On dense orbits in the space of subequivalence relations

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Abstract

We first explain how to endow the space of subequivalence relations of any non-singular countable equivalence relation with a Polish topology, extending the framework of Kechris' recent monograph on subequivalence relations of probability measure-preserving countable equivalence relations. We then restrict to p.m.p. equivalence relations and discuss dense orbits therein for the natural action of the full group and of the automorphism group of the relation. Our main result is a characterization of the subequivalence relations having a dense orbit in the space of subequivalence relations of the ergodic hyperfinite p.m.p. equivalence relation. We also show that in this setup, all full groups orbits are meager. We finally provide a few Borel complexity calculations of natural subsets in spaces of subequivalence relations using a natural metric we call the uniform metric. This answers some questions from an earlier version of Kechris' monograph.

1 Introduction

Measured group theory can roughly be defined as the study of countable groups through the partitions into orbits associated to their free probability measure preserving (p.m.p.) actions on standard probability space. A fundamental notion there is that of that of an *orbit subgroup*: a countable group Λ is an orbit subgroup of another countable group Γ when Λ admits a free p.m.p. action all of whose orbits are contained in those of a free p.m.p. action of Γ. In this more flexible framework, while the Ornstein-Weiss theorem tells us that all countable infinite amenable subgroups are orbit subgroups of one another [OW80], the Gaboriau-Lyons theorem characterizes non amenable groups as being exactly those which admit the free group on two generators as an orbit subgroup. Their result has had many applications, allowing to extend results which were only known for groups containing free subgroups to the optimal class of all non-amenable groups, see e.g. [IKT09, Sew14].

This motivates the following more general question: given a partition of the space into orbits, what are the possible subpartitions into orbits? In more precise terms, let us call an equivalence relation on a standard probability space (X, μ) non-singular when it comes from an action of a countable group which preserves μ null-sets, and p.m.p. when it comes from an action of a countable group which actually preserves the measure μ. We can now ask our precise question: given a non-singular equivalence relation, what are its possible subequivalence relations? A fundamental tool towards answering such questions is Kechris' remarkable result that the space of subequivalence relations can be endowed with a natural Polish topology when the ambient equivalence relation is p.m.p., allowing the use of many tools from descriptive set theory. We first extend his result to the non-singular case.

Theorem A. *Let $\mathcal{R}$ be a non-singular equivalence relation on a standard probability space. Then its space $\mathrm{Sub}(\mathcal{R})$ of subequivalence relations is a Polish space for the topology induced by the measure algebra of $\mathcal{R}$.*

The measure that we put on $\mathcal{R}$ is the usual measure obtained by integrating cardinality of vertical fibers, which allows us to make sense of its measure algebra. Note that this means that we are identifying subequivalence relations which coincide on a conull set. We refer the reader to Section 2 for details. Our description of the topology on the space of subequivalence relations was already known to Kechris in the p.m.p. case, see [Kec, Sec. 4.4(1)].

We actually provide two proofs of Theorem A. The first one is very short and uses the fact that in a measure algebra, converging sequences always admit subsequences whose limit is actually equal to their lim inf (see Proposition 2.3). This observation provides a unifying viewpoint to some special cases noted by Kechris, see [Kec, Thm. 5.1 and Thm. 18.3]. The second proof is somehow more natural in that we directly show that the various axioms of being a subequivalence relation define closed subsets in the measure algebra. Along the way, we study a natural non-Hausdorff topology on the measure algebra that we call the lower topology. This topology has some nice continuity property which also allow us to give a slick proof of the fact that the map which associates to a subset of an equivalence relation the equivalence relation it generates is Baire class one (see Proposition 3.19, which is a natural generalization of [Kec, Prop. 19.1]).

Our next result is in the p.m.p. setup, and was motivated by the question of dense orbits in the space of subequivalence relations, asked by Kechris in his monograph. Indeed, it is very natural to try to understand subequivalence relations up to isomorphism, and this is precisely what the action of the automorphism group of $\mathcal{R}$ on subequivalence relations of $\mathcal{R}$ encodes. As it turns out, when $\mathcal{R}$ is the ergodic hyperfinite p.m.p. equivalence relation we obtain the following complete characterization of dense orbits for the natural actions of both the full group $[\mathcal{R}]$ and the automorphism group $\mathrm{Aut}(\mathcal{R})$.

Theorem B (see Cor. 4.13). *For a subequivalence relation $\mathcal{S}$ of the hyperfinite ergodic p.m.p. equivalence relation $\mathcal{R}_0$, the following are equivalent:*

(i) $\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}_0$;

(ii) The $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$;

(iii) The $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$.

In this theorem, $\mathcal{S}$ being everywhere of infinite index in $\mathcal{R}$ means that for every $A \subseteq X$ of positive measure, the restriction of $\mathcal{S}$ to A has infinite index in the restriction of $\mathcal{R}$ to A (almost every $\mathcal{R}_{\restriction A}$-class splits into infinitely many $\mathcal{S}_{\restriction A}$-classes; see the end of Section 4.1 for more on this).

Example 1.1. View $\mathcal{R}_0$ as the cofinite equivalence relation on $X = \{0,1\}^{\mathbb{N}}$ endowed with the probability measure $\mu = (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes \mathbb{N}}$, namely $((x_n), (y_n)) \in \mathcal{R}_0$ iff there is $N \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geqslant N$. Define $\mathcal{S}$ by $((x_n), (y_n)) \in \mathcal{S}$ iff there is $N \in \mathbb{N}$ such that $x_{2n} = y_{2n}$ for all $n \geqslant N$. Then $\mathcal{S}$ is both aperiodic and has everywhere infinite index in $\mathcal{R}_0$ (see Proposition 4.6), so by the above theorem it has a dense orbit in $\mathrm{Sub}(\mathcal{R}_0)$.

Example 1.2. View $\mathcal{R}_0$ as the equivalence relation on $X = \{0,1\}^{\mathbb{Z}^2}$ endowed with the probability measure $\mu = (\frac{1}{2}(\delta_0 + \delta_1))^{\otimes \mathbb{Z}^2}$ generated by the Bernoulli shift of $\mathbb{Z}^2$. Then since

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the shift is mixing, the subequivalence relation $\mathcal{S}$ generated by $\mathbb{Z} \leqslant \mathbb{Z}^2$ is ergodic. By construction $\mathcal{S}$ is of infinite index, so since it is ergodic it is everywhere of infinite index. By the above theorem $\mathcal{S}$ has a dense orbit in $\mathrm{Sub}(\mathcal{R}_0)$. Note that by [Kec, Thm. 10.1], ergodic subequivalence relations actually form a G_δ set, so the Baire category theorem provides us with a dense G_δ subset consisting of *ergodic* subequivalence relations of $\mathcal{R}_0$ satisfying the equivalent conditions of Theorem B.

Our approach to Theorem B is very much inspired by a joint work in preparation with Fima, Mukherjee and Patri where we similarly characterize dense orbits in the space of von Neumann subalgebras of the hyperfinite type II_1 factor. In this setup, we have to rely on a difficult result of Popa which characterizes the subalgebras which have a sequence of unitary conjugates converging to the trivial subalgebra, see Lem. 2.3 from [Pop19]. Here, we similarly first need to understand which subequivalence relations have a sequence of translates converging to the trivial subequivalence relations $\Delta_X = \{(x, x) \colon x \in X\}$, which is the content of our next result.

Theorem C (see Thm. 4.7)**.** *Let $\mathcal{R}$ be an aperiodic p.m.p. equivalence relation on a standard probability space (X, μ). Let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be a subequivalence relation. The following are equivalent:*

(i) $\mathcal{S}$ has everywhere infinite index in $\mathcal{R}$;

(ii) the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains Δ_X;

(iii) the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains Δ_X.

The proof of Theorem C relies on an inductive construction, using a countable dense subgroup of the full group and the fact that such a subgroup acts highly transitively on almost every orbit, a result due to Eisenmann and Glasner [EG16, Thm. 1.19].

Once this is done, we prove Theorem B using the fact that finite equivalence relations are dense in $\mathrm{Sub}(\mathcal{R}_0)$ and that they can always be translated inside any aperiodic subequivalence relation. This allows us to apply Theorem C on a fundamental domain of the finite subequivalence relation we want to approximate. A more general statement on the closed subspace of hyperfinite subequivalence relations can actually be proved this way, see Corollary 4.12.

Using Ioana's intertwining of subequivalence relations [Ioa12, Lem. 1.7] and the two above examples of subequivalence relations, we moreover have the following result, which again follows from a more general statement on the space hyperfinite subequivalence relations (see Theorem 4.16).

Theorem D (see Cor. 4.17)**.** *Let $\mathcal{R}_0$ be the ergodic p.m.p. hyperfinite equivalence relation. Then all $[\mathcal{R}_0]$-orbits are meager in $\mathrm{Sub}(\mathcal{R}_0)$.*

It then follows from Mycielcki's theorem (see e.g. [Gao09, Thm. 5.3.1]) that $\mathcal{R}_0$ contains a continuum of aperiodic subequivalence relations of infinite index which are in pairwise disjoint full group orbits. Moreover, by [Kec, Prop. 10.1] and Example 1.2, such a continuum can be found inside ergodic subequivalence relations. We leave the following question open.

Question 1.3. Can $\mathrm{Sub}(\mathcal{R}_0)$ contain a comeager $\mathrm{Aut}(\mathcal{R}_0)$-orbit?

This question can also be asked for equivalence relations $\mathcal{R}$ different from $\mathcal{R}_0$, although the existence of dense $\mathrm{Aut}(\mathcal{R})$- or $[\mathcal{R}]$-orbits in their space of subequivalence relations is already an open problem (see the remark preceding Corollary 4.13 for some examples where there are no dense orbits though). Also note that if $\mathcal{R}$ is a type II_∞ or type III ergodic non-singular equivalence relation, it is not hard to show that there are always dense full group orbits in $\mathrm{Sub}(\mathcal{R})$ (see the remark right before Section 4.4 for a sketch of proof), but the question of comeager orbits is completely open there.

Our paper concludes with a few complexity calculations, answering some questions from an earlier version of [Kec]. Let us highlight here a way of understanding Kechris' uniform topology on the space of subequivalence relations which we use: given two non-singular equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, denote by $\mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)$ the set of Borel subsets A of X such that $\mathcal{R}_{1 \restriction A} = \mathcal{R}_{2 \restriction A}$, and define

$$d_u(\mathcal{R}_1, \mathcal{R}_2) = 1 - \sup_{A \in \mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)} \mu(A)$$

This defines a metric that we propose to call the uniform metric since it induces the uniform topology defined in [Kec, Sec. 4.6], although it looks finer a priori (to see that the two topologies coincide, one needs to use [IKT09, Thm. 1]).

Let us finish by outlining the paper: along with basic preliminaries, Section 2 provides a first proof of Theorem A using the $\liminf$. Section 3 contains the second proof, which relies in a more direct way on the topology of the measure algebra, using as well the lower topology that we define and study therein. Section 4 is devoted to the proof of Theorem B and D in their more general form, and contains in particular the proof of Theorem C in Subsection 4.2. Finally, Section 5 contains our complexity computations, which use the uniform metric.

2 The Polish topology on the space of subequivalence relations in the non-singular setup

The main purpose of this section is to extend the topology on the space of subequivalence relations from [Kec] to the non-singular setup. To do so, we use the framework of measure algebras, which does yield the right topology for the probability measure-preserving case as noted in [Kec, Sec. 4.4(1)]. This direct approach also sheds a new light on some results from [Kec], such as Theorem 5.1 and Proposition 4.27.

2.1 The measure algebra of a standard σ-finite space

Let (Y, λ) be a standard σ-finite measured space, i.e. a standard Borel space endowed with a σ-finite atomless measure. Such spaces are always isomorphic either to $\mathbb{R}$ endowed with the Lebesgue measure or a finite length interval, also endowed with the Lebesgue measure. The **measure algebra** of (Y, λ) is the space $\mathrm{MAlg}(Y, \lambda)$ of all Borel subsets of X, where we identify $A, B \subseteq X$ Borel as soon as $\lambda(A \triangle B) = 0$.

In order to endow $\mathrm{MAlg}(Y, \lambda)$ with a complete metric, we first have to choose some finite measure μ in the same class as λ, i.e. sharing the same measure zero sets. Note that this does not change the definition of the measure algebra. We can then equip the measure algebra $\mathrm{MAlg}(Y, \lambda) = \mathrm{MAlg}(X, \mu)$ with the metric $d_\mu(A, B) = \mu(A \triangle B)$.

Lemma 2.1. *Let μ and μ' be two finite measures on Y in the same equivalence class. Then d_μ and $d_{\mu'}$ are uniformly equivalent: given any $\epsilon > 0$, there is $\delta > 0$ such that for all $A, B \in \mathrm{MAlg}(Y, \mu)$, $d_\mu(A, B) < \delta$ implies $d_{\mu'}(A, B) < \epsilon$ and vice-versa.*

Proof. By symmetry, given $\epsilon > 0$ it suffices to find a δ such that the direct implication holds. But this is a direct consequence of the fact that μ and μ' are in the same class: indeed by [Coh13, Lem. 4.2.1], there actually is $\delta > 0$ such that whenever $\mu(C) < \delta$, we must have $\mu'(C) < \epsilon$. $\square$

In particular, the above lemma yields that even when λ is infinite, its measure algebra is endowed with a natural topology, independent of the choice of some μ finite in the class of λ. The fact that that d_μ is complete is usually proved via the following lemma, which is important to us on its own.

Lemma 2.2. *Let (A_n) be a sequence of elements of $\mathrm{MAlg}(Y, \mu)$ such that $\sum_n \mu(A_n \triangle A_{n+1}) < +\infty$. Let $\liminf A_n = \bigcup_N \bigcap_{n \geqslant N} A_n$, then the sequence (A_n) converges to $\liminf A_n$.*

Proof. By the Borel-Cantelli lemma, for almost all $x \in X$ there is N such that for all $n \geqslant N$, $x \notin A_n \triangle A_{n+1}$. Let $\epsilon > 0$, we then find N and a set X_0 such that $\mu(X_0) > 1 - \epsilon$ and for all $x \in X$ and all $n \geqslant N$, we have $x \notin A_n \triangle A_{n+1}$. Now observe that for all $n \geqslant N$, we have $A_n \cap X_0 = \liminf A_n \cap X_0$, in particular $A_n \triangle \liminf A_n \subseteq X \setminus X_0$ which has measure less than ϵ as wanted. $\square$

Proposition 2.3. *Let (A_n) be a sequence of elements of $\mathrm{MAlg}(Y, \mu)$ which converges to some $A \in \mathrm{MAlg}(Y, \mu)$. Then we can find a subsequence (A_{n_k}) such that*

$$A = \liminf A_{n_k} = \bigcup_N \bigcap_{k \geqslant N} A_{n_k}$$

Proof. Since (A_n) converges, it is Cauchy, in particular it admits a subsequence $(A_{n_k})_k$ such that for all $k \in \mathbb{N}$, $d_\mu(A_{n_k}, A_{n_{k+1}}) < \frac{1}{2^k}$. We thus have by the previous lemma $A_{n_k} \to \liminf A_{n_k} = A$. $\square$

Proposition 2.4. *The metric d_μ is complete, and $\mathrm{MAlg}(Y, \mu)$ is separable.*

Proof. Let (A_n) be a Cauchy sequence, it suffices to show that it has a convergent subsequence. But because the sequence is Cauchy we may find a subsequence $(A_{n_k})_k$ such that for all $k \in \mathbb{N}$, $\mu(A_{n_k} \triangle A_{n_{k+1}}) < 2^{-k}$. Then $\sum_k \mu(A_{n_k} \triangle A_{n_{k+1}}) < +\infty$ and we get the desired result by applying the previous lemma.

Towards proving separability, first note that since (Y, μ) is standard, we may as well assume it is an interval of finite length endowed with the Lebesgue measure. The regularity of the Lebesgue measure can then be used to show that finite unions of open intervals with rational endpoints are dense in the measure algebra, thus yielding the desired countable dense subset. $\square$

2.2 The space of subequivalence relations

Given an action of a countable group Γ by Borel bijections on X standard Borel, we obtain the associated equivalence relation $\mathcal{R}_\Gamma = \{(x, \gamma x) \colon x \in X, \gamma \in \Gamma\}$ whose equivalence classes are exactly the Γ-orbits. This equivalence relation is a Borel subset of $X \times X$, and its classes are countable: $\mathcal{R}_\Gamma$ is by definition a **countable Borel equivalence relation** (**CBER**, which can conveniently be read as "seeber"). Conversely, the Feldman-Moore theorem ensures us that every CBER comes from a Borel action of a countable group, see for instance [KM04, Thm. 1.3].

Given such an equivalence relation, define its **Borel pseudo full group** as the set of all partial bijection $\varphi \colon \operatorname{dom} \varphi \subseteq X \to \operatorname{rng} \varphi \subseteq X$ such that for all $x \in A$, we have $(x, \varphi(x)) \in \mathcal{R}$. By definition, the **Borel full group** is the group made of elements of the Borel pseudo full group whose domain and range is equal to X.

If (X, μ) is now a standard probability space, we say that a CBER is **non-singular** (or that μ is quasi-invariant, or that $\mathcal{R}$ is null-preserving) if for all φ in its Borel pseudo full group, we have $\mu(\operatorname{dom} \varphi) = 0$ iff $\mu(\operatorname{rng} \varphi) = 0$. The main (and only by the Feldman-Moore theorem) source of such equivalence relations is the following:

Definition 2.5. An action of a countable group Γ by Borel bijections on X is called **non-singular** (or null-preserving, or quasi-preserving μ) if for all $A \subseteq X$ Borel, we have $\mu(A) = 0$ iff $\mu(\gamma A) = 0$.

Remark. By Lemma 2.1, non-singular actions yield action by uniformly continuous homeomorphisms on the measure algebra $\operatorname{MAlg}(X, \mu)$.

Given $\Gamma \curvearrowright (X, \mu)$ non-singular the associated CBER $\mathcal{R}_\Gamma$ is non-singular, and conversely any non-singular CBER arises in this manner (see [KM04, Prop. 8.1]).

Now let $\mathcal{R}$ be a non-singular CBER. We equip $\mathcal{R}$ with the left measure M defined by

$$M(A) = \int_X |A_x| \, d\mu(x)$$

for all $A \subseteq \mathcal{R}$ Borel. Let m be an equivalent probability measure, then the measure algebra of $\mathcal{R}$ with respect to M is equal to that of $\mathcal{R}$ with respect to m. This measure is σ-finite since $\mathcal{R}$ can be covered by the graphs of the elements of a countable group Γ such that $\mathcal{R} = \mathcal{R}_\Gamma$ (each such graph has measure 1).

It then follows from Proposition 2.4 that $\operatorname{MAlg}(\mathcal{R}, M)$ is a Polish space. Moreover by Lemma 2.1 the topology (and actually the uniform structure) do not depend on the choice of m. Since the measure m is not canonical, we will most of the time write our topological space as $\operatorname{MAlg}(\mathcal{R}, M)$.

A Borel subset of $\mathcal{R}$ is called a Borel subequivalence relation if it defines an equivalence relation on X. An element of $\operatorname{MAlg}(\mathcal{R}, M)$ is called a subequivalence relation if its equivalence class *contains* an equivalence relation. Denote by $\operatorname{Sub}(\mathcal{R}) \subseteq \operatorname{MAlg}(\mathcal{R}, M)$ the space of subequivalence relations of $\mathcal{R}$. We can now easily prove the following result of Kechris (our proof is moreover motivated by [Kec, Thm. 5.1], which we have essentially recast as Proposition 2.3).

Theorem 2.6 (Kechris, [Kec, Sec. 4.4.(1)]). *The space* $\operatorname{Sub}(\mathcal{R})$ *of subequivalence relations of* $\mathcal{R}$ *is a closed subset of* $\operatorname{MAlg}(\mathcal{R}, M)$, *hence Polish.*

Proof. Observe that the $\liminf$ of any sequence of equivalence relations is an equivalence relation. By Proposition 2.3 the result follows. $\qquad\square$

Observe that we did not even use that $\mathcal{R}$ was non-singular in the above. However, we need non-singularity for our space to encode the right notion of equality of subequivalence relations as follows.

Proposition 2.7. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be two Borel subequivalence relations of a non-singular CBER $\mathcal{R}$, suppose $M(\mathcal{S}_1 \bigtriangleup \mathcal{S}_2) = 0$. Then there is a full measure $\mathcal{S}_1$ and $\mathcal{S}_2$-invariant set onto which $\mathcal{S}_1$ and $\mathcal{S}_2$ are the same.*

Proof. By definition for almost all x we have $[x]_{\mathcal{S}_1} = [x]_{\mathcal{S}_2}$. Since $\mathcal{S}_1$ is non singular we find a smaller $\mathcal{S}_1$-invariant set X_0 for which the same conclusion holds (if $\mathcal{S}_1 = \mathcal{R}_\Gamma$ we let $X_0 = \bigcap_{\gamma \in \Gamma} \gamma X_1$, where $X_1 = \{x \in X \colon [x]_{\mathcal{S}_1} = [x]_{\mathcal{S}_2}\}$). This finishes the proof. $\square$

We will also see in the next section that non-singularity is important if we want to concretely see our space of subequivalence relations as a closed set, i.e. to say that the axioms of subequivalence relations each define closed sets.

3 Strong and lower topologies

In this section, we first study the strong topology on the space of subequivalence relations as defined in [Kec], which will be useful towards establishing our orbit density result (Theorem B). We then use a lower topology on the measure algebra so as to obtain a natural proof of the fact that the space of subequivalence relations is closed in the measure algebra, and of the fact that the map which takes a graphing to the equivalence it generates is Baire class one [Kec, Prop. 19.1].

3.1 Topologies on measure algebras of σ-finite spaces

We first go back to measure algebras of σ-finite spaces, which we now write as (Y, M), motivated by our previous example where Y is a non-singular CBER.

Proposition 3.1. *Let (Y, M) be a standard σ-finite space. Let (Y_n) be a countable family of finite measure subsets of Y such that $\bigcup_n Y_n = Y$. Then the map $\Phi : \mathrm{MAlg}(Y, M) \to \prod_n \mathrm{MAlg}(Y_n, M_{Y_n})$ which takes A to the sequence $(A \cap Y_n)$ is a homeomorphism onto its image.*

Proof. Observe that the probability measure

$$m(A) := \sum_n \frac{1}{2^{n+1} M(Y_n)} M(A \cap Y_n)$$

is equivalent to M and thus the associated metric d_m induces the topology of $\mathrm{MAlg}(Y, M)$. Now the space $\prod_n \mathrm{MAlg}(Y_n, M_{Y_n})$ can be endowed with a compatible metric $\tilde{d}$ given by

$$\tilde{d}((A_n), (B_n)) = \sum_n \frac{1}{2^{n+1} M(Y_n)} M(A_n \bigtriangleup B_n),$$

It is then straightforward to check that Φ is an isometry for the metric d_m on its domain and $\tilde{d}$ on its range. $\square$

We now go on to obtain a more precise version of this result, following Kechris' approach to the *strong topology* on $\mathrm{Sub}(\mathcal{R})$.

Observe that every element φ of the Borel pseudo full group of a CBER $\mathcal{R}$ is completely determined by its *graph*, namely the set $\{(x, \varphi(x))\}$, which is a Borel subset of $\mathcal{R}$. From now on, *we identify elements of the Borel pseudo full group to their graph*. Doing this up to measure zero when $\mathcal{R}$ is nonsingular, we arrive at the following definition.

Definition 3.2. The **pseudo full group** of a non-singular equivalence relation $\mathcal{R}$ on (X, μ) is the quotient of the Borel pseudo full group by the equivalence relation which identifies φ_1, φ_2 when $M(\varphi_1 \triangle \varphi_2) = 0$. It is denoted by $[[\mathcal{R}]]$, and for ease of notation we still write its elements as φ's. Finally the **full group** of $\mathcal{R}$ denotes all the elements of $[[\mathcal{R}]]$ with full measure domain and range.

It is a direct consequence of the definition of M that $M(\varphi_1 \triangle \varphi_2) = 0$ iff $\mu(\mathrm{dom}\, \varphi_1 \triangle \mathrm{dom}\, \varphi_2) = 0$ and for almost all $x \in \mathrm{dom}\, \varphi_1 \cap \mathrm{dom}\, \varphi_2$, we have $\varphi_1(x) = \varphi_2(x)$. The Borel pseudo full group is stable under countable increasing unions and arbitrary countable intersections, so it is stable under taking $\liminf$. Following the same approach as for the proof of Theorem 2.6, one obtains that $[[\mathcal{R}]]$ is a closed subset of $\mathrm{MAlg}(\mathcal{R}, M)$, in particular it is Polish. Note however that $[\mathcal{R}]$ is not closed for this topology (see Example 3.6).

We now make a modification of Kechris' uniquely generating sequences of involutions. Let us say that an element φ of the Borel pseudo-full group is a **moving partial involution** if $\mathrm{dom}\, \varphi = \mathrm{rng}\, \varphi$ and for all $x \in \mathrm{dom}\, \varphi$, we both have $\varphi(x) \neq x$ and $\varphi(\varphi(x)) = x$.

Definition 3.3. A sequence (φ_n) of moving partial involutions of the Borel pseudo-full group of a CBER $\mathcal{R}$ is called a **uniquely generating sequence of moving partial involutions** if

$$\mathcal{R} \setminus \Delta_X = \bigsqcup_n \varphi_n,$$

where $\Delta_X = \{(x, x) \colon x \in X\}$ is the equality relation.

Proposition 3.4. *Every CBER $\mathcal{R}$ admits a uniquely generating sequence of moving partial involutions.*

Proof. Let (T_n) be a uniquely generating sequence of involutions for $\mathcal{R}$ as in [Kec, Prop. 4.6], and let φ_n be the restriction of T_n to its support. $\square$

Remark. The disjointness of the graphs of the φ_n's simplifies a bit the arguments we carry out in this paper since it guarantees that all the $\varphi_i(x)$ are distinct whenever they are defined (see in particular the proof of (i)$\Rightarrow$ii in Theorem 4.7). The disjointness also allows us to upgrade Proposition 3.1 and obtain that the map

$$\mathrm{MAlg}(\mathcal{R}, M) \to \mathrm{MAlg}(\Delta_X, M_{\Delta_X}) \times \prod_{n \in \mathbb{N}} \mathrm{MAlg}(\varphi_n, M_{\varphi_n}))$$

which takes A to the sequence $(A \cap Y_n)$ is not only a homeomorphism onto its image, but actually surjective.

The following corollary is a slight reformulation of [Kec, Thm. 4.13] which identifies the strong and the weak topology on the space of subequivalence relations.

Corollary 3.5. *Let $\mathcal{R}$ be a non-singular equivalence relation, let (φ_n) be a uniquely generating sequence of moving partial involutions. Then the map*

$$\mathrm{Sub}(\mathcal{R}) \to \prod_{n \in \mathbb{N}} \mathrm{MAlg}(\varphi_n, M_{\varphi_n}))$$

which takes $\mathcal{S}$ to the sequence $(\mathcal{S} \cap \varphi_n)_n$ is a homeomorphism onto its image.

Proof. This is a direct consequence of the definition of uniquely generating sequences of moving partial involutions and of Proposition 3.1 once we note that every equivalence relation contains Δ_X. $\square$

Following the above corollary and Kechris' terminology, we now call **strong topology** the topology induced by d_m on $\mathrm{Sub}(\mathcal{R})$, which is Polish by Theorem 2.6.

3.2 Lower topologies on measure algebras

Let us first work a bit more on the relationship between M and m. Observe that M defines a lower-semi continuous function on $\mathrm{MAlg}(Y, M)$ (possibly taking $+\infty$ as a value). Indeed if $M(A) > \alpha$, we have a $\delta > 0$ such that $M(C) < M(A) - \alpha$ whenever C is a subset of A satisfying $m(C) < \delta$, and so as soon as $m(A \setminus B) < \delta$ we have $M(B) > \alpha$. In particular for every $k \in \mathbb{N}$ the set of elements with measure $\leqslant k$ is closed.

Let us denote by $\mathrm{MAlg}_f(\mathcal{R}, M)$ the F_σ subset of elements of $\mathrm{MAlg}(\mathcal{R}, M)$ with finite M-measure. It can be endowed with the metric $d_M(A, B) = M(A \triangle B)$ which clearly refines the topology induced by d_m. Moreover, the proof of Proposition 2.4 can easily be adapted to show that d_M is complete separable (see [LM22, Lem. 2.1] for a complete proof). Let us see why d_m and d_M yield different topologies on $\mathrm{MAlg}_f(\mathcal{R}, M)$.

Example 3.6. Going back to the case $Y = \mathcal{R}$, observe that if T in the full group of $\mathcal{R}$ is aperiodic, then $T^n \to \emptyset$ for d_m while it stays in the sphere around 1 for d_M. In particular d_m does not refine d_M (here again we identified an element of the full group to its graph). Let us also mention that $[\mathcal{R}]$ is d_M-closed, and that the metric induced by d_M on $[\mathcal{R}]$ is the well-known uniform metric $d_u(S, T) = \mu(\{x \in X : S(x) \neq T(x)\})$, endowing $[\mathcal{R}]$ with a Polish group topology.

Despite the above example, M and m share the same *lower topologies* when restricted to $\mathrm{MAlg}_f(Y, M)$. These lower topologies are not Hausdorff, however they have nice continuity properties.

Definition 3.7. Let (Y, M) be a σ-finite measured space (the measure M can be finite). We endow its measure algebra with the **lower M-topology** which is defined by taking as neighborhoods of an element $A \in \mathrm{MAlg}(Y, M)$ all sets of the form

$$\{B \in \mathrm{MAlg}(Y, M) : M(A \setminus B) < \epsilon\}$$

for some $\epsilon > 0$.

This neighborhood system does satisfy the axiom that implies that it is a basic neighborhood system for the topology it induces, namely every neighborhood V contains a smaller neighborhood all of whose elements admit V as a neighborhood (axiom (V_{IV}) in [Bou98, Chap. 1, §1.2]). This is simply because if $M(A \setminus B) < \frac{\epsilon}{2}$ and $M(B \setminus C) < \frac{\epsilon}{2}$ then $M(A \setminus C) < \epsilon$).

Note that the only neighborhood of the emptyset is the whole measure algebra, while Y is contained in every neighborhood of every point. Moreover, $\{\emptyset\}$ is closed, and more generally M is lower semi-continuous for the lower M-topology.

Lemma 3.8. *If M is a σ-finite measure on Y and m is an equivalent finite measure, then the measures M and m induce the same lower topology on the space $\mathrm{MAlg}_f(Y, M)$ of finite M-measure subsets.*

Proof. Take A such that $M(A) < +\infty$. Because M and m, when restricted to A, are equivalent, we find δ such that every subset of A of m measure less than δ has M-measure less than ϵ. Then $\{B : m(A \setminus B) < \delta\} \subseteq \{B : M(A \setminus B) < \epsilon\}$. So the lower m-topology refines the lower M-topology. The other inequality works the same. □

Let us note that the space of infinite measure subsets is open for the lower M-topology, while it is not for the lower m-topology. Indeed if A has infinite measure, the set of B such that $M(A \setminus B) < 1$ does not contain any m-neighborhood, because there are subsets of A of very small m-measure but still with infinite measure.

We will now list various interesting properties of the lower topology, and then relate it to the usual topology in the last two lemmas.

Lemma 3.9. *Let M be a σ-finite measure on Y. Then the intersection map $(A_1, A_2) \mapsto A_1 \cap A_2$ is continuous, where we put the M-lower topology everywhere.*

Proof. Let $\epsilon > 0$. Note that $(A_1 \cap A_2) \setminus (B_1 \cap B_2) \subseteq ((A_1 \cap A_2) \setminus B_1) \cup ((A_1 \cap A_2) \setminus B_2) \subseteq (A_1 \setminus B_1) \cup (A_2 \setminus B_2)$, so if if $M(A_1 \setminus B_1) < \epsilon$ and $M(A_2 \setminus B_2) < \epsilon$, then $M((A_1 \cap A_2) \setminus (B_1 \cap B_2)) < 2\epsilon$ as wanted. □

Lemma 3.10. *Let m be a finite measure on Y. Then the countable union map $(A_n)_{n \in \mathbb{N}} \mapsto \bigcup_{n \in \mathbb{N}} A_n$ is continuous for the m-lower topology everywhere.*

Proof. Let $(A_n)_{n \in \mathbb{N}}$ be a family of elements of the measure algebra, then since m is finite we may find N such that $\mu((\bigcup_n A_n) \setminus \bigcup_{n < N} A_n) < \epsilon$. Now if we take $B_0, ..., B_{N-1}$ such that $m(A_i \setminus B_i) < \epsilon/N$, and any $B_N, B_{N+1}, ...,$ we see that $\mu((\bigcup_n A_n) \setminus (\bigcup_n B_n)) < 2\epsilon$ as wanted. □

Remark. Note that the countable union map is *not* continuous if we put the usual topology everywhere (indeed X can easily be obtained as a limit of reunions of sequences of sets which converge coordinatewise to $\emptyset$). Here in nevertheless a positive result for the usual topology which will be useful later on.

Lemma 3.11. *Let m be a finite measure on Y, let $(Y_j)_{j \in \mathbb{N}}$ be a disjoint family of measurable subsets of Y. Then the restriction of the countable union map to the set*

$$\{(A_n) \in \mathrm{MAlg}(Y, m)^{\mathbb{N}} : \forall n \in \mathbb{N}, A_n \subseteq Y_n\},$$

is continuous for the d_m-topology.

Proof. Since the Y_n's are disjoint and m is finite, given any $\epsilon > 0$ we can find N such that $m(\bigcup_{n > N} Y_n) < \epsilon$, in particular if (A_n) satisfies $A_n \subseteq Y_n$ for every n then its union is up to an ϵ error independent of $(A_n)_{n > N}$. The conclusion thus follows from the continuity of finite unions for the d_m-topology. □

We now relate the Hausdorff topology associated to the metric d_m and to the corresponding lower m-topology.

Lemma 3.12. *Let m be a finite measure on Y, denote by τ_m^l the lower m-topology and by τ_m the topology induced by the metric d_m. Then the identity map $(\mathrm{MAlg}(Y, m), \tau_l^m) \to (\mathrm{MAlg}(Y, m), \tau_m)$ is Baire class one.*

Proof. We begin by noting some useful continuity properties for the lower m-topology. By definition, for every $A \in \mathrm{MAlg}(Y, m)$ the map $B \mapsto m(A \setminus B)$ is upper semi-continuous for the lower topology. Since $m(B) = m(X) - m(X \setminus B)$, we deduce that m defines a lower semi-continuous map $(\mathrm{MAlg}(Y, m), \tau_m^l) \to \mathbb{R}^+$. By Lemma 3.9, for every $A \in \mathrm{MAlg}(Y, m)$ the map $B \mapsto m(A \cap B)$ is thus lower semi-continuous as well.

We finally observe that

$$d_m(A, B) = m(A \triangle B) = m(A) + m(B) - 2m(A \cap B),$$

so $d_m(A, \cdot)$ is an Baire class one function on $(\mathrm{MAlg}(Y, \nu), \tau_m^l) \to \mathbb{R}^+$, being the sum of two Baire class one functions. In particular, d_m-open balls are F_σ for τ_m^l. Since every τ_m-open set is a countable union of open balls by separability, the conclusion follows. $\square$

We conclude this section by a lemma which interwines the two topologies we have defined in presence of a finite measure.

Lemma 3.13. *Let m be a finite measure on Y. The space of all (A, B) such that $A \subseteq B$ is closed if we put on the first coordinate the m-lower topology, and on the second the usual measure algebra topology.*

Proof. Suppose that $A \not\subseteq B$, let $\epsilon = m(A \setminus B)$. Then if $m(A \setminus A') < \epsilon/2$ and $m(B' \triangle B) < \epsilon/2$, we still have $A' \not\subseteq B'$. $\square$

3.3   Seeing that the space of subequivalence relations is closed

We now provide a more concrete proof that the space of subequivalence relations is Polish by seeing that the conditions which make an equivalence relation define closed sets in $\mathrm{MAlg}(\mathcal{R}, M)$ for the topology induced by d_m. First note that reflexivity means containing Δ_X, and hence defines a closed set by Lemma 3.13.

Let us fix a non-singular equivalence relation $\mathcal{R}$, we recall that m is some finite measure equivalent to the measure M obtained by integrating the cardinalities of vertical fibers. Observe that the flip $\sigma : (x, y) \mapsto (y, x)$ quasi-preserves M (this is one characterization of non-singularity, see [KM04, Prop. 8.2]), in particular it quasi-preserves m and hence induces a homeomorphism on $\mathrm{MAlg}(\mathcal{R}, M)$.

This implies that the space of **graphings** on $\mathcal{R}$ (symmetric subrelations of $\mathcal{R}$) is closed in $\mathrm{MAlg}(\mathcal{R}, M)$, because it is the set of fixed points of σ. In particular, we recover the following fact from [Kec, Chap. 18].

Proposition 3.14. *The space of graphings is Polish for the topology induced by the measure algebra $\mathrm{MAlg}(\mathcal{R}, M)$.* $\square$

Remark. This could also be seen through the same approach as for Theorem 2.6, noting that the liminf of any sequence of graphings is a graphing.

To see that the space of subequivalence is closed, we need to deal with transitivity. Observe that on the space of measurable subsets of $\mathcal{R}$, we have an operation $\circ$ defined by $A \circ B := \{(x,z) : \exists y \in X : (x,y) \in A \text{ and } (y,z) \in B\}$. Note that $\circ$ extends the composition of elements of the pseudo full group of $\mathcal{R}$, and that it is well defined only because we are in the non-singular setup.

Lemma 3.15. *On the pseudo full group of $\mathcal{R}$, the composition is continuous for the M-lower topology, and hence for the m-lower topology.*

Proof. Note that the M-measure of the graph of an element of the pseudo full group is the μ-measure of its domain, so by Lemma 3.8, we only need to check the continuity for the lower M-topology.

Now fix $\varphi_0, \psi_0 \in [[\mathcal{R}]]$, let $\epsilon > 0$, find δ such that whenever $\mu(A) < \delta$ we have $\mu(\psi_0^{-1}(A)) < \epsilon/2$. Take φ such that $M(\varphi \setminus \varphi_0) < \delta$, then $\mu(\{x \in \text{dom } (\varphi_0) : \varphi(x) \neq \varphi_0(x)\}) < \delta$ (in the definition we consider that $\varphi_0(x)$ is different from $\varphi(x)$ if $x \notin \text{dom } \varphi$). Then take ψ such that $M(\psi \setminus \psi_0) < \epsilon/2$, the set $X_1 = \{x \in \text{dom } \psi_0 : \psi(x) = \psi_0(x)\}$ contains dom ψ_0 up to an $\epsilon/2$ error, and

$$\psi_{\restriction X_1}^{-1}(\{x \in \text{dom } (\varphi_0) : \varphi(x) \neq \varphi_0(x)\}) = \psi_{0 \restriction X_1}^{-1}(\{x \in \text{dom } (\varphi_0) : \varphi(x) \neq \varphi_0(x)\})$$

has thus measure at most $\epsilon/2$. We conclude that the set of $x \in \text{dom } \varphi_0\psi_0$ such that $\varphi\psi(x) = \varphi_0\psi_0(x)$ contains dom $\varphi_0\psi_0$ up to an $\epsilon/2 + \epsilon/2 = \epsilon$ error as wanted. The fact that the continuity also holds for the m-lower topology is then a direct consequence of Lemma 3.8 since elements of $[[\mathcal{R}]]$ have M-measure at most 1. $\qquad\square$

Lemma 3.16. *The map $A, B \mapsto A \circ B$ is continuous if we put everywhere the lower m-topology.*

Proof. Fixing a countable group Γ such that $\mathcal{R} = \mathcal{R}_\Gamma$, the desired continuity follows from the previous lemma along with the formula

$$A \circ B = \bigcup_{\gamma, \gamma' \in \Gamma} \gamma \cap A \circ (\gamma' \cap B)$$

and the fact that finite intersections and countable unions are continuous maps for the lower topology (Lemma 3.9 and 3.10). $\qquad\square$

We thus arrive at the desired result saying that the space of transitive subrelations is closed.

Proposition 3.17. *The space of $A \subseteq \mathcal{R}$ such that $A \circ A \subseteq A$ is closed in the Polish topology on $\mathrm{MAlg}(\mathcal{R}, M)$.*

Proof. This follows directly from the previous lemma along with Lem. 3.13. $\qquad\square$

Theorem 3.18. $\mathrm{Sub}(\mathcal{R})$ *is a closed subset of $\mathrm{MAlg}(\mathcal{R}, M)$ for the d_m-topology.*

Proof. We already observed that being reflexive exactly means containing Δ_X and thus is a closed condition by Lemma 3.13. The fact that symmetry is a closed condition is a reformulation of the fact that graphings form a closed set (Proposition 3.14). Finally transitivity is a closed condition by the previous proposition. So $\mathrm{Sub}(\mathcal{R})$ is the intersection of three closed subsets, hence closed itself. $\qquad\square$

We also have the following nice consequence of our work on the lower topology, which generalizes Proposition 19.1 from [Kec].

Proposition 3.19. *The map which takes a measurable subset of $\mathcal{R}$ to the equivalence relation it generates is continuous in the lower topology, in particular it is Baire class one for the strong topology (induced by d_m).*

Proof. This map associates to a $A \subseteq \mathcal{R}$ the equivalence relation $\bigcup_{n \in \mathbb{N}} (\Delta_X \cup A \cup \sigma(A))^{\circ n}$, so it is a continuous map by the continuity of σ, of countable unions (Lemma 3.10) and of composition (Lemma 3.16). The statement about the strong topology then follows directly from Lemma 3.12. □

We can finally obtain the following strengthening of Proposition 4.27 from [Kec].

Corollary 3.20. *The map which takes a sequence $(\mathcal{R}_n)$ to the equivalence relation $\bigvee_n \mathcal{R}_n$ it generates is continuous in the lower topology, in particular it is Baire class one for the strong topology.*

Proof. This is a direct consequence of the previous result along with the fact that taking countable unions is continuous in the lower topology (Lemma 3.10). □

4 Dense orbits for hyperfinite subequivalence relations in the p.m.p. case

In the present section, we exclusively work in the probability measure preserving setup, so $\mathcal{R}$ is a p.m.p. equivalence relation on (X, μ). Recall that $\mathrm{Aut}(\mathcal{R})$ is the group of all $T \in \mathrm{Aut}(X, \mu)$ such that $T \times T(\mathcal{R}) = \mathcal{R}$. Observe that $\mathrm{Aut}(\mathcal{R})$ acts on $\mathrm{MAlg}(\mathcal{R}, M)$ by $T \cdot (x, y) = (T(x), T(y))$ and preserves the measure M. The group $\mathrm{Aut}(\mathcal{R})$ contains the full group of $\mathcal{R}$, and it is a Polish group for the topology induced by the group $\mathrm{Aut}(\mathcal{R}, M)$ of all measure-preserving bijections of $(\mathcal{R}, M)$ (see [Kec10, Prop. 6.3] where $\mathrm{Aut}(\mathcal{R})$ is denoted $N[\mathcal{R}]$). The trivial subequivalence relation will play an important role; we denote it by $\Delta_X = \{(x, x) \colon x \in X\}$ (given our identification of full groups elements to their graphs, we could also write it as the identity map id_X).

4.1 Preliminaries on conditional measures and index

In order to work in the p.m.p. setup, we have to understand which sets can be taken to other sets by (pseudo) full group elements up to measure zero. This is done here through the concept of *conditional measures*, a low-tech version of the ergodic decomposition which was already present in Dye's founding paper [Dye59]. We already used them in a previous paper on non-ergodic p.m.p. equivalence relations, see [LM16, Sec. 2], but we recently gave a more detailed exposition for general full groups in a joint work with Slutsky which we use here as a reference (see Appendix D in [LMS23]).

Definition 4.1. Let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ), denote by $M_{\mathcal{R}}$ the closed subalgebra of (X, μ) consisting of $[\mathcal{R}]$-invariant subsets, and by $\mathbb{E}_{\mathcal{R}}$ the projection $\mathrm{L}^2(X, \mu) \to \mathrm{L}^2(X, M_{\mathcal{R}}, \mu)$. Given $A \in \mathrm{MAlg}(X, \mu)$, its $M_{\mathcal{R}}$-**conditional measure** $\mu_{\mathcal{R}}$ is the $M_{\mathcal{R}}$-measurable function
$$\mu_{\mathcal{R}}(A) = \mathbb{E}_{M_{\mathcal{R}}}(\chi_A).$$

It can be checked that $\mu_{\mathcal{R}}$ takes values in $[0,1]$, satisfies the usual axioms of a measure, and that elements of $[\mathcal{R}]$ preserve $\mu_{\mathcal{R}}$ (see [LMS23, Prop. D.6]). Say that $\mathcal{R}$ is **ergodic** when $M_{\mathcal{R}} = \{\emptyset, X\}$, then $\mathcal{R}$ is ergodic iff $\mu_{\mathcal{R}} = \mu$. By [LMS23, Prop. D.10], we moreover have:

Lemma 4.2. *Let $A, B \in \mathrm{MAlg}(X, \mu)$, let $\mathcal{R}$ be a p.m.p. equivalence relation on (X, μ). The following are equivalent:*

(i) *there is $\varphi \in [[\mathcal{R}]]$ such that $\mathrm{dom}\,\varphi = A$ and $\mathrm{rng}\,\varphi = B$;*

(ii) *$\mu_{\mathcal{R}}(A) = \mu_{\mathcal{R}}(B)$.*

We finally note the following important consequence of aperiodicity (having only infinite classes) for a p.m.p. equivalence relation (it is actually a characterization, but we don't need that).

Proposition 4.3 (Maharam's Lemma, see [LMS23, Thm. D.12]). *Let $\mathcal{R}$ be a p.m.p. aperiodic equivalence relation, let $A \in \mathrm{MAlg}(X, \mu)$, let $f : X \to [0, 1]$ be an $M_{\mathcal{R}}$-measurable function such that $0 \leqslant f \leqslant \mu_{\mathcal{R}}(A)$. Then there is $B \subseteq A$ such that $\mu_{\mathcal{R}}(B) = f$.*

We finish this section by recalling some important definitions on the index of subequivalence relations.

Definition 4.4. Let $\mathcal{R}$ be a non-singular equivalence relation on (X, μ). We say that $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$

- has **infinite index** in $\mathcal{R}$ if for almost all $x \in X$, the $\mathcal{R}$-equivalence class of x contains infinitely many distinct $\mathcal{S}$-classes.

- has **finite index** in $\mathcal{R}$ if for almost all $x \in X$, the $\mathcal{R}$-equivalence class of x is the reunion of a finite set of $\mathcal{S}$-classes.

- has **everywhere infinite index** in $\mathcal{R}$ if for all $A \subseteq X$ such that $\mu(A) > 0$, the restriction of $\mathcal{S}$ to A has infinite index in the restriction of $\mathcal{R}$ to A.

Remark. It might not be clear at first sight that having infinite index is not the same as having everywhere infinite index. Here is the simplest example of an infinite index $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$, not everywhere of infinite index: take $\mathcal{R}$ ergodic, let $A \subseteq X$ of positive non full measure, and let $\mathcal{S} = \mathcal{R}_{\restriction A} \sqcup \Delta_{X \setminus A}$.

Let us note that when $\mathcal{R}$ is ergodic, since the number of $\mathcal{S}$-classes inside the $\mathcal{R}$-class of $x \in X$ is $\mathcal{R}$-invariant, it is constant almost everywhere, and hence $\mathcal{S}$ either has finite or infinite index in $\mathcal{R}$. Similarly if $\mathcal{S}$ is ergodic and of infinite index, then it is everywhere of infinite index because the $\mathcal{S}$-class of almost every $x \in X$ intersects A. We finally note the following non-ergodic way of producing everywhere infinite index subequivalence relations.

Lemma 4.5. *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be non-singular aperiodic equivalence relations on respective standard probability spaces (X_1, μ_1) and (X_2, μ_2). Then $\mathcal{R}_1 \times \Delta_{X_2}$ is everywhere of infinite index in $\mathcal{R}_1 \times \mathcal{R}_2$.*

Proof. Let $A \subseteq X_1 \times X_2$ of positive measure, by Fubini's theorem for almost all $(x_1, x_2) \in A$ the vertical section $A_{x_1} = \{x_2' \in X_2 : (x_1, x_2' \in A)\}$ has positive measure, and hence by aperiodicity the $\mathcal{R}_2$-class of x_2 intersects A_{x_1} in an infinite set. This implies that the $(\mathcal{R}_1 \times \mathcal{R}_2)_{\restriction A}$-class of (x_1, x_2) contains infinitely many $(\mathcal{R}_1 \times \Delta_{X_2})_{\restriction A}$ classes. $\square$

Proposition 4.6. *The equivalence relation $\mathcal{S}$ from Example 1.1 is everywhere of infinite index in $\mathcal{R}_0$.*

Proof. The even-odd partition of $\mathbb{N}$ induces a bijection $\{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ under which $\mathcal{R}_0$ becomes $\mathcal{R}_0 \times \mathcal{R}_0$ and $\mathcal{S}$ becomes $\mathcal{R}_0 \times \Delta_{\{0,1\}^{\mathbb{N}}}$, so the result follows from the previous lemma. $\square$

4.2   Approximating the diagonal

Recall that Δ_X denotes the equality relation on X. We will now characterize equivalence relations whose orbit closure contains Δ_X, mirroring Popa's result on asymptotic orthogonalization of subalgebras of a finite factor [Pop19, Lem. 2.3].

Theorem 4.7. *Let $\mathcal{R}$ be an aperiodic p.m.p. equivalence relation. Let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be a subequivalence relation. The following are equivalent:*

(i) $\mathcal{S}$ has everywhere infinite index in $\mathcal{R}$;

(ii) the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains Δ_X;

(iii) the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains Δ_X.

Proof. The implication (ii) $\implies$ (iii) is clear since $[\mathcal{R}] \leqslant \mathrm{Aut}(\mathcal{R})$.

Let us then check that (iii) implies (i) by proving the contrapositive: assuming that $\mathcal{S}$ does not have infinite index everywhere, we need to show that the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ does not contain Δ_X.

By assumption, we have a subset $A \subseteq X$ of positive measure such that the restriction $\mathcal{S}_{\restriction A}$ has finite index in $\mathcal{R}_{\restriction A}$. Shrinking A further if necessary, we may assume the index of the restriction of $\mathcal{S}$ to A in the restriction of $\mathcal{R}$ to A is constant equal to k. Let $N \in \mathbb{N}$ such that $1/N < \mu(A)$.

By aperiodicity and Maharam's lemma with $f = \frac{1}{2kN}$ (see Proposition 4.3), we can partition X in $2kN$ pieces of equal $\mathcal{R}$-conditional measure. Using Lemma 4.2, we thus have $B \subseteq X$ and $\varphi_1, ..., \varphi_{2kN} \in [[\mathcal{R}]]$ with domain B such that $\varphi_1 = \mathrm{id}_B$ and the sets $\varphi_1(B), ..., \varphi_{2kN}(B)$ partition X.

Let B_0 be the set of $x \in B$ such that there are at least $k+1$ distinct indices $i \in \{1, \ldots, 2kN\}$ such that $\varphi_i(x) \in A$. Write $A_0 = \{x \in A : \exists i, \varphi_i^{-1}(x) \in B_0\}$, then since A_0 is covered by the disjoint translates of B_0 we have $\mu(A_0) \leqslant 2kN\mu(B_0)$. Letting $A_1 = A \setminus A_0$, we again have that A_1 is covered by the disjoint translates of $B_1 = B \setminus B_0$, but by definition for every $x \in B_1$ there are at most k indices $i \in \{1, \ldots, 2kN\}$ such that $\varphi(x) \in A_1$, so that

$$\mu(A_1) \leqslant k\mu(B_1) \leqslant \frac{k}{2kN}.$$

Since $A = A_0 \sqcup A_1$, we then have $\mu(A) \leqslant \frac{k}{2kN} + 2kN\mu(B_0)$. But $1/N < \mu(A)$ so $1/2N < 2kN\mu(B_0)$ and hence

$$\mu(B_0) \geqslant \frac{1}{2kN^2}.$$

Now for all $x \in B_0$, because the index of $\mathcal{S}_{\restriction A}$ in $\mathcal{R}_{\restriction A}$ is k, we must have two distinct i, j such that $(\varphi_i(x), \varphi_j(x)) \in \mathcal{S}$. In particular,

$$\sum_{1 \leqslant i < j \leqslant 2N} M(\varphi_i^{-1}\varphi_j \cap \mathcal{S}) \geqslant \frac{1}{2kN^2}.$$

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The latter estimate will actually be valid for every translate of $\mathcal{S}$ by $T \in \mathrm{Aut}(\mathcal{R})$ because $T(A)$ will still satisfy $1/N < \mu(T(A))$, so for every $T \in \mathrm{Aut}(\mathcal{R})$,

$$\sum_{1 \leqslant i < j \leqslant 2N} M(\varphi_i^{-1}\varphi_j \cap T \cdot \mathcal{S}) \geqslant \frac{1}{2kN^2}.$$

This inequality defines a closed set of subequivalence relations, but it is not satisfied by Δ_X, which finishes the proof of (iii)⇒(i).

We finally prove that (i) implies ii. We fix $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ everywhere of infinite index. Let us also fix a uniquely generating sequence (φ_k) of moving partial involutions of $\mathcal{R}$ as provided by Proposition 3.4. Applying Proposition 3.1 to the sequence of M-finite measure subsets φ_k, it suffices to show that given some $k \in \mathbb{N}$ and $\epsilon > 0$, we can find $T \in [\mathcal{R}]$ such that for all $i \in \{1, \ldots, k\}$,

$$M((T^{-1} \cdot \mathcal{S}) \cap \varphi_i) < \epsilon.$$

We will actually do better and construct $T \in [\mathcal{R}]$ such that for all $i \in \{1, \ldots, k\}$, $M((T^{-1} \cdot \mathcal{S}) \cap \varphi_i) = 0$. Observe that it now suffices (and it is necessary) to build T such that that for almost all $x \in \mathrm{dom}\, \varphi_i$ and all $i \in \{1, \ldots, k\}$,

$$(T(x), T\varphi_i(x)) \notin \mathcal{S}.$$

Our construction makes a crucial use of a result of Eisenmann and Glasner: letting $\Gamma = \{\gamma_n \colon n \in \mathbb{N}\}$ be a dense subgroup of $[\mathcal{R}]$, by [EG16, Prop. 1.19] we have that Γ acts highly transitively on the orbit of almost every $x \in X$, and we may as well restrict ourselves to the set of all such x's.

We can now start building T inductively as the increasing union of elements ψ_n of the pseudo full group defined as follows.

First, ψ_0 is the restriction of γ_0 to the set of all $x \in X$ such that for all $i \in \{1, \ldots, k\}$, we have $(\gamma_0 x, \gamma_0\varphi_i(x)) \notin \mathcal{S}$. Then, assuming ψ_n has been built, we extend it as ψ_{n+1} by letting $\psi_{n+1}(x) = \gamma_{n+1}x$ if $x \notin \mathrm{dom}\, \psi_n$ and $\gamma_{n+1}x \notin \mathrm{rng}\, \psi_n$ and for all $i \in \{1, \ldots, k\}$

- if $\psi_n\varphi_i(x)$ is defined then $(\gamma_{n+1}x, \psi_n\varphi_i(x)) \notin \mathcal{S}$.

- if $\psi_n\varphi_i(x)$ is not defined then $(\gamma_{n+1}x, \gamma_{n+1}\varphi_i(x)) \notin \mathcal{S}$

Let $\psi = \bigcup_n \psi_n$, we need to argue that ψ has full domain. Assume not, observe first that for almost all x in the complement of the domain of ψ, the $\mathcal{R}$-class of x intersects of the complement of the range of ψ, because otherwise we have a positive measure $\mathcal{R}$-invariant set such that ψ^{-1} takes it into a subset of itself of smaller measure, contradicting that $\mathcal{R}$ is measure-preserving.

Now take $x \in X \setminus \mathrm{dom}\, \psi$ as above. Let $i_1, \ldots, i_l \in \{1, \ldots, k\}$ be the indices such that $\psi\varphi_{i_1}(x), \ldots, \psi\varphi_{i_l}(x)$ is defined, denote by $j_1 \ldots j_{k-l}$ the remaining indices. Since $\mathcal{S}$ has infinite index in $X \setminus \mathrm{rng}\, \psi$ there is $z \in [x]_\mathcal{R} \cap X \setminus \mathrm{rng}\, \psi$ and $z_1, \ldots, z_{k-l} \in [x]_\mathcal{R} \cap X \setminus \mathrm{rng}\, \psi$ such that

- for all $m \in \{1, \ldots, l\}$ we have $(z, \psi_n\varphi_{i_m}(x)) \notin \mathcal{S}$ and

- for all $m \in \{1, \ldots, k-l\}$ we have $(z, z_m) \notin \mathcal{S}$.

By high transitivity, there are infinitely many $\gamma \in \Gamma$ such that

$$\gamma x = z \text{ and } \forall m \in \{1, \cdots, k-l\}, \quad \gamma \varphi_{j_m}(x) = z_m.$$

This shows that almost all $x \in X \setminus \operatorname{dom} \varphi$ there is $n \in \mathbb{N}$ such that $\gamma_{n+1} x \notin \operatorname{rng} \psi$ and for all $i \in \{1, \ldots, k\}$, we have :

- if $\psi \varphi_i(x)$ is defined then $(\gamma_{n+1} x, \psi \varphi_i(x)) \notin \mathcal{S}$, and

- if $\psi \varphi_i(x)$ is not defined then $(\gamma_{n+1} x, \gamma_{n+1} \varphi_i(x)) \notin \mathcal{S}$.

Let n be the first integer such that the set of $x \in X \setminus \operatorname{dom} \varphi$ satisfying the above two conditions is non null. By the definition of ψ, this set should be contained in the domain of ψ_{n+1} and hence of ψ, a contradiction.

So ψ is everywhere defined; we just need to check that the full group element $T = \psi$ is as wanted. Let $x \in X$, take $n \geqslant -1$ such that $T(x)$ was defined at stage $n+1$ as $T(x) = \psi_{n+1}(x) = \gamma_{n+1} x$.

Let $i \in \{1, \ldots, k\}$. If $\psi_n \varphi_i(x)$ was already defined, then $(\psi_n \varphi_i(x), \gamma_{n+1} x) \notin \mathcal{S}$ by construction, which means that $(T \varphi_i(x), T(x)) \notin \mathcal{S}$. If $\psi_n \varphi_i(x)$ was not defined, there are two possibilites:

- Either we defined at the same time $\psi_{n+1}(\varphi_i(x)) = \gamma_{n+1} \varphi_i(x)$, in which case the second condition $(\gamma_{n+1} x, \gamma_{n+1} \varphi_i(x)) \notin \mathcal{S}$ guarantees $(\psi_n \varphi_i(x), \gamma_{n+1} x) \notin \mathcal{S}$ and hence $(T \varphi_i(x), T(x)) \notin \mathcal{S}$.

- Or we defined $\psi_{m+1}(\varphi_i(x)) = \gamma_{m+1}(\varphi_i(x))$ at a later stager $m > n$, but then the first condition $(\gamma_{m+1} \varphi_i(x), \psi_m \varphi_i \varphi_i(x)) \notin \mathcal{S}$ and the fact that φ_i is involutive imply that $(\psi_{m+1} \varphi_i(x), \psi_m(x)) \notin \mathcal{S}$ and hence $(T \varphi_i(x), T(x)) \notin \mathcal{S}$.

Since in all cases we reached the desired conclusion $(T \varphi_i(x), T(x)) \notin \mathcal{S}$, the proof is finished. $\qquad \square$

4.3 Dense orbits in the space of hyperfinite subequivalence relations

By definition a CBER is called **finite** when all its equivalence classes are finite.

Definition 4.8. Let $\mathcal{R}$ be a CBER, say that $\mathcal{R}$ is hyperfinite if it can be written as an increasing union of finite Borel subequivalence relations.

In the measured context, hyperfiniteness can be characterized in full group terms as follows: $\mathcal{R}$ is **hyperfinite** iff for all $T_1, \ldots, T_n \in [\mathcal{R}]$ and $\epsilon > 0$, after throwing away a set X' of measure ϵ, the equivalence relation generated by the restrictions $T_{1 \restriction X \setminus X'}, \ldots, T_{n \restriction X \setminus X'}$ is finite.

Remark. The above definition is Dye's original notion of *approximate finiteness* for full groups [Dye59] (see also [LM14, Prop. 1.57] or [KM04, Lem. 10.4] for a proof of the aforementionned equivalence).

Definition 4.9. Given a non-singular equivalence relation $\mathcal{R}$ on (X, μ), let us denote by $\operatorname{Sub}_{\text{hyp}}(\mathcal{R})$ its space of hyperfinite subequivalence relations.

Using lim inf's as in the proof of Theorem 2.6 and the fact that the class of non-singular hyperfinite equivalence relations is stable under countable increasing unions, one can show that that $\mathrm{Sub_{hyp}}(\mathcal{R})$ is closed in $\mathrm{Sub}(\mathcal{R})$ (see [Kec, Thm. 8.1]). Let us observe that this can also be seen as a consequence of the characterization via approximate finiteness: if $\mathcal{S}$ is not hyperfinite as witnessed by some $\epsilon > 0$ and $T_1, ..., T_n$ its in full group, any $\mathcal{S}'$ which contains the T_i's in its full group up to an $\epsilon/2n$ error will also fail to be hyperfinite.

Before we state our main result on dense orbits in $\mathrm{Sub_{hyp}}(\mathcal{R})$, we need a preparatory well-known lemma on finite subequivalence relations.

Lemma 4.10. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation. Then two finite subequivalence relations $\mathcal{S}_1$ and $\mathcal{S}_2$ of $\mathcal{R}$ are in the same $[\mathcal{R}]$-orbit iff for all $n \in \mathbb{N}$*

$$\mu(\{x \in \mathbb{N} \colon |[x]_{\mathcal{S}_1}| = n\}) = \mu(\{x \in \mathbb{N} \colon |[x]_{\mathcal{S}_2}| = n\}). \tag{1}$$

Proof. The direct implication is clear. Assume conversely that (1) holds. Let us fix some notation by letting, for $i \in \{1, 2\}$ and $n \geqslant 1$

$$X_n^i = \{x \in \mathbb{N} \colon |[x]_{\mathcal{S}_i}| = n\}.$$

Our assumption then becomes: the equality $\mu(X_n^1) = \mu(X_n^2)$ holds for every $n \geqslant 1$. Let $<$ be a Borel linear order on X. We can then define for all $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$ the Borel set
$$Y_{n,j}^i = \{x \in X_n^i \colon x \text{ is the } j\text{'th element of } [x]_{\mathcal{S}_i}\}$$
For $i \in \{1, 2\}$, and $j_1, j_2 \in \{1, \ldots, n\}$, we have $\varphi_{n,j_1,j_2}^i : Y_{n,j_1}^i \to Y_{n,j_2}^i$ defined by taking $x \in Y_{n,j_1}^i$ to the j_2'th element of its $\mathcal{S}_i$-class, then $\varphi_{n,j_1,j_2}^i \in [[\mathcal{S}_i]] \subseteq [[\mathcal{R}]]$. In particular since $\mathcal{R}$ is p.m.p., we have that $\mu(Y_{n,j_1}^i) = \mu(Y_{n,j_2}^i)$. Since X_n^i is partitioned by $(Y_{n,j}^i)_{j=1}^n$, we conclude
$$n\mu(Y_{n,1}^1) = \mu(X_n^1) = \mu(X_n^2) = n\mu(Y_{n,1}^2)$$
so that $\mu(Y_{n,1}^1) = \mu(Y_{n,1}^2)$.

Since $\mathcal{R}$ is ergodic, for every $n \in \mathbb{N}$ we can fix some $\varphi_n \in [[\mathcal{R}]]$ such that $\mathrm{dom}\, \varphi_n = Y_{n,1}^1$ and $\mathrm{rng}\, \varphi_n = Y_{n,1}^2$. We then extend simultaneously these φ_n as $T \in [\mathcal{R}]$ by letting for every $n \geqslant 1$, $j \in \{1, \ldots n\}$ and $x \in X_{n,j}^1$:

$$T(x) = \varphi_{n,1,j}^2 \varphi_n \varphi_{n,j,1}^1(x)$$

In other words, given $x \in X$ whose $\mathcal{S}_1$-class has cardinality n, we look at the first element y of the $\mathcal{S}_1$-class of x, and then $T(x)$ is the element of the $\mathcal{S}_2$-class of $\varphi_n(y)$ which is in the same position as x. Then by construction $T \cdot \mathcal{S}_1 = \mathcal{S}_2$ as wanted. $\square$

Theorem 4.11. *Let $\mathcal{R}$ be a p.m.p. ergodic equivalence relation, let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$ be aperiodic and have everywhere infinite index. Then the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\mathrm{Sub_{hyp}}(\mathcal{R})$.*

Proof. It follows from the definition of hyperfiniteness that finite equivalence relations are dense in $\mathrm{Sub_{hyp}}(\mathcal{R})$, so it suffices to approximate every finite subequivalence relation of $\mathcal{R}$ by an element of the $[\mathcal{R}]$-orbit of S. Let $\mathcal{R}_0$ be such a finite equivalence relation. For each n let
$$X_n = \{x \in X : |[x]_{\mathcal{R}_0}| = n\}.$$

Because $\mathcal{S}$ is aperiodic, we can find an element of the $[\mathcal{R}]$-orbit of $\mathcal{R}_0$ contained in $\mathcal{S}$. Indeed, by Maharam's lemma we first have a partition of X into pieces $(Y_{j,n})_{1 \leqslant j \leqslant n}$ such

that $\mu_{\mathcal{S}}(Y_{j,n}) = \frac{\mu(X_n)}{n}$. We then have for every $2 \leqslant j \leqslant n$ some $\varphi_{j,n} \in [[\mathcal{S}]]$ such that dom $\varphi_{j,n} = Y_{1,n}$ and rng $\varphi_{j,n} = Y_{j,n}$.

Now let $Y_1 = \bigsqcup_n Y_{1,n}$, $\psi_1 = \mathrm{id}_{Y_1}$ and for all $j \geqslant 2$, $\psi_j = \bigsqcup_{n \geqslant j} \varphi_{j,n}$. Note that the ranges of the ψ_j's partition X. Denoting by $\mathcal{S}_0$ the equivalence relation generated by all the ψ_j's, we then have

$$\mathcal{S}_0 = \bigsqcup_n \bigsqcup_{i,j=1}^n \varphi_{n,i} \times \varphi_{n,j}(\Delta_{Y_{1,n}}) = \bigsqcup_{i,j} \psi_i \times \psi_j(\Delta_{Y_1} \cap (\mathrm{dom}\ \psi_i \times \mathrm{dom}\ \psi_j)),$$

In particular the $\mathcal{S}_0$-class of every $x \in \bigsqcup_{j=1}^n Y_{n,j}$ has cardinality n, and so by Lemma 4.10 we can fix some $T \in [\mathcal{R}]$ such that $T \cdot \mathcal{S}_0 = \mathcal{R}_0$.

By Theorem 4.7, we find a sequence $(T_k)_k$ in the full group of the restriction of $\mathcal{R}$ to Y_1 such that $T_k \cdot \mathcal{S}_{|Y_1} \to \Delta_{Y_1}$. We then let $\tilde{T}_k(x) = \psi_j T_k \psi_j^{-1}(x)$ for all $x \in \mathrm{rng}\ \psi_j$.

By construction $\tilde{T}_k \cdot \mathcal{S} = \bigsqcup_{i,j} \psi_i \times \psi_j(T_k \cdot \mathcal{S}_{|Y_1} \cap (\mathrm{dom}\ \psi_i \times \mathrm{dom}\ \psi_j))$, so by Lemma 3.11

$$\tilde{T}_k \cdot \mathcal{S} \to \bigsqcup_{i,j} \psi_i \times \psi_j(\Delta_{Y_1} \cap (\mathrm{dom}\ \psi_i \times \mathrm{dom}\ \psi_j)) = \mathcal{S}_0$$

which yields the desired result since we then have $T\tilde{T}_k \cdot \mathcal{S} \to T \cdot \mathcal{S}_0 = \mathcal{R}_0$. $\qquad\square$

Remark. By [IKT09, Cor. 5.4 (ii)], if $\mathcal{R}$ is aperiodic and comes from a measure-preserving action of a property (T) countable group, then there are no dense orbits in the space of subequivalence relations because any $\mathcal{S}$ with a dense orbit would have to have finite index in some restriction of $\mathcal{R}$, and hence cannot contain Δ_X in its orbit by Theorem 4.7. The general fact is that there cannot be dense orbits in the space of subequivalence relations of $\mathcal{R}$ as soon as $\mathcal{R}$ is not *approximable* as defined by Gaboriau and Tucker-Drob in [GT16]. Indeed if $\mathcal{S}$ has a dense orbit Lemma 2.2 provides a sequence $\mathcal{S}_n$ such that $\mathcal{R} = \liminf \mathcal{S}_n$, while non approximability forces some restriction of $\bigcap_{n \geqslant N} \mathcal{S}_n$ to coincide with $\mathcal{R}$ on a positive measure subset A, contradicting the density of the orbit of $\mathcal{S}$ by Theorem C. For more examples of non approximable equivalence relations, the reader can consult the paper of Gaboriau and Tucker-Drob, where they obtain for instance a quantitative version of non-approximability for some equivalence relations coming from actions of product groups (see [GT16, Thm. 2.4]).

Corollary 4.12. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, let $\mathcal{S} \in \mathrm{Sub}(\mathcal{R})$. The following are equivalent:*

  (i) *$\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}$;*

  (ii) *$\mathcal{S}$ is aperiodic and the closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains Δ_X;*

  (iii) *$\mathcal{S}$ is aperiodic and the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains Δ_X;*

  (iv) *The closure of the $[\mathcal{R}]$-orbit of $\mathcal{S}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$;*

  (v) *The closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.*

Proof. The equivalence of the first three items (i), (ii) and (iii) is a direct consequence of Theorem 4.7. The implication (i) $\implies$ (iv) is exactly Theorem 4.11, and (iv) clearly implies (v). Finally, let us prove (v) $\implies$ (iii): assume the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of

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$\mathcal{S}$ contains $\mathrm{Sub}_{hyp}(\mathcal{R})$. Since Δ_X is hyperfinite, we have that Δ_X belongs to the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S}$, so we only have to show $\mathcal{S}$ is aperiodic. Assume by contradiction that $\mathcal{S}$ is not aperiodic, then for some $n \in \mathbb{N}$ we have $\mu(\{x \in X : |[x]_{\mathcal{S}}| \leqslant n\}) > 0$. Let $\delta = \mu(\{x \in X : |[x]_{\mathcal{S}}| \leqslant n\})$, then $\mathcal{S}$ belongs to the $\mathrm{Aut}(\mathcal{R})$-invariant set $\mathbf{B}$ of all $\mathcal{S}' \in \mathrm{Sub}(\mathcal{R})$ such that

$$\sup_{T_1,\ldots,T_{n+1} \in [\mathcal{R}]} M(\mathcal{S}' \cap (T_1 \cup \cdots \cup T_{n+1})) \leqslant \delta n + (1 - \delta)(n + 1).$$

Note that the intersection map is continuous for the lower M-topology by Lemma 3.9 and that M is lower semi-continuous for the lower M-topology. Since arbitrary supremums of lower semi-continuous function are lower semi-continuous, the $\mathrm{Aut}(\mathcal{R})$-invariant set $\mathbf{B}$ is closed for the lower M-topology, in particular it is closed for the strong topology. Since the closure of the $\mathrm{Aut}(\mathcal{R})$-orbit of $\mathcal{S} \in \mathbf{B}$ contains $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$, we have $\mathbf{B} \subseteq \mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.

However $\mathbf{B}$ is disjoint from the set of aperiodic subequivalence relations since for $\mathcal{T}$ aperiodic we can find $T_1, \ldots, T_{n+1}$ in the full group of $\mathcal{T}$ with disjoint graphs, so that $M(\mathcal{T} \cap (T_1 \sqcup T_2 \sqcup \cdots \sqcup T_{n+1})) = n + 1$. Since $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ contains aperiodic subequivalence relations, we reached the desired contradiction. So $\mathcal{S}$ is aperiodic and hence (iii) holds as wanted. $\qquad\square$

Corollary 4.13. *For a subequivalence $\mathcal{S}$ of the hyperfinite ergodic p.m.p. equivalence relation $\mathcal{R}_0$, the following are equivalent:*

(i) $\mathcal{S}$ is aperiodic and has everywhere infinite index in $\mathcal{R}_0$;

(ii) $\mathcal{S}$ is aperiodic and the closure of the $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ contains Δ_X;

(iii) $\mathcal{S}$ is aperiodic and the closure of the $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ contains Δ_X;

(iv) The $[\mathcal{R}_0]$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$;

(v) The $\mathrm{Aut}(\mathcal{R}_0)$-orbit of $\mathcal{S}$ is dense in $\mathrm{Sub}(\mathcal{R}_0)$.

Proof. This is a direct consequence of the previous corollary since every subequivalence relation of a hyperfinite equivalence relation is hyperfinite. $\qquad\square$

Remark. In the non-singular ergodic type II_∞ or type III case, one can show that there are always dense orbits in the space of subequivalence relations. Let us sketch the proof: first enumerate a dense subset of $\mathrm{Sub}(\mathcal{R})$ as $(\mathcal{S}_n)$ and note that there is a sequence (φ_n) of elements of $[[\mathcal{R}]]$ such that for all n, $\mathrm{dom}\,\varphi_n = X$ but $X = \bigsqcup_n \mathrm{rng}\,\varphi_n$. Then the equivalence relation $\mathcal{S} = \bigsqcup_n \varphi_n \cdot \mathcal{S}_n$ has a dense full group orbit, as one can see by approximating each φ_n by a full group element in the d_M metric.

4.4 Meagerness of full group orbits

In this section, we use Ioana's intertwining for subequivalence relations in order to show that full groups orbits are always meager in the space of hyperfinite subequivalence relations. In what follows, we use the **uniform metric** on the full group of a p.m.p. equivalence relation $\mathcal{R}$, defined by

$$d_u(T_1, T_2) = \mu(\{x \in X : T_1(x) \neq T_2(x)\}) = d_M(T_1, T_2),$$

which is biinvariant, complete and separable, thus endowing $[\mathcal{R}]$ with a Polish group topology. We will use without mention the well-known fact that if $M(\mathcal{S} \cap T) > 1 - \epsilon$, then there is $S \in [\mathcal{S}]$ such that $d_u(S, T) < \epsilon$ (to see this, note that $\mathcal{S} \cap T \in [[\mathcal{S}]]$, hence it can be extended to an element S of the full group of $\mathcal{S}$ by Lemma 4.2).

Definition 4.14 (Ioana)**.** Let $\mathcal{S}, \mathcal{T}$ be two subequivalence relations of a p.m.p. equivalence relation $\mathcal{R}$, write $\mathcal{S} \prec \mathcal{T}$ if there is no sequence (T_n) in the full group of $\mathcal{T}$ such that for all $U_1, U_2 \in [\mathcal{R}]$, we have

$$M(\mathcal{S} \cap U_1 T_n U_2) \to 0.$$

We leave it to the reader to check that $\prec$ is $[\mathcal{R}]$-invariant, meaning that if $\mathcal{S} \prec \mathcal{T}$ and $U_1, U_2 \in [\mathcal{R}]$, then $U_1 \cdot \mathcal{S} \prec U_2 \cdot \mathcal{T}$. Also note that we always have $\mathcal{S} \prec \mathcal{S}$.

Lemma 4.15. *The relation $\prec$ defines a F_σ subset of $\mathrm{Sub}(\mathcal{R}) \times \mathrm{Sub}(\mathcal{R})$.*

Proof. We show that the complement of $\prec$ in $\mathrm{Sub}(\mathcal{R}) \times \mathrm{Sub}(\mathcal{R})$ is G_δ. Let us fix (U_i) dense in $\mathcal{R}$. It is then not hard to check that $\mathcal{S} \not\prec \mathcal{T}$ iff there is a sequence (T_n) in the full group of T such that $M(\mathcal{S} \cap U_i T_n U_j) \to 0$ for all $i, j \in \mathbb{N}$ (this is essentially the first step of the proof of [Ioa12, Lem. 1.7]). It follows that $\mathcal{S} \not\prec \mathcal{T}$ if and only if for every $k \in \mathbb{N}$ and $\epsilon > 0$, we can find $T \in [\mathcal{T}]$ such that

$$\forall i, j \in \{1, \dots, k\}, \quad M(\mathcal{S} \cap U_i T U_j) < \epsilon. \tag{2}$$

Now by density of (U_i), we finally have $\mathcal{S} \not\prec \mathcal{T}$ iff for every $k \in \mathbb{N}$ and $\epsilon > 0$, there is $l \in \mathbb{N}$ such that $M(\mathcal{T} \cap U_l) > 1 - \epsilon$ and for all $i, j \in \{1, \dots, k\}$:

$$M(\mathcal{S} \cap U_i U_l U_j) < \epsilon.$$

It is now straightforward to check that this last condition defines a G_δ set, so we are done. $\qquad\square$

Theorem 4.16. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation, consider the $[\mathcal{R}]$-action on the Polish space $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ of hyperfinite subequivalence relations of $\mathcal{R}$. Then all $[\mathcal{R}]$-orbits in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ are meager.*

Proof. Assume by contradiction there is a non meager $[\mathcal{R}]$-orbit. Since there is a dense $[\mathcal{R}]$-orbit in $\mathrm{Sub}(\mathcal{R})$, the topological 0-1 law yields that there is a comeager orbit. Denoting by E the equivalence relation generated by the $[\mathcal{R}]$-action on $\mathrm{Sub}(\mathcal{R})$, we deduce that E is comeager in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R}) \times \mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$.

By [Dye59, Thm. 4], the p.m.p. equivalence relation $\mathcal{R}$ contains an ergodic hyperfinite subequivalence relation $\mathcal{S}$. Let $\mathcal{T}$ be an aperiodic subequivalence relation of $\mathcal{S}$ with diffuse ergodic decomposition such as the one coming from example 1.1 once we identify $\mathcal{S}$ to $\mathcal{R}_0$. The following claim will essentially finish our proof.

Claim. *We have $\mathcal{S} \not\prec \mathcal{T}$.*

Proof of the claim. Suppose $\mathcal{S} \prec \mathcal{T}$. Then by [Ioa12, Lem. 1.7], there is a nonzero $\varphi \in [[\mathcal{R}]]$ and $k \in \mathbb{N}$ such that if $A = \mathrm{dom}\,\varphi$ and $B = \mathrm{rng}\,\varphi$, then every $\varphi \cdot \mathcal{S}_{\restriction A}$-class is contained in the union of at most k classes of $\mathcal{T}_{\restriction B}$. Since $\mathcal{T}$ has diffuse ergodic decomposition, we can find $C_1, \dots, C_{k+1}$ partitioning B which are all $\mathcal{T}_{\restriction B}$-invariant and have measure $\frac{1}{(k+1)\mu(B)}$.

Since $\mathcal{S}$ is ergodic, the $\varphi \cdot \mathcal{S}_{\restriction A}$-class of almost every $x \in B$ intersects all the C_i, and since they are $\mathcal{T}_{\restriction B}$-invariant, we conclude that almost every $\varphi \cdot \mathcal{S}_{\restriction A}$-class cannot be contained in less than $k + 1$ classes of $\mathcal{T}_{\restriction B}$, a contradiction. $\qquad\square_{\mathrm{claim}}$

Since $\prec$ is $[\mathcal{R}]$-invariant, its complement also is. This complement is moreover G_δ by Lemma 4.15, and by the above claim it contains the subset $[\mathcal{R}] \cdot \mathcal{S} \times [\mathcal{R}] \cdot \mathcal{T}$, which is dense in $\mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R}) \times \mathrm{Sub}_{\mathrm{hyp}}(\mathcal{R})$ by Theorem 4.11. So the complement of $\prec$ is comeager, and it should thus intersect E. This means that one can find a subequivalence relation $\mathcal{S}$ and $T \in [\mathcal{R}]$ such that $\mathcal{S} \not\prec T \cdot \mathcal{S}$, a contradiction. $\qquad\square$

Remark. Our proof is inspired by the Glasner-Weiss proof that $\mathrm{Aut}(X, \mu)$ does not have comeager conjugacy classes, replacing Rokhlin's lemma by Theorem 4.11 and Del Junco's disjointness by Ioana's interwining relation $\not\prec$ (see [GW08, Sec. 3]).

Corollary 4.17. *Let $\mathcal{R}_0$ be the hyperfinite ergodic p.m.p. equivalence relation. Then all the $[\mathcal{R}_0]$-orbits are meager in $\mathrm{Sub}(\mathcal{R}_0)$*

Proof. Again this follows directly from the above theorem since every subequivalence relation of a hyperfinite equivalence relation is hyperfinie. $\qquad\square$

5   The uniform metric and complexity calculations

We begin this final section by introducing a natural metric on the space of all non-singular equivalence relations. Given two non-singular equivalence relations $\mathcal{R}_1$ and $\mathcal{R}_2$, denote by $\mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)$ the set of all $A \in \mathrm{MAlg}(X, \mu)$ such that $\mathcal{R}_{1\restriction A} = \mathcal{R}_{2\restriction A}$. Define

$$d_u(\mathcal{R}_1, \mathcal{R}_2) = 1 - \sup_{A \in \mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)} \mu(A) = \inf_{A \in \mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)} 1 - \mu(A).$$

Let us check that this is indeed a metric. Symmetry is clear. For Hausdorffness, if $d_u(\mathcal{R}_1, \mathcal{R}_2) = 0$ then we find (X_n) with $\mu(X_n) \geqslant 1 - 2^{-n}$ and $\mathcal{R}_{1\restriction X_n} = \mathcal{R}_{2\restriction X_n}$. By the Borel-Cantelli lemma almost every $x \in X$ is in all but finitely many X_n, which yields a full measure set restricted to which $\mathcal{R}_1$ and $\mathcal{R}_2$ coincide.

We finally need to show that the triangle inequality holds. Observe that if $A_1 \in \mathfrak{C}(\mathcal{R}_1, \mathcal{R}_2)$ and $A_2 \in \mathfrak{C}(\mathcal{R}_2, \mathcal{R}_3)$, then $A_1 \cap A_2 \in \mathfrak{C}(\mathcal{R}_1, \mathcal{R}_3)$, moreover

$$X \setminus (A_1 \cap A_2) \subseteq (X \setminus A_1) \cup (X \setminus A_2),$$

so by taking measures and infimums we get the desired triangle inequality.

Lemma 5.1. *The uniform metric refines the strong topology.*

Proof. The uniform metric clearly refines the uniform topology defined in [Kec, Sec. 4.6], which in turn refines the strong topology. $\qquad\square$

Remark. The results from [IKT09, Thm. 1] applied to the subequivalence relation $\mathcal{R}_1 \cap \mathcal{R}_2$ yield that the uniform metric actually induces the uniform topology.

Proposition 5.2. *Suppose $\mathcal{R}$ is an aperiodic non-singular equivalence relation. There is a continuous reduction from the set of finite subsets of $\mathbb{N}$ to the complement of the set of subequivalence relations which are of infinite index, and to the set of finite index subequivalence relations, both for the uniform metric.*

Proof. Let (X_n) be a partition of X into sets which intersect almost every $\mathcal{R}$-class (e.g. obtained via Maharam's lemma as sets of $\mathcal{R}$-conditional measure constant equal to 2^{-n-1}).

Let $B \subseteq \mathbb{N}$. We associate to B the set $Y_B = \bigcup_{n \in \mathbb{N} \setminus B} X_n$ and let the reduction be $B \mapsto \mathcal{R}_B$ where

$$\mathcal{R}_B = (\mathcal{R} \cap Y_B \times Y_B) \sqcup \bigcup_{n \in B} \mathcal{R} \cap (X_n \times X_n)$$

The continuity of our reduction for the uniform metric is clear from the definition (if two subsets B and B' coincide on the first n integers, then the associated equivalence relations coincide on $\bigcup_{k < n} X_k$). Moreover, if B is finite we see that $\mathcal{R}_B$ has finite index equal to $|B| + 1$, and if not it has infinite index as wanted. $\qquad\square$

Corollary 5.3. *Let $\mathcal{R}$ be a non-singular aperiodic equivalence relation. The set of subequivalence relations of $\mathcal{R}$ with infinite index is G_δ-hard in the uniform metric, in particular it is G_δ-complete in the strong topology.*

The next statements are all about the strong topology.

Proposition 5.4. *Let $\mathcal{R}$ be a nonsingular aperiodic equivalence relation. The space of finite index subequivalence relations is $F_{\sigma\delta}$-hard if and only if E has infinitely many ergodic components, otherwise it is F_σ-complete.*

Proof. Denote by $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ the space of finite index subequivalence relations. Note that by ergodicity, if E has only finitely many ergodic components, then the space of finite index subequivalence relations is equal to the reunion over $k \in \mathbb{N}$ of the spaces $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ of all subequivalence relations whose index is uniformly less than k.

Let us show that for each $k \in \mathbb{N}$, the set $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is closed. Suppose $\mathcal{S} \notin \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$, then there is a positive measure set of $x \in X$ such that $[x]_\mathcal{R}$ contains at least $k + 1$ distinct $\mathcal{S}$ classes. We thus find $A \subseteq X$ of positive measure and $\varphi_1, ... \varphi_k \in [[\mathcal{R}]]$ with common domain A and such that $A, \mathrm{rng}\,\varphi_1, \ldots, \mathrm{rng}\,\varphi_k$ are and $\mathcal{S}$ is disjoint from the reunion of the graphs of the $\varphi_i \varphi_j^{-1}$. In other words we find a finite subequivalence relation of $\mathcal{R}$ which is disjoint from $\mathcal{S}$ (after of course removing Δ_X) and all whose non trivial classes have cardinal $\geqslant k + 1$.

Now let $\mathcal{S}_n \to \mathcal{S}$, since

$$M(\varphi_1 \sqcup \cdots \sqcup \varphi_k \cap \mathcal{S}_n) = \int_A |\{i \in \{1, \ldots, k\} : (x, \varphi_i(x)) \in \mathcal{S}_n\}| \, d\mu(x) \to 0,$$

for n large enough there will be a positive measure set of $x \in A$ such that for all $i \in \{1, \ldots, k\}$, $(x, \varphi_i(x)) \notin \mathcal{S}_n$, in particular $\mathcal{S}_n$ has somewhere index $> k$. So $\mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is closed and hence $\mathrm{Sub}_{[<\infty]}(\mathcal{R}) = \bigcup_{k \in \mathbb{N}} \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is F_σ. Moreover, $\mathrm{Sub}_{[<\infty]}(\mathcal{R}) = \bigcup_{k \in \mathbb{N}} \mathrm{Sub}_{[\leqslant k]}(\mathcal{R})$ is F_σ-complete by virtue of the preceding proposition.

Now if $\mathcal{R}$ has infinitely many ergodic components, let (X_n) be a partition of X into $\mathcal{R}$-invariant sets of positive measure. Denote by $\mathcal{P}_f(\mathbb{N}) \subseteq \{0, 1\}^\mathbb{N}$ the set of finite subsets of $\mathbb{N}$, viewed as finitely supported functions $\mathbb{N} \to \{0, 1\}$.

For every $n \in \mathbb{N}$, let $\Phi_n : \to \mathrm{Sub}(\mathcal{R}_{\upharpoonright X_n})$ be the reduction from the previous proposition, and for a sequence (x_n) of elements of $\{0, 1\}^\mathbb{N}$, let $\Phi(x_n) = \bigsqcup_n \Phi_n(x_n) \in \mathrm{Sub}(\mathcal{R})$. By Lemma 3.11 Φ is continuous, and by construction $\Phi^{-1}(\mathrm{Sub}_{[<\infty]}(\mathcal{R})) = \prod_{n \in \mathbb{N}} \mathcal{P}_f(\mathbb{N})$. Being a (countable) infinite product of F_σ-hard sets, the latter is $F_{\sigma\delta}$-hard, so the set of finite index subequivalence relations is $F_{\sigma\delta}$-hard. $\qquad\square$

Remark. It follows from [Kec, Prop. 9.4] that $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ is always $F_{\sigma\delta}$ (the proof in the nonsingular case works the same), so when $\mathcal{R}$ is aperiodic and has infinitely many ergodic components, $\mathrm{Sub}_{[<\infty]}(\mathcal{R})$ is actually $F_{\sigma\delta}$-complete.

We finally use a similar approach to show that the set of finite subequivalence relations is always $F_{\sigma\delta}$-complete (for $\mathcal{R}$ aperiodic), by first directly showing that the space of finite subequivalence relations if F_σ-hard. In the remainder of the paper, we set

$$\Gamma_0 = \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z} = \{f : \mathbb{N} \to \mathbb{Z}/2\mathbb{Z} \mid f(n) = 0 \text{ for all but finitely many } n \in \mathbb{N}\}.$$

Lemma 5.5. *The space* $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$ *of finite subgroups of* Γ_0 *is an* F_σ-*complete subset of the space* $\mathrm{Sub}(\Gamma_0)$ *of subgroups of* Γ_0.

Proof. The set of finite subgroups is countable, in particular it is F_σ. It is F_σ-complete via the continuous reduction $B \subseteq \mathbb{N} \mapsto \Lambda_B := \bigoplus_{n \in B} \mathbb{Z}/2\mathbb{Z} \leqslant \Gamma$ of the F_σ-complete set of finite subsets of $\mathbb{N}$ to $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$. $\square$

Remark. The above lemma can also be proven by first noting that by the Baire category theorem, the countable dense set $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)$ cannot be G_δ in the perfect zero-dimensional Polish space of subgroups of Γ_0, and then applying Wadge's theorem (see [Kec95, Thm. 22.10]).

The following lemma is probably well-known and true as well in the non-singular case, but for simplicity we stick to the p.m.p. setup.

Lemma 5.6 (Dye). *For every aperiodic p.m.p. equivalence relation* $\mathcal{R}$, *there is a free* Γ_0-*action which induces a subequivalence relation of* $\mathcal{R}$.

Proof. By Maharam's lemma, we may build by induction a partition $(A_n)_{n \geqslant 0}$ of X into subsets such that for all $n \in \mathbb{N}$, their conditional measure satisfies $\mu_{\mathcal{R}}(A_n) = \frac{1}{2^{n+1}}$. For $n \in \mathbb{N}$, let now $B_n = X \setminus \bigsqcup_{m \leqslant n} A_m$, which also satisfies $\mu_{\mathcal{R}}(A_n) = \frac{1}{2^{n+1}}$. Moreover, for every n we have $B_n = A_{n+1} \sqcup B_{n+1}$ by construction.

By Lemma 4.2, for every $n \in \mathbb{N}$ there is $\varphi_n \in [[\mathcal{R}]]$ such that $\dim \varphi_n = A_n$ and $\mathrm{rng}\, \varphi_n = B_n$. We will extend these to involutions so as to have a free Γ_0-action. Letting $G_n = \{f \in \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}: \forall m > n, f(m) = 0\}$, we can write Γ_0 as the increasing union $\Gamma_0 = \bigcup_n G_n$, and we have $G_n = \langle s_0, \ldots, s_n \rangle$, where $s_i(k) = 0$ if $i \neq k$, and $s_i(k) = 1$ if $i = k$. Our inductive construction of the action will be so that for all n,

$$X = \bigsqcup_{g \in G_n} g \cdot B_n \tag{3}$$

We begin by letting $s_0 \cdot x = \varphi_0(x)$ if $x \in A_0$, and $s_0 \cdot x = \varphi_0^{-1}(x)$ if $x \in B_0 = X \setminus A_0$, then $X = B_0 \sqcup s_0 \cdot B_0$. And then, assuming that the action of G_n has been defined so that $X = \bigsqcup_{g \in G_n} g \cdot B_n$, we begin by defining the s_{n+1} action on B_n by letting $s_{n+1} \cdot x = \varphi_{n+1}(x)$ if $x \in A_{n+1}$ and $s_{n+1}(x) = \varphi_{n+1}^{-1}(x)$ if $x \in B_{n+1}$. Since we want s_{n+1} to commute with G_n and we have $X = \bigsqcup_{g \in G_n} g \cdot B_n$, we are then forced to extend s_{n+1} to the whole of x by letting, for every $x \in B_n$ and every $g \in G_n$,

$$s_{n+1}(g \cdot x) = g \cdot s_{n+1} \cdot x.$$

It is then not hard to check that this does define a G_{n+1} action with $X = \bigsqcup_{g \in G_{n+1}} g \cdot B_{n+1}$, which provides us a Γ_0-action by induction in the end. This action is free since Equation 3 implies that the action restricted to each G_n is free, which finishes the proof. $\square$

Remark. It follows from [Dye59, Thm. 3] that when $\mathcal{R}$ is p.m.p. hyperfinite and aperiodic, one can even ask that $\mathcal{R}$ is equal to the equivalence relation generated by the Γ_0-action. This led Dye to formulate the question wether any p.m.p. equivalence relation should come the free action of some countable group (see the paragraph right before Lemma 6.5 in [Dye59]). This question was answered in the negative 40 years later by Furman [Fur99].

Given a non-singular equivalence relation $\mathcal{R}$, we denote by $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ the space of finite subequivalence relations of $\mathcal{R}$.

Proposition 5.7. *Let $\mathcal{R}$ be an aperiodic p.m.p. equivalence relation. Then $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ is $F_{\sigma\delta}$-complete.*

Proof. Partition X into (X_n) where each X_n has positive measure. The restriction of $\mathcal{R}$ to each X_n is aperiodic, by the previous lemma it thus contains a subequivalence relation coming from a free Γ_0-action α_n. Now consider the map $\prod_n \mathrm{Sub}(\Gamma_0) \to S(E)$ which takes any sequence (λ_n) to the equivalence relation $\bigsqcup_n \mathcal{R}_{\alpha_n(\Lambda_n)}$. This is a topological embedding by Lemma 3.11, and it reduces the $F_{\sigma\delta}$-complete set $\mathrm{Sub}_{\mathrm{fin}}(\Gamma_0)^{\mathbb{N}}$ to the set of finite equivalence relations, which is thus $F_{\sigma\delta}$-hard. Since by [Kec, Thm. 8.5], $\mathrm{Sub}_{\mathrm{fin}}(\mathcal{R})$ is $F_{\sigma\delta}$, it is $F_{\sigma\delta}$-complete as wanted. $\square$

Remark. Endowing $\mathrm{Sub}(\mathcal{R})$ with its strong topology and the uniform metric, we get a Polish topometric space in the sense of Ben Yaacov [BY08], and our remark preceding Corollary 4.12 can be upgraded to the fact that if $\mathcal{R}$ has property (T), then $\mathcal{R}$ is a metrically isolated point of $\mathrm{Sub}(\mathcal{R})$ (see also [GT16] for more examples). I don't know whether the action of the full group or automorphism group of the hyperfinite ergodic p.m.p. equivalence relation admits a metrically generic orbit, as characterized in [BYM23].

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