CONVERGENCE AND COLLAPSING OF CAT(0)-LATTICES

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ABSTRACT. We study the theory of convergence for CAT(0)-lattices (that is groups Γ acting geometrically on proper, geodesically complete CAT(0)-spaces) and their quotients (CAT(0)-orbispaces). We describe some splitting and collapsing phenomena, explaining precisely how these action can degenerate to a possibly non-discrete limit action. Finally, we prove a compactness theorem for the class of compact CAT(0)-homology orbifolds, and some applications: an isolation result for flat orbispaces and an entropy-pinching theorem.

CONTENTS

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1. Introduction

The theory of CAT(0)-groups, i.e. groups admitting a geometric action on some CAT(0)-space, whose roots can be traced back to the '70s with the works of Gromoll-Wolf [GW71] and Lawson-Yau [LY72] on fundamental groups of compact, nonpositively curved manifolds, has flourished in the last twenty years, with stunning developments in geometric group theory and applications to low dimensional topology; let us just mention all the research opened by the rank rigidity conjecture for CAT(0)-spaces and the methods coming from the theory of cubulable groups, with application to the virtually Haken conjecture.

The works of Caprace and Monod [CM09a], [CM09b] shed some new light on the structure of CAT(0)-spaces possessing a geometric action; as we will see, their results, together with the deep, recent insights on the topology of locally compact, geodesically complete, CAT(κ)-spaces provided by Lytchak and Nagano in [LN19], [LN18], will be of great importance for our work.

In this paper, which should be thought as a companion paper of [CS23], and pursues the study of packed CAT(0)-spaces and discrete group actions on such spaces initiated in [CS21] and [CS22], we investigate the convergence of geometric actions of groups on CAT(0)-spaces along the line of the classical theory of convergence in Riemannian geometry. Namely, we will be interested in *uniform* CAT(0)-*lattices*, that is discrete, cocompact isometry groups Γ of some CAT(0)-space X; one equivalently says that Γ acts *geometrically* on X. Moreover, throughout this paper all CAT(0)-spaces will be assumed to be *proper* and *geodesically complete*, which ensures many desirable geometric properties, such as the equality of topological dimension and Hausdorff dimension, the existence of a canonical measure μ_X, etc. (see Section 2 for fundamentals on CAT(0)-spaces). Our goal is to describe precisely the structure of equivariant Gromov-Hausdorff limits of sequences of (possibly collapsing) geometric actions on CAT(0)-spaces $\Gamma_n \curvearrowright X_n$, and of the corresponding quotients $M_n = \Gamma_n \backslash X_n$. As an application, we will present a compactness result (Corollary D) and some rigidity and pinching theorems (Corollaries E, F below); see also some stability results in Section 4 (Theorem 4.14 and Corollaries 4.15, 4.18).

To begin with, let us denote by

$$\mathrm{CAT}_0(D_0)$$

the class of uniform CAT(0)-lattices[1] (Γ, X) with $\mathrm{diam}(\Gamma\backslash X) \leq D_0$. We will also say that Γ acts D_0-*cocompactly* on X when $\mathrm{diam}(\Gamma\backslash X) \leq D_0$, and that the space X is D_0-*cocompact*. Notice that the constant D_0 is there simply to fix a scale, as the metric of any cocompact CAT(0)-space X can be renormalized in order that it becomes D_0-cocompact. The lattice Γ is called *nonsingular* if there exists at least a point $x \in X$ with trivial stabilizer: this is a mild assumption on the action which rules out pathological (although differently interesting) cases, see for instance [CS22, Example 1.4]

[1] We emphasize that by *uniform* CAT(0)-*lattice* we mean a CAT(0)-group Γ with a fixed faithful geometric action of Γ on some CAT(0)-space X, which explains the notation (Γ, X).

and [BK90, Theorem 7.1]. This condition is automatically satisfied for instance when the the lattice is torsion-free or X is a homology manifold (see Section 2.1 and [CS22]). We call the quotient metric space $M = \Gamma\backslash X$ a CAT(0)-*orbispace* and we will say that the CAT(0)-orbispace M is *nonsingular* if Γ acts nonsingularly on X. Notice that if Γ is torsion-free then M is a locally CAT(0)-space.

The starting point of our study is to understand when a family of uniform CAT(0)-lattices admits a limit (possibly not a lattice), i.e. the precompactness of our class $\mathrm{CAT}_0(D_0)$. The noton of convergence that we will use is the *equivariant Gromov-Hausdorff convergence* (*à la Fukaya*, cp. [Fuk86]). We refer to Section 4 for the precise definition; let us just mention here that, denoting by $B_\Gamma(x, r)$ the subset of elements $\gamma \in \Gamma$ moving $x \in X$ less than r, saying that a sequence of lattices (Γ_j, X_j) converges towards a limit action $(\Gamma_\infty, X_\infty)$ simply means that there exist Gromov-Hausdorff ε-approximations $f_\varepsilon : B_{X_j}(x_j, \frac{1}{\varepsilon}) \to B_{X_\infty}(x_\infty, \frac{1}{\varepsilon})$ between larger and larger balls of X_j, X_∞ centered at basepoints x_j and x_∞, which are ε-equivariant with respect to maps $\phi_\varepsilon : B_{\Gamma_j}(x_j, \frac{1}{\varepsilon}) \to B_{\Gamma_\infty}(x_\infty, \frac{1}{\varepsilon})$ (that is, with an equivariancy error smaller than ε), for $\varepsilon \to 0$. For sequences of lattices which are D_0-cocompact, the limit does not depend on the basepoints x_j, x_∞; moreover, this is equivalent (up to subsequences) to taking the ultralimit group Γ_ω acting on the ultralimit metric space X_ω defined using any non-principal ultrafilter ω, provided that the space X_ω is proper, see [Cav22a] and §4.1. Now, the answer to the existence of limits for a family of CAT(0)-lattices is simple and comes from the very definition of equivariant GH-convergence: *any family of isometry groups $\Gamma_j < \mathrm{Isom}(X_j)$ (sub-)converges to some limit isometry group Γ_∞ of a space X_∞, as soon as the spaces X_j converge to X_∞ with respect to the pointed Gromov-Hausdorff distance*, see Proposition 4.4). Hence, the precompactness of a sequence in $\mathrm{CAT}_0(D_0)$ reduces to precompactness of the underlying spaces.

Recall that a metric space X is said to satisfy the *P_0-packing condition at scale r_0* (for short, X is *(P_0, r_0)-packed*) if every ball of radius $3r_0$ in X contains at most P_0 points that are $2r_0$-separated. For geodesically complete CAT(0)-spaces, the convexity of the distance function implies that a packing condition at some scale r_0 yields an explicit, uniform control of the packing function $\mathrm{Pack}(R, r)$ of X (that is, the maximum number of disjoint r-balls that one can pack in any R-ball), see [CS21, Theorem 4.2]. This is exactly Gromov's classical condition for precompactness of a family of spaces (also known as "uniform compactness of r-balls", cp. [Gro81], or "geometrical boundedness" [DY05]). Moreover, it is easy to see that any compact metric space, as well as any metric space X admitting an uniform lattice, is (P_0, r_0)-packed for some constants P_0 and r_0 (cp. the proof of [Cav22a, Lemma 5.4]). Summarizing: the class $\mathrm{CAT}_0(D_0)$ has a natural filtration

$$\mathrm{CAT}_0(D_0) = \bigcup_{P_0, r_0} \mathrm{CAT}_0(P_0, r_0, D_0) \tag{1}$$

where $\mathrm{CAT}_0(P_0, r_0, D_0)$ is the subset of $\mathrm{CAT}_0(D_0)$ made of lattices (Γ, X) such that X is (P_0, r_0)-packed, and from the packing assumption one can deduce the following:

Proposition (Proposition 4.7)**.** *A subset $\mathcal{F} \subseteq \mathrm{CAT}_0(D_0)$ is precompact with respect to the equivariant pointed Gromov-Hausdorff convergence if and only if there exist $P_0, r_0 > 0$ such that $\mathcal{F} \subseteq \mathrm{CAT}_0(P_0, r_0, D_0)$.*

We stress the "only if" part in the above statement: for a family of CAT(0)-lattices with uniformily bounded codiameter, the uniform packing assumption at some fixed scale is a necessary condition in order to converge. In view of this, this assumption is crucial for studying equivariant Gromov-Hausdorff limits and is, in a sense, minimal.
This (P_0, r_0)-packing condition should be thought as a weak, local substitute of a lower bound on the curvature; however, it is much weaker than assuming the curvature bounded below in the sense of Alexandrov, or than a lower bound of the Ricci curvature in the Riemannian case, and even of a $\mathrm{CD}(\kappa, n)$ condition. Indeed, the Bishop-Gromov's comparison theorem for Riemannian n-manifolds with $\mathrm{Ric}_X \geq -(n-1)\kappa$ for $\kappa \geq 0$ (or its generalization to $\mathrm{CD}(\kappa, n)$ spaces, see for instance [Stu06]) yields a doubling condition

$$\frac{\mu(B_X(x, 2r))}{\mu(B_X(x, r))} \leq C(\kappa, n, r) \tag{2}$$

from which the packing condition at scale $r_0 \leq r/4$ easily follows. Notice however that he class of $\mathrm{CD}(\kappa, n)$ metric measure spaces which are CAT(0) restricts to topological manifolds, by [KKK22], whereas our class contains non-manifolds. As proved in [CS21], for geodesically complete CAT(0)-spaces, the packing condition at some scale is the same as a uniform upper bound of the canonical measure of all r-balls, a condition sometimes called *macroscopic scalar curvature bounded below* cp. [Gut10], [Sab20].

Coming to the problem of describing the possible limits of lattices in our class, the first important distinction is between collapsing and non-collapsing sequences. We define the *free systole* of a lattice Γ as

$$\mathrm{sys}^\diamond(\Gamma, X) = \inf_{x \in X} \inf_{g \in \Gamma^* \setminus \Gamma^\diamond} d(x, gx)$$

where $\Gamma^* = \Gamma \setminus \{\mathrm{id}\}$ and $\Gamma^\diamond$ is the subset of elliptic elements of Γ.
For nonsingular lattices, when assuming a bound on the diameter, the smallness of the free systole is quantitatively equivalent to the smallness of the *diastole* of the lattice, that is the invariant

$$\mathrm{dias}(\Gamma, X) = \sup_{x \in X} \inf_{g \in \Gamma^*} d(x, gx)$$

(see Proposition 2.9). Accordingly, a lattice (Γ, X) is said to be *ε-collapsed* if $\mathrm{sys}^\diamond(\Gamma, X) < \varepsilon$, and a sequence $(\Gamma_j, X_j)_{j \in \mathbb{N}}$ is said to converge *collapsing* to $(\Gamma_\infty, X_\infty)$ if $\limsup_{j \to +\infty} \mathrm{sys}^\diamond(\Gamma_j, X_j) = 0$; the sequence will be called *non-collapsing* otherwise.
The collapsing condition for uniform CAT(0)-lattices turns out to be equivalent to the the fact that the dimension of the limit orbispace $M_\infty = \Gamma_\infty \backslash X_\infty$ is strictly smaller than the dimension of the orbispaces $M_j = \Gamma_j \backslash X_j$, as we will prove in Theorem 4.16, Section 4.6; this is a very intuitive but nontrivial result, which follows a-posteriori from the analysis of both the collapsing and non-collapsing case. Therefore, we can also say unambiguously that the orbispaces M_j converge collapsing when $\mathrm{sys}^\diamond(\Gamma_j, X_j) \to 0$.

The following result shows that limits of CAT(0)-lattices may well be non-discrete, both in the collapsing and in the non-collapsing case; they however always define a closed, *totally disconnected* group of isometries of a canonical factor of the limit space (which is another source of interest in the theory of totally disconnected groups):

Theorem A (Limits of CAT(0)-lattices, Theorems 4.9 & 4.10)**.**
Let $(\Gamma_j, X_j)_{j\in\mathbb{N}} \subseteq \mathrm{CAT}_0(D_0)$ be a sequence of lattices converging in the equivariant Gromov-Hausdorff distance to a limit isometry group $(\Gamma_\infty, X_\infty)$, and let $M_j = \Gamma_j\backslash X_j$ the corresponding orbispaces. Then:
– X_∞ is a proper, geodesically complete $\mathrm{CAT}(0)$-space,
– $\Gamma_\infty < \mathrm{Isom}(X_\infty)$ is a closed and D_0-cocompact group,
– the M_j's converge to $M_\infty = \Gamma_\infty\backslash X_\infty$ in the Gromov-Hausdorff topology.

Moreover, if the sequence $(\Gamma_j, X_j)_{j\in\mathbb{N}}$ is:

i) non-collapsing, *then the group Γ_∞ is totally disconnected; in addition, if each Γ_j is nonsingular, then Γ_∞ is discrete, isomorphic to Γ_j for $j \gg 0$, and M_∞ is equivariantly homotopic equivalent to M_j for $j \gg 0$;*
ii) collapsing, *then X_∞ splits isometrically as $X'_\infty \times \mathbb{R}^\ell$ with $\ell \geq 1$, Γ_∞ is not discrete with identity component $\Gamma^\circ_\infty \cong \mathbb{R}^\ell$, acting as the group of translation of the factor $\mathbb{R}^\ell$; finally $M_\infty = \Gamma'_\infty\backslash X'_\infty$ where $\Gamma'_\infty = \Gamma_\infty/\Gamma^\circ_\infty$ is closed and totally disconnected.*

If the limit M_∞ is a Riemannian manifold, then the equivariant homotopy equivalence between M_j and M_∞ can be promoted to homeomorphism, as we will see in Section 4.5 (Corollary 4.14).

Remarks (Non-discrete limits)**.**
i) A sequence of (singular) lattices can converge to a non-discrete group *even without collapsing*: an example can be found in [BK90]. In that paper the authors build an infinite, ascending chain of lattices Γ_j of a regular tree T with bounded valency, with same compact quotient, containing torsion subgroups $G_j < \Gamma_j$ whose order tends to infinity.
The sequence (Γ_j, T) is non-collapsing (as T does not split) and the limit group Γ_∞ is a totally disconnected, non-discrete subgroup of $\mathrm{Isom}(T)$ (since if it was discrete, there would be a bound on the order of its finite subgroups, by [BH13, Corollary II.2.8], contradicting the fact that the limit contains arbitrarily large finite subgroups too).

ii) In the collapsing case, even assuming that the lattices are nonsingular or torsion-free, one cannot improve in general the conclusion saying that the totally disconnected group Γ'_∞ is discrete: see Example 3.6 and Remark 4.13.

By Theorem A, the class $\mathrm{CAT}_0(D_0)$ is not closed, because of the existence of either singular or collapsing sequences, which make the limit group non-discrete (while it is not difficult to see that the conditions of being D_0-cocompact and (P_0, r_0)-packed are stable under Gromov-Hausdorff limits). The problem of determining a natural extension of $\mathrm{CAT}_0(D_0)$ which forms a compact class will be considered by the first author in [Cav23].
However, as we will see in a moment, the limit M_∞ of the quotient orbispaces $M_j = \Gamma_j\backslash X_j$ might still be a reasonable CAT(0)-orbispace, even when the

sequence is collapsing. For this, we need to understand more deeply how collapsing occurs in our class, and it is exactly the purpose of the following. The main idea to give M_∞ a CAT(0)-orbispace structure is that, for a collapsing sequence, the spaces X_j split as $Y_j \times \mathbb{R}^\ell$ and we would like to prove a corresponding splitting of the groups $\Gamma_j = \Gamma_{Y_j} \times \mathbb{Z}^\ell$, with each factor preserving the product decomposition; then, we would show that collapsing only happens in the Euclidean factor via the action of $\mathbb{Z}^\ell$, and disintegrate the action of the continuous part Γ_∞° of the limit group Γ_∞ on $X_\infty = Y_\infty \times \mathbb{R}^\ell$. Unfortunately, this picture breaks down for general nonsingular CAT(0)-lattices: as opposite to the Riemannian case (where for any lattice of a nonpositively curved Riemannian manifold with Euclidean factor $\mathbb{R}^\ell$ it is always possible to virtually split a free abelian subgroup of rank ℓ), we will see in §3.2, Example 3.6, an example of a collapsing sequence of lattices (Γ_j, X_j) belonging to $\mathrm{CAT}_0(P_0, r_0, D_0)$, where X_j is the product of a tree T with $\mathbb{R}^2$, but Γ_j does not split any abelian group of positive rank (not even virtually). Actually, the splitting of a CAT(0)-space X as $X' \times \mathbb{R}^\ell$ forces a lattice Γ of X to act preserving the components (i.e. every $g \in \Gamma$ acts as (g', g''), where $g' \in \mathrm{Isom}(X')$ and $g'' \in \mathrm{Isom}(\mathbb{R}^\ell)$), but, in general, the projections of Γ on $\mathrm{Isom}(X')$ and $\mathrm{Isom}(\mathbb{R}^n)$ are not discrete, as [CM19, Example 1] shows; this is precisely the obstruction to virtually splitting the $\mathbb{Z}^\ell$-factor from Γ.
On the other hand, the group splitting always holds (virtually) for sufficiently collapsed uniform lattices of CAT(0)-spaces whose full isometry group is Lie:

Theorem B (Group splitting in the Lie setting, extract from Thm. 3.7)**.**
Given P_0, r_0, D_0, there exist $\sigma_0^* = \sigma_0^*(P_0, r_0, D_0) > 0$ and $I_0 = I_0(P_0, r_0, D_0)$ such that the following holds. Let X be a proper, geodesically complete, CAT(0)-space whose isometry group is a Lie group, and let $X = Y \times \mathbb{R}^n$ be the splitting of the maximal Euclidean factor of X: then, every uniform lattice Γ of X has a normal, finite index subgroup $\check{\Gamma}$ which splits as $\check{\Gamma} = \check{\Gamma}_Y \times \mathbb{Z}^n$ where $\check{\Gamma}_Y$ (resp. $\mathbb{Z}^n$) acts discretely on Y (resp. $\mathbb{R}^n$).
If Γ is nonsingular then $\check{\Gamma}$ is nonsingular too.
Moreover, if X is (P_0, r_0)-packed and Γ is D_0-cocompact then $\mathrm{sys}^\diamond(\Gamma_Y, Y) \geq \sigma_0^*$ and $\check{\Gamma}$ can be chosen of index $[\Gamma : \check{\Gamma}] \leq I_0$.

In this situation, $M = \Gamma \backslash X$ finitely covers (in the sense of orbispaces) the orbispace $\check{M} := \check{\Gamma} \backslash X$, which splits as a metric product $\check{N} \times \mathbb{T}^n$, where $\check{N} = \check{\Gamma}_Y \backslash Y$ and $\mathbb{T}^n = \mathbb{Z}^n \backslash \mathbb{R}^n$ is a torus. We will say in this case that the orbispace M *splits virtually.*

The first part of Theorem 3.7 is a pretty natural consequence of Caprace and Monod's work on isometry groups of CAT(0)-spaces. On the other hand, the control of the index of $\check{\Gamma}$ in Γ and the bound of the diastole on the non-Euclidean factor follow from a finiteness theorem proved in [CS23] and a subtle splitting result which will be recalled in Section 3.1, Theorem 3.1. Both estimates are crucial to understand Gromov-Hausdorff limit of the quotient spaces $M_j = \Gamma_j \backslash X_j$ when the actions collapse. As a result, we obtain a complete description of how nonsingular lattices collapse, in the Lie setting:

Theorem C (Collapsing in the nonsingular, Lie setting)**.**
Assume that a sequence of nonsingular lattices $(\Gamma_j, X_j)_{j \in \mathbb{N}} \subseteq \mathrm{CAT}_0(D_0)$ converge collapsing to $(\Gamma_\infty, X_\infty)$, and that the groups $\mathrm{Isom}(X_j)$ are Lie.

Let $X_j = Y_j \times \mathbb{R}^j$ be the canonical splitting of the Euclidean factor of X_j, let $X_\infty = X'_\infty \times \mathbb{R}^\ell$ be the splitting given by Theorem A, with $\ell = \dim(\Gamma^\circ_\infty) \geq 1$, and let $k_\infty = \lim_j k_j \geq \ell$ be the dimension of the Euclidean factor of X_∞. Then:

- *the lattice Γ_j has a normal subgroup of finite index $\check{\Gamma}_j$ splitting as $\check{\Gamma}_{Y_j} \times \mathbb{Z}^{k_j}$;*
- *the orbispace $M_j = \Gamma_j \backslash X_j$ virtually splits as $\check{N}_j \times \mathbb{T}^{k_j}_j$, where $\check{N}_j = \Gamma_{Y_j} \backslash Y_j$ is a nonsingular $\mathrm{CAT}(0)$-orbispace and $\mathbb{T}^{k_j} = \mathbb{Z}^{k_j} \backslash \mathbb{R}^{k_j}$ is a flat torus.*

Moreover:

(i) the $\check{N}_j$'s converge non-collapsing to a nonsingular orbispace $\check{N}_\infty$;
(ii) the tori $\mathbb{T}^{k_j}$ converge to a flat torus $\mathbb{T}^{k'_\infty}$, with $k'_\infty = k_\infty - \ell$;
(iii) the M_j's converge to the nonsingular orbispace $M_\infty = \Gamma'_\infty \backslash X'_\infty$, where $\Gamma'_\infty = \Gamma_\infty / \Gamma^\circ_\infty$ is discrete;
(iv) M_∞ is the quotient of $\check{N}_\infty \times \mathbb{T}^{k'_\infty}$ by a finite group of isometries Λ_∞.

Finally, if $(\Gamma_\infty, X_\infty) \in \mathrm{CAT}_0(P_0, r_0, D_0)$ and $(\check{\Gamma}_{Y_\infty}, Y_\infty)$ is the limit of the $(\check{\Gamma}_{Y_j}, Y_j)$'s, then $\mathrm{sys}^\diamond(\check{\Gamma}_{Y_\infty}, Y_\infty) \geq \sigma^*_0(P_0, r_0, D_0)$ and $|\Lambda_\infty| \leq I_0(P_0, r_0, D_0)$ (here σ^*_0 and I_0 are the same as in Theorem B).

In other words, the theorem says that the collapsing occurs in this class only by possibly shrinking to a point a flat torus (virtual) fiber. This result is new even for collapsing sequences of compact Riemannian manifolds M with nonpositive sectional curvature $-\kappa \leq k(M) \leq 0$. Notice that also the fact that k_∞ equals the dimension of the Euclidean factor of X_∞ is not trivial and follows by Theorem A, see Corollary 4.18.

An interesting case where the above convergence theorem applies is the one of CAT(0)-*homology orbifolds.* A metric space M is a CAT(0)-homology orbifold if it can be realized as the quotient $M = \Gamma \backslash X$ of a proper CAT(0)-space X which is a homology manifold (with respect to the induced topology) by a discrete group of isometries. We denote respectively by

$$\mathcal{HO}\text{-}\mathrm{CAT}_0(D_0), \qquad \mathcal{HO}\text{-}\mathrm{CAT}_0(P_0, r_0, D_0)$$

the class of compact CAT(0)-homology orbifolds $M = \Gamma \backslash X$ with $\mathrm{diam}(M) \leq D_0$, and the subclass of those such that X is, moreover, (P_0, r_0)-packed.
Notice that, as a consequence of (1) we still have

$$\mathcal{HO}\text{-}\mathrm{CAT}_0(D_0) = \bigcup_{P_0, r_0} \mathcal{HO}\text{-}\mathrm{CAT}_0(P_0, r_0, D_0).$$

The class of CAT(0)-homology orbifolds is interesting for many reasons. First remark that CAT(0)-homology manifolds are always geodesically complete (cp. [BH13, Proposition II.5.12]); also, their lattices always yield nonsingular orbispaces (see discussion in Section 2.1).
More importantly, the class $\mathcal{HO}\text{-}\mathrm{CAT}_0(P_0, r_0, D_0)$ contains all nonpositively curved Riemannian manifolds with (uniformly) bounded sectional curvature. The advantage over Riemannian manifolds is that being a homology manifold (as well as the CAT(0) and the (P_0, r_0)-packing conditions) is a property which is stable under Gromov-Hausdorff convergence (see [CS22, Theorem 6.1] and [LN18, Lemma 3.3]); while limits of of Riemannian manifolds, even without collapsing, yield at best manifolds with $C^{1,\alpha}$-metrics, for which the Riemannian curvature is not defined).

Finally, the full isometry group of a CAT(0)-homology manifold is always a Lie group, as we will prove in Section 2.1, Proposition 2.4, which allows us to apply the results of Theorem C in this class and show the following:

Corollary D (Gromov-Hausdorff compactness).
For all fixed P_0, r_0 and D_0, the class $\mathcal{HO}$-$\mathrm{CAT}_0(P_0, r_0, D_0)$ is compact with respect to the Gromov-Hausdorff topology.

Actually, among the most natural classes of metric spaces containing all Riemannian manifolds with bounded sectional curvature, $\mathcal{HO}$-$\mathrm{CAT}_0(P_0, r_0, D_0)$ is the smallest class we know to be compact under Gromov-Hausdorff limits. Comparing with the classical compactness theorems in Riemannian geometry (with bounded sectional curvature, e.g. [Gro07], or bounded Ricci curvature as in [And90], [AC92]), the novelty here is that, besides the wider generality of application (metric spaces, bounded packing instead of bounded curvature, torsion allowed), we do not assume any lower bound neither on the injectivity radius nor on the volume, thus allowing convergence to spaces of smaller dimension.
We stress the fact that, in order to obtain a closed class, it is necessary to consider also quotients by discrete groups *with torsion*. Indeed, a sequence of compact, nonpositively curved manifolds with uniformly bounded curvature does not converge, generally, to a locally CAT(0)-manifold or even to a locally CAT(0)-homology manifold, as the following simple example shows.

Example. Let M_j be the quotient of $\mathbb{R}^2$ by the discrete group Γ_j generated by $\{t, g_j\}$, where t is the translation of length 1 along the x-axis and g_j is the glide reflection with axis equal to the y-axis, with translation length $\frac{1}{j}$. Each M_j is a smooth, flat 2-manifold homeomorphic to the Klein bottle. However, the M_j's converge collapsing to a metric space M_∞ which is the quotient of $\mathbb{R}^2$ by the limit Γ_∞ of the groups $\langle t, g_j \rangle$. The limit space M_∞ is homeomorphic to the closed interval $[0, 1]$, which is not a homology manifold since it has boundary points; but it clearly is a (flat) Riemannian orbifold.

Remark. One can sharpen Corollary D in the Riemannian setting. Suppose that $M_j = \Gamma_j \backslash X_j \in \mathcal{HO}$-$\mathrm{CAT}_0(P_0, r_0, D_0)$ is a sequence of Riemannian orbifolds converging to M_∞: then, M_∞ is isometric to a quotient $\Gamma'_\infty \backslash X'_\infty$, where X'_∞ is a *topological* manifold. This follows by the same argument of the proof of Corollary D and [LN18, Theorem 1.3].

The following corollaries are two direct applications of the theory developed so far, and follow a classical scheme of proof: we assume to have a sequence of counterexamples and pass to the limit, then we obtain a contradiction by using, respectively, a well-known result of Adams and Ballmann about virtually abelian lattices of CAT(0)-spaces, and a continuity result for the entropy (which we will prove in the Appendix).

Corollary E (Isolation of $\mathbb{R}^n$ among cocompact CAT(0)-spaces).
There exists $\varepsilon = \varepsilon(D_0, n) > 0$ such that every D_0-cocompact, geodesically complete CAT(0)*-space X satisfying $d_{\mathrm{pGH}}(X, \mathbb{R}^n) < \varepsilon$ is isometric to $\mathbb{R}^n$.*

Notice that, in particular, this results implies that there do not exist arbitrarily small, D_0-periodic deformations of the Euclidean metric of $\mathbb{R}^n$ which remain nonpositively curved, unless these deformations are flat.

Finally, recall that the (*packing*, or *covering*) *entropy* of a proper CAT(0)-space X is the asymptotic invariant defined as

$$\mathrm{Ent}(X) = \limsup_{R\to+\infty} \frac{1}{R}\log(\mathrm{Pack}(R,r))$$

This can be equivalently defined as the critical exponent of any uniform lattice of X, and does not depend on the small, chosen radius r, see Section 2.3. It is well-known that for every Hadamard manifold X possessing a torsionless, uniform lattice Γ, $\mathrm{Ent}(X)$ coincides with the topological entropy of the geodesic flow on the unit tangent bundle of the quotient manifold $M = \Gamma\backslash X$ [Man79]. Then, we have:

Corollary F (Entropy-rigidity of $\mathbb{R}^n$)**.**
There exists $h{=}h(P_0, r_0, D_0){>}0$ with the following property.
Let X be a D_0-cocompact, geodesically complete CAT(0)*-space which is (P_0, r_0)-packed: if* $\mathrm{Ent}(X){<}h$*, then X is isometric to $\mathbb{R}^n$, for some n.*

As a consequence, any non-flat, compact Riemannian manifold M with sectional curvature $-\kappa{\le}K_M{\le}0$ and $\mathrm{diam}(M){\le}D$ has topological entropy greater than a positive, universal constant $h_0{=}h_0(\kappa, D)$ (for $h_0{=}h(P_0, r_0, D)$, where (P_0, r_0) are the packing constant deduced from the lower curvature bound given by (2)).

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2. Preliminaries on CAT(0)-spaces

We fix here some notation and recall some facts about CAT(0)-spaces.
Throughout the paper X will be a *proper* metric space with distance d. The open (resp. closed) ball in X of radius r, centered at x, will be denoted by $B_X(x, r)$ (resp. $\overline{B}_X(x, r)$); we will often drop the subscript X when the space is clear from the context.
A *geodesic* in a metric space X is an isometry $c\colon [a, b] \to X$, where $[a, b]$ is an interval of $\mathbb{R}$. The *endpoints* of the geodesic c are the points $c(a)$ and $c(b)$; a geodesic with endpoints $x, y \in X$ is also denoted by $[x, y]$. A *geodesic ray* is an isometry $c\colon [0, +\infty) \to X$ and a geodesic line is an isometry $c\colon \mathbb{R} \to X$.
A metric space X is called *geodesic* if for every two points $x, y \in X$ there is a geodesic with endpoints x and y.
A metric space X is called CAT(0) if it is geodesic and every geodesic triangle $\Delta(x, y, z)$ is thinner than its Euclidean comparison triangle $\overline{\Delta}(\bar{x}, \bar{y}, \bar{z})$: that is, for any couple of points $p \in [x, y]$ and $q \in [x, z]$ we have $d(p, q) \le d(\bar{p}, \bar{q})$ where $\bar{p}, \bar{q}$ are the corresponding points in $\overline{\Delta}(\bar{x}, \bar{y}, \bar{z})$ (see for instance [BH13] for the basics of CAT(0)-geometry). As a consequence, every CAT(0)-space is *uniquely geodesic*: for every two points x, y there exists a unique geodesic with endpoints x and y.

A CAT(0)-space X is *geodesically complete* if every geodesic $c\colon [a,b] \to X$ can be extended to a geodesic line. For instance, if a CAT(0)-space is a homology manifold then it is always geodesically complete, see [BH13, II.5.12].
The *boundary at infinity* of a CAT(0)-space X (that is, the set of equivalence classes of geodesic rays, modulo the relation of being asymptotic), endowed with the Tits distance, will be denoted by ∂X, see [BH13, Chapter II.9].
A subset C of X is said to be *convex* if for all $x, y \in C$ the geodesic $[x, y]$ is contained in C. Given a subset $Y \subseteq X$ we denote by $\mathrm{Conv}(Y)$ the *convex closure* of Y, that is the smallest closed convex subset containing Y. If C is a convex subset of a CAT(0)-space X then it is itself CAT(0), and its boundary at infinity ∂C naturally and isometrically embeds in ∂X.

We will denote by $\mathrm{HD}(X)$ and $\mathrm{TD}(X)$ the Hausdorff and the topological dimension of a metric space X, respectively. By [LN19] we know that if X is a proper and geodesically complete CAT(0)-space then every point $x \in X$ has a well defined integer dimension in the following sense: there exists $n_x \in \mathbb{N}$ such that every small enough ball around x has Hausdorff dimension equal to n_x. This defines a *stratification of* X into pieces of different integer dimensions: namely, if X^k denotes the subset of points of X with dimension k, then

$$X = \bigcup_{k \in \mathbb{N}} X^k.$$

The *dimension* of X is the supremum of the dimensions of its points: it coincides with the *topological dimension* of X, cp. [LN19, Theorem 1.1].
Calling $\mathcal{H}^k$ the k-dimensional Hausdorff measure, the formula

$$\mu_X := \sum_{k \in \mathbb{N}} \mathcal{H}^k \llcorner X^k$$

defines a *canonical measure* on X which is locally positive and locally finite.

2.1. **Isometry groups and orbispaces.**

The *translation length* of an isometry g of a CAT(0)-space X is defined as

$$\ell(g) := \inf_{x \in X} d(x, gx).$$

When the infimum is realized, the isometry g is called *elliptic* if $\ell(g) = 0$ and *hyperbolic* otherwise. The *minimal set of* g, $\mathrm{Min}(g)$, is defined as the subset of points of X where g realizes its translation length; notice that if g is elliptic then $\mathrm{Min}(g)$ is the subset of points fixed by g. An isometry is called *semisimple* if it is either elliptic or hyperbolic; a subgroup Γ of isometries of X is called *semisimple* if all of its elements are semisimple.
Let $\mathrm{Isom}(X)$ be the group of isometries of X, endowed with the compact-open topology: as X is proper, it is a topological, locally compact group. A subgroup Γ of isometries is *discrete* if it is discrete as a subset of Isom(X) (with respect to the compact-open topology). A *uniform lattice* of X is a discrete isometry group Γ such that the quotient metric space $\Gamma\backslash X$ is compact; then, we call $\mathrm{diam}(\Gamma\backslash X)$ the *codiameter* of Γ. We will often say that Γ is D-cocompact if $\mathrm{diam}(\Gamma\backslash X) \leq D$, and that the space X is *cocompact* if it admits a uniform lattice. Notice that a cocompact CAT(0)-space X is always proper.

The action of Γ on X (and, by extension, the group Γ itself) is called *non-singular* if there exists a point $x_0 \in X$ such that the pointwise stabilizer $\mathrm{Stab}_\Gamma(x_0)$ is trivial, and *singular* otherwise; see [CS22, Example 1.4] and [CS23, Remark 5.3] for examples of this phenomenon.
Finally, when dealing with isometry groups of CAT(0)-spaces with torsion, a difficulty is that there may exist nontrivial elliptic isometries which act as the identity on open sets. Following [DLHG90, Chapter 11], a subgroup Γ of $\mathrm{Isom}(X)$ will be called *rigid* (or *slim*, with the terminology used in [CS22]) if for all $g \in \Gamma$ the subset $\mathrm{Fix}(g)$ has empty interior.
Notice that a torsion-free group is trivially rigid, as well as any discrete group acting on a CAT(0)-homology manifold, as proved in [CS22, Lemma 2.1]). Also, every rigid isometry group Γ is automatically nonsingular since, by Baire's theorem, the union of fixed-point sets of all elliptic elements of Γ has empty interior, cp. [CS22, Proposition 2.11].

We will call the quotient metric space $M = \Gamma \backslash X$ of a proper, geodesically complete, CAT(0)-space X by a discrete group of isometries Γ, a CAT(0)-*orbispace*. We will also say that the CAT(0)-orbispace M is *nonsingular* (resp. *rigid*) if Γ acts nonsingularly (resp. rigidly) on X. Notice that if Γ is torsion-free then M is a locally CAT(0)-space.
When restricting our attention to groups Γ acting *rigidly* on X (i.e. such that every $g \in \Gamma$ acting as the identity on an open subset is trivial) then our definition of CAT(0)-orbispace is equivalent to the notion of *rigid, developable orbispace* as defined in [DLHG90, Ch.11] through an orbifold atlas, with CAT(0) universal covering in the sense of orbispaces. However, our results will apply to all CAT(0)-orbispaces as defined above (with some distinction between the singular and nonsingular cases).

Remark 2.1. Notice that a CAT(0)-orbispace has more structure than simply the structure of a metric space, including also the data of a group action on some CAT(0)-space. In this sense, it is worth to stress that (as opposite to the case of *Riemannian* orbifolds) two lattices Γ_1, Γ_2 of different CAT(0)-spaces X_1, X_2 may give the same metric quotient M.
An example of this (for CAT(1)-orbispaces) is the *k-triplex* X constructed in [Nag22]: this is a proper, geodesically complete, purely k-dimensional CAT(1)-space (actually, a topological sphere) admitting a lattice $\Gamma \cong \mathbb{Z}_3$ such that $M = \Gamma \backslash X$ is isometric to the CAT(1)-orbispace $M' = \mathbb{Z}_2 \backslash \mathbb{S}^k$, with $\mathbb{Z}_2$ acting as a rotation of π.
An example for CAT(0)-orbispaces can be costructed by taking a ramified covering of a flat torus $T = \mathbb{Z}^2 \backslash \mathbb{R}^2$: let X be obtained by taking two copies $T_i \setminus \gamma_i$ of T minus a geodesic segment γ, and then glueing together the resulting couples of boundaring edges (γ_1^+, γ_1^-) with (γ_2^-, γ_2^+). This yields a (metric) ramified covering $f : X \to T$ from a topological surface of genus 2 to the initial torus, with ramification locus given by the endpoints of γ. Notice that X is a metric space which has an involutive isometry σ such that $\langle \sigma \rangle \backslash X = T$, and it can be seen as the quotient of a CAT(0)-space $\tilde{X}$ by a lattice Γ containing an elliptic element $\tilde{\sigma}$ of order 2 (and all its conjugates). Calling $\tilde{\Gamma}$ the subgroup of $\mathrm{Isom}(\tilde{X})$ generated by $\tilde{\sigma}$ and Γ, we have that $\tilde{\Gamma} \backslash \tilde{X}$ is also isometric to T.

On the other hand, the underlying metric space $M = \Gamma\backslash X$ determines X and Γ when Γ acts freely (since in this case X is just the universal cover of M), or when X is Riemannian, as the following result shows:

Proposition 2.2. *Let Γ and Γ' be discrete isometry groups of two Hadamard manifolds X, X' respectively. If $\Gamma\backslash X$ is isometric to $\Gamma'\backslash X'$ then there exists an isomorphism $\varphi\colon \Gamma \to \Gamma'$ and a φ-equivariant isometry $F : X \to X'$.*

Proof. The space $M = \Gamma\backslash X = \Gamma'\backslash X$ admits two orbifold structures induced by the two actions of Γ and Γ'. By [Lan20, Lemma 2.2] the two orbifold structures coincide since locally the actions of Γ on X and Γ' on X' are isometrically equivariant. Since both X and X' are simply connected, we deduce that they are both orbifold universal coverings of M (see for instance [Lan20, Theorem 2.9], originally proved in [Thu97]). By the universal property of the orbifold universal covering we get a diffeomorphism $F\colon X \to X'$ commuting with the projections π and π' on M. By construction F is also equivariant with respect to an isomorphism between Γ and Γ'. Finally, again by [Lan20, Lemma 2.2] and the commutation property $\pi' \circ F = \pi$, we get that F is a local isometry and therefore a global one. □

Strictly speaking, in view of the discussion above, the quotient metric space $M = \Gamma\backslash X$ should only be called the *support* of the orbispace; we will continue to use the sloppy terminology CAT(0)-*orbispace* both for the support and for the metric space with its uniformizing global chart $X \to M$, as far as the statements involved are clear. For this reason, we will say that two CAT(0)-orbispaces $M = \Gamma/X$ and $M' = \Gamma'/X'$ are:

– *equivariantly homotopy equivalent*, if there exist an isomorphism $\varphi\colon \Gamma \to \Gamma'$ and a φ-equivariant homotopy equivalence[2] $F\colon X \to X'$;
– *isometric as orbispaces* if, moreover, the map F is an isometry.

2.2. **Lie isometry groups.**

In general, the full isometry group $\mathrm{Isom}(X)$ of a CAT(0)- space is not a Lie group, for instance in the case of regular trees. When a CAT(0)-space X admits a uniform lattice Γ then $\mathrm{Isom}(X)$ is known to have more structure, as proved by P.-E.Caprace and N.Monod:

Proposition 2.3 ([CM09b, Thm.1.6 & Add.1.8], [CM09a, Cor.3.12])**.**
Let X be a proper, geodesically complete, CAT(0)*-space, admitting a uniform lattice. Then X splits isometrically as $M \times \mathbb{R}^n \times N$, where M is a symmetric space of noncompact type and $\mathcal{D} := \mathrm{Isom}(N)$ is totally disconnected.*
Moreover

$$\mathrm{Isom}(X) \cong \mathcal{S} \times \mathcal{E}_n \times \mathcal{D}$$

where $\mathcal{S}$ is a semi-simple Lie group with trivial center and without compact factors, and $\mathcal{E}_n = \mathrm{Isom}(\mathbb{R}^n)$.

[2] that is, a map satisfying $F(gx) = \varphi(g)F(x)$ for every $x \in X$ and every $g \in \Gamma$, admitting moreover a φ^{-1}-equivariant homotopy inverse $G : X' \to X$ (i.e. such that $F \circ G$ and $G \circ F$ are homotopic to the identity through, respectively, Γ-equivariant and Γ'-equivariant homotopies).

In particular, under the assumptions of Proposition 2.3, the group $\mathrm{Isom}(X)$ is a Lie group if and only if the factor $\mathcal{D}$ is discrete. On the other hand, $\mathrm{Isom}(X)$ is always a Lie group when X is a CAT(0)-homology manifold:

Proposition 2.4. *Let X be a locally compact,* CAT(0)*-homology manifold. Then* $\mathrm{Isom}(X)$ *is a Lie group.*

Proof. If the dimension of X is smaller than or equal to 2 then X is a topological manifold. Otherwise we can find a $\mathrm{Isom}(X)$-invariant, locally finite subset E of X such that $X \setminus E$ is a connected topological manifold (see [LN18, Theorem 1.2]). The Hausdorff dimension of $X \setminus E$ is at most the Hausdorff dimension of X, which coincides with its topological dimension, say n, as recalled at the beginning of Section 2; therefore, $\mathrm{HD}(X \setminus E) = n$. The classical work about the Hilbert-Smith Conjecture implies that the group $\mathrm{Isom}(X)$ is a Lie group if and only if it does not contain a subgroup isomorphic to some p-adic group $\mathbb{Z}_p$, see for instance [Lee97]. Now, a theorem of Repovş and Şçepin asserts that $\mathbb{Z}_p$ cannot act effectively by Lipschitz homeomorphisms on any finite dimensional Riemannian manifold. The proof in [RS97] adapts perfectly to our setting; indeed, if we suppose that the p-adic group acts on X by isometries we can apply [Yan60] as in [RS97] to deduce that $\mathrm{HD}(X \setminus E) \geq n + 2$, giving a contradiction. □

2.3. The packing condition and Margulis' Lemma.

Let X be a metric space and $r > 0$. A subset Y of X is called *r-separated* if $d(y, y') > r$ for all $y, y' \in Y$. Given $x \in X$ and $0 < r \leq R$ we denote by $\mathrm{Pack}(\overline{B}(x, R), r)$ the maximal cardinality of a $2r$-separated subset of $\overline{B}(x, R)$. Moreover we denote by $\mathrm{Pack}(R, r)$ the supremum of $\mathrm{Pack}(\overline{B}(x, R), r)$ among all points of X. Given $P_0, r_0 > 0$ we say that X satisfies the P_0-packing condition at scale r_0 (or that it is (P_0, r_0)-packed, for short) if $\mathrm{Pack}(3r_0, r_0) \leq P_0$. We will simply say that X *is packed* if it satisfies a P_0-packing condition at scale r_0, for some $P_0, r_0 > 0$.

The packing condition should be thought as a metric, weak replacement of a Ricci curvature lower bound. Actually, by Bishop-Gromov's Theorem, for a n-dimensional Riemannian manifold a lower bound on the Ricci curvature $\mathrm{Ric}_X \geq -(n-1)\kappa$, $\kappa \geq 0$, implies a uniform estimate of the packing function at any fixed scale r_0, that is

$$\mathrm{Pack}(3r_0, r_0) \leq \frac{v_{\mathbb{H}^n_\kappa}(3r_0)}{v_{\mathbb{H}^n_\kappa}(r_0)} \tag{3}$$

where $v_{\mathbb{H}^n_\kappa}(r)$ is the volume of a ball of radius r in the n-dimensional space form with constant curvature $-\kappa$.

Also remark that every metric space admitting a cocompact action is packed (for some P_0, r_0), see the proof of [Cav22a, Lemma 5.4].

The packing condition has many interesting geometric consequences for complete, geodesically complete CAT(0)-spaces, as showed in [CS24], [CS21] [Cav21], [Cav22b], [CS22], [CS23]. Here, we just recall the following uniform bounds on the canonical measure of balls, and of the entropy and dimension:

Proposition 2.5 ([CS21, Theorem 4.9])**.**

Let X be a complete, geodesically complete, (P_0, r_0)-packed, CAT(0)*-space. Then X is proper, and*

(i) there exist functions $v, V\colon (0, +\infty) \to (0, +\infty)$ depending only on P_0, r_0 such that for all $x \in X$ and $R > 0$ we have

$$v(R) \leq \mu_X(\overline{B}(x, R)) \leq V(R);$$

(ii) the entropy of X is bounded above in terms of P_0 and r_0, namely

$$\mathrm{Ent}(X) \leq \frac{\log(1 + P_0)}{r_0};$$

(iii) the dimension of X is at most $n_0 := P_0/2$.

Recall that the (*packing*, or *covering*) *entropy* of a proper CAT(0)-space X is the asymptotic invariant defined as

$$\mathrm{Ent}(X) = \limsup_{R \to +\infty} \frac{1}{R} \log(\mathrm{Pack}(R, r))$$

and does not depend neither on the point x (by the triangular inequality) nor on the small, chosen radius r (by [CS21, Corollary 4.8]). It can be equivalently defined as the critical exponent of any uniform lattice Γ of X, as proved in [Cav22b]), that is

$$\mathrm{Ent}(X) = \lim_{R \to +\infty} \frac{1}{R} \log(\#(\Gamma x \cap B(x, R))) \tag{4}$$

Let now Γ be a subgroup of $\mathrm{Isom}(X)$. For $x \in X$ and $r \geq 0$ we set

$$\overline{\Sigma}_r(\Gamma, X, x) := \{g \in \Gamma \text{ s.t. } d(x, gx) \leq r\} \tag{5}$$

$$\overline{\Gamma}_r(X, x) := \langle \overline{\Sigma}_r(\Gamma, X, x) \rangle \tag{6}$$

The groups $\overline{\Gamma}_r(X, x)$ are sometimes called *"almost stabilizers"*; when the context is clear we will simply write $\overline{\Sigma}_r(\Gamma, x)$ or $\overline{\Sigma}_r(x)$ and $\overline{\Gamma}_r(x)$.
Notice that, since X is assumed to be proper and Γ acts by isometries, a subgroup Γ is *discrete* if and only if the orbit Γx is discrete and $\mathrm{Stab}_\Gamma(x)$ is finite for some (or, equivalently, for all) $x \in X$. This is in turn equivalent to asking that the subsets $\overline{\Sigma}_r(x)$ are finite for all $x \in X$ and all $r \geq 0$.
The following remarkable version of the Margulis' Lemma, due to Breuillard-Green-Tao, is another important consequence of a packing condition at some fixed scale. It clarifies the structure of the almost stabilizers $\overline{\Gamma}_r(x)$ for small r. We decline it for geodesically complete CAT(0)-spaces:

Proposition 2.6 ([BGT11, Corollary 11.17]).
Given $P_0, r_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(P_0, r_0) > 0$ such that the following holds. Let X be a proper, geodesically complete, (P_0, r_0)-packed, CAT(0)*-space and let Γ be a discrete subgroup of* $\mathrm{Isom}(X)$*: then, for every $x \in X$ and every $0 \leq \varepsilon \leq \varepsilon_0$, the almost stabilizer $\overline{\Gamma}_\varepsilon(x)$ is virtually nilpotent.*

We will often refer to the constant $\varepsilon_0 = \varepsilon_0(P_0, r_0)$ as the *Margulis' constant*. The conclusion of Proposition 2.6 can be improved for cocompact groups, as in this case the group $\overline{\Gamma}_\varepsilon(x)$ is *virtually abelian* (cp. [BH13, Theorem II.7.8]; indeed, a cocompact group of a CAT(0)-space is always semisimple).

2.4. **Discrete, virtually abelian groups.**
Recall that the (abelian, or Prüfer) *rank* of an abelian group A, denoted $\mathrm{rk}(A)$, is the maximal cardinality of a subset $S \subset A$ of $\mathbb{Z}$-linear independent elements. The rank of a *virtually abelian group* G, $\mathrm{rk}(G)$, is the rank of any free abelian subgroup A of finite index in G (notice that if A' is a finite index subgroup of an abelian group A, then A and A' have same rank, so $\mathrm{rk}(G)$ is well defined).
Among CAT(0)-spaces, the Euclidean space $\mathbb{R}^k$ and its discrete groups play a special role. In the following, we will denote by $\mathcal{E}_k$ the group of isometries of $\mathbb{R}^k$, and by $\mathrm{Transl}(\mathbb{R}^k)$ the normal subgroup of all translations.
A *crystallographic group* is a discrete, cocompact group G of isometries of $\mathbb{R}^k$. The simplest and most important of them, in view of Bieberbach's Theorem, are *Euclidean lattices*: i.e. free abelian crystallographic groups. It is well known that a lattice must act by translations on $\mathbb{R}^k$ (see for instance [Far81]); so, alternatively, a lattice $\mathcal{L}$ can be seen as the set of linear combinations with integer coefficients of k independent vectors $b_1, \ldots, b_k$ (we will make no difference between a lattice and this representation); the integer k is called the *rank* of the lattice. For our purposes, the content of the famous Bieberbach's Theorems can be stated as follows:

Proposition 2.7 (Bieberbach's Theorem)**.**
There exists $J(k)$, only depending on k, such that the following holds true. For any crystallographic group G of $\mathbb{R}^k$ the subgroup $\mathcal{L}(G) = G \cap \mathrm{Transl}(\mathbb{R}^k)$ is a normal subgroup of index at most $J(k)$, in particular a lattice.

The subgroup $\mathcal{L}(G)$ is called the *maximal lattice* of G.
An almost immediate consequence of Bieberbach' theorem is the following lemma, which will be crucial in characterizing the limits of collapsing sequences in Section 4.4. For this we set $r_k := \sqrt{2\sin\left(\frac{\pi}{J(k)}\right)}$.

Lemma 2.8. *Let G be a crystallographic group of $\mathbb{R}^k$, and let $0 < r < r_k$. If $g \in G$ moves all points of $B_{\mathbb{R}^k}(O, \frac{1}{r})$ less than r, then g is a translation.*

Proof. Let $g(x) = Ax + b$ be any nontrivial isometry of G, with $A \in O(k)$. If $A = \mathrm{Id}$ there is nothing to prove. Otherwise Proposition 2.7 implies that g^j is a translation for some $j \leq J(k)$. Since $g^j(x) = A^j x + \sum_{i=0}^{j-1} A^i b$, we deduce that $A^j = \mathrm{Id}$. Let us consider the (nontrivial) subspace V orthogonal to $\mathrm{Fix}(A)$. We can decompose V in the ortogonal sum of a subspace V_0, on which A acts as a reflection, and of some planes V_i, each of which A acts on as a nontrivial rotation with angle ϑ_i. Since A has order at most $J(k)$, we have $\vartheta_i \geq \frac{2\pi}{J(k)}$.
If V_0 is not trivial then take a vector v_0 of length 1 in V_0. Denote by $b_0 = \lambda v_0$ the projection of b on the line spanned by v_0. For all $t \in \mathbb{R}$ it holds

$$d(g(tv_0), tv_0) = \| -2tv_0 + b\| \geq |\lambda - 2t|.$$

So, for $r < \sqrt{2}$, we can always find t_r with $|t_r| < \frac{1}{r}$ such that $|\lambda - 2t_r| > r$; but the point $t_r v_0$ contradicts the assumptions on g.
On the other hand, if V_0 is trivial, then there is at least a plane V_i on which A acts as a nontrivial rotation, say V_1. Let b_1 be the projection of b on V_1. We choose $v_1 \in V_1$ of norm 1 such that $b_1 = \lambda(Av_1 - v_1)$ for some $\lambda \geq 0$.

The length of $Av_1 - v_1$ is at least $2\sin\left(\frac{\pi}{J(k)}\right)$, because $\vartheta_1 \geq \frac{2\pi}{J(k)}$, therefore $d(g(tv_1), tv_1) \geq 2t\sin\left(\frac{\pi}{J(k)}\right)$ $\forall t \geq 0$. Again, if $r < \sqrt{2\sin\left(\frac{\pi}{J(k)}\right)}$ we can find t_r with $|t_r| \leq \frac{1}{r}$ such that $t_r v_1$ contradicts the assumptions on g. □

2.5. **Finiteness results.**
The *systole* (or *minimal displacement*) and the *free-systole* of a discrete group Γ *at a point* $x \in X$ are defined respectively as

$$\operatorname{sys}(\Gamma, x) := \inf_{g \in \Gamma^*} d(x, gx), \qquad \operatorname{sys}^\diamond(\Gamma, x) := \inf_{g \in \Gamma^* \setminus \Gamma^\diamond} d(x, gx),$$

where $\Gamma^* = \Gamma \setminus \{\mathrm{id}\}$ and $\Gamma^\diamond$ is the subset of all elliptic isometries of Γ. Accordingly, the *(global) systole* and the *free-systole of* Γ are defined as

$$\operatorname{sys}(\Gamma, X) = \inf_{x \in X} \operatorname{sys}(\Gamma, x), \qquad \operatorname{sys}^\diamond(\Gamma, X) = \inf_{x \in X} \operatorname{sys}^\diamond(\Gamma, x).$$

as opposite to the *diastole* and the *free-diastole* of Γ, which are

$$\operatorname{dias}(\Gamma, X) = \sup_{x \in X} \operatorname{sys}(\Gamma, x), \qquad \operatorname{dias}^\diamond(\Gamma, X) = \sup_{x \in X} \operatorname{sys}^\diamond(\Gamma, x).$$

Notice that the action of Γ is nonsingular if and only if $\operatorname{dias}(\Gamma, X) > 0$.
While obviuosly one always has, by definition,

$$\operatorname{sys}(\Gamma, X) \leq \operatorname{sys}^\diamond(\Gamma, X) \leq \operatorname{dias}^\diamond(\Gamma, X)$$

$$\operatorname{sys}(\Gamma, X) \leq \operatorname{dias}(\Gamma, X) \leq \operatorname{dias}^\diamond(\Gamma, X)$$

the free systole and the diastole are quantitatively equivalent (for small values) for nonsingular, cocompact actions on packed CAT(0)-spaces with bounded codiameter. More precisely:

Proposition 2.9 (Theorem 3.1, [CS23]). *Given P_0, r_0 and D_0, there exists $A = A(P_0, r_0) > 0$, $B = B(P_0, D_0) > 1$ such that the following holds true: for any proper, geodesically complete, (P_0, r_0)-packed,* CAT(0)*-space X, and any discrete, nonsingular, D_0-cocompact group $\Gamma < \operatorname{Isom}(X)$ we have*

$$B^{-1/\operatorname{dias}(\Gamma, X)} \leq \operatorname{sys}^\diamond(\Gamma, X) \leq A \cdot \operatorname{dias}(\Gamma, X)$$

provided that the free systole and the diastole are both smaller than $\min\{r_0, \varepsilon_0\}$. *(Here, ε_0 is the Margulis' constant given by Proposition 2.6).*

This equivalence will be useful when studying the convergence of non-collapsing sequences of groups with elliptic elements.

We conclude recalling two of the main findings of [CS23], which we will be crucial in our convergence results. The first is a general finiteness theorem for uniformily packed, D_0-cocompact CAT(0)-lattices Γ, up to abstract isomorphism. The second one is more subtle and also requires a uniform lower bound of the diastole: it bounds the number of such groups Γ up to isomorphism of marked groups. In order to state it correctly, recall that a *marked group* is a group Γ endowed with a generating set Σ; two marked groups (Γ, Σ) and (Γ', Σ') are said to be *equivalent* if there exists a group isomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi(\Sigma) = \Sigma'$. It is well known (cp. for instance

[Ser03], [Gro07]) that if X is a geodesic metric space and $\Gamma < \mathrm{Isom}(X)$ is discrete and D_0-cocompact, then for all $D \geq D_0$ the subset

$$\overline{\Sigma}_{2D}(x) := \{g \in \Gamma \text{ s.t. } d(x, gx) \leq 2D\} \tag{7}$$

is a generating set for Γ, that is $\overline{\Gamma}_{2D}(x) = \Gamma$, for every $x \in X$ (recall the definition (6)) of $\overline{\Gamma}_r(x)$): we call this a *$2D$-short generating set of Γ at x.*

Theorem 2.10 ([CS23, Theorem A & Prop.5.1])**.** *Let P_0, r_0 and D_0 be fixed. There exist only finitely many nonsingular D_0-cocompact lattices (Γ, X) of proper, geodesically complete,* CAT(0)*-spaces which are (P_0, r_0)-packed, up to group isomorphisms.*
Moreover, for every fixed $D \geq D_0$ and $s > 0$, there exist only finitely many marked groups (Γ, Σ), up to equivalence of marked groups, such that:
– *Γ is a D_0-cocompact lattice of a proper, geodesically complete, (P_0, r_0)-packed,* CAT(0)*-space X satisfying* $\mathrm{sys}(\Gamma, x) \geq s$ *for some $x \in X$,*
– *$\Sigma = \overline{\Sigma}_{2D}(x)$ is a $2D$-short generating set of Γ at x.*

3. Splitting under collapsing

3.1. Space splitting.

We record here one of the main results of [CS23]: if a uniform lattice Γ of a packed, CAT(0)-space X is sufficiently collapsed (that is, the free-systole is sufficiently small), then X splits a non-trivial Euclidean factor.
Recall that the dimension of any (proper, geodesically complete) (P_0, r_0)-packed CAT(0)-space X is bounded above by $n_0 = P_0/2$, by Proposition 2.5. Recall also the Margulis's constant ε_0 given by Proposition 2.6, which depends only on the packing constants P_0, r_0. Then, we set

$$J_0 := \max_{k \in \{0, \dots, n_0\}} J(k) + 1 \tag{8}$$

where $J(k)$ is the constant appearing in Bieberbach's theorem (Proposition 2.7), bounding the index of the maximal lattice of any crystallographic group in dimension k. Notice that J_0 depends only on n_0, so ultimately only on P_0.
Finally, for a discrete subgroup $\Gamma < \mathrm{Isom}(X)$, recall the definition (6) of almost-stabilizer $\overline{\Gamma}_r(x) < \Gamma$ generated by $\overline{\Sigma}_r(x)$ given in Section 2.1.

Theorem 3.1 (see [CS23], Splitting Theorem 4.1 and its proof)**.**
Given P_0, r_0 and D_0, there exists a function $\sigma_{P_0,r_0,D_0} : (0, \varepsilon_0] \to (0, \varepsilon_0]$ (depending only on the parameters P_0, r_0, D_0) such that the following holds. Let X be a proper, geodesically complete, (P_0, r_0)-packed, CAT(0)*-space, and $\Gamma < \mathrm{Isom}(X)$ be discrete and D_0-cocompact. For every chosen $\varepsilon \in (0, \varepsilon_0]$, if there exists $x_0 \in X$ such that $\mathrm{rk}(\overline{\Gamma}_\sigma(x_0)) \geq 1$, where $\sigma = \sigma_{P_0,r_0,D_0}(\varepsilon)$, then:*

(i) the space X splits isometrically and Γ-invariantly as $Y \times \mathbb{R}^k$, with $k \geq 1$;
(ii) there exists $\varepsilon^* \in (\sigma_{P_0,r_0,D_0}(\varepsilon), \varepsilon)$ such that the rank of the virtually abelian subgroups $\overline{\Gamma}_{\varepsilon^*}(x)$ is exactly k, for all $x \in X$;
(iii) for every $x \in X$ there exists $y \in Y$ such that $\overline{\Gamma}_{\varepsilon^*}(x)$ preserves $\{y\} \times \mathbb{R}^k$;
(iv) for every $x \in X$ the projection of $\overline{\Gamma}_{\varepsilon^*}(x)$ on $\mathrm{Isom}(\mathbb{R}^k)$ is a crystallographic group, whose maximal lattice is generated by the projection of a subset $\Sigma \subset \Sigma_{4J_0 \cdot \varepsilon^*}(x)$;
(v) for every $x \in X$ the closure of the projection of $\overline{\Gamma}_{\varepsilon^*}(x)$ on $\mathrm{Isom}(Y)$ is compact and totally disconnected;

(vi) for every $x \in X$ the group $\overline{\Gamma}_{\varepsilon^*}(x)$ contains a finite index, free abelian subgroup A of rank k which is commensurated in Γ (hence, $\overline{\Gamma}_{\varepsilon^*}(x)$ is itself commensurated in Γ).

In particular, the above properties hold if $\mathrm{sys}^\diamond(\Gamma, X) \leq \sigma_{P_0,r_0,D_0}(\varepsilon)$.

Here, by Γ-*invariant splitting* we mean that every isometry of Γ preserves the product decomposition. By [BH13, Proposition I.5.3.(4)] we can see Γ as a subgroup of $\mathrm{Isom}(Y) \times \mathrm{Isom}(\mathbb{R}^k)$. In particular it is meaningful to talk about the projection of A on $\mathrm{Isom}(Y)$ and $\mathrm{Isom}(\mathbb{R}^k)$. Recall that a subgroup $G < \Gamma$ is said to be *commensurated in* Γ if the groups G and $\gamma G \gamma^{-1}$ are commensurable in Γ for every $\gamma \in \Gamma$ (i.e. the intersection $G \cap \gamma G \gamma^{-1}$ has finite index in G for every $\gamma \in \Gamma$).

Let us define, for the following, the constant (only depending on P_0, r_0, D_0)

$$\sigma_0^* := \sigma_{P_0,r_0,D_0}(\varepsilon_0) \tag{9}$$

Assertion (iii) yields, for any $x \in X$, a slice $\{y\} \times \mathbb{R}^k$ preserved by $\overline{\Gamma}_{\sigma_0^*}(x)$ (as $\sigma_0^* < \varepsilon_0^* < \varepsilon_0$), provided that $\mathrm{sys}^\diamond(\Gamma, X) \leq \sigma_0^*$ so that X splits as $Y \times \mathbb{R}^k$. For our convergence theorems, we need to strengthen this conclusion and to show that, for at least one specific point x_0, the slice can be chosen to be the one passing through x_0. This is proved by the following:

Proposition 3.2. *Let X be a proper, geodesically complete, (P_0, r_0)-packed* $\mathrm{CAT}(0)$*-space, and let Γ be a discrete, D_0-cocompact subgroup of* $\mathrm{Isom}(X)$*. Assume that* $\mathrm{sys}^\diamond(\Gamma, X) \leq \sigma_0^*$*, so that X splits as $Y \times \mathbb{R}^k$ by Theorem 3.1. Then there exists $x_0 = (y_0, v) \in X$ such that $\overline{\Gamma}_{\sigma_0^*}(x_0)$ preserves $\{y_0\} \times \mathbb{R}^k$.*

The proof is based on a maximality argument similar to [CS22, Theorem 3.1]. We just need a very basic fact:

Lemma 3.3 ([CS22, Lemma 3.3])**.** *For every $x \in X$ and $r > 0$ there exists an open set $U \ni x$ such that $\overline{\Sigma}_r(y) \subseteq \overline{\Sigma}_r(x)$ for all $y \in U$.*

Proof of Proposition 3.2. We introduce a partial order on X defined by

$$x \preceq x' \text{ if and only if } \overline{\Sigma}_{\sigma_0^*}(x) \subseteq \overline{\Sigma}_{\sigma_0^*}(x').$$

We can show that there is a maximal element as in the proof of [CS22, Theorem 3.1, Step 3], which we report here for the reader's convenience: Lemma 3.3 implies that the function $x \mapsto \#\overline{\Sigma}_{\sigma_0^*}(x)$ is upper semicontinuous and clearly Γ-invariant, so (by cocompactness) it has a maximum and it is enough to take a point x where the maximum is realized.

Apply Theorem 3.1.(iii) to the point x: there exists $y_0 \in Y$ such that $\overline{\Gamma}_{\sigma_0^*}(x)$ preserves $\{y_0\} \times \mathbb{R}^k$. If $x \in \{y_0\} \times \mathbb{R}^k$, then we set $x_0 = x$ and there is nothing more to prove. Otherwise call x_0 the projection of x on the closed, convex, $\overline{\Gamma}_{\sigma_0^*}(x)$-invariant subset $\{y_0\} \times \mathbb{R}^k$. Let $c \colon [0, d(x, x_0)] \to X$ be the geodesic $[x, x_0]$ and set

$$T = \sup\{t \in [0, d(x, x_0)] \text{ s.t. } \overline{\Gamma}_{\sigma_0^*}(c(t)) = \overline{\Gamma}_{\sigma_0^*}(x)\}.$$

By definition $T \geq 0$ and we claim that $T = d(x, x_0)$ and it is a maximum.

In fact, assume $\overline{\Gamma}_{\sigma_0^*}(c(t)) = \overline{\Gamma}_{\sigma_0^*}(x)$. So, $\overline{\Gamma}_{\sigma_0^*}(c(t + t')) \subseteq \overline{\Gamma}_{\sigma_0^*}(c(t)) = \overline{\Gamma}_{\sigma_0^*}(x)$ for all $t' > 0$ small enough, by Lemma 3.3. On the other hand, for every $g \in \overline{\Gamma}_{\sigma_0^*}(x)$ we have $d(x_0, gx_0) \leq d(x, gx)$ (since g acts on Y fixing y_0).

Therefore, by the convexity of the displacement functions we deduce that $\overline{\Gamma}_{\sigma_0^*}(x) \subseteq \overline{\Gamma}_{\sigma_0^*}(c(t+t'))$ too. This shows that the supremum defining T is not realized, unless $T = d(x, x_0)$. Let now $0 \leq t_n < T$ such that $t_n \to T$, so the points $c(t_n)$ converge to $c(T)$. By Lemma 3.3 we get $\overline{\Sigma}_{\sigma_0^*}(c(t_n)) \subseteq \overline{\Sigma}_{\sigma_0^*}(c(T))$ for all n big enough. But $\overline{\Sigma}_{\sigma_0^*}(c(t_n)) = \overline{\Sigma}_{\sigma_0^*}(x)$ is maximal for $\preceq$, therefore $\overline{\Sigma}_{\sigma_0^*}(x) = \overline{\Sigma}_{\sigma_0^*}(c(t_n)) = \overline{\Sigma}_{\sigma_0^*}(c(T))$ for all n big enough. This implies that $T = d(x, x_0)$ is actually a maximum. Hence, $\overline{\Gamma}_{\sigma_0^*}(x_0) = \overline{\Gamma}_{\sigma_0^*}(x)$, and x_0 is the point we are looking for. $\square$

A consequence of Theorem 3.1 is the following control of the almost stabilizers, which will be useful in studying converging sequences. Recall the function $\sigma_{P_0,r_0,D_0} : (0, \varepsilon_0] \to (0, \varepsilon_0]$ given by Theorem 3.1, where ε_0 is the Margulis constant, and the constant J_0 introduced at the beginning of this section. Then:

Corollary 3.4. *Let X be a proper, geodesically complete, (P_0, r_0)-packed* CAT(0)*-space, and let Γ be a discrete, D_0-cocompact subgroup of* $\mathrm{Isom}(X)$. *Let $\sigma(\Gamma, X) := \sigma_{P_0,r_0,D_0}(\varepsilon^\diamond)$ be the constant obtained for $\varepsilon^\diamond = \min\left\{\varepsilon_0, \frac{\mathrm{sys}^\diamond(\Gamma,X)}{4J_0}\right\}$. Then*

(i) the almost stabilizers $\overline{\Gamma}_{\sigma(\Gamma,X)}(x)$ are finite, for all $x \in X$;

(ii) there exists $x_0 \in X$ such that $\overline{\Gamma}_{\sigma(\Gamma,X)}(x_0)$ fixes x_0.

Observe that, since by (i) the group $\overline{\Gamma}_{\sigma(\Gamma,X)}(x)$ is finite, then it has a fixed point (as it acts on a CAT(0)-space, cp. [BH13, Chapter II, Corollary 2.8]). Here we are saying that for at least one specific point x_0 we have that $\overline{\Gamma}_{\sigma(\Gamma,X)}(x_0)$ fixes exactly x_0.

Proof of Corollary 3.4. We first prove (i), and set, for short, $\sigma = \sigma(\Gamma, X)$. Suppose that there exists $x \in X$ such that $\overline{\Gamma}_\sigma(x)$ is not finite: since $\overline{\Gamma}_\sigma(x)$ is finitely generated and virtually abelian, then it contains a hyperbolic isometry, so $\mathrm{rk}(\overline{\Gamma}_\sigma(x)) \geq 1$. Theorem 3.1 implies that X splits isometrically and Γ-invariantly as $Y \times \mathbb{R}^k$, with $k \geq 1$, and that there exists $\varepsilon^* \in (\sigma, \varepsilon^\diamond)$ such that $\overline{\Gamma}_{\varepsilon^*}(x)$ projects on $\mathrm{Isom}(\mathbb{R}^k)$ as a crystallographic group. Moreover, we can find elements of $\overline{\Sigma}_{4J_0 \cdot \varepsilon^*}(x)$ that project to translations on $\mathrm{Isom}(\mathbb{R}^k)$. But these elements are hyperbolic isometries of Γ with translation length at most $4J_0 \cdot \varepsilon^* < \mathrm{sys}^\diamond(\Gamma, X)$, a contradiction.
Assertion (ii) follows from an argument similar to that of Proposition 3.2: we consider the same partial order $x \preceq x' \Leftrightarrow \overline{\Sigma}_\sigma(x) \subseteq \overline{\Sigma}_\sigma(x')$, and find a maximal element x. The group $\overline{\Gamma}_\sigma(x)$ being finite, the closed, convex set $\mathrm{Fix}(\overline{\Gamma}_\sigma(x))$ is not empty. If $x \in \mathrm{Fix}(\overline{\Gamma}_\sigma(x))$, then $x_0 = x$ and there is nothing more to prove. Otherwise we call x_0 the projection of x on $\mathrm{Fix}(\overline{\Gamma}_\sigma(x))$, consider again the geodesic $c \colon [0, d(x, x_0)] \to [x, x_0] \subset X$ and we show as in Proposition 3.2 that

$$T := \sup\{t \in [0, d(x, x_0)] \text{ s.t. } \overline{\Gamma}_\sigma(c(t)) = \overline{\Gamma}_\sigma(x)\} = d(x, x_0)$$

by Lemma 3.3 and the convexity of the displacement function. Moreover, again by Lemma 3.3 and by the maximality of x it follows that $T = d(x, x_0)$ is a maximum, hence $\overline{\Gamma}_\sigma(x_0) = \overline{\Gamma}_\sigma(x)$, and x_0 is the announced fixed point. $\square$

Finally, we record here another corollary of the splitting theorem, which will be used in studying the Euclidean factor of the limit of a sequence of collapsing actions in Section 4.6.

Theorem 3.5 ([CS23], Renormalization Theorem E).
Given P_0, r_0, D_0, there exist $s_0 = s_0(P_0, r_0, D_0) > 0$ and $\Delta_0 = \Delta_0(P_0, D_0)$ such that the following holds. Let Γ be a discrete and D_0-cocompact isometry group of a proper, geodesically complete, (P_0, r_0)-packed CAT(0)*-space X: then Γ admits also a faithful, discrete, Δ_0-cocompact action by isometries on a* CAT(0)*-space X' isometric to X, such that* $\mathrm{sys}^\diamond(\Gamma, X') \geq s_0$.
Moreover, the action of Γ on X is nonsingular if and only if the action on X' is nonsingular.

3.2. **Group splitting: a counterexample.**
The Splitting Theorem 3.1 might suggest that, under the assumption that the action is sufficiently collapsed, then also the group Γ virtually splits a non-trivial free abelian subgroup. Unfortunately, while this is always the case for non-positively curved Riemannian manifolds by the work of Eberlein [Ebe83] (indeed, in that case it is always possible to split virtually a free abelian subgroup whose rank coincides exactly with the rank of the maximal Euclidean factor of the space), this is no longer true for discrete, cocompact groups of general, geodesically complete CAT(0)-spaces, as [CM19, Example 1] shows. More examples of this phenomenon can be found in [LM21].
In this section, we present a basic example, constructed in [LM21], showing that, no matter how the action is collapsed, the virtual splitting cannot be proved under our hypotheses.

Example 3.6. We construct a sequence of proper, geodesically complete, CAT(0)-spaces X_j, all (P_0, r_0)-packed for the same constants $P_0, r_0 > 0$, and a sequence of discrete, nonsingular groups $\Gamma_j < \mathrm{Isom}(X_j)$ which are all D_0-cocompact for some $D_0 > 0$ such that:

(a) each Γ_j has no non-trivial, virtually normal, abelian subgroups;
(b) $\mathrm{sys}^\diamond(\Gamma_j, X_j) \to 0$ as $j \to +\infty$.

Recall that a subgroup A of a group Γ is said to be *virtually normal* if there exists a finite index subgroup Γ' such that $A \cap \Gamma'$ is normal in Γ'.
Observe that (b) gives strong restrictions to the spaces X_j and the groups Γ_j in virtue of Theorem 3.1; indeed, the spaces X_j will split a Euclidean factor and the groups Γ_j will have a commensurated non-trivial, abelian subgroup. On the other hand, the condition that Γ virtually splits a free abelian factor of positive rank is equivalent to have a virtually normal, non trivial, free abelian subgroup, by [CM19, Theorem 2]; so, these groups Γ_j do not split.

The idea behind the example is very simple.
We start as in [LM21] from the lattices $\mathcal{L}_+, \mathcal{L}_-$ of $\mathbb{Z}^2 = \langle a, b \mid [a, b] = 1\rangle$ respectively generated by $\mathcal{B}_+ = \{a^2b^{-1}, ab^2\}$ and by $\mathcal{B}_- = \{a^2b, a^{-1}b^2\}$, and define the (torsion-free) HNN-extension

$$\Gamma = \langle a, b, t \mid [a, b] = 1,\ ta^2b^{-1}t^{-1} = a^2b,\ tab^2t^{-1} = a^{-1}b^2\rangle$$

that is $\Gamma = \mathbb{Z}^2 * \varphi$, where $\varphi : \mathcal{L}_+ \to \mathcal{L}_-$ is given by $\varphi(a^2b^{-1}) = a^2b$ and $\varphi(ab^2) = a^{-1}b^2$. We denote by T its Bass-Serre tree: this is a locally finite tree, since $[\mathbb{Z}^2 : \mathcal{L}_\pm] = 5$. The group Γ acts naturally on T and on $\mathbb{R}^2$, with

a, b acting on the latter as orthogonal translations of length 1 and t as a rotation around the origin O of angle $\arccos(\frac{3}{5})$.
Let us consider the diagonal action of Γ on the (proper, geodesically complete) CAT(0)-space $X := T \times \mathbb{R}^2$, which is $(P_0, 1)$-packed for some $P_0 > 0$. First notice that this action is faithful. In fact, the pointwise stabilizer of the axis of t on T is $\bigcap_{n\in\mathbb{Z}} t^n \mathbb{Z}^2 t^{-n}$, which is trivial. Indeed, suppose that $w \in \bigcap_{n\in\mathbb{Z}} t^n \mathbb{Z}^2 t^{-n}$, that is $w = t^n z_n t^{-n}$ for some sequence $(z_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^2$. Then $d_{\mathbb{R}^2}(wO, O) = d_{\mathbb{R}^2}(t^n z_n O, O) = d_{\mathbb{R}^2}(z_n O, O)$ for every n, as t fixes O. By discreteness of the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ we can assume that $z_n = z \in \mathbb{Z}^2$ for infinitely many n's; therefore, there exist integers $n \neq m$ such that $w = t^n z t^{-n} = t^m z t^{-m}$, i.e. $z = t^k z t^{-k}$ for some $k \neq 0$. This implies that $zO = t^k zO$, but since t acts as a rotation by an angle which is an irrational multiple of π, then t^k does not fix any point different from O. It follows that z, and in turn w, is necessarily trivial.
Moreover, it is straightforward to check that the action of Γ on X is discrete, while both the actions of Γ on the two factors are not.
Finally, the action of Γ on X is nonsingular (since Γ is torsion-free) and 1-cocompact, since it is on both factors.
It remains to check that Γ has no virtually normal, nontrivial, abelian subgroups. Suppose Γ virtually normalizes an abelian subgroup A of rank $k \geq 1$. By [CM19, Theorem 2.(ii)] the space $X = \mathbb{R}^2 \times T$ splits as $Y \times \mathbb{R}^k$ and a finite index subgroup Γ' of Γ splits as a product $\Gamma_Y \times \mathbb{Z}^k$, where Γ_Y acts discretely on Y and $\mathbb{Z}^k$ acts as a lattice on $\mathbb{R}^k$. By [FL08, Theorem 1.1] we deduce that either $Y = T$ (if $k = 2$) or $Y = \mathbb{R} \times T$ (if $k = 1$). The first case has to be excluded since in that case the projection of Γ on $\mathbb{R}^2$ would be discrete, containing the discrete group Γ_Y as finite index subgroup. In the second case we would find a finite index subgroup of the projection of Γ on $\mathbb{R}^2$ that preserves a line, which is clearly not possible.
We now take $X_j = \frac{1}{j} \cdot \mathbb{R}^2 \times T$ and define $\Gamma_j := \Gamma$ with its natural action on X_j induced by the action of Γ on X. The sequence of groups Γ_j acting on X_j clearly satisfies the conditions (a) and (b).

The behaviour of this sequence will be further studied in Section 4.4, Example 4.13.

3.3. **Group splitting: when the isometry group is Lie.**

The purpose of this section is to prove that a uniform lattice Γ of a CAT(0)-space X with a nontrivial Euclidean factor $\mathbb{R}^k$ virtually splits a free abelian subgroup of rank k, provided that $\mathrm{Isom}(X)$ is a Lie group. By Proposition 2.4, this holds, in particular, for all CAT(0)-homology manifolds.
The group splitting stems almost immediately from [CM19]. However, for our convergence results, we need a more precise result, that is to bound the index in Γ of the subgroup $\check{\Gamma}$ which splits $\mathbb{Z}^k$.

Theorem 3.7. *Given P_0, r_0, D_0, there exist I_0 and $\sigma_0^* > 0$ (the same as in Proposition 3.2), only depending on P_0, r_0, D_0, such that the following holds. Let X be a proper, geodesically complete, (P_0, r_0)-packed,* CAT(0)*-space whose isometry group is a Lie group, and let $X = Y \times \mathbb{R}^n$ be the splitting of the maximal Euclidean factor of X. Then for every discrete, D_0-cocompact group $\Gamma < \mathrm{Isom}(X)$ we have:*

(i) Γ *has a normal, finite index subgroup* $\check{\Gamma}$ *which splits as* $\check{\Gamma} = \check{\Gamma}_Y \times \mathbb{Z}^n$*;*
(ii) *the projections of* Γ *on* $\mathrm{Isom}(Y)$ *and* $\mathrm{Isom}(\mathbb{R}^n)$ *are discrete, as well as the factors* $\check{\Gamma}_Y$ *and* $\mathbb{Z}^n$ *of* $\check{\Gamma}$*;*
(iii) $\mathrm{sys}^\diamond(\Gamma_Y, Y) \geq \sigma_0^*$.

Moreover if Γ *is nonsingular then* $\check{\Gamma}$ *can be chosen of index* $[\Gamma : \check{\Gamma}] \leq I_0$ *and* $\check{\Gamma}_Y$ *is nonsingular. As a result, if* Γ *is nonsingular,* $M = \Gamma\backslash X$ *is the quotient by a finite group of isometries* $\Lambda \cong \Gamma/\check{\Gamma}$ *with* $|\Lambda| \leq I_0$ *of a space* $\check{M}$ *which splits isometrically as* $\check{N} \times \mathbb{T}^n$*, where* $\check{N} = \check{\Gamma}_Y\backslash Y$ *and* $\mathbb{T}^n$ *is a flat* n*-dimensional torus.*

We also say that Γ and M *virtually split* as $\check{\Gamma}_Y \times \mathbb{Z}^n$ and $\check{N} \times \mathbb{T}^n$, respectively.

Proof. By Proposition 2.3, the space X splits isometrically as $M \times N \times \mathbb{R}^n$ and $\mathrm{Isom}(X) \cong \mathcal{S} \times \mathcal{D} \times \mathcal{E}_n$ where $\mathcal{S} \cong \mathrm{Isom}(M)$ is a semi-simple Lie group with trivial center and no compact factors, $\mathcal{D} \cong \mathrm{Isom}(N)$ is totally disconnected and $\mathcal{E}_n \cong \mathrm{Isom}(\mathbb{R}^n)$. Since $\mathrm{Isom}(X)$ is assumed to be a Lie group then $\mathcal{D}$ must be discrete. Let $\Gamma_{\mathcal{S}}, \Gamma_{\mathcal{D}}$ and $\Gamma_{\mathcal{E}_n}$ be the projections of Γ on $\mathcal{S}, \mathcal{D}$ and $\mathcal{E}_n$ respectively. By [CM19, Theorem 2(i)] there exists a free abelian subgroup A of Γ of rank n which is commensurated in Γ, and its projection $A_{\mathcal{E}_n}$ on $\mathcal{E}_n$ is a crystallographic group. Arguing as in the proof of that theorem, we also deduce that the image $A_{\mathcal{S}}$ of A on $\mathcal{S}$ is finite by Borel density. On the other hand, the projection $A_{\mathcal{D}}$ of A on $\mathcal{D}$ is an abelian, commensurated subgroup of $\Gamma_{\mathcal{D}}$. Since $\Gamma_{\mathcal{D}}$ acts cocompactly, hence minimally, on the factor N, and every irreducible factor of N is not Euclidean, Proposition 4 of [CM19] implies that $A_{\mathcal{D}}$ either fixes a point or acts minimally on $\mathcal{D}$; but an abelian group cannot act minimally on a proper, non-trivial, CAT(0)-space without Euclidean factors, therefore $A_{\mathcal{D}}$ fixes a point in N. This implies that $A_{\mathcal{D}}$ is precompact, therefore finite, since $\mathcal{D}$ is discrete.
Let now $\{a_1, \ldots, a_n\}$ be a generating set of A. For every $i = 1, \ldots, n$ we write $a_i = (s_i, d_i, \tau_i) \in \mathcal{S} \times \mathcal{D} \times \mathcal{E}_n$. The discussion above implies that there exists $K \in \mathbb{N}^*$ such that $a_i^K = (\mathrm{id}, \mathrm{id}, \tau_i^K)$ for all i, where τ_i is a translation of $\mathbb{R}^n$. Therefore the group

$$B = \{(\mathrm{id}, \mathrm{id}, \tau) \in \Gamma \text{ s.t. } \tau \in \mathrm{Transl}(\mathbb{R}^n)\}$$

is a normal, free abelian subgroup of rank n of Γ. It then follows by [CM19, Theorem 2.(ii)] that Γ has a normal, finite index subgroup $\check{\Gamma}$ which splits as $\check{\Gamma} = \check{\Gamma}_Y \times \mathbb{Z}^n$, where $\check{\Gamma}_Y$ is a discrete group of $Y := M \times N$, and $\mathbb{Z}^n$ acts as a lattice on $\mathbb{R}^n$. It is then immediate that also the projections Γ_Y and $\Gamma_{\mathbb{R}^n}$ of Γ act discretely on the respective factors. This proves (i) and (ii).
Moreover, Y is again a proper, geodesically complete, (P_0, r_0)-packed, CAT(0)-space and Γ_Y acts D_0-cocompactly on Y. Since Y has no Euclidean factors, Theorem 3.1 implies that $\mathrm{sys}^\diamond(\Gamma_Y, Y) \geq \sigma_0^* = \sigma_{P_0, r_0, D_0}(\varepsilon_0)$, where ε_0 is the Margulis constant, which proves (iii).
Finally, by Theorem 2.10 the possible isomorphism types of nonsingular groups Γ satisfying the assumptions of Theorem 3.7 are finite, so the index of $\check{\Gamma}$ in Γ can be supposed uniformly bounded by a constant I_0 only depending on P_0, r_0 and D_0. The last assertion is consequence of (i) and of this bound. The fact that $\check{\Gamma}_Y$ is nonsingular if Γ is nonsingular is obvious. $\square$

We remark that we do not know examples of proper, geodesically complete, CAT(0)-space X whose isometry group is Lie, and possessing a singular, discrete group $\Gamma < \mathrm{Isom}(X)$.

4. Convergence and collapsing

We now move to the problem of convergence of CAT(0)-group actions. Namely, the basic objects we are interested are *isometry groups of pointed spaces* (which we will restrict to uniform lattices of CAT(0)-spaces, from Section 4.2 on), that is triples

$$(\Gamma, X, x)$$

where X is a proper metric space, $x \in X$ is a basepoint and Γ is a *closed* (but not necessarily discrete, nor cocompact) subgroup of $\mathrm{Isom}(X)$.
An *equivariant isometry* between isometric actions of pointed spaces (Γ, X, x) and (Λ, Y, y) is an isometry $F\colon X \to Y$ such that

– $F(x) = y$;
– $F_*\colon \mathrm{Isom}(X) \to \mathrm{Isom}(Y)$ defined by $F_*(g) = F \circ g \circ F^{-1}$ yields an isomorphism between Γ and Λ.

We recall the definition of equivariant Gromov-Hausdorff convergence of isometric actions, as introduced by Fukaya in [Fuk86, Ch.1] and [FY92, §3].

Definition 4.1. Let $(\Gamma, X, x), (\Lambda, Y, y)$ be isometry groups of pointed spaces. Given $\varepsilon > 0$, an *equivariant ε-approximation* from (Γ, X, x) to (Λ, Y, y) is a triple (f, ϕ, ψ) where:

- $f\colon B(x, \frac{1}{\varepsilon}) \to B(y, \frac{1}{\varepsilon})$ is a map such that $f(x) = y$ and satisfying
  - $|d(f(x_1), f(x_2)) - d(x_1, x_2)| < \varepsilon \ \forall x_1, x_2 \in B(x, \frac{1}{\varepsilon})$;
  - $\forall y_1 \in B(y, \frac{1}{\varepsilon}) \ \exists x_1 \in B(x, \frac{1}{\varepsilon})$ such that $d(f(x_1), y_1) < \varepsilon$;
- $\phi\colon \Sigma_{\frac{1}{\varepsilon}}(\Gamma, x) \to \Sigma_{\frac{1}{\varepsilon}}(\Lambda, y)$ is a map satisfying $d(f(gx_1), \phi(g)f(x_1)) < \varepsilon$ $\forall g \in \Sigma_{\frac{1}{\varepsilon}}(\Gamma, x)$ and $\forall x_1 \in B(x, \frac{1}{\varepsilon})$ such that $gx_1 \in B(x, \frac{1}{\varepsilon})$;
- $\psi\colon \Sigma_{\frac{1}{\varepsilon}}(\Lambda, y) \to \Sigma_{\frac{1}{\varepsilon}}(\Gamma, x)$ is a map satisfying $d(f(\psi(g)x_1), gf(x_1)) < \varepsilon$ $\forall g \in \Sigma_{\frac{1}{\varepsilon}}(\Lambda, y)$ and $\forall x_1 \in B(x, \frac{1}{\varepsilon})$ such that $\psi(g)x_1 \in B(x, \frac{1}{\varepsilon})$.

Definition 4.2. The *equivariant, pointed Gromov-Hausdorff distance* between two isometry groups of pointed spaces (Γ, X, x) and (Λ, Y, y)

$$d_{\text{eq-pGH}}((\Gamma, X, x), (\Lambda, Y, y))$$

is the infimum of ε for which there exists an equivariant ε-approximation from (Γ, X, x) to (Λ, Y, y) and an equivariant ε-approximation from (Λ, Y, y) to (Γ, X, x). [3] A sequence of isometry groups of pointed spaces (Γ_n, X_n, x_n) *converges in the equivariant, pointed Gromov-Hausdorff sense* to (Γ, X, x) if for every $\varepsilon > 0$ there exists $n_\varepsilon \geq 0$ such that if $n \geq n_\varepsilon$ then

$$d_{\text{eq-pGH}}((\Gamma_n, X_n, x_n), (\Gamma, X, x)) < \varepsilon.$$

In this case we write $(\Gamma_n, X_n, x_n) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma, X, x)$.

[3] The definition is slightly different from the one in [Fuk86] and it might not define a true distance. However the two definitions are comparable and in particular they define the same convergence.

For trivial groups $\Gamma, \Lambda, \Gamma_n, \Lambda_n$ we recover the usual definition of pointed, Gromov-Hausdorff distance and convergence (denoted d_{pGH} and $\xrightarrow[\mathrm{pGH}]{}$); moreover, for compact spaces with uniformily bounded diameter this reduces to the usual Gromov-Hausdorff distance/convergence d_{GH} (in the following, sometimes abbrev. GH-*distance*).

In the sequel we will follow an equivalent approach which uses ultralimits, which is more adapted to our purposes. We will present it in the next subsection, and then recall the relations with the usual notion of Gromov-Hausdorff convergence, referring to [Jan17] and [Cav22a]; see also [DK18], [CS21] for more details on ultralimits.
On the other hand, we will state our results in terms of Gromov-Hausdorff convergence, which is more commonly used in literature.

4.1. **Equivariant Gromov-Hausdorff convergence and ultralimits.**
A *non-principal ultrafilter* ω is a finitely additive measure on $\mathbb{N}$ such that $\omega(A) \in \{0, 1\}$ for every $A \subseteq \mathbb{N}$ and $\omega(A) = 0$ for every finite subset of $\mathbb{N}$. We will write ω-*a.s.* or *for* ω-*a.e.*(j) in the usual measure theoretic sense. Given a bounded sequence (a_j) of real numbers and a non-principal ultrafilter ω there exists a unique $a \in \mathbb{R}$ such that the set $\{j \in \mathbb{N} \text{ s.t. } |a_j - a| < \varepsilon\}$ has ω-measure 1 for every $\varepsilon > 0$ (cp. [DK18, Lemma 10.25]). The real number a is then called *the ultralimit of the sequence* a_j and is denoted by ω-$\lim a_j$.

A non-principal ultrafilter ω being fixed, for any sequence of pointed metric spaces (X_j, x_j) one can define the *ultralimit pointed metric space*

$$(X_\omega, x_\omega) = \omega\text{-}\lim(X_j, x_j)$$

– first, one says that a sequence (y_j), where $y_j \in X_j$ for every j, is *admissible* if there exists M such that $d(x_j, y_j) \leq M$ for ω-a.e.(j);
– then, one defines (X_ω, x_ω) as set of admissible sequences (y_j) modulo the relation $(y_j) \sim (y'_j)$ if and only if ω-$\lim d(y_j, y'_j) = 0$.
The point of X_ω defined by the class of the sequence (y_j) is denoted by $y_\omega = \omega$-$\lim y_j$. Finally, the formula

$$d(\omega\text{-}\lim y_j, \omega\text{-}\lim y'_j) := \omega\text{-}\lim d(y_j, y'_j)$$

defines a metric on X_ω which is called the *ultralimit distance* on X_ω.
Using a non-principal ultrafilter ω, one can also talk of limits of isometries and of isometry groups of pointed metric spaces. A sequence of isometries g_j of pointed metric spaces (X_j, x_j) is *admissible* if there exists $M \geq 0$ such that $d(g_j x_j, x_j) \leq M$ ω-a.s.. Any such sequence defines a limit isometry $g_\omega = \omega$-$\lim g_j$ of X_ω by the formula (see [DK18, Lemma 10.48]):

$$g_\omega y_\omega := \omega\text{-}\lim g_j y_j.$$

Finally, given sequence of isometry groups of pointed spaces (Γ_j, X_j, x_j) we can define the ultralimit group $\Gamma_\omega = \omega\text{-}\lim \Gamma_j$ as

$$\Gamma_\omega = \{\omega\text{-}\lim g_j \text{ s.t. } g_j \in \Gamma_j \text{ for } \omega\text{-a.e.}(j)\}$$

In particular the elements of Γ_ω are ultralimits of admissible sequences.

Lemma 4.3 ([Cav22a], Lemma 3.7)**.** *The composition of admissible sequences of isometries is an admissible sequence of isometries and the limit of the composition is the composition of the limits.*

Therefore, one has a well-defined composition law on Γ_ω: if $g_\omega = \omega\text{-}\lim g_j$ and $h_\omega = \omega\text{-}\lim h_j$ we set $g_\omega \circ h_\omega := \omega\text{-}\lim(g_j \circ h_j)$. With this operation Γ_ω becomes a true group of isometries of X_ω.
Notice that Γ_ω depends, in general, on the chosen basepoints $x_j \in X_j$ (since this choice selects the admissible isometries), and that if X_ω is proper then Γ_ω is always a *closed* subgroup of isometries of X_ω [Cav22a, Proposition 3.8].
In conclusion, a non-principal ultrafilter ω being given, for any sequence of isometry groups of pointed spaces (Γ_j, X_j, x_j) there exists an *ultralimit isometry group of a pointed space*

$$(\Gamma_\omega, X_\omega, x_\omega) = \omega\text{-}\lim(\Gamma_j, X_j, x_j).$$

We now turn to the equivalence between the ultralimit approach and the best known notion of Gromov-Hausdorff convergence. The relation between the Gromov-Hausdorff convergence (simple, pointed, and equivariant) and the convergence via ultralimits theory is resumed by the following result:

Proposition 4.4 ([Cav22a, Proposition 3.13 & Corollary 3.14])**.**
Let (Γ_j, X_j, x_j) be a sequence of isometry groups of pointed spaces:

(i) if $(\Gamma_j, X_j, x_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty, x_\infty)$, then $(\Gamma_\omega, X_\omega, x_\omega) \cong (\Gamma_\infty, X_\infty, x_\infty)$ for every non-principal ultrafilter ω;

(ii) reciprocally, if ω is a non-principal ultrafilter and the ultralimit (X_ω, x_ω) is proper then $(\Gamma_{j_k}, X_{j_k}, x_{j_k}) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\omega, X_\omega, x_\omega)$ for some subsequence $\{j_k\}$.

Moreover, if for every non-principal ultrafilter ω the ultralimit $(\Gamma_\omega, X_\omega, x_\omega)$ is equivariantly isometric to the same isometry group of pointed space (Γ, X, x), with X proper, then $(\Gamma_j, X_j, x_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma, X, x)$.

When restricted to the case $\Gamma_j = (1)$ for all j (resp. when the spaces X_j are compact), this statement also explains the standard relations between pointed Gromov-Hausdorff convergence (resp. simple Gromov-Hausdorff convergence) and ultralimit convergence, as studied for instance in [Jan17]. In particular, this shows that if $(X_j, x_j) \underset{\text{pGH}}{\longrightarrow} (X_\infty, x_\infty)$, then *any* family of isometry groups $\Gamma_j < \text{Isom}(X_j)$ (sub-)converges to some limit isometry group Γ_∞ of X_∞ with respect to the pointed, equivariant GH-convergence (since, for any choice of ω, the ultralimit $X_\omega = X_\infty$ is proper by (i), and then there exists $(j_k)_{k\in\mathbb{N}}$ such that $(\Gamma_{j_k}, X_{j_k}, x_{j_k}) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\omega, X_\omega, x_\omega)$ by (ii)).

A sequence (Γ_j, X_j, x_j) is called *D-cocompact* if each Γ_j is D-cocompact, and *uniformly cocompact* if it is D-cocompact for some D. It is not difficult to show that the ultralimit of a sequence of isometry groups of pointed spaces does not depend on the choice of the basepoints, if the isometry groups are uniformly cocompact:

Lemma 4.5. *Let (Γ_j, X_j, x_j) be a sequence of isometry groups of pointed spaces, all D-cocompact, and let $(x'_j)_{j\in\mathbb{N}}$ be a different sequence of basepoints. Then, for every non-principal ultrafilter ω, the ultralimit of (Γ_j, X_j, x_j) is equivariantly isometric to the ultralimit of (Γ_j, X_j, x'_j).*

Therefore, when considering the convergence of uniformily cocompact isometric actions, we will often omit the basepoint, if unnecessary for our arguments.
Finally, we remark that the equivariant pointed Gromov-Hausdorff convergence of a sequence of isometric actions with uniformly bounded codiameter implies the pointed Gromov-Hausdorff convergence of the quotients.

Lemma 4.6 ([Fuk86, Theorem 2.1])**.**
Let (Γ_j, X_j) be a sequence of D-cocompact isometry groups of pointed spaces. If $(\Gamma_j, X_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty)$ then $\Gamma_j\backslash X_j =: M_j \underset{\text{GH}}{\longrightarrow} M_\infty := \Gamma_\infty\backslash X_\infty$.

Of course, even if the groups Γ_j are all discrete, the group Γ_∞ may be not discrete and the structure of the quotient $\Gamma_\infty\backslash X_\infty$ is not clear at all.

4.2. Convergence of CAT(0)-lattices.

Our next goal is to explain what happens in our specific setting, where the X_j are all proper, geodesically complete CAT(0)-spaces, and the Γ_j are uniformly cocompact. With this purpose, we now precisely define the following classes of isometry groups we are interested in:

- $\text{CAT}_0(D_0)$: this is the class of isometry groups of spaces (Γ, X) where X is a proper, geodesically complete, CAT(0)-space, and Γ is a D_0-cocompact, lattice of X;
- $\text{CAT}_0(P_0, r_0, D_0)$: the subclass of $\text{CAT}_0(D_0)$ made of the lattices (Γ, X) such that X is, moreover, (P_0, r_0)-packed.

We will denote by $\mathcal{O}\text{-CAT}_0(D_0)$, $\mathcal{O}\text{-CAT}_0(P_0, r_0, D_0)$ the respective classes of quotients $M = \Gamma\backslash X$: these are compact CAT(0)-orbispaces, with (Γ, X) respectively in $\text{CAT}_0(D_0)$ and $\text{CAT}_0(P_0, r_0, D_0)$.

We will say that a sequence of proper metric spaces X_j (and, by extension, a sequence of isometric actions (Γ_j, X_j, x_j)) is *uniformly packed* if there exists (P_0, r_0) such that every X_j is (P_0, r_0)-packed. The proof of [Cav22a, Lemma 5.4] implies that

$$\text{CAT}_0(D_0) = \bigcup_{P_0, r_0} \text{CAT}_0(P_0, r_0, D_0).$$

More precisely, we have:

Proposition 4.7. *A subset $\mathcal{F} \subseteq \text{CAT}_0(D_0)$ is precompact (with respect to the equivariant pointed Gromov-Hausdorff convergence) if and only if there exist $P_0, r_0 > 0$ such that $\mathcal{F} \subseteq \text{CAT}_0(P_0, r_0, D_0)$.*

Proof. The "if" direction follows from the fact that the class of proper, geodesically complete CAT(0)-spaces which are (P_0, r_0)-packed is precompact[4], by [CS21, Theorem 6.4]: that is, every sequence of pointed spaces (X_j, x_j) in this class (sub)-convergences to some proper metric space (X_∞, x_∞) (which is still geodesically complete, CAT(0) and (P_0, r_0)-packed). The conclusion then follows from the discussion after Proposition 4.4, showing that then any sequence of isometry groups Γ_j of the pointed spaces

[4] and even compact, by Corollary 6.7 of [CS21], since the dimension of any such space is bounded by $n_0 = P_0/2$

(X_j, x_j) subconverges in the pointed, equivariant GH-convergence to some isometry group Γ_∞ of (X_∞, x_∞).
The other direction is provided by [Cav22a, Lemma 5.8], which shows that if (Γ_j, X_j) is a sequence of D_0-cocompact isometry groups of spaces which converges in the equivariant pointed Gromov-Hausdorff sense to some isometry group $(\Gamma_\infty, X_\infty)$, then the sequence (Γ_j, X_j) is uniformly packed. $\square$

Let us now consider a sequence of lattices (Γ_j, X_j) in $\mathrm{CAT}_0(D_0)$, such that $(\Gamma_j, X_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty)$. We will always denote by $M_j = \Gamma_j \backslash X_j$ and $M_\infty = \Gamma_\infty \backslash X_\infty$ the quotient spaces. Our goal is to describe the limit group Γ_∞ acting on X_∞ and the quotient $M_\infty = \Gamma_\infty \backslash X_\infty$.

Definition 4.8 (Standard setting of convergence)**.**
We say that we are in the *standard setting of convergence* when we have a sequence of lattices (Γ_j, X_j) in $\mathrm{CAT}_0(D_0)$ such that $(\Gamma_j, X_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty)$.
Notice that, in this setting:

(i) we can always assume that the (Γ_j, X_j)'s belong to some $\mathrm{CAT}_0(P_0, r_0, D_0)$, for some constants P_0, r_0 (by Proposition 4.7);
(ii) the limit space X_∞ is a proper, geodesically complete CAT(0)-space which is still (P_0, r_0)-packed (since this class of spaces is closed under ultralimits and compact with respect to the pointed Gromov-Hausdorff convergence, by [CS21, Theorem 6.1]);
(iii) the limit group Γ_∞ may well be non-discrete, but it is always a closed subgroup of $\mathrm{Isom}(X_\infty)$ (as recalled after Lemma 4.3, by [Cav22a]), in particular M_∞ is always a genuine metric space;
(iv) the quotients $M_j = \Gamma_j \backslash X_j \underset{\text{GH}}{\longrightarrow} M_\infty = \Gamma_\infty \backslash X_\infty$ (by Lemma 4.6).

A note of attention should be put on (iv): the convergence of the quotient orbispaces $M_j \underset{\text{GH}}{\longrightarrow} M_\infty$ does not imply in general [5] the convergence of the isometry groups $(\Gamma_j, X_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty)$ (as this is false even for constant sequences, by Remark 2.1).
Moreover, we will say, for short, that we are in the *nonsingular* setting of convergence (resp. in the *Lie setting*) if each group Γ_j is nonsingular (resp. all isometry groups $\mathrm{Isom}(X_j)$ are Lie).
Finally, a sequence of lattices $(\Gamma_j, X_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty)$ in the standard setting of convergence (equivalently, the sequence of orbispaces $M_j \underset{\text{GH}}{\longrightarrow} M_\infty$) will be called:

- *non-collapsing*, if $\limsup_{j\to+\infty} \mathrm{sys}^\diamond(\Gamma_j, X_j) > 0$;
- *collapsing*, if $\liminf_{j\to+\infty} \mathrm{sys}^\diamond(\Gamma_j, X_j) = 0$.

These cases are not apparently mutually exclusive, but indeed they are. Actually, the two cases are respectively equivalent to the the conditions that

[5] This is false even for sequences of Riemannian orbifolds with uniformly bounded sectional curvature, contrarily to what asserted in [Fuk86, Proposition 2.7] (where the non-collapsing assumption is forgotten). For instance, the product of a closed Riemannian manifold $M = \Gamma \backslash X$ with a torus $T_j = \Gamma_j \backslash \mathbb{R}^2$ converges to M by taking $\Gamma_j = \varepsilon_j \mathbb{Z}^2$ with $\varepsilon_j \to 0$, but $(\Gamma \times \Gamma_j, X \times \mathbb{R}^2)$ does not converge to (Γ, X).

the dimension of the limit $M_\infty = \Gamma_\infty \backslash X_\infty$ equals the dimension of the quotients M_j or decreases, as we will prove in Subsection 4.6 (Theorem 4.16). This is a very intuitive but not trivial result, and will follow by a careful analysis of both the collapsed and non-collapsed case.

4.3. **Convergence without collapsing.**

Theorem 4.9. *Assume* $(\Gamma_j, X_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty)$ *in the standard setting, without collapsing. Then:*

(i) the limit group Γ_∞ *is closed and totally disconnected;*
(ii) moreover, if all the the groups Γ_j *are nonsingular, then* Γ_∞ *is discrete and isomorphic to* Γ_j *for* j *big enough.*

Further, the limit of the orbispaces $M_j{=}\Gamma_j\backslash X_j$ *is isometric to* $M_\infty{=}\Gamma_\infty\backslash X_\infty$ *and, in case (ii),* M_j *is equivariantly homotopy equivalent to* M_∞ *for* $j \gg 0$.

Proof of Theorem 4.9.
We choose a subsequence $\{j_h\}$ be for which $\lim_{h\to+\infty} \operatorname{sys}^\diamond(\Gamma_{j_h}, X_{j_h}){>}0$ exists, and we choose a non-principal ultrafilter ω with $\omega(\{j_h\}) = 1$ (which is always possible, see [Jan17, Lemma 3.2]). Therefore we assume that

$$\operatorname{sys}^\diamond(\Gamma_j, X_j) \geq \varepsilon > 0 \text{ for } \omega\text{-a.e.}(j).$$

By Proposition 4.4, it is enough to show the thesis for the ultralimit group Γ_ω of X_ω, which coincide respectively with Γ_∞ and X_∞.
By the remark (i) after 4.8, we may assume that all the (Γ_j, X_j)'s belong to the class $\mathrm{CAT}_0(P_0, r_0, D_0)$, for suitable P_0, r_0, and that $\varepsilon \leq \varepsilon_0$.
Let then $\sigma = \sigma_{P_0,r_0,D_0}(\frac{\varepsilon}{4J_0})$ be the constant provided by Theorem 3.1.
By Lemma 4.5 the limit triple does not depend on the choice of the basepoints, so we may assume that the basepoints x_j are those provided by Corollary 3.4(ii), that is the almost stabilizers $\overline{\Gamma}_\sigma(x_j) < \Gamma_j$ fix the basepoint x_j for ω-a.e.(j). We have to show that $\Gamma_\omega^\circ = \{\mathrm{id}\}$.
It is clear that if $g_\omega = \omega\text{-}\lim g_j \in \overline{\Sigma}_{\frac{\sigma}{2}}(x_\omega)$ then $g_j \in \overline{\Sigma}_\sigma(x_j)$ for ω-a.e.(j). Therefore, $g_j x_j = x_j$ by our choice of x_j, so $g_\omega x_\omega = x_\omega$. Notice that the subset $\overline{\Sigma}_{\frac{\sigma}{2}}(x_\omega) \subset \Gamma_\omega$ has non-empty interior: in fact, calling $\varphi_{x_\omega} : \Gamma_\omega \to X_\omega$ the continuous map $\varphi_{x_\omega}(g_\omega) = g_\omega x_\omega$, the open set $U = \varphi_{x_\omega}^{-1}(B(x_\omega, \frac{\sigma}{2}))$ is contained in $\overline{\Sigma}_{\frac{\sigma}{2}}(x_\omega)$. Therefore $U \cap \Gamma_\omega^\circ$ is an open neighbourhood of the identity in Γ_ω°, and it generates Γ_ω°; it then follows that $\langle \Sigma_{\frac{\sigma}{2}}(x_\omega)\rangle \cap \Gamma_\omega^\circ = \Gamma_\omega^\circ$. In particular every element of Γ_ω° fixes x_ω. The set $\mathrm{Fix}(\Gamma_\omega^\circ)$ is convex, closed, non-empty and Γ_ω-invariant because Γ_ω° is a normal subgroup of Γ_ω. The action of Γ_ω on X_ω is clearly D_0-cocompact, hence minimal, so $\mathrm{Fix}(\Gamma_\omega^\circ) = X_\omega$ and $\Gamma_\omega^\circ = \{\mathrm{id}\}$. The facts that Γ_ω is closed and that M_∞ is the limit of the M_j's are just the remarks (iii)&(iv) after Definition 4.8.
To prove part (ii) first recall that by Theorem 2.9, up to replacing ε with a smaller number (and possibly changing again the basepoint by Lemma 4.5), we also have $\operatorname{sys}(\Gamma_j, x_j) \geq \varepsilon > 0$ for ω-a.e.(j). It is then easy to deduce that $\operatorname{sys}(\Gamma_\omega, x_\omega) \geq \varepsilon$ too (actually, if $g_\omega = \omega\text{-}\lim g_j$ is any element of Γ_ω, then $d(x_j, g_j x_j) \geq \varepsilon$ for ω-a.e.(j), hence $d(x_\omega, g_\omega x_\omega) \geq \varepsilon$). Since X_ω is proper, this is enough to show that Γ_ω is discrete, because the orbit $\Gamma_\omega x_\omega$ is discrete and the stabilizer of x_ω is trivial.

Finally, choose $D_0 < D < D_0+1$ such that $d(x_\omega, g_\omega x_\omega) \neq 2D$ for all $g_\omega \in \Gamma_\omega$, and consider the generating sets $\Sigma_j := \overline{\Sigma}_{2D}(x_j)$ and $\Sigma_\omega := \overline{\Sigma}_{2D}(x_\omega)$ for Γ_j and Γ_ω, respectively. It is classical that the spaces X_j and X_ω are respectively quasi-isometric to the Cayley graphs of $\mathrm{Cay}(\Gamma_j, \Sigma_j)$ and $\mathrm{Cay}(\Gamma_\omega, \Sigma_\omega)$ with the respective word metrics d_{Σ_j} and d_{Σ_ω}, namely

$$\frac{1}{2D} \cdot d(gx_j, g'x_j) \leq d_{\Sigma_j}(g, g') \leq \frac{1}{2(D - D_0)} d(gx_j, g'x_j) + 1$$

and analogously for d_{Σ_ω} and X_ω. From this, one can prove exactly as in [Cav22a, Proposition 7.6] that

$$\big(\mathrm{Cay}(\Gamma_\omega, \Sigma_\omega), \{\mathrm{id}\}\big) = \omega\text{-}\lim \big(\mathrm{Cay}(\Gamma_j, \Sigma_j), \{\mathrm{id}\}\big).$$

But, since by Theorem 2.10 we only have finitely many marked groups (Γ_j, Σ_j), this clearly implies that the (Γ_j, Σ_j)'s must be all isomorphic for j big enough, in particular isomorphic to $(\Gamma_\omega, \Sigma_\omega)$.
The second to last assertion follows from the fact that the class of proper, geodesically complete, (P_0, r_0)-packed, pointed CAT(0)-spaces (X_j, x_j) is closed under ultralimits (cp. [CS21, Theorem 6.1]) and from Lemma 4.6; clearly, the diameter of the limit $\Gamma_\omega \backslash X_\omega$ is still bounded by D_0.
Finally, by [CS23, Theorem 5.4], any isomorphism $\varphi_j : \Gamma_j \to \Gamma_\omega$ between nonsingular uniform CAT(0)-lattices can be upgraded to φ-equivariant homotopy equivalence $f_j : X_j \to X_\omega$, which proves that M_j and M_ω are equivariantly homotopy equivalent. □

4.4. Convergence with collapsing.

Theorem 4.10. *Assume* $(\Gamma_j, X_j) \underset{\text{eq-pGH}}{\longrightarrow} (\Gamma_\infty, X_\infty)$ *in the standard setting, with collapsing, and let* Γ_∞° *be the identity component of* Γ_∞*. Then:*
(i) X_∞ *splits isometrically and* Γ_∞*-invariantly as* $X'_\infty \times \mathbb{R}^\ell$*, for some* $\ell \geq 1$*;*
(ii) $\Gamma_\infty^\circ = \{\mathrm{id}\} \times \mathrm{Transl}(\mathbb{R}^\ell)$*, and* $X'_\infty = \Gamma_\infty^\circ \backslash X_\infty$*.*
The space X'_∞ *is proper, geodesically complete,* (P_0, r_0)*-packed and* CAT(0)*. The orbispaces* $M_j = \Gamma_j \backslash X_j$ *converge to* $M_\infty = \Gamma_\infty \backslash X_\infty$*, which is isometric to the quotient of* X'_∞ *by the closed, totally disconnected group* $\Gamma'_\infty := \Gamma_\infty / \Gamma_\infty^\circ$*.*

Recall that all the spaces under consideration, being (P_0, r_0) packed, have dimension $\leq n_0 = P_0/2$, by Proposition 2.5. Also, recall the constant J_0 introduced in Section 3.1, which bounds the index of the canonical lattice $\mathcal{L}(G)$ in any crystallographic group G of $\mathbb{R}^k$, for $k \leq n_0$.
To prove Theorem 4.10, we will use the following general fact, whose proof is straightforward and left to the reader:

Lemma 4.11. *Let* $(\Gamma_j, X_j, x_j), (\Gamma'_j, X'_j, x'_j,)$ *be two sequences as above,* ω *be a non-principal ultrafilter. Then the ultralimit of* $(\Gamma_j \times \Gamma'_j, X_j \times X'_j, (x_j, x'_j))$ *is equivariantly isometric to* $(\Gamma_\omega \times \Gamma'_\omega, X_\omega \times X'_\omega, (x_\omega, x'_\omega))$*.*

We will also need the following, whose proof is included for completeness:

Lemma 4.12. *Let* A *be a group of translations of* $\mathbb{R}^k$*. Then* $A \cong \mathbb{R}^\ell \times \mathbb{Z}^d$ *with* $\ell + d \leq k$*. Moreover, there is a corresponding* A*-invariant metric factorization of* $\mathbb{R}^k$ *as* $\mathbb{R}^\ell \times \mathbb{R}^{k-\ell}$*, such that the connected component* $A^\circ \cong \mathbb{R}^\ell$ *can be identified with* $\mathrm{Transl}(\mathbb{R}^\ell)$*.*

Proof. Let τ_v denote the translation of $\mathbb{R}^k$ by a vector v. Consider the connected component A° of the identity of A: it is an abelian, connected group of translations. The group A° acts transitively on the the vector subspace $V = \{v \in \mathbb{R}^k \text{ s.t. } \tau_v \in A^\circ\}$; moreover, as A° acts by translations, the stabilizer of any point v is trivial, therefore A° can be identified with $V \cong \mathbb{R}^\ell$. We can split $\mathbb{R}^k$ isometrically as $V \oplus V^\perp$, where $V^\perp$ is the orthogonal complement of V. Since A acts by translations, each isometry $a \in A$ can be decomposed as (a_1, a_2), with $a_1 \in V$ and $a_2 \in V^\perp$. Notice that the isometry $(\mathrm{id}, a_2) \in A$ because the action of A° on V is transitive; so, if $(a_1, a_2) \in A$, also the isometry (a_1, id) belongs to A. In other words, $A = A^\circ \times A^\perp$, where $A^\perp$ is the projection of A on $\mathrm{Isom}(V^\perp)$, Notice that $A^\perp = A/A^\circ$ is totally disconnected, hence discrete and free abelian, so it is isomorphic to some $\mathbb{Z}^d$ with $d \leq k - \ell$, acting on the factor $\mathbb{R}^{k-\ell}$. □

Proof of Theorem 4.10.
As we did in the proof of Theorem 4.9 we choose a non-principal ultrafilter ω such that $\omega\text{-}\lim \mathrm{sys}^\diamond(\Gamma_j, X_j) = 0$, and by Proposition 4.4 it is enough to show the thesis for the ultralimit $(\Gamma_\omega, X_\omega) = (\Gamma_\infty, X_\infty)$.
Again, by remark (i) after 4.8, we may assume that all the (Γ_j, X_j)'s belong to $\mathrm{CAT}_0(P_0, r_0, D_0)$, for some constants P_0, r_0.
By our choice of the ultrafilter ω, we have $\mathrm{sys}^\diamond(\Gamma_j, X_j) \leq \sigma_0^*$ for ω-a.e.(j), where $\sigma_0^* = \sigma_{P_0,r_0,D_0}(\varepsilon_0)$ is the constant provided by Theorem 3.1 for $\varepsilon = \varepsilon_0$, the Margulis constant; so X_j splits as $Y_j \times \mathbb{R}^{k_j}$ with $k_j \geq 1$ for ω-a.e.(j). By Lemma 4.5 the limit does not depend on the choice of the basepoints, so we can assume that the basepoints are the points $x_j = (y_j, v_j)$ provided by Proposition 3.2, that is such that $\overline{\Gamma}_{\sigma_0^*}(x_j)$ preserves the slice $\{y_j\} \times \mathbb{R}^{k_j}$. Moreover, if $k_\omega = \omega\text{-}\lim k_j$, we have $k_j = k_\omega$ for ω-a.e.(j) and $k_\omega \leq n_0$.
Now, choose a positive $\varepsilon \leq \frac{1}{2} \min\left\{\sigma_0^*, \sqrt{2\sin\left(\frac{\pi}{J_0}\right)}\right\}$ Then, by Lemma 2.8, we deduce that every element g of a crystallographic group of $\mathbb{R}^k$, for $k \leq n_0$, moving every point of $B_{\mathbb{R}^k}(O, \frac{1}{2\varepsilon})$ less than 2ε is a translation. We consider the open subset of Γ_ω defined by

$$U(x_\omega, \varepsilon) := \left\{g_\omega \in \Gamma_\omega \text{ s.t. } d(y_\omega, g_\omega y_\omega) < \varepsilon \text{ for all } y_\omega \in B_{X_\omega}(x_\omega, 1/\varepsilon)\right\}.$$

Since $\langle U(x_\omega, \varepsilon)\rangle$ is an open subgroup of Γ_ω containing the identity, we have $\langle U(x_\omega, \varepsilon)\rangle \cap \Gamma_\omega^\circ = \Gamma_\omega^\circ$, for every $\varepsilon > 0$. Remark that if $g_\omega = \omega\text{-}\lim g_j$ is in $U(x_\omega, \varepsilon)$, then for ω-a.e.(j) the isometry g_j belongs to

$$U(x_j, 2\varepsilon) := \left\{g_j \in \Gamma_j \text{ s.t. } d(y_j, g_j y_j) < 2\varepsilon \text{ for all } y_j \in B_{X_j}(x_j, 1/(2\varepsilon))\right\}.$$

Every element $g_j \in U(x_j, 2\varepsilon)$ belongs to $\overline{\Gamma}_{\sigma_0^*}(x_j)$ since $2\varepsilon \leq \sigma_0^*$, therefore it acts on $X_j = Y_j \times \mathbb{R}^{k_\omega}$ as $g_j = (g_j', g_j'')$, where g_j' fixes y_j (because of our choice of basepoints); moreover, g_j'' is a global translation of $\mathbb{R}^{k_\omega}$, since it moves the points of $B_{\mathbb{R}^{k_\omega}}(O, \frac{1}{2\varepsilon})$ less than 2ε. An application of Lemma 4.11 says that also X_ω splits isometrically as $Y_\omega \times \mathbb{R}^{k_\omega}$, where Y_ω is the ultralimit of the (Y_j, y_j)'s. Moreover clearly Γ_ω preserves the product decomposition. In particular every element of $g_\omega \in \Gamma_\omega^\circ$ can be written as a product $u_\omega(1) \cdots u_\omega(n)$, with each $u_\omega(k)$ in $U(x_\omega, \varepsilon)$; as $u_\omega(k) = \omega\text{-}\lim u_j(k)$ with $u_j(k) = (u_j(k)', u_j(k)'') \in U(x_j, 2\varepsilon)$, where $u_j(k)'$

fixes y_j and $u_j(k)''$ is a translation, it follows that also g_ω can be written as $(g'_\omega, g''_\omega) \in \mathrm{Isom}(Y_\omega) \times \mathrm{Isom}(\mathbb{R}^{k_\omega})$ where $g'_\omega y_\omega = y_\omega$ and g''_ω is a global translation (being the ultralimit of Euclidean translations).
Let us call $p\colon \Gamma_\omega \to \mathrm{Isom}(Y_\omega)$ the projection map. The group $p(\Gamma^\circ_\omega)$ is normal in $p(\Gamma_\omega)$. The set of fixed points $\mathrm{Fix}(p(\Gamma^\circ_\omega))$ is closed, convex, non-empty and $p(\Gamma_\omega)$-invariant (since $p(\Gamma^\circ_\omega)$ is normal in $p(\Gamma_\omega)$). Since Γ_ω is clearly D_0-cocompact, so it is $p(\Gamma_\omega)$. Then the action of $p(\Gamma_\omega)$ on Y_ω is minimal (cp. [CM09b, Lemma 3.13]) which implies that $\mathrm{Fix}(p(\Gamma^\circ_\omega)) = Y_\omega$, that is $p(\Gamma^\circ_\omega) = \{\mathrm{id}\}$. We conclude that Γ°_ω is a connected subgroup of $\{\mathrm{id}\} \times \mathrm{Transl}(\mathbb{R}^{k_\omega})$.
By Lemma 4.12, $\mathbb{R}^{k_\omega}$ splits isometrically as $\mathbb{R}^\ell \times \mathbb{R}^{k_\omega - \ell}$ and Γ°_ω can be identified with the subgroup of translations of the factor $\mathbb{R}^\ell$, for some $\ell \leq k_\omega$. Setting $X'_\omega := (Y_\omega \times \mathbb{R}^{k_\omega-\ell})$, this is still a proper, geodesically complete, (P_0, r_0)-packed, CAT(0)-space, and clearly $X'_\omega = \Gamma^\circ_\omega \backslash X_\omega$. Notice that, since Γ°_ω is normal in Γ_ω, then the splitting $X_\omega = X'_\omega \times \mathbb{R}^\ell$ is Γ_ω-invariant.
Let us now show that Γ°_ω is non-trivial, hence $\ell \geq 1$. Actually, if Γ°_ω was trivial then Γ_ω would be totally disconnected, and by [Cap09, Corollary 3.3], we would have $\mathrm{sys}^\diamond(\Gamma_\omega, X_\omega) > 0$. However, we are able to contruct hyperbolic isometries of Γ_ω with arbitrarily small translation length, which will prove that Γ°_ω is non-trivial. Indeed, fix any $\lambda > 0$ and an error $\delta > 0$. By the collapsing assumption we can find hyperbolic isometries $g_j \in \Gamma_j$ with $\ell(g_j) \leq \delta$, for ω-a.e.(j). By D_0-cocompactness, up to conjugating g_j we can suppose g_j has an axis at distance at most D_0 from x_j. Take a power m_j of g_j such that $\lambda < \ell(g_j^{m_j}) \leq \lambda + \delta$. Then, the sequence $(g_j^{m_j})$ is admissible and defines a hyperbolic element of Γ_ω whose translation length is between λ and $\lambda + \delta$. By the arbitrariness of λ and δ we conclude.
Finally, the quotients $M_j = \Gamma_j \backslash X_j$ converge to $M_\infty = \Gamma_\infty \backslash X_\infty = \Gamma_\omega \backslash X_\omega$, by the remark (iv) after 4.8 and $\Gamma_\omega \backslash X_\omega = (\Gamma_\omega / \Gamma^\circ_\omega) \backslash (\Gamma^\circ_\omega \backslash X_\omega) = \Gamma'_\omega \backslash X'_\omega$.
It is easy to see that the totally disconnected group $\Gamma_\omega / \Gamma^\circ_\omega$ is closed as a subgroup of $\mathrm{Isom}(X'_\omega)$. □

Remark 4.13. In full generality, the conclusion of Theorem 4.10 cannot be improved saying that the totally disconnected group $\Gamma_\infty / \Gamma^\circ_\infty$ is discrete.
Indeed, let (X_j, Γ_j) be the sequence considered in the Example 3.6, where none of the groups Γ_j have non-trivial, abelian, virtually normal subgroups. The sequence converges in the pointed equivariant Gromov-Hausdorff sense to a limit isometric action $(X_\infty, \Gamma_\infty)$, where $X_\infty = \mathbb{R}^2 \times T$ and the connected component of Γ_∞ is $\Gamma^\circ_\infty = \mathrm{Transl}(\mathbb{R}^2) \times \{\mathrm{id}\}$, so $X'_\infty = T$. The group $\Gamma_\infty / \Gamma^\circ_\infty$, acting on T, contains the infinite order, elliptic isometries induced by a and b, so it is not discrete. In particular Γ_∞ is not a Lie group.
On the other hand observe that, in general, if Γ_∞ is a Lie group, then one can use the same proof of [SRZ22, Lemma 58] to produce a non-trivial, nilpotent, normal subgroup of Γ_j, for j big enough. This means that the existence of non-trivial, nilpotent, normal subgroups in Γ_j for j big enough, and the corresponding splitting of the groups Γ_j, is strictly related to the structure of the limit group.

In Section 4.7 we will see a more precise relation between the metric limit M_∞ and the approximants M_j in the standard, nonsingular setting of convergence in the collapsing case, provided that the isometry groups $\mathrm{Isom}(X_j)$ are Lie.

4.5. **Riemannian limits.**
Recall that any Riemannian, geodesically complete CAT(0)-space X is a *Hadamard manifold*, i.e. a complete simply connected manifold with nonpositive sectional curvature. If the limit M_∞ of a sequence of CAT(0)-orbispaces is a Riemannian manifold, then the equivariant homotopy equivalence between M_j and M_∞ of Theorem 4.9 can be promoted to homeomorphism. For this, we need the following preliminary fact:

Theorem 4.14. *Let X be a n-dimensional Hadamard manifold. There exists $\varepsilon = \varepsilon(X, D_0) > 0$ such that if a proper, geodesically complete, D_0-cocompact* CAT(0)*-space X' satisfies*

$$d_{\text{p-GH}}((X, x), (X', x')) < \varepsilon$$

(for some choice of basepoints x, x') then X' is a topological manifold.

Proof. Reasoning by contradiction, take a sequence of (proper, geodesically complete) pointed CAT(0)-spaces (X_j, x_j) admitting D_0-cocompact lattices Γ_j, and converging in pointed GH-distance to a Hadamard manifold (X, x), and suppose that the X_j's are definitely not topological manifolds. Since $(X_j, x_j) \underset{\text{pGH}}{\longrightarrow} (X, x)$, then (up to a subsequence) we may assume that the lattices (Γ_j, X_j) converge to some D_0-cocompact (possibly non-discrete) isometry group Γ of X. Then, by Proposition 4.7, the lattices (Γ_j, X_j) belong definitely to $\mathrm{CAT}_0(P_0, r_0, D_0)$, for some P_0 and r_0.
Moreover, by D_0-cocompactness, the distance of x_j to the stratum $X_j^{n_j}$ of X_j of maximal dimension is uniformily bounded above; then, the dimension of X_j is exactly n for $j \gg 0$, by [CS21, Proposition 6.5].
Now, using the terminology of [LN19], every point of X is $(n, 0)$-strained.
Then, we set $\delta := \frac{1}{50n^2}$ and we claim that every point of $\overline{B}(x_j, 2D_0)$ is (n, δ)-strained if j is larger than some constant j_δ. Actually, suppose that we could find a sequence of points $(y_j)_{j \in \mathbb{N}}$, each one in $\overline{B}(x_j, 2D_0)$, not (n, δ)-strained. This sequence defines a point $y \in X$, which is $(n, 0)$-strained and in particular also (n, δ)-strained. Since all the balls $\overline{B}(x_j, 2D_0)$ are uniformly packed, we can use [LN19, Lemma 7.8] to deduce that also the points y_j would be (n, δ)-strained, a contradiction. So, the fact that all points of $\overline{B}(x_j, 2D_0)$ are (n, δ)-strained implies that $B(x_j, 2D_0)$ is locally biLipschitz homeomorphic to $\mathbb{R}^n$ by [LN19, Corollary 11.8]. In particular, $B(x_j, 2D_0)$ is a topological manifold, and by D_0-cocompactness we conclude that the whole X_j are topological manifolds, which gives a contradiction. □

Corollary 4.15. *Let (Γ, X) be a torsionless, uniform lattice of a Hadamard manifold. Then, there exists $\varepsilon = \varepsilon(\Gamma, X) > 0$ such that the following holds: for every other* CAT(0)*-lattice (Γ', X'), if $d_{\text{eq-pGH}}((\Gamma, X), (\Gamma', X')) < \varepsilon$ then $M' = \Gamma' \backslash X'$ is homeomorphic to $M = \Gamma \backslash X$.*

Proof. Let $D_0 = \mathrm{diam}(\Gamma \backslash X)$ and suppose, by contradiction, that there exists a a sequence of lattices (Γ_j, X_j) converging to (Γ, X), with $M_j = \Gamma_j \backslash X_j$ not homeomorphic to M. Then, $\mathrm{diam}(\Gamma_j \backslash X_j) \leq 2D_0$ definitely, and by Proposition 4.7 the (Γ_j, X_j)'s and their limit (Γ, X) belong to $\mathrm{CAT}_0(P_0, r_0, 2D_0)$, for some P_0 and r_0. Since the limit group is discrete by assumption, we deduce that we are in the noncollapsing setting, by Theorem 4.10. Moreover, by Corollary 4.14, we know that the X_j's are topological manifolds for $j \gg 0$, thus we are in the nonsingular setting of convergence. Then, by Theorem 4.9, we conclude that Γ_j is isomorphic to Γ for $j \gg 0$, in particular Γ_j is torsion-free as well. It follows that the M_j's are locally CAT(0)-spaces which converge in the GH-distance to the Riemannian manifold M. Since we are in the noncollapsing case, the injectivity radius of all the the spaces M_j is uniformily bounded away from zero: then, applying [Nag02, Theorem 1.8] (with δ that can be taken arbitrarily small since M is a Riemannian manifold), implies that M_j is homeomorphic to M for $j \gg 0$. □

4.6. **Limit dimension and limit Euclidean factor.**
If $X \in \mathrm{CAT}_0(P_0, r_0, D_0)$ then $\dim(X) \leq n_0 = P_0/2$, by Proposition 2.5. Moreover, if $(X_j, x_j) \underset{\mathrm{pGH}}{\longrightarrow} (X_\infty, x_\infty)$ then, by [CS21, Theorem 6.5]

$$\dim(X_\infty) \leq \liminf_{j \to +\infty} \dim(X_j).$$

The following theorem precisely relates the collapsing (as defined in 4.8) in the standard setting of convergence to the dimension of the limit quotients, and is a direct consequence of the convergence Theorems 4.9 & 4.10.

Theorem 4.16 (Characterization of collapsing)**.**
Let $(\Gamma_j, X_j) \in \mathrm{CAT}_0(D_0)$ be a sequence of lattices converging in the pointed equivariant GH-distance to $(\Gamma_\infty, X_\infty)$, and let $M_j = \Gamma_j \backslash X_j$, $M_\infty = \Gamma_\infty \backslash X_\infty$. Then:

(i) the sequence is non-collapsing if and only if $\mathrm{TD}(M_\infty) = \lim_{j \to +\infty} \mathrm{TD}(M_j)$;
(ii) the sequence is collapsing if and only if $\mathrm{TD}(M_\infty) < \lim_{j \to +\infty} \mathrm{TD}(M_j)$.

Moreover, in the above characterizations, the topological dimension TD can be replaced by the Hausdorff dimension HD.

We just need the following additional fact in order to establish the persistence of the dimension.

Lemma 4.17. *Let X be a proper, geodesically complete, CAT(0)-space and $\Gamma < \mathrm{Isom}(X)$ be closed, cocompact and totally disconnected. Then the quotient metric space $M = \Gamma \backslash X$ satisfies*

$$\mathrm{HD}(M) = \mathrm{HD}(X) = \mathrm{TD}(X) = \mathrm{TD}(M).$$

Proof. The quotient map $p \colon X \to M$ is open because it is the quotient under a group action. Moreover the fiber $p^{-1}([y])$ is discrete for all $[y] \in M$. Indeed, suppose there are points $g_j y$ accumulating to $x \in X$, with $g_j \in \Gamma$. By Ascoli-Arzelà we can extract a subsequence, which we denote again (g_j), converging to $g_\infty \in \Gamma$ with respect to the compact-open topology. Since X is geodesically complete, then the stabilizer $\mathrm{Stab}_\Gamma(y)$ is open (see [CM09b, Theorem 6.1]). In particular, $g_\infty^{-1} g_j y = y$ and we have $g_j y = g_\infty y$ for $j \gg 0$. This shows that the fibers are discrete. Then, by [Eng78, Theorem 1.12.7], we deduce that $\mathrm{TD}(X) = \mathrm{TD}(M)$. On the other hand, since the projection p is 1-Lipschitz, we have $\mathrm{HD}(M) \leq \mathrm{HD}(X)$. Moreover, $\mathrm{HD}(X) = \mathrm{TD}(X)$ because of [LN19, Theorem 1.1]. As $\mathrm{TD}(M) \leq \mathrm{HD}(M)$ always holds, we get

$$\mathrm{TD}(M) \leq \mathrm{HD}(M) \leq \mathrm{HD}(X) = \mathrm{TD}(X) = \mathrm{TD}(M). \quad \square$$

Proof of Theorem 4.16.
As the groups are all D_0-cocompact, by [CS21, Proposition 6.5] we have

$$\mathrm{TD}(X_\infty) = \mathrm{HD}(X_\infty) = \lim_{j\to+\infty} \mathrm{HD}(X_j) = \lim_{j\to+\infty} \mathrm{TD}(X_j),$$

so the limit exists. Moreover, $\mathrm{TD}(M_j) = \mathrm{HD}(M_j) = \mathrm{TD}(X_j) = \mathrm{HD}(X_j)$ for every j by Lemma 4.17. Therefore, we also have

$$\mathrm{TD}(X_\infty) = \mathrm{HD}(X_\infty) = \lim_{j\to+\infty} \mathrm{HD}(M_j) = \lim_{j\to+\infty} \mathrm{TD}(M_j).$$

Suppose first that the sequence is non-collapsing. In this case M_∞ is isometric to the quotient of X_∞ by a closed, cocompact, totally disconnected group by Theorem 4.9. Hence $\mathrm{TD}(M_\infty) = \mathrm{TD}(X_\infty) = \lim_{j\to+\infty} \mathrm{TD}(M_j)$ by Lemma 4.17, and similarly for the Hausdorff dimension.
Suppose now that the sequence is collapsing. Theorem 4.10 says that M_∞ is the quotient of a proper, geodesically complete, CAT(0)-space X'_∞ of dimension strictly smaller than the dimension of X_∞, by a closed, cocompact, totally disconnected group. Again Lemma 4.17 then implies that

$$\mathrm{TD}(M_\infty) = \mathrm{HD}(M_\infty) < \mathrm{HD}(X_\infty) = \mathrm{TD}(X_\infty) = \lim_{j\to+\infty} \mathrm{TD}(M_j)$$

and similarly for the Hausdorff dimension. □

Notice that since we proved that $\lim_{j\to+\infty} \mathrm{TD}(M_j)$ exists, Theorem 4.16 excludes to have sequences (Γ_j, X_j, x_j) converging with mixed behaviour (that is, such that along some subsequence the convergence is collapsed, and along other subsequences it is non-collapsed).

Another consequence of the analysis of convergence in the nonsingular setting, combined with the renormalization Theorem 3.5, is the persistence of the Euclidean factor at the limit:

Corollary 4.18 (Euclidean factor of limits)**.**
Let $(\Gamma_j, X_j) \subseteq \mathrm{CAT}_0(D_0)$ be a sequence of nonsingular lattices converging to $(\Gamma_\infty, X_\infty)$, and let k_j be the dimension of the Euclidean factor of X_j. Then $k_\infty := \lim_{j\to+\infty} k_j$ exists and equals the dimension of the Euclidean factor of X_∞.

Proof. By Proposition 4.4 it is sufficient to prove that for any ultrafilter ω the ultralimit $k_\omega = \omega\text{-}\lim k_j$ is the same and equals the dimension of the Euclidean factor of X_ω. By the renormalization Theorem 3.5 and by Theorem 2.9, up to passing to an action of the groups Γ_j on new spaces X'_j we may assume that the sequence is non-collapsing. Notice that this does not change the isometry type of the limit space X_ω because the spaces X'_j provided by the renormalization Theorem 3.5 are isometric to X_j. Then, we can apply the convergence Theorem 4.9 and deduce that the limit group Γ_ω is discrete (and cocompact). Now, by [CM19, Theorem 2], for any CAT(0)-space X possessing a discrete, cocompact group Γ, the Euclidean rank coincides with the maximum rank of a free abelian commensurated subgroup $A < \Gamma$. But this maximum rank is stable under ultralimits in the standard, nonsingular setting of convergence without collapsing, since by Theorem 4.9 the groups Γ_j are all isomorphic to Γ_ω for $j \gg 0$. □

In contrast, if the quotient spaces $M_j = \Gamma_j\backslash X_j$, $M_\infty = \Gamma_\infty\backslash X_\infty$ split (or virtually split) metrically maximal dimensional tori $\mathbb{T}^{k_j}$ and $\mathbb{T}^{k_\infty}$, the dimension k_∞ of the limit tori in the collapsing case is generally different to the limit of the k_j's.

4.7. **Collapsing when the isometry groups are Lie.**
We prove here the convergence Theorem C stated in the introduction, where we suppose to have a converging sequence of nonsingular CAT(0)-orbifolds $M_j = \Gamma\backslash X_j$ where all the groups $\mathrm{Isom}(X_j)$ are Lie. This includes, for instance, all collapsing sequences of nonpositively curved Riemannian manifolds or, more generally, of CAT(0)-homology manifolds.
For the proof, we need the following

Lemma 4.19. *Let (Γ_j, X_j, x_j) be a sequence of isometry groups of pointed spaces, and let ω be a non-principal ultrafilter. If $\check{\Gamma}_j$ are subgroups of Γ_j of index $[\Gamma_j : \check{\Gamma}_j] \leq I$ for ω-a.e.(j), then $[\Gamma_\omega : \check{\Gamma}_\omega] \leq I$.*

Proof. Suppose to find $g_{\omega,1}, \ldots, g_{\omega,I+1} \in \Gamma_\omega$ such that $g_{\omega,m}\Gamma'_\omega \neq g_{\omega,n}\Gamma'_\omega$ for all different $m, n \in \{1, \ldots, I+1\}$. We write $g_{\omega,m} = \omega\text{-}\lim g_{j,m}$. We can find different $m, n \in \{1, \ldots, I+1\}$ such that $g_{j,m}^{-1} g_{j,n} \in \Gamma'_j$ for ω-a.e.(j). By definition this implies that $g_{\omega,m}^{-1} g_{\omega,n} \in \Gamma'_\omega$, a contradiction. $\square$

Proof of Theorem C.
Let ω be any non-principal ultrafilter, and let $(\Gamma_\omega, X_\omega)$ be the ultralimit of the (Γ_j, X_j)'s. We know that $\omega\text{-}\lim \mathrm{sys}^\diamond(\Gamma_j, X_j) = 0$. Therefore, the first assertion about M_j follows from Theorem 3.7: for $j \gg 0$, the space X_j splits as $Y_j \times \mathbb{R}^{k_j}$, where $\mathbb{R}^{k_j}$ is the Euclidean factor, and there is a normal subgroup $\check{\Gamma}_j = \check{\Gamma}_{Y_j} \times \mathbb{Z}^{k_j} \triangleleft \Gamma_j$ of index $[\Gamma_j : \check{\Gamma}_j] \leq I_0$ such that $\check{M}_j = \check{\Gamma}_j\backslash X_j = \big(\check{\Gamma}_{Y_j}\backslash Y_j\big) \times \big(\mathbb{Z}^{k_j}\backslash\mathbb{R}^{k_j}\big) = \check{N}_j \times \mathbb{T}^{k_j}$ and $\mathrm{sys}^\diamond(\check{\Gamma}_{Y_j}, Y_j) \geq \sigma_0^*$.
Let $(\check{\Gamma}_\omega, X_\omega)$, $(\check{\Gamma}_{Y_\omega}, Y_\omega)$ and $(A_\omega, \mathbb{R}^{k_\omega})$ be the ultralimits of the groups $\check{\Gamma}_j$, $\check{\Gamma}_{Y_j}$ and $\mathbb{Z}^{k_j}$ acting respectively on X_j, Y_j and $\mathbb{R}^{k_j}$. By Lemma 4.11 we have corresponding splittings $X_\omega = Y_\omega \times \mathbb{R}^{k_\omega}$ and $\check{\Gamma}_\omega = \check{\Gamma}_{Y_\omega} \times A_\omega$, with $k_\omega = \omega\text{-}\lim k_j$; moreover, by Corollary 4.18, $k_\omega = k_\infty$ is the dimension of the Euclidean factor of X_ω.
The groups $\check{\Gamma}_{Y_j}$ act nonsingularly without collapsing on Y_j by Theorem 3.7, which implies that $\check{\Gamma}_{Y_\omega}$ acts discretely on Y_ω, by Theorem 4.9(ii). Therefore, the $\check{N}_j$'s converge without collapsing to a nonsingular orbispace $\check{N}_\omega = \check{\Gamma}_{Y_\omega}\backslash Y_\omega$. This proves (i).
Moreover, since $[\Gamma_j : \check{\Gamma}_j] \leq I_0$, Lemma 4.19 implies that $[\Gamma_\omega : \check{\Gamma}_\omega] \leq I_0$ too. Therefore the subgroup $\check{\Gamma}_\omega$ is open in Γ_ω and $\check{\Gamma}_\omega^\circ = \Gamma_\omega^\circ = \mathbb{R}^\ell$, where $\ell \geq 1$ is provided by Theorem 4.10. On the other hand, A_ω acts cocompactly by translations on $\mathbb{R}^{k_\omega}$, so Lemma 4.12 implies that A_ω is isomorphic to the group $\mathbb{Z}^{k_\omega - \ell'} \times \mathbb{R}^{\ell'}$ acting on $\mathbb{R}^{k_\omega} \cong \mathbb{R}^{k_\omega - \ell'} \times \mathbb{R}^{\ell'}$, for some $\ell' \geq 0$. As $\check{\Gamma}_{Y_\omega}$ acts discretely on Y_ω, it follows that $\check{\Gamma}_\omega^\circ = A_\omega^\circ = \mathbb{R}^{\ell'}$, therefore $\ell = \ell'$. Moreover, since the groups $\mathbb{Z}^{k_j}$ converge to $A_\omega = \mathbb{Z}^{k_\omega - \ell} \times \mathbb{R}^\ell$, and the factors act separately by translations on the factors $\mathbb{R}^{k_\omega - \ell}$ and $\mathbb{R}^\ell$ of $\mathbb{R}^{k_\omega}$, it is clear from Lemma 4.6 that the tori $\mathbb{T}^{k_j} = \mathbb{Z}^{k_j}\backslash\mathbb{R}^{k_j}$ converge to the limit torus $\mathbb{Z}^{k_\omega - \ell}\backslash\mathbb{R}^{k_\omega - \ell} = \mathbb{T}^{k_\omega - \ell}$. This proves (ii).

Now, the group $\check{\Gamma}'_\omega := \check{\Gamma}_\omega/\check{\Gamma}^\circ_\omega = \check{\Gamma}_{Y_\omega} \times \mathbb{Z}^{k_\omega-\ell}$ has index $\leq I_0$ in $\Gamma'_\omega = \Gamma_\omega/\Gamma^\circ_\omega$; and, since the first one is discrete, it follows that also Γ'_ω is discrete. By Theorem 4.10 it follows that the spaces M_j converge to $M_\omega = \Gamma'_\omega\backslash X'_\omega$, where $X'_\omega = \Gamma^\circ_\omega\backslash X_\omega$, hence M_ω is a nonsingular orbispace, proving (iii).
Finally, remark that since $\check{\Gamma}_j$ is normal in Γ_j for all j, then $\check{\Gamma}_\omega$ is normal in Γ_ω (cp. [CS24, Lemma 6.14.(iii)]), and in turns this implies that $\check{\Gamma}'_\omega \triangleleft \Gamma'_\omega$. Therefore, this space M_ω is the quotient of $\check{M}_\omega := \check{\Gamma}'_\omega\backslash X'_\omega$ by the group $\Lambda_\omega = \Gamma'_\omega/\check{\Gamma}'_\omega$ which has cardinality $\leq I_0$.
Moreover Y_ω has no Euclidean factors because of Corollary 4.18, so $\mathrm{sys}^\diamond(\check{\Gamma}_{Y_\omega}) \geq \sigma^*_0$ by Theorem 3.7. This proves (iv). □

As a result of the convergence theorems proved in this section, we obtain the compactness Corollary D for CAT(0)-homology orbifolds. Recall the class $\mathcal{HO}$-$\mathrm{CAT}_0(P_0, r_0, D_0)$ defined in the introduction, consisting of all the quotients $M = \Gamma\backslash X$ with (Γ, X) belonging to $\mathrm{CAT}_0(P_0, r_0, D_0)$, where X is supposed, in addition, to be a homology manifold.

Proof of Corollary D.
Let $M_j = \Gamma_j\backslash X_j$ be a sequence in $\mathcal{HO}$-$\mathrm{CAT}_0(P_0, r_0, D_0)$. Since the X_j are homology manifolds, then Γ_j is nonsingular and $\mathrm{Isom}(X_j)$ is a Lie group for every j, as recalled in Section 2.5 and by Proposition 2.4.
By [CS21, Theorem 6.1], the class $\mathrm{CAT}_0(P_0, r_0, D_0)$ is compact with respect to the Gromov-Hausdorff convergence. Moreover, if $(X_j, x_j) \underset{\mathrm{pGH}}{\longrightarrow} (X_\infty, x_\infty)$, then X_∞ is again a homology manifold, by [LN18, Lemma 3.3].
Now, by Theorem 4.16, the sequence (Γ_j, X_j) is either non-collapsing, or collapsing. In the first case, by Theorem 4.9 the quotients M_j converge to some space $M_\infty = \Gamma_\infty\backslash X_\infty$, for a discrete subgroup Γ_∞ of $\mathrm{Isom}(X_\infty)$, hence it belongs to $\mathcal{HO}$-$\mathrm{CAT}_0(P_0, r_0, D_0)$. In the second case, the M_j's converge to a lower dimensional limit space M_∞, which by Theorem C is isometric to the quotient of the space X'_∞ (a proper, geodesically complete, (P_0, r_0)-packed CAT(0)-space, by Theorem 4.10) by the discrete, D_0-cocompact group Γ'_∞. Moreover, if $X_\infty = Y_\infty \times \mathbb{R}^{k_\infty}$ is the splitting of the Euclidean factor, we have $X'_\infty = Y_\infty \times \mathbb{R}^{k_\infty-\ell}$ by the proof of Theorem 4.10 and by Corollary 4.18. By the Kunneth's formula it then follows that Y_∞, and in turn X'_∞, are homology manifolds; therefore, M_∞ still belongs to $\mathcal{HO}$-$\mathrm{CAT}_0(P_0, r_0, D_0)$. □

4.8. Isolation of Euclidean spaces and entropy rigidity.

We prove in this section the rigidity and pinching Corollaries E & F stated in the introduction.

Proof of Corollary E.
If the thesis is false, there exists a sequence of lattices $(\Gamma_j, X_j) \in \mathrm{CAT}_0(D_0)$ such that the X_j's converge to $\mathbb{R}^n$ in the pointed GH-distance, but with X_j not isometric to $\mathbb{R}^n$ for every j. By Propositions 4.4 and 4.7, there exist $P_0, r_0 > 0$ such that $(\Gamma_j, X_j) \in \mathrm{CAT}_0(P_0, r_0, D_0)$ for all j, and, up to extracting a subsequence, we may assume that the sequence (Γ_j, X_j) converges in the equivariant pointed GH-distance to $(\Gamma, \mathbb{R}^n)$ for some closed subgroup $\Gamma < \mathrm{Isom}(\mathbb{R}^n)$. By the renormalization procedure of Theorem 3.5, which does not change the isometry type of the spaces X_j and keeps the diameter of the quotients $\Gamma_j\backslash X_j$ bounded by Δ_0, we may assume that

the sequence is noncollapsing. Moreover, the spaces X_j are topological manifold for $j \gg 0$, by Corollary 4.14. Therefore we are in the nonsingular, non-collapsed setting of convergence, and we deduce from Theorem 4.9(ii) that the limit Γ is discrete, and Γ_j is isomorphic to Γ for $j \gg 0$. In particular Γ_j is virtually abelian of rank n. Then, [AB98, Corollary C] says that X_j is isometric to $\mathbb{R}^n$, leading to the contradiction. □

We conclude by proving Corollary F. The idea is that, if we assume to have a sequence of uniformly packed and uniformly cocompact CAT(0)-spaces with smaller and smaller entropy, we can pass to the limit and, by a a continuity argument, deduce that the entropy of the limit is zero; then we conclude again from [AB98, Corollary C] saying that a space with zero entropy has to be flat.
However, the continuity argument is delicate: it is known in some cases, under the assumption that the group actions are torsion-free (see [Rev08, Proposition 38] and [Cav22a]). Here we will use a continuity result (Proposition A.1) for the entropy of spaces admitting nonsingular lattices, possibly with torsion, which will be proved in the Appendix, bypassing the techniques of [Rev08] and working directly at the level of universal covers.

Proof of Corollary F.
Reasoning by contradiction, assume that there exists a sequence $(\Gamma_j, X_j) \in \mathrm{CAT}_0(P_0, r_0, D_0)$ such that $\mathrm{Ent}(X_j) \to 0$ but X_j is not flat for every j, and (up to a subsequence) let $(\Gamma_\infty, X_\infty)$ be the limit. After the renormalization of the X_j's given by Theorem 3.5, which does not affect the packing constants and keeps the diameters of $\Gamma_j \backslash X_j$ uniformly bounded by a constant $\Delta_0 = \Delta_0(P_0, D_0)$, we may assume that the sequence is noncollapsing. Moreover, by definition, the renormalization does not change the entropy of X_j either (as the renormalized spaces are isometric to X_j). We therefore are in the noncollapsing, nonsingular setting, so we deduce from Theorem 4.9 that the limit group Γ_∞ is still discrete, nonsingular and isomorphic to Γ_j for $j \gg 0$. Better, by Proposition 2.9 we can choose basepoints $x_j \in X_j$ such that $\mathrm{sys}(\Gamma_j, x_j)$ (not only the free-systoles) are all larger than some constant $s_0' = s_0'(P_0, r_0, \Delta_0) > 0$, and the same lower bound on the systole passes to the limit at a limit basepoint $x_\infty \in X_\infty$. We are now in position to apply Proposition A.1 to infer that $\mathrm{Ent}(X_\infty) = \lim_{j \to +\infty} \mathrm{Ent}(X_j) = 0$. The group Γ_∞ is amenable, because it has subexponential growth, and so it is Γ_j for $j \gg 0$. Therefore X_j, for $j \gg 0$, is isometric to some Euclidean space $\mathbb{R}^n$ by [AB98, Corollary C], which gives the contradiction. □

Appendix A. Continuity of entropy under GH-approximations

Proposition A.1. *There exists $\varepsilon = \varepsilon(s_0, D_0, \delta) > 0$ such that if Γ and Γ' are D_0-lattices of proper, geodesic, simply connected spaces X, X' respectively, with $\min\{\mathrm{sys}(\Gamma, x), \mathrm{sys}(\Gamma', x')\} \geq s_0$ and $d_{\text{eq-pGH}}((\Gamma, X, x), (\Gamma', X', x')) < \varepsilon$ then*

$$\left| \frac{\mathrm{Ent}(X)}{\mathrm{Ent}(X')} - 1 \right| \leq \delta. \tag{10}$$

We divide the proof in several steps. We will first prove that the groups Γ, Γ' have canonical generating sets which make them isomorphic as marked groups, and the marked isomorphism is given by the approximation morphisms. Then, we quantify the distortion of this isomorphism, showing that it is a $(1+\delta, C)$-quasi-isometry, which will imply (10).
The value of ε appearing in the statement can be taken equal to $\frac{1}{3L}$, where

$$L = \max\left\{8D_0, \frac{1}{s_0}, \frac{2}{\delta}\left(D_0 + \sqrt{D_0^2 + \frac{\delta}{3}}\right)\right\}.$$

Proof of Proposition A.1.
Let (f, ϕ, ψ) be an equivariant ε-approximation between (Γ, X, x) and (Γ', X', x') for $\varepsilon = \frac{1}{3L}$, with $f(x) = x'$ according to the Definitions 4.1&4.2. Recall the subset $\overline{\Sigma}_r(\Gamma, X, x)$ defined in (5), and set, for short,

$$\overline{\Sigma}_r = \overline{\Sigma}_r(\Gamma, X, x), \qquad \overline{\Sigma}'_r = \overline{\Sigma}_r(\Gamma', X', x').$$

Notice that, $\overline{\Sigma}_L$ and $\overline{\Sigma}'_L$ are generating sets for Γ and Γ' respectively, since $L > 2D_0$ (see for instance [Ser03, Appendix to §3]).

Lemma A.2. *The map $\phi : \overline{\Sigma}_{3L} \to \overline{\Sigma}'_{3L}$ is a bijection with inverse ψ, and sends the identity element of Γ in the identity of Γ'.*

Proof. To show the surjectivity, let $g' \in \overline{\Sigma}'_{3L}$. Then, setting $g = \psi(g')$, we have that $g \in \overline{\Sigma}_{3L}$, $d(f(gx), g'x') < \varepsilon$ and $d(f(gx), \phi(\psi(g'))x') < \varepsilon$, by Definition 4.1. Hence

$$d(\phi(g)x', g'x') < 2\varepsilon < s_0$$

which implies that $\phi(g) = g'$ since $\operatorname{sys}(\Gamma', x') \geq s_0$. Assume now that $g_1, g_2 \in \overline{\Sigma}_{3L}$ satisfy $\phi(g_1) = \phi(g_2) = g' \in \Sigma'_{3L}$. Then, by Definition 4.1

$$d(g_i x, \psi(g')x) < d(f(g_i x), f(\psi(g')x)) + \varepsilon, \text{ for } i = 1, 2.$$

On the other hand, by Definition 4.1, $d(f(g_i x), \phi(g_i)f(x)) = d(f(g_i x), g'x') < \varepsilon$ and $d(f(\psi(g')x), g'x') < \varepsilon$, therefore we deduce that

$$d(g_i x, \psi(g')x) \leq d(f(g_i x), g'x') + d(f(\psi(g')x), g'x') + \varepsilon < 3\varepsilon \leq s_0,$$

which implies that $g_1 = \psi(g') = g_2$ as $\operatorname{sys}(\Gamma, x) \geq s_0$. So, ϕ is a bijection with inverse ψ. Finally $d(f(e \cdot x), \phi(e)f(x)) = d(x', \phi(e)x') < \varepsilon < s_0$, therefore $\phi(e) = e$ necessarily. $\square$

Lemma A.3. *The maps ϕ, ψ induce isomorphisms $\tilde{\phi} : \Gamma \to \Gamma'$, $\tilde{\psi} : \Gamma' \to \Gamma$ inverse to each other (which coincide with ϕ and ψ on $\overline{\Sigma}_L$ and $\overline{\Sigma}'_L$).*

Proof. Let us first show that the map $\phi \colon \overline{\Sigma}_L \to \Gamma'$ satisfies

$$\phi(s_1 s_2 s_3) = \phi(s_1)\phi(s_2)\phi(s_3), \text{ for every } s_1, s_2, s_3 \in \overline{\Sigma}_L \tag{11}$$

and that the same holds for the map $\psi \colon \overline{\Sigma}'_L \to \Gamma$. Indeed, applying several times the Definition 4.1 we find

$$d(\phi(s_2 s_3)f(x), f(s_2 s_3 x)) < \varepsilon,$$
$$d(f(s_2 s_3 x), \phi(s_2)f(s_3 x)) < \varepsilon,$$
$$d(\phi(s_2)f(s_3 x), \phi(s_2)\phi(s_3)f(x)) = d(f(s_3 x), \phi(s_3)f(x)) < \varepsilon$$

and by the triangular inequality we get $d(\phi(s_2s_3)x', \phi(s_2)\phi(s_3)x') < 3\varepsilon \leq s_0$. Therefore $\phi(s_2s_3) = \phi(s_2)\phi(s_3)$, as $\mathrm{sys}(\Gamma', x') \geq s_0$. Arguing in a similar way we also deduce that $\phi(s_1s_2s_3) = \phi(s_1)\phi(s_2s_3) = \phi(s_1)\phi(s_2)\phi(s_3)$.
The proof for the map ψ is identical.
Let now $\tilde{\phi}\colon \mathbb{F}(\overline{\Sigma}_L) \to \Gamma'$ be the homomorphism from the free group with basis $\overline{\Sigma}_L$ to Γ' defined by extension of $s_i \mapsto \phi(s_i)$ for every $s_i \in \overline{\Sigma}_L$. Recall that by [Ser03, App., Ch.3], the group Γ admits a presentation $\Gamma = \langle \overline{\Sigma}_L | \mathcal{R} \rangle$ (that is, Γ is isomorphic to the quotient of the free group $\mathbb{F}(\overline{\Sigma}_L)$ by the normal closure of $\mathcal{R}$), where every relator in $\mathcal{R}$ has length at most 3 in the alphabet $\overline{\Sigma}_L$. Then, if $w = s_1s_2s_3 \in \mathcal{R}$, with each $s_i \in \overline{\Sigma}_L$, we deduce by (11) that

$$\tilde{\phi}(g) = \tilde{\phi}(s_1s_2s_3) = \phi(s_1)\phi(s_2)\phi(s_3) = \phi(s_1s_2s_3) = \phi(e)$$

which is the identity in Γ' by Lemma A.2. This shows that the map $\tilde{\phi}$ induces a homomorphism $\Gamma \to \Gamma'$, still denoted by $\tilde{\phi}$, which coincides with ϕ on $\overline{\Sigma}_L$. Similarly, ψ induces a homomorphism $\tilde{\psi}\colon \Gamma' \to \Gamma$ such that $\tilde{\psi} = \psi$ on $\overline{\Sigma}'_L$.
To conclude that $\tilde{\phi}$ is an isomorphism with inverse $\tilde{\psi}$ is enough to remark that they coincide with ϕ and ψ on the generating sets $\overline{\Sigma}_L$, $\overline{\Sigma}'_L$ respectively, and that ϕ and ψ are inverse to each other on these sets, by Lemma A.2. □

Lemma A.4. *If $d(x, gx) \geq L$ and $d(x', g'x') \geq L$ we have:*

$$d(x', \tilde{\phi}(g)x') \leq d(x, gx)(1 + \delta) + 2(D_0 + \varepsilon)$$
$$d(x, \tilde{\psi}(g)x) \leq d(x', g'x')(1 + \delta) + 2(D_0 + \varepsilon)$$

Proof. Let $d = d(x, gx) \geq L$, let $c = [x, gx]$ be a geodesic between x and gx, and set $n := \lceil \frac{2d}{L} \rceil$. Let us define $x_i := c(i\frac{L}{2})$ for $i = 0, \ldots, n-1$ and $x_n := gx$. For every $i = 0, \ldots, n$ we choose $g_i \in \Gamma$ such that $d(g_ix, x_i) \leq 2D_0$, with $g_0 = \mathrm{id}$ and $g_n = g$. We define $h_i = g_{i-1}^{-1}g_i$ for $i = 1, \ldots, n$ and we observe that $g = h_1 \cdots h_n$. Notice that each h_i belongs to $\overline{\Sigma}_L$ as

$$d(x, h_ix) \leq \frac{L}{2} + 4D_0 \leq L.$$

By definition of $\tilde{\phi}$ we then have $\tilde{\phi}(g) = \phi(h_1) \cdots \phi(h_n)$, and the usual application of Definition 4.1 yields

$$d(x', \phi(h_i)x') \leq d(f(x), f(h_ix)) + \varepsilon \leq d(x, h_ix) + 2\varepsilon$$

therefore

$$\begin{aligned}
d(x', \tilde{\phi}(g)x') &\leq d(x', \phi(h_1)x') + \ldots + d(x', \phi(h_n)x') \\
&\leq d(x, h_1x) + \ldots + d(x, h_nx) + 2n\varepsilon \\
&\leq d(x, x_1) + \ldots + d(x_{n-1}, x_n) + 2n(D_0 + \varepsilon) \\
&= d + 2n(D_0 + \varepsilon) = d\left(1 + \frac{2n}{d}(D_0 + \varepsilon)\right) \\
&\leq d(x, gx)\left(1 + \frac{4}{L}(D_0 + \varepsilon)\right) + 2(D_0 + \varepsilon) \\
&\leq d(x, gx)(1 + \delta) + 2(D_0 + \varepsilon).
\end{aligned}$$

since $n \leq \frac{2d}{L} + 1$, and $\frac{4}{L}(D_0 + \varepsilon) \leq \delta$ by our choice of L.
The proof for $\tilde{\psi}$ is the same. □

End of proof of Proposition A.1.
Set $B^*(x,R) = B(x,R) - \overline{B}(x,L)$, and observe that we can equivalently compute the entropy of X as

$$\mathrm{Ent}(X) = \lim_{R\to+\infty} \frac{1}{R} \log \# \left(\Gamma x \cap B^*(x,R)\right).$$

By Lemma A.4, for every element $g \in \Gamma$ such that $gx \in B^*(x,R)$ we have $d(x', \tilde{\phi}(g)x') \leq (1+\delta)R + 2(D_0+\varepsilon)$. Observe moreover that the points $\tilde{\phi}(g)x'$ are different for different g's in Γ. Therefore

$$\# \left(\Gamma x \cap B^*(x,R)\right) \leq \# \left(\Gamma' x' \cap B(x', (1+\delta)R + 2(D_0+\varepsilon))\right)$$

which yields

$$\mathrm{Ent}(X) \leq (1+\delta) \cdot \lim_{R\to+\infty} \frac{1}{R} \log \# \left(\Gamma' x' \cap B(x',R)\right) = (1+\delta)\cdot \mathrm{Ent}(X').$$

Reversing the roles of X and X' we deduce that $\mathrm{Ent}(X') \leq (1+\delta)\mathrm{Ent}(X)$, which proves (10) as $\frac{1}{1+\delta} > 1-\delta$. $\square$

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