The Gapeev-Shiryaev Conjecture

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The Gapeev-Shiryaev conjecture (originating in [4] and [5]) can be broadly stated as follows: Monotonicity of the signal-to-noise ratio implies monotonicity of the optimal stopping boundaries. The conjecture was originally formulated both within (i) sequential testing problems for diffusion processes (where one needs to decide which of the two drifts is being indirectly observed) and (ii) quickest detection problems for diffusion processes (where one needs to detect when the initial drift changes to a new drift). In this paper we present proofs of the Gapeev-Shiryaev conjecture both in (i) the sequential testing setting (under Lipschitz/Hölder coefficients of the underlying SDEs) and (ii) the quickest detection setting (under analytic coefficients of the underlying SDEs). The method of proof in the sequential testing setting relies upon a stochastic time change and pathwise comparison arguments. Both arguments break down in the quickest detection setting and get replaced by arguments arising from a stochastic maximum principle for hypoelliptic equations (satisfying Hörmander's condition) that is of independent interest. Verification of the Gapeev-Shiryaev conjecture establishes the fact that sequential testing and quickest detection problems with monotone signal-to-noise ratios are amenable to known methods of solution.

1. Introduction

The Gapeev-Shiryaev conjecture (originating in [4] and [5]) can be broadly stated as follows: Monotonicity of the signal-to-noise ratio implies monotonicity of the optimal stopping boundaries. The conjecture was originally formulated both within (i) sequential testing problems for diffusion processes [4] (where one needs to decide which of the two drifts is being indirectly observed) and (ii) quickest detection problems for diffusion processes [5] (where one needs to detect when the initial drift changes to a new drift). Both (i) and (ii) have a large number of applications and the importance of the conjectured implication follows from the well-known fact that optimal stopping problems with monotone optimal stopping boundaries are amenable to known methods of solution (see [11] and the references therein). The purpose of the present paper is to present proofs of the Gapeev-Shiryaev conjecture both in (i) the sequential testing setting (under Lipschitz/Hölder coefficients of the underlying SDEs) and (ii) the quickest detection setting (under analytic coefficients of the underlying SDEs). The solution found under (ii) also answers a related question that was left open in [1].

The sequential testing problem is recalled in Section 2. The problem has a long history and we refer to [7] and the references therein for fuller historical details. The Gapeev-Shiryaev

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conjecture in this setting is proved in Section 3 (Theorem 2). Pathwise comparison arguments attempted to derive the conjecture in [4] are inconclusive for a number of reasons (see (2.28) in [4] upon recalling (2.9), (2.11), (2.24) in [4]). We show in the proof of Theorem 2 that such a pathwise comparison becomes conclusive if one first applies a stochastic time change. Similar time-change arguments have been used earlier in [10] and more recently in [1]. In essence this is possible because the posterior probability ratio process Φ (defined in (2.9) and solving (2.15)+(2.16) when coupled with the observed process X under a new probability measure) is driftless. The question of how to tackle the problem when the corresponding process has a non-zero drift has been left open in [1]. In Remark 3 we recall a variety of known sufficient conditions for pathwise uniqueness of the time-changed SDE that is needed in Theorem 2 to make the pathwise comparison applicable.

The quickest detection problem is recalled in Section 4. The Gapeev-Shiryaev conjecture in this setting is proved in Section 5 (Theorem 6). Pathwise comparison arguments attempted to derive the conjecture in [5] are inconclusive for a number of reasons (see (4.6) in [5] upon recalling (2.9)-(2.11) and the equation below (4.5) in [5]). Moreover, on closer inspection one sees that the stochastic time change applied in the sequential testing proof of Theorem 2 does not reduce the quickest detection problem to a tractable form where similar pathwise comparison arguments would be applicable. In essence this is due to the fact that the posterior probability distribution ratio process Φ (defined in (4.9) and solving (4.16)+(4.17) when coupled with the observed process X under a new probability measure) is no longer driftless. The question of how to tackle the problem thus reduces to the question raised in [1]. For these reasons we are led to employ a different method of proof in Theorem 6 which is based on a stochastic maximum principle for hypoelliptic equations (satisfying Hörmander's condition) that is of independent interest. This is achieved by passing to the canonical infinitesimal generator equation of (Φ, X), characterising all trap curves for (Φ, X) at which Hörmander's condition fails, proving the Gapeev-Shiryaev conjecture in the absence of trap curves for (Φ, X), and then devising an approximating procedure by varying the drift of Φ that captures the Gapeev-Shiryaev conjecture in the presence of trap curves for (Φ, X) as well. To ensure that the trap curves of (Φ, X) have a global character we assume that the coefficients of the underlying SDEs are analytic. In Remark 7 we also briefly address C^∞ coefficients which are not necessarily analytic. The proof of Theorem 6 then shows that the Gapeev-Shiryaev conjecture is true in the absence of trap curves for (Φ, X) having a local character.

Verification of the Gapeev-Shiryaev conjecture establishes the fact that sequential testing and quickest detection problems with monotone signal-to-noise ratios are amenable to known methods of solution and therefore tractable (in the sense that the optimal stopping boundaries can be characterised as unique solutions to nonlinear Volterra/Fredholm integral equations). This broad conclusion has numerous theoretical/practical applications. The verification also confirms deep insights that the papers [4] and [5] have brought to light in this regard.

2. Sequential testing: Problem formulation

In this section we recall the sequential testing problem under consideration. The Gapeev-Shiryaev conjecture in this setting will be studied in the next section.

1. We consider a Bayesian formulation of the problem where it is assumed that one observes

a sample path of the diffusion process X having a drift coefficient equal to either μ_0 or μ_1 with prior probabilities $1-\pi$ and π respectively. The problem is to detect the true drift coefficient as soon as possible and with minimal probabilities of the wrong terminal decisions. This problem belongs to the class of sequential testing problems (see [7] and the references therein for fuller historical details).

2. Standard arguments imply that the previous setting can be realised on a probability space $(\Omega, \mathcal{F}, \mathsf{P}_\pi)$ with the probability measure P_π decomposed as follows

$$\mathsf{P}_\pi = (1-\pi)\mathsf{P}_0 + \pi \mathsf{P}_1 \tag{2.1}$$

for $\pi \in [0,1]$ where P_i is the probability measure under which the observed diffusion process X has drift μ_i for $i=0,1$. This can be formally achieved by introducing an unobservable random variable θ taking values 0 and 1 with probabilities $1-\pi$ and π under P_π and assuming that X after starting at some point $x \in I\!\!R$ solves the stochastic differential equation

$$dX_t = \big[\mu_0(X_t) + \theta\big(\mu_1(X_t) - \mu_0(X_t)\big)\big]\, dt + \sigma(X_t)\, dB_t \tag{2.2}$$

driven by a standard Brownian motion B that is independent from θ under P_π for $\pi \in [0,1]$. We assume that the real-valued functions μ_0, μ_1 and $\sigma > 0$ are continuous and that either $\mu_1 > \mu_0$ or $\mu_1 < \mu_0$ on $I\!\!R$. The state space of X will be assumed to be $I\!\!R$ for simplicity and the same arguments will also apply to smaller subsets/subintervals of $I\!\!R$.

3. Being based upon the continued observation of X, the problem is to test sequentially the hypotheses $H_0 : \theta = 0$ and $H_1 : \theta = 1$ with minimal loss. For this, we are given a sequential decision rule (τ, d_τ), where τ is a stopping time of X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \,|\, 0 \le s \le t)$ of X for $t \ge 0$), and d_τ is an $\mathcal{F}_\tau^X$-measurable random variable taking values 0 and 1. After stopping the observation of X at time τ, the terminal decision function d_τ takes value i if and only if the hypothesis H_i is to be accepted for $i = 0,1$. With constants $a > 0$ and $b > 0$ given and fixed, the problem then becomes to compute the risk function

$$V(\pi) = \inf_{(\tau, d_\tau)} \mathsf{E}_\pi\big[\tau + aI(d_\tau = 0, \theta = 1) + bI(d_\tau = 1, \theta = 0)\big] \tag{2.3}$$

for $\pi \in [0,1]$ and find the optimal decision rule $(\tau_*, d^*_{\tau_*})$ at which the infimum in (2.3) is attained. Note that $\mathsf{E}_\pi(\tau)$ in (2.3) is the expected waiting time until the terminal decision is made, and $\mathsf{P}_\pi(d_\tau = 0, \theta = 1)$ and $\mathsf{P}_\pi(d_\tau = 1, \theta = 0)$ in (2.3) are probabilities of the wrong terminal decisions respectively. Note also that the linear combination on the right-hand side of (2.3) represents the Lagrangian and once the problem has been solved in this form it will also lead to the solution of the constrained problems where upper bounds are imposed on the probabilities of the wrong terminal decisions.

4. To tackle the sequential testing problem (2.3) we consider the *posterior probability* process $\mathit{\Pi} = (\mathit{\Pi}_t)_{t \ge 0}$ of H_1 given X that is defined by

$$\mathit{\Pi}_t = \mathsf{P}_\pi(\theta = 1 \,|\, \mathcal{F}_t^X) \tag{2.4}$$

for $t \ge 0$. Noting that $\mathsf{P}_\pi(d_\tau = 0, \theta = 1) = \mathsf{E}_\pi[(1-d_\tau)\Pi_\tau]$ and $\mathsf{P}_\pi(d_\tau = 1, \theta = 0) = \mathsf{E}_\pi[d_\tau (1-\Pi_\tau)]$, and defining $\tilde{d}_\tau = I(a\Pi_\tau \ge b(1-\Pi_\tau))$ for any given (τ, d_τ), it is easily seen that the problem (2.3) is equivalent to the optimal stopping problem

$$V(\pi) = \inf_{\tau} \mathsf{E}_\pi \big[\tau + M(\Pi_\tau) \big] \tag{2.5}$$

where the infimum is taken over all stopping times τ of X and $M(\pi) = a\pi \wedge b(1-\pi)$ for $\pi \in [0,1]$. Letting τ_* denote the optimal stopping time in (2.5), and setting $c = b/(a+b)$, these arguments also show that the optimal decision function in (2.3) is given by $d^*_{\tau_*} = 0$ if $\Pi_{\tau_*} < c$ and $d^*_{\tau_*} = 1$ if $\Pi_{\tau_*} \geq c$. Thus to solve the initial problem (2.3) it is sufficient to solve the optimal stopping problem (2.5).

5. The *signal-to-noise ratio* in the problem (2.3) is defined by

$$\rho(x) = \frac{\mu_1(x) - \mu_0(x)}{\sigma(x)} \tag{2.6}$$

for $x \in I\!R$. If ρ is constant, then $\mathit{\Pi}$ is known to be a one-dimensional Markov (diffusion) process so that the optimal stopping problem (2.5) can be tackled using established techniques both in infinite and finite horizon (see [13, Section 21]). If ρ is not constant, then Π fails to be a Markov process on its own, however, the enlarged process $(\mathit{\Pi}, X)$ is Markov and this makes the optimal stopping problem (2.5) inherently two-dimensional and therefore more challenging.

6. To connect the process $\mathit{\Pi}$ in the problem (2.5) to the observed process X we consider the *likelihood ratio* process $L = (L_t)_{t \geq 0}$ defined by

$$L_t = \frac{d\mathsf{P}_{1,t}}{d\mathsf{P}_{0,t}} \tag{2.7}$$

where $\mathsf{P}_{0,t}$ and $\mathsf{P}_{1,t}$ denote the restrictions of the probability measures P_0 and P_1 to $\mathcal{F}_t^X$ for $t \geq 0$. By the Girsanov theorem one finds that

$$L_t = \exp\Big(\int_0^t \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma^2(X_s)}\, dX_s - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_s) - \mu_0^2(X_s)}{\sigma^2(X_s)}\, ds \Big) \tag{2.8}$$

for $t \geq 0$. A direct calculation based on (2.1) shows that the *posterior probability ratio* process $\mathit{\Phi} = (\mathit{\Phi}_t)_{t \geq 0}$ of θ given X that is defined by

$$\mathit{\Phi}_t = \frac{\mathit{\Pi}_t}{1 - \mathit{\Pi}_t} \tag{2.9}$$

can be expressed in terms of L (and hence X as well) as follows

$$\mathit{\Phi}_t = \Phi_0\, L_t \tag{2.10}$$

for $t \geq 0$ where $\mathit{\Phi}_0 = \pi/(1-\pi)$.

7. Changing the measure P_π for $\pi \in [0,1]$ to P_0 in the problem (2.5) provides crucial simplifications of the setting which makes the subsequent analysis possible. Recalling that

$$\frac{d\mathsf{P}_{\pi,\tau}}{d\mathsf{P}_{0,\tau}} = \frac{1-\pi}{1-\Pi_\tau} \tag{2.11}$$

where $\mathsf{P}_{\pi,\tau}$ denotes the restriction of the measure P_π to $\mathcal{F}_\tau^X$ for $\pi \in [0,1)$ and a stopping time τ of X, one finds that

$$V(\pi) = (1-\pi)\,\hat{V}(\pi) \tag{2.12}$$

where the value function $\hat{V}$ is given by

$$\hat{V}(\pi) = \inf_\tau \mathsf{E}_0\Big[\int_0^\tau \big(1+\Phi_t^{\pi/(1-\pi)}\big)\,dt + \hat{M}\big(\Phi_\tau^{\pi/(1-\pi)}\big)\Big] \tag{2.13}$$

for $\pi \in [0,1)$ with $\hat{M}(\varphi) = a\varphi \wedge b$ for $\varphi \in [0,\infty)$ and the infimum in (2.13) is taken over all stopping times τ of X (see proofs of Lemma 1 and Proposition 2 in [7] for fuller details). Recall from (2.9) that Φ starts at $\Phi_0 = \pi/(1-\pi)$ and this dependence on the initial point is indicated by a superscript $\pi/(1-\pi)$ to Φ in (2.13) above for $\pi \in [0,1)$. Moreover, from (2.8) and (2.10) we see that under P_0 we have

$$\Phi_t = \Phi_0 \,\exp\Big(\int_0^t \rho(X_s)\,dB_s - \frac{1}{2}\int_0^t \rho^2(X_s)\,ds\Big) \tag{2.14}$$

for $t \geq 0$ where ρ is given by (2.6) above. Hence by Itô's formula we find that the stochastic differential equations for (Φ, X) under P_0 read as follows

$$d\Phi_t = \rho(X_t)\Phi_t\,dB_t \tag{2.15}$$

$$dX_t = \mu_0(X_t)\,dt + \sigma(X_t)\,dB_t \tag{2.16}$$

where (2.16) follows from (2.2) upon recalling that θ equals 0 under P_0.

8. To tackle the resulting optimal stopping problem (2.13) for the strong Markov process (Φ, X) solving (2.15)+(2.16) we will enable (Φ, X) to start at any point (φ, x) in $[0,\infty)\times I\!\!R$ under the probability measure $\mathsf{P}_{\varphi,x}^0$ (where we move 0 from the subscript to a superscript for notational convenience) so that the optimal stopping problem (2.13) extends as

$$\hat{V}(\varphi, x) = \inf_\tau \mathsf{E}_{\varphi,x}^0\Big[\int_0^\tau \big(1+\Phi_t\big)\,dt + \hat{M}\big(\Phi_\tau\big)\Big] \tag{2.17}$$

for $(\varphi, x) \in [0,\infty)\times I\!\!R$ with $\mathsf{P}_{\varphi,x}^0((\Phi_0, X_0) = (\varphi, x)) = 1$ where the infimum in (2.17) is taken over all stopping times τ of (Φ, X). In this way we have reduced the initial sequential testing problem (2.3) to the optimal stopping problem (2.17) for the strong Markov process (Φ, X) solving the system (2.15)+(2.16) under the measure $\mathsf{P}_{\varphi,x}^0$ with $(\varphi, x) \in [0,\infty)\times I\!\!R$. Note that the optimal stopping problem (2.17) is inherently two-dimensional.

3. Sequential testing: Proof of the GS conjecture

In this section we present a proof of the Gapeev-Shiryaev (GS) conjecture in the sequential testing problem (2.3).

1. Recall that (2.3) is equivalent to the optimal stopping problem (2.17) for the strong Markov process (Φ, X) solving (2.15)+(2.16). Looking at (2.17) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$C = \{\, (\varphi, x) \in [0,\infty)\times I\!\!R \mid \hat{V}(\varphi, x) < \hat{M}(\varphi)\,\} \tag{3.1}$$

$$(3.2) \qquad D = \{\, (\varphi,x) \in [0,\infty)\times I\!R \mid \hat{V}(\varphi,x) = \hat{M}(\varphi) \,\}$$

respectively. It then follows by [13, Corollary 2.9] that the first entry time of the process (Φ, X) into the (closed) set D defined by

$$(3.3) \qquad \tau_D = \inf\{\, t \geq 0 \mid (\Phi_t, X_t) \in D \,\}$$

is optimal in (2.17) whenever $\mathsf{P}_{\varphi,x}(\tau_D < \infty) = 1$ for all $(\varphi,x) \in [0,\infty)\times[0,\infty)$ and $\hat{V}$ is continuous (or upper semicontinuous).

2. The Bolza formulated problem (2.17) can be Lagrange reformulated by applying the Itô-Tanaka formula to $\hat{M}$ composed with Φ. This yields

$$(3.4) \qquad \hat{V}(\varphi,x) = \inf_{\tau} \mathsf{E}^0_{\varphi,x}\Big[\int_0^{\tau} (1+\Phi_t)\, dt - \frac{a}{2}\, \ell^{b/a}_{\tau}(\Phi) \Big] + \hat{M}(\varphi)$$

for $(\varphi,x) \in [0,\infty)\times I\!R$ where $\ell^{b/a}_{\tau}(\Phi)$ is the local time of Φ at b/a and τ given by

$$(3.5) \qquad \ell^{b/a}_{\tau}(\Phi) = \mathsf{P}\text{-}\lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^{\tau} I\big(\tfrac{b}{a} - \varepsilon \leq \Phi_t \leq \tfrac{b}{a} + \varepsilon\big)\, d\langle \Phi, \Phi\rangle_t$$

and the infimum in (3.4) is taken over all stopping times τ of (Φ, X) (see Proposition 3 in [7] for details). The Lagrange reformulation (3.4) of the optimal stopping problem (2.17) reveals the underlying rationale for continuing vs stopping in a clearer manner. Indeed, recalling that the local time process $t \mapsto \ell^{b/a}_t(\Phi)$ strictly increases only when Φ_t is at b/a, and that $\ell^{b/a}_t(\Phi) \sim \sqrt{t}$ is strictly larger than $\int_0^t (1+\Phi_s)\, ds \sim t$ for small t, we see from (3.4) that it should never be optimal to stop at $\varphi = b/a$ and the incentive for stopping should increase the further away Φ_t gets from b/a for $t \geq 0$. These informal conjectures can be formalised by applying the Itô-Tanaka formula to $\varphi \mapsto |\varphi - b/a|$ composed with $\Phi^{b/a}$ and showing that

$$(3.6) \qquad \{\, (\varphi,x) \in [0,\infty)\times I\!R \mid \varphi = b/a \,\} \subseteq C$$

(see Lemma 9 in [7] for details).

3. Moving from the vertical line $\varphi = b/a$ outwards one can formally define the (least) boundaries between C and D by setting

$$(3.7) \qquad b_0(x) = \sup\big\{\, \varphi \in \big[0, \tfrac{b}{a}\big) \mid (\varphi,x) \in D \,\big\} \;\;\&\;\; b_1(x) = \inf\big\{\, \varphi \in \big(\tfrac{b}{a}, \infty\big] \mid (\varphi,x) \in D \,\big\}$$

for every $x \in I\!R$ given and fixed. Clearly $b_0(x) < b/a < b_1(x)$ for all $x \in I\!R$ and the supremum and infimum in (3.7) are attained since D is closed when $\hat{V}$ is continuous (or upper semicontinuous). Moreover, the boundaries b_0 and b_1 separate the sets C and D entirely in the sense that

$$(3.8) \qquad C = \{\, (\varphi,x) \in [0,\infty)\times I\!R \mid b_0(x) < \varphi < b_1(x) \,\}$$

$$(3.9) \qquad D = \{\, (\varphi,x) \in [0,\infty)\times I\!R \mid 0 \leq \varphi \leq b_0(x) \;\text{ or }\; b_1(x) \leq \varphi < \infty \,\}.$$

This can be established by noting that

$$(3.10) \qquad \varphi \mapsto \hat{V}(\varphi,x) \;\text{ is increasing and concave on }\; [0,\infty)$$

for every $x \in I\!R$ given and fixed, both properties being evident from (2.17) and the explicit (Markovian) dependence of $\varPhi$ on its initial point as seen from (2.14). Concavity of $\varphi \mapsto \hat{V}(\varphi,x)$ combined with non-negativity and piecewise linearity of $\varphi \mapsto \hat{M}(\varphi)$ in (2.17) implies that if $(\varphi,x) \in D$ with $\varphi < b/a$ and $\varphi_1 < \varphi$ then $(\varphi_1,x) \in D$ as well as that if $(\varphi,x) \in D$ with $\varphi > b/a$ and $\varphi_2 > \varphi$ then $(\varphi_2,x) \in D$. This establishes (3.8) and (3.9) as claimed.

4. The optimal stopping boundary in the problem (2.17) is the topological boundary between the continuation set C and the stopping set D. The previous arguments show that the optimal stopping boundary can be described by the graphs of two functions b_0 and b_1 as stated in (3.8) and (3.9) above. The GS conjecture deals with their *monotonicity* which makes the optimal stopping problem (2.17) amenable to known methods of solution.

Remark 1 (The GS conjecture). The following implication has been conjectured in [4]:

(3.11) *If $\mu_1 > \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto b_0(x)$ is decreasing/increasing and $x \mapsto b_1(x)$ is increasing/decreasing. Similarly, if $\mu_1 < \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto b_0(x)$ is increasing/decreasing and $x \mapsto b_1(x)$ is decreasing/increasing.*

Note that the monotonicity of $x \mapsto b_0(x)$ and $x \mapsto b_1(x)$ addressed in (3.11) can be inferred from the monotonicity of $x \mapsto \hat{V}(\varphi,x)$ for every $\varphi \in [0,\infty)$ given and fixed. Indeed, if $x \mapsto \hat{V}(\varphi,x)$ is increasing and $(\varphi,x) \in D$ then $0 = \hat{V}(\varphi,x) - \hat{M}(\varphi) \le \hat{V}(\varphi,y) - \hat{M}(\varphi) \le 0$ so that $V(\varphi,y) - \hat{M}(\varphi) = 0$ and hence $(\varphi,y) \in D$ for all $y \ge x$. Combined with (3.8)+(3.9) above this shows that if $x \mapsto \hat{V}(\varphi,x)$ is increasing for every $\varphi \in [0,\infty)$ given and fixed, then $x \mapsto b_0(x)$ is increasing and $x \mapsto b_1(x)$ is decreasing. Similarly, using the same arguments one finds that if $x \mapsto \hat{V}(\varphi,x)$ is decreasing for every $\varphi \in [0,\infty)$ given and fixed, then $x \mapsto b_0(x)$ is decreasing and $x \mapsto b_1(x)$ is increasing. It follows therefore that in order to establish (3.11) it is enough to show that if $\mu_1 > \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi,x)$ is decreasing/increasing, and if $\mu_1 < \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi,x)$ is increasing/decreasing, both for every $\varphi \in [0,\infty)$ given and fixed.

5. From (2.15) we see that X is present in the diffusion coefficient of $\varPhi$ and this makes the monotonicity of $x \mapsto \hat{V}(\varphi,x)$ in (2.17) more challenging to establish (most often such monotonicity fails). The separation of variables which naturally occurs in the diffusion coefficient of $\varPhi$ being equal to $\rho(x)\varphi$ for $(\varphi,x) \in [0,\infty)\times I\!R$ suggests to apply of a stochastic time change which will remove dependence on the x variable in the diffusion coefficient of the time-changed process $\hat{\varPhi}$. This can be achieved with the clock set as the inverse of the additive functional with a density function equal to ρ^2 composed with a marginal variable of X. Applying the new clock to X solving (2.16) then shows that the time-changed process $\hat{X}$ solves

$$d\hat{X}_t = \Big(\frac{\mu_0}{\rho^2}\Big)(\hat{X}_t)\,dt + \Big(\frac{\sigma}{\rho}\Big)(\hat{X}_t)\,d\tilde{B}_t \tag{3.12}$$

where $\hat{X}_0 = x$ in $I\!R$ and $\tilde{B}$ is a standard Brownian motion. One then hopes that the time-changed version of (2.17) has a favourable form and we will see below that this is the case indeed. Fuller details of all these arguments are given in the proof below.

Theorem 2. *If pathwise uniqueness holds for the stochastic differential equation* (3.12)*, then the GS conjecture* (3.11) *is true.*

Proof. Recall that in order to establish (3.11) it is enough to show that if $\mu_1 > \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi, x)$ is decreasing/increasing, and if $\mu_1 < \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi, x)$ is increasing/decreasing, both for every $\varphi \in [0, \infty)$ given and fixed.

1. Motivated by the desire to apply a stochastic time change in (2.17) as described above, consider the additive functional $A = (A_t)_{t \geq 0}$ defined by

$$A_t = \int_0^t \rho^2(X_s)\, ds \tag{3.13}$$

and note that $t \mapsto A_t$ is continuous and strictly increasing with $A_0 = 0$ and $A_t \uparrow A_\infty$ as $t \uparrow \infty$. Hence the same properties hold for its inverse $T = (T_t)_{t \geq 0}$ defined by

$$T_t = A_t^{-1} \tag{3.14}$$

for $t \in [0, A_\infty)$. Because A is adapted to $(\mathcal{F}_t^X)_{t \geq 0}$ it follows that each T_t is a stopping time with respect to $(\mathcal{F}_t^X)_{t \geq 0}$ so that $T = (T_t)_{t \geq 0}$ defines a time change relative to $(\mathcal{F}_t^X)_{t \geq 0}$. Since (Φ, X) is a strong Markov process we know by the well-known result dating back to [16] that the time-changed process $(\hat{\Phi}, \hat{X}) = ((\hat{\Phi}_t, \hat{X}_t))_{t \geq 0}$ defined by

$$(\hat{\Phi}_t, \hat{X}_t) = (\Phi_{T_t}, X_{T_t}) \tag{3.15}$$

for $t \geq 0$ is a Markov process under $\mathsf{P}^0_{\varphi,x}$ for $(\varphi, x) \in [0, \infty) \times I\!R$. Moreover, from (3.13) one can read off that the infinitesimal generator of $(\hat{\Phi}, \hat{X})$ is given by

$$I\!L_{\hat{\Phi}, \hat{X}} = \frac{1}{\rho^2(x)}\, I\!L_{\Phi, X} \tag{3.16}$$

where $I\!L_{\Phi, X}$ is the infinitesimal generator of (Φ, X). Finally, in addition to (3.13) it is easily seen using (3.14) that we have

$$T_t = \int_0^t \frac{1}{\rho^2(\hat{X}_s)}\, ds \tag{3.17}$$

for $t \geq 0$.

2. Recalling that (Φ, X) solves (2.15)+(2.16) we find that

$$\begin{aligned} \hat{\Phi}_t &= \Phi_{T_t} = \Phi_0 + \int_0^{T_t} \rho(X_s)\, \Phi_s\, dB_s \\ &= \Phi_0 + \int_0^t \rho(X_{T_s})\, \Phi_{T_s}\, dB_{T_s} = \hat{\Phi}_0 + \int_0^t \hat{\Phi}_s\, d\tilde{B}_s \end{aligned} \tag{3.18}$$

$$\begin{aligned} \hat{X}_t &= X_{T_t} = X_0 + \int_0^{T_t} \mu_0(X_s)\, ds + \int_0^{T_t} \sigma(X_s)\, dB_s \\ &= X_0 + \int_0^t \mu_0(X_{T_s})\, dT_s + \int_0^t \sigma(X_{T_s})\, dB_{T_s} \end{aligned} \tag{3.19}$$

$$= X_0 + \int_0^t \mu_0(\hat{X}_s)\,\frac{1}{\rho^2(\hat{X}_s)}\,ds + \int_0^t \sigma(\hat{X}_s)\,\frac{1}{\rho(\hat{X}_s)}\,d\tilde{B}_s$$

where the process $\tilde{B} = (\tilde{B}_t)_{t\ge 0}$ is defined by

$$\tilde{B}_t = \int_0^t \rho(X_{T_s})\,dB_{T_s} = \int_0^{T_t} \rho(X_s)\,dB_s = M_{T_t} \tag{3.20}$$

upon setting $M_t = \int_0^t \rho(X_s)\,dB_s$ for $t \ge 0$. Since $M = (M_t)_{t\ge 0}$ is a continuous local martingale with respect to $(\mathcal{F}_t^X)_{t\ge 0}$ it follows that $\tilde{B} = (\tilde{B}_t)_{t\ge 0}$ is a continuous local martingale with respect to $(\hat{\mathcal{F}}_t^X)_{t\ge 0}$ where $\hat{\mathcal{F}}_t^X := \mathcal{F}_{T_t}^X$ for $t \ge 0$. Note moreover that $\langle \tilde{B}, \tilde{B}\rangle_t = \langle M_T, M_T\rangle_t = \langle M, M\rangle_{T_t} = \int_0^{T_t} \rho^2(X_s)\,ds = A_{T_t} = t$ for $t \ge 0$. Hence by Lévy's characterisation theorem (see e.g. [14, p. 150]) we can conclude that $\tilde{B}$ is a standard Brownian motion with respect to $(\hat{\mathcal{F}}_t^X)_{t\ge 0}$. It follows therefore that (3.18)+(3.19) can be written as

$$d\hat{\Phi}_t = \hat{\Phi}_t\,d\tilde{B}_t \tag{3.21}$$

$$d\hat{X}_t = \hat{\mu}(\hat{X}_t)\,dt + \hat{\sigma}(\hat{X}_t)\,d\tilde{B}_t \tag{3.22}$$

under $\mathsf{P}^0_{\varphi,x}$ for $(\varphi, x) \in [0,\infty)\times I\!R$ where we set $\hat{\mu} := \mu_0/\rho^2$ and $\hat{\sigma} = \sigma/\rho$. This shows that $\hat{\Phi}$ and $\hat{X}$ are fully decoupled diffusion processes (driven by the same Brownian motion) where $\hat{\Phi}_t = \Phi_0\, e^{\tilde{B}_t - t/2}$ is a geometric Brownian motion for $t \ge 0$ and (3.22) establishes (3.12) above as claimed. Recalling known sufficient conditions (see e.g. [15, pp 166–173]) we formally see that the system (3.21)+(3.22) has a unique weak solution and hence by the well-known result (see e.g. [15, pp 158–163]) we can conclude that $(\hat{\Phi}, \hat{X})$ is a (time-homogeneous) strong Markov process under $\mathsf{P}^0_{\varphi,x}$ for $(\varphi, x) \in [0,\infty)\times I\!R$.

3. Making use of the previous facts we can now derive a time-changed version of the optimal stopping problem (2.17) as follows. For this, recall that $\tau = T_\sigma$ is a stopping time of (Φ, X) if and only if $\sigma = A_\tau$ is a stopping time of $(\hat{\Phi}, \hat{X})$. Thus, if either τ or σ is given, we can form σ or τ respectively, and using (3.17) note that

$$\begin{aligned}\mathsf{E}^0_{\varphi,x}\Big[\int_0^\tau \big(1+\Phi_t\big)\,dt + \hat{M}\big(\Phi_\tau\big)\Big] &= \mathsf{E}^0_{\varphi,x}\Big[\int_0^{T_\sigma} \big(1+\Phi_t\big)\,dt + \hat{M}\big(\Phi_{T_\sigma}\big)\Big] \\ = \mathsf{E}^0_{\varphi,x}\Big[\int_0^\sigma \big(1+\Phi_{T_t}\big)\,dT_t + \hat{M}\big(\hat{\Phi}_\sigma\big)\Big] &= \mathsf{E}^0_{\varphi,x}\Big[\int_0^\sigma \big(1+\hat{\Phi}_t\big)\,\frac{1}{\rho^2(\hat{X}_t)}\,dt + \hat{M}\big(\hat{\Phi}_\sigma\big)\Big].\end{aligned} \tag{3.23}$$

Taking the infimum over all τ and/or σ on both sides of (3.23) we see that the time-changed version of (2.17) reads as follows

$$\hat{V}(\varphi, x) = \inf_\sigma \mathsf{E}^0_{\varphi,x}\Big[\int_0^\sigma \big(1+\hat{\Phi}_t\big)\,\frac{1}{\rho^2(\hat{X}_t)}\,dt + \hat{M}\big(\hat{\Phi}_\sigma\big)\Big] \tag{3.24}$$

for $(\varphi, x) \in [0,\infty)\times I\!R$ where the infimum is taken over all stopping times σ of $(\hat{\Phi}, \hat{X})$.

4. To examine monotonicity of $x \mapsto \hat{V}(\varphi, x)$ for $\varphi \in [0,\infty)$ given and fixed, note that the pathwise uniqueness of solution to (3.22) assumed combined with the existence of a weak

solution to (3.22) established implies the existence of a strong solution to (3.22) (cf. [17]). It follows therefore that for any standard Brownian motion $\tilde{B}$ given and fixed, the solution X_t^x to (3.22) starting at $x \in I\!R$ can be realised as a deterministic/measurable functional of x and $(\tilde{B}_s)_{0 \le s \le t}$ for $t \ge 0$. Moreover, solving (3.21) in closed form with the same $\tilde{B}$ as in (3.22), we know that the solution Φ_t^φ starting at $\varphi \in [0, \infty)$ is given by $\varphi\, e^{\tilde{B}_t - t/2}$ for $t \ge 0$. Finally, the pathwise uniqueness for (3.22) implies that $\hat{X}_t^x \le \hat{X}_t^y$ almost surely for $t \ge 0$ whenever $x \le y$ in $I\!R$. Indeed, this is evident by pathwise uniqueness itself (through equality) if $x = y$, while if $x < y$ then setting $\tilde{X}_t^y := \hat{X}_t^y$ for $t \le \tau$ and $\tilde{X}_t^y := \hat{X}_t^x$ for $t > \tau$ where $\tau = \inf\{\, t \ge 0 \mid \hat{X}_t^y = \hat{X}_t^x \,\}$, it is easily verified that $\tilde{X}^y$ solves (3.22) after staring at y but differs from $\hat{X}^y$ if $\hat{X}_t^x \le \hat{X}_t^y$ fails with strictly positive probability for some $t > 0$. Combining these facts we see that

$$\begin{aligned} (3.25) \qquad \mathsf{E}^0_{\varphi,x}\Big[\int_0^\sigma \big(1+\hat{\Phi}_t\big) \frac{1}{\rho^2(\hat{X}_t)}\, dt + \hat{M}\big(\hat{\Phi}_\sigma\big) \Big] &= \mathsf{E}_0\Big[\int_0^\sigma \big(1+\hat{\Phi}_t^\varphi\big) \frac{1}{\rho^2(\hat{X}_t^x)}\, dt + \hat{M}\big(\hat{\Phi}_\sigma^\varphi\big) \Big] \\ \le \mathsf{E}_0\Big[\int_0^\sigma \big(1+\hat{\Phi}_t^\varphi\big) \frac{1}{\rho^2(\hat{X}_t^y)}\, dt + \hat{M}\big(\hat{\Phi}_\sigma^\varphi\big) \Big] &= \mathsf{E}^0_{\varphi,y}\Big[\int_0^\sigma \big(1+\hat{\Phi}_t\big) \frac{1}{\rho^2(\hat{X}_t)}\, dt + \hat{M}\big(\hat{\Phi}_\sigma\big) \Big] \end{aligned}$$

for all $x \le y$ in $I\!R$ and any stopping time σ of $(\hat{\Phi}, \hat{X})$ whenever $z \mapsto \rho^2(z)$ is decreasing on $I\!R$. Taking the infimum over all such σ on both sides of (3.25) we find using (3.24) that $\hat{V}(\varphi, x) \le \hat{V}(\varphi, y)$ for all $x \le y$ in $I\!R$ whenever $z \mapsto \rho^2(z)$ is decreasing on $I\!R$. Noting from (2.6) that $z \mapsto \rho^2(z)$ is decreasing if $z \mapsto \rho(z)$ is decreasing or increasing on $I\!R$ when $\mu_1 > \mu_0$ or $\mu_1 < \mu_0$ respectively, we see that this completes the proof when $z \mapsto \rho^2(z)$ is decreasing. Reversing the inequality in (3.25) and arguing in exactly the same way completes the proof when $z \mapsto \rho^2(z)$ is increasing as well. $\square$

Remark 3. There is a variety of known sufficient conditions for pathwise uniqueness of the stochastic differential equation (3.12). For example, if μ_0/ρ^2 is (locally) Lipschitz and $\sigma/|\rho|$ is (locally) 1/2-Hölder, then the pathwise uniqueness holds for (3.12) and the GS conjecture is true (cf. [17]). Similarly, if μ_0/ρ^2 and $\sigma/|\rho| \ge \varepsilon > 0$ are (locally) bounded and measurable (both being satisfied if μ_0, μ_1 and $\sigma > 0$ are continuous with $\mu_1 > \mu_0$ or $\mu_1 < \mu_0$ as assumed throughout), and $\sigma/|\rho|$ is of bounded variation on any compact interval, then the pathwise uniqueness holds for (3.12) and the GS conjecture is true (cf. [9]). For further details of these and related arguments see [15, Sections 39-41] and [8, Section 5.5]. Note that ρ in (3.12) can be replaced by $|\rho|$ in these two (and similar other) implications if $\tilde{B}$ is replaced by $-\tilde{B}$ in both (3.21) and (3.22). The latter replacement corresponds to viewing (3.22) by means of $-\hat{X}$ and $-\hat{\mu}$ rather than $\hat{X}$ and $\hat{\mu}$ respectively. In terms of the initial problem when $\mu_1 < \mu_0$ it means that multiplying both sides of (2.2) by -1 we can view $-X$ as the observed process driven by the standard Brownian motion $-B$ with drifts $\tilde{\mu}_i(x) := -\mu_i(-x)$ for $i = 1, 2$ and the diffusion coefficient $\tilde{\sigma}(x) := \sigma(-x)$ for $x \in I\!R$, so that $\tilde{\mu}_1 > \tilde{\mu}_0$ which in turn implies that the resulting signal-to-noise ratio $\tilde{\rho} := (\tilde{\mu}_1 - \tilde{\mu}_0)/\tilde{\sigma}$ is strictly positive again.

4. Quickest detection: Problem formulation

In this section we recall the quickest detection problem under consideration. The Gapeev-Shiryaev conjecture in this setting will be studied in the next section.

1. We consider a Bayesian formulation of the problem where it is assumed that one observes a sample path of the diffusion process X whose drift coefficient μ_0 changes to another drift coefficient μ_1 at some random/unobservable time θ taking value 0 with probability $\pi \in [0,1]$ and being exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$. The problem is to detect the unknown time θ as accurately as possible (neither too early nor too late). This problem belongs to the class of quickest detection problems (see [6] and the references therein for fuller historical details).

2. Standard arguments imply that the previous setting can be realised on a probability space $(\Omega, \mathcal{F}, \mathsf{P}_\pi)$ with the probability measure P_π decomposed as follows

$$\mathsf{P}_\pi = \pi\, \mathsf{P}^0 + (1-\pi) \int_0^\infty \lambda e^{-\lambda t}\, \mathsf{P}^t\, dt \tag{4.1}$$

for $\pi \in [0,1]$ where P^t is the probability measure under which the observed process X undergoes the change of drift at time $t \in [0,\infty)$. The unobservable time θ is a non-negative random variable satisfying $\mathsf{P}_\pi(\theta = 0) = \pi$ and $\mathsf{P}_\pi(\theta > t \,|\, \theta > 0) = e^{-\lambda t}$ for $t > 0$. Thus $\mathsf{P}^t(X \in \cdot\,) = \mathsf{P}_\pi(X \in \cdot \,|\, \theta = t)$ is the probability law of a diffusion process whose drift μ_0 changes to drift μ_1 at time $t > 0$. To remain consistent with this notation we also denote by P^∞ the probability measure under which the observed process X undergoes no change of its drift. Thus $\mathsf{P}^\infty(X \in \cdot\,) = \mathsf{P}_\pi(X \in \cdot \,|\, \theta = \infty)$ is the probability law of a diffusion process with drift μ_0 at all times.

3. The observed process X after starting at some point $x \in I\!R$ solves the stochastic differential equation

$$dX_t = \big[\mu_0(X_t) + I(t \ge \theta)\big(\mu_1(X_t) - \mu_0(X_t)\big)\big]\, dt + \sigma(X_t)\, dB_t \tag{4.2}$$

driven by a standard Brownian motion B that is independent from θ under P_π for $\pi \in [0,1]$. We assume that the real-valued functions μ_0, μ_1 and $\sigma > 0$ are continuous and that either $\mu_1 > \mu_0$ or $\mu_1 < \mu_0$ on $I\!R$. The state space of X will be assumed to be $I\!R$ for simplicity and the same arguments will also apply to smaller subsets/subintervals of $I\!R$.

4. Being based upon continuous observation of X, the problem is to find a stopping time τ_* of X (i.e. a stopping time with respect to the natural filtration $\mathcal{F}_t^X = \sigma(X_s \,|\, 0 \le s \le t)$ of X for $t \ge 0$) that is 'as close as possible' to the unknown time θ. More precisely, the problem consists of computing the value function

$$V(\pi) = \inf_\tau \Big[\mathsf{P}_\pi(\tau < \theta) + c\, \mathsf{E}_\pi(\tau - \theta)^+\Big] \tag{4.3}$$

and finding the optimal stopping time τ_* at which the infimum in (4.3) is attained for $\pi \in [0,1]$ and $c > 0$ given and fixed. Note in (4.3) that $\mathsf{P}_\pi(\tau < \theta)$ is the probability of the *false alarm* and $\mathsf{E}_\pi(\tau - \theta)^+$ is the expected *detection delay* associated with a stopping time τ of X for $\pi \in [0,1]$. Note also that the linear combination on the right-hand side of (4.3) represents the Lagrangian and once the problem has been solved in this form it will also lead to the solution of the constrained problem where an upper bound is imposed on either the probability of the false alarm or the expected detection delay when the other probability is minimised.

5. To tackle the optimal stopping problem (4.3) we consider the *posterior probability distribution* process $\varPi = (\varPi_t)_{t \ge 0}$ of θ given X that is defined by

$$\varPi_t = \mathsf{P}_\pi(\theta \le t \,|\, \mathcal{F}_t^X) \tag{4.4}$$

for $t \ge 0$. The right-hand side of (4.3) can then be rewritten to read

$$V(\pi) = \inf_\tau \mathsf{E}_\pi \Big(1 - \varPi_\tau + c \int_0^\tau \varPi_t \, dt \Big) \tag{4.5}$$

for $\pi \in [0,1]$ where the infimum is taken over all stopping times τ of X.

6. The *signal-to-noise ratio* in the problem (4.3) is defined by

$$\rho(x) = \frac{\mu_1(x) - \mu_0(x)}{\sigma(x)} \tag{4.6}$$

for $x \in I\!R$. If ρ is constant, then $\varPi$ is known to be a one-dimensional Markov (diffusion) process so that the optimal stopping problem (4.5) can be tackled using established techniques both in infinite and finite horizon (see [13, Section 22]). If ρ is not constant, then Π fails to be a Markov process on its own, however, the enlarged process $(\varPi, X)$ is Markov and this makes the optimal stopping problem (4.5) inherently two-dimensional and therefore more challenging.

7. To connect the process $\varPi$ to the observed process X we consider the *likelihood ratio* process $L = (L_t)_{t \ge 0}$ defined by

$$L_t = \frac{d\mathsf{P}_t^0}{d\mathsf{P}_t^\infty} \tag{4.7}$$

where P_t^0 and P_t^∞ denote the restrictions of the probability measures P^0 and P^∞ to $\mathcal{F}_t^X$ for $t \ge 0$. By the Girsanov theorem one finds that

$$L_t = \exp\Big(\int_0^t \frac{\mu_1(X_s) - \mu_0(X_s)}{\sigma^2(X_s)} \, dX_s - \frac{1}{2} \int_0^t \frac{\mu_1^2(X_s) - \mu_0^2(X_s)}{\sigma^2(X_s)} \, ds \Big) \tag{4.8}$$

for $t \ge 0$. A direct calculation based on (4.1) shows that the *posterior probability distribution ratio* process $\varPhi = (\varPhi_t)_{t \ge 0}$ of θ given X that is defined by

$$\varPhi_t = \frac{\varPi_t}{1 - \varPi_t} \tag{4.9}$$

can be expressed in terms of L (and hence X as well) as follows

$$\varPhi_t = e^{\lambda t} L_t \Big(\varPhi_0 + \lambda \int_0^t \frac{ds}{e^{\lambda s} L_s} \Big) \tag{4.10}$$

for $t \ge 0$ where $\varPhi_0 = \pi/(1-\pi)$.

8. Changing the measure P_π for $\pi \in [0,1]$ to P^∞ in the problem (4.5) provides crucial simplifications of the setting which makes the subsequent analysis possible. Recalling that

$$\frac{d\mathsf{P}_{\pi,\tau}}{d\mathsf{P}_\tau^\infty} = e^{-\lambda \tau} \, \frac{1-\pi}{1-\varPi_\tau} \tag{4.11}$$

where P_τ^∞ and $\mathsf{P}_{\pi,\tau}$ denote the restrictions of measures P^∞ and P_π to $\mathcal{F}_\tau^X$ respectively for $\pi \in [0,1)$ and a stopping time τ of X, one finds that

$$V(\pi) = (1-\pi)\big[1 + c\hat{V}(\pi)\big] \tag{4.12}$$

where the value function $\hat{V}$ is given by

$$\hat{V}(\pi) = \inf_\tau \mathsf{E}^\infty\Big[\int_0^\tau e^{-\lambda t}\Big(\varPhi_t^{\pi/(1-\pi)} - \frac{\lambda}{c}\Big)\, dt\Big] \tag{4.13}$$

for $\pi \in [0,1)$ and the infimum in (4.13) is taken over all stopping times τ of X (see proofs of Lemma 1 and Proposition 2 in [6] for fuller details). Recall from (4.9) that $\varPhi$ starts at $\varPhi_0 = \pi/(1-\pi)$ and this dependence on the initial point is indicated by a superscript $\pi/(1-\pi)$ to $\varPhi$ in (4.13) above for $\pi \in [0,1)$. Moreover, from (4.8) we see that under P^∞ we have

$$L_t = \exp\Big(\int_0^t \rho(X_s)\, dB_s - \frac{1}{2}\int_0^t \rho^2(X_s)\, ds\Big) \tag{4.14}$$

for $t \geq 0$. Hence by Itô's formula we see that L under P^∞ solves

$$dL_t = \rho(X_t) L_t\, dB_t \tag{4.15}$$

with $L_0 = 1$. Applying Itô's formula in (4.10) then shows that the stochastic differential equations for $(\varPhi, X)$ under P^∞ read as follows

$$d\varPhi_t = \lambda(1+\varPhi_t)\, dt + \rho(X_t)\varPhi_t\, dB_t \tag{4.16}$$

$$dX_t = \mu_0(X_t)\, dt + \sigma(X_t)\, dB_t \tag{4.17}$$

where (4.17) follows from (4.2) upon recalling that θ equals ∞ under P^∞.

9. To tackle the resulting optimal stopping problem (4.13) for the strong Markov process $(\varPhi, X)$ solving (4.16)+(4.17) we will enable $(\varPhi, X)$ to start at any point (φ, x) in $[0,\infty)\times I\!R$ under the probability measure $\mathsf{P}_{\varphi,x}^\infty$ so that the optimal stopping problem (4.13) extends as

$$\hat{V}(\varphi, x) = \inf_\tau \mathsf{E}_{\varphi,x}^\infty\Big[\int_0^\tau e^{-\lambda t}\Big(\varPhi_t - \frac{\lambda}{c}\Big)\, dt\Big] \tag{4.18}$$

for $(\varphi, x) \in [0,\infty)\times I\!R$ with $\mathsf{P}_{\varphi,x}^\infty((\varPhi_0, X_0) = (\varphi, x)) = 1$ where the infimum in (4.18) is taken over all stopping times τ of $(\varPhi, X)$. In this way we have reduced the quickest detection problem (4.3) to the optimal stopping problem (4.18) for the strong Markov process $(\varPhi, X)$ solving the system (4.16)+(4.17) under the measure $\mathsf{P}_{\varphi,x}^\infty$ with $(\varphi, x) \in [0,\infty)\times I\!R$. Note that the optimal stopping problem (4.18) is inherently two-dimensional.

5. Quickest detection: Proof of the GS conjecture

In this section we present a proof of the Gapeev-Shiryaev (GS) conjecture in the quickest detection problem (4.3).

1. Recall that (4.3) is equivalent to the optimal stopping problem (4.18) for the strong Markov process $(\varPhi, X)$ solving (4.16)+(4.17). Looking at (4.18) we may conclude that the (candidate) continuation and stopping sets in this problem need to be defined as follows

$$C = \{ (\varphi, x) \in [0,\infty) \times I\!R \mid \hat{V}(\varphi, x) < 0 \} \tag{5.1}$$

$$D = \{ (\varphi, x) \in [0,\infty) \times I\!R \mid \hat{V}(\varphi, x) = 0 \} \tag{5.2}$$

respectively. It then follows by [13, Corollary 2.9] that the first entry time of the process $(\varPhi, X)$ into the (closed) set D defined by

$$\tau_D = \inf\{ t \geq 0 \mid (\varPhi_t, X_t) \in D \} \tag{5.3}$$

is optimal in (4.18) whenever $\mathsf{P}_{\varphi,x}(\tau_D < \infty) = 1$ for all $(\varphi, x) \in [0,\infty) \times [0,\infty)$ and $\hat{V}$ is continuous (or upper semicontinuous). In this implication note that the Lagrange formulated problem (4.18) can be Mayer reformulated by embedding the two-dimensional Markov process $(\varPhi, X)$ into the four-dimensional Markov process $(T, \varPhi, X, I)$ where $T_t = t$ and $I_t = \int_0^t e^{-\lambda T_s}(\varPhi_s - \lambda/c)\, ds$ for $t \geq 0$ (see [13, Chapter III] for fuller details).

2. Since the integrand in (4.18) is strictly negative for $\varphi < \lambda/c$ it is clear that this region of the state space is contained in C (otherwise the first exit times of $(\varPhi, X)$ from a sufficiently small neighbourhood would violate stopping at once). Expanding on this argument further one can formally define the (least) boundary between C and D by setting

$$b(x) = \inf\{ \varphi \geq 0 \mid (\varphi, x) \in D \} \tag{5.4}$$

for every $x \in I\!R$ given and fixed. Clearly $b(x) \geq \lambda/c$ and the infimum in (5.4) is attained since D is closed when $\hat{V}$ is continuous (or upper semicontinuous). Moreover, the boundary b separates the sets C and D entirely in the sense that

$$C = \{ (\varphi, x) \in [0,\infty) \times I\!R \mid \varphi < b(x) \} \tag{5.5}$$

$$D = \{ (\varphi, x) \in [0,\infty) \times I\!R \mid \varphi \geq b(x) \}. \tag{5.6}$$

This can be established by noting that

$$\varphi \mapsto \hat{V}(\varphi, x) \text{ is increasing on } [0,\infty) \tag{5.7}$$

for every $x \in I\!R$ given and fixed, which is evident from (4.18) and the explicit (Markovian) dependence of $\varPhi$ on its initial point as seen from (4.10). Indeed, if $(\varphi, x) \in D$ and $\psi \geq \varphi$ then by (5.7) we have $0 = \hat{V}(\varphi, x) \leq \hat{V}(\psi, x) \leq 0$ so that $\hat{V}(\psi, x) = 0$ and hence $(\psi, x) \in D$ establishing (5.5) and (5.6) as claimed.

3. The optimal stopping boundary in the problem (4.18) is the topological boundary between the continuation set C and the stopping set D. The previous arguments show that the optimal stopping boundary can be described by the graph of a function b as stated in (5.5) and (5.6) above. The GS conjecture deals with its *monotonicity* which makes the optimal stopping problem (4.18) amenable to known methods of solution.

Remark 4 (The GS conjecture). The following implication has been conjectured in [5]:

(5.8) *If* $\mu_1 > \mu_0$ *and* $x \mapsto \rho(x)$ *is increasing/decreasing, then* $x \mapsto b(x)$

is increasing/decreasing. Similarly, if $\mu_1 < \mu_0$ *and* $x \mapsto \rho(x)$ *is increasing/decreasing, then* $x \mapsto b(x)$ *is decreasing/increasing.*

Note that the monotonicity of $x \mapsto b(x)$ addressed in (5.8) can be inferred from the monotonicity of $x \mapsto \hat{V}(\varphi, x)$ for every $\varphi \in [0, \infty)$ given and fixed. Indeed, if $x \mapsto \hat{V}(\varphi, x)$ is increasing and $(\varphi, x) \in D$ then $0 = \hat{V}(\varphi, x) \le \hat{V}(\varphi, y) \le 0$ so that $V(\varphi, y) = 0$ and hence $(\varphi, y) \in D$ for all $y \ge x$. Combined with (5.5)+(5.6) above this shows that if $x \mapsto \hat{V}(\varphi, x)$ is increasing for every $\varphi \in [0, \infty)$ given and fixed, then $x \mapsto b(x)$ is decreasing. Similarly, using the same arguments one finds that if $x \mapsto \hat{V}(\varphi, x)$ is decreasing for every $\varphi \in [0, \infty)$ given and fixed, then $x \mapsto b(x)$ is increasing. It follows therefore that in order to establish (5.8) it is enough to show that if $\mu_1 > \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi, x)$ is decreasing/increasing, and if $\mu_1 < \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi, x)$ is increasing/decreasing, both for every $\varphi \in [0, \infty)$ given and fixed.

4. From (4.16) we see that X is present in the diffusion coefficient of Φ and this makes the monotonicity of $x \mapsto \hat{V}(\varphi, x)$ in (4.18) more challenging to establish (most often such monotonicity fails). Moreover, on closer inspection one sees that the stochastic time change (3.14) applied in the proof of Theorem 2 above does not reduce the problem (4.18) to a tractable form where similar pathwise comparison arguments would be applicable. This is due to the existence of a non-zero drift term in (4.16) that was absent in (2.15) above. For these reasons we are led to employ a different method of proof which is based on a stochastic maximum principle for hypoelliptic equations (satisfying Hörmander's condition) that is of independent interest. In essence this is possible due to the fact that

$$\varphi \mapsto \hat{V}(\varphi, x) \text{ is concave on } [0, \infty) \tag{5.9}$$

for every $x \in I\!R$ given and fixed, which is evident from the structure of (4.18) due to the fact that Φ is a linear (Markovian) functional of its initial point as seen from (4.10) above. Fuller details of the method employed are given in the proof below.

To exclude degenerate cases (in which Hörmander's condition fails) we are led to consider curves in the state space $[0, \infty) \times I\!R$ of the process (Φ, X) that can be represented as the graphs of functions from $I\!R$ to $[0, \infty)$. Thus each such a curve γ can be identified with the graph $\{(\gamma(x), x) \mid x \in I\!R\}$ of a (continuous) function $\gamma : I\!R \to [0, \infty)$.

Definition 5 (Trap). *A curve* γ *is said to be a trap for* (Φ, X) *if* (Φ, X) *after starting (or entering) at any point of* γ *remains in* γ *forever.*

We will see in the proof below that a necessary and sufficient condition for the existence of a trap γ for (Φ, X) when μ_0, μ_1, σ are analytic is that the function F defined by

$$F(x) = \int_0^x \frac{\rho(y)}{\sigma(y)}\, dy \tag{5.10}$$

for $x \in I\!R$ satisfies the nonlinear differential equation

$$\frac{1}{2}\Big(\sigma^2 F'' + (\mu_0 + \mu_1) F'\Big) - \lambda = \frac{\lambda}{\kappa}\, e^{-F} \tag{5.11}$$

on $I\!R$ for some $\kappa > 0$, and in this case we have

$$\gamma(x) = \kappa\, e^{F(x)} \tag{5.12}$$

for $x \in I\!R$. Thus, if the equality (5.11) fails at one point in $I\!R$ at least, then no curve γ can be a trap for (Φ, X) . Most often this is the case although not always. For example, if $\mu_0 = 0$ and $\mu_1(x) = \sigma^2(x) = 2\lambda(1+e^{-x})$ for $x \in I\!R$ then $\rho/\sigma = (\mu_1-\mu_0)/\sigma^2 = 1$ and $F(x) = x$ for $x \in I\!R$ so that (5.11) is satisfied with $\kappa = 1$ on $I\!R$ and the curve $\gamma(x) = e^x$ for $x \in I\!R$ is a trap for (Φ, X) in this case.

Theorem 6. *If* μ_0 , μ_1 , σ *are analytic on* $I\!R$ *, then the GS conjecture* (5.8) *is true.*

Proof. Recall that in order to establish (5.8) it is enough to show that if $\mu_1 > \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi, x)$ is decreasing/increasing, and if $\mu_1 < \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing, then $x \mapsto \hat{V}(\varphi, x)$ is increasing/decreasing, both for every $\varphi \in [0, \infty)$ given and fixed.

Part I: In this part we assume that no curve γ is a trap for (Φ, X) (i.e. the equality (5.11) fails at one point in $I\!R$ at least for every $\kappa > 0$ given and fixed). Under this hypothesis we divide the proof in five further parts as follows.

1. *Free-boundary problem.* Recalling (4.18) and setting $L(\varphi) = \varphi - \lambda/c$ for $\varphi \in [0, \infty)$, standard Markovian results of optimal stopping (cf. [13, Subsection 7.2]) imply that $\hat{V}$ and b solve the free-boundary problem

$$I\!L_{\Phi,X}\hat{V} - \lambda\hat{V} = -L \ \text{ in } \ C \tag{5.13}$$

$$\hat{V} = 0 \ \text{ at } \ \partial C \ \text{ (instantaneous stopping)} \tag{5.14}$$

$$\hat{V}_\varphi = \hat{V}_x = 0 \ \text{ at } \ \partial_r D \ \text{ (smooth fit)} \tag{5.15}$$

where $I\!L_{\Phi,X}$ is the infinitesimal generator of (Φ, X) given by

$$I\!L_{\Phi,X} = \lambda(1+\varphi)\partial_\varphi + \mu_0\partial_x + \varphi\rho\sigma\,\partial_{\varphi x} + \frac{1}{2}\varphi^2\rho^2\partial_{\varphi\varphi} + \frac{1}{2}\sigma^2\partial_{xx} \tag{5.16}$$

and $\partial_r D$ denotes the set of boundary points of C that are (probabilistically) regular for D (see [3, Section 2 & Theorem 8] for fuller details). The strong Markov process (Φ, X) solves the system (4.16)+(4.17) driven by a single Brownian motion B so that $I\!L_{\Phi,X}$ is a *degenerate* parabolic differential operator and regularity of $\hat{V}$ in C indicated in (5.13) above cannot be inferred from the classic existence and uniqueness results for parabolic or elliptic equations (cf. [13, p. 131]). Instead we will derive this regularity by disclosing the hypoelliptic structure of $I\!L_{\Phi,X}$ that in turn will also establish that (Φ, X) is a strong Feller process as used in (5.15) above. The first step in this direction consists of reducing $I\!L_{\Phi,X}$ to its canonical form which is simpler to deal with.

2. *Canonical equation.* To reduce $I\!L_{\Phi,X}$ to its canonical form, set

$$U_t := F(X_t) - \log \Phi_t \tag{5.17}$$

for $t \geq 0$ where the function $F : I\!\!R \rightarrow I\!\!R$ is defined by (5.10) above. One can then verify using Itô's formula that

$$dU_t = a(U_t, X_t)\, dt \tag{5.18}$$

$$dX_t = \mu_0(X_t)\, dt + \sigma(X_t)\, dB_t \tag{5.19}$$

with $(U_0, X_0) = (u, x)$ under $\mathsf{P}^\infty_{u,x}$ where

$$a(u,x) = f(x) + e^u g(x) \tag{5.20}$$

$$f(x) = \frac{1}{2}\Big(\sigma^2\Big(\frac{\rho}{\sigma}\Big)' + (\mu_0+\mu_1)\frac{\rho}{\sigma}\Big)(x) - \lambda \tag{5.21}$$

$$g(x) = \lambda\, e^{-F(x)} \tag{5.22}$$

for u and x in $I\!\!R$. From (5.18)+(5.19) we see that (U,X) is a strong Markov process under P^∞ with the infinitesimal generator given by

$$I\!\!L_{U,X} = a\partial_u + \mu_0\partial_x + \frac{1}{2}\sigma^2\, \partial_{xx}\, . \tag{5.23}$$

Note that the process U is of bounded variation (the substitution $R_t := e^{U_t}$ transforms (5.18) into a Bernoulli equation which is solvable in a closed form). The differential operator $I\!\!L_{U,X}$ from (5.23) is a canonical version of the differential operator $I\!\!L_{\Phi,X}$ from (5.16) and it is clear from (5.17) that $I\!\!L_{\Phi,X}$ and $I\!\!L_{U,X}$ are C^∞-diffeomorphic. Hence to establish that $I\!\!L_{\Phi,X}$ is hypoelliptic it is sufficient to establish that $I\!\!L_{U,X}$ is hypoelliptic. We do the latter in the next step by verifying that $I\!\!L_{U,X}$ satisfies Hörmander's condition.

3. *Hypoellipticity (Hörmander's condition).* To verify that $I\!\!L_{U,X}$ from (5.23) satisfies Hörmander's condition (4.41) in [12], note that in the notation of that paper we have

$$I\!\!L_{U,X} = D_0 + D_1^2 \tag{5.24}$$

with $D_0 = a\partial_u + b\partial_x \sim [a;b]$ and $D_1 = (\sigma/\sqrt{2})\partial_x \sim [0;\sigma/\sqrt{2}]$ where a is given by (5.20) above and $b = \mu_0 - \sigma\sigma_x/2$. A direct calculation shows that

$$\begin{aligned}[D_1, D_0] &= (\sigma/\sqrt{2})a_x\partial_u + (\sigma/\sqrt{2})b_x\partial_x - b(\sigma_x/\sqrt{2})\partial_x \\ &\sim [(\sigma/\sqrt{2})a_x; (\sigma/\sqrt{2})b_x - b(\sigma_x/\sqrt{2})] \\ &=: [(\sigma/\sqrt{2})a_x; f^1]\end{aligned} \tag{5.25}$$

$$\begin{aligned}[D_1, [D_1, D_0]] &= (\sigma/\sqrt{2})((\sigma/\sqrt{2})a_x)_x\partial_u + (\sigma/\sqrt{2})f^1_x\partial_x - f^1(\sigma_x/\sqrt{2})\partial_x \\ &\sim [(\sigma/\sqrt{2})((\sigma/\sqrt{2})a_x)_x; (\sigma/\sqrt{2})f^1_x\partial_x - f^1(\sigma_x/\sqrt{2})] \\ &=: [(\sigma/\sqrt{2})((\sigma/\sqrt{2})a_x)_x; f^2]\end{aligned} \tag{5.26}$$

where f^1 and f^2 are functions of x in $I\!\!R$. Continuing by induction we find that

$$\begin{aligned}&[D_1, [D_1, \ldots, [D_1, D_0] \ldots] \\ &= (\sigma/\sqrt{2})(\ldots((\sigma/\sqrt{2})a_x)_x \ldots)_x\partial_u + (\sigma/\sqrt{2})f^{n-1}_x\partial_x - f^{n-1}(\sigma_x/\sqrt{2})\partial_x\end{aligned} \tag{5.27}$$

$$\sim [(\sigma/\sqrt{2})(\ldots((\sigma/\sqrt{2})a_x)_x\ldots)_x; (\sigma/\sqrt{2})f_x^{n-1} - f^{n-1}(\sigma_x/\sqrt{2})]$$
$$=: [(\sigma/\sqrt{2})(\ldots((\sigma/\sqrt{2})a_x)_x\ldots)_x; f^n]$$

where f^n is a function of x in $I\!R$ for $n \geq 1$ and $f^0 := b$. Since $\sigma > 0$ in D_1 hence we see that Hörmander's condition $\dim Lie(D_0, D_1) = 2$ holds at a point if inductively $a \neq 0$ or $(\sigma/\sqrt{2})a_x \neq 0$ or $(\sigma/\sqrt{2})((\sigma/\sqrt{2})a_x)_x \neq 0$ or $\ldots$ or $(\sigma/\sqrt{2})(\ldots((\sigma/\sqrt{2})a_x)_x\ldots)_x \neq 0$ at that point for some $n \geq 1$ (corresponding to the number of ∂_x in the expression). We claim that this must be true at all points since otherwise $a(u_0, x_0) = 0$ and $\partial_x^n a(u_0, x_0) = 0$ for all $n \geq 1$, and because $x \mapsto a(u_0, x) = f(x) - e^{u_0} g(x)$ is analytic (due to μ_0, μ_1, $\sigma > 0$ being analytic), we would be able to conclude by Taylor expansion that $a(u_0, x) = 0$ for all x belonging to an open interval containing $x_0 \in I\!R$ with $u_0 \in I\!R$ given and fixed. Applying the same argument to the boundary points of the interval and continuing in exactly the same way by (transfinite) induction if needed, we would be able to conclude that $a(u_0, x) = 0$ for all $x \in I\!R$. Recalling (5.18) this would mean that $U_t = u_0$ for all $t > 0$ when $U_0 = u_0$ so that by (5.17) we would be able to conclude that $\varPhi_t = \gamma(X_t)$ for all $t \geq 0$ where γ is given by (5.12) above with $\kappa = e^{-u_0}$. This would mean that the curve γ is a trap for $(\varPhi, X)$ which in turn is a contradiction with the hypothesis that such traps do not exist. This shows that Hörmander's condition $\dim Lie(D_0, D_1) = 2$ holds for $I\!L_{U,X}$ from (5.23) as claimed. Recalling (5.17) it follows therefore by Corollary 7 in [12] that $\hat{V}$ from (4.18) belongs to C^∞ on C as indicated in (5.13) above. Note that in exactly the same way one can verify that the backward time-space differential operator $-\partial_t + I\!L_{\varPhi,X}$ satisfies the parabolic Hörmander condition and hence by Corollary 9 in [12] we can conclude that $(\varPhi, X)$ is a strong Feller process as stated following (5.16) above.

4. *Stochastic maximum principle.* By (5.16) we see that (5.13) reads

$$\lambda(1+\varphi)\hat{V}_\varphi + \mu_0 \hat{V}_x + \varphi\rho\sigma\hat{V}_{\varphi x} + \frac{1}{2}\varphi^2\rho^2\hat{V}_{\varphi\varphi} + \frac{1}{2}\sigma^2\hat{V}_{xx} - \lambda\hat{V} = -L \tag{5.28}$$

in C. Differentiating both sides of (5.28) with respect to x and setting

$$U := \hat{V}_x \tag{5.29}$$

we find that U solves

$$\begin{aligned}\big(\lambda(1+\varphi)+\varphi(\rho\sigma)'\big)U_\varphi + \big(\mu_0+\sigma\sigma'\big)U_x + \varphi\rho\sigma U_{\varphi x} + \frac{1}{2}\varphi^2\rho^2 U_{\varphi\varphi}\\ + \frac{1}{2}\sigma^2 U_{xx} + (\mu_0'-\lambda)U = -\varphi^2\rho\rho'\hat{V}_{\varphi\varphi}\end{aligned} \tag{5.30}$$

in C. Setting

$$I\!L_{\tilde{\varPhi},\tilde{X}} = \big(\lambda(1+\varphi)+\varphi(\rho\sigma)'\big)\partial_\varphi + \big(\mu_0+\sigma\sigma'\big)\partial_x + \varphi\rho\sigma\partial_{\varphi x} + \frac{1}{2}\varphi^2\rho^2\partial_{\varphi\varphi} + \frac{1}{2}\sigma^2\partial_{xx} \tag{5.31}$$

we see that (5.30) can be rewritten as follows

$$I\!L_{\tilde{\varPhi},\tilde{X}} U - rU = -H \text{ in } C \tag{5.32}$$

where we set $r = \lambda - \mu_0'$ and

$$H = \varphi^2 \rho\rho' \hat{V}_{\varphi\varphi} \tag{5.33}$$

in C. From (5.9) and (5.33) we see that

$$\text{sign}(H) = -\,\text{sign}(\rho\rho') \tag{5.34}$$

in C. Without loss of generality consider the case in the sequel when $\mu_1 > \mu_0$ and $x \mapsto \rho(x)$ is increasing (note that other cases can be derived using exactly the same arguments). Then $\rho\rho' \geq 0$ so that by (5.34) we have

$$H \leq 0 \tag{5.35}$$

in C. Standard arguments (see e.g. [15, pp 158-163 & 166-173]) show ˘2pt that $I\!\!L_{\tilde{\Phi},\tilde{X}}$ is the infinitesimal generator of a strong Markov process $(\tilde{\Phi}, \tilde{X})$ which can be characterised as a unique weak solution to the system of stochastic differential equations

$$d\tilde{\Phi}_t = \big(\lambda(1+\tilde{\Phi}_t)+\tilde{\Phi}_t(\rho\sigma)'(\tilde{X}_t)\big)\,dt + \tilde{\Phi}_t\rho(\tilde{X}_t)\,d\tilde{B}_t \tag{5.36}$$

$$d\tilde{X}_t = \big(\mu_0(\tilde{X}_t)+(\sigma\sigma')(\tilde{X}_t)\big)\,dt + \sigma(\tilde{X}_t)\,d\tilde{B}_t \tag{5.37}$$

under a probability measure $\tilde{\mathsf{P}}_{\varphi,x}$ such that $\tilde{\mathsf{P}}_{\varphi,x}\big((\tilde{\Phi}_0, \tilde{X}_0) = (\varphi, x)\big) = 1$ for ˘2pt $(\varphi, x) \in [0,\infty)\times I\!\!R$ where $\tilde{B}$ is a standard Brownian motion. Note that the affine and linear placement of $\tilde{\Phi}_t$ in the drift and diffusion coefficient of (5.36) respectively ensures that that vertical line $\varphi = 0$ is an entrance boundary of $(\tilde{\Phi}, \tilde{X})$ for $[0,\infty)\times I\!\!R$ (meaning that the first component $\tilde{\Phi}$ remains in $(0,\infty)$ after starting at any non-negative point).

The previous conclusions suggest to consider the stopping time

$$\sigma_{D^0} = \inf\{\, t \geq 0 \mid (\tilde{\Phi}_t, \tilde{X}_t) \in D^0 \,\} \tag{5.38}$$

where D^0 denotes the interior of D. Then it is well ˘1pt known (cf. [2, Theorem 11.4, p. 62]) that $(\tilde{\Phi}_{\sigma_{D^0}}, \tilde{X}_{\sigma_{D^0}})$ on $\{\sigma_{D^0} < \infty\}$ belongs to the set $\partial_r D^0$ of boundary points of C that are (probabilistically) regular for D^0. Since $\partial_r D^0$ is contained in the set $\partial_r D$ of boundary points of C that are (probabilistically) regular for D, it follows that $(\tilde{\Phi}_{\sigma_{D^0}}, \tilde{X}_{\sigma_{D^0}})$ on $\{\sigma_{D^0} < \infty\}$ belongs to the set $\partial_r D$ and hence by the second equality in (5.15) upon recalling (5.29) we can conclude that the equality holds

$$U(\tilde{\Phi}_{\sigma_{D^0}}, \tilde{X}_{\sigma_{D^0}}) = 0 \tag{5.39}$$

$\tilde{\mathsf{P}}_{\varphi,x}$-almost surely on $\{\sigma_{D^0} < \infty\}$ for any $(\varphi, x) \in [0,\infty)\times I\!\!R$ given and fixed. Suppose that $U(\varphi, x) > 0$ for some $(\varphi, x) \in C$ and consider the stopping time

$$\nu = \inf\{\, t \geq 0 \mid (\tilde{\Phi}_t, \tilde{X}_t) \in Z \,\} \tag{5.40}$$

where Z denotes the set of all points in the closure of C at which U equals zero. Then $\nu \leq \sigma_{D^0}$ and by Itô's formula and the optional sampling theorem we find that

$$U(\varphi, x) = \tilde{\mathsf{E}}_{\varphi,x}\Big[e^{-r(\nu\wedge\tau_n)}U(\tilde{\Phi}_{\nu\wedge\tau_n}, \tilde{X}_{\nu\wedge\tau_n})\Big] + \tilde{\mathsf{E}}_{\varphi,x}\Big[\int_0^{\nu\wedge\tau_n} e^{-rt}H(\tilde{\Phi}_t, \tilde{X}_t)\,dt\Big] \tag{5.41}$$

for $n \geq 1$ where we use (5.32) above and $(\tau_n)_{n\geq 1}$ is a localising sequence of stopping times for the continuous local martingale arising from Itô's formula. Since $U(\tilde{\Phi}_\nu, \tilde{X}_\nu) = 0$ with $U(\tilde{\Phi}_t, \tilde{X}_t) \geq 0$ for $t \in [0, \nu]$ with $\nu < \infty$, we see by Fatou's lemma that

$$0 = \tilde{\mathsf{E}}_{\varphi,x}\Big[e^{-r\nu}U(\tilde{\Phi}_\nu, \tilde{X}_\nu)\Big] \geq \limsup_{n\to\infty} \tilde{\mathsf{E}}_{\varphi,x}\Big[e^{-r(\nu\wedge\tau_n)}U(\tilde{\Phi}_{\nu\wedge\tau_n}, \tilde{X}_{\nu\wedge\tau_n})\Big] \tag{5.42}$$

when $\nu < \infty$ and U is bounded on $C \setminus Z$ with $r > 0$ i.e. $\mu_0' < \lambda$. Letting $n \to \infty$ in (5.41) and using (5.42) we find by the monotone convergence theorem that

$$U(\varphi, x) \leq \tilde{\mathsf{E}}_{\varphi,x}\bigg[\int_0^\nu e^{-rt}H(\tilde{\Phi}_t, \tilde{X}_t)\,dt\bigg] \leq 0 \tag{5.43}$$

where in the final inequality we use (5.35) above. Since $U(\varphi, x) > 0$ this is a contradiction and hence $U(\varphi, x) \leq 0$ for all $(\varphi, x) \in C$. Recalling (5.29) this shows that $x \mapsto \hat{V}(\varphi, x)$ is decreasing on $I\!R$ for every $\varphi \in [0, \infty)$ given and fixed. This completes the proof in the special case when when ν is finite valued and U is bounded on $C \setminus Z$ with $r > 0$ i.e. $\mu_0' < \lambda$.

5. *Localisation.* The general case can be reduced to the special case of finite valued ν and bounded U by approximating the optimal stopping problem (4.18) with a sequence of optimal stopping problems having bounded continuation sets C_n which approximate the continuation set C alongside the pointwise convergence of the approximating value functions $\hat{V}^n$ to the value function $\hat{V}$ as $n \to \infty$. For instance, this can be achieved using the same arguments as above by instantaneously reflecting X downwards at n and upwards at $-n$ for any $n \geq 1$ given and fixed while keeping the remaining probabilistic characteristics of (Φ, X) unchanged. Indeed, approximating $\hat{V}^n$ and $\hat{V}$ by taking their infima over all stopping times $\tau \leq \tau_n$ instead, where τ_n denotes the first hitting time of X to either n or $-n$, we see that the resulting/approximating function $\hat{V}_n$ is the same for both $\hat{V}^n$ and $\hat{V}$ because (Φ, X) remains unchanged on $[0, \tau_n]$ for $n \geq 1$. Moreover, noting that the 'negative' integrand $e^{-\lambda t}(\lambda/c)$ in $\hat{V}^n$ and $\hat{V}$ integrates to a finite value $1/c$ over all $t \in [0, \infty)$, it is easily verified using the monotone convergence theorem with $\tau_n \uparrow \infty$ as $n \to \infty$ that $\hat{V}_n - R_n \leq \hat{V}^n \leq \hat{V}_n$ with $\hat{V}_n \to \hat{V}$ and $R_n \to 0$ pointwise as $n \to \infty$. Letting $n \to \infty$ in the previous two inequalities we thus see that $\hat{V}^n \to \hat{V}$ pointwise as claimed. Applying then the first part of the proof above when $\mu_0' < \lambda$ i.e. $r > 0$ to the approximating value function $\hat{V}^n$ of $\hat{V}$ upon using that C_n and therefore U^n as well are bounded (because the vertical component $[-n, n]$ of the state space is bounded while the 'negative' integrand in $\hat{V}^n$ globally integrates to a finite value as pointed out above), and noting that U^n equals zero at the horizontal lines $x = n$ and $x = -n$ (due to instantaneous reflection) so that the corresponding stopping time ν_n is finite valued (because $\tilde{X}$ solving (5.37) exits $[-n, n]$ with probability one), we can conclude that each $x \mapsto \hat{V}^n(\varphi, x)$ is decreasing on $I\!R$ for every $n \geq 1$ and $\varphi \in [0, \infty)$ given and fixed. Hence passing to the pointwise limit as $n \to \infty$ we obtain that $x \mapsto \hat{V}(\varphi, x)$ is decreasing as claimed for every $\varphi \in [0, \infty)$ given and fixed. The case $\mu_0' \geq \lambda$ can be reduced to the case $r > 0$ by replacing X with $S(X)$ where S is the scale function of X (characterised as a strictly increasing analytic solution to $I\!L_X S = 0$). This has the effect of setting the initial drift μ_0 of the observed diffusion process $S(X)$ equal to 0, so that $r = \lambda - \mu_0' = \lambda > 0$, which makes the arguments above applicable to $S(X)$ in place of X. This completes the proof in the general case.

Part II: In this part we allow that a curve γ is a trap for (Φ, X) (i.e. the equality (5.11) holds on $I\!R$ for some $\kappa > 0$ given and fixed). Under this hypothesis we divide the proof in two further parts as follows.

6. Replacing λ by $\lambda_\varepsilon := \lambda+\varepsilon$ for $\varepsilon > 0$ we claim that the equality (5.11) fails at one point in $I\!R$ at least for every $\kappa_\varepsilon > 0$ given and fixed. To verify the claim suppose that (5.11) holds on $I\!R$ for both λ and κ as well as λ_ε and κ_ε for some $\kappa_\varepsilon > 0$, i.e.

$$G_1(x) - \lambda = \frac{\lambda}{\kappa}\, G_2(x) \tag{5.44}$$

$$G_1(x) - \lambda_\varepsilon = \frac{\lambda_\varepsilon}{\kappa_\varepsilon}\, G_2(x) \tag{5.45}$$

for all $x \in I\!R$ where $G_1 = (1/2)\big(\sigma^2 F'' + (\mu_0+\mu_1)F'\big)$ and $G_2 = e^{-F}$ with F from (5.10) above. Differentiating with respect to x in both (5.44) and (5.45) we find that

$$\frac{\lambda}{\kappa} = \frac{G_1'(x)}{G_2'(x)} = \frac{\lambda_\varepsilon}{\kappa_\varepsilon} \tag{5.46}$$

for all $x \in I\!R$. Replacing $\lambda_\varepsilon/\kappa_\varepsilon$ by λ/κ in (5.45) and using that (5.44) holds for $x \in I\!R$, we see that ε must be equal to zero which is a contradiction, establishing the claim. Thus it follows that if we replace λ by λ_ε for $\varepsilon > 0$ in the kinematics of $\Phi^{(\lambda)}$ from (4.10) above, and consider the optimal stopping problem (4.18) with $\Phi^{(\lambda_\varepsilon)}$ in place of $\Phi^{(\lambda)}$ while retaining the same $\lambda > 0$ in the rest of the integrand, then no curve γ is a trap for $(\Phi^{(\lambda_\varepsilon)}, X)$ (because the equality (5.11) with λ_ε in place of λ fails at one point in $I\!R$ at least for every $\kappa_\varepsilon > 0$ given and fixed) so that Part I of the proof above is applicable in exactly the same way (note that the equality of λ appearing in (4.10) and (4.18) has played no role in the arguments).

7. To realise the idea expressed in the previous part, set

$$\Phi_t^{(\lambda)} = e^{\lambda t} L_t \Big(\varphi + \lambda \int_0^t \frac{ds}{e^{\lambda s} L_s}\Big) \tag{5.47}$$

for $t \ge 0$ where $\lambda > 0$ and $\varphi \in [0,\infty)$ as in (4.10) above. Note that

$$\lambda \mapsto \Phi_t^{(\lambda)} \text{ is increasing on } [0,\infty) \tag{5.48}$$

for every $t \ge 0$ given and fixed (because $s \le t$ in (5.47) above). Set

$$\hat{V}^{(\lambda_\varepsilon)}(\varphi,x) = \inf_\tau \mathsf{E}_{\varphi,x}^\infty \Big[\int_0^\tau e^{-\lambda t}\Big(\Phi_t^{(\lambda_\varepsilon)} - \frac{\lambda}{c}\Big)\, dt\Big] \tag{5.49}$$

for $(\varphi,x) \in [0,\infty)\times I\!R$ with $\mathsf{P}_{\varphi,x}^\infty((\Phi_0^{(\lambda_\varepsilon)}, X_0) = (\varphi,x)) = 1$, where $\lambda_\varepsilon := \lambda+\varepsilon$ with ˇ1pt $\lambda > 0$ and $\varepsilon \ge 0$, and the infimum in (5.49) is taken over all stopping times τ of $(\Phi^{(\lambda_\varepsilon)}, X)$ as in (4.18) above. Note that $\hat{V}^{(\lambda)} = \hat{V}$ for every $\lambda > 0$ where $\hat{V}$ is given in (4.18) above.

We claim that the following relation holds

$$\lim_{\varepsilon \downarrow 0} \hat{V}^{(\lambda_\varepsilon)}(\varphi,x) = \hat{V}^{(\lambda)}(\varphi,x) \tag{5.50}$$

for all $(\varphi, x) \in [0,\infty) \times I\!R$. For this, fix any $(\varphi, x) \in [0,\infty) \times I\!R$ and note that

$$\hat{V}^{(\lambda_\varepsilon)}(\varphi, x) \le \mathsf{E}^{\infty}_{\varphi,x}\Big[\int_0^{\tau_D \wedge n} e^{-\lambda t} \Big(\varPhi_t^{(\lambda_\varepsilon)} - \frac{\lambda}{c} \Big)\, dt \Big] \tag{5.51}$$

for all $\varepsilon > 0$ and $n \ge 1$ where the stopping time τ_D is optimal for $V^{(\lambda)}(\varphi, x)$. Letting $\varepsilon \downarrow 0$ in (5.51) and using the dominated convergence theorem we find that

$$\limsup_{\varepsilon \downarrow 0} \hat{V}^{(\lambda_\varepsilon)}(\varphi, x) \le \mathsf{E}^{\infty}_{\varphi,x}\Big[\int_0^{\tau_D \wedge n} e^{-\lambda t} \Big(\varPhi_t^{(\lambda)} - \frac{\lambda}{c} \Big)\, dt \Big] \tag{5.52}$$

for all $n \ge 1$. Letting $n \to \infty$ in (5.52) and using the monotone convergence theorem we get

$$\limsup_{\varepsilon \downarrow 0} \hat{V}^{(\lambda_\varepsilon)}(\varphi, x) \le \hat{V}^{(\lambda)}(\varphi, x)\,. \tag{5.53}$$

Moreover, letting τ_{D_ε} denote the optimal stopping time for $\hat{V}^{(\lambda_\varepsilon)}(\varphi, x)$ and recalling (5.48) above, we can conclude that

$$\begin{aligned} \hat{V}^{(\lambda)}(\varphi, x) &\le \mathsf{E}^{\infty}_{\varphi,x}\Big[\int_0^{\tau_{D_\varepsilon}} e^{-\lambda t} \Big(\varPhi_t^{(\lambda)} - \frac{\lambda}{c} \Big)\, dt \Big] \\ &\le \mathsf{E}^{\infty}_{\varphi,x}\Big[\int_0^{\tau_{D_\varepsilon}} e^{-\lambda t} \Big(\varPhi_t^{(\lambda_\varepsilon)} - \frac{\lambda}{c} \Big)\, dt \Big] = \hat{V}^{(\lambda_\varepsilon)}(\varphi, x) \end{aligned} \tag{5.54}$$

for all $\varepsilon > 0$. Letting $\varepsilon \downarrow 0$ in (5.54) we get

$$\hat{V}^{(\lambda)}(\varphi, x) \le \liminf_{\varepsilon \downarrow 0} \hat{V}^{(\lambda_\varepsilon)}(\varphi, x)\,. \tag{5.55}$$

Combining (5.53) and (5.55) we obtain (5.50) as claimed.

Applying Part I of the proof above to each $\hat{V}^{(\lambda_\varepsilon)}$ from (5.49) with $\varepsilon > 0$ given and fixed, we find that $x \mapsto \hat{V}^{(\lambda_\varepsilon)}(\varphi, x)$ is decreasing/increasing or increasing/decreasing depending on whether $\mu_1 > \mu_0$ and $x \mapsto \rho(x)$ is increasing/decreasing or $\mu_1 < \mu_0$ and $x \mapsto \rho(x)$ is decreasing/increasing respectively for every $\varphi \in [0,\infty)$ given and fixed. Recalling (5.50) we see that the same monotonicity properties hold for $x \mapsto \hat{V}^{(\lambda)}(\varphi, x)$ for every $\varphi \in [0,\infty)$ given and fixed as claimed. This completes the proof $\square$

Remark 7. One could also encounter localised versions of the degenerate cases which are not covered by Theorem 6 when μ_0 , μ_1 , σ are C^∞ but not analytic on $I\!R$. The equality (5.11) may then hold only on a subinterval I of $I\!R$ (including a singleton) for some $\kappa > 0$ and the curve γ given by (5.12) for $x \in I$ (representing the points at which Hörmander's condition fails) is a (local) trap for $(\varPhi, X)$ only while X belongs to I . We will not study the degenerate cases (either global or local) in the present paper. Note that the proof of Theorem 6 remains valid when μ_0 , μ_1 , σ are C^∞ but not analytic on $I\!R$, and consequently the GS conjecture is true, if the equality (5.11) fails at all points in $I\!R$ for any $\kappa > 0$.

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