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MULTIPLICATIVE POLYNOMIAL EQUATIONS IN INFINITELY MANY VARIABLES

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Abstract. This paper describes infinite sets of polynomial equations in infinitely many variables with the property that the existence of a solution or even an approximate solution for every finite subset of the equations implies the existence of a solution for the infinite set of equations.

1. Finitely many implies infinite many

In mathematics there are theorems asserting that, for certain classes of equations, if every finite subset of an infinite set of the equations has a solution, then the infinite set of equations has a solution. For example, if every finite subset of an infinite set of linear equations in n variables (that is, equations of the form $\sum_{j=1}^{n} a_{i,j}x_j = b_i$) has a solution, then the infinite set of linear equations in n variables has a solution. For infinite sets of linear equations in infinitely many variables, in which each equation contains only finitely many variables (that is, equations of the form $\sum_{j=1}^{\infty} a_{i,j}x_j = b_i$ with $a_{i,j} \neq 0$ for only finitely many j), a "finitely solvable implies infinitely solvable" theorem is also true. For a survey of related results, see Nathanson [4].

There are analogous results for linear equations that contain infinitely many variables. The pair (p, q) is *Hölder conjugate* or, simply, *conjugate* if $p > 1$ and $q > 1$ are real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

or if $p = 1$ and $q = \infty$. For $1 \leq p < \infty$, let

$$\ell^p = \left\{ \mathbf{a} = (a_j)_{j=1}^{\infty} : \|\mathbf{a}\|_p = \left(\sum_{j=1}^{\infty} |a_j|^p \right)^{1/p} < \infty \right\}$$

and, for $p = \infty$, let

$$\ell^{\infty} = \left\{ \mathbf{a} = (a_j)_{j=1}^{\infty} : \|\mathbf{a}\|_{\infty} = \sup\{|a_j| : j = 1, 2, 3, \ldots\} < \infty \right\}$$

be the classical Lebesgue spaces of infinite sequences of real or complex numbers.

Recall Hölder's inequality: If (p, q) is a conjugate pair and

$$\mathbf{a} = (a_j)_{j=1}^{\infty} \in \ell^p \qquad \text{and} \qquad \mathbf{x} = (x_j)_{j=1}^{\infty} \in \ell^q$$

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then

$$\mathbf{ax} = (a_j x_j)_{j=1}^{\infty} \in \ell^1$$

and

$$\sum_{j=1}^{\infty} |a_j x_j| = \|\mathbf{ax}\|_1 \leq \|\mathbf{a}\|_p \|\mathbf{x}\|_q .$$

Thus, the infinite series

$$(\mathbf{a}, \mathbf{x}) = \sum_{j=1}^{\infty} a_j x_j$$

converges absolutely and it makes sense to ask if there is a solution $\mathbf{x} \in \ell^q$ of the linear equation in infinitely many variables

$$L(\mathbf{x}) = (\mathbf{a}, \mathbf{x}) = \sum_{j=1}^{\infty} a_j x_j = b. \tag{1}$$

We retain the usual ambiguity between sequences of numbers and sequences of variables. An *exact solution* or, simply, a *solution* of equation (1) is a sequence $\mathbf{x} = (x_j)_{j=1}^{\infty} \in \ell^q$ such that $L(\mathbf{x}) = b$. Equation (1) has an *approximate solution* if, for every $\varepsilon > 0$, there exists $\mathbf{x}_\varepsilon \in \ell^q$ such that

$$|L(\mathbf{x}_\varepsilon) - b| < \varepsilon.$$

Let $\mathbf{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers and $\mathbf{N}^k$ the set of k-tuples of positive integers. Extending a classical result of F. Riesz [5, pp. 61–62] (see also Banach [2, p. 47]), Abian and Eslami [1] proved the following beautiful theorem.

Theorem 1. *Let (p, q) be a conjugate pair with $p > 1$ and $q > 1$ and let $\mathbf{a}_i = (a_{i,j})_{j=1}^{\infty} \in \ell^p$ for all $i \in \mathbf{N}$, Consider the linear equation in infinitely many variables*

$$L_i(\mathbf{x}) = \sum_{j=1}^{\infty} a_{i,j} x_j = b_i.$$

Let $M > 0$. If, for every finite subset S of $\mathbf{N}$, there is a sequence $\mathbf{x}_S \in \ell^q$ such that $\|\mathbf{x}_S\|_q \leq M$ and $L_i(\mathbf{x}_S) = b_i$ for all $i \in S$, then there is a sequence $\mathbf{x} \in \ell^q$ such that $\|\mathbf{x}\|_q \leq M$ and $L_i(\mathbf{x}) = b_i$ for all $i \in \mathbf{N}$.

Abian and Eslami proved their theorem only for $p > 1$ and for countably many equations with real coefficients and real solutions, but the result also holds for $(p, q) = (1, \infty)$ and for uncountably many equations with complex coefficients and complex solutions. In Appendix A we prove Riesz's theorem and derive Theorem 1 from it.

It is important to observe that Theorem 1 is not true without the condition that the norm of $\|\mathbf{x}_S\|_q$ is uniformly bounded by M for all finite sets S. The following example is essentially due to Helly [3].

Let $\mathbf{a} = (a_j)_{j=1}^{\infty} \in \ell^p$ with $a_j \neq 0$ for all $j \in \mathbf{N}$. For all $i \in \mathbf{N}$, define the sequence $\mathbf{a}_i = (a_{i,j})_{j=1}^{\infty} \in \ell^p$ as follows:

$$a_{i,j} = \begin{cases} 0 & \text{if } j < i \\ a_i & \text{if } j \geq i. \end{cases}$$

For all $i \in \mathbf{N}$, consider the linear equation in infinitely many variables

$$L_i(\mathbf{x}) = \sum_{j=1}^{\infty} a_{i,j} x_j = \sum_{j=i}^{\infty} a_j x_j = 1.$$

We obtain the "triangular" system of equations

$$\begin{aligned}
a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_ix_i + a_{i+1}x_{i+1} + \cdots \quad &= 1 \\
a_2x_2 + a_3x_3 + \cdots + a_ix_i + a_{i+1}x_{i+1} + \cdots \quad &= 1 \\
a_3x_3 + \cdots + a_ix_i + a_{i+1}x_{i+1} + \cdots \quad &= 1 \\
&\vdots \\
a_ix_i + a_{i+1}x_{i+1} + \cdots \quad &= 1 \\
a_{i+1}x_{i+1} + \cdots \quad &= 1
\end{aligned}$$

For all $i \geq 1$, if $L_i(\mathbf{x}) = L_{i+1}(\mathbf{x}) = 1$, then $a_i x_i = 0$ and so $x_i = 0$ (because $a_i \neq 0$). It follows that the infinite set of equations has no solution. However, every finite subset of these equations is solvable.

Here is one solution. For all $r \in \mathbf{N}$, the sequence $\mathbf{x}_r = (x_{r,j})_{j=1}^{\infty} \in \ell^q$ defined by

$$x_{r,j} = \begin{cases} 1/a_r & \text{if } j = r \\ 0 & \text{if } j \neq r \end{cases}$$

is a solution of the finite set of equations $\{L_i(\mathbf{x}) = 1 : i = 1, \ldots, r\}$. We have

$$\|\mathbf{x}_r\|_q = \frac{1}{|a_r|} \qquad \text{and} \qquad \lim_{r\to\infty} \|\mathbf{x}_r\|_q = \lim_{r\to\infty} \frac{1}{|a_r|} = \infty.$$

Thus, the sequence of solutions $(\mathbf{x}_r)_{r=1}^{\infty}$ is not uniformly bounded in ℓ^q.

If $\mathbf{x}_r$ is any solution of the finite set of the first r equations, then

$$1 = \sum_{j=r}^{\infty} a_j x_j = \sum_{j=1}^{\infty} a_{r,j} x_j = \|\mathbf{a}_r \mathbf{x}_r\|_1 \leq \|\mathbf{a}_r\|_p \|\mathbf{x}_r\|_q$$

and so

$$\|\mathbf{x}_r\|_q \geq \frac{1}{\|\mathbf{a}_r\|_p}.$$

Because $\mathbf{a} \in \ell^p$, we have $\lim_{r\to\infty} \|\mathbf{a}_r\|_p = 0$ and so the sequence of solutions $(\mathbf{x}_r)_{r=1}^{\infty}$ is not uniformly bounded in ℓ^q.

The following result extends Theorem 1 to approximately solvable systems of linear equations.

Theorem 2. *Let (p, q) be a conjugate pair. Let I be an infinite set and let $\mathbf{a}_i = (a_{i,j})_{j=1}^{\infty} \in \ell^p$ for all $i \in I$. Consider the linear equation in infinitely many variables*

$$L_i(\mathbf{x}) = \sum_{j=1}^{\infty} a_{i,j} x_j = b_i.$$

Let $M > 0$. If, every finite subset S of I and for every $\varepsilon > 0$, the finite set of linear inequalities

$$\{|L_i(\mathbf{x}) - b_i| \leq \varepsilon : i \in S\}$$

has a solution $\mathbf{x}_{S,\varepsilon} \in \ell^q$ with $\|\mathbf{x}_{S,\varepsilon}\|_q \leq M$, then the infinite set of linear equations

$$\{L_i(\mathbf{x}) = b_i : i \in I\}$$

has an exact solution $\mathbf{x} \in \ell^q$ with $\|\mathbf{x}\|_q \leq M$.

In this paper we generalize Theorem 2 to certain infinite sets of polynomial equations in infinitely many variables and prove (this is our main result) that the existence of norm-bounded or even sequentially bounded approximate solutions to all finite subsets of the set of polynomial equations is sufficient to guarantee the existence of an exact solution to the infinite set (Theorems 4 and 5). The theorems on linear equations are special cases of the polynomial results.

Our results apply to polynomials with coefficients and solutions in the field of real numbers and also in the field of complex numbers. However, the "finitely many implies infinitely many" paradigm is not true for all subfields of the complex numbers. It is not true, for example, for the field $\mathbf{Q}$ of rational numbers. Let $(b_i)_{i\in I}$ be an infinite sequence of irrational numbers and consider the infinite set of linear equations $\{x_i = b_i : i \in I\}$. For every $\varepsilon > 0$, every finite subset and, indeed, the infinite set of the equations has an approximate solution in $\mathbf{Q}$, but neither the infinite set nor any nonempty finite subset of the equations has an exact solution in $\mathbf{Q}$.

2. Polynomials in infinitely many variables

Let D be a positive integer, let $d \in \{1, 2, \ldots, D\}$, and let

$$1 \leq D < q \leq \infty.$$

If $q < \infty$, then

$$\frac{1}{q/(q-d)} + \frac{1}{q/d} = 1$$

and so the real numbers $q/(q-d) > 1$ and $q/d > 1$ form a conjugate pair. For $q = \infty$, we define

$$\frac{q}{(q-d)} = 1 \qquad \text{and} \qquad \frac{q}{d} = \infty.$$

If

$$\mathbf{a} = (a_j)_{j=1}^{\infty} \in \ell^{q/(q-d)}$$

and

$$\mathbf{x} = (x_j)_{j=1}^{\infty} \in \ell^{q}$$

then

$$\mathbf{x}^d = \left(x_j^d\right)_{j=1}^{\infty} \in \ell^{q/d}.$$

From Hölder's inequality we obtain

$$\mathbf{a}\mathbf{x}^d = \left(a_j x_j^d\right)_{j=1}^{\infty} \in \ell^{1}$$

and

$$\sum_{j=1}^{\infty} \left|a_j x_j^d\right| = \left\|\mathbf{a}\mathbf{x}^d\right\|_1 \leq \|\mathbf{a}\|_{q/(q-d)} \left\|\mathbf{x}^d\right\|_{q/d}.$$

Thus, the infinite series

$$\left(\mathbf{a}, \mathbf{x}^d\right) = \sum_{j=1}^{\infty} a_j x_j^d$$

converges absolutely.

For all $k \in \{1, 2, \ldots, D\}$, let $\mathcal{D}_k$ be the set of all k-tuples of positive integers whose sum is at most D, that is,

$$\mathcal{D}_k = \left\{(d_1, \ldots, d_k) \in \mathbf{N}^k : d_1 + d_2 + \cdots + d_k \leq D\right\}. \tag{2}$$

This is a finite set of cardinality $\binom{D}{k}$. The set $\bigcup_{k=1}^{D} \mathcal{D}_k$ has cardinality $\sum_{k=1}^{D} \binom{D}{k} = 2^D - 1$. For all $\Delta = (d_1, d_2, \ldots, d_k) \in \mathcal{D}_k$ and $J = (j_1, j_2, \ldots, j_k) \in \mathbf{N}^k$, we define the monomial

$$x_J^\Delta = x_{j_1}^{d_1} x_{j_2}^{d_2} \cdots x_{j_k}^{d_k} \tag{3}$$

of degree

$$|\Delta| = d_1 + d_2 + \cdots + d_k \le D.$$

For example, let $D = 3$. There are seven sets of monomials. For $k = 1$ and $J = (j_1) \in \mathbf{N}^1$, we have

$$\Delta \in \mathcal{D}_1 = \{(1), (2), (3)\} \quad \text{and} \quad x_J^\Delta \in \left\{ x_{j_1},\ x_{j_1}^2,\ x_{j_1}^3 \right\}.$$

For $k = 2$ and $J = (j_1, j_2) \in \mathbf{N}^2$, we have

$$\Delta \in \mathcal{D}_2 = \{(1,1), (1,2), (2,1)\} \quad \text{and} \quad x_J^\Delta \in \left\{ x_{j_1} x_{j_2},\ x_{j_1} x_{j_2}^2,\ x_{j_1}^2 x_{j_2} \right\}.$$

For $k = 3$ and $J = (j_1, j_2, j_3) \in \mathbf{N}^3$, we have

$$\Delta \in \mathcal{D}_3 = \{(1,1,1)\} \quad \text{and} \quad x_J^\Delta = x_{j_1} x_{j_2} x_{j_3}.$$

Monomials do not have a unique representation in the form x_J^Δ. For example, with $D = 3$, let $k = 2$ and $J = (1,1) \in \mathbf{N}^2$. If $\Delta = (1,2) \in \mathcal{D}_2$, then $x_J^\Delta = x_1 x_1^2 = x_1^3$. If $\Delta = (2,1) \in \mathcal{D}_2$, then $x_J^\Delta = x_1^2 x_1 = x_1^3$. If $D = 3$, $k = 1$, $J = (1) \in \mathbf{N}^1$, and $\Delta = (3) \in \mathcal{D}_1$, then $x_J^\Delta = x_1^3$.

We do not need unique representation of monomials, but would have unique representation if we considered only k-tuples $(j_1, j_2, \ldots, j_k) \in \mathbf{N}^k$ such that $j_1 < j_2 < \cdots < j_k$.)

We consider only polynomials with zero constant term. In this paper, "polynomial" means polynomial with zero constant term.

A *polynomial of degree at most D in infinitely many variables* $x_1, x_2, x_3, \ldots$ is an infinite series of the form

$$P(\mathbf{x}) = \sum_{k=1}^{D} \sum_{\Delta \in \mathcal{D}_k} \sum_{J \in \mathbf{N}^k} a_{\Delta, J}\ x_J^\Delta \tag{4}$$

with coefficients $a_{\Delta,J} \in \mathbf{R}$. For example, a polynomial of degree at most 3 in infinitely many variables is of the form

$$\begin{aligned} P(\mathbf{x}) &= \sum_{j_1=1}^{\infty} a_{(1),(j_1)} x_{j_1} + \sum_{j_1=1}^{\infty} a_{(2),(j_1)} x_{j_1}^2 + \sum_{j_1=1}^{\infty} a_{(3),(j_1)} x_{j_1}^3 \\ &\quad + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} a_{(1,1),(j_1,j_2)} x_{j_1} x_{j_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} a_{(1,2),(j_1,j_2)} x_{j_1} x_{j_2}^2 \\ &\quad + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} a_{(2,1),(j_1,j_2)} x_{j_1}^2 x_{j_2} + \sum_{j_1=1}^{\infty} \sum_{j_2=1}^{\infty} \sum_{j_3=1}^{\infty} a_{(1,1,1),(j_1,j_2,j_3)} x_{j_1} x_{j_2} x_{j_3}. \end{aligned}$$

3. Multiplicative polynomials

Let

$$P(\mathbf{x}) = \sum_{k=1}^{D} \sum_{\Delta \in \mathcal{D}_k} \sum_{J \in \mathbf{N}^k} a_{\Delta, J}\ x_J^\Delta$$

be a polynomial of degree at most D in infinitely many variables $\mathbf{x} = (x_j)_{j=1}^\infty$ with coefficients $a_{\Delta,J}$. We consider polynomials whose coefficients $a_{\Delta,J}$ are *multiplicative* in the following sense: For all $d \in \{1, 2, \ldots, D\}$ there is a sequence

$$\mathbf{a}_d = (a_{d,j})_{j=1}^\infty \in \ell^{q/(q-d)}$$

such that, if

$$\Delta = (d_1, d_2, \ldots, d_k) \in \mathcal{D}_k \qquad \text{and} \qquad J = (j_1, j_2, \ldots, j_k) \in \mathbf{N}^k$$

then

$$a_{\Delta,J} = a_{(d_1,d_2,\ldots,d_k),(j_1,j_2,\ldots,j_k)} = a_{d_1,j_1} a_{d_2,j_2} \cdots a_{d_k,j_k}.$$

This gives the monomial factorization

$$\begin{aligned} a_{\Delta,J} x_J^\Delta &= (a_{d_1,j_1} a_{d_2,j_2} \cdots a_{d_k,j_k}) \left(x_{j_1}^{d_1} x_{j_2}^{d_2} \cdots x_{j_k}^{d_k} \right) \\ &= \left(a_{d_1,j_1} x_{j_1}^{d_1} \right) \left(a_{d_2,j_2} x_{j_2}^{d_2} \right) \cdots \left(a_{d_k,j_k} x_{j_k}^{d_k} \right) \end{aligned}$$

and so

$$\begin{aligned} \sum_{J \in \mathbf{N}^k} a_{\Delta,J}\ x_J^\Delta &= \sum_{(j_1,j_2,\ldots,j_j) \in \mathbf{N}^k} \left(a_{d_1,j_1} x_{j_1}^{d_1} \right) \left(a_{d_2,j_2} x_{j_2}^{d_2} \right) \cdots \left(a_{d_k,j_k} x_{j_k}^{d_k} \right) \\ &= \left(\sum_{j_1=1}^\infty a_{d_1,j_1} x_{j_1}^{d_1} \right) \left(\sum_{j_2=1}^\infty a_{d_2,j_2} x_{j_2}^{d_2} \right) \cdots \left(\sum_{j_k=1}^\infty a_{d_k,j_k} x_{j_k}^{d_k} \right) \\ &= \left(\mathbf{a}_{d_1}, \mathbf{x}^{d_1} \right) \left(\mathbf{a}_{d_2}, \mathbf{x}^{d_2} \right) \cdots \left(\mathbf{a}_{d_k}, \mathbf{x}^{d_k} \right). \end{aligned}$$

For $\mathbf{x} = (x_j)_{j=1}^\infty \in \ell^q$, the rearrangement is justified by the absolute convergence of the k infinite series $\left(\mathbf{a}_{d_1}, \mathbf{x}^{d_1} \right), \left(\mathbf{a}_{d_2}, \mathbf{x}^{d_2} \right), \ldots, \left(\mathbf{a}_{d_k}, \mathbf{x}^{d_k} \right)$. We obtain

$$\begin{aligned} P(\mathbf{x}) &= \sum_{k=1}^D \sum_{\Delta \in \mathcal{D}_k} \sum_{J \in \mathbf{N}^k} a_{\Delta,J}\ x_J^\Delta \\ &= \sum_{k=1}^D \sum_{(d_1,d_2,\ldots,d_k) \in \mathcal{D}_k} \left(\sum_{j_1=1}^\infty a_{d_1,j_1} x_{j_1}^{d_1} \right) \left(\sum_{j_2=1}^\infty a_{d_2,j_2} x_{j_2}^{d_2} \right) \cdots \left(\sum_{j_k=1}^\infty a_{d_k,j_k} x_{j_k}^{d_k} \right) \\ &= \sum_{k=1}^D \sum_{(d_1,d_2,\ldots,d_k) \in \mathcal{D}_k} \left(\mathbf{a}_{d_1}, \mathbf{x}^{d_1} \right) \left(\mathbf{a}_{d_2}, \mathbf{x}^{d_2} \right) \cdots \left(\mathbf{a}_{d_k}, \mathbf{x}^{d_k} \right). \end{aligned}$$

This is a finite sum, and so the series $P(\mathbf{x})$ converges absolutely for all $\mathbf{x} \in \ell^q$. Thus, for all $b \in \mathbf{R}$, the *multiplicative polynomial equation*

$$P(\mathbf{x}) = b$$

is well-defined, and we can ask if this equation has a solution $\mathbf{x} \in \ell^q$.

Here is an example of a multiplicative polynomial. Let $1 \leq d \leq D < q \leq \infty$ and so

$$1 \leq \frac{q}{q-1} \leq \frac{q}{q-d} \leq \frac{q}{q-D}.$$

For all $s \in \mathbf{C}$ with $\Re(s) > (q-d)/q$, the Riemann zeta function

$$\zeta\left(\frac{qs}{q-d} \right) = \sum_{j=1}^\infty \frac{1}{j^{qs/(q-d)}} = \sum_{j=1}^\infty \left(\frac{1}{j^s} \right)^{q/(q-d)}$$

converges absolutely. Consider the sequence

$$\mathbf{a}_d = \left(\frac{1}{j^s}\right)_{j=1}^{\infty} \in \ell^{q/(q-d)}$$

and the polynomial

$$(\mathbf{a}_d, \mathbf{x}^d) = \sum_{j=1}^{\infty} \frac{x_j^d}{j^s}.$$

Let

$$\Delta = (d_1, d_2, \dots, d_k) \in \mathcal{D}_k \qquad \text{and} \qquad J = (j_1, j_2, \dots, j_k) \in \mathbf{N}^k.$$

For all $h \in \{1, \dots, k\}$, let $s_h \in \mathbf{C}$ satisfy $\Re(s_h) > (q-d)/q$ and let $\mathbf{a}_{d_h} = (j^{-s_h})_{j=1}^{\infty}$. Then

$$\begin{aligned}\sum_{J \in \mathbf{N}^k} a_{\Delta,J}\, x_J^{\Delta} &= \left(\mathbf{a}_{d_1}, \mathbf{x}^{d_1}\right)\left(\mathbf{a}_{d_2}, \mathbf{x}^{d_2}\right) \cdots \left(\mathbf{a}_{d_k}, \mathbf{x}^{d_k}\right) \\ &= \sum_{(j_1,\dots,j_k) \in \mathbf{N}^k} \frac{x_{j_1}^{d_1} \cdots x_{j_k}^{d_k}}{j_1^{s_1} \cdots j_k^{s_k}}.\end{aligned}$$

The associated multiplicative polynomial of degree at most D is

$$\begin{aligned}P(\mathbf{x}) &= P(\mathbf{x}, s_1, \dots, s_h) \\ &= \sum_{k=1}^{D} \sum_{\Delta \in \mathcal{D}_k} \sum_{J \in \mathbf{N}^k} a_{\Delta,J}\, x_J^{\Delta} \\ &= \sum_{k=1}^{D} \sum_{(d_1,\dots,d_k) \in \mathcal{D}_k} \sum_{(j_1,\dots,j_k) \in \mathbf{N}^k} \frac{x_{j_1}^{d_1} \cdots x_{j_k}^{d_k}}{j_1^{s_1} \cdots j_k^{s_k}}.\end{aligned}$$

For $D = 1$ and $q = 2$ and for complex numbers s with $\Re(s) > 1/2$, the linear polynomial in infinitely many variables is the Dirichlet series

$$P(\mathbf{x}, s) = \sum_{n=1}^{\infty} \frac{x_n}{n^s}.$$

The analogue of Theorem 1 for linear equations is the following "finitely many implies infinitely many" solvability result for multiplicative polynomial equations.

Theorem 3. *Let D be a positive integer and let $1 \leq D < q \leq \infty$. Let I be an infinite set. For all $i \in I$ and $d \in \{1, 2, \dots, D\}$, let*

$$\mathbf{a}_{i,d} = (a_{i,d,j})_{j=1}^{\infty} \in \ell^{q/(q-d)}.$$

For $\Delta = (d_1, d_2, \dots, d_k) \in \mathcal{D}_k$ and $J = (j_1, j_2, \dots, j_k) \in \mathbf{N}^k$, let

$$a_{i,\Delta,J} = a_{i,d_1,j_1} a_{i,d_2,j_2} \cdots a_{i,d_k,j_k}.$$

For all $i \in I$, the sequences $\mathbf{a}_{i,d}$ determine the multiplicative polynomial equation

$$\begin{aligned} P_i(\mathbf{x}) &= \sum_{k=1}^{D} \sum_{\Delta \in \mathcal{D}_k} \sum_{J \in \mathbf{N}^k} a_{i,\Delta,J} \ x_J^{\Delta} \\ &= \sum_{k=1}^{D} \sum_{(d_1,d_2,\ldots,d_k) \in \mathcal{D}_k} \left(\mathbf{a}_{i,d_1}, \mathbf{x}^{d_1}\right) \left(\mathbf{a}_{i,d_2}, \mathbf{x}^{d_2}\right) \cdots \left(\mathbf{a}_{i,d_k}, \mathbf{x}^{d_k}\right) \\ &= b_i. \end{aligned}$$

Let $M > 0$. If, for every finite subset S of I, the finite set of polynomial equations $\{P_i(\mathbf{x}) = b_i : i \in S\}$ has a solution $\mathbf{x}_S \in \ell^q$ with $\|\mathbf{x}_S\|_q \leq M$, then the infinite set of polynomial equations $\{P_i(\mathbf{x}) = b_i : i \in I\}$ has a solution $\mathbf{x} \in \ell^q$ with $\|\mathbf{x}\|_q \leq M$.

4. Approximate finite implies exact infinite

The set $\{P_i(\mathbf{x}) = b_i : i \in S\}$ of polynomial equations in infinitely many variables $\mathbf{x} = (x_j)_{j=1}^{\infty}$ has an *approximate solution* if, for every $\varepsilon > 0$, there exists a sequence $\mathbf{x}_\varepsilon$ such that $|P_i(\mathbf{x}_\varepsilon) - b_i| \leq \varepsilon$ for all $i \in S$. We shall prove that an infinite set of multiplicative polynomial equations has an exact solution if every finite subset of the equations has a norm-bounded approximate solution. This is the main result in this paper and immediately implies Theorem 3.

Theorem 4. *Let D be a positive integer and let $1 \leq D < q \leq \infty$. For all $k \in \mathbf{N}$, let*

$$\mathcal{D}_k = \left\{(d_1, \ldots, d_k) \in \mathbf{N}^k : d_1 + d_2 + \cdots + d_k \leq D\right\}.$$

Let I be an infinite set and let $\mathbf{x} = (x_j)_{j=1}^{\infty}$. For all $i \in I$ and $d \in \{1, 2, \ldots, D\}$, let

$$\mathbf{a}_{i,d} = (a_{i,d,j})_{j=1}^{\infty} \in \ell^{q/(q-d)}.$$

Then

$$\left(\mathbf{a}_{i,d}, \mathbf{x}^d\right) = \sum_{j=1}^{\infty} a_{i,d,j} x_j^d.$$

For $\Delta = (d_1, d_2, \ldots, d_k) \in \mathcal{D}_k$ and $J = (j_1, j_2, \ldots, j_k) \in \mathbf{N}^k$, let

$$a_{i,\Delta,J} = a_{i,d_1,j_1} a_{i,d_2,j_2} \cdots a_{i,d_k,j_k}.$$

. For all $i \in I$, the finite set of sequences $\{\mathbf{a}_{i,d}\}_{d=1}^{D}$ determines the multiplicative polynomial equation

$$\begin{aligned} P_i(\mathbf{x}) &= \sum_{k=1}^{D} \sum_{\Delta \in \mathcal{D}_k} \sum_{J \in \mathbf{N}^k} a_{i,\Delta,J} \ x_J^{\Delta} \\ &= \sum_{k=1}^{D} \sum_{(d_1,d_2,\ldots,d_k) \in \mathcal{D}_k} \left(\mathbf{a}_{i,d_1}, \mathbf{x}^{d_1}\right) \left(\mathbf{a}_{i,d_2}, \mathbf{x}^{d_2}\right) \cdots \left(\mathbf{a}_{i,d_k}, \mathbf{x}^{d_k}\right) \\ &= b_i. \end{aligned}$$

Let $M > 0$. If, for every $\varepsilon > 0$ and every finite subset S of I, the finite set of polynomial inequalities

$$\{|P_i(\mathbf{x}) - b_i| \leq \varepsilon : i \in S\}$$

has a solution $\mathbf{x}_{S,\varepsilon} \in \ell^q$ with $\|\mathbf{x}_{S,\varepsilon}\|_q \leq M$, then the infinite set of polynomial equations

$$\{P_i(\mathbf{x}) = b_i : i \in I\}$$

has an exact solution $\mathbf{x} \in \ell^q$ *with* $\|\mathbf{x}\|_q \leq M$.

We begin with two results whose statements and proofs are valid in both the real and complex cases.

For $M > 0$, the closed interval $[-M, M] = \{x \in \mathbf{R} : |x| \leq M\}$ and the closed ball $B_M = \{x \in \mathbf{C} : |x| \leq M\}$ are compact. For polynomial equations in $\mathbf{R}$ we use the compact topological space

$$\Omega = \prod_{j=1}^{\infty} [-M, M]$$

and for polynomial equations in $\mathbf{C}$ we use the compact topological space

$$\Omega = \prod_{j=1}^{\infty} B_M.$$

Lemma 1. *The set*

$$X_{q,M} = \{\mathbf{x} \in \ell^q : \|\mathbf{x}\|_q \leq M\}$$

is a compact subset of the topological space Ω.

Proof. If $\mathbf{x} = (x_j)_{j=1}^{\infty} \in \ell^q$ and $\|\mathbf{x}\|_q \leq M$, then $|x_j| \leq M$ for all $j \in \mathbf{N}$ and so $X_{q,M} \subseteq \Omega$. Because Ω is compact, it suffices to prove that $X_{q,M}$ is closed in the product topology on Ω, or equivalently, that the complement of $X_{q,M}$ is open. The complement of $X_{q,M}$ is the set

$$\begin{aligned} Y_{q,M} &= \Omega \setminus X_{q,M} \\ &= \{\mathbf{y} \in \Omega : \mathbf{y} \notin \ell^q\} \bigcup \{\mathbf{y} \in \Omega : \mathbf{y} \in \ell^q \text{ and } \|\mathbf{y}\|_q > M\} \\ &= \left\{ \mathbf{y} = (y_j)_{j=1}^{\infty} : \sum_{j=1}^{\infty} |y_j|^q > M^q \right\}. \end{aligned}$$

Let $\mathbf{y} = (y_j)_{j=1}^{\infty} \in Y_{q,M}$. There exist $\varepsilon > 0$ and $N \in \mathbf{N}$ such that

$$\sum_{j=1}^{N} |y_j|^q > M^q + \varepsilon.$$

Because the function $f(t) = |t|^q$ is continuous, there exists $\delta > 0$ such that $|t - y_j| < \delta$ implies

$$|t|^q > |y_j|^q - \frac{\varepsilon}{2N}$$

for all $j \in \{1, 2, \ldots, N\}$. The set

$$U = \left\{ \mathbf{z} = (z_j)_{j=1}^{\infty} \in \Omega : |z_j - y_j| < \delta \text{ for all } j \in \{1, 2, \ldots, N\} \right\}$$

is an open neighborhood of $\mathbf{y}$ in Ω. For all $\mathbf{z} = (z_j)_{j=1}^{\infty} \in U$ we have

$$\begin{aligned} \sum_{j=1}^{\infty} |z_j|^q \geq \sum_{j=1}^{N} |z_j|^q \; &> \sum_{j=1}^{N} \left(|y_j|^q - \frac{\varepsilon}{2N} \right) \\ &= \sum_{j=1}^{N} |y_j|^q - \frac{\varepsilon}{2} > M^q + \frac{\varepsilon}{2} \\ &> M^q \end{aligned}$$

and so $\mathbf{z} \in Y_{q,M}$. It follows that $U \subseteq Y_{q,M}$ and so $Y_{q,M}$ is an open subset of Ω. This completes the proof. □

Lemma 2. *Let (p,q) be a conjugate pair and let $\mathbf{a} = (a_j)_{j=1}^{\infty} \in \ell^p$. Let $M > 0$. The linear functional f on $X_{q,M}$ defined by*

$$f(\mathbf{x}) = (\mathbf{a}, \mathbf{x}) = \sum_{j=1}^{\infty} a_j x_j$$

is continuous with respect to the product topology on $X_{q,M}$ as a subspace of Ω.

Proof. Let U be an open subset of $\mathbf{R}$ or $\mathbf{C}$. We shall prove $f^{-1}(U)$ is open in $X_{q,M}$ as a subspace of Ω.

If $\mathbf{x} = (x_j)_{j=1}^{\infty} \in f^{-1}(U)$, then $f(\mathbf{x}) \in U$ and there exists $\varepsilon > 0$ such that U contains the open set

$$\{t : |t - f(\mathbf{x})| < \varepsilon\} \subseteq U.$$

Because $\mathbf{a} \in \ell^p$, the series $\sum_{j=1}^{\infty} |a_j|^p$ converges and there is an integer $N_{\mathbf{a}}$ such that

$$\sum_{j=N+1}^{\infty} |a_j|^p < \left(\frac{\varepsilon}{3M}\right)^p$$

for all $N \geq N_{\mathbf{a}}$. Because $\mathbf{x} \in \ell^q$, the series $\sum_{j=1}^{\infty} |x_j|^q$ converges and there is an integer $N_{\mathbf{x}}$ such that

$$\sum_{j=N+1}^{\infty} |x_j|^q < \left(\frac{\varepsilon}{3M}\right)^q$$

for all $N \geq N_{\mathbf{x}}$. Choose $N \geq \max(N_{\mathbf{a}}, N_{\mathbf{x}})$ and let $\delta > 0$ satisfy

$$\delta \sum_{j=1}^{N} |a_j| < \frac{\varepsilon}{3}.$$

The set

$$V' = \left\{\mathbf{y} = (y_j)_{j=1}^{\infty} \in \Omega : |y_j - x_j| < \delta \text{ for all } j \in \{1, 2, \ldots, N\}\right\}$$

is an open neighborhood of $\mathbf{x}$ in the topological space Ω and so

$$\begin{aligned} V &= V' \cap X_{q,M} \\ &= \left\{\mathbf{y} = (y_j)_{j=1}^{\infty} \in X_{q,M} : |y_j - x_j| < \delta \text{ for all } j \in \{1, 2, \ldots, N\}\right\} \end{aligned}$$

is an open neighborhood of $\mathbf{x}$ in $X_{q,M}$. We shall prove that $V \subseteq f^{-1}(U)$.

Let $\mathbf{y} = (y_j)_{j=1}^{\infty} \in V$. We define

$$\tilde{x_j} = \begin{cases} 0 & \text{if } j \leq N \\ x_j & \text{if } j \geq N+1 \end{cases}$$

and

$$\tilde{y_j} = \begin{cases} 0 & \text{if } j \leq N \\ y_j & \text{if } j \geq N+1 \end{cases}$$

and

$$\tilde{\mathbf{x}} = (\tilde{x_j})_{j=1}^{\infty} \in \ell^q \qquad \text{and} \qquad \tilde{\mathbf{y}} = (\tilde{y_j})_{j=1}^{\infty} \in \ell^q.$$

Then

$$\|\tilde{\mathbf{x}}\|_q \leq \|\mathbf{x}\|_q \leq M \qquad \text{and} \qquad \|\tilde{\mathbf{y}}\|_q \leq \|\mathbf{y}\|_q \leq M.$$

We define

$$\tilde{a_j} = \begin{cases} 0 & \text{if } j \leq N \\ a_j & \text{if } j \geq N+1 \end{cases}$$

and

$$\tilde{\mathbf{a}} = (\tilde{a_j})_{j=1}^{\infty} \in \ell^p.$$

Then

$$\|\tilde{\mathbf{a}}\|_p \leq \frac{\varepsilon}{3M}.$$

We have

$$\begin{aligned} f(\mathbf{y}) - f(\mathbf{x}) = (\mathbf{a}, \mathbf{y} - \mathbf{x}) &= \sum_{j=1}^{\infty} a_j(y_j - x_j) \\ &= \sum_{j=1}^{N} a_j(y_j - x_j) + \sum_{j=N+1}^{\infty} a_j(y_j - x_j). \end{aligned}$$

Applying the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned} |f(\mathbf{y}) - f(\mathbf{x})| &\leq \sum_{j=1}^{N} |a_j| \; |y_j - x_j| + \sum_{j=N+1}^{\infty} |a_j(y_j - x_j)| \\ &\leq \delta \sum_{j=1}^{N} |a_j| + \|(\tilde{\mathbf{a}}, \tilde{\mathbf{y}} - \tilde{\mathbf{x}})\|_1 \\ &< \frac{\varepsilon}{3} + \|\tilde{\mathbf{a}}\|_p \, \|\tilde{\mathbf{y}} - \tilde{\mathbf{x}}\|_q \\ &\leq \frac{\varepsilon}{3} + \|\tilde{\mathbf{a}}\|_p \left(\|\tilde{\mathbf{y}}\|_q + \|\tilde{\mathbf{x}})\|_q \right) \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3M}(M + M) \\ &= \varepsilon. \end{aligned}$$

Therefore, $f(\mathbf{y}) \in U$ and $V \subseteq f^{-1}(U)$. Thus, the set $f^{-1}(U)$ is open in $X_{q,M}$ and the function $f(\mathbf{x}) = (\mathbf{a}, \mathbf{x})$ is continuous on $X_{q,M}$. This completes the proof. $\square$

We can now prove Theorem 4.

Proof. For all $i \in I$ we have the multiplicative polynomial

$$P_i(\mathbf{x}) = \sum_{k=1}^{D} \sum_{(d_1, d_2, \ldots, d_k) \in \mathcal{D}_k} \left(\mathbf{a}_{i,d_1}, \mathbf{x}^{d_1}\right) \left(\mathbf{a}_{i,d_2}, \mathbf{x}^{d_2}\right) \cdots \left(\mathbf{a}_{i,d_k}, \mathbf{x}^{d_k}\right).$$

By Lemma 1, the set $X_{q,M}$ is a compact subset of Ω. By Lemma 2, the linear functionals $\left(\mathbf{a}_{i,d_1}, \mathbf{x}^{d_1}\right)$ are continuous on $X_{q,M}$. Finite sums of finite products of continuous functions are continuous, and so the multiplicative polynomials $P_i(\mathbf{x})$ are continuous functions on $X_{q,M}$ for all $i \in I$. It follows that, for all $i \in I$ and $\varepsilon > 0$, the approximation set

$$F_{i,\varepsilon} = \{\mathbf{x} \in X_{q,M} : |P_i(\mathbf{x}) - b_i| \leq \varepsilon\}$$

is a closed subset of the compact set $X_{q,M}$. For every finite subset S of I, the set of polynomial inequalities

$$\{|P_i(\mathbf{x}) - b_i| \leq \varepsilon : i \in S\}$$

has a solution $\mathbf{x}_{S,\varepsilon} \in \ell^q$ with $\|\mathbf{x}_{S,\varepsilon}\|_q \leq M$ and so the set of closed sets $\{F_{i,\varepsilon} : i \in I \text{ and } \varepsilon > 0\}$ has the finite intersection property. Therefore,

$$\bigcap_{\substack{i \in I \\ \varepsilon > 0}} F_{i,\varepsilon} = \{\mathbf{x} \in X_{q,M} : P_i(\mathbf{x}) = b_i \text{ for all } i \in I\}$$

is nonempty and the infinite set of polynomial equations has an exact solution. This completes the proof Theorem 4. □

Theorem 2 for infinitely many linear equations in infinitely many variables is the special case of Theorem 4 with $D = 1$. Theorem 4 implies Theorem 3 and Theorem 2 implies Theorem 1.

We have the following refinement of Theorem 4.

Theorem 5. *With the hypotheses of Theorem 4, let*

$$\mathbf{m} = (m_j)_{j=1}^{\infty} \in \ell^q \qquad \text{with } m_j \geq 0 \text{ for all } j \in \mathbf{N}.$$

If, for every $\varepsilon > 0$ and every finite subset S of I, the finite set of polynomial inequalities

$$\{|P_i(\mathbf{x}) - b_i| \leq \varepsilon : i \in S\}$$

has a solution $\mathbf{x}_{S,\varepsilon} = (x_{S,\varepsilon,j})_{j=1}^{\infty} \in \ell^q$ with $|x_{S,\varepsilon,j}| \leq m_j$ for all $j \in \mathbf{N}$, then the infinite set of polynomial equations

$$\{P_i(\mathbf{x}) = b_i : i \in I\}$$

has an exact solution $\mathbf{x} = (x_j)_{j=1}^{\infty} \in \ell^q$ with $|x_j| \leq m_j$ for all $j \in \mathbf{N}$.

Proof. Let $M = \|\mathbf{m}\|_q$. For polynomial equations in $\mathbf{R}$ we use the compact space

$$X_{\mathbf{m}} = \prod_{j=1}^{\infty} [-m_j, m_j]$$

and for polynomial equations in $\mathbf{C}$ we use the compact space

$$X_{\mathbf{m}} = \prod_{j=1}^{\infty} B_{m_j}.$$

If $\mathbf{x} \in X_{\mathbf{m}}$, then $\mathbf{x} \in \ell^q$ and $\|\mathbf{x}\|_q \leq M$, and so

$$X_{\mathbf{m}} \subseteq X_{q,M} = \{\mathbf{x} \in \ell^q : \|\mathbf{x}\|_q \leq M\}.$$

By Lemma 1, the set $X_{q,M}$ is a compact subset of Ω. The set $X_{\mathbf{m}}$ is a closed subset of $X_{q,M}$ and so $X_{\mathbf{m}}$ is compact. By Lemma 2, for all $\mathbf{a} \in \ell^p$, the linear functional $f(\mathbf{x}) = (\mathbf{a}, \mathbf{x})$ is continuous on $X_{q,M}$, and so its restriction to $X_{\mathbf{m}}$ is continuous. It follows that the multiplicative polynomials $P_i(\mathbf{x})$ are continuous functions on $X_{\mathbf{m}}$ for all $i \in I$, and so the approximation set

$$F_{i,\varepsilon} = \{\mathbf{x} \in X_{\mathbf{m}} : |P_i(\mathbf{x}) - b_i| \leq \varepsilon\}$$

is a nonempty closed subset of the compact set $X_{\mathbf{m}}$. These sets have the finite intersection property and so

$$\bigcap_{\substack{i \in I \\ \varepsilon > 0}} F_{i,\varepsilon} = \{\mathbf{x} \in X_{q,M} : P_i(\mathbf{x}) = b_i \text{ for all } i \in I\}$$

is nonempty. This completes the proof □

5. Open problems

. The results in this paper suggest several questions.

(1) Are Theorems 3, 4, and 5 true for infinite sets of polynomial equations in which the polynomials are not multiplicative?
(2) Are there classes $\mathcal{C}$ of infinite sets of polynomial equations in infinitely many variables for which there is an integer $S = S(\mathcal{C})$ such that, if every finite set of at most S equations has a solution or an approximate solution, then the infinite set of equations has a solution?
(3) Let E be a subfield of the complex numbers. Let $\{P_i(\mathbf{x}) = b_i : i \in I\}$ be an infinite set of polynomial equations in which every finite subset of the equations has an exact solution or an approximate solution in E.
    (a) For what subfields E of the complex numbers is it true that the infinite set of equations have an exact solution in E?
    (b) For what sets of polynomial equations might this be true?
(4) For what subfields E of the complex numbers do we have "approximate finite implies infinite exact" for all infinite sets of polynomial equations with all scalars in E?
(5) The "finitely many implies infinitely many" paradigm applies to linear equations and multiplicative polynomial equations with bounded or sequentially bounded norms (Theorems 3, 4, and 5). For equations of this special type, countably many implies finitely many. But there may be other classes of equations for which this paradigm does not hold, but a stronger condition ("countably many implies uncountably many") is true. The problem is to determine if such "new" classes of equations exist.

    Is there an uncountably infinite set of equations such that every countably infinite subset of the equations has an exact or an approximate solution, but the uncountably infinite set of equations has no exact solution?

Appendix A. The theorems of F. Riesz and Abian-Eslami

Theorem 6. *Let (p,q) be a conjugate pair and let $M > 0$. Let $\mathbf{a}_i = (a_{i,j})_{j=1}^{\infty} \in \ell^p$ for all $i \in \mathbf{N}$. The following are equivalent.*

(a) *For all $r \in \mathbf{N}$, there exists $\mathbf{x}_r = (x_{r,j})_{j=1}^{\infty} \in \ell^q$ such that*

$$\|\mathbf{x}_r\|_q \leq M \qquad \text{and} \qquad \sum_{j=1}^{\infty} a_{i,j} x_{r,j} = b_i \qquad \text{for all } i \in \{1,2,3,\ldots,r\}.$$

(b) *For all $r \in \mathbf{N}$ and $h_1, \ldots, h_r \in \mathbf{R}$,*

$$\left| \sum_{i=1}^{r} h_i b_i \right| \leq M \left(\sum_{j=1}^{\infty} \left| \sum_{i=1}^{r} h_i a_{i,j} \right|^p \right)^{1/p}.$$

(c) *There exists $\mathbf{x} = (x_j)_{j=1}^{\infty} \in \ell^q$ such that*

$$\|\mathbf{x}\|_q \leq M \qquad \text{and} \qquad \sum_{j=1}^{\infty} a_{i,j} x_= b_i \qquad \text{for all } i \in \mathbf{N}.$$

F. Riesz proved the equivalence of (b) and (c) and Abian-Eslami proved (a) implies (c).

Proof. First we prove that (a) implies (b). For all $h_1, \ldots, h_r \in \mathbf{R}$,

$$\sum_{i=1}^{r} h_i \mathbf{a}_i = \sum_{i=1}^{r} h_i (a_{i,j})_{j=1}^{\infty} = \left(\sum_{i=1}^{r} h_i a_{i,j} \right)_{j=1}^{\infty} \in \ell^p$$

and

$$\left\| \sum_{i=1}^{r} h_i \mathbf{a}_i \right\|_p = \left(\sum_{j=1}^{\infty} \left| \sum_{i=1}^{r} h_i a_{i,j} \right|^p \right)^{1/p}.$$

We have $\mathbf{x}_r = (x_{r,j})_{j=1}^{\infty} \in \ell^q$ and so, by Hólder's inequality,

$$\left(\sum_{i=1}^{r} h_i \mathbf{a}_i \right) \mathbf{x}_r = \left(\sum_{i=1}^{r} h_i a_{i,j} x_j \right)_{j=1}^{\infty} \in \ell^1$$

and

$$\left\| \left(\sum_{i=1}^{r} h_i \mathbf{a}_i \right) \mathbf{x}_r \right\|_1 = \sum_{j=1}^{\infty} \left| \sum_{i=1}^{r} h_i a_{i,j} x_j \right| < \infty.$$

For every positive integer N, let

$$g(N) = \sum_{j=N+1}^{\infty} \left| \sum_{i=1}^{r} h_i a_{i,j} x_j \right|.$$

We have

$$\lim_{N \to \infty} g(N) = 0$$

and

$$\begin{aligned}
\left|\sum_{i=1}^{r} h_i b_i\right| &= \left|\sum_{i=1}^{r} h_i \left(\sum_{j=1}^{\infty} a_{i,j} x_{r,j}\right)\right| \\
&\leq \left|\sum_{i=1}^{r} h_i \sum_{j=1}^{N} a_{i,j} x_{r,j}\right| + \left|\sum_{i=1}^{r} h_i \sum_{j=N+1}^{\infty} a_{i,j} x_{r,j}\right| \\
&\leq \left|\sum_{j=1}^{N} \sum_{i=1}^{r} h_i a_{i,j} x_{r,j}\right| + \sum_{j=N+1}^{\infty} \left|\sum_{i=1}^{r} h_i a_{i,j} x_{r,j}\right| \\
&\leq \sum_{j=1}^{N} \left|\sum_{i=1}^{r} h_i a_{i,j} x_{r,j}\right| + g(N) \\
&\leq \sum_{j=1}^{\infty} \left|\sum_{i=1}^{r} h_i a_{i,j} x_{r,j}\right| + g(N) \\
&= \left\| \left(\sum_{i=1}^{r} h_i \mathbf{a}_i\right) \mathbf{x}_r \right\|_1 + g(N) \\
&\leq \left\| \sum_{i=1}^{r} h_i \mathbf{a}_i \right\|_p \|\mathbf{x}_r\|_q + g(N) \\
&\leq M \left(\sum_{j=1}^{\infty} \left| \sum_{i=1}^{r} h_i a_{i,j} \right|^p \right)^{1/p} + g(N).
\end{aligned}$$

Because this inequality is valid for all N and because $\lim_{N\to\infty} g(N) = 0$, it follows that (a) implies (b).

Next we prove that (b) implies (c). Let W be the vector subspace of ℓ^p spanned by the set $\{\mathbf{a}_i : i \in \mathbf{N}\}$. Let $h_1, \ldots, h_r, h'_1, \ldots, h'_r$ be scalars such that

$$\sum_{i=1}^{r} h_i \mathbf{a}_i = \sum_{i=1}^{r} h'_i \mathbf{a}_i.$$

Then

$$\sum_{i=1}^{r} (h_i - h'_i) \mathbf{a}_i = \mathbf{0}$$

and

$$\begin{aligned}
\left|\sum_{i=1}^{r} h_i b_i - \sum_{i=1}^{r} h'_i b_i\right| &= \left|\sum_{i=1}^{r} (h_i - h'_i) b_i\right| \\
&\leq M \left(\sum_{j=1}^{\infty} \left| \sum_{i=1}^{r} (h_i - h'_i) a_{i,j} \right|^p \right)^{1/p} \\
&= M \left\| \sum_{i=1}^{r} (h_i - h'_i) \mathbf{a}_i \right\|_p = 0.
\end{aligned}$$

It follows that there is a well-defined linear functional f on W such that

$$f\left(\sum_{i=1}^{r} h_i \mathbf{a}_i\right) = \sum_{i=1}^{r} h_i b_i.$$

In particular, $f(\mathbf{a}_i) = b_i$. Because

$$\left| f\left(\sum_{i=1}^{r} h_i \mathbf{a}_i\right)\right| = \left|\sum_{i=1}^{r} h_i b_i\right| \leq M \left\|\sum_{i=1}^{r} h_i \mathbf{a}_i\right\|_p$$

for all $\sum_{i=1}^{r} h_i \mathbf{a}_i \in W$, the linear functional f has norm $\|f\| \leq M$. By the Hahn-Banach theorem, there is a bounded linear functional F on ℓ^p such that $F(\mathbf{w}) = f(\mathbf{w})$ for all $\mathbf{w} \in W$ and $\|F\| \leq M$.

For every bounded linear functional F on ℓ^p there is a sequence $\mathbf{x} \in \ell^q$ such that $F(\mathbf{a}) = (\mathbf{a}, \mathbf{x})$ for all $\mathbf{a} \in \ell^p$. For all $i \in \mathbf{N}$ we have

$$b_i = f(\mathbf{a}_i) = F(\mathbf{a}_i) = (\mathbf{a}_i, \mathbf{x}) = \sum_{j=1}^{\infty} a_{i,j} x_j.$$

Thus, (b) implies (c).

The proof that (c) implies (a) is immediate. □

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