0 \in \widehat{U}$ and let $\widehat{\zeta}_0 = \widehat{F}(\widehat{u}_0)$. Then, the equation

$$\widehat{F}(\widehat{u}) - \widehat{\zeta}_0 = 0 \tag{3.17}$$

has the solution $\widehat{u}_0$ that satisfies the hypothesis:

$$\begin{aligned}
&D\widehat{F}(\widehat{u}_0) \text{ is a Fredholm operator of } W \text{ onto } Z \text{ with index } m\,,\\
&\qquad dim\ Ker(D\widehat{F}(\widehat{u}_0)) = q + m\,.
\end{aligned} \tag{3.18}$$

For instance, we can choose $\widehat{u}_0$ such that $dim\ Ker(D\widehat{F}(\widehat{u}_0)) = \max$.

4. Proof of the main theorem on the extended systems, Theorem 3.5

4.1. **Some properties of the extended systems associated to $D_uF(\lambda, u)$ and $DF(\lambda, u)$.** Throughout this Section, let us consider the hypotheses of Theorem 3.5.

Lemma 4.1. *The statements (i) and (ii) are equivalent, and the same is true for (iii) and (iv).*

(i) x is the unique solution of (2.2) and $\theta_0 = B(x) \in \mathbb{R}^{q+m}$.

(ii) Let θ_0 be a fixed element in $\mathbb{R}^{q+m}$. Let us consider Ψ and (3.4). x is a solution of (3.4). $D\Psi(x)$ is an isomorphism of X onto Y.

(iii) Assume (i). $DG(x)$ is a Fredholm operator with index $q+m$, $Range(DG(x)) = Z$ and $Ker(DG(x)) \cap Ker(B) = \{0\}$.

(iv) Assume (ii). $\Phi_G(x, \cdot)$ is an isomorphism of X onto Y.

The lemma remains true if we replace B by a nonlinear B_N in the definition (3.3) of Ψ and we keep B in the definition of $\Phi_G(x, \cdot)$.

Proof. These results are obtained by replacing x_0 with x in the formulation of problem (2.4) as it is introduced by Crouzeix and Rappaz in [18]. □

Remark 4.2. We formulate only the results for $DF(\lambda, u)$ since those for $D_uF(\lambda, u)$ are similar.

Corollary 4.3. *Under the conditions of Lemma 4.1 (iv), we have:*

$$dim\ Ker(DG(x)) = q + m\,. \tag{4.1}$$

$$B \text{ is an isomorphism of } Ker(DG(x)) \text{ onto } \mathbb{R}^{q+m}\,, \tag{4.2}$$

$$\chi_1, \dots, \chi_{q+m} \text{ are linearly independent on } Ker(DG(x))\,. \tag{4.3}$$

$$X = Ker(DG(x)) \oplus Ker(B)\,, \tag{4.4}$$

Proof. Let us prove (4.1). Consider the elements $(\delta_i^{q+m}, 0) \in Y$, $i = 1, \dots, q+m$. Since $\Phi_G(x, \cdot)$ is an isomorphism of X onto Y, it follows that there exist $x_1, \dots, x_{q+m} \in X$ such that $\Phi_G(x, x_i) = (\delta_i^{q+m}, 0)$, so $B(x_i) = \delta_i^{q+m}$ (otherwise said, $\chi_j(x_i) = \delta_{ij}$) and $DG(x)x_i = 0$. We have $x_i \neq 0$ since $\Phi_G(x, x_i) = (0, 0)$ otherwise. Suppose that there exist $\beta_i \in \mathbb{R}$, $i = 1, \dots, q+m$, where $\beta_i \neq 0$ for at least one i, such that $\sum_{i=1}^{q+m} \beta_i x_i = 0$. For every j, $j = 1, \dots, q+m$, we obtain $0 = \chi_j(0) = \chi_j(\sum_{i=1}^{q+m} \beta_i x_i) = \sum_{i=1}^{q+m} \beta_i \chi_j(x_i)$ $= \beta_j \chi_j(x_j) = \beta_j$. So our supposition is false. Hence $\{x_1, \dots, x_{q+m}\}$ is a linearly independent subset of $Ker(DG(x))$. So $dim\ Ker(DG(x)) \geq q+m$. Suppose that $dim\ Ker(DG(x)) > q+m$. Then, we have another element x', $x' \neq 0$, such that $x_1, \dots, x_{q+m}$, x' are linearly independent and $DG(x)x' = 0$. Let $\varsigma' = B(x')$. Suppose that $\varsigma' = 0$. Since $\Phi_G(x, \cdot)$ is an isomorphism of X onto Y, there results that $x' = 0$, so it remains $\varsigma' \neq 0$. Since $\varsigma' \in \mathbb{R}^{q+m}$, there exist $\beta_i \in \mathbb{R}$, $i = 1, \dots, q+m$, such that $\varsigma' = \sum_{i=1}^{q+m} \beta_i \delta_i^{q+m}$. We have $(\varsigma', 0) = \sum_{i=1}^{q+m} \beta_i (\delta_i^{q+m}, 0)$, $\beta_i \in \mathbb{R}$. We obtain $\Phi_G(x, (x' - \sum_{i=1}^{q+m} \beta_i x_i)) = (\varsigma', 0) - \sum_{i=1}^{q+m} \beta_i (\delta_i^{q+m}, 0) = 0$. $\Phi_G(x, \cdot)$ is an isomorphism of X onto Y so $Ker(\Phi_G(x, \cdot) = \{0\}$. It follows that

$x' - \sum_{i=1}^{q+m} \beta_i x_i = 0$ which contradicts the supposition that $x_1, \ldots, x_{q+m}$, x' are linearly independent. It remains (4.1).

From (4.1), it follows that B is an isomorphism of $Ker(DG(x))$ onto $\mathbb{R}^{q+m}$.

It follows that $\chi_1, \ldots, \chi_{q+m}$ are linearly independent on $Ker(DG(x))$.

Let us define $X_1 = Ker(DG(x)) \oplus Ker(B)$. Suppose that there exists $\overline{x} \in X$ and $\overline{x} \notin X_1$. Let $(\varsigma', \varsigma'') = \Phi_G(x, \overline{x})$ $(*)$, where $\varsigma' \in \mathbb{R}^{q+m}$ and $\varsigma'' \in Z$. Since $\Phi_G(x, \cdot)$ is an isomorphism of X onto Y, there results that, for $(\varsigma', 0) \in Y$, there exists $x' \in X$ such that $\Phi_G(x, x') = (\varsigma', 0)$ and, for $(0, \varsigma'') \in Y$, there exists $x'' \in X$ such that $\Phi_G(x, x'') = (0, \varsigma'')$. We obtain that $x' \in Ker(DG(x))$ and $x'' \in Ker(B)$, so $\overline{\xi} = x' + x'' \in X_1$, and $\Phi_G(x, \overline{\xi}) = (\varsigma', \varsigma'')$. So we obtain two solutions $\overline{x}$, $\overline{\xi}$, $\overline{x} \neq \overline{\xi}$, for $(*)$. This contradicts the uniqueness of $\overline{x}$ in $(*)$. It remains (4.4).

□

Lemma 4.4. *Under the hypotheses and the conclusions of Lemma 4.1 (iii) or (iv), we have:*

(i) $y \in Ker(DG(x))$, $y \neq 0$, if and only if $\exists\ \theta_{q+m} \neq 0$ such that $B(y) = \theta_{q+m}$ and $DG(x)y = 0$.

(ii) Let $q + m = dim\ Ker(DG(x))$. Then, $y_1, \ldots, y_{q+m} \in Ker(DG(x))$, $y_i \neq 0$, $i = 1, \ldots, q+m$, form a basis for $Ker(DG(x))$ if and only if $\exists\ \hat{\theta}_1, \ldots, \hat{\theta}_{q+m}$, $\hat{\theta}_i \neq 0$, $i = 1, \ldots, q+m$, which form a basis for $\mathbb{R}^{q+m}$, such that $B(y_i) = \hat{\theta}_i$ and $DG(x)y_i = 0$.

Proof. $\Phi_G(x, \cdot)$ is an isomorphism of X onto Y.

□

Lemma 4.5. *Replacing $\Phi_G(x, \cdot)$ by $\Phi_H(x, \cdot)$ in Lemma 4.1 (iii) and (iv), Corollary 4.3, Lemma 4.4, we obtain similar results related to $H(\lambda, u, \cdot)$ and $\bar{\mathcal{B}}$ instead of $DG(x)$ and B.*

4.2. **The properties of kernel of $DF(\lambda_0, u_0)$ and of $D_uF(\lambda_0, u_0)$.** Let us replace x by $x_0 = (0, \lambda_0, u_0)$ and y_i by $y_{i,0} = (0, \mu_{i,0}, w_{i,0})$ in Subsection 4.1.

Lemma 4.6. *Under the hypotheses of Theorem 3.5 (ii), we have:*

$$\text{(4.5)} \quad \textit{A basis of } Ker(DG(x_0)) \textit{ is } \{(0, \mu_{1,0}, w_{1,0}), \ldots, (0, \mu_{q+m,0}, w_{q+m,0})\},$$

$$\text{(4.6)} \quad Ker(DG(x_0)) = \{(f, \lambda, u) \in X; f = 0, (\lambda, u) \in Ker(DF(\lambda_0, u_0))\},$$

$$\text{(4.7)} \quad \textit{A basis of } Ker(DF(\lambda_0, u_0)) \textit{ is } \{(\mu_{1,0}, w_{1,0}), \ldots, (\mu_{q+m,0}, w_{q+m,0})\},$$

$$\text{(4.8)} \quad dim\ Ker(DF(\lambda_0, u_0)) = q + m,$$

$$\text{(4.9)} \quad \textit{A basis of } Ker(H(\lambda_0, u_0, \cdot)) \textit{ is } \{(0, v_{1,0}), \ldots, (0, v_{n,0})\},$$

$$\text{(4.10)} \quad Ker(H(\lambda_0, u_0, \cdot)) = \{(e, v) \in \Delta; e = 0, v \in Ker(D_uF(\lambda_0, u_0))\},$$

$$\text{(4.11)} \quad \textit{A basis of } Ker(D_uF(\lambda_0, u_0)) \textit{ is } \{v_{1,0}, \ldots, v_{n,0}\},$$

$$\text{(4.12)} \quad dim\ Ker(D_uF(\lambda_0, u_0)) = n,$$

$\chi_1, \ldots, \chi_{q+m}$ are linearly independent on $Ker(DG(x_0))$,

$\bar{\chi}_1, \ldots, \bar{\chi}_n$ are linearly independent on $Ker(H(\lambda_0, u_0, \cdot))$.

Proof. Observe that $y_{i,0} = (0, \mu_{i,0}, w_{i,0})$, where $(\mu_{i,0}, w_{i,0}) \in Ker(DF(\lambda_0, u_0))$.

$\Phi_G(x_0, \cdot)$ is an isomorphism of X onto Y, so we have (4.1) and (4.2). Hence, we obtain (4.5) and (4.6).

There results that the set $\{(\mu_{1,0}, w_{1,0}), \ldots, (\mu_{q+m,0}, w_{q+m,0})\}$ is a linearly independent subset of $Ker(DF(\lambda_0, u_0))$. Suppose that $dim\, Ker(DF(\lambda_0, u_0)) > q+m$. So there exists $(\mu', w') \in Ker(DF(\lambda_0, u_0))$, $(\mu', w') \neq 0$, such that $\{(\mu_{1,0}, w_{1,0}), \ldots, (\mu_{q+m,0}, w_{q+m,0}), (\mu', w')\}$ is a linearly independent subset of $Ker(DF(\lambda_0, u_0))$. Hence $(0, \mu', w') \in Ker(DG(x_0))$, so $dim\, Ker(DG(x_0)) > q + m$, which contradicts (4.1). It remains that (4.7) and (4.8) are true.

□

4.3. **The decompositions of space Z.**

Lemma 4.7. *Under the hypotheses of Theorem 3.5 (ii), the elements $\bar{a}_1$, …, $\bar{a}_q$ form a linearly independent set and the elements $\bar{b}_1, \ldots, \bar{b}_n$ form a linearly independent set.*

Proof. Suppose that $\bar{a}_1$, …, $\bar{a}_q$ form a linearly dependent set, therefore there exists $f' = ((f')^1, \ldots, (f')^q) \in \mathbb{R}^q$, $f' \neq 0$ and $\sum_{i=1}^{q} (f')^i \bar{a}_i = 0$. Then, for an element $(\mu, w) \in Ker(DF(\lambda_0, u_0))$, we have $(f', \mu, w) \in Ker(DG(x_0))$, with $f' \neq 0$. This is a contradiction. □

Lemma 4.8. *Under the hypotheses of Theorem 3.5 (ii), we have:*

$$Z = Z_1 \oplus Range(DF(\lambda_0, u_0))\,. \tag{4.13}$$

$$Z = Z_3 \oplus Range(D_u F(\lambda_0, u_0))\,. \tag{4.14}$$

Proof. We have $DG(x_0)y = DF(\lambda_0, u_0)(\mu, w) - \sum_{i=1}^{q} g^i \bar{a}_i$, where $-\sum_{i=1}^{q} g^i \bar{a}_i \in Z_1$, $g_i \in \mathbb{R}$, $i = 1, \ldots, q$. So

$$Range(DG(x_0)) = \{z \in Z; z = z_1 + z_2, z_1 \in Z_1, z_2 \in Range(DF(\lambda_0, u_0))\}\,. \tag{4.15}$$

Let us prove that

$$Z_1 \cap Range(DF(\lambda_0, u_0)) = \{0\}\,. \tag{4.16}$$

Suppose that (4.16) is not true, so there exists $a \in Z$, $a \neq 0$, such that $a \in Z_1$ and $a \in Range(DF(\lambda_0, u_0))$. So there exits $g = (g^1, \ldots, g^q) \in \mathbb{R}^q$ and $(\lambda, u) \in \mathbb{R}^m \times W$ such that $a = \sum_{i=1}^{q} g^i \bar{a}_i$ and $a = DF(\lambda_0, u_0)(\lambda, u)$. It follows that $DF(\lambda_0, u_0)(\lambda, u) - \sum_{i=1}^{q} g^i \bar{a}_i = 0$, i.e. $(g, \lambda, u) \in Ker(DG(x_0))$. From (4.6), there results that $g = 0$, so $a = 0$. This contradicts the supposition that $a \neq 0$, hence (4.16).

From (4.15) and (4.16), it follows that

$$Range(DG(x_0)) = Z_1 \oplus Range(DF(\lambda_0, u_0))\,. \tag{4.17}$$

Since $\Phi_G(x_0, \cdot)$ is an isomorphism of X onto Y, it follows follows that $Range(\Phi_G(x_0, \cdot)) = Y$, therefore

$$Range(DG(x_0)) = Z\,. \tag{4.18}$$

From (4.18) and (4.17), there results (4.13).

□

Lemma 4.9. *Under the hypotheses of Theorem 3.5 (ii), we have:*

$$\text{codim } Z_2 = q\,, \quad Z_2 \text{ is closed in } Z\,, \tag{4.19}$$

$$\text{codim } Z_4 = n\,, \quad Z_4 \text{ is closed in } Z\,. \tag{4.20}$$

Proof. From (4.13) and Lemma 4.7, we have $codim\ Z_2 = dim\ Z_1 = q$. We obtain (4.19).

□

4.4. **Properties of the Fréchet derivatives and the consequences on the qualitative aspects.**

Lemma 4.10. *Under the hypotheses of Theorem 3.5 (ii), we have:*
(a) $DF(\lambda_0, u_0)$ is a Fredholm operator of $\mathbb{R}^m \times W$ onto Z with index m;
(b) $D_uF(\lambda_0, u_0)$ is a Fredholm operator of W onto Z with index zero.

Proof. The proof follows from (4.8), (4.19) and (4.12), (4.20).

□

4.5. **The proof of implication (i) $\Rightarrow$ (ii) of Theorem 3.5.** From the implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) of Lemma 4.1 and from Remark 4.2, we deduce that x_0 is a solution of (3.4), $\Phi_G(x_0, \cdot)$ is an isomorphism of X onto Y and $\Phi_H(x_0, \cdot)$ is an isomorphism of Δ onto Σ. From (3.8), we have the equations $\Phi_G(x_0, y_{i,0}) - [\delta_i^{q+m}, 0]^T = 0$, $\Phi_H(x_0, \mathrm{z}_{k,0}) - [\delta_k^n, 0]^T = 0$. From (3.12), (3.13), (2.3) and (3.6), we deduce $g_{i,0} = 0$, $e_{k,0} = 0$. The proof is complete by applying Lemma 3.3.

4.6. **The proof of implication (ii) $\Rightarrow$ (i) of Theorem 3.5.** The proof follows from the implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) of Lemma 4.1 where x is x_0, from Remark 4.2, from Corollary 4.3 with x is x_0, from Lemma 4.4 to Lemma 4.10.

5. The main result on the existence of a bifurcation problem

5.1. **Introduction.** We formulate some sufficient conditions for the existence of an equation which has a bifurcation point that satisfies the hypothesis (1.2). We also obtain the existence of a bifurcation problem for which a given problem is a perturbation.

In the sequel, in Sections 5 - 10, we keep the notations from the preceding sections, but we do not consider the equations they define. Equations (1.1), (2.2) and (3.8), hypothesis (1.2) and Theorem 3.5 will be related to a function S_0 obtained as a perturbation of S. Let us fix these:

Let W and Z be real Banach spaces. They are both infinite-dimensional spaces or they are both finite-dimensional spaces with $dim\ W = dim\ Z$. Let $m \geq 1$, $p \geq 2$. Let $F : \mathbb{R}^m \times W \to Z$ be a nonlinear function of class C^p. Let $q \geq 1$, $n \geq 1$, $\bar{a}_1, \dots, \bar{a}_q$, $\bar{b}_1, \dots, \bar{b}_n \in Z \backslash \{0\}$. Let us consider G, H defined in (2.1), (3.5) and $B \in L(X, \mathbb{R}^{q+m})$, $\bar{\mathcal{B}} \in L(\Delta, \mathbb{R}^n)$. Let us take some points $\widetilde{s}_0$, $\widetilde{\phi'_0} \in \Gamma$. Let S and Φ be the functions defined in (3.7), for some $\theta_0 \in \mathbb{R}^{q+m}$, and (3.10), respectively.

Let us construct the function S from (3.7), assuming the existence of the elements that allow this construction. We seek a function S_0 of the same form (3.7) such that the equation $S_0(s) = 0$ has a solution s_0 and both S_0 and s_0 satisfy the statement (ii) of Theorem 3.5.

We started with the following analysis. For the function S from (3.7), consider a point $\widetilde{s}_0$ of the form of s_0, that is, $\widetilde{f}_0 = 0$, $\widetilde{g}_{i,0} = 0$, $\widetilde{e}_{k,0} = 0$. Assume that $DS(\widetilde{s}_0)$ is an isomorphism of Γ onto Σ. Under an additional hypothesis, according to Theorem 3.1 [18], Theorem IV.3.1 [29] and Theorem I.2.1 [14], equation $S(s) = 0$ (that is, (3.8)) has a solution s in a neighborhood of $\widetilde{s}_0$. We are interested in the case that the solution s is of the form of s_0 from Theorem 3.5 (ii) (a). Let $\widetilde{\varrho}_0 = S(\widetilde{s}_0)$. Applying the inverse function theorem, S is a local C^p-diffeomorphism at $\widetilde{s}_0$ from $\widetilde{\mathcal{U}}$ onto $\widetilde{\mathcal{V}}$, where $\widetilde{\mathcal{U}}$ is a neighborhood of $\widetilde{s}_0$ and $\widetilde{\mathcal{V}}$ is a neighborhood of $\widetilde{\varrho}_0$. Let us take Φ defined in (3.10). $\Phi(x, \cdot)$ is an isomorphism of Γ onto Σ for x from some neighborhood of $\widetilde{x}_0$. For each $s \in \widetilde{\mathcal{U}}$, we can consider the equation in $\phi' \in \Gamma$,

$$S(s) - \Phi(x, \phi') = 0\,. \tag{5.1}$$

Let us consider now this equation for $(s, \phi') \in \widetilde{\mathcal{U}} \times \Gamma$. Let us observe that the form of the left hand side of (5.1) can be compared, especially the rows that correspond to $\Phi_G(x, y_i)$ and $\Phi_H(x, \mathrm{z}_k)$, with the form of the function $S(s)$ defined in (3.7). We then ask if there exists a solution (s, ϕ') of (5.1) which allows us to find the function S_0 and the solution s_0.

We obtain the function S_0 as a perturbation of S. The main result is in Theorem 5.4.

The idea of the proof came when we read the proofs of Theorem 6C.1 and Theorem 6D.1, from [21]. Here, some mappings $G_{[21]}$ and $g_{[21]}$ are constructed and, under adequate hypotheses, the existence of a solution of the equation $g_{[21]}(x) \in G_{[21]}(x)$ is proved (using the contraction mapping principle for set-valued mappings, Theorem 5E.2 [21]). For the formulation of the results and of the proofs, we also take into account Graves' theorem 5D.2 [21] together with one of its consequences and the techniques from the proofs of Theorem 3.1 [18], Theorem IV.3.1 [29] and Theorem I.2.1 [14].

The analysis of (5.1) and the [21] method mentioned above lead us to the construction of the mappings $\mathcal{G}(s, \phi')$ and $\mathcal{Q}(s, \phi')$ from (5.7) and (5.8). The aim of this construction is that equation $\mathcal{Q}(s, \phi') \in \mathcal{G}(s, \phi')$ and a solution $(\bar{s}, \bar{\phi}')$ of this one generate a function S_0 of the form (3.7) and a solution s_0, with $f_0 = 0$, $g_{i,0} = 0$, $e_{k,0} = 0$, $i = 1, \ldots, q + m$, $k = 1, \ldots, n$, of the equation $S_0(s) = 0$. S_0 and s_0 satisfy the statements (a) and (b) of Theorem 3.5 (ii).

We obtain

$$S(s_0) - \Phi(x_0, \phi'_0) \ni 0\,, \tag{5.2}$$

that is, (s_0, ϕ'_0) is a solution of the equation

$$S(s) - \Phi(x, \phi') \ni 0\,. \tag{5.3}$$

This is equation (5.26). Relation (5.2) allows us to get S_0 and s_0. Corollary 5.10 tells us that equation $S_0(s) = 0$ provides a bifurcation problem $F(\lambda, u) - \varrho = 0$ for which the given problem (1.1) is a perturbation.

5.2. **The definition of the mappings $\mathcal{G}(s,\phi')$ and $\mathcal{Q}(s,\phi')$. - The points $\widetilde{s}_0$ and $\widetilde{\phi}'_0$**

We denote $\widetilde{s}_0=(\widetilde{x}_0,\widetilde{y}_{i,0},\widetilde{z}_{k,0})$ and $\widetilde{\phi}'_0=(\widetilde{y}'_0,\widetilde{y}'_{i,0},\widetilde{z}'_{k,0})\in\Gamma$.
$\widetilde{s}_0=(\widetilde{x}_0,\widetilde{y}_{1,0},\ldots,\widetilde{y}_{q+m,0},\widetilde{z}_{1,0},\ldots,\widetilde{z}_{n,0})$, $\widetilde{\phi}'_0=(\widetilde{y}'_0,\widetilde{y}'_{1,0},\ldots,\widetilde{y}'_{q+m,0},\widetilde{z}'_{1,0},\ldots,\widetilde{z}'_{n,0})$.
$\widetilde{x}_0=(\widetilde{f}_0,\widetilde{\lambda}_0,\widetilde{u}_0)$, $\widetilde{y}_{i,0}=(\widetilde{g}_{i,0},\widetilde{\mu}_{i,0},\widetilde{w}_{i,0})$, $\widetilde{z}_{k,0}=(\widetilde{e}_{k,0},\widetilde{v}_{k,0})$,
$\widetilde{y}'_0=(\widetilde{g}'_0,\widetilde{\mu}'_0,\widetilde{w}'_0)$, $\widetilde{y}'_{i,0}=(\widetilde{g}'_{i,0},\widetilde{\mu}'_{i,0},\widetilde{w}'_{i,0})$, $\widetilde{z}'_{k,0}=(\widetilde{e}'_{k,0},\widetilde{v}'_{k,0})$
and $\widetilde{f}_0=0$, $\widetilde{g}_{i,0}=0$, $\widetilde{e}_{k,0}=0$, $\widetilde{y}'_0=0$, $\widetilde{\phi}'_0=0$, $i=1,\ldots,q+m$, $k=1,\ldots,n$.

Let us take the value $\widetilde{\theta}_0=B(\widetilde{x}_0)\in\mathbb{R}^{q+m}$ for θ_0 in the definition (3.3) of Ψ and in the definition (3.7) of S. Let Φ be defined by (3.10).

Let us denote
$\xi(g',g'_i,e'_k)=\widetilde{\phi}'_0+((g',0,0),(g'_1,0,0),\ldots,(g'_{q+m},0,0),(e'_1,0,0),\ldots,(e'_n,0,0))$,
that is, $\xi(g',g'_i,e'_k)$ is $\widetilde{\phi}'_0$ where $\widetilde{g}'_0$, $\widetilde{g}'_{i,0}$, $\widetilde{e}'_{k,0}$ are replaced by $\widetilde{g}'_0+g'$, $\widetilde{g}'_{i,0}+g'_i$, $\widetilde{e}'_{k,0}+e'_k$ respectively. We have

$$\Phi(x,\xi(g',g'_i,e'_k))=\begin{bmatrix} B(g',0,0) \\ -\sum_{\imath=1}^{q}(g')^{\imath}\bar{a}_{\imath} \\ B(g'_i,0,0) \\ -\sum_{\ell=1}^{q}(g'_i)^{\ell}\bar{a}_{\ell} \\ \bar{\mathcal{B}}(e'_k,0) \\ -\sum_{\jmath=1}^{n}(e'_k)^{\jmath}\bar{b}_{\jmath}\end{bmatrix}. \tag{5.4}$$

- The mappings $\mathcal{G}$ and $\mathcal{Q}$

Let us fix a real α, $0<\alpha<1$, $\alpha\neq\frac{1}{2}$.

Let us define

$$\mathcal{H}_S(s)=S(\widetilde{s}_0)+DS(\widetilde{s}_0)(s-\widetilde{s}_0)\,, \tag{5.5}$$

$$\mathcal{H}_\Phi(x,\phi')=\Phi(\widetilde{x}_0,\widetilde{\phi}'_0)+D\Phi(\widetilde{x}_0,\widetilde{\phi}'_0)((x,\phi')-(\widetilde{x}_0,\widetilde{\phi}'_0))\,. \tag{5.6}$$

$\mathcal{H}_S$ is a strict first-order approximation of S at $\widetilde{s}_0$.

$\mathcal{H}_\Phi$ is a strict first-order approximation of Φ at $(\widetilde{x}_0,\widetilde{\phi}'_0)$.

$\mathcal{G}:\Gamma\times\Gamma\to\Sigma$,

$$\begin{aligned}\mathcal{G}(s,\phi')&=\frac{1}{2}S(s)-\frac{1}{2}\Phi(x,\phi')+\frac{1}{2}\mathcal{H}_S(s)-\frac{1}{2}\mathcal{H}_\Phi(x,\phi')\\ &-(1-\alpha)\Phi(\widetilde{x}_0,\xi(f,g_i,e_k))+(1-\alpha)\Phi(\widetilde{x}_0,\xi(g',g'_i,e'_k))\,.\end{aligned} \tag{5.7}$$

$\mathcal{Q}:\Gamma\times\Gamma\to\Sigma$,

$$\begin{aligned}\mathcal{Q}(s,\phi')&=-\frac{1}{2}S(s)+\frac{1}{2}\Phi(x,\phi')+\frac{1}{2}\mathcal{H}_S(s)-\frac{1}{2}\mathcal{H}_\Phi(x,\phi')\\ &+\alpha\Phi(\widetilde{x}_0,\xi(f,g_i,e_k))-\alpha\Phi(\widetilde{x}_0,\xi(g',g'_i,e'_k))\,.\end{aligned} \tag{5.8}$$

Observe that we have

$$\Phi(x,\xi(f,g_i,e_k))-\Phi(x,\xi(g',g'_i,e'_k))=\Phi(\widetilde{x}_0,\xi(f,g_i,e_k))-\Phi(\widetilde{x}_0,\xi(g',g'_i,e'_k))\,,$$

$$\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)=S(\widetilde{s}_0)\,,\ \mathcal{Q}(\widetilde{s}_0,\widetilde{\phi}'_0)=0\,. \tag{5.9}$$

Lemma 5.1. *$\mathcal{G}$ is surjective.*

Proof. Using (5.7) and (5.16), we have

$$\mathcal{G}(\widetilde{s}_0, \phi') = S(\widetilde{s}_0) - \Phi(\widetilde{x}_0, \phi') + (1-\alpha)\Phi(\widetilde{x}_0, \xi(g', g_i', e_k')) \,.$$

From Lemma 3.4, we have that $\Phi(\widetilde{x}_0, \cdot)$ is an isomorphism of Γ onto Σ. Hence $\forall\, \zeta \in \Sigma$, $\exists\, (\widetilde{s}_0, \phi') \in \Gamma \times \Gamma$ such that $\mathcal{G}(\widetilde{s}_0, \phi') = \zeta$ and $\mathcal{G}$ is surjective. $\square$

We use the surjectivity of $\mathcal{G}$ in the Proof of Theorem 5.4, Subsection 5.5. We mention that we can skip this condition since we work with set-valued mappings; for details for the methodology, see [21].

- The operator $A = \mathcal{A}$

$$\mathcal{A} = D\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}_0') \,. \tag{5.10}$$

We write $\mathcal{A}(\alpha)$ when it is necessary to consider α as a variable.

$$\begin{aligned} D\mathcal{G}(s, \phi')(\overline{s}, \overline{\phi}') &= \frac{1}{2} DS(s)\overline{s} - \frac{1}{2} D\Phi(x, \phi')(\overline{x}, \overline{\phi}') - \frac{1}{2} D\Phi(\widetilde{x}_0, \widetilde{\phi}_0')(\overline{x}, \overline{\phi}') \\ &+ \frac{1}{2} DS(\widetilde{s}_0)\overline{s} - (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{f}, \overline{g}_i, \overline{e}_k)) + (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{g}', \overline{g}_i', \overline{e}_k')) \,. \end{aligned} \tag{5.11}$$

$$\begin{aligned} \mathcal{A}(\overline{s}, \overline{y}') &= DS(\widetilde{s}_0)\overline{s} - D\Phi(\widetilde{x}_0, \widetilde{\phi}_0')(\overline{x}, \overline{\phi}') \\ &-(1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{f}, \overline{g}_i, \overline{e}_k)) + (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{g}', \overline{g}_i', \overline{e}_k')) \,, \end{aligned} \tag{5.12}$$

where

$$DS(\widetilde{s}_0)\overline{s} = \begin{bmatrix} B(\overline{x}) \\ DG(\widetilde{x}_0)\overline{x} \\ B(\overline{y}_i) \\ D^2F(\widetilde{\lambda}_0, \widetilde{u}_0)((\widetilde{\mu}_{i,0}, \widetilde{w}_{i,0}), (\overline{\lambda}, \overline{u})) + DG(\widetilde{x}_0)\overline{y}_i \\ \bar{\mathcal{B}}(\overline{z}_k) \\ D_{(\lambda,u)}(D_u F(\widetilde{\lambda}_0, \widetilde{u}_0)\widetilde{v}_{k,0})(\overline{\lambda}, \overline{u}) + H(\widetilde{\lambda}_0, \widetilde{u}_0, \overline{z}_k) \end{bmatrix} , \tag{5.13}$$

and

(5.14)

$$D\Phi(\widetilde{x}_0, \widetilde{\phi}_0')(\overline{x}, \overline{\phi}') = \begin{bmatrix} B(\overline{y}') \\ D^2F(\widetilde{\lambda}_0, \widetilde{u}_0)((\widetilde{\mu}_0', \widetilde{w}_0'), (\overline{\lambda}, \overline{u})) + DG(\widetilde{x}_0)\overline{y}' \\ B(\overline{y}_i') \\ D^2F(\widetilde{\lambda}_0, \widetilde{u}_0)((\widetilde{\mu}_{i,0}', \widetilde{w}_{i,0}'), (\overline{\lambda}, \overline{u})) + DG(\widetilde{x}_0)\overline{y}_i' \\ \bar{\mathcal{B}}(\overline{z}_k') \\ D_{(\lambda,u)}(D_u F(\widetilde{\lambda}_0, \widetilde{u}_0)\widetilde{v}_{k,0}')(\overline{\lambda}, \overline{u}) + H(\widetilde{\lambda}_0, \widetilde{u}_0, \overline{z}_k') \end{bmatrix} ,$$

for all $i = 1, \ldots, q+m$, $k = 1, \ldots, n$. We have $D^2F(\widetilde{\lambda}_0, \widetilde{u}_0)((\widetilde{\mu}_0', \widetilde{w}_0'), (\overline{\lambda}, \overline{u})) = 0$, $D^2F(\widetilde{\lambda}_0, \widetilde{u}_0)((\widetilde{\mu}_{i,0}', \widetilde{w}_{i,0}'), (\overline{\lambda}, \overline{u})) = 0$, $D_{(\lambda,u)}(D_u F(\widetilde{\lambda}_0, \widetilde{u}_0)\widetilde{v}_{k,0}')(\overline{\lambda}, \overline{u}) = 0$.

- Condition $A = \mathcal{A}$ is surjective By fixing $\overline{y}'$, the condition is verified immediately.

- The definitions of $reg\,\mathcal{A}$, κ and γ

$reg\,\mathcal{A}$ is defined as in (2.7). We take $\kappa = reg\,\mathcal{A}$. Define $\gamma_\Psi = \widetilde{\gamma}(\Psi, \widetilde{x}_0, X, Y)$, $\gamma = \gamma_S = \widetilde{\gamma}(S, \widetilde{s}_0, \Gamma, \Sigma)$, $\gamma_{S_0} = \widetilde{\gamma}(S_0, \widetilde{s}_0, \Gamma, \Sigma)$, where we use (2.11). Observe that γ do not depend on α.

Lemma 5.2.

$$\kappa \leq \gamma \,. \tag{5.15}$$

Proof. We have $\widetilde{\phi}'_0 = 0$. (5.14) gives

$$\Phi(\widetilde{x}_0, \overline{\phi}') = D\Phi(\widetilde{x}_0, \widetilde{\phi}'_0)(\overline{x}, \overline{\phi}')\ , \tag{5.16}$$

and we replace $D\Phi(\widetilde{x}_0, \widetilde{\phi}'_0)(\overline{x}, \overline{\phi}')$ by $\Phi(\widetilde{x}_0, \overline{\phi}')$ in the expression (5.12) of $\mathcal{A}$.

We denote

$$DS_\alpha(\widetilde{s}_0)\overline{s} = DS(\widetilde{s}_0)\overline{s} - (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{f}, \overline{g}_i, \overline{e}_k))\,, \tag{5.17}$$

$$\Phi_\alpha(\widetilde{x}_0, \overline{\phi}') = \Phi(\widetilde{x}_0, \overline{\phi}') - (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{g}', \overline{g}'_i, \overline{e}'_k))\,. \tag{5.18}$$

Hence

$$\mathcal{A}(\overline{s}, \overline{y}') = DS_\alpha(\widetilde{s}_0)\overline{s} - \Phi_\alpha(\widetilde{x}_0, \overline{\phi}')\,. \tag{5.19}$$

For $y \in \Sigma$, we have the sets $\mathcal{E}_y$, $\mathcal{E}^0_y$ such that

$\mathcal{E}_y = \mathcal{A}^{-1}(y) = (DS_\alpha(\widetilde{s}_0) - \Phi_\alpha(\widetilde{x}_0, \cdot))^{-1}(y) = \{(\overline{s}, \overline{\phi}') | DS_\alpha(\widetilde{s}_0)\overline{s} - \Phi_\alpha(\widetilde{x}_0, \overline{\phi}') = y\} \supseteq \{(\overline{s}, \overline{\phi}') | DS_\alpha(\widetilde{s}_0)\overline{s} = y,\ \overline{\phi}' = 0\} = \{(\overline{s}, \overline{\phi}') | \overline{s} = DS_\alpha(\widetilde{s}_0)^{-1}(y),\ \overline{\phi}' = 0\} = \mathcal{E}^0_y$.

$$\begin{aligned} reg\,\mathcal{A} &= \sup_{\|y\|\leq 1} d(0, \mathcal{A}^{-1}(y)) = \sup_{\|y\|\leq 1} d(0, \mathcal{E}_y) \\ &\leq \sup_{\|y\|\leq 1} d(0, \mathcal{E}^0_y) = \sup_{\|y\|\leq 1} d(0, \{\overline{s} | \overline{s} = DS_\alpha(\widetilde{s}_0)^{-1}(y)\}) \\ &= \sup_{\|y\|\leq 1} \|DS_\alpha(\widetilde{s}_0)^{-1}(y)\| = \|DS(\widetilde{s}_0)^{-1}\|_{L(\Sigma,\Gamma)}\,. \end{aligned} \tag{5.20}$$

□

- The definition of $\mu = L(\varepsilon)$

We denote $\widetilde{\mathcal{G}}_3(s, \phi') = \frac{1}{2}S(s) - \frac{1}{2}\Phi(x, \phi')$.

$D\widetilde{\mathcal{G}}_3(s, \phi')(\overline{s}, \overline{\phi}') = \frac{1}{2}DS(s)\overline{s} - \frac{1}{2}D\Phi(x, \phi')(\overline{x}, \overline{\phi}')$

$\psi_1((s, \phi'), (\overline{s}, \overline{\phi}'))$

$=$

$\|\widetilde{\mathcal{G}}_3(s, \phi') - \widetilde{\mathcal{G}}_3(\overline{s}, \overline{\phi}') - D\widetilde{\mathcal{G}}_3(\widetilde{s}_0, \widetilde{\phi}'_0)((s, \phi') - (\overline{s}, \overline{\phi}'))\|$

$=$

$\| \int_0^1 [D\widetilde{\mathcal{G}}_3((\overline{s}, \overline{\phi}') + t((s, \phi') - (\overline{s}, \overline{\phi}'))) - D\widetilde{\mathcal{G}}_3(\widetilde{s}_0, \widetilde{\phi}'_0)] \cdot ((s, \phi') - (\overline{s}, \overline{\phi}'))dt\|$,

where we use some standard techniques from [14, 18, 29, 49].

$\psi_2((s, \phi'), (\overline{s}, \overline{\phi}')) =$

$\sum_{\imath=1}^{q} |f^\imath - \overline{f}^\imath| \|\bar{a}_\imath\| + \sum_{i=1}^{q+m} \sum_{\ell=1}^{q} |g_i^\ell - \overline{g}_i^\ell| \|\bar{a}_\ell\| + \sum_{k=1}^{n} \sum_{\jmath=1}^{n} |e_k^\jmath - \overline{e}_k^\jmath| \|\bar{b}_\jmath\|$ $+ \sum_{\imath=1}^{q} |(g')^\imath - (\overline{g}')^\imath| \|\bar{a}_\imath\| + \sum_{i=1}^{q+m} \sum_{\ell=1}^{q} |(g'_i)^\ell - (\overline{g}'_i)^\ell| \|\bar{a}_\ell\| + \sum_{k=1}^{n} \sum_{\jmath=1}^{n} |(e'_k)^\jmath - (\overline{e}'_k)^\jmath| \|\bar{b}_\jmath\|$.

Take $\mu = L(\varepsilon) = \widetilde{L}(\mathcal{G}, (\widetilde{s}_0, \widetilde{\phi}'_0), (s, \phi'), \varepsilon, \Gamma \times \Gamma, \Sigma)$, where we use (2.12). We obtain $\mu = L(\varepsilon) = \widetilde{L}(\widetilde{\mathcal{G}}_3, (\widetilde{s}_0, \widetilde{\phi}'_0), (s, \phi'), \varepsilon, \Gamma \times \Gamma, \Sigma)$.

Define $L_\Psi(\varepsilon) = \widetilde{L}(\Psi, \widetilde{x}_0, x, \varepsilon, X, Y)$, $L_S(\varepsilon) = \widetilde{L}(S, \widetilde{s}_0, s, \varepsilon, \Gamma, \Sigma)$ and $L_{S_0}(\varepsilon) = \widetilde{L}(S_0, \widetilde{s}_0, s, \varepsilon, \Gamma, \Sigma)$. We have

$$L_S(\varepsilon) = \sup_{(s,\phi') \in \mathbb{B}_\varepsilon(\widetilde{s}_0, \widetilde{\phi}'_0),\, \phi'=0} \|DS(\widetilde{s}_0) - DS(s)\|_{L(\Gamma,\Sigma)}\ . \tag{5.21}$$

Lemma 5.3.

$$\frac{1}{2}L_S(\varepsilon) \leq L(\varepsilon)\ . \tag{5.22}$$

Proof. See Appendix A. Let us use (A.1). We have $\widetilde{\phi}'_0 = 0$. Take $\phi' = 0$. We have

$$\begin{aligned} &\frac{1}{2}\|DS(\widetilde{s}_0) - DS(s)\| \\ = &\sup_{\|(\overline{s},\overline{\phi}')\|\leq 1,\,\overline{\phi}'=0} \|D\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)(\overline{s},\overline{\phi}') - D\mathcal{G}(s,0)(\overline{s},\overline{\phi}')\| \\ \leq &\sup_{\|(\overline{s},\overline{\phi}')\|\leq 1} \|D\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)(\overline{s},\overline{\phi}') - D\mathcal{G}(s,0)(\overline{s},\overline{\phi}')\| \\ &= \|D\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0) - D\mathcal{G}(s,0)\|\ . \end{aligned}$$

$$\frac{1}{2}L_S(\varepsilon) \leq \sup_{(s,\phi')\in\mathbb{B}_\varepsilon(\widetilde{s}_0,\widetilde{\phi}'_0),\,\phi'=0} \|D\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0) - D\mathcal{G}(s,0)\|_{L(\Gamma\times\Gamma,\Sigma)}$$

$$\leq \sup_{(s,\phi')\in\mathbb{B}_\varepsilon(\widetilde{s}_0,\widetilde{\phi}'_0)} \|D\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0) - D\mathcal{G}(s,\phi')\|_{L(\Gamma\times\Gamma,\Sigma)} = \mu = L(\varepsilon)\ .$$

□

5.3. The theorem on the existence of a bifurcation problem.

Theorem 5.4. *Let W and Z be real Banach spaces. They are both infinite-dimensional spaces or they are both finite-dimensional spaces with dim W = dim Z. Let $m \geq 1$, $p \geq 2$. Let $F : \mathbb{R}^m \times W \to Z$ be a nonlinear function of class C^p. Let $q \geq 1$, $n \geq 1$, $\bar{a}_1,\ldots,\bar{a}_q$, $\bar{b}_1,\ldots,\bar{b}_n \in Z\backslash\{0\}$. Let us consider G, H defined in (2.1), (3.5) and $B \in L(X,\mathbb{R}^{q+m})$, $\bar{\mathcal{B}} \in L(\Delta,\mathbb{R}^n)$. Let us take the points $\widetilde{s}_0$ and $\widetilde{\phi}'_0$ introduced above. Let S and Φ be the functions defined in (3.7), where θ_0 is replaced by $\widetilde{\theta}_0 = B(\widetilde{x}_0)$, and (3.10), respectively. Let us consider the above definitions (5.7) and (5.8) of $\mathcal{G}$ and $\mathcal{Q}$ and the related entities.*

Assume that for some ε and α, $\varepsilon > 0$ and $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, α arbitrarily small, we have

$$2\kappa L(\varepsilon) + 2\kappa\alpha\widehat{a} < 1\,, \tag{5.23}$$

where $\kappa \geq reg\,\mathcal{A}$ and $\widehat{a} = \max\{\ \|\bar{a}_1\|,\ldots,\|\bar{a}_q\|,\ \|\bar{b}_1\|,\ldots,\|\bar{b}_n\|\ \}$.

Let $c = \kappa^{-1} - L(\varepsilon)$ and $M > \|D\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)\|_{L(\Gamma\times\Gamma,\Sigma)}$.

Then, there exist positive constants a^* and $b^* = ca^*$ such that

$$d((s,\phi'),\mathcal{G}^{-1}(\zeta)) \leq \frac{\kappa}{1-\kappa\mu}\|\zeta - \mathcal{G}(s,\phi')\|\,, \tag{5.24}$$

for $((s,\phi'),\zeta) \in \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}'_0) \times \mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0))$.

Let $\delta = \|\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)\|_\Sigma = \|S(\widetilde{s}_0)\|_\Sigma$. Assume

$$\delta < \frac{1}{2}ca^*\,. \tag{5.25}$$

Assume that $D\Psi(\widetilde{x}_0)$ is an isomorphism of X onto Y and $DS(\widetilde{s}_0)$ is an isomorphism of Γ onto Σ.

Then, there exists a solution $(\hat{s}_0, \hat{\phi}'_0) \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$ *of the equation*

$$S(s) - \Phi(x, \phi') \ni 0\,, \tag{5.26}$$

where $\hat{s}_0 = (\hat{x}_0, \hat{y}_{1,0}, \ldots, \hat{y}_{q+m,0}, \hat{\mathrm{z}}_{1,0}, \ldots, \hat{\mathrm{z}}_{n,0})$,
$\hat{x}_0 = (\hat{f}_0, \hat{\lambda}_0, \hat{u}_0)$, $\hat{y}_{i,0} = (\hat{g}_{i,0}, \hat{\mu}_{i,0}, \hat{w}_{i,0})$, $\hat{\mathrm{z}}_{k,0} = (\hat{e}_{k,0}, \hat{v}_{k,0})$,
$\hat{f}_0 = 0$, $\hat{g}_{i,0} = 0$, $\hat{e}_{k,0} = 0$, $i = 1, \ldots, q+m$, $k = 1, \ldots, n$.
$\hat{\phi}'_0 = (\hat{y}'_0, \hat{y}'_{1,0}, \ldots, \hat{y}'_{q+m,0}, \hat{\mathrm{z}}'_{1,0}, \ldots, \hat{\mathrm{z}}'_{n,0})$,
$\hat{y}'_0 = (\hat{g}'_0, \hat{\mu}'_0, \hat{w}'_0)$, $\hat{y}'_{i,0} = (\hat{g}'_{i,0}, \hat{\mu}'_{i,0}, \hat{w}'_{i,0})$, $\hat{\mathrm{z}}'_{k,0} = (\hat{e}'_{k,0}, \hat{v}'_{k,0})$
and $\hat{g}'_0 = 0$, $\hat{g}'_{i,0} = 0$, $\hat{e}'_{k,0} = 0$, $i = 1, \ldots, q+m$, $k = 1, \ldots, n$.

Let $s_0 = (x_0, y_{1,0}, \ldots, y_{q+m,0}, \mathrm{z}_{1,0}, \ldots, \mathrm{z}_{n,0})$ *with* $(f_0, \lambda_0, u_0) = x_0 = \hat{x}_0$, $y_{i,0} = \hat{y}_{i,0} - \hat{y}'_{i,0}$, $\mathrm{z}_{k,0} = \hat{\mathrm{z}}_{k,0} - \hat{\mathrm{z}}'_{k,0}$ *and* $i = 1, \ldots, q+m$, $k = 1, \ldots, n$. $s_0 \in \mathbb{B}_{a^*}(\widetilde{s}_0)$.

Let us fix $\hat{x}_0$ *and* $\hat{y}'_0$ *(whose existence is demonstrated) in* $\Phi_G(x, y')$ *from the first line of* $\Phi(x, \phi')$ *in (5.26). Let us take* $\theta_0 = \widetilde{\theta}_0 + B(\hat{y}'_0)$ *and* $\varrho = DF(\hat{\lambda}_0, \hat{u}_0)(\hat{\mu}'_0, \hat{w}'_0)$ *and consider them as constants. Let us denote by* S_0 *the function* S *from (3.7) formulated for this* θ_0 *and for* $F(\lambda, u) - \varrho$ *instead of* $F(\lambda, u)$.

Then, s_0 *is the solution of the equation*

$$S_0(s) = 0\ . \tag{5.27}$$

Equation (5.27) is of the form of equation (3.8).

Then, the component (λ_0, u_0) *of* s_0 *is a solution of the equation*

$$F(\lambda, u) - \varrho = 0\,, \tag{5.28}$$

$(\lambda_0, u_0) \in \mathbb{B}_{a^*}(\widetilde{\lambda}_0, \widetilde{u}_0)$.

Let us replace κ *by* γ *in (5.23) and, instead of (5.23), consider*

$$2\gamma L(\varepsilon) + 2\kappa\alpha\widehat{a} < 1\,. \tag{5.29}$$

Then, s_0 *is the unique solution of the equation (5.27) in* $\mathbb{B}_a(\widetilde{s}_0)$ *for every* $a \geq a^*$ *that satisfies* $\gamma L_S(a) < 1$. *The system (5.27) and its solution* s_0 *verify the assertions (a) and (b) of the statement (ii) of Theorem 3.5.*

Then, the component (λ_0, u_0) *of* s_0 *is the unique solution of the equation (5.28) that satisfies hypothesis (1.2) and the rest of the hypotheses of the statement (i) of Theorem 3.5 in* $\mathbb{B}_a(\widetilde{\lambda}_0, \widetilde{u}_0)$ *for every* $a \geq a^*$ *that satisfies* $\gamma L_S(a) < 1$. *The solution* (λ_0, u_0) *is a bifurcation point of problem (5.28).*

We have

$$\|\varrho\| \leq \frac{a^*}{2\gamma} + \|S(\widetilde{s}_0)\| < (\frac{1}{\gamma} - \frac{1}{2}L(\varepsilon))a^*\,, \tag{5.30}$$

$$\|\varrho\| \leq (\|DF(\widetilde{\lambda}_0, \widetilde{u}_0)\| + L_F(a^*))a^*\,, \tag{5.31}$$

$$\|s_0 - r\|_\Gamma \leq [\gamma/(1 - \gamma L_S(a)] \cdot \|S_0(r)\|_\Sigma\,, \forall r \in \mathbb{B}_a(\widetilde{s}_0)\,. \tag{5.32}$$

The proof of Theorem 5.4 is given after some remarks.

The following conditions (5.33) and (5.34) do not depend on α.

Corollary 5.5. *(i) Let* α_0 *be an arbitrarily fixed positive number,* $0 < \alpha_0 < 0.1$. *Let us take*

$$\kappa = \sup_{0<\alpha\leq\alpha_0} reg\,\mathcal{A}(\alpha)\,,\ M > \sup_{0<\alpha\leq\alpha_0} \|D\mathcal{G}(\widetilde{s}_0, \widetilde{y}'_0)\| = \sup_{0<\alpha\leq\alpha_0} \|\mathcal{A}(\alpha)\|\,.$$

Instead of (5.23), we can assume that for some ε, $\varepsilon > 0$, we have

$$2\kappa L(\varepsilon) < 1\,. \tag{5.33}$$

(ii) Instead of (5.23), we can assume that for some ε, $\varepsilon > 0$, we have

$$2\gamma L(\varepsilon) < 1\,. \tag{5.34}$$

Proof. Let us observe that $L(\varepsilon)$ do not depend on α. We have (5.33). There exists $0 < \alpha < \alpha_0$ ($0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$), such that (5.23) holds.

□

5.4. **Preliminaries related to the metric regularity property.** We now formulate some estimates of the radii of some balls included in the neighborhoods U and V from Corollary 2.5 related to the metric regularity property of the function f at $\bar{x}$ for $\bar{y}$.

Lemma 5.6. *Let us retain the hypotheses of Graves' theorem 2.4.*

Let $\widetilde{a}$ be such that $0 < \widetilde{a} < \varepsilon$ and $int\mathbb{B}_{\widetilde{a}}(\bar{x})$ is the maximal (for the inclusion) open ball, with center $\bar{x}$ and radius $\widetilde{a}$, contained in U.

Let a be fixed close to $\widetilde{a}$, $0 < a < \widetilde{a}$.

Let us fix b, $0 < b \leq c\tau$.

Let us take $a^* = \min\{a, \frac{b}{c}\}$ and $b^* = ca^*$. We denote $U^* = \mathbb{B}_{a^*}(\bar{x})$, $V^* = \mathbb{B}_{b^*}(f(\bar{x}))$.

The relation (2.9) holds for all $(x, y) \in U^* \times V^*$.

Proof. From the definition of U, we deduce that $U \subset \mathbb{B}_\varepsilon(\bar{x})$, so $0 < \widetilde{a} < \varepsilon$.

From the definition of V, there results that we can choose V of the form $\mathbb{B}_{c\tau}(f(\bar{x}))$, $int\mathbb{B}_{c\tau}(f(\overline{x}))$ or $\mathbb{B}_b(f(\overline{x}))$, with $0 < b < c\tau$.

The value $\frac{b}{c}$ is taken related to the conclusion of Graves' theorem (Theorem 2.4), to the relation (2.10) in Theorem 2.6 and to linear openess ([21]).

For any U^* and V^* such that $U^* \subseteq U$, $V^* \subseteq V$, we have (2.9) for $(x, y) \in U^* \times V^*$.

□

Consider $L(\varepsilon)$ as in [14, 18, 29] but with another radius.

Lemma 5.7. *Let us retain the hypotheses of Graves' theorem 2.4. Take $\mu = L(\varepsilon) = \widetilde{L}(f, \bar{x}, x, \varepsilon, X, Y)$ (where we use (2.12)). Let $M > \|Df(\bar{x})\|$. Let us consider the neighborhoods U and V from Corollary 2.5.*

(i) Relation (2.9) is satisfied for any $(x, y) \in \mathbb{B}_a(\bar{x}) \times V$, where $\mathbb{B}_a(\bar{x}) \subset U$ and $V \supseteq \mathbb{B}_b(\bar{y})$, with $a \geq a^* > a_h^*$ and $b \geq b^* > b_h^*$. $0 < a < \varepsilon$. τ, a^*, b^* are given by

$$\tau < \frac{L(\varepsilon) + M}{L(\varepsilon) + M + c} \cdot \varepsilon\,, \tag{5.35}$$

(We can impose even $\leq$ because of the increase with M.)

$$a_U^* = \frac{c\tau}{L(\varepsilon) + M}\,, \quad b_V^* = c\tau\,, \tag{5.36}$$

$$a^* = \min\{a_U^*, \frac{b_V^*}{c}\}\,, \quad b^* = ca^*\,. \tag{5.37}$$

(ii) Assume that $2\kappa L(\varepsilon) < 1$. Relation (2.9) holds for any $(x, y) \in \mathbb{B}_{a_h^*}(\bar{x}) \times V$, where $\mathbb{B}_{a_h^*}(\bar{x}) \subset U$ and $V = \mathbb{B}_{b_h^*}(\bar{y})$. τ, a_h^*, b_h^* are given by

$$\tau < \frac{1 + 2\kappa M}{2 + 2\kappa M} \cdot \varepsilon \,, \tag{5.38}$$

$$a_U^* = \frac{\tau}{1 + 2\kappa M} \,, \quad b_V^* = \frac{\tau}{2\kappa} < c\tau \,, \tag{5.39}$$

$$a_h^* = \min\{a_U^*, \kappa b_V^*\} \,, \quad b_h^* = c a_h^* \,. \tag{5.40}$$

Proof. **Proof of (i)**

We consider b_V^* from (5.36) and $V = \mathbb{B}_{b_V^*}(\bar{y})$. We seek a_U^* such that

$\|f(x) - f(\bar{x})\| \leq \|f(x) - f(\bar{x}) - Df(\bar{x})(x - \bar{x})\| + \|Df(\bar{x})(x - \bar{x})\| \leq L(\varepsilon)\|x - \bar{x}\| + \|Df(\bar{x})\|\|x - \bar{x}\| \leq (L(\varepsilon) + \|Df(\bar{x})\|)a_U^* \leq (L(\varepsilon) + M)a_U^*$ and we impose the condition for a_U^*

$$(L(\varepsilon) + M)a_U^* \leq c\tau \,. \tag{5.41}$$

We take a_U^* from (5.36). We have $\mathbb{B}_{a_U^*}(\bar{x}) \subset U$.

We must verify the condition $\mathbb{B}_\tau(x) \subset \mathbb{B}_\varepsilon(\bar{x})$ for all $x \in U$.

If $U = \mathbb{B}_{a_U^*}(\bar{x})$, we have: let $x' \in \mathbb{B}_\tau(x)$. $\|x' - \bar{x}\| \leq \|x' - x\| + \|x - \bar{x}\| \leq \tau + a_U^* = \tau + \frac{c\tau}{L(\varepsilon)+M} < \varepsilon$. So the condition (5.35) for τ.

Instead of a_U^* and $\mathbb{B}_{a_U^*}(\bar{x})$, let us consider the value $\frac{b_V^*}{c}$ and $\mathbb{B}_{\frac{b_V^*}{c}}(\bar{x})$ related to the conclusion of Graves' theorem (Theorem 2.4), to the relation (2.10) in Theorem 2.6 and to linear openess ([21]).

We take a^* from (5.37). We have $\mathbb{B}_{a^*}(\bar{x}) \subset U$.

If $a^* = \frac{b_V^*}{c}$, it is not necessary to modify τ from (5.35).

From the definition of U, we deduce that $U \subset \mathbb{B}_\varepsilon(\bar{x})$, so $0 < a^* < \varepsilon$.

Linear openness gives us $\mathbb{B}_{b^*}(\bar{y}) = [f(\bar{x}) + ca^* int\mathbb{B}] \cap V$ (see also Theorem 2.6). Hence (5.37) for b^*.

Proof of (ii)

We have $2\kappa L(\varepsilon) < 1$, so

$$c = \frac{1 - \kappa L(\varepsilon)}{\kappa} > \frac{1 - \frac{1}{2}}{\kappa} = \frac{1}{2\kappa} \,. \tag{5.42}$$

or $\frac{1}{c} < 2\kappa$. We also have $\kappa < \frac{1}{c}$ ([21]).

We also consider the following estimates:

Using (5.36), we take b_V^* from (5.39). We take $V = \mathbb{B}_{b_V^*}(\bar{y})$.

The relation (5.41) gives

$\|f(x) - f(\bar{x})\| \leq \|f(x) - f(\bar{x}) - Df(\bar{x})(x - \bar{x})\| + \|Df(\bar{x})(x - \bar{x})\| \leq L(\varepsilon)\|x - \bar{x}\| + \|Df(\bar{x})\|\|x - \bar{x}\| \leq (L(\varepsilon) + \|Df(\bar{x})\|)a_U^* \leq (L(\varepsilon) + M)a_U^*$ and we impose the condition for a_U^*

$$(L(\varepsilon) + M)a_U^* < (\frac{1}{2\kappa} + M)a_U^* \leq \frac{\tau}{2\kappa} < c\tau \,. \tag{5.43}$$

We take a_U^* from (5.39). We have $\mathbb{B}_{a_U^*}(\bar{x}) \subset U$.

If $U = \mathbb{B}_{a_U^*}(\bar{x})$, we have: using (5.39), we obtain τ from (5.38).

We have $\kappa < \frac{1}{c}$. In order to avoid the dependence on the index "h" in Theorem 7.7, we take κ instead of $\frac{1}{c}$. More precisely, we take κb_V^* instead of $\frac{b_V^*}{c}$. κb_V^* do not depend on "h". $\kappa b_V^* < \frac{b_V^*}{c}$.

We take a_h^* from (5.40). We have $\mathbb{B}_{a_h^*}(\bar{x}) \subset U$.

If $a_h^* = \kappa b_V^*$, it is not necessary to modify τ from (5.38).

Linear openness ([21]) gives us $\mathbb{B}_{b_h^*}(\bar{y}) = [f(\bar{x}) + ca_h^* int\mathbb{B}] \cap V$ (see also Theorem 2.6). Hence (5.40) for b_h^*.

□

Lemma 5.8. *In Theorems 5.4, 7.6 and 7.7, we can consider τ, a^* and b^* from the general case as in Lemmas 5.6 and 5.7 by replacing f by $\mathcal{G}$. In Theorem 7.7, as in Lemma 5.7 (ii), ε, τ and a_h^* do not depend on the parameter h (they are constants).*

5.5. **Proof of Theorem 5.4.** Let us observe first that (5.23) is equivalent to

$$\frac{1}{c}(\frac{1}{2}L(\varepsilon) + \alpha\widehat{a}) < \frac{1}{2}\,. \tag{5.44}$$

Since $\mathcal{G}$ is surjective (see Lemma 5.1), $\mathcal{G}^{-1}$ is a set-valued mapping, so we can apply the framework of Subsection 2.2.

- (i) The verification of the assumptions of Graves' theorem 2.4

In Theorem 2.4, in Corollary 2.5, in Lemma 5.6 and in Lemma 5.7, let us replace X, Y, f, x, $\bar{x}$, y, $\bar{y} = f(\bar{x})$, ε, A, κ, μ, c, τ, U, V, $d(\cdot,\cdot)$, $\|\cdot\|$ by $\Gamma \times \Gamma$, Σ, $\mathcal{G}$, (s,ϕ'), $(\widetilde{s}_0,\widetilde{\phi}'_0)$, ζ, $\widetilde{\zeta}_0 = \mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)$, ε, $\mathcal{A}$, κ, $L(\varepsilon)$, c, τ, U, V, $d(\cdot,\cdot)$, $\|\cdot\|$, respectively.

Here and in Lemmas 5.6, 5.7, we use the same notations M, τ, a^*, b^*.

- Condition $A = \mathcal{A}$ is surjective is verified above.
- Condition $\kappa \geq reg\, A$ is verified since we take $\kappa = reg\,\mathcal{A}$.
- Let us introduce $\mu = L(\varepsilon) = \widetilde{L}(\mathcal{G}, (\widetilde{s}_0,\widetilde{\phi}'_0), (s,\phi'), \varepsilon, \Gamma\times\Gamma, \Sigma)$.
- Condition $\kappa\mu < 1$ results from (5.23).
- Let us verify condition (2.8) of Graves' theorem 2.4.

We have

$\|\mathcal{G}(s,\phi') - \mathcal{G}(\overline{s},\overline{\phi}') - \mathcal{A}((s,\phi') - (\overline{s},\overline{\phi}'))\| = \psi_1((s,\phi'),(\overline{s},\overline{\phi}'))$.

As in the proofs of Theorem 3.1 [18], Theorem IV.3.1 [29] and Theorem I.2.1 [14], we obtain:

$\mathcal{G}(s,\phi') - \mathcal{G}(\overline{s},\overline{\phi}') - \mathcal{A}((s,\phi') - (\overline{s},\overline{\phi}'))$

$=$

$\int_0^1 [D\mathcal{G}((\overline{s},\overline{\phi}') + t((s,\phi') - (\overline{s},\overline{\phi}'))) - D\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)]\cdot((s,\phi') - (\overline{s},\overline{\phi}'))dt$

Relation (5.45) holds with $\mu = L(\varepsilon) = \widetilde{L}(\mathcal{G}, (\widetilde{s}_0,\widetilde{\phi}'_0), (s,\phi'), \varepsilon, \Gamma\times\Gamma, \Sigma)$.

$$\|\mathcal{G}(s,\phi') - \mathcal{G}(\overline{s},\overline{\phi}') - \mathcal{A}((s,\phi') - (\overline{s},\overline{\phi}'))\| \leq L(\varepsilon)\|(s,\phi') - (\overline{s},\overline{\phi}')\|\,. \tag{5.45}$$

($\mathcal{G}$ is strictly differentiable at $(\widetilde{s}_0,\widetilde{\phi}'_0)$ (see definition, pages 31 and 275, [21])).

Then, the conclusions and the consequences of Graves' theorem 2.4 hold for $\mathcal{G}$ and $(\widetilde{s}_0,\widetilde{\phi}'_0)$.

- (ii) The formulation of the consequence of Graves' theorem, Corollary 2.5

Let us take a positive $\tau < \varepsilon$. From Corollary 2.5 for $\mathcal{G}$, there results that there exist a neighborhood U of $(\widetilde{s}_0,\widetilde{\phi}'_0)$ and a neighborhood V of $\mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)$ such that (i) $\mathbb{B}_\tau(s,\phi') \subset \mathbb{B}_\varepsilon(\widetilde{s}_0,\widetilde{\phi}'_0)$ for all $(s,\phi') \in U$ and $\|\mathcal{G}(s,\phi')$ $-\ \mathcal{G}(\widetilde{s}_0,\widetilde{\phi}'_0)\| < c\tau$ for $(s,\phi') \in U$, (ii) $\|\zeta - \mathcal{G}(s,\phi')\| \leq c\tau$ for $\zeta \in V$, where

$(s,\phi') \in U$, (iii) $\mathcal{G}$ is the metrically regular at $(\widetilde{s}_0, \widetilde{\phi}'_0)$ for $\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)$ with constant $\frac{\kappa}{1-\kappa\mu}$, that is,

$$d((s,\phi'), \mathcal{G}^{-1}(\zeta)) \leq \frac{\kappa}{1-\kappa\mu} \|\zeta - \mathcal{G}(s,\phi')\| \,, \tag{5.46}$$

for $((s,\phi'), \zeta) \in U \times V$.

We can consider τ, a^* and b^* from the general case as in Lemma 5.6 or from the formulas in Lemma 5.7 for $\mathcal{G}$. Hence, we have (5.46). The following discussion is true for every choice.

In order to fix a choice, we use Lemma 5.7 (i). $\mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0) \subset U$, $\mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)) \subseteq V$.

- (iii) The imposition of some conditions

In order to have $\mathcal{Q}(s,\phi') \in \mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0))$, for $(s,\phi') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$, we impose the following two conditions (5.47) and (5.49).

We first assure that $0 \in \mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0))$ and $\|0 - \mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)\| < \frac{b^*}{2}$ by imposing

$$\delta < \frac{1}{2} \cdot b^* \,, \tag{5.47}$$

in other words, the assumption (5.25) (since $b^* = ca^*$).

For $(s,\phi') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$, we have

$$\begin{aligned} \|\mathcal{Q}(s,\phi')\| &\leq \psi_1((s,\phi'), (\widetilde{s}_0, \widetilde{\phi}'_0)) + \alpha\psi_2((s,\phi'), (\widetilde{s}_0, \widetilde{\phi}'_0)) \\ &\leq (\frac{1}{2}L(a^*) + \alpha\widehat{a}) \|(\widetilde{s}_0, \widetilde{\phi}'_0) - (s,\phi')\| \,. \end{aligned} \tag{5.48}$$

Let us impose the condition

$$\|\mathcal{Q}(s,\phi')\| < \frac{1}{2} \cdot b^* \,, \ \forall (s,\phi') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0) \,. \tag{5.49}$$

If $\frac{a^*}{b^*}(\frac{1}{2}L(a^*) + \alpha\widehat{a}) < \frac{1}{2}$, then we have (5.49). We have $b^* = ca^*$ and $L(a^*) \leq L(\varepsilon)$. There results that if (5.44) is satisfied, that is, if $\lambda < \frac{1}{2}$ is satisfied, where λ is defined below, in (5.54), then we have (5.49).

Let $z \in \mathbb{B}_{\frac{b^*}{2}}(0)$. Then, $\|z - \mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)\| \leq \|z - 0\| + \|0 - \mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)\| \leq \frac{b^*}{2} + \|\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)\| < \frac{b^*}{2} + \frac{b^*}{2} = b^*$, so $z \in int\mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0))$.

Taking condition (5.49) and replacing $z = \mathcal{Q}(s,\phi')$, the previous relation leads us to

$$\|\mathcal{Q}(s,\phi') - \mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)\| < b^* \,, \tag{5.50}$$

for $(s,\phi') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$. In other words, $\mathcal{Q}(s,\phi') \in \mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0))$, for $(s,\phi') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$.

Then, (5.46) holds for $((s,\phi'), \zeta) \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0) \times \mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0))$.

We obtain, by replacing $\zeta = \mathcal{Q}(\bar{s}, \bar{\phi}')$ in (5.46),

$$d((s,\phi'), \mathcal{G}^{-1}(\mathcal{Q}(\bar{s}, \bar{\phi}')) \leq \frac{\kappa}{1-\kappa\mu} \|\mathcal{Q}(\bar{s}, \bar{\phi}') - \mathcal{G}(s,\phi')\| \,, \tag{5.51}$$

for (s,ϕ'), $(\bar{s}, \bar{\phi}') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$.

- (iv) The definition of the mapping T_0 for the contraction mapping principle for set-valued mappings

Let us define the mappings:

$\zeta \in \mathbb{B}_{b^*}(\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0))$, $\Theta : \zeta \mapsto \mathcal{G}^{-1}(\zeta)$,

$(s, \phi') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$, $T_0 : (s, \phi') \mapsto \Theta(\mathcal{Q}(s, \phi'))$.

Observe that *dom* $\mathcal{G}^{-1} = \Sigma$, where we use Lemma 5.1. We extend the definition of these mappings to the spaces Γ and Σ.

$T_0 : \Gamma \times \Gamma \rightrightarrows \Gamma \times \Gamma$, $T_0 : (s, \phi') \mapsto \mathcal{G}^{-1}(\mathcal{Q}(s, \phi'))$.

- (v) The verification of the assumptions of the contraction mapping principle for set-valued mappings, Theorem 2.3

In Theorem 2.3, let us replace X, ρ, T, x, $\bar{x}$, a, λ, $d(\cdot,\cdot)$, $e(\cdot,\cdot)$, u, v by $\Gamma \times \Gamma$, ρ, T_0, (s, ϕ'), $(\widetilde{s}_0, \widetilde{\phi}'_0)$, a^*, λ, $d(\cdot,\cdot)$, $e(\cdot,\cdot)$, (s, ϕ'), $(\overline{s}, \overline{\phi}')$ respectively.

- The verification of the condition $gphT_0 \cap (\mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0) \times \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0))$ **is closed**

For brevity, only for this verification, we keep some notations from the general case from Subsection 2.2. So, instead of a^*, (s, ϕ'), $(\widetilde{s}_0, \widetilde{\phi}'_0)$, ζ, $\Gamma \times \Gamma$, Σ, we keep a, x, $\bar{x}$, y, X, Y respectively. Let us verify that $gphT_0 \cap (\mathbb{B}_a(\bar{x}) \times \mathbb{B}_a(\bar{x}))$ is closed.

The graph of the continuous mapping $\mathcal{G}$, $gph\ \mathcal{G} = \{(x, y) \in X \times Y | y = \mathcal{G}(x)\} = \{(x, y) \in X \times Y | y \in \mathcal{G}(x)\}$ is closed in $X \times Y$. Here, we identify the function $\mathcal{G} : X \to Y$ with the set-valued mapping $\mathcal{G} : X \rightrightarrows Y$ such that $\mathcal{G}$ is single-valued at every point of *dom* $\mathcal{G}$.

$\mathcal{G}^{-1}(y) = \{x \in X | y \in \mathcal{G}(x)\}$.

$gph\ \mathcal{G}^{-1} = \{(y, x) \in Y \times X | (x, y) \in gph\ \mathcal{G}\}$.

Hence the graph of $\mathcal{G}^{-1}$, $gph\ \mathcal{G}^{-1}$, is closed in $Y \times X$.

$Range(\mathcal{Q}) = Y$.

$gph\ \mathcal{Q} = \{(x, y) \in X \times Y | y = \mathcal{Q}(x)\}$.

$gph\ \mathcal{G}^{-1} = \{(y, x) \in Range(\mathcal{Q}) \times X | y \in \mathcal{G}(x)\}$.

$gph\ \mathcal{G}^{-1} = \{(y, x) \in Range(\mathcal{Q}) \times X | (x', y) \in gph\ \mathcal{Q},\ y \in \mathcal{G}(x)\}$.

$gph\ \mathcal{G}^{-1}(\mathcal{Q}(\cdot)) = \{(x', x) \in X \times X | (x', y) \in gph\ \mathcal{Q},\ y \in \mathcal{G}(x)\}$.

Proof of $gphT_0 \cap (\mathbb{B}_a(\bar{x}) \times \mathbb{B}_a(\bar{x}))$ **is closed**

We prove using sequences for the formulation

$gph\ \mathcal{G}^{-1}(\mathcal{Q}(\cdot)) = \{(x', x) \in X \times X | y = \mathcal{Q}(x'),\ y \in \mathcal{G}(x)\}$.

Let $(x'_\ell, x_\ell) \in gph\ \mathcal{G}^{-1}(\mathcal{Q}(\cdot))$, $(x'_\ell, x_\ell) \to (x', x)$ as $\ell \to \infty$, $(x', x) \in X \times X$.

Define $y_\ell = \mathcal{Q}(x'_\ell)$. From $(x'_\ell, x_\ell) \in gph\ \mathcal{G}^{-1}(\mathcal{Q}(\cdot))$, we have $y_\ell \in \mathcal{G}(x_\ell)$.

Since $\mathcal{Q}$ is continuous, $x'_\ell \to x'$ implies $y_\ell \to \mathcal{Q}(x')$, so $(y_\ell, x_\ell) \to (\mathcal{Q}(x'), x)$ as $\ell \to \infty$.

$y_\ell \in \mathcal{G}(x_\ell)$ implies that $(x_\ell, y_\ell) \in gph\ \mathcal{G}$. Since $gph\ \mathcal{G}$ is closed in $X \times Y$, there results that $(x, \mathcal{Q}(x')) \in gph\ \mathcal{G}$ so $\mathcal{Q}(x') \in \mathcal{G}(x)$.

Let $y = \mathcal{Q}(x')$. We have $y \in \mathcal{G}(x)$. Hence $(x', x) \in gph\ \mathcal{G}^{-1}(\mathcal{Q}(\cdot))$ and $gph\ \mathcal{G}^{-1}(\mathcal{Q}(\cdot))$ is closed in $X \times X$.

We now take the intersection with $\mathbb{B}_a(\bar{x}) \times \mathbb{B}_a(\bar{x})$.

There results that $gphT_0 \cap (\mathbb{B}_a(\bar{x}) \times \mathbb{B}_a(\bar{x}))$ is closed, that is, $gphT_0 \cap (\mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0) \times \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0))$ is closed.

- An estimate for $d((\widetilde{s}_0, \widetilde{\phi}'_0), T_0(\widetilde{s}_0, \widetilde{\phi}'_0))$ **from the condition I of Theorem 2.3**

Using (5.51), we have $d((\widetilde{s}_0, \widetilde{\phi}'_0), T_0(\widetilde{s}_0, \widetilde{\phi}'_0)) = d((\widetilde{s}_0, \widetilde{\phi}'_0), \mathcal{G}^{-1}(\mathcal{Q}(\widetilde{s}_0, \widetilde{\phi}'_0)))$ $\leq \frac{\kappa}{1-\kappa\mu} \|\mathcal{Q}(\widetilde{s}_0, \widetilde{\phi}'_0) - \mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)\| = \frac{\kappa}{1-\kappa\mu} \|S(\widetilde{s}_0)\| = \frac{\kappa}{1-\kappa\mu}\delta = \frac{1}{c}\delta$.

- An estimate for $e(T_0(s, \phi') \cap \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0), T_0(\overline{s}, \overline{\phi}'))$ **from the condition II of Theorem 2.3**

We obtain

$$\|\mathcal{Q}(s,\phi') - \mathcal{Q}(\overline{s},\overline{\phi}')\| \leq \psi_1((s,\phi'),(\overline{s},\overline{\phi}')) + \alpha\psi_2((s,\phi'),(\overline{s},\overline{\phi}')) \,. \tag{5.52}$$

We have the equivalence $(\hat{s},\hat{\phi}') \in \mathcal{G}^{-1}(\mathcal{Q}(s,\phi')) \cap \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0') \Leftrightarrow$ $\mathcal{Q}(s,\phi') \in \mathcal{G}(\hat{s},\hat{\phi}')$ and $(\hat{s},\hat{\phi}') \in \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')$. We remember that $\mathcal{Q}$ is a function and $\frac{\kappa}{1-\kappa\mu} = \frac{1}{c}$.

Using (5.51), we have $e(T_0(s,\phi') \cap \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0'), T_0(\overline{s},\overline{\phi}'))$
$\leq \sup\{d((\hat{s},\hat{\phi}'), T_0(\overline{s},\overline{\phi}')) | (\hat{s},\hat{\phi}') \in T_0(s,\phi') \cap \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')\}$
$= \sup\{d((\hat{s},\hat{\phi}'), \mathcal{G}^{-1}(\mathcal{Q}(\overline{s},\overline{\phi}'))) | (\hat{s},\hat{\phi}') \in \mathcal{G}^{-1}(\mathcal{Q}(s,\phi')) \cap \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')\}$
$\leq \sup\{\frac{\kappa}{1-\kappa\mu}\|\mathcal{Q}(\overline{s},\overline{\phi}') - \mathcal{G}(\hat{s},\hat{\phi}')\| \mid (\hat{s},\hat{\phi}') \in \mathcal{G}^{-1}(\mathcal{Q}(s,\phi')) \cap \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')\}$
$\leq \sup\{\frac{\kappa}{1-\kappa\mu}\|\mathcal{Q}(\overline{s},\overline{\phi}') - \mathcal{G}(\hat{s},\hat{\phi}')\| \mid \mathcal{Q}(s,\phi') \in \mathcal{G}(\hat{s},\hat{\phi}'), (\hat{s},\hat{\phi}') \in \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')\}$
$= \frac{\kappa}{1-\kappa\mu}\|\mathcal{Q}(\overline{s},\overline{\phi}') - \mathcal{Q}(s,\phi')\| \leq \frac{\kappa}{1-\kappa\mu}(\frac{1}{2}L(\varepsilon) + \alpha\widehat{a})\|(\overline{s},\overline{\phi}') - (s,\phi')\|,$
for all (s,ϕ'), $(\overline{s},\overline{\phi}') \in \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')$.

- (vi) The formulation of some conditions related to conditions I and II from Theorem 2.3

Condition II from Theorem 2.3 leads to

$$\frac{1}{c}(\frac{1}{2}L(\varepsilon) + \alpha\widehat{a}) \leq \lambda \,. \tag{5.53}$$

We take $\lambda = \frac{1}{c}(\frac{1}{2}L(\varepsilon) + \alpha\widehat{a})$. From (5.44), we have $0 < \lambda < \frac{1}{2}$. Hence,

$$\lambda = \frac{1}{c}(\frac{1}{2}L(\varepsilon) + \alpha\widehat{a}) < \frac{1}{2} \,. \tag{5.54}$$

Condition I from Theorem 2.3 leads to

$$\frac{1}{c}\delta < a^*(1-\lambda) \quad \text{or} \quad \delta < ca^*(1-\lambda) \,. \tag{5.55}$$

We have $b^* = ca^*$ from (5.37) and $\frac{1}{2} < 1 - \lambda$. Taking into account conditions (5.47) and (5.55), we impose

$$\delta < \min\{ca^*(1-\lambda), \frac{1}{2}b^*\} = \frac{1}{2}b^* = \frac{1}{2}ca^* \,, \tag{5.56}$$

that is, assumption (5.25).

With λ given by (5.54), conditions I and II, from Theorem 2.3, are verified.

- (vii) The formulation of the conclusion of the contraction mapping principle for set-valued mappings

The assumptions of Theorem 2.3 are verified, hence T_0 has a fixed point in $\mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')$; that is, there exists $(\bar{s},\bar{\phi}') \in \mathbb{B}_{a^*}(\widetilde{s}_0,\widetilde{\phi}_0')$ such that $(\bar{s},\bar{\phi}') \in T_0(\bar{s},\bar{\phi}')$.

$$(\bar{s},\bar{\phi}') \in T_0(\bar{s},\bar{\phi}') \Leftrightarrow (\bar{s},\bar{\phi}') \in \mathcal{G}^{-1}(\mathcal{Q}(\bar{s},\bar{\phi}'))$$

$$\Leftrightarrow \mathcal{Q}(\bar{s},\bar{\phi}') \in \mathcal{G}(\bar{s},\bar{\phi}') \Leftrightarrow \mathcal{G}(\bar{s},\bar{\phi}') - \mathcal{Q}(\bar{s},\bar{\phi}') \ni 0 \Leftrightarrow$$

$$S(\bar{s}) - \Phi(\bar{x},\bar{\phi}') - \Phi(\widetilde{x}_0,\xi(\bar{f},\bar{g}_i,\bar{e}_k)) + \Phi(\widetilde{x}_0,\xi(\bar{g}',\bar{g}_i',\bar{e}_k')) \ni 0 \,, \tag{5.57}$$

$$\Leftrightarrow$$

$$S(\hat{s}_0) - \Phi(\hat{x}_0,\hat{\phi}_0') \ni 0 \,, \tag{5.58}$$

that is, $(\hat{s}_0, \hat{\phi}'_0)$ is a solution of the equation (5.26), where $\hat{s}_0 = (\hat{x}_0, \hat{y}_{1,0}, \dots, \hat{y}_{q+m,0}, \hat{\mathrm{z}}_{1,0}, \dots, \hat{\mathrm{z}}_{n,0})$, $\hat{\phi}'_0 = (\hat{y}'_0, \hat{y}'_{1,0}, \dots, \hat{y}'_{q+m,0}, \hat{\mathrm{z}}'_{1,0}, \dots, \hat{\mathrm{z}}'_{n,0})$ and $\hat{x}_0 = (\hat{f}_0, \hat{\lambda}_0, \hat{u}_0) = (0, \bar{\lambda}, \bar{u})$, $\hat{y}_{i,0} = (\hat{g}_{i,0}, \hat{\mu}_{i,0}, \hat{w}_{i,0}) = (0, \bar{\mu}_i, \bar{w}_i)$, $\hat{\mathrm{z}}_{k,0} = (\hat{e}_{k,0}, \hat{v}_{k,0}) = (0, \bar{v}_k)$, $\hat{y}'_0 = (\hat{g}'_0, \hat{\mu}'_0, \hat{w}'_0) = (0, \bar{\mu}', \bar{w}')$, $\hat{y}'_{i,0} = (\hat{g}'_{i,0}, \hat{\mu}'_{i,0}, \hat{w}'_{i,0}) = (0, \bar{\mu}'_i, \bar{w}'_i)$, $\hat{\mathrm{z}}'_{k,0} = (\hat{e}'_{k,0}, \hat{v}'_{k,0}) = (0, \bar{v}'_k)$, for $i = 1, \dots, q+m$, $k = 1, \dots, n$.

It also results $(\hat{s}_0, \hat{\phi}'_0) \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$. Indeed, we have:
$\|(\widetilde{s}_0, \widetilde{\phi}'_0) - (\hat{s}_0, \hat{\phi}'_0)\| \le \|(\widetilde{s}_0, \widetilde{\phi}'_0) - (\hat{s}_0, \hat{\phi}'_0)\| + \|\widetilde{f}_0 - \bar{f}\| + \sum_{i=1}^{q+m} \|\widetilde{g}_{i,0} - \bar{g}_i\| + \sum_{k=1}^{n} \|\widetilde{e}_{k,0} - \bar{e}_k\| + \|\widetilde{g}'_0 - \bar{g}'\| + \sum_{i=1}^{q+m} \|\widetilde{g}'_{i,0} - \bar{g}'_i\| + \sum_{k=1}^{n} \|\widetilde{e}'_{k,0} - \bar{e}'_k\| = \|(\widetilde{s}_0, \widetilde{\phi}'_0) - (\bar{s}, \bar{\phi}')\| \le a^*$.

(5.58) becomes

$$\begin{bmatrix} B(\hat{x}_0) - \widetilde{\theta}_0 - B(\hat{y}'_0) \\ G(\hat{x}_0) - DG(\hat{x}_0)\hat{y}'_0 \\ B(\hat{y}_{i,0} - \hat{y}'_{i,0}) - \delta_i^{q+m} \\ DG(\hat{x}_0)(\hat{y}_{i,0} - \hat{y}'_{i,0}) \\ \bar{\mathcal{B}}(\hat{\mathrm{z}}_{k,0} - \hat{\mathrm{z}}'_{k,0}) - \delta_k^n \\ H(\lambda_0, u_0, (\hat{\mathrm{z}}_{k,0} - \hat{\mathrm{z}}'_{k,0})) \end{bmatrix} \ni 0 \,. \tag{5.59}$$

Let us fix $\hat{x}_0$ and $\hat{y}'_0$ in $\Phi_G(x, y')$ from the first line of $\Phi(x, \phi')$ in (5.26). Let us take $\theta_0 = \widetilde{\theta}_0 + B(\hat{y}'_0)$ and $\varrho = DF(\hat{\lambda}_0, \hat{u}_0)(\hat{\mu}'_0, \hat{w}'_0)$. S_0 denotes the function S from (3.7) formulated for this θ_0 and for $F(\cdot,\cdot) - \varrho$ instead of F. Let us use Remark 9.3.

Then, (5.59) gives

$$\begin{bmatrix} B(\hat{x}_0) - \theta_0 \\ G(\hat{x}_0) - \varrho \\ B(\hat{y}_{i,0} - \hat{y}'_{i,0}) - \delta_i^{q+m} \\ D(G(\hat{x}_0) - \varrho)(\hat{y}_{i,0} - \hat{y}'_{i,0}) \\ \bar{\mathcal{B}}(\hat{\mathrm{z}}_{k,0} - \hat{\mathrm{z}}'_{k,0}) - \delta_k^n \\ \bar{H}(\varrho, \lambda_0, u_0, (\hat{\mathrm{z}}_{k,0} - \hat{\mathrm{z}}'_{k,0})) \end{bmatrix} = 0 \,, \tag{5.60}$$

where $\bar{H}$ is H for $F(\cdot,\cdot) - \varrho$ instead of $F(\cdot,\cdot)$,

$$\bar{H} : Z \times \mathbb{R}^m \times W \times \Delta \to Z, \ \bar{H}(\varrho, \lambda, u, \mathrm{z}) = D_u(F(\lambda, u) - \varrho)v - \sum_{k=1}^{n} e^k \bar{b}_k \,.$$

Let $s_0 = (x_0, y_{1,0}, \dots, y_{q+m,0}, \mathrm{z}_{1,0}, \dots, \mathrm{z}_{n,0})$ with $x_0 = \hat{x}_0$, $y_{i,0} = \hat{y}_{i,0} - \hat{y}'_{i,0}$, $\mathrm{z}_{k,0} = \hat{\mathrm{z}}_{k,0} - \hat{\mathrm{z}}'_{k,0}$ and $i = 1, \dots, q+m$, $k = 1, \dots, n$.

Relation (5.60) means that s_0 is a solution of the system (5.27) verifying the assertion (a) of the statement (ii) of Theorem 3.5.

It also results $s_0 \in \mathbb{B}_{a^*}(\widetilde{s}_0)$. Indeed, we have:
$\|s_0 - \widetilde{s}_0\| = \|\hat{x}_0 - \widetilde{x}_0\| + \sum_{i=1}^{q+m} \|(\hat{y}_{i,0} - \hat{y}'_{i,0}) - \widetilde{y}_{i,0}\| + \sum_{k=1}^{n} \|(\hat{\mathrm{z}}_{k,0} - \hat{\mathrm{z}}'_{k,0}) - \widetilde{\mathrm{z}}_{k,0}\| \le \|\hat{x}_0 - \widetilde{x}_0\| + \sum_{i=1}^{q+m} \|\hat{y}_{i,0} - \widetilde{y}_{i,0}\| + \sum_{k=1}^{n} \|\hat{\mathrm{z}}_{k,0} - \widetilde{\mathrm{z}}_{k,0}\| + \sum_{i=1}^{q+m} \|\hat{y}'_{i,0} - \widetilde{y}'_{i,0}\| + \sum_{k=1}^{n} \|\hat{\mathrm{z}}'_{k,0} - \widetilde{\mathrm{z}}'_{k,0}\| \le \|(\hat{s}_0, \hat{\phi}'_0) - (\widetilde{s}_0, \widetilde{\phi}'_0)\| \le a^*$.

- (viii) The existence of some isomorphisms

We have $\gamma_{S_0} = \gamma$, $L_S(\varepsilon) = L_{S_0}(\varepsilon)$. $DS_0(\widetilde{s}_0) = DS(\widetilde{s}_0)$ is an isomorphism of Γ onto Σ. We have $\kappa \le \gamma$ and $\frac{1}{2}L_S(\varepsilon) \le L(\varepsilon)$. Then,
$\gamma L_S(a^*) \le 2\gamma \frac{1}{2} L_S(\varepsilon) \le 2\gamma L(\varepsilon) < 1$.

$L_S(a^*) < \frac{1}{\gamma}$, so $DS_0(s_0) = DS(s_0)$ is an isomorphism of Γ onto Σ.

$\gamma_\Psi \leq \gamma$ (this is verified using the definition), $L_\Psi(a^*) \leq L_S(a^*)$.

$\gamma_\Psi L_\Psi(a^*) \leq \gamma L_S(a^*) < 1$, hence $D\Psi_0(x_0) = D\Psi(x_0)$ is an isomorphism of X onto Y.

The system (5.27) and its solution s_0 verify the assertion (b) of the statement (ii) of Theorem 3.5.

As in the proofs of Theorem 3.1 [18], Theorem IV.3.1 [29] and Theorem I.2.1 [14], it follows that s_0 is the unique solution of the equation (5.27) in $\mathbb{B}_a(\widetilde{s}_0)$ for every $a \geq a^*$ that satisfies $\gamma L_S(a) < 1$.

We now use Theorem 3.5. Then, the component (λ_0, u_0) of s_0 is a solution of the equation (5.28) that satisfies hypothesis (1.2) and the rest of the hypotheses of the statement (i) of Theorem 3.5 in $\mathbb{B}_a(\widetilde{\lambda}_0, \widetilde{u}_0)$.

Assume that there exist two different such solutions in $\mathbb{B}_a(\widetilde{\lambda}_0, \widetilde{u}_0)$. Then, equation (5.27) has two different solutions in $\mathbb{B}_a(\widetilde{s}_0)$. This contradicts the uniqueness of s_0 in $\mathbb{B}_a(\widetilde{s}_0)$. Then, (λ_0, u_0) is the unique solution of the equation (5.28) that satisfies hypothesis (1.2) and the rest of the hypotheses of the statement (i) of Theorem 3.5 in $\mathbb{B}_a(\widetilde{\lambda}_0, \widetilde{u}_0)$ for every $a \geq a^*$ that satisfies $\gamma L_S(a) < 1$.

- (ix) Some estimates

We have $2\gamma L_{S_0}(a^*) \leq 1$ and $DS_0(\widetilde{s}_0)$ is an isomorphism of Γ onto Σ. Then, the hypotheses of Lemma 2.2 are satisfied in our situation. We get: for any $\zeta \in \operatorname{int} \mathbb{B}_{\frac{a^*}{2\gamma}}(S_0(\widetilde{s}_0))$, the equation $S_0(s) = \zeta$ has a unique solution s in $\mathbb{B}_{a_1}(\widetilde{s}_0)$, where $a_1 = 2\gamma \|S_0(\widetilde{s}_0) - \zeta\| \leq a^*$.

For our proof, we take $\zeta = 0$. We obtain $2\gamma \|S_0(\widetilde{s}_0)\| \leq a^*$. We have $\|\varrho\| = \|S_0(\widetilde{s}_0) - S(\widetilde{s}_0)\| \leq \|S_0(\widetilde{s}_0)\| + \|S(\widetilde{s}_0)\| \leq \|S_0(\widetilde{s}_0)\| + \|S(\widetilde{s}_0)\| \leq \frac{a^*}{2\gamma} + \|S(\widetilde{s}_0)\|$. Hence, (5.30) holds. We also use (5.25) and $\frac{a^*}{2\gamma} + \frac{1}{2} c a^* = (\frac{1}{\gamma} - \frac{1}{2} L(\varepsilon)) a^*$.

We also have $\|\varrho\| = \|DF(\hat{\lambda}_0, \hat{u}_0)(\hat{\mu}'_0, \hat{w}'_0)\| \leq (\|DF(\widetilde{\lambda}_0, \widetilde{u}_0)\| + L_F(a^*)) a^*$. Then, we have (5.31).

Relation (5.32) is obtained as in the proofs of Theorem 3.1 [18], Theorem IV.3.1 [29] and Theorem I.2.1 [14].

5.6. **Some consequences.**

Corollary 5.9. *Theorem 5.4 holds for those a^* and b^* given by Lemma 5.6, under the conditions that (5.50) is verified, that is, $\|\mathcal{Q}(s, \phi') - \mathcal{G}(\widetilde{s}_0, \widetilde{\phi}'_0)\| < b^*$, for $(s, \phi') \in \mathbb{B}_{a^*}(\widetilde{s}_0, \widetilde{\phi}'_0)$, and it is satisfied condition I from Theorem 2.3, that is, (5.55) or $\frac{1}{c}\delta < a^*(1 - \lambda)$, with λ given by (5.54) or $\lambda = \frac{1}{c}(\frac{1}{2} L(\varepsilon) + \alpha \widehat{a}) < \frac{1}{2}$.*

Corollary 5.10. *Assume that $(\widetilde{\lambda}_0, \widetilde{u}_0)$ belongs to a solution branch of equation (1.1). $(\widetilde{\lambda}_0, \widetilde{u}_0)$ can be a regular or a nonregular solution. Assume the hypotheses of Theorem 5.4. $\Psi(\widetilde{x}_0) = 0$ in δ. If $\varrho \neq 0$, then, the given problem (1.1) is a perturbation of the bifurcation problem (5.28). If $\varrho = 0$, then, the bifurcation point (λ_0, u_0) belongs to the solution branch of equation (1.1).*

Proof. Equation (5.27) provides a bifurcation problem (5.28) for which the given problem (1.1) is a perturbation.

□

Corollary 5.11. *$\varrho = 0$ if and only if $(\hat{\mu}_0', \hat{w}_0') = \sum_{i=1}^{q+m} \beta_i\ ((\hat{\mu}_{i,0}, \hat{w}_{i,0}) - (\hat{\mu}_{i,0}', \hat{w}_{i,0}'))$, $\beta_i \in \mathbb{R}$.*

5.7. **The plan of the study for the next sections.** In Section 9, we investigate the formulation of S_0 from (5.27) using the function $F(\lambda, u) - DF(\lambda, u)(\hat{\mu}_0', \hat{w}_0')$ (where $(\hat{\mu}_0', \hat{w}_0')$ is fixed) instead of the function $F(\lambda, u) - DF(\hat{\lambda}_0, \hat{u}_0)(\hat{\mu}_0', \hat{w}_0')$ (where $(\hat{\lambda}_0, \hat{u}_0)$ and $(\hat{\mu}_0', \hat{w}_0')$ are fixed and $\varrho = DF(\hat{\lambda}_0, \hat{u}_0)(\hat{\mu}_0', \hat{w}_0')$ $=$ *constant*).

In Sections 6 - 8, for simplicity, we study only the case $\varrho = DF(\hat{\lambda}_0, \hat{u}_0)(\hat{\mu}_0', \hat{w}_0')$ $=$ *constant* from Theorem 5.4.

6. The existence of a class of equivalent maps

We prove that there exists a class (family) of maps C^p - equivalent (right equivalent) at (λ_0, u_0) to $F(\cdot) - \varrho$ at (λ_0, u_0) and that satisfies hypothesis (1.2) in (λ_0, u_0).

We use the definition of equivalence of maps from [70, 71]. The following discussion has the references [2, 15, 31, 32, 33, 41, 42, 66, 70, 71].

Consider the equation (5.28).

Let φ_0 be a local C^p - diffeomorphism from some open neighborhood $\mathcal{U}(\lambda_0, u_0) = \mathcal{U}_1(\lambda_0) \times \mathcal{U}_2(u_0)$ of (λ_0, u_0) in $\mathbb{R}^m \times W$ onto some open neighborhood $\widehat{\mathcal{U}}(\widehat{\lambda}_0, \widehat{u}_0) = \widehat{\mathcal{U}}_1(\widehat{\lambda}_0) \times \widehat{\mathcal{U}}_2(\widehat{u}_0)$ of $(\widehat{\lambda}_0, \widehat{u}_0) = (\lambda_0, u_0)$ in $\mathbb{R}^m \times W$ of the form $(\widehat{\lambda}, \widehat{u}) = \varphi_0(\lambda, u) = (\Lambda(\lambda), \varphi_*(\lambda, u))$, $\Lambda(\lambda) \in \mathbb{R}^m$, $\varphi_*(\lambda, u) \in W$, satisfying the following assumptions: $\varphi_0(\lambda_0, u_0) = (\widehat{\lambda}_0, \widehat{u}_0)$, $D_u\varphi_*(\lambda_0, u_0)$ is bijective.

Let Π be a local C^p - diffeomorphism from some open neighborhood $\mathcal{U}_3(0)$ of 0 in $\mathbb{R}^q$ onto some open neighborhood $\widehat{\mathcal{U}}_3(0)$ of 0 in $\mathbb{R}^q$. $\Pi(0) = 0$.

Let $\bar{\Pi}$ be a local C^p - diffeomorphism from some open neighborhood $\mathcal{U}_4(0)$ of 0 in $\mathbb{R}^n$ onto some open neighborhood $\widehat{\mathcal{U}}_4(0)$ of 0 in $\mathbb{R}^n$. $\bar{\Pi}(0) = 0$.

$(\widehat{\lambda}, \widehat{u}) = \varphi_0(\lambda, u) = (\Lambda(\lambda), \varphi_*(\lambda, u))$, $(\lambda, u) = \varphi_0^{-1}(\widehat{\lambda}, \widehat{u})$.

$\widehat{f} = \Pi(f)$, $f = \Pi^{-1}(\widehat{f}) = ((\Pi^{-1})_1(\widehat{f}), \ldots, (\Pi^{-1})_q(\widehat{f}))$.

$\widehat{e} = \bar{\Pi}(e)$, $e = \bar{\Pi}^{-1}(\widehat{e}) = ((\bar{\Pi}^{-1})_1(\widehat{e}), \ldots, (\bar{\Pi}^{-1})_n(\widehat{e}))$.

Let φ be the local C^p - diffeomorphism from $\mathcal{U}_3(0) \times \mathcal{U}_1(\lambda_0) \times \mathcal{U}_2(u_0)$ onto $\widehat{\mathcal{U}}_3(0) \times \widehat{\mathcal{U}}_1(\widehat{\lambda}_0) \times \widehat{\mathcal{U}}_2(\widehat{u}_0)$ given by $\varphi(f, \lambda, u) = (\Pi(f), \varphi_0(\lambda, u))$.

$(\widehat{f}, \widehat{\lambda}, \widehat{u}) = \varphi(f, \lambda, u) = (\Pi(f), \varphi_0(\lambda, u)) = (\Pi(f), \Lambda(\lambda), \varphi_*(\lambda, u))$.

$(f, \lambda, u) = \varphi^{-1}(\widehat{f}, \widehat{\lambda}, \widehat{u}) = (\Pi^{-1}(\widehat{f}), \varphi_0^{-1}(\widehat{\lambda}, \widehat{u}))$

$D(\varphi_0^{-1})(\widehat{\lambda}, \widehat{u}) = D\varphi_0(\lambda, u)^{-1}$, where $(\widehat{\lambda}, \widehat{u}) = \varphi_0(\lambda, u)$, for all (λ, u) in a small neighborhood of (λ_0, u_0).

$D(\Pi^{-1})(\widehat{f}) = D\Pi(f)^{-1}$, where $\widehat{f} = \Pi(f)$, for all f in a small neighborhood of 0.

$D(\bar{\Pi}^{-1})(\widehat{e}) = D\bar{\Pi}(e)^{-1}$, where $\widehat{e} = \bar{\Pi}(e)$, for all e in a small neighborhood of 0.

$D(\varphi^{-1})(\widehat{f}, \widehat{\lambda}, \widehat{u}) = D\varphi(f, \lambda, u)^{-1}$, where $(\widehat{f}, \widehat{\lambda}, \widehat{u}) = \varphi(f, \lambda, u)$, for all (f, λ, u) in a small neighborhood of $(0, \lambda_0, u_0)$.

When we write $(\lambda, u) \in \mathcal{U}_1(\lambda_0) \times \mathcal{U}_2(u_0)$, without "$\forall$", related to a relation, we understand that (λ, u) is in the maximal neighborhood (related to inclusion) contained in $\mathcal{U}_1(\lambda_0) \times \mathcal{U}_2(u_0)$ such that that relation holds and so on.

Let us retain $F(\lambda, u) - \varrho$ from (5.28).

Using the definition for C^p equivalence (right equivalence) of maps on open subsets of Banach spaces, from [70, 71], related to [2, 15, 31, 32, 33, 41, 42, 66], let us consider a map $\widehat{F} : \mathbb{R}^m \times W \to Z$, of class C^p, defined locally by

$$\widehat{F}(\widehat{\lambda}, \widehat{u}) = F(\varphi_0^{-1}(\widehat{\lambda}, \widehat{u})) - \varrho\,,\ (\widehat{\lambda}, \widehat{u}) \in \widehat{\mathcal{U}}_1(\widehat{\lambda}_0) \times \widehat{\mathcal{U}}_2(\widehat{u}_0)\,, \tag{6.1}$$

so $\widehat{F}(\varphi_0(\lambda, u)) = F(\lambda, u) - \varrho$, $(\lambda, u) \in \mathcal{U}_1(\lambda_0) \times \mathcal{U}_2(u_0)$,

$$\widehat{F}(\varphi_0(\lambda, u)) = F(\lambda, u) - \varrho\,,\ (\lambda, u) \in \mathcal{U}_1(\lambda_0) \times \mathcal{U}_2(u_0)\,, \tag{6.2}$$

that is, $\widehat{F}$ is C^p - equivalent (right equivalent) at $(\widehat{\lambda}_0, \widehat{u}_0)$ to $F - \varrho$ at (λ_0, u_0).

We have $\widehat{F}(\varphi_0(\lambda_0, u_0)) = \widehat{F}(\lambda_0, u_0) = 0$ so $(\widehat{\lambda}_0, \widehat{u}_0)$ is the solution of

$$\widehat{F}(\widehat{\lambda}, \widehat{u}) = 0\,, \tag{6.3}$$

Let us prove that $\widehat{F}$ satisfies hypothesis (1.2) in (λ_0, u_0).

Let us consider a map $\widehat{G} : \mathbb{R}^q \times \mathbb{R}^m \times W \to Z$, of class C^p, defined locally, for $(\widehat{f}, \widehat{\lambda}, \widehat{u}) \in \widehat{\mathcal{U}}_3(0) \times \widehat{\mathcal{U}}(\widehat{\lambda}_0, \widehat{u}_0)$, by

$$\widehat{G}(\widehat{f}, \widehat{\lambda}, \widehat{u}) = F(\varphi_0^{-1}(\widehat{\lambda}, \widehat{u})) - \varrho - \sum_{i=1}^{q} (\Pi^{-1})_i(\widehat{f})\bar{a}_i\,, \tag{6.4}$$

or $\widehat{G}(\varphi(f, \lambda, u)) = G(f, \lambda, u) - \varrho$, $(f, \lambda, u) \in \mathcal{U}_3(0) \times \mathcal{U}(\lambda_0, u_0)$, that is, $G - \varrho$ is C^p - equivalent (right equivalent) at $(0, \lambda_0, u_0)$ to $\widehat{G}$ at $(0, \widehat{\lambda}_0, \widehat{u}_0)$.

We have $\widehat{G}(\widehat{f}, \widehat{\lambda}, \widehat{u}) = G(\varphi^{-1}(\widehat{f}, \widehat{\lambda}, \widehat{u})) - \varrho = G(\Pi^{-1}(\widehat{f}), \varphi_0^{-1}(\widehat{\lambda}, \widehat{u})) - \varrho =$ $F(\varphi_0^{-1}(\widehat{\lambda}, \widehat{u})) - \sum_{i=1}^{q} (\Pi^{-1})_i(\widehat{f})\bar{a}_i - \varrho = \widehat{F}(\widehat{\lambda}, \widehat{u}) - \sum_{i=1}^{q} (\Pi^{-1})_i(\widehat{f})\bar{a}_i$.

Let us introduce some new notations.

$(\widehat{g}, \widehat{\mu}, \widehat{w}) = D\varphi(f, \lambda, u)(g, \mu, w) = (D\Pi(f)g, D\varphi_0(\lambda, u)(\mu, w))$.

$(g, \mu, w) = D\varphi(f, \lambda, u)^{-1}(\widehat{g}, \widehat{\mu}, \widehat{w}) = (D\Pi(f)^{-1}\widehat{g}, D\varphi_0(\lambda, u)^{-1}(\widehat{\mu}, \widehat{w}))$.

$D\Pi(f)g = \widehat{g}$, $g = D\Pi(f)^{-1}\widehat{g}$, with $D\Pi(f)^{-1}\widehat{g} = D(\Pi^{-1})(\widehat{f})\widehat{g}$ if $\widehat{f} = \Pi(f)$, for all f in a small neighborhood of 0.

$D\varphi_0(\lambda, u)(\mu, w) = (D\Lambda(\lambda)\mu, D\varphi_*(\lambda, u)(\mu, w)) = (\widehat{\mu}, \widehat{w})$

$(\widehat{g}_i, \widehat{\mu}_i, \widehat{w}_i) = D\varphi(f, \lambda, u)(g_i, \mu_i, w_i)$.

$(0, \widehat{\mu}_{i,0}, \widehat{w}_{i,0}) = D\varphi(0, \lambda_0, u_0)(0, \mu_{i,0}, w_{i,0}) = (D\Pi(0)0, D\varphi_0(\lambda_0, u_0)(\mu_{i,0}, w_{i,0}))$ $= (0, D\Lambda(\lambda_0)\mu_{i,0}, D\varphi_*(\lambda_0, u_0)(\mu_{i,0}, w_{i,0}))$.

$(\mu_{i,0}, w_{i,0}) = D\varphi_0(\lambda_0, u_0)^{-1}(\widehat{\mu}_{i,0}, \widehat{w}_{i,0})$, $0 = D\Pi(0)^{-1}0$.

$\widehat{v} = D_u\varphi_*(\lambda, u)v$, $\widehat{v}_k = D_u\varphi_*(\lambda, u)v_k$, $\widehat{v}_{k,0} = D_u\varphi_*(\lambda_0, u_0)v_{k,0}$, $v_{k,0} =$ $D_u\varphi_*(\lambda_0, u_0)^{-1}\widehat{v}_{k,0}$, $e_k = D(\bar{\Pi}^{-1})(0)\widehat{e}_k$, $0 = D\bar{\Pi}(0)^{-1}0$.

We have

$D(G(f, \lambda, u) - \varrho)(g, \mu, w) = D\widehat{G}(\varphi(f, \lambda, u))D\varphi(f, \lambda, u)(g, \mu, w) = D\widehat{G}(\widehat{f}, \widehat{\lambda}, \widehat{u})(\widehat{g}, \widehat{\mu}, \widehat{w})$ $= D\widehat{F}(\widehat{\lambda}, \widehat{u})(\widehat{\mu}, \widehat{w}) - \sum_{i=1}^{q} D(\Pi^{-1})_i(\widehat{f})\widehat{g}\bar{a}_i$

$D(F(\lambda, u) - \varrho)(\mu, w) = D\widehat{F}(\varphi_0(\lambda, u))D\varphi_0(\lambda, u)(\mu, w) = D\widehat{F}(\widehat{\lambda}, \widehat{u})(\widehat{\mu}, \widehat{w})$.

$D_u(F(\lambda, u) - \varrho)v = D_u\widehat{F}(\varphi_0(\lambda, u))v = D_u\widehat{F}(\Lambda(\lambda), \varphi_*(\lambda, u))v = D_{\varphi_*}\widehat{F}(\varphi_0(\lambda, u))D_u\varphi_*(\lambda, u)v$ $= D_{\varphi_*}\widehat{F}(\widehat{\lambda}, \widehat{u})\widehat{v} = D_{\widehat{u}}\widehat{F}(\widehat{\lambda}, \widehat{u})\widehat{v}$.

Let us consider a map $\widehat{H} : \mathbb{R}^m \times W \times \Delta \to Z$, of class C^{p-1}, defined locally, for $(\widehat{e}, \widehat{v}) \in \widehat{\mathfrak{U}}_4(0) \times \widehat{\mathfrak{U}}_2(\widehat{u}_0)$, by

$$\widehat{H}(\widehat{\lambda}, \widehat{u}, (\widehat{e}, \widehat{v})) = D_{\varphi_*}\widehat{F}(\widehat{\lambda}, \widehat{u})\widehat{v} - \sum_{k=1}^{n} D(\bar{\Pi}^{-1})_k(0)\widehat{e}\bar{b}_k \,. \tag{6.5}$$

We have

$$\begin{aligned}
&B(\Pi^{-1}(\widehat{f}), \varphi_0^{-1}(\widehat{\lambda}, \widehat{u})) - \theta_0 = B(f, \lambda, u) - \theta_0 \\
&\widehat{G}(\widehat{f}, \widehat{\lambda}, \widehat{u}) = G(f, \lambda, u) - \varrho \\
&B(D\Pi(f)^{-1}\widehat{g}_i, D\varphi_0(\lambda, u)^{-1}(\widehat{\mu}_i, \widehat{w}_i)) - \delta_i^{q+m} = B(g_i, \mu_i, w_i) - \delta_i^{q+m} \\
&D\widehat{G}(\widehat{f}, \widehat{\lambda}, \widehat{u})(\widehat{g}_i, \widehat{\mu}_i, \widehat{w}_i) = D(G(f, \lambda, u) - \varrho)(g_i, \mu_i, w_i) \\
&\bar{\mathfrak{B}}(D(\bar{\Pi}^{-1})(0)\widehat{e}_k, D_u\varphi_*(\lambda_0, u_0)^{-1}\widehat{v}_k) - \delta_k^n = \bar{\mathfrak{B}}(e_k, v_k) - \delta_k^n \\
&\widehat{H}(\widehat{\lambda}, \widehat{u}, (\widehat{e}_k, \widehat{v}_k)) = H(\lambda, u, (e_k, v_k))
\end{aligned} \tag{6.6}$$

Let us define

$$\begin{aligned}
&\widehat{B}_N(\overline{f}, \overline{\lambda}, \overline{u}) = B(\Pi^{-1}(\overline{f}), \varphi_0^{-1}(\overline{\lambda}, \overline{u})) \\
&\widehat{B}(\overline{g}, \overline{\mu}, \overline{w}) = B(D\Pi(0)^{-1}\overline{g}, D\varphi_0(\lambda_0, u_0)^{-1}(\overline{\mu}, \overline{w})) \\
&\widehat{\bar{\mathfrak{B}}}(\overline{e}, \overline{v}) = \bar{\mathfrak{B}}(D(\bar{\Pi}^{-1})(0)\overline{e}, D_u\varphi_*(\lambda_0, u_0)^{-1}\overline{v})
\end{aligned} \tag{6.7}$$

and

(6.8)

$$\widehat{S} : \Gamma \to \Sigma,\ \widehat{S}(\overline{s}) = \begin{bmatrix} \widehat{B}_N(\overline{f}, \overline{\lambda}, \overline{u}) - \theta_0 \\ \widehat{G}(\overline{f}, \overline{\lambda}, \overline{u}) \\ \widehat{B}(\overline{g}_i, \overline{\mu}_i, \overline{w}_i) - \delta_i^{q+m} \\ D\widehat{G}(\overline{f}, \overline{\lambda}, \overline{u})(\overline{g}_i, \overline{\mu}_i, \overline{w}_i) \\ \widehat{\bar{\mathfrak{B}}}(\overline{e}_k, \overline{v}_k) - \delta_k^n \\ \widehat{H}(\overline{\lambda}, \overline{u}, (\overline{e}_k, \overline{v}_k)) \end{bmatrix}, \quad \begin{aligned} &i = 1, \ldots, q+m\,, \\ &k = 1, \ldots, n\,, \end{aligned}$$

Let $\overline{s}_0 = (\overline{x}_0, \overline{y}_{1,0}, \ldots, \overline{y}_{q+m,0}, \overline{z}_{1,0}, \ldots, \overline{z}_{n,0})$, $\overline{x}_0 = (\overline{f}_0, \overline{\lambda}_0, \overline{u}_0) = (0, \widehat{\lambda}_0, \widehat{u}_0)$, $\overline{y}_{i,0} = (\overline{g}_{i,0}, \overline{\mu}_{i,0}, \overline{w}_{i,0}) = (0, \widehat{\mu}_{i,0}, \widehat{w}_{i,0})$, $\overline{z}_{k,0} = (\overline{e}_{k,0}, \overline{v}_{k,0}) = (0, \widehat{v}_{k,0})$, $i = 1, \ldots, q+m$, $k = 1, \ldots, n$.

From (6.6), we obtain

$$\widehat{S}(\overline{s}_0) = S(s_0) = 0\,. \tag{6.9}$$

Theorem 6.1. *There exists a class of maps C^p - equivalent (right equivalent) at (λ_0, u_0) to $F(\cdot) - \varrho$ at (λ_0, u_0) and that satisfies hypothesis (1.2) in (λ_0, u_0).*

Proof. There exists a local C^p - diffeomorphism φ_0, as defined above, such that (6.2) takes place. We now apply Corollary 3.6 for the equation $\widehat{S}(\overline{s}) = 0$ and for its solution $\overline{s}_0$. Hence, $\widehat{F}$ satisfies hypothesis (1.2) in (λ_0, u_0). Taking all the local C^p - diffeomorphism of the form of φ_0, $\widehat{F}$ becomes a representative of a class of maps C^p - equivalent (right equivalent) at (λ_0, u_0) to $F(\cdot) - \varrho$ at (λ_0, u_0) and that satisfies hypothesis (1.2) in (λ_0, u_0). ∎

7. Bifurcation of the solutions of approximate equations

In this Section, under the hypothesis that the exact problem has a bifurcation point, we study the existence of an equation, defined on closed subspaces of the given spaces, which has a bifurcation point that approximates the exact one and has the same type as this one.

7.1. **Approximate spaces and functions.** Let us consider a closed subspace W_h of W and a closed subspace Z_h of Z. The spaces W_h and Z_h are both finite-dimensional spaces, with $dim\ W_h = dim\ Z_h$, or they are both infinite-dimensional spaces. Let $F_h \in C^p(\mathbb{R}^m \times W_h, Z_h)$ be an approximation of the function F that defines equation (1.1). h is a positive parameter.

The mathematical entities are the same in the exact case and in the approximate case, depending on the Banach spaces only. This is why, in the approximate case, we maintain the notations from the exact case and adjoin an index h. The entities W, Z, F, (λ, u), X, Y, x, $x_0 = (f_0, \lambda_0, u_0)$, $\widetilde{x}_0$, ..., from the infinite-dimensional case, become W_h, Z_h, F_h, (λ_h, u_h), X_h, Y_h, x_h, $x_{0h} = (f_{0h}, \lambda_{0h}, u_{0h})$, $\widetilde{x}_{0h}$, ..., respectively, in the approximate case.

Let $\pi_h^W : W \to W_h$ and $\pi_h^Z \in L(Z, Z_h)$ be two operators. As a conclusion of the conditions used to relate the exact spaces and the approximate spaces in the literature [6, 7, 8, 9, 11, 12, 13, 17, 18, 23, 29, 39, 43, 55, 62, 67, 68], we assume that π_h^Z has the following property

$$\|\bar{a} - \pi_h^Z \bar{a}\|_Z \le C\|\bar{a}\|_Z,\ \forall \bar{a} \in Z,\ \text{where } C < 1\,. \tag{7.1}$$

Let $\bar{a}_{ih} = \pi_h^Z \bar{a}_i$, $i = 1, \ldots, q$, $\bar{b}_{kh} = \pi_h^Z \bar{b}_k$, $k = 1, \ldots, n$, $Z_{1,h} = sp\ \{\bar{a}_{1h}, \ldots, \bar{a}_{qh}\}$, $Z_{3,h} = sp\ \{\bar{b}_{1h}, \ldots, \bar{b}_{nh}\}$. Let I_N be the identity operator on $\mathbb{R}^N$, $N \ge 1$, and $\widetilde{f}_h = I_q f$, $\widetilde{g}_h = I_q g \in \mathbb{R}^q$, $\widetilde{e}_h = I_n e \in \mathbb{R}^n$, $\widetilde{\lambda}_h = I_m \lambda$, $\widetilde{\mu}_h = I_m \mu \in \mathbb{R}^m$, $\widetilde{u}_h = \pi_h^W u$, $\widetilde{w}_h = \pi_h^W w$, $\widetilde{v}_h = \pi_h^W v \in W_h$. Since $f_0 = 0$, $g_{i,0} = 0$, $e_{k,0} = 0$, we take $\widetilde{f}_{0h} = 0$, $\widetilde{g}_{i,0,h} = 0$, $\widetilde{e}_{k,0,h} = 0$.

Lemma 7.1. *(i) For every $\bar{a} \in Z$, $\bar{a} \ne 0$, we have $\pi_h^Z \bar{a} \ne 0$. (ii) Assume that the elements $\bar{a}_1$, ..., $\bar{a}_k \in Z$ form a linearly independent set. Then, the elements $\bar{a}_{1h}, \ldots, \bar{a}_{kh} \in Z_h$ form a linearly independent set.*

Proof. (i) If there exists $\bar{a} \ne 0$ such that $\pi_h^Z \bar{a} = 0$, then we deduce from (7.1) that $1 \le C$. This contradicts the condition $C < 1$.

(ii) Suppose that $\bar{a}_{1h}$, ..., $\bar{a}_{qh}$ form a linearly dependent set, therefore there exists $f' = ((f')^1, \ldots, (f')^q) \in \mathbb{R}^q$, $f' \ne 0$ and $\sum_{i=1}^q (f')^i \bar{a}_{ih} = 0$. Then, $\sum_{i=1}^q (f')^i \bar{a}_i \ne 0$. This contradicts the condition (i) since $\pi_h^Z(\sum_{i=1}^q (f')^i \bar{a}_i)$ $= \sum_{i=1}^q (f')^i \bar{a}_{ih} = 0$.

□

Introduce the function of class C^p,

$$G_h : X_h \to Z_h,\ G_h(x_h) = F_h(\lambda_h, u_h) - \sum_{i=1}^q f_h^i \bar{a}_{ih}\,,$$

and the function of class C^{p-1},

$$H_h : \mathbb{R}^m \times W_h \times \Delta_h \to Z_h,\ H_h(\lambda_h, u_h, \mathrm{z}_h) = D_u F_h(\lambda_h, u_h) v_h - \sum_{k=1}^n e_h^k \bar{b}_{kh}\,.$$

Introduce two operators $B_h,\ B_h \in L(X_h, \mathbb{R}^{q+m})$ and $\bar{\mathcal{B}}_h \in L(\Delta_h, \mathbb{R}^n)$ that approximate the operators B and $\bar{\mathcal{B}}$, respectively. $\widetilde{\theta}_{0h} = B_h(\widetilde{x}_{0h})$ and

$$\Psi_h : X_h \to Y_h,\ \Psi_h(x_h) = \begin{bmatrix} B_h(x_h) - \widetilde{\theta}_{0h} \\ G_h(x_h) \end{bmatrix} . \tag{7.2}$$

We have $D\Psi_h(x_h)y_h = [B_h(y_h), DG_h(x_h)y_h]^T$. We also define

$$\Phi_{G,h}(x_h, \cdot) : X_h \to Y_h,\ \Phi_{G,h}(x_h, y_h) = [B_h(y_h), DG_h(x_h)y_h]^T , \tag{7.3}$$

$$\Phi_{H,h}(x_h, \cdot) : \Delta_h \to \Sigma_h,\ \Phi_{H,h}(x_h, \mathrm{z}_h) = [\bar{\mathcal{B}}_h(\mathrm{z}_h), H_h(\lambda_h, u_h, \mathrm{z}_h)]^T . \tag{7.4}$$

Remark 7.2. In [18], the approximate equations are constructed on the same spaces as the exact equation and the exact operator B is maintained in the approximate case.

7.2. **Some sufficient conditions for $\Phi_{G,h}(\widetilde{x}_{0h}, \cdot)$ and $\Phi_{H,h}(\widetilde{x}_{0h}, \cdot)$ to be isomorphisms.** Assume the following hypotheses:

(7.5) there exists an isomorphism $\mathcal{J}$ of W onto Z such that $\mathcal{J}(W_h) = Z_h$,

(7.6) there exists $\eta_1 > 0$ such that, for every $x_h \in X_h,$ we have
$$\|Bx_h - B_h x_h\|_{\mathbb{R}^{q+m}} \le \eta_1 \|x_h\|_{X_h} ,$$

(7.7) there exists $\eta_2 > 0$ such that, for every $(\lambda_h, u_h) \in \mathbb{R}^m \times W_h,$ we have
$$\|\pi_h^Z DF(\lambda_0, u_0)(\lambda_h, u_h) - DF_h(\widetilde{\lambda}_{0h}, \widetilde{u}_{0h})(\lambda_h, u_h)\|_{Z_h} \le \eta_2 \|(\lambda_h, u_h)\|_{\mathbb{R}^m \times W_h} ,$$

(7.8) there exists $\eta_3 > 0$ such that, for every $\mathrm{z}_h \in \Delta_h,$ we have
$$\|\bar{\mathcal{B}}\mathrm{z}_h - \bar{\mathcal{B}}_h \mathrm{z}_h\|_{\mathbb{R}^n} \le \eta_3 \|\mathrm{z}_h\|_{\Delta_h} ,$$

(7.9) there exists $\eta_4 > 0$ such that, for every $v_h \in W_h,$ we have
$$\|\pi_h^Z D_u F(\lambda_0, u_0)v_h - D_u F_h(\widetilde{\lambda}_{0h}, \widetilde{u}_{0h})v_h\|_{Z_h} \le \eta_4 \|v_h\|_{W_h} .$$

Lemma 7.3. *([60]) Let E, F be two Banach spaces and let $T, S \in L(E, F)$. If the operator T is bijective and $\|T^{-1}\|_{L(F,E)}\|T - S\|_{L(E,F)} < 1$, then the operator S is bijective and $\|S^{-1}\|_{L(F,E)} \le (1 - q)^{-1}\ \|T^{-1}\|_{L(F,E)}$, $\forall q \in \mathbb{R}$ that satisfies $\|T^{-1}\|_{L(F,E)}\|T - S\|_{L(E,F)} \le q < 1$.*

The following Theorem 7.4 and its proof are the adaptation, to our conditions, of Theorem XIV.1.1 and of its proof from [43].

Let us denote $q_{G,1} = C(\|DF(\lambda_0, u_0)\|_{L(\mathbb{R}^m \times W,Z)} + \sum_{i=1}^{q} \|\bar{a}_i\|_Z + \|\mathcal{J}\|_{L(W,Z)})$, $q_{G,2} = \eta_1 + \eta_2 + C\|\mathcal{J}\|_{L(W,Z)}$, $q_G = q_{G,h} = (q_{G,1} + q_{G,2})\ \|\Phi_G(x_0, \cdot)^{-1}\|_{L(Y,X)}$, $q_{H,1} = C(\|D_u F(\lambda_0, u_0)\|_{L(W,Z)} + \sum_{k=1}^{n} \|\bar{b}^k\|_Z + \|\mathcal{J}\|_{L(W,Z)})$, $q_{H,2} = \eta_3 + \eta_4 + C\|\mathcal{J}\|_{L(W,Z)}$, $q_H = q_{H,h} = (q_{H,1} + q_{H,2})\ \|\Phi_H(x_0, \cdot)^{-1}\|_{L(\Sigma,\Delta)}$.

Theorem 7.4. *(i) Suppose that Hypotheses (1.2), (7.1), (7.5) - (7.7) hold. If*

$$q_G = q_{G,h} = (q_{G,1} + q_{G,2})\|\Phi_G(x_0, \cdot)^{-1}\|_{L(Y,X)} < 1 , \tag{7.10}$$

then $\Phi_{G,h}(\widetilde{x}_{0h}, \cdot)$ is an isomorphism of X_h onto Y_h and

$$\|\Phi_{G,h}(\widetilde{x}_{0h}, \cdot)^{-1}\|_{L(Y_h,X_h)} \le (1 - q_G)^{-1}\|\Phi_G(x_0, \cdot)^{-1}\|_{L(Y,X)} . \tag{7.11}$$

(ii) Suppose that Hypotheses (1.2), (7.1), (7.5), (7.8), (7.9) hold. If

$$q_H = q_{H,h} = (q_{H,1} + q_{H,2})\|\Phi_H(x_0,\cdot)^{-1}\|_{L(\Sigma,\Delta)} < 1\,, \tag{7.12}$$

then $\Phi_{H,h}(\widetilde{x}_{0h},\cdot)$ *is an isomorphism of* Δ_h *onto* Σ_h *and*

$$\|\Phi_{H,h}(\widetilde{x}_{0h},\cdot)^{-1}\|_{L(\Sigma_h,\Delta_h)} \le (1-q_H)^{-1}\|\Phi_H(x_0,\cdot)^{-1}\|_{L(\Sigma,\Delta)}\,. \tag{7.13}$$

Proof. Since (1.2) holds, from the discussion on (2.6), as in [18], we have that $\Phi_G(x_0,\cdot)$ is an isomorphism of X onto Y. From Theorem 3.5(ii) and Lemma 3.3, $\Phi_H(x_0,\cdot)$ is an isomorphism of Δ onto Σ.

(i) Define $\widehat{\mathcal{J}} : X \to Y$, $\widehat{\mathcal{J}}x = [I_{q+m}(f,\lambda), \mathcal{J}u]^T$. From the hypothesis (7.5), there results that $\widehat{\mathcal{J}}$ is an isomorphism of X onto Y such that $\widehat{\mathcal{J}}(X_h) = Y_h$.

Let $\mathcal{J}_h$ be the restriction of $\mathcal{J}$ to W_h and let $\widehat{\mathcal{J}}_h$ be the restriction of $\widehat{\mathcal{J}}$ to X_h, $\widehat{\mathcal{J}}_h x = [I_{q+m}|_{X_h}(f,\lambda), \mathcal{J}_h u]^T$. We have $\mathcal{J}_h(W_h) = Z_h$ and $\widehat{\mathcal{J}}_h(X_h) = Y_h$. Let $\Pi_h = (I_{q+m}, \pi_h^Z) \in L(Y, Y_h)$.

Let $\mathcal{T} = (\mathcal{T}_B,\, \mathcal{T}_G)$, $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{T}}_B,\, \widetilde{\mathcal{T}}_G) \in L(X,Y)$ and $\widetilde{\mathcal{T}}_h = (\widetilde{\mathcal{T}}_{h,B},\, \widetilde{\mathcal{T}}_{h,G})$, $\mathcal{T}_h = (\mathcal{T}_{h,B},\, \mathcal{T}_{h,G}) \in L(X_h, Y_h)$ defined by

$$\begin{aligned} \mathcal{T} &= \Phi_G(x_0,\cdot) = \widehat{\mathcal{J}} + (\Phi_G(x_0,\cdot) - \widehat{\mathcal{J}})\,, \\ \widetilde{\mathcal{T}} &= \widehat{\mathcal{J}} + \Pi_h(\Phi_G(x_0,\cdot) - \widehat{\mathcal{J}})\,, \\ \widetilde{\mathcal{T}}_h &= \widehat{\mathcal{J}}_h + \Pi_h(\Phi_G(x_0,\cdot) - \widehat{\mathcal{J}}_h)\,, \\ \mathcal{T}_h &= \Phi_{G,h}(\widetilde{x}_{0h},\cdot) = \widehat{\mathcal{J}}_h + (\Phi_{G,h}(\widetilde{x}_{0h},\cdot) - \widehat{\mathcal{J}}_h)\,. \end{aligned} \tag{7.14}$$

Define the first component of the operators $\mathcal{T}$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{T}}_h$, $\mathcal{T}_h$:
$\mathcal{T}_B x = Bx = I_{q+m}(f,\lambda) + (Bx - I_{q+m}(f,\lambda))$,
$\widetilde{\mathcal{T}}_B x = I_{q+m}(f,\lambda) + I_{q+m}(Bx - I_{q+m}(f,\lambda))$,
$\widetilde{\mathcal{T}}_{h,B} x = I_{q+m}(f,\lambda) + I_{q+m}(Bx - I_{q+m}(f,\lambda))$,
$\mathcal{T}_{h,B} x = B_h x = I_{q+m}(f,\lambda) + (B_h x - I_{q+m}(f,\lambda))$.

Define the second component of the operators $\mathcal{T}$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{T}}_h$, $\mathcal{T}_h$:
$\mathcal{T}_G x = DG(x_0)x = \mathcal{J}u + (DF(\lambda_0,u_0)(\lambda,u) - \sum_{i=1}^q f^i \bar{a}_i - \mathcal{J}u)$,
$\widetilde{\mathcal{T}}_G x = \mathcal{J}u + \pi_h^Z DF(\lambda_0,u_0)(\lambda,u) - \pi_h^Z(\sum_{i=1}^q f^i \bar{a}_i) - \pi_h^Z \mathcal{J}u$,
$\widetilde{\mathcal{T}}_{h,G} x = \mathcal{J}_h u + \pi_h^Z DF(\lambda_0,u_0)(\lambda,u) - \sum_{i=1}^q f^i \bar{a}_{ih} - \pi_h^Z \mathcal{J}_h u$,
$\mathcal{T}_{h,G} x = DG_h(\widetilde{x}_{0h})x = \mathcal{J}_h u + (DF_h(\widetilde{\lambda}_{0h},\widetilde{u}_{0h})(\lambda,u) - \sum_{i=1}^q f^i \bar{a}_{ih} - \mathcal{J}_h u)$,
where $\pi_h^Z(\sum_{i=1}^q f^i \bar{a}_i) = \sum_{i=1}^q f^i \pi_h^Z \bar{a}_i = \sum_{i=1}^q f^i \bar{a}_{ih}$.

Let $\bar{x} \in X$. We denote $\mathcal{L}_{G,0}\bar{x} = DF(\lambda_0,u_0)(\bar{\lambda},\bar{u}) - \sum_{i=1}^q \bar{f}^i \bar{a}_i - \mathcal{J}\bar{u}$. We have, using (7.1), that $\|(\mathcal{T} - \widetilde{\mathcal{T}})\bar{x}\|_Y \le \|(I_Z - \pi_h^Z)\mathcal{L}_{G,0}\bar{x}\|_Z \le C\|\mathcal{L}_{G,0}\bar{x}\|_Z$. Using (7.10), there results

$$\|\mathcal{T} - \widetilde{\mathcal{T}}\|_{L(X,Y)} = \sup_{x\in X, \|x\|_X \le 1} \|(\mathcal{T} - \widetilde{\mathcal{T}})x\|_Y \le q_{G,1} < \|\mathcal{T}^{-1}\|_{L(Y,X)}^{-1}\,. \tag{7.15}$$

Applying Lemma 7.3, since $\mathcal{T} = \Phi_G(x_0,\cdot)$ is an isomorphism of X onto Y, there results that $\widetilde{\mathcal{T}}$ is an isomorphism of X onto Y and

$$\|\widetilde{\mathcal{T}}^{-1}\|_{L(Y,X)} \le (1 - q_{G,1}\|\mathcal{T}^{-1}\|_{L(Y,X)})^{-1}\|\mathcal{T}^{-1}\|_{L(Y,X)}\,. \tag{7.16}$$

If $x_h' = \widetilde{\mathcal{T}}^{-1}\zeta_h$, then $\widetilde{\mathcal{T}}x_h' = \zeta_h$ or $\widehat{\mathcal{J}}x_h' + \Pi_h(\Phi_G(x_0,x_h') - \widehat{\mathcal{J}}x_h') = \zeta_h$ or $\widehat{\mathcal{J}}x_h' = \zeta_h - \Pi_h(\Phi_G(x_0,x_h') - \widehat{\mathcal{J}}x_h')$, where $\zeta_h \in Y_h$ and $\Pi_h(\Phi_G(x_0,x_h') - \widehat{\mathcal{J}}x_h') \in Y_h$. So $x_h' = \widehat{\mathcal{J}}^{-1}(\zeta_h - \Pi_h(\Phi_G(x_0,x_h') - \widehat{\mathcal{J}}x_h')) \in X_h$, since $\widehat{\mathcal{J}}(X_h)$

$= Y_h$. There results that the operator $\widetilde{\mathfrak{T}}$ has the property that if $\zeta_h \in Y_h$, then $\widetilde{\mathfrak{T}}^{-1}\zeta_h \in X_h$.

Let us consider the operator $\widetilde{\mathfrak{T}}_h$. We have: $\forall\ x_h \in X_h$, $\widetilde{\mathfrak{T}}_h x_h = \widetilde{\mathfrak{T}} x_h$. It follows that the operator $\widetilde{\mathfrak{T}}_h$ has a continuous inverse that coincides with $\widetilde{\mathfrak{T}}^{-1}$ on Y_h and

$$\|\widetilde{\mathfrak{T}}_h^{-1}\|_{L(Y_h,X_h)} \le \|\widetilde{\mathfrak{T}}^{-1}\|_{L(Y,X)}\,. \tag{7.17}$$

It follows that $\widetilde{\mathfrak{T}}_h$ is an isomorphism of X_h onto Y_h.

Let x_h be an arbitrary element in X_h. We have $\|(\widetilde{\mathfrak{T}}_h - \mathfrak{T}_h)x_h\|_{Y_h} = \|(B|_{X_h} - B_h)x_h\|_{\mathbb{R}^{q+m}} + \|[\pi_h^Z DF(\lambda_0, u_0) - DF_h(\widetilde{\lambda}_{0h}, \widetilde{u}_{0h})](\lambda_h, u_h) + (I_Z - \pi_h^Z)\mathfrak{J}_h u_h\|_{Z_h}$. Using (7.1), (7.6) and (7.7), there results

$$\|\widetilde{\mathfrak{T}}_h - \mathfrak{T}_h\|_{L(X_h,Y_h)} = \sup_{x_h \in X_h, \|x_h\|_{X_h} \le 1} \|(\widetilde{\mathfrak{T}}_h - \mathfrak{T}_h)x_h\|_{Y_h} \le q_{G,2}\,. \tag{7.18}$$

We have $\|\widetilde{\mathfrak{T}}_h^{-1}\|_{L(Y_h,X_h)}\|\widetilde{\mathfrak{T}}_h - \mathfrak{T}_h\|_{L(X_h,Y_h)} \le q_{G,2}\|\widetilde{\mathfrak{T}}_h^{-1}\|_{L(Y_h,X_h)} \le q_{G,2}\|\widetilde{\mathfrak{T}}^{-1}\|_{L(Y,X)}$ $\le r < 1$, where $r = (1 - q_{G,1}\|\mathfrak{T}^{-1}\|_{L(Y,X)})^{-1} q_{G,2}\|\mathfrak{T}^{-1}\|_{L(Y,X)}$, using (7.17), (7.16) and (7.10). Applying Lemma 7.3, since $\widetilde{\mathfrak{T}}_h$ is an isomorphism of X_h onto Y_h, it follows that $\mathfrak{T}_h = \Phi_{G,h}(\widetilde{x}_{0h}, \cdot)$ is an isomorphism of X_h onto Y_h and

$$\|\mathfrak{T}_h^{-1}\|_{L(Y_h,X_h)} \le (1 - r)^{-1}\|\widetilde{\mathfrak{T}}_h^{-1}\|_{L(Y_h,X_h)}\,,$$

whence, using (7.17) and (7.16), we deduce

$$\|\mathfrak{T}_h^{-1}\|_{L(Y_h,X_h)} \le [1 - (q_{G,1} + q_{G,2})\|\Phi_G(x_0, \cdot)^{-1}\|_{L(Y,X)}]^{-1}\|\mathfrak{T}^{-1}\|_{L(Y,X)}\,,$$

so (7.11).

□

Corollary 7.5. *Let us adjoin the index h to C and η_k in order to indicate the dependence on h. Assume that*

$$\lim_{h\to 0} C_h = 0\,,\ \lim_{h\to 0} \eta_{k,h} = 0\,,\ k = 1, 2, 3, 4\,. \tag{7.19}$$

Then, there exists a real $h_0 > 0$ such that for all h, $h \le h_0$, $q_{G,h} < 1$ and $q_{H,h} < 1$.

7.3. **Existence of an approximate bifurcation problem.**

Theorem 7.6. *Assume that the exact problem (1.1) has a bifurcation point (λ_0, u_0) that satisfies hypothesis (1.2). Assume that, for some fixed h, the hypotheses of Theorem 7.4 and of Theorem 5.4 for the approximate case are satisfied. Then, there exists an approximate equation (7.20),*

$$F_h(\lambda_h, u_h) - \varrho_h = 0\,, \tag{7.20}$$

which is of the form of (5.28). The solution (λ_{0h}, u_{0h}) of (7.20) has the same type as the solution (λ_0, u_0) of equation (1.1). (λ_{0h}, u_{0h}) satisfies the hypothesis (1.2). The radii a_h^* and b_h^* depend on h. Theorem 6.1 holds, that is, there exists a class of maps C^p - equivalent (right equivalent) at $(\widehat{\lambda}_{0h}, \widehat{u}_{0h}) = (\lambda_{0h}, u_{0h})$ to $F_h(\cdot) - \varrho_h$ at (λ_{0h}, u_{0h}) and that satisfies the hypothesis (1.2) in $(\widehat{\lambda}_{0h}, \widehat{u}_{0h}) = (\lambda_{0h}, u_{0h})$.

Proof. The construction related to (λ_0, u_0), from [18], presented in Section 2, and the construction from Section 3 lead to the statement (i) of Theorem 3.5 and, then, equivalently, to the statement (ii) of Theorem 3.5.

Using Lemma 3.3, there results that $\Phi_G(x_0, \cdot) = D\Psi(x)$ is an isomorphism of X onto Y, $\Phi_H(x_0, \cdot)$ is an isomorphism of Δ onto Σ.

Using Theorem 7.4 and Lemma 3.3, we obtain that $D\Psi_h(\widetilde{x}_{0h})$ is an isomorphism of X_h onto Y_h and $DS_h(\widetilde{s}_{0,h})$ is an isomorphism of Γ_h onto Σ_h.

We now apply Theorem 5.4 for the case of the approximate formulation. □

In the following Theorem, we formulate conditions similar to those from Theorem IV.3.2, page 304, and Theorem IV.3.7, page 312, [29], and Corollary 3.1, page 52, [18].

Theorem 7.7. *Assume that the exact problem has a bifurcation point that satisfies the statement (i) of Theorem 3.5. Assume the hypotheses of Corollary 7.5. Let α be an arbitrarily small fixed positive number, $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$. Let h_0 be the real from Corollary 7.5. Consider, for each h, $h \leq h_0$, the mappings $\mathcal{G}_h$ and $\mathcal{Q}_h$ given by (5.7) and (5.8).*

For all $h \leq h_0$, we take $\kappa_h = reg\,\mathcal{A}_h$ and $M_h > \|D\mathcal{G}_h(\widetilde{s}_{0,h}, \widetilde{\phi}'_{0,h})\|$. Let $L_h(\varepsilon) = \widetilde{L}(\mathcal{G}_h, (\widetilde{s}_{0,h}, \widetilde{\phi}'_{0,h}), (s_h, \phi'_h), \varepsilon, \Gamma_h \times \Gamma_h, \Sigma_h)$, where we use (2.12). Define $\kappa = \sup_{h \leq h_0} \kappa_h$, $c_h = \frac{1-\kappa L_h(\varepsilon)}{\kappa}$, $\widehat{a} = \sup_{h \leq h_0} \max\{\ \|a_{1h}\|, \ldots, \|a_{qh}\|, \|b_{1h}\|, \ldots, \|b_{nh}\|\ \}$, $M = \sup_{h \leq h_0} M_h$ and $\delta_h = \|\mathcal{G}_h(\widetilde{s}_{0,h}, \widetilde{y}'_{0h})\|$.

Assume that

$$\lim_{h \to 0} \delta_h = 0\,, \tag{7.21}$$

$$\lim_{\beta \to 0} (\sup_{h \leq h_0} L_h(\beta)) = 0\,. \tag{7.22}$$

Assume that $D\Psi_h(\widetilde{x}_{0h})$ is an isomorphism of X_h onto Y_h and $DS_h(\widetilde{s}_{0,h})$ is an isomorphism of Γ_h onto Σ_h.

Then, there exists $h_1 > 0$, $h_1 \leq h_0$, such that $\forall h \leq h_1$, Theorem 5.4 and Theorem 7.6 are applied with ε, τ and $a^* = a^*_h$ that do not depend on h (they are constants). The reals τ, a^*, b^* and the condition (5.25) from Theorem 5.4 are given by τ, a^*_h, b^*_h, from (5.38) - (5.40) for $\mathcal{G}_h$, and (7.23) respectively,

$$\delta_h \leq \frac{1}{2} \cdot \frac{1}{2\kappa} \cdot a^*_h < \frac{1}{2} \cdot b^*_h\ ,\ \forall h \leq h_1\,. \tag{7.23}$$

Let s_{0h} be the solution of the equation (5.27) written in this case for a fixed index h. We have

$$\|(\lambda_0, u_0) - (\lambda_{0h}, u_{0h})\| \leq C_0\|s_0\| + [\gamma_h/(1-\gamma_h L_{S_h}(a)]\cdot\|S_{0,h}(\widetilde{s}_{0,h})\|_{\Sigma_h}\,, \tag{7.24}$$

where $S_{0,h}$ is S_0 in the approximate case. C_0 results using (7.1) for $\|s_0 - \widetilde{s}_{0,h}\| \leq C_0\|s_0\|$.

Proof of Theorem 7.7

Proof. (7.22) implies that we can take an $\varepsilon > 0$ such that

$$2\kappa L_h(\varepsilon) + 2\kappa\alpha\widehat{a} < 1\ ,\ \forall h \leq h_0\,, \tag{7.25}$$

that is, (5.23), $\forall h \leq h_0$.

Remark that ε do not depend on h (for $h \leq h_0$).

For each c_h, we have (5.42).

(7.21) assures that condition (5.25) is satisfied. Then, we apply Theorem 5.4.

We use (5.38) - (5.40) from Lemma 5.7 (ii) applied to $\mathcal{G}_h$. ε, τ and a_h^* do not depend on h (they are constants).

Let us remark that, for condition I from Theorem 2.3, using (5.42), we propose that

$$\frac{1}{c_h}\delta_h < 2\kappa\delta_h \leq \frac{1}{2}a^* < a^*(1-\lambda_h)\,. \tag{7.26}$$

From (7.26), for condition I, we impose (7.23).

We have

$\|(\lambda_0, u_0) - (\lambda_{0h}, u_{0h})\| \leq \|s_0 - s_{0h}\| \leq \|s_0 - \widetilde{s}_{0,h}\| + \|\widetilde{s}_{0,h} - s_{0h}\| \leq C_0\|s_0\| + \|\widetilde{s}_{0,h} - s_{0h}\|$. We use (5.32). We obtain (7.24).

□

Lemma 7.8. *Let $\delta_h = \|\mathcal{G}_h(\widetilde{s}_{0,h}, \widetilde{y}'_{0h})\| = \|S_h(\widetilde{s}_{0,h})\|_{\Sigma_h}$. If $\Psi_h(\widetilde{x}_{0h}) = 0$ in δ_h, then, (7.21) is equivalent to the following conditions, for $i = 1, \ldots, q + m$, $k = 1, \ldots, n$,*

$$\lim_{h\to 0}(\Phi_{G,h}(\widetilde{x}_{0h}, \widetilde{y}_{i,0,h}) - [\delta_i^{q+m}, 0]^T) = 0\,, \;\lim_{h\to 0}(\Phi_{H,h}(\widetilde{x}_{0h}, \widetilde{z}_{k,0,h}) - [\delta_k^n, 0]^T) = 0\,,$$

Corollary 7.9. *$q_h = q$, $n_h = n$, where the index "h" indicates the approximate case.*

Proof. $q + m$ and n are fixed by B and $\bar{\mathcal{B}}$.

□

Corollary 5.10 gives:

Corollary 7.10. *Assume that $(\widetilde{\lambda}_{0h}, \widetilde{u}_{0h})$ belongs to a solution branch of equation (1.3). $(\widetilde{\lambda}_0, \widetilde{u}_0)$ can be a regular or a nonregular solution. Assume the hypotheses of Theorem 7.6 or of Theorem 7.7. $\Psi_h(\widetilde{x}_{0h}) = 0$ in δ_h. If $\varrho_h \neq 0$, then, the given problem (1.3) is a perturbation of the bifurcation problem (7.20) (or of (1.4)). If $\varrho_h = 0$, then, the bifurcation point (λ_{0h}, u_{0h}) belongs to the solution branch of equation (1.3).*

Remark 7.11. We proved that there exists an approximate bifurcation problem (7.20) (or (1.4)) that preserves the type of the bifurcation point of (1.1). The given problem (1.3) is a perturbation of (7.20) (or (1.4)) (when $\|\varrho_h\|$ is small enough.).

8. The Dirichlet problem for the stationary Navier-Stokes equations

In the particular case of the stationary Navier-Stokes equations, we maintain the position from Section 1, where we state that we do not discuss if the exact bifurcation point is generic or not. As a consequence of the results from Section 7, in a certain configuration, if the model of a stationary flow has a generic or an ungeneric bifurcation point, then there exists (at least) one approximate equation that has a bifurcation point of the same type as the exact model. The hypothesis of the existence of an ungeneric bifurcation point cannot be excluded, see [25, 45, 63]. For example, we do not exclude transcritical bifurcation and nonsymmetric pitchfork bifurcation from our discussion. If they exit, then they are regained in the approximate case by some perturbed approximate equation of the form (7.20) (or (1.4)). For this discussion, other references are mentioned in Section 1.

Here, we show a modality to place the Dirichlet problem for the stationary Navier-Stokes equations in the framework of the preceding sections.

8.1. **The setting from** [29]**.** In order to apply the results for the case of the Dirichlet problem for the stationary Navier-Stokes equations, formulated in primitive variables, approximated by finite element method (with discontinuous pressure), we use the setting of this problem from Section IV.4.1, [29], in the framework of Section IV.3.3, [29]. We only indicate the connection to the problem from Section IV.4.1, [29].

Let Ω be a bounded and connected open subset of $\mathbb{R}^N$ $(N = 2, 3)$ with a Lipschitz - continuous boundary $\partial\Omega$. In the sequel, $\mathbf{u}$ is the velocity, p is the kinematic pressure and ν is the kinematic viscosity. We take $N = 2, 3$, $\mathcal{X} = H_0^1(\Omega)^N \times L_0^2(\Omega)$, $\mathcal{Y} = H^{-1}(\Omega)^N$. $\lambda = 1/\nu > 0$. λ is the bifurcation parameter, $\mathbf{u} = (u_1, \ldots, u_N) \in H_0^1(\Omega)^N$, $\mathbf{x} = (x_1, \ldots, x_N) \in \Omega$. Let $\mathcal{T}_S \in L(\mathcal{Y}, \mathcal{X})$ be the Stokes operator that associates to $\mathbf{f} \in \mathcal{Y}$ the solution $(\mathbf{u}, p) = \mathcal{T}_S\mathbf{f}$ of the homogeneous Stokes problem,

$$
\begin{aligned}
&-\triangle\mathbf{u} + \mathbf{grad}\, p = \mathbf{f} \text{ in } \Omega\,,\\
&div\, \mathbf{u} = 0 \text{ in } \Omega\,,\\
&\mathbf{u} = 0 \text{ on } \partial\Omega\,.
\end{aligned}
\tag{8.1}
$$

The functions F and G, from Sections IV.3.1 and IV.4.1, [29], are F_{NS} and G_{NS}, where $F_{NS} : (0,\infty)\times\mathcal{X} \to \mathcal{X}$, $F_{NS}(\lambda, v) = v + \mathcal{T}_S G_{NS}(\lambda, v)$, and $G_{NS} : (0,\infty)\times\mathcal{X} \to \mathcal{Y}$, $G_{NS}(\lambda, v) = \lambda\cdot(\sum_{j=1}^N v_j(\partial\mathbf{v}/\partial x_j) - \mathbf{f})$ for $v = (\mathbf{v}, q)$. G_{NS} is of class C^∞ and has bounded derivatives of all order on all bounded subsets of $\mathcal{X}$. For $(\mathbf{u}, p), (\overline{\mathbf{u}}, \overline{p}), (\overline{\overline{\mathbf{u}}}, \overline{\overline{p}}) \in \mathcal{X}$, we have $D_{(\mathbf{u},p)}G_{NS}(\lambda, \mathbf{u}, p)(\overline{\mathbf{u}}, \overline{p}) = \lambda \cdot \sum_{j=1}^N (u_j(\partial\overline{\mathbf{u}}/\partial x_j) + \overline{u}_j(\partial\mathbf{u}/\partial x_j))$, $D^2_{(\mathbf{u},p)(\mathbf{u},p)}G_{NS}(\lambda, \mathbf{u}, p)((\overline{\mathbf{u}}, \overline{p}), (\overline{\overline{\mathbf{u}}}, \overline{\overline{p}})) = \lambda \cdot \sum_{j=1}^N (\overline{\overline{u}}_j(\partial\overline{\mathbf{u}}/\partial x_j) + \overline{u}_j(\partial\overline{\overline{\mathbf{u}}}/\partial x_j))$.

Let us fix λ. $(\mathbf{u}, p)$ is a solution of the homogeneous Navier-Stokes problem if and only if $u = (\mathbf{u}, \lambda p)$ is a solution of $F_{NS}(\lambda, u) = 0$ ([29]).

Let h be a positive parameter tending to zero. For each h, let Γ_h and E_h be two finite-dimensional spaces such that$\Gamma_h \subset H^1(\Omega)^N$, $E_h \subset L^2(\Omega)$ and assume that E_h contains the constant functions.

Let $\Gamma_{0h} = \Gamma_h \cap H_0^1(\Omega)^N$, $M_h = E_h \cap L_0^2(\Omega)$, $\mathcal{X}_h = \Gamma_{0h} \times M_h$. Assume:

(a) There exists an operator $r_h \in L([H^2(\Omega) \cap H_0^1(\Omega)]^N, \Gamma_{0h})$ and an integer ℓ such that

$$
\|\mathbf{v} - r_h\mathbf{v}\|_{1,\Omega} \le C_r \cdot h^m \cdot \|\mathbf{v}\|_{m+1,\Omega},\ \forall\mathbf{v} \in H^{m+1}(\Omega)^N,\ 1 \le m \le \ell\,.
\tag{8.2}
$$

(b) There exists an operator $s_h \in L(L^2(\Omega), E_h)$ such that

$$
\|q - s_h q\|_{0,\Omega} \le C_s \cdot h^m \cdot \|q\|_{m,\Omega},\ \forall q \in H^m(\Omega),\ 0 \le m \le \ell\,.
\tag{8.3}
$$

These assumptions (a) and (b) together with the the uniform inf-sup condition are the hypotheses H_1, H_2 and H_3 for the approximation from [29].

Equation $F_{NS}(\lambda, u) = 0$ is approximated by $\mathcal{F}_{NS,h}(\lambda, u_h) = 0$ ([29]), $u_h = (\mathbf{u}_h, \lambda p_h)$, where $\mathcal{F}_{NS,h} : \mathbb{R} \times \mathcal{X}_h \to \mathcal{X}_h$, $\mathcal{F}_{NS,h}(\lambda, u_h) = u_h + \mathcal{T}_{S,h} G_{NS}(\lambda, u_h)$ and $\mathcal{T}_{S,h} \in L(\mathcal{Y}, \mathcal{X}_h)$ is the approximate Stokes operator that associates to $\mathbf{f} \in \mathcal{Y}$ the solution $(\mathbf{u}_h, p_h) = \mathcal{T}_{S,h}\mathbf{f}$ of the finite element method approximation of problem (8.1) ([29]),

$$(8.4) \qquad (grad\, \mathbf{u}_h, grad\, \mathbf{w}_h) - (p_h, div\, \mathbf{w}_h) = \langle \mathbf{f}, \mathbf{w}_h \rangle, \ \forall\, \mathbf{w}_h \in \Gamma_{0h}\,,$$

$$(8.5) \qquad (div\, \mathbf{u}_h, \mu_h) = 0, \ \forall\, \mu_h \in M_h\,.$$

According to Theorem II.1.8, page 125, and to the proof of Theorem IV.4.1, page 317, [29], we have:

$$(8.6) \qquad \lim_{h \to 0} \|(\mathcal{T}_S - \mathcal{T}_{S,h})\mathbf{f}\|_{\mathcal{X}} = 0\,, \ \forall\, \mathbf{f} \in \mathcal{Y}\,.$$

According to the proof of Theorem IV.4.1, page 317, [29], we have:

$$(8.7) \qquad \sum_{j=1}^{N} (u_j(\partial \overline{\mathbf{u}}/\partial x_j) + \overline{u}_j(\partial \mathbf{u}/\partial x_j)) \in L^{3/2}(\Omega)^N\,,$$

where $\mathbf{u}$, $\overline{\mathbf{u}} \in H_0^1(\Omega)^N$, and, for $\mathcal{Z} = L^{3/2}(\Omega)^N \subset \mathcal{Y}$,

$$(8.8) \qquad \lim_{h \to 0} \|\mathcal{T}_S - \mathcal{T}_{S,h}\|_{L(\mathcal{Z},\mathcal{X})} = 0\,.$$

8.2. **The supplementary variabile q.** Let us write the homogeneous Navier-Stokes problem as a problem of the form (1.1). λ is variable. We introduce a new variable $q = \lambda p$ and we write the problem from [29] in the following form

$$(8.9) \qquad \begin{aligned} & p - \lambda^{-1} q = 0\,, \\ & (\mathbf{u}, q) + \mathcal{T}_S G_{NS}(\lambda, \mathbf{u}, p) = 0\,, \end{aligned}$$

or

$$(8.10) \qquad \begin{aligned} & p + \widetilde{\mathcal{T}}_S \widetilde{G}_{NS}(\lambda, q) = 0\,, \\ & (\mathbf{u}, q) + \mathcal{T}_S G_{NS}(\lambda, \mathbf{u}, p) = 0\,, \end{aligned}$$

where $\mathcal{T}_*(p, \mathbf{f}) = \begin{bmatrix} \widetilde{\mathcal{T}}_S p \\ \mathcal{T}_S \mathbf{f} \end{bmatrix} = \begin{bmatrix} I_1 p \\ \mathcal{T}_S \mathbf{f} \end{bmatrix}$,

$\mathcal{G}_*(\lambda, p, \mathbf{u}, q) = \begin{bmatrix} -\lambda^{-1} q \\ G_{NS}(\lambda, \mathbf{u}, p) \end{bmatrix} = \begin{bmatrix} \widetilde{G}_{NS}(\lambda, q) \\ G_{NS}(\lambda, \mathbf{u}, p) \end{bmatrix}$.

I_1, I_2 are the identity operators on $L_0^2(\Omega)$, $H_0^1(\Omega)^N$ respectively.

$\mathcal{T}_*^{-1}(\bar{a}_i)$ are linearly independent if and only if $\bar{a}_i$ are linearly independent.

We denote $\mathcal{W} = L_0^2(\Omega) \times H_0^1(\Omega)^N \times L_0^2(\Omega)$. Let $\mathcal{F}_* : (0, \infty) \times \mathcal{W} \to \mathcal{W}$, $\mathcal{F}_*(\lambda, p, \mathbf{u}, q) = (p, \mathbf{u}, q) + \mathcal{T}_* \mathcal{G}_*(\lambda, p, \mathbf{u}, q)$. $(\lambda, \mathbf{u}, q) = (\lambda, \mathbf{u}, \lambda p)$ is a solution of $F_{NS}(\lambda, \mathbf{u}, q) = 0$ is equivalent to say that $(\lambda, p, \mathbf{u}, q)$ is a solution of the following problem

$$(8.11) \qquad \mathcal{F}_*(\lambda, p, \mathbf{u}, q) = 0\,.$$

$\mathcal{F}_*$ is approximated by $\mathcal{F}_{*,h} : (0, \infty) \times \mathcal{W}_h \to \mathcal{W}_h$, $\mathcal{F}_{*,h}(\lambda, p, \mathbf{u}, q) = (p, \mathbf{u}, q) + \mathcal{T}_{*,h} \mathcal{G}_*(\lambda, p, \mathbf{u}, q)$, $\mathcal{W}_h = M_h \times \Gamma_{0h} \times M_h$.

8.3. **The verification of the hypotheses of Theorem 7.4.** Let us replace F, $\mathbb{R}^m \times W$, Z, (λ, u), (λ_0, u_0), in problem (1.1) and in hypothesis (1.2), by $\mathcal{F}_*$, $\mathbb{R} \times \mathcal{W}$, $\mathcal{W}$, $(\lambda, p, \mathbf{u}, q)$, $(\lambda_0, p_0, \mathbf{u}_0, q_0)$ respectively. We say that we study problem (8.11) under hypothesis (1.2) and we want to prove that the results of Theorem 7.4 hold for the approximation (**??**) of (8.11). For this, it suffices to verify hypotheses (7.1), (7.5) - (7.9).

Let $\pi_h^{\mathcal{W}} = \pi_h^Z = \pi_h^W = (s_h, r_h, s_h)$ and $I = (I_1, I_2, I_1)$. We take

$$\widetilde{\lambda}_{0h} = \lambda_0\,,\ \widetilde{p}_{0h} = s_h p_0\,, \widetilde{\mathbf{u}}_{0h} = r_h \mathbf{u}_0\,, \widetilde{q}_{0h} = s_h q_0\,. \tag{8.12}$$

We have $\widetilde{\lambda}_{0h} = \lambda_0$. We denote $\sigma_0 = (\lambda_0, p_0, \mathbf{u}_0)$, $\widetilde{\sigma}_{0h} = (\widetilde{\lambda}_{0h}, \widetilde{p}_{0h}, \widetilde{\mathbf{u}}_{0h})$, $\sigma_0^{NS} = (\lambda_0, \mathbf{u}_0, p_0)$, $\widetilde{\sigma}_{0h}^{NS} = (\widetilde{\lambda}_{0h}, \widetilde{\mathbf{u}}_{0h}, \widetilde{p}_{0h})$.

Let us first verify (7.9). We have

(8.13) there exists $\eta_4 > 0$ such that, for every $v_h \in W_h$, we have

$$\begin{aligned}
&\|\pi_h^Z D_u F(\lambda_0, u_0) v_h - D_u F_h(\widetilde{\lambda}_{0h}, \widetilde{u}_{0h}) v_h\|_{Z_h}\\
&= \|\pi_h^{\mathcal{W}} D_{(p,\mathbf{u},q)} \mathcal{F}_*(\sigma_0, q_0)(\overline{p}, \overline{\mathbf{u}}, \overline{q}) - D_{(p,\mathbf{u},q)} \mathcal{F}_{*,h}(\widetilde{\sigma}_{0h}, \widetilde{q}_{0h})(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&\le \|(\pi_h^{\mathcal{W}} - I)(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&+\|\pi_h^{\mathcal{W}} \begin{bmatrix} \widetilde{\mathcal{T}}_S D_q \widetilde{G}_{NS}(\lambda_0, q_0)\overline{q} \\ \mathcal{T}_S D_{(\mathbf{u},p)} G_{NS}(\sigma_0^{NS})(\overline{\mathbf{u}}, \overline{p}) \end{bmatrix} - \begin{bmatrix} \tilde{\mathcal{T}_{S,h}} D_q \widetilde{G}_{NS}(\widetilde{\lambda}_{0h}, \widetilde{q}_{0h})\overline{q} \\ \mathcal{T}_{S,h} D_{(\mathbf{u},p)} G_{NS}(\widetilde{\sigma}_{0h}^{NS})(\overline{\mathbf{u}}, \overline{p}) \end{bmatrix} \|_{\mathcal{W}_h}\\
&= \|(\pi_h^{\mathcal{W}} - I)(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&+\|(r_h, s_h)\mathcal{T}_S D_{(\mathbf{u},p)} G_{NS}(\sigma_0^{NS})(\overline{\mathbf{u}}, \overline{p}) - \mathcal{T}_{S,h} D_{(\mathbf{u},p)} G_{NS}(\widetilde{\sigma}_{0h}^{NS})(\overline{\mathbf{u}}, \overline{p})\|_{\mathcal{X}_h}\\
&\le \|(\pi_h^{\mathcal{W}} - I)(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&+\|((r_h, s_h) - (I_2, I_1))\mathcal{T}_S D_{(\mathbf{u},p)} G_{NS}(\sigma_0^{NS})(\overline{\mathbf{u}}, \overline{p})\|_{\mathcal{X}_h}\\
&+\|(\mathcal{T}_S - \mathcal{T}_{S,h}) D_{(\mathbf{u},p)} G_{NS}(\sigma_0^{NS})(\overline{\mathbf{u}}, \overline{p})\|_{\mathcal{X}_h}\\
&+\|\mathcal{T}_{S,h}(D_{(\mathbf{u},p)} G_{NS}(\sigma_0^{NS})(\overline{\mathbf{u}}, \overline{p}) - D_{(\mathbf{u},p)} G_{NS}(\widetilde{\sigma}_{0h}^{NS})(\overline{\mathbf{u}}, \overline{p}))\|_{\mathcal{X}_h}\\
&\le \eta_4 \|(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h} = \eta_4 \|v_h\|_{W_h}\,,
\end{aligned}$$

where η_4 is obtained using (8.12), (8.2), (8.3), (8.7) si (8.8).

Let us verify (7.7). We have

(8.14) there exists $\eta_2 > 0$ such that, for every $(\lambda_h, u_h) \in \mathbb{R}^m \times W_h$, we have

$$\begin{aligned}
&\|\pi_h^Z DF(\lambda_0, u_0)(\lambda_h, u_h) - DF_h(\widetilde{\lambda}_{0h}, \widetilde{u}_{0h})(\lambda_h, u_h)\|_{Z_h}\\
&= \|\pi_h^{\mathcal{W}} D\mathcal{F}_*(\sigma_0, q_0)(\overline{\lambda}, \overline{p}, \overline{\mathbf{u}}, \overline{q}) - D\mathcal{F}_{*,h}(\widetilde{\sigma}_{0h}, \widetilde{q}_{0h})(\overline{\lambda}, \overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&\le \|\pi_h^{\mathcal{W}} \begin{bmatrix} \widetilde{\mathcal{T}}_S D_\lambda \widetilde{G}_{NS}(\lambda_0, q_0)\overline{\lambda} \\ \mathcal{T}_S D_\lambda G_{NS}(\sigma_0^{NS})\overline{\lambda} \end{bmatrix} - \begin{bmatrix} \tilde{\mathcal{T}_{S,h}} D_\lambda \widetilde{G}_{NS}(\widetilde{\lambda}_{0h}, \widetilde{q}_{0h})\overline{\lambda} \\ \mathcal{T}_{S,h} D_\lambda G_{NS}(\widetilde{\sigma}_{0h}^{NS})\overline{\lambda} \end{bmatrix} \|_{\mathcal{W}_h}\\
&+\|\pi_h^{\mathcal{W}} D_{(p,\mathbf{u},q)} \mathcal{F}_*(\sigma_0, q_0)(\overline{p}, \overline{\mathbf{u}}, \overline{q}) - D_{(p,\mathbf{u},q)} \mathcal{F}_{*,h}(\widetilde{\sigma}_{0h}, \widetilde{q}_{0h})(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&= \|s_h \widetilde{\mathcal{T}}_S D_\lambda \widetilde{G}_{NS}(\lambda_0, q_0)\overline{\lambda} - \tilde{\mathcal{T}_{S,h}} D_\lambda \widetilde{G}_{NS}(\widetilde{\lambda}_{0h}, \widetilde{q}_{0h})\overline{\lambda}\|_{\mathcal{M}_h}\\
&+\|(r_h, s_h)\mathcal{T}_S G_{NS}(\overline{\lambda}, \mathbf{u}_0, p_0) - \mathcal{T}_{S,h} G_{NS}(\overline{\lambda}, \widetilde{\mathbf{u}}_{0h}, \widetilde{p}_{0h})\|_{\mathcal{X}_h}\\
&+\|\pi_h^{\mathcal{W}} D_{(p,\mathbf{u},q)} \mathcal{F}_*(\sigma_0, q_0)(\overline{p}, \overline{\mathbf{u}}, \overline{q}) - D_{(p,\mathbf{u},q)} \mathcal{F}_{*,h}(\widetilde{\sigma}_{0h}, \widetilde{q}_{0h})(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&\le \|((r_h, s_h) - (I_2, I_1))\mathcal{T}_S G_{NS}(\overline{\lambda}, \mathbf{u}_0, p_0))\|_{\mathcal{X}_h}\\
&+\|(\mathcal{T}_S - \mathcal{T}_{S,h}) G_{NS}(\overline{\lambda}, \mathbf{u}_0, p_0)\|_{\mathcal{X}_h} + \|\mathcal{T}_{S,h}(G_{NS}(\overline{\lambda}, \mathbf{u}_0, p_0) - G_{NS}(\overline{\lambda}, \widetilde{\mathbf{u}}_{0h}, \widetilde{p}_{0h}))\|_{\mathcal{X}_h}\\
&+\|\pi_h^{\mathcal{W}} D_{(p,\mathbf{u},q)} \mathcal{F}_*(\sigma_0, q_0)(\overline{p}, \overline{\mathbf{u}}, \overline{q}) - D_{(p,\mathbf{u},q)} \mathcal{F}_{*,h}(\widetilde{\sigma}_{0h}, \widetilde{q}_{0h})(\overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathcal{W}_h}\\
&\le \eta_2 \|(\overline{\lambda}, \overline{p}, \overline{\mathbf{u}}, \overline{q})\|_{\mathbb{R} \times \mathcal{W}_h} = \eta_2 \|(\lambda_h, u_h)\|_{\mathbb{R}^m \times W_h}\,,
\end{aligned}$$

where η_2 is obtained using (8.13), (8.12), (8.2), (8.3) and (8.6) for $\mathbf{f} = G_{NS}(1, \mathbf{u}_0, p_0)$.

Equation (8.11) has the form of equation (1.1). Assume that (8.11) satisfies the hypotheses of Theorem 7.6. Then the corresponding equation (7.20), with $\varrho_h = (\widetilde{\varrho}_h, \overline{\varrho}_h, \widehat{\varrho}_h)$, is

$$\mathcal{F}_{*,h}(\lambda_h, p_h, \mathbf{u}_h, q_h) - \varrho_h = 0\,, \tag{8.15}$$

or

$$(p_h, \mathbf{u}_h, q_h) + \mathcal{T}_{*,h}\mathcal{G}_*(\lambda_h, p_h, \mathbf{u}_h, q_h) - (\widetilde{\varrho}_h, \overline{\varrho}_h, \widehat{\varrho}_h) = 0\,, \tag{8.16}$$

or

$$p_h - \lambda_h^{-1} q_h = \widetilde{\varrho}_h\,, \tag{8.17}$$

$$\begin{aligned}(grad\,(\mathbf{u}_h - \overline{\varrho}_h), grad\,\mathbf{w}_h) - ((q_h - \widehat{\varrho}_h), div\,\mathbf{w}_h) \\ = \lambda_h \langle \mathbf{f} - \sum_{j=1}^{N} u_{h,j}\frac{\partial \mathbf{u}_h}{\partial x_j}, \mathbf{w}_h\rangle,\ \forall\, \mathbf{w}_h \in \Gamma_{0h}\,,\end{aligned} \tag{8.18}$$

$$(div\,(\mathbf{u}_h - \overline{\varrho}_h), \mu_h) = 0,\ \forall\, \mu_h \in M_h\,. \tag{8.19}$$

With the same settings, the framework of Section 10.2.3 (and also of Section 10.2.2, rewritten for spectral Galerkin approximations), from [55], allows the approximate Stokes operator $\mathcal{T}_{S,h}$ to be constructed using spectral methods.

9. A complement to Theorem 5.4

In this section, we investigate if we can fix only $\hat{y}'_0$ in (5.26), in Theorem 5.4.

Let $F : \mathbb{R}^m \times W \to Z$ be a nonlinear function of class C^p. Let $(\mu', w') \in \mathbb{R}^m \times W$. Let us define

$$\widetilde{F} : \mathbb{R}^m \times W \to Z,\ \widetilde{F}(\lambda, u) = F(\lambda, u) - DF(\lambda, u)(\mu', w')\,,$$

and $\widetilde{G} : X \to Z$, $\widetilde{H} : \mathbb{R}^m \times W \times \Delta \to Z$, $\widetilde{\Psi} : X \to Y$, where $\widetilde{G}$, $\widetilde{H}$, $\widetilde{\Psi}$ are obtained by replacing F by $\widetilde{F}$ in the definitions of G, H, Ψ respectively.

Let us define $\widetilde{S} : \Gamma \to \Sigma$, $\widetilde{S}(s) = S(s)$, and $\widetilde{\Phi} : \Gamma \times \Gamma \to \Sigma$,

$$\widetilde{\Phi}(s, \phi') = \Phi(s, \phi') + \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ D^2F(\lambda, u)((\mu', w'), ((\mu_i, w_i) - (\mu'_i, w'_i))) \end{bmatrix} \\ \begin{bmatrix} 0 \\ D_u(DF(\lambda, u)(\mu', w'))(v_k - v'_k) \end{bmatrix} \end{bmatrix},$$

for all $i = 1, \ldots, q + m$, $k = 1, \ldots, n$.

In the definitions from Subsections 5.1 and 5.2, let us replace F, G, H, Ψ, S, Φ by $\widetilde{F}$, $\widetilde{G}$, $\widetilde{H}$, $\widetilde{\Psi}$, $\widetilde{S}$, $\widetilde{\Phi}$ respectively.

Corollary 9.1. *Assume the hypotheses of Theorem 5.4 where we replace F, G, H, S, Φ by $\widetilde{F}$, $\widetilde{G}$, $\widetilde{H}$, $\widetilde{\Psi}$, $\widetilde{S}$, $\widetilde{\Phi}$ respectively. Then (5.26) becomes*

$$\widetilde{S}(s_0) - \widetilde{\Phi}(s_0, \phi'_0) \ni 0\,, \tag{9.1}$$

Let us fix $\hat{y}'_0$ (whose existence is demonstrated) in (9.1). Let us take $\theta_0 = \widetilde{\theta}_0 + B(\hat{y}'_0)$. Let us define

$$\widetilde{F}_0 : \mathbb{R}^m \times W \to Z,\ \widetilde{F}_0(\lambda, u) = F(\lambda, u) - DF(\lambda, u)(\hat{\mu}'_0, \hat{w}'_0)\,, \tag{9.2}$$

and $\widetilde{G}_0 : X \to Z$, $\widetilde{H}_0 : \mathbb{R}^m \times W \times \Delta \to Z$, $\widetilde{\Psi}_0 : X \to Y$, where $\widetilde{G}_0$, $\widetilde{H}_0$, $\widetilde{\Psi}_0$ are obtained by replacing F by $\widetilde{F}_0$ in the definitions of G, H, Ψ respectively. Let $\widetilde{S}_0 : \Gamma \to \Sigma$ be S where we replace F by $\widetilde{F}_0$ in (3.7) and we use θ_0 defined above.

Then, s_0 is the solution of the equation

$$\widetilde{S}_0(s) = 0\ . \tag{9.3}$$

Equation (9.3) is of the form of equation (3.8).

Then, the component (λ_0, u_0) of s_0 is a solution of the equation

$$F(\lambda, u) - DF(\lambda, u)(\hat{\mu}'_0, \hat{w}'_0) = 0\,, \tag{9.4}$$

$(\lambda_0, u_0) \in \mathbb{B}_{a^*}(\widetilde{\lambda}_0, \widetilde{u}_0)$.

Assume that $D\widetilde{\Psi}_0(\widetilde{x}_0)$ is an isomorphism of X onto Y and $D\widetilde{S}_0(\widetilde{s}_0)$ is an isomorphism of Γ onto Σ. Under some other additional conditions like in Theorem 5.4, we have:

Then, s_0 is the unique solution of the equation (9.3) in some $\mathbb{B}_a(\widetilde{s}_0)$ with $a \geq a^*$. The system (9.3) and its solution s_0 verify the assertions (a) and (b) of the statement (ii) of Theorem 3.5.

Then, the component (λ_0, u_0) of s_0 is the unique solution of the equation (9.4) that satisfies hypothesis (1.2) and the rest of the hypotheses of the statement (i) of Theorem 3.5 in some $\mathbb{B}_a(\widetilde{\lambda}_0, \widetilde{u}_0)$ with $a \geq a^*$. The solution (λ_0, u_0) is a bifurcation point of problem (9.4).

Proof. The proof of Theorem 5.4 remains valid. We only mention that (5.57) has the formulation

$$\widetilde{S}(\bar{s}) - \widetilde{\Phi}(\bar{x}, \bar{\phi}') - \widetilde{\Phi}(\widetilde{x}_0, \xi(\bar{f}, \bar{g}_i, \bar{e}_k)) + \widetilde{\Phi}(\widetilde{x}_0, \xi(\bar{g}', \bar{g}'_i, \bar{e}'_k)) \ni 0\,,$$

$$\Leftrightarrow$$

$$\begin{bmatrix} B(0, \bar{\lambda}, \bar{u}) - \widetilde{\theta}_0 - B(0, \bar{\mu}', \bar{w}') \\ F(\bar{\lambda}, \bar{u}) - DF(\bar{\lambda}, \bar{u})(\bar{\mu}', \bar{w}') \\ B((0, \bar{\mu}_i, \bar{w}_i) - (0, \bar{\mu}'_i, \bar{w}'_i)) - \delta_i^{q+m} \\ D(F(\bar{\lambda}, \bar{u}) - DF(\bar{\lambda}, \bar{u})(\bar{\mu}', \bar{w}'))((\bar{\mu}_i, \bar{w}_i) - (\bar{\mu}'_i, \bar{w}'_i))) \\ \bar{\mathcal{B}}((0, \bar{v}_k) - (0, \bar{v}'_k)) - \delta_k^n \\ D_u(F(\lambda, u) - DF(\bar{\lambda}, \bar{u})(\bar{\mu}', \bar{w}'))(\bar{v}_k - \bar{v}'_k) \end{bmatrix} \ni 0\,,$$

□

Lemma 9.2. *Assume that $DS(\widetilde{s}_0)$ is an isomorphism of Γ onto Σ. If*

$$\gamma \| DS(\widetilde{s}_0) - D\widetilde{S}_0(\widetilde{s}_0) \|_{L(\Gamma, \Sigma)} < 1\,, \tag{9.5}$$

then $D\widetilde{S}_0(\widetilde{s}_0)$ is an isomorphism of Γ onto Σ.

Proof. This results from Lemma 3.1 [18], Theorem 6A.1 [21] and Lemma IV.3.3 [29].

□

Remark 9.3. Instead of a constant ϱ in the equation (5.28), we have obtained ϱ in the form of a function $\varrho(\lambda, u) = DF(\lambda, u)(\hat{\mu}_0', \hat{w}_0')$ in the equation (9.4).

10. Future Work

We have formulated some sufficient conditions for the existence of an approximate equation (7.20) that has a bifurcation point of the same type as the bifurcation point of a given exact equation (1.1).

In a further research that will continue the present one, we have the following purposes:

(i) to prove that, given a function F and an approximation F_h for this, under some conditions, if there exists ϱ_h (which is zero or nonzero) so that the equation

$$F_h(\lambda_h, u_h) - \varrho_h = 0 \tag{10.1}$$

has a bifurcation point, then there exists ϱ such that the equation

$$F(\lambda, u) - \varrho = 0 \tag{10.2}$$

has a bifurcation point of the same type as the bifurcation point of (10.1). This idea is inspired by a result from [43] where Kantorovich and Akilov prove that, given the linear operators that define an exact equation and an approximate equation, under certain hypotheses, if the approximate operator is an isomorphism, then the exact operator is an isomorphism.

(ii) to formulate some algorithms so that, by studying the approximate equation

$$F_h(\lambda_h, u_h) = 0\,, \tag{10.3}$$

to decide if there exists ϱ_h such that the equation (10.1) has a bifurcation point, to determine ϱ_h (only if this is necessary) and to determine the type of the bifurcation point of (10.1) and hence of (10.2). In the study for (ii), we will extend the methods introduced in [5].

In this way, we can reduce the study of the qualitative aspects of a bifurcation problem on infinite-dimensional Banach spaces to the study of an approximate problem. Finally, the study can be performed on a computer.

Appendix A. A formulation of $D\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}_0')(\overline{s}, \overline{\phi}') - D\mathcal{G}(s, \phi')(\overline{s}, \overline{\phi}')$

$$\Upsilon(\widetilde{s}_0, \widetilde{\phi}_0', s, \phi', \overline{s}, \overline{\phi}') = D\mathcal{G}(\widetilde{s}_0, \widetilde{\phi}_0')(\overline{s}, \overline{\phi}') - D\mathcal{G}(s, \phi')(\overline{s}, \overline{\phi}') \tag{A.1}$$

$$= \frac{1}{2}DS(\widetilde{s}_0)\overline{s} - \frac{1}{2}D\Phi(\widetilde{x}_0, \widetilde{\phi}_0')(\overline{x}, \overline{\phi}') - (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{f}, \overline{g}_i, \overline{e}_k)) + (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{g}', \overline{g}_i', \overline{e}_k'))$$

$$-\frac{1}{2}DS(s)\overline{s} + \frac{1}{2}D\Phi(x, \phi')(\overline{x}, \overline{\phi}') + (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{f}, \overline{g}_i, \overline{e}_k)) - (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{g}', \overline{g}_i', \overline{e}_k'))$$

$$\begin{aligned} &= \frac{1}{2}(DS(\widetilde{s}_0)\overline{s} - DS(s)\overline{s}) \\ &-\frac{1}{2}(D\Phi(\widetilde{x}_0, \widetilde{\phi}_0')(\overline{x}, \overline{\phi}') - D\Phi(x, \phi')(\overline{x}, \overline{\phi}')) \\ &-(1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{f}, \overline{g}_i, \overline{e}_k)) + (1-\alpha)\Phi(\widetilde{x}_0, \xi(\overline{g}', \overline{g}_i', \overline{e}_k')) \end{aligned}$$

$$+(1-\alpha)\Phi(\widetilde{x}_0,\xi(\overline{f},\overline{g}_i,\overline{e}_k))-(1-\alpha)\Phi(\widetilde{x}_0,\xi(\overline{g}',\overline{g}'_i,\overline{e}'_k))\,.$$

$$\Upsilon(\widetilde{s}_0,\widetilde{\phi}'_0,s,\phi',\overline{s},\overline{\phi}')=\frac{1}{2}(\left[\begin{array}{l} B(\overline{x}) \\ DG(\widetilde{x}_0)\overline{x} \\ B(\overline{y}_i) \\ D^2F(\widetilde{\lambda}_0,\widetilde{u}_0)((\widetilde{\mu}_{i,0},\widetilde{w}_{i,0}),(\overline{\lambda},\overline{u}))+DG(\widetilde{x}_0)\overline{y}_i \\ \bar{\mathcal{B}}(\overline{z}_k) \\ D_{(\lambda,u)}(D_uF(\widetilde{\lambda}_0,\widetilde{u}_0)\widetilde{v}_{k,0})(\overline{\lambda},\overline{u})+H(\widetilde{\lambda}_0,\widetilde{u}_0,\overline{z}_k) \end{array}\right]$$

$$-\left[\begin{array}{l} B(\overline{x}) \\ DG(x)\overline{x} \\ B(\overline{y}_i) \\ D^2F(\lambda,u)((\mu_i,w_i),(\overline{\lambda},\overline{u}))+DG(x)\overline{y}_i \\ \bar{\mathcal{B}}(\overline{z}_k) \\ D_{(\lambda,u)}(D_uF(\lambda,u)v_k)(\overline{\lambda},\overline{u})+H(\lambda,u,\overline{z}_k) \end{array}\right])$$

$$-\frac{1}{2}(\left[\begin{array}{l} B(\overline{y}') \\ D^2F(\widetilde{\lambda}_0,\widetilde{u}_0)((\widetilde{\mu}'_0,\widetilde{w}'_0),(\overline{\lambda},\overline{u}))+DG(\widetilde{x}_0)\overline{y}' \\ B(\overline{y}'_i) \\ D^2F(\widetilde{\lambda}_0,\widetilde{u}_0)((\widetilde{\mu}'_{i,0},\widetilde{w}'_{i,0}),(\overline{\lambda},\overline{u}))+DG(\widetilde{x}_0)\overline{y}'_i+A \\ \bar{\mathcal{B}}(\overline{z}'_k) \\ D_{(\lambda,u)}(D_uF(\widetilde{\lambda}_0,\widetilde{u}_0)\widetilde{v}'_{k,0})(\overline{\lambda},\overline{u})+H(\widetilde{\lambda}_0,\widetilde{u}_0,\overline{z}'_k)+B \end{array}\right]$$

$$-\left[\begin{array}{l} B(\overline{y}') \\ D^2F(\lambda,u)((\mu',w'),(\overline{\lambda},\overline{u}))+DG(x)\overline{y}' \\ B(\overline{y}'_i) \\ D^2F(\lambda,u)((\mu'_i,w'_i),(\overline{\lambda},\overline{u}))+DG(x)\overline{y}'_i+C \\ \bar{\mathcal{B}}(\overline{z}'_k) \\ D_{(\lambda,u)}(D_uF(\lambda,u)v'_k)(\overline{\lambda},\overline{u})+H(\lambda,u,\overline{z}'_k)+D \end{array}\right])\,.$$

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References

[1] E. L. Allgower, K. Georg, *Introduction to Numerical Continuation Methods*, SIAM, Philadelphia, 2003.

[2] P. Ashwin, K. Böhmer, Z. Mei, *A numerical Liapunov-Schmidt method with applications to Hopf bifurcation on a square*, Math. Comp., **64**, 210 (1995), 649-670.

[3] W.J.Beyn, *On discretization of bifurcation problems.* In [51], 46-73.

[4] W.J.Beyn, *Defining equations for singular solutions and numerical applications.* In [47], 42-56.

[5] C.-L. Bichir, *Contributions on the Study of Hydrodynamic Stability*, Ph. D. Thesis, Institute of Mathematics "Simion Stoilow" of the Romanian Academy, Bucharest, and University of Piteşti, 2002 (in Romanian).

[6] P. B. Bochev, M. D. Gunzburger, *Least-Squares Finite Element Methods*, Springer, New York, 2009.

[7] K. Böhmer, *On hybrid methods for bifurcation studies for general operator equations.* In B. Fiedler (Ed.), *Ergodic theory, analysis and efficient simulation of dynamical systems*, 73-107, Springer, Berlin, Heidelberg, New York, 2001.

[8] K. Böhmer, S. Dahlke, *Stability and convergence for discretization methods with applications to wavelet Galerkin schemes*, Inter. J. of Pure Appl. Math., **6** 4 (2003), 465-489.

[9] K. Böhmer, *Numerical Methods for Nonlinear Elliptic Differential Equations*, Oxford University Press, 2010.

[10] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2010.

[11] F. Brezzi, J. Rappaz, P. A. Raviart, *Finite-dimensional approximation of nonlinear problems, I. Branches of nonsingular solutions*, Numer. Math., **36** (1980), 1-25.

[12] F. Brezzi, J. Rappaz, P. A. Raviart, *Finite-dimensional approximation of nonlinear problems, II. Limit points*, Numer. Math., **37** (1981), 1-28.

[13] F. Brezzi, J. Rappaz, P. A. Raviart, *Finite-dimensional approximation of nonlinear problems, III. Simple bifurcation points*, Numer. Math., **38** (1981), 1-30.

[14] G. Caloz, J. Rappaz, *Numerical analysis for nonlinear and bifurcation problems.* In P. G. Ciarlet, J. L. Lions (Eds.), *Handbook of Numerical Analysis*, vol. V, 487-637, North-Holland, Amsterdam, 1997.

[15] D. Chillingworth, *A Global Genericity Theorem for Bifurcations in Variational Problems*, J. Funct. Anal., **35** (1980), 251-278.

[16] K. A. Cliffe, A. Spence, *The calculation of high order singularities in the finite Taylor problem.* In [47], 129-144.

[17] K. A. Cliffe, A. Spence, S. J. Tavener, *The numerical analysis of bifurcation problems with application to fluid mechanics*, Acta Numerica, **9** (2000), 39-131.

[18] M. Crouzeix, J. Rappaz, *On Numerical Approximation in Bifurcation Theory*, Masson-Paris, Springer-Berlin, 1990.

[19] E. Doedel, *Lecture Notes on Numerical Analysis of Nonlinear Equations*, 2007, http://cmvl.cs.concordia.ca/publications/notes.ps.gz, from the Home Page of the AUTO Web Site, http://indy.cs.concordia.ca/auto/.

[20] A. L. Dontchev, *The Graves Theorem Revisited*, J. Convex Anal. **3** (1996), no. 1, 45-53.

[21] A. L. Dontchev, R. T. Rockafellar, *Implicit Functions and Solution Mappings. A View from Variational Analysis*, Springer, New York, 2009.

[22] P.G.Drazin, W.H.Reid, *Hydrodynamic Stability*, Cambridge University Press, 2004.

[23] H. C. Elman, D. J. Silvester, A. J. Wathen, *Finite Elements and Fast Iterative Solvers: with Applications in Incompressible Fluid Dynamics*, Oxford University Press, 2005.

[24] M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, Springer, New York, 2011.

[25] C. Foiaş, R. Temam, Remarques sur les équations de Navier-Stokes stationnaires et les phénomènes successifs de bifurcation, Ann. Scuola Norm. Sup. Pisa, Ser. IV, **5**, 1 (1978), 29-63.

[26] A. Georgescu, *Hydrodynamic Stability Theory*, Kluwer, Dordrecht, 1985.

[27] A. Georgescu, C.-L. Bichir, *Master - disciple discussions on Navier - Stokes equations, hydrodynamic stability theory, bifurcation theory and extended systems*, 1997-2002.

[28] A. Georgescu, M. Moroianu, I. Oprea, *Bifurcation Theory. Principles and Applications*, Applied and Industrial Mathematics Series, **1**, University of Piteşti, 1999 (in Romanian).

[29] V. Girault, P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, Springer, Berlin, 1986.

[30] K. Glashoff, E. L. Allgower, H. O. Peitgen (eds.), *Numerical Solution of Nonlinear Equations*, Springer, Berlin, 1981.

[31] M. Golubitsky, D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Vol. 1, Springer, New York, 1985.

[32] M. Golubitsky, I. Stewart, D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Vol. 2, Springer, New York, 1988.

[33] W.J.F. Govaerts, *Numerical methods for Bifurcations of Dynamical Equilibria*, SIAM, Philadelphia, 2000.

[34] A. Griewank, G. W. Reddien, *Characterization and computation of generalized turning points*, SIAM J. Numer. Anal., **21** (1984), 176-185.

[35] A. Griewank, G. W. Reddien, *The approximation of generalized turning points by projection methods with superconvergence to the critical parameter*, Numer. Math., **48** (1986), 591-606.

[36] V. L. Hansen, *Fundamental Concepts in Modern Analysis*, World Scientific, Singapore, 1999.

[37] M. Hermann, W. Middelmann, P. Kunkel, *Augmented Systems for the Computation of Singular Points in Banach Space Problems*, Z. Angew. Math. Mech., **78** (1998), 39-50.

[38] M. Hermann, W. Middelmann, *Numerical Analysis of Perturbed Bifurcation Problems*, Preprint Math/94/2 (1998), $http://www.researchgate.net/publication/$ $2302254\_Numerical\_Analysis\_of\_Perturbed\_Bifurcation\_Problems.$

[39] C.Ilioi, *Optimization Problems and Algorithms for Approximate Solutions*, Editura Academiei Române, Bucharest, 1980 (in Romanian).

[40] A. D. Jepson, A. Spence, *Singular points and their computation.* In [47], 195-209.

[41] A. D. Jepson, A. Spence, *On a reduction process for nonlinear equations*, SIAM J. Math. Anal., **20** (1989), 39-56.

[42] A. D. Jepson, A. Spence, K. A. Cliffe, *The numerical solution of nonlinear equations having several parameters. Part III: Equations with Z_2-symmetry*, SIAM J. Numer. Anal., **28** (1991), 809-832.

[43] L.V.Kantorovich, G.P.Akilov, *Functional Analysis*, Ed. Ştiinţifică şi Enciclopedică, Bucharest, 1986 (Romanian translation from the Second Russian Edition, NAUKA, Moscow, 1977).

[44] H.B.Keller, *Numerical Methods in Bifurcation Problems*, Springer, Berlin, Heidelberg, New York, Tokyo, 1987.

[45] H. Kielhöfer, *Bifurcation Theory. An Introduction with Applications to Partial Differential Equations*, Springer, New York, Dordrecht, Heidelberg, London, 2012.

[46] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley & Sons, New York. 1978.

[47] T. Küpper, H. D. Mittelmann, H. Weber (eds.), *Numerical Methods for Bifurcation Problems*, Proceedings of the Conference at the University of Dortmund, August 22-26, 1983, Birkhäuser, Basel, 1984.

[48] Yu. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer, New York, 1998.

[49] S. Lang, *Real and Functional Analysis*, Springer, New York, 1993.

[50] R. Menzel, *Numerical determination of multiple bifurcation points.* In [47], 310-318.

[51] H. D. Mittelmann, H. Weber (eds.), *Bifurcation Problems and their Numerical Solution*, Proceedings of the Workshop at the University of Dortmund, January 15-17, 1980, Birkhäuser, Basel, 1980.

[52] Moore, G., *The numerical treatment of non-trivial bifurcation points*, Numer. Funct. Anal. Optim., **2** (1980), 441-472.

[53] G. Moore, A. Spence, *The calculation of turning points of nonlinear equations*, SIAM J. Numer. Anal., **17** (1980), 567-576.

[54] I. Moret, *The approximation of generalized turning points by Krylov subspace methods*, Numer. Math., **68** (1994), 241-353.

[55] A.Quarteroni, A.Valli, *Numerical Approximation of Partial Differential Equations*, Springer, Berlin, 2008.

[56] R. Seydel, *Numerical computation of branch points in ordinary differential equations*, Numer. Math., **32** (1979), 51-68.

[57] R. Seydel, *Numerical computation of branch points in nonlinear equations*, Numer. Math., **33** (1979), 339-352.

[58] R. Seydel, *Nonlinear Computation*, invited lecture and paper presented at the Distinguished Plenary Lecture session on Nonlinear Science in the 21st Century, 4th IEEE International Workshops on Cellular Neural Networks and Applications, and Nonlinear Dynamics of Electronic Systems, Sevilla, June, 26, 1996.

[59] Gh. Sireţchi, *Functional Analysis*, Multiplication Centre of Bucharest University, 1982 (in Romanian).

[60] Gh. Sireţchi, *Concrete Spaces in Functional Analysis*, Multiplication Centre of Bucharest University, 1982 (in Romanian).

[61] Gh. Sireţchi, *Mathematical Analysis*, **3**, 3, Multiplication Centre of Bucharest University, 1983 (in Romanian).

[62] R.Temam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland, Amsterdam, 1979.

[63] R.Temam, *Navier-Stokes equations and nonlinear functional analysis*, SIAM, Philadelphia, 1995.

[64] R.Temam, *Some developments on Navier-Stokes equations in the second half of the 20th century*, in *Dévelopment des Mathématiques au cours de la seconde moitié du XXčme sičcle*, J.P. Pier (Ed.), Birkhauser, 1999.

[65] H. Weber, *On the numerical computation of secondary bifurcation problems.* In [30], 407-426.

[66] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer, New York, 1986.

[67] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/A: Linear Monotone Operators*, Springer, New York, 1990.

[68] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B: Nonlinear Monotone Operators*, Springer, New York, 1990.

[69] E. Zeidler, *Nonlinear Functional Analysis and its Applications III: Variational Methods and Optimization*, Springer, New York, 1985.

[70] E. Zeidler, *Nonlinear Functional Analysis and its Applications IV: Applications to Mathematical Physic*, Springer, New York, 1988.

[71] E. Zeidler, *Applied Functional Analysis. Main Principles and Their Applications*, Springer, New York, 1995.

Independent researcher, Apartment 5, Flat V1, 74 Nicolae Bălcescu Street, Galaţi, 800001, Romania

Email address: catalinliviubichir@yahoo.com