A nonvariational form of the Neumann problem for the Poisson equation

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Abstract: We present a nonvariational setting for the Neumann problem for the Poisson equation for solutions that are Hölder continuous and that may have infinite Dirichlet integral. We introduce a distributional normal derivative on the boundary for the solutions that extends that for harmonic functions that has been introduced in a previous paper and we solve the nonvariational Neumann problem for data in the interior with a negative Schauder exponent and for data on the boundary that belong to a certain space of distributions on the boundary.

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1 Introduction

We plan to consider the Neumann problem for the Poisson equation for Hölder continuous solutions. By the classical examples of Prym [24] and Hadamard [9] on harmonic functions in the unit ball of the plane that are continuous up to the boundary and have infinite Dirichlet integral, *i.e.*, whose gradient is not square-summable, we cannot expect that the solutions of the Poisson equation

$$\Delta u = f$$

in a bounded open subset Ω of $\mathbb{R}^n$ have a finite Dirichlet integral, not even in case $f = 0$ and the boundary of Ω is smooth. For a discussion on this point we refer to Maz'ya and Shaposnikova [16], Bottazzini and Gray [2] and Bramati, Dalla Riva and Luczak [3].

In case $f = 0$, $\alpha \in]0,1[$ and for Ω of class $C^{1,\alpha}$, one can introduce a notion of normal derivative ∂_ν on the boundary $\partial\Omega$ in the sense of distributions of a α-Hölder continuous harmonic function u in Ω (which may have infinite Dirichlet integral) and introduce a space

$$V^{-1,\alpha}(\partial\Omega)$$

of distributions on the boundary (cf. Definition 5.20) such that the Neumann problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega\,, \\ \partial_\nu u = g & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

can be solved for u in the space $C^{0,\alpha}(\overline{\Omega})$ of α-Hölder continuous functions for all data $g \in V^{-1,\alpha}(\partial\Omega)$ that satisfy a compatibility condition that generalizes the classical one (cf. [13, §20]).

In this paper, we consider the space $C^{-1,\alpha}(\overline{\Omega})$ of sums of α-Hölder continuous functions and of first order partial distributional derivatives of α-Hölder continuous functions in Ω and we introduce a distributional normal derivative on $\partial\Omega$ for functions u in the space $C^{0,\alpha}(\overline{\Omega})_\Delta$ of functions in $C^{0,\alpha}(\overline{\Omega})$ such that the distributional Laplace operator Δu belongs to $C^{-1,\alpha}(\overline{\Omega})$ and that extends the above mentioned notion of normal derivative ∂_ν (see Definition 5.11). Then we show that if we choose f in the space $C^{-1,\alpha}(\overline{\Omega})$ and g in $V^{-1,\alpha}(\partial\Omega)$ that satisfy a compatibility condition that generalizes the classical one, then we can solve the Neumann problem

$$\begin{cases} \Delta u = f & \text{in } \Omega\,, \\ \partial_\nu u = g & \text{on } \partial\Omega\,, \end{cases} \tag{1.2}$$

for u in the space $C^{0,\alpha}(\overline{\Omega})_\Delta$ (cf. Theorem 6.8).

Here we mention that the Schauder space with negative exponent $C^{-1,\alpha}(\overline{\Omega})$ has been known for a long time and has been used in the analysis of elliptic and parabolic partial differential equations (cf. Triebel [28], Gilbarg and Trudinger [8], Vespri [29], Lunardi and Vespri [17], Dalla Riva, the author and Musolino [5], [10]).

Our approach here develops from that of [13] and holds in the (nonseparable) spaces of Hölder continuous functions, but could be extended to different function spaces whose distributional gradient has no summability properties and differs from the so-called transposition method of Lions and Magenes [15, Chapt. II, §6], Rŏitberg and Sěftel' [25], Aziz and Kellog [1] which exploit the form of the dual of a Sobolev space of functions and which accordingly are suitable in a Sobolev space setting.

The paper is organized as follows. Section 2 is a section of preliminaries and notation. In Section 3, we show that one can canonically embed the space $C^{-1,\alpha}(\overline{\Omega})$ into the dual of $C^{1,\alpha}(\overline{\Omega})$. In Section 4 we first summarize the properties of the distributional harmonic volume potential and then we prove the continuity statement for the distributional volume potential with densities of class $C^{-1,\alpha}(\overline{\Omega})$ of Proposition 4.28 that complements previously known results (cf. [10, Thm. 3.6], Dalla Riva, Musolino and the author [5, Thm. 7.19]).

In Section 5, we introduce a distributional form of normal derivative for Hölder continuous solutions of the Poisson equation (cf. Definition 5.11). Here we note that we cannot exploit the first Green Identity in order to introduce a distributional normal derivative on the boundary as done by Lions and Magenes [15], Nečas [22, Chapt. 5], Nedelec and Planchard [23, p. 109], Costabel [4], McLean [18, Chapt. 4], Mikhailov [19], Mitrea, Mitrea and Mitrea [21, §4.2]. Indeed, we need to take normal derivatives of functions for which we have no information on the integrability of the gradient. It is interesting to note that whereas in a Sobolev space setting one needs to require that Δu is at least locally integrable function (cf. Costabel [4, Lem. 3.2]), in the Hölder space setting above, Δu is required to be in the space of distributions $C^{-1,\alpha}(\overline{\Omega})$.

In Section 6, we solve the Neumann problem (1.2). In the appendix at the end of the paper, we have collected some classical results on the classical harmonic volume potential in Hölder and Schauder spaces.

2 Preliminaries and notation

Unless otherwise specified, we assume throughout the paper that

$$n \in \mathbb{N} \setminus \{0,1\}\,,$$

where $\mathbb{N}$ denotes the set of natural numbers including 0. $|A|$ denotes the operator norm of a matrix A with real (or complex) entries, A^t denotes the transpose matrix of A. Let Ω be an open subset of $\mathbb{R}^n$. $C^1(\Omega)$ denotes the set of continuously differentiable functions from Ω to $\mathbb{R}$. Let $s \in \mathbb{N} \setminus \{0\}$, $f \in \left(C^1(\Omega)\right)^s$. Then Df denotes the Jacobian matrix of f.

For the (classical) definition of open set of class C^m or of class $C^{m,\alpha}$ and of the Hölder and Schauder spaces $C^{m,\alpha}(\overline{\Omega})$ on the closure $\overline{\Omega}$ of an open set Ω and of the Hölder and Schauder spaces $C^{m,\alpha}(\partial\Omega)$ on the boundary $\partial\Omega$ of an open set Ω for some $m \in \mathbb{N}$, $\alpha \in]0,1]$, we refer for example to Dalla Riva, the author and Musolino [5, §2.3, §2.6, §2.7, §2.11, §2.13, §2.20]. If $m \in \mathbb{N}$, $C^m_b(\overline{\Omega})$ denotes the space of m-times continuously differentiable functions from Ω to $\mathbb{R}$ such that all the partial derivatives up to order m have a bounded continuous extension to $\overline{\Omega}$ and we set

$$\|f\|_{C^m_b(\overline{\Omega})} \equiv \sum_{|\eta|\leq m} \sup_{x\in\overline{\Omega}} |D^\eta f(x)| \qquad \forall f \in C^m_b(\overline{\Omega})\,.$$

If $\alpha \in]0,1]$, then $C^{m,\alpha}_b(\overline{\Omega})$ denotes the space of functions of $C^m_b(\overline{\Omega})$ such that the partial derivatives of order m are α-Hölder continuous in Ω. Then we equip $C^{m,\alpha}_b(\overline{\Omega})$ with the norm

$$\|f\|_{C^{m,\alpha}_b(\overline{\Omega})} \equiv \|f\|_{C^m_b(\overline{\Omega})} + \sum_{|\eta|=m} |D^\eta f|_\alpha \qquad \forall f \in C^{m,\alpha}_b(\overline{\Omega})\,,$$

where $|D^\eta f|_\alpha$ denotes the α-Hölder constant of the partial derivative $D^\eta f$ of order η of f in Ω. If Ω is bounded, we obviously have $C^m_b(\overline{\Omega}) = C^m(\overline{\Omega})$ and $C^{m,\alpha}_b(\overline{\Omega}) = C^{m,\alpha}(\overline{\Omega})$. Then $C^{m,\alpha}_{\mathrm{loc}}(\overline{\Omega})$ denotes the space of those functions $f \in C^m(\overline{\Omega})$ such that $f_{|\overline{\Omega\cap\mathbb{B}_n(0,\rho)}}$ belongs to $C^{m,\alpha}(\overline{\Omega\cap\mathbb{B}_n(0,\rho)})$ for all $\rho \in]0,+\infty[$. The space of real valued functions of class C^∞ with compact support in an open set Ω of $\mathbb{R}^n$ is denoted $\mathcal{D}(\Omega)$. Then its dual $\mathcal{D}'(\Omega)$ is known to be the space of distributions in Ω. The support of a function is denoted by the abbreviation 'supp'.

If Ω is a bounded open subset of class C^1 of $\mathbb{R}^n$, then Ω is known to have a finite number $\varkappa^+$ connected components and the exterior

$$\Omega^- \equiv \mathbb{R}^n \setminus \overline{\Omega}$$

of Ω is known to have a finite number $\varkappa^- + 1$ connected components. Then, the (bounded) connected components of Ω are denoted by $\Omega_1, \ldots, \Omega_{\varkappa^+}$, the unbounded connected component of Ω^- is denoted by $(\Omega^-)_0$, and the bounded connected components of Ω^- are denoted by $(\Omega^-)_1, \ldots, (\Omega^-)_{\varkappa^-}$ (cf. *e.g.*, [5, Lem. 2.38]). We denote by ν_Ω or simply by ν the outward unit normal of Ω on $\partial\Omega$. Then $\nu_{\Omega^-} = -\nu_\Omega$ is the outward unit normal of Ω^- on $\partial\Omega = \partial\Omega^-$.

Now let $\alpha \in]0,1]$, $m \in \mathbb{N}$. If Ω is a bounded open subset of $\mathbb{R}^n$ of class $C^{\max\{m,1\},\alpha}$, then we find convenient to consider the dual $(C^{m,\alpha}(\partial\Omega))'$ of $C^{m,\alpha}(\partial\Omega)$ with its usual (normable) topology and the corresponding duality pairing $< \cdot, \cdot >$ and we say that the elements of $(C^{m,\alpha}(\partial\Omega))'$ are distributions in $\partial\Omega$. Since $C^{m,\alpha}(\partial\Omega)$ is easily seen to be dense in $C^m(\partial\Omega)$, the transpose mapping of the canonical injection of $C^{m,\alpha}(\partial\Omega)$ into $C^m(\partial\Omega)$ is a continuous injective operator from $(C^m(\partial\Omega))'$ into $(C^{m,\alpha}(\partial\Omega))'$.

Also, if X is a vector subspace of the space $L^1(\partial\Omega)$ of Lebesgue integrable functions on $\partial\Omega$, we find convenient to set

$$X_0 \equiv \left\{ f \in X : \int_{\partial\Omega} f \, d\sigma = 0 \right\} . \tag{2.1}$$

Similarly, if X is a vector subspace of $(C^{m,\alpha}(\partial\Omega))'$, we find convenient to set

$$X_0 \equiv \left\{ f \in (C^{m,\alpha}(\partial\Omega))' : < f, 1 >= 0 \right\} . \tag{2.2}$$

Morever, we retain the standard notation for the Lebesgue spaces L^p for $p \in [1,+\infty]$ (cf. *e.g.*, Folland [7, Chapt. 6], [5, §2.1]) and m_n denotes the n dimensional Lebesgue measure.

If Ω is a bounded open subset of $\mathbb{R}^n$, then we find convenient to consider the dual $(C^{m,\alpha}(\overline{\Omega}))'$ of $C^{m,\alpha}(\overline{\Omega})$ with its usual (normable) topology and the corresponding duality pairing $< \cdot, \cdot >$ and we say that the elements of $(C^{m,\alpha}(\overline{\Omega}))'$ are distributions in $\overline{\Omega}$. Since $C^{m,\alpha}(\overline{\Omega})$ is easily seen to be dense in $C^m(\overline{\Omega})$, the transpose mapping of the canonical injection of $C^{m,\alpha}(\overline{\Omega})$ into $C^m(\overline{\Omega})$ is a continuous injective operator from $(C^m(\overline{\Omega}))'$ into $(C^{m,\alpha}(\overline{\Omega}))'$. Let $r_{|\Omega}$ be the restriction map from $\mathcal{D}(\mathbb{R}^n)$ to $C^{m,\alpha}(\overline{\Omega})$. Then we can associate to each $\mu \in (C^{m,\alpha}(\overline{\Omega}))'$ the distribution $r^t_{|\Omega}\mu \in \mathcal{D}'(\mathbb{R}^n)$, where $r^t_{|\Omega}$ denotes the transpose map of $r^t_{|\Omega}$. The following Lemma is well known and is an immediate consequence of the Hölder inequality.

Lemma 2.3 *Let $m \in \mathbb{N}$, $\alpha \in]0,1[$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the canonical inclusion $\mathcal{J}$ from the Lebesgue space $L^1(\Omega)$ of integrable functions in Ω to $(C^{m,\alpha}(\overline{\Omega}))'$ that takes f to the functional $\mathcal{J}[f]$ defined by*

$$< \mathcal{J}[f], v >\equiv \int_\Omega f v \, d\sigma \qquad \forall v \in C^{m,\alpha}(\overline{\Omega}) \,, \tag{2.4}$$

is linear continuous and injective.

As customary, we say that $\mathcal{J}[f]$ is the 'distribution that is canonically associated to f' and we omit the indication of the inclusion map $\mathcal{J}$ when no ambiguity can arise. By Lemma 2.3, the space $C^{0,\alpha}(\overline{\Omega})$ is continuously embedded into $(C^{m,\alpha}(\overline{\Omega}))'$.

We now summarize the definition and some elementary properties of the Schauder space $C^{-1,\alpha}(\overline{\Omega})$ by following the presentation of Dalla Riva, the author and Musolino [5, §2.22].

Definition 2.5 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $\alpha \in]0,1]$. Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote by $C^{-1,\alpha}(\overline{\Omega})$ the subspace*

$$\left\{ f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j \, : \, f_j \in C^{0,\alpha}(\overline{\Omega}) \ \forall j \in \{0,\dots,n\} \right\} \,,$$

of the space of distributions $\mathcal{D}'(\Omega)$ in Ω.

According to the above definition, the space $C^{-1,\alpha}(\overline{\Omega})$ is the image of the linear and continuous map

$$\Xi \, : \, (C^{0,\alpha}(\overline{\Omega}))^{n+1} \to \mathcal{D}'(\Omega)$$

that takes an $(n+1)$-tuple $(f_0,\dots,f_n)$ to $f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j$. Let π denote the canonical projection

$$\pi \, : \, (C^{0,\alpha}(\overline{\Omega}))^{n+1} \to (C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi \tag{2.6}$$

of $(C^{0,\alpha}(\overline{\Omega}))^{n+1}$ onto the quotient space $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$. Let $\tilde{\Xi}$ be the unique linear injective map from $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$ onto the image $C^{-1,\alpha}(\overline{\Omega})$ of Ξ such that

$$\Xi = \tilde{\Xi} \circ \pi \,. \tag{2.7}$$

Then, $\tilde{\Xi}$ is a linear bijection from $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$ onto $C^{-1,\alpha}(\overline{\Omega})$.

Since $(C^{0,\alpha}(\overline{\Omega}))^{n+1}$ is a Banach space and $\mathrm{Ker}\,\Xi$ is a closed subspace of the Banach space $(C^{0,\alpha}(\overline{\Omega}))^{n+1}$, we know that $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$ is a Banach space (cf. *e.g.*, [5, Thm. 2.1]). We endow $C^{-1,\alpha}(\overline{\Omega})$ with the norm induced by $\tilde{\Xi}$, i.e., we set

$$\|f\|_{C^{-1,\alpha}(\overline{\Omega})} \equiv \inf\Bigg\{ \sum_{j=0}^{n} \|f_j\|_{C^{0,\alpha}(\overline{\Omega})} : \qquad f = f_0 + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f_j \,, \ f_j \in C^{0,\alpha}(\overline{\Omega}) \ \forall j \in \{0, \dots, n\} \Bigg\}. \tag{2.8}$$

By definition of the norm $\|\cdot\|_{C^{-1,\alpha}(\overline{\Omega})}$, the linear bijection $\tilde{\Xi}$ is an isometry of the space $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$ onto the space $(C^{-1,\alpha}(\overline{\Omega}), \|\cdot\|_{C^{-1,\alpha}(\overline{\Omega})})$. Since the quotient $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$ is a Banach space, it follows that $(C^{-1,\alpha}(\overline{\Omega}), \|\cdot\|_{C^{-1,\alpha}(\overline{\Omega})})$ is also a Banach space.

Since Ξ is continuous from $(C^{0,\alpha}(\overline{\Omega}))^{n+1}$ to $\mathcal{D}'(\Omega)$, a fundamental property of the quotient topology implies that the map $\tilde{\Xi}$ is continuous from the quotient space $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$ to $\mathcal{D}'(\Omega)$ (cf. *e.g.*, [5, Prop. A.5]).

Hence, $(C^{-1,\alpha}(\overline{\Omega}), \|\cdot\|_{C^{-1,\alpha}(\overline{\Omega})})$ is continuously embedded into $\mathcal{D}'(\Omega)$. Also, the definition of the norm $\|\cdot\|_{C^{-1,\alpha}(\overline{\Omega})}$ implies that $C^{0,\alpha}(\overline{\Omega})$ is continuously embedded into $C^{-1,\alpha}(\overline{\Omega})$ and that the partial derivation $\frac{\partial}{\partial x_j}$ is continuous from $C^{0,\alpha}(\overline{\Omega})$ to $C^{-1,\alpha}(\overline{\Omega})$ for all $j \in \{1, \dots, n\}$. Generically,the elements of $C^{-1,\alpha}(\overline{\Omega})$ are not integrable functions, but distributions in Ω. We also point out the validity of the following elementary but useful lemma.

Lemma 2.9 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $\alpha \in]0,1]$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let X be a normed space. Let L be a linear map from $C^{-1,\alpha}(\Omega)$ to X. Then L is continuous if and only if the map*

$$L \circ \Xi$$

is continuous on $C^{0,\alpha}(\overline{\Omega})^{N+1}$.

Proof. If L is continuous, then so is the composite map $L \circ \Xi$. Conversely, if $L \circ \Xi$ is continuous, we note that

$$L \circ \Xi = L \circ \tilde{\Xi} \circ \pi$$

Then a fundamental property of the quotient topology implies that the map $L \circ \tilde{\Xi}$ is continuous on the quotient $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$. Since $\tilde{\Xi}$ is an isometry from $(C^{0,\alpha}(\overline{\Omega}))^{n+1}/\mathrm{Ker}\,\Xi$ onto $C^{-1,\alpha}(\partial\Omega)$, its inverse map is continuous and accordingly

$$L = L \circ \tilde{\Xi} \circ \left(\tilde{\Xi}\right)^{(-1)}$$

is continuous. □

We now define a linear functional $\mathcal{I}_\Omega$ on $C^{-1,\alpha}(\overline{\Omega})$ which extends the integration in Ω to all elements of $C^{-1,\alpha}(\overline{\Omega})$ as in [5, Prop. 2.89].

Proposition 2.10 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $\alpha \in]0,1]$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then there exists one and only one linear and continuous operator $\mathcal{I}_\Omega$ from the space $C^{-1,\alpha}(\overline{\Omega})$ to $\mathbb{R}$ such that*

$$\mathcal{I}_\Omega[f] = \int_\Omega f_0 \, dx + \int_{\partial\Omega} \sum_{j=1}^n (\nu_\Omega)_j f_j \, d\sigma \tag{2.11}$$

for all $f = f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j \in C^{-1,\alpha}(\overline{\Omega})$. Moreover,

$$\mathcal{I}_\Omega[f] = \int_\Omega f \, dx \qquad \forall f \in C^{0,\alpha}(\overline{\Omega}) \, .$$

3 An embedding theorem of $C^{-1,\alpha}(\overline{\Omega})$ into the dual of $C^{1,\alpha}(\overline{\Omega})$

We plan to show that we can extend all distributions of $C^{-1,\alpha}(\overline{\Omega})$, which are elements of the dual of $\mathcal{D}(\Omega)$, to elements of the dual of $C^{1,\alpha}(\overline{\Omega})$. We first observe that in the specific case in which

$$f = f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j$$

with $f_0 \in C^{0,\alpha}(\overline{\Omega})$ and $f_j \in C^{1,\alpha}(\overline{\Omega})$ for all $j \in \{1, \dots, n\}$, we have $f \in C^{0,\alpha}(\overline{\Omega})$ and the Divergence Theorem implies that

$$\begin{aligned} \int_\Omega f v \, dx &= \int_\Omega f_0 v + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j v \, dx \\ &= \int_\Omega f_0 v \, dx + \int_{\partial\Omega} \sum_{j=1}^n (\nu_\Omega)_j f_j v \, d\sigma - \sum_{j=1}^n \int_\Omega f_j \frac{\partial v}{\partial x_j} \, dx \end{aligned}$$

for all $v \in C^{1,\alpha}(\overline{\Omega})$. Hence, it makes sense to define a 'canonical' extension of some $f \in C^{-1,\alpha}(\overline{\Omega})$, that is a linear functional on $\mathcal{D}(\Omega)$, to the whole of $C^{1,\alpha}(\overline{\Omega})$ by taking the right hand side of the above equality also in the case in which $f_j \in C^{0,\alpha}(\overline{\Omega})$ for all $j \in \{1, \dots, n\}$. We do so by means of the following statement.

Proposition 3.1 *Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $\alpha \in]0,1[$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. Then the following statements hold.*

(i) If $(f_0, \dots, f_n) \in C^{0,\alpha}(\overline{\Omega})^{n+1}$ and

$$f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j = 0$$

in the sense of distributions in Ω*, then*

$$\int_\Omega f_0 v\,dx + \int_{\partial\Omega} \sum_{j=1}^n (\nu_\Omega)_j f_j v\,d\sigma - \sum_{j=1}^n \int_\Omega f_j \frac{\partial v}{\partial x_j}\,dx = 0 \qquad \forall v \in C^{1,\alpha}(\overline{\Omega})\,.$$

(ii) *There exists one and only one linear and continuous extension operator* $E^\sharp$ *from* $C^{-1,\alpha}(\overline{\Omega})$ *to* $\left(C^{1,\alpha}(\overline{\Omega})\right)'$ *such that*

$$\begin{aligned} &< E^\sharp[f], v > \\ &\qquad = \int_\Omega f_0 v\,dx + \int_{\partial\Omega} \sum_{j=1}^n (\nu_\Omega)_j f_j v\,d\sigma - \sum_{j=1}^n \int_\Omega f_j \frac{\partial v}{\partial x_j}\,dx \quad \forall v \in C^{1,\alpha}(\overline{\Omega}) \end{aligned} \tag{3.2}$$

for all $f = f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j \in C^{-1,\alpha}(\overline{\Omega})$*. Moreover,*

$$E^\sharp[f]_{|\Omega} = f\,,\ \ \text{i.e.,}\ \ < E^\sharp[f], v > = < f, v > \qquad \forall v \in \mathcal{D}(\Omega) \tag{3.3}$$

for all $f \in C^{-1,\alpha}(\overline{\Omega})$ *and*

$$< E^\sharp[f], v > = < f, v > \qquad \forall v \in C^{1,\alpha}(\overline{\Omega}) \tag{3.4}$$

for all $f \in C^{0,\alpha}(\overline{\Omega})$.

Proof. (i) Since all components of the vector valued function $(f_1, \ldots, f_n)$ and its distributional divergence $-f_0$ belong to $C^{0,\alpha}(\overline{\Omega})$, which is continuously embedded into the Lebesgue space $L^2(\Omega)$ of square integrable functions in Ω, there exists a sequence $\{(f_{l1}, \ldots, f_{ln})\}_{l\in\mathbb{N}}$ in $\left(C^\infty(\overline{\Omega})\right)^n$ such that

$$\begin{aligned} &\lim_{l\to\infty} (f_{l1}, \ldots, f_{ln}) = (f_1, \ldots, f_n) \qquad \text{in } \left(L^2(\Omega)\right)^n\,, \\ &\lim_{l\to\infty} \sum_{j=1}^n \frac{\partial}{\partial x_j} f_{lj} = \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j \qquad \text{in } L^2(\Omega)\,, \\ &\lim_{l\to\infty} \sum_{j=1}^n (\nu_\Omega)_j f_{lj} = \sum_{j=1}^n (\nu_\Omega)_j f_j \qquad \text{in } \left(H^{1/2}(\partial\Omega)\right)' \end{aligned}$$

where $\left(H^{1/2}(\partial\Omega)\right)'$ is the dual of the space $H^{1/2}(\partial\Omega)$ of traces on the boundary of the Sobolev space $H^1(\Omega)$ of functions in $L^2(\Omega)$ which have first order distributional derivatives in $L^2(\Omega)$ (cf., e.g., Tartar [26, p. 101]). Then we have

$$\begin{aligned} &\int_\Omega f_0 v\,dx + \int_{\partial\Omega} \sum_{j=1}^n (\nu_\Omega)_j f_j v\,d\sigma - \sum_{j=1}^n \int_\Omega f_j \frac{\partial v}{\partial x_j}\,dx \\ &\qquad = \lim_{l\to\infty} \Bigg\{ \int_\Omega f_{l0} v\,dx + \int_{\partial\Omega} \sum_{j=1}^n (\nu_\Omega)_j f_{lj} v\,d\sigma - \sum_{j=1}^n \int_\Omega f_{lj} \frac{\partial v}{\partial x_j}\,dx \end{aligned}$$

$$
\begin{aligned}
&= \lim_{l\to\infty}\left\{\int_\Omega f_{l0} v\,dx + \int_\Omega \sum_{j=1}^n \frac{\partial}{\partial x_j}(f_{lj} v)\,dx - \sum_{j=1}^n \int_\Omega f_{lj}\frac{\partial v}{\partial x_j}\,dx\right\} \\
&= \lim_{l\to\infty}\left\{\int_\Omega f_{l0} v\,dx + \int_\Omega \sum_{j=1}^n \frac{\partial f_{lj}}{\partial x_j} v\,dx\right\} \\
&= \int_\Omega f_0 v\,dx + \int_\Omega \sum_{j=1}^n \frac{\partial f_j}{\partial x_j} v\,dx \\
&= \int_\Omega f_0 v\,dx + \int_\Omega -f_0 v\,dx = 0 \qquad \forall v \in C^{1,\alpha}(\overline{\Omega})\,.
\end{aligned}
$$

(ii) Let L be the operator from $C^{0,\alpha}(\overline{\Omega})^{n+1}$ to $\left(C^{1,\alpha}(\overline{\Omega})\right)'$ that takes $(f_0,\dots,f_n)$ to the functional that is defined by the right-hand side of (3.2). By (i), we have $\mathrm{Ker}\,\Xi \subseteq \mathrm{Ker}\,L$. Since the operator Ξ from $C^{0,\alpha}(\overline{\Omega})^{n+1}$ to $C^{-1,\alpha}(\overline{\Omega})$ is surjective, the Homomorphism Theorem for linear maps between vector spaces implies the existence of a unique linear map $E^\sharp$ from $C^{-1,\alpha}(\overline{\Omega})$ to $\left(C^{1,\alpha}(\overline{\Omega})\right)'$ such that $L = E^\sharp \circ \Xi$, i.e., such that (3.2) holds true (cf. *e.g.*, [5, Thm. A.1]). Then we note that if $f = f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j$, we have

$$
\begin{aligned}
|<E^\sharp[f], v>| &\le m_n(\Omega)\|f_0\|_{C^{0,\alpha}(\overline{\Omega})}\|v\|_{C^{0,\alpha}(\overline{\Omega})} \\
&\quad + m_{n-1}(\partial\Omega)\sum_{j=1}^n \|f_j\|_{C^{0,\alpha}(\overline{\Omega})}\|v\|_{C^{0,\alpha}(\overline{\Omega})} + m_n(\Omega)\sum_{j=1}^n \|f_j\|_{C^{0,\alpha}(\overline{\Omega})}\|v\|_{C^{1,\alpha}(\overline{\Omega})}\,,
\end{aligned}
$$

for all $v \in C^{1,\alpha}(\overline{\Omega})$. Then Lemma 2.9 implies the continuity of $E^\sharp$. Equality (3.3) is an immediate consequence of (3.2) and equality (3.4) follows by taking $f_0 = f$, $f_1 = \cdots = f_n = 0$ in (3.2) (see also Lemma 2.3). □

4 The distributional harmonic volume potential

Since we are going to exploit the layer potential theoretic method, we introduce the fundamental solution S_n of the Laplace operator. Namely, we set

$$
S_n(\xi) \equiv \begin{cases} \frac{1}{s_n}\ln|\xi| & \forall \xi \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n = 2\,, \\ \frac{1}{(2-n)s_n}|\xi|^{2-n} & \forall \xi \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n > 2\,, \end{cases}
$$

where s_n denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0,1)$. If $n \ge 2$, then there exists $\varsigma \in]0,+\infty[$ such that

$$
\sup_{\xi\in\mathbb{R}^n\setminus\{0\}} |\xi|^{|\eta|+(n-2)}|D^\eta S_n(\xi)| \le \varsigma^{|\eta|}|\eta|! \qquad \forall \eta \in \mathbb{N}^n\setminus\{0\}\,, \tag{4.1}
$$

(cf. reference [14, Lem. A.6] of the author and Musolino). Let $\alpha \in]0,1]$ and $m \in \mathbb{N}$. If Ω is a bounded open subset of $\mathbb{R}^n$, then we can consider the restriction map $r_{|\overline{\Omega}}$ from $\mathcal{D}(\mathbb{R}^n)$ to $C^{m,\alpha}(\overline{\Omega})$. Then the transpose map $r^t_{|\overline{\Omega}}$ is linear and continuous from $(C^{m,\alpha}(\overline{\Omega}))'$ to $\mathcal{D}'(\mathbb{R}^n)$. Moreover, if $\mu \in (C^{m,\alpha}(\overline{\Omega}))'$, then $r^t_{|\overline{\Omega}}\mu$ has compact support. Hence, it makes sense to consider the convolution of $r^t_{|\overline{\Omega}}\mu$ with the fundamental solution of the Laplace operator. Thus we are now ready to introduce the following known definition.

Definition 4.2 *Let $\alpha \in]0,1]$, $m \in \mathbb{N}$. Let Ω be a bounded open subset of $\mathbb{R}^n$. If $\mu \in (C^{m,\alpha}(\overline{\Omega}))'$, then the (distributional) volume potential relative to S_n and μ is the distribution*

$$\mathcal{P}_\Omega[\mu] = (r^t_{|\overline{\Omega}}\mu) * S_n \in \mathcal{D}'(\mathbb{R}^n)\,.$$

By the definition of convolution, we have

$$\begin{aligned}< (r^t_{|\overline{\Omega}}\mu) * S_n, \varphi > &=< r^t_{|\overline{\Omega}}\mu(y), < S_n(\eta), \varphi(y+\eta) >> \\ &=< r^t_{|\overline{\Omega}}\mu(y), \int_{\mathbb{R}^n} S_n(\eta)\varphi(y+\eta)\,d\eta >=< r^t_{|\overline{\Omega}}\mu(y), \int_{\mathbb{R}^n} S_n(x-y)\varphi(x)\,dx >\end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$. In general, $(r^t_{|\overline{\Omega}}\mu) * S_n$ is not a function, *i.e.* $(r^t_{|\overline{\Omega}}\mu) * S_n$ is not a distribution that is associated to a locally integrable function in $\mathbb{R}^n$. However, this is the case if for example μ is associated to a function of $L^\infty(\Omega)$, *i.e.*, $\mu = \mathcal{J}[f]$ with $f \in L^\infty(\Omega)$ (see Lemma 2.3 with any choice of $m \in \mathbb{N}$, $\alpha \in]0,1]$). Indeed,

$$\begin{aligned}< (r^t_{|\overline{\Omega}}\mu) * S_n, \varphi > &=< (r^t_{|\overline{\Omega}}\mathcal{J}[f]) * S_n, \varphi > \\ &=< r^t_{|\overline{\Omega}}\mathcal{J}[f](y), \int_\Omega S_n(x-y)\varphi(x)\,dx > \\ &=< \mathcal{J}[f](y), r_{|\overline{\Omega}} \int_{\mathbb{R}^n} S_n(x-y)\varphi(x)\,dx > \\ &= \int_\Omega f(y) \int_{\mathbb{R}^n} S_n(x-y)\varphi(x)\,dx\,dy = \int_{\mathbb{R}^n}\int_\Omega S_n(x-y)f(y)\,dy\varphi(x)\,dx \\ &=< \int_\Omega S_n(x-y)f(y)\,dy, \varphi(x) >\end{aligned}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and thus the (distributional) volume potential relative to S_n and μ is associated to the function

$$\int_\Omega S_n(x-y)f(y)\,dy \qquad \text{a.a. } x \in \mathbb{R}^n\,, \tag{4.3}$$

that is locally integrable in $\mathbb{R}^n$ (cf. *e.g.*, Theorem A.1 of the Appendix) and that with some abuse of notation we still denote by the symbol $\mathcal{P}_\Omega[\mathcal{J}[f]]$ or even more simply by the symbol $\mathcal{P}_\Omega[f]$. We also note that under the assumptions of Definition 4.2, classical properties of the convolution of distributions imply that

$$\Delta\left((r^t_{|\overline{\Omega}}\mu) * S_n\right) = (r^t_{|\overline{\Omega}}\mu) * (\Delta S_n) = (r^t_{|\overline{\Omega}}\mu) * \delta_0 = (r^t_{|\overline{\Omega}}\mu) \quad \text{in } \mathcal{D}'(\mathbb{R}^n)\,, \tag{4.4}$$

where δ_0 is the Dirac measure with mass at 0. We now present a classical formula for the function that represents the restriction of the distributional volume potential $(r^t_{|\overline{\Omega}}\mu) * S_n$ to $\mathbb{R}^n \setminus \operatorname{supp}\mu$ (and thus to $\mathbb{R}^n \setminus \overline{\Omega}$) by means of the following statement. For the convenience of the reader, we include a proof.

Proposition 4.5 *Let $\tau \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution with compact support $\operatorname{supp}\tau$. Then the real valued function θ from $\mathbb{R}^n \setminus \operatorname{supp}\tau$ that is defined by*

$$\theta(x) \equiv < \tau(y), S_n(x-y) > \qquad \forall x \in \mathbb{R}^n \setminus \operatorname{supp}\tau \tag{4.6}$$

is of class C^∞ and the restriction of $\tau * S_n$ to $\mathbb{R}^n \setminus \operatorname{supp}\tau$ is associated to the function θ. Namely,

$$< \tau * S_n, \varphi > = \int_{\mathbb{R}^n \setminus \overline{\Omega}} < \tau(y), S_n(x-y) > \varphi(x)\, dx \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^n \setminus \operatorname{supp}\tau)\,. \tag{4.7}$$

[Here we note that the symbol $< \tau(y), S_n(x-y) >$ in (4.6) means

$$< \tau(y), \omega(y) S_n(x-y) >\,,$$

where $\omega \in \mathcal{D}(\mathbb{R}^n \setminus \{x\})$ and ω equals 1 in an open neighborhood of $\operatorname{supp}\tau$.] Moreover, θ is harmonic.

Proof. Since τ is a distribution in $\mathbb{R}^n$ with compact support and $S_n(x-\cdot)$ is of class C^∞ in $\mathbb{R}^n \setminus \{x\}$ for all $x \in \mathbb{R}^n \setminus \operatorname{supp}\tau$, the differentiablity theorem for distributions with compact support in $\mathbb{R}^n$ applied to test functions depending on a parameter implies that the function θ is of class C^∞ in $\mathbb{R}^n \setminus \operatorname{supp}\tau$ (cf. *e.g.*, Treves [27, Thm. 27.2]). We now fix $\varphi \in \mathcal{D}(\mathbb{R}^n \setminus \operatorname{supp}\tau)$ and we prove equality (4.7).

Let $\Omega^\sharp$ be an open neighborhood of $\operatorname{supp}\tau$ such that $\overline{\Omega^\sharp} \cap \operatorname{supp}\varphi = \emptyset$. By the known sequential density of $\mathcal{D}(\Omega^\sharp)$ in the space of compactly supported distributions in $\Omega^\sharp$, there exists a sequence $\{\tau_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}(\Omega^\sharp)$ such that

$$\lim_{j\to\infty} \tau_j = \tau \qquad \text{in } (C^\infty(\Omega^\sharp))'_b\,, \tag{4.8}$$

and accordingly in $(C^\infty(\mathbb{R}^n))'_b$, where $(C^\infty(\Omega^\sharp))'_b$ and $(C^\infty(\mathbb{R}^n))'_b$ denote the dual of $C^\infty(\Omega^\sharp)$ with the topology of uniform convergence on the bounded subsets of $C^\infty(\Omega^\sharp)$ and the dual of $C^\infty(\mathbb{R}^n)$ with the topology of uniform convergence on the bounded subsets of $C^\infty(\mathbb{R}^n)$, respectively (cf. *e.g.*, Treves [27, Thm. 28.2]).

Then the above mentioned differentiablity theorem for distributions with compact support in $\mathbb{R}^n$ applied to test functions depending on a parameter implies that the function $< \tau_j(y), S_n(\cdot - y) >$ is of class C^∞ in $\mathbb{R}^n \setminus \operatorname{supp}\tau$ for each $j \in \mathbb{N}$. By the definition of convolution and the convergence of (4.8) in $(C^\infty(\mathbb{R}^n))'_b$ we have

$$< \tau * S_n, \varphi > = < \tau(y), < S_n(\eta), \varphi(y+\eta) >> \tag{4.9}$$

$$
\begin{aligned}
&= \lim_{j\to\infty} < \tau_j(y), < S_n(\eta), \varphi(y+\eta) >> \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^n} \tau_j(y) \int_{\mathbb{R}^n} S_n(\eta)\varphi(y+\eta)\, d\eta\, dy \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^n} \tau_j(y) \int_{\mathbb{R}^n} S_n(x-y)\varphi(x)\, dx\, dy \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tau_j(y) S_n(x-y)\, dy \varphi(x)\, dx \\
&= \lim_{j\to\infty} \int_{\mathbb{R}^n} < \tau_j(y), S_n(x-y) > \varphi(x)\, dx
\end{aligned}
$$

Next we turn to show that the sequence $\{< \tau_j(y), S_n(x-y) >\}_{j\in\mathbb{N}}$ converges uniformly to $< \tau(y), S_n(x-y) >$ in $x \in \operatorname{supp}\varphi$. Since $\Omega^\sharp$ has a strictly positive distance from $\operatorname{supp}\varphi$, the set

$$
\{S_n(x-\cdot) :\, x \in \operatorname{supp}\varphi\}
$$

is bounded in $C^\infty(\Omega^\sharp)$ and accordingly

$$
\lim_{j\to\infty} < \tau_j(y), S_n(x-y) >=< \tau, S_n(x-y) >
$$

uniformly in $x \in \operatorname{supp}\varphi$ (see Treves [27, Chapt. 10, Ex. I, Chapt. 14] for the definition of topology of $C^\infty(\Omega^\sharp)$ and of bounded subsets of $C^\infty(\Omega^\sharp)$). Hence,

$$
\lim_{j\to\infty} \int_{\mathbb{R}^n} < \tau_j(y), S_n(x-y) > \varphi(x)\, dx = \int_{\mathbb{R}^n} < \tau(y), S_n(x-y) > \varphi(x)\, dx
$$

and equality (4.9) implies that equality (4.7) holds true. Moreover, known properties of the convolution imply that

$$
\Delta\,(\tau * S_n) = \tau * (\Delta S_n) = \tau * \delta_0 = \tau \quad \text{in } \mathcal{D}'(\mathbb{R}^n)\,. \tag{4.10}
$$

Since τ vanishes in $\mathbb{R}^n \setminus \operatorname{supp}\tau$, the Weyl lemma implies that the function that represents the restriction of $\tau * S_n$ to $\mathbb{R}^n \setminus \operatorname{supp}\tau$ is real analytic and harmonic.

□

By applying Proposition 4.5 to $\tau = (r^t_{|\overline{\Omega}}\mu)$, we obtain a formula for the function that represents the restriction of the distributional volume potential $(r^t_{|\overline{\Omega}}\mu) * S_n$ to $\mathbb{R}^n \setminus \overline{\Omega}$. Under the assumptions of Definition 4.2, we set

$$
\begin{aligned}
\mathcal{P}^+_\Omega[\mu] \quad &\equiv \quad \Big((r^t_{|\overline{\Omega}}\mu) * S_n\Big)_{|\Omega} \qquad \text{in } \Omega\,, \\
\mathcal{P}^-_\Omega[\mu](x) \quad &\equiv \quad \Big((r^t_{|\overline{\Omega}}\mu) * S_n\Big)_{|\Omega^-} \qquad \text{in } \Omega^-\,.
\end{aligned}
\tag{4.11}
$$

$\mathcal{P}_\Omega^+[\mu]$ is a distribution in Ω (which may be a function under some extra assumption on μ). Instead, Proposition 4.5 implies that $\mathcal{P}_\Omega^-[\mu]$ is associated to the function

$$< (r^t_{|\overline{\Omega}}\mu)(y), S_n(x-y) > \qquad \forall x \in \Omega^- \,,$$

which is real analytic and harmonic in Ω^-. In accordance with the current literature, we use the same symbol for a function and for the distribution that is associated to the function, when no ambiguity can arise.

Next we introduce the following statement that generalizes the known condition for classical harmonic volume potentials to be harmonic at infinity. For the convenience of the reader, we include a proof.

Proposition 4.12 *Let $\tau \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution with compact support* $\operatorname{supp}\tau$*. Let θ be the function from $\mathbb{R}^n \setminus \operatorname{supp}\tau$ to $\mathbb{R}$ that is defined by (4.6) and that represents the restriction of $\tau * S_n$ to $\mathbb{R}^n \setminus \operatorname{supp}\tau$. Then the following statements hold.*

(i) θ is harmonic in $\mathbb{R}^n \setminus \operatorname{supp}\tau$.

(ii) If $n \geq 3$, then θ is harmonic at infinity. In particular, $\lim_{\xi\to\infty}\theta(\xi)$ equals 0.

(iii) If $n = 2$, then θ is harmonic at infinity if and only if $< \tau, 1 >= 0$. If such a condition holds, then $\lim_{\xi\to\infty}\theta(\xi)$ equals 0.

Proof. (i) holds by Proposition 4.5. Since $\operatorname{supp}\tau$ is compact, there exists a bounded open neighborhood $\Omega^\dagger$ of $\operatorname{supp}\tau$. Then the restriction $\tau_{|\Omega^\dagger}$ of τ to $\Omega^\dagger$ is a distribution in $\Omega^\dagger$ with compact support equal to $\operatorname{supp}\tau$.

Since $\tau_{|\Omega^\dagger}$ has compact support, there exists a unique $\tau_1 \in \left(C^\infty(\Omega^\dagger)\right)'$ such that

$$< \tau_1, \psi >=< \tau, \psi > \qquad \forall \psi \in \mathcal{D}(\Omega^\dagger)$$

(cf. *e.g.*, Treves [27, proof of Thm. 24.2]) and accordingly there exist a compact subset K_1 of $\Omega^\dagger$ that contains $\operatorname{supp}\tau$, $c_{\tau,K_1} \in]0,+\infty[$ and $m \in \mathbb{N}$ such that

$$| < \tau, \psi > | \leq c_{\tau,K_1} \sup_{|\gamma|\leq m} \sup_{K_1} |D^\gamma \psi| \qquad \forall \psi \in C^\infty(\Omega^\dagger)\,. \tag{4.13}$$

If $x \in \mathbb{R}^n \setminus \Omega^\dagger$, then there exist a bounded open neighborhood W_τ of K_1 and a bounded open neighborhood W_x of x such that $W_\tau \cap W_x = \emptyset$. Possibly replacing W_τ with $W_\tau \cap \Omega^\dagger$, we can assume that $W_\tau \subseteq \Omega^\dagger$. Next we take $\omega_{x,\tau} \in C^\infty(\mathbb{R}^n)$ such that $\omega_{x,\tau}$ equals 1 in W_τ and ω equals 0 in W_x. Then $\omega_{x,\tau}(y)S_n(x-y)$ is of class C^∞ in the variable $y \in \mathbb{R}^n$ and

$$\begin{aligned} |\theta(x)| &= | < \tau(y), S_n(x-y) > | = | < \tau(y), \omega_{x,\tau}(y) S_n(x-y) > | \\ &\leq c_{\tau,K_1} \sup_{|\gamma|\leq m} \sup_{y\in K_1} |D^\gamma_y(\omega_{x,\tau}(y)S_n(x-y))| \\ &= c_{\tau,K_1} \sup_{|\gamma|\leq m} \sup_{y\in K_1} |D^\gamma_y(S_n(x-y))| \qquad \forall x \in \mathbb{R}^n \setminus \Omega^\dagger\,. \end{aligned} \tag{4.14}$$

Now let $r_0 \in]0,+\infty[$ be such $\operatorname{supp}\tau \subseteq \mathbb{B}_n(0,r_0)$. By the definition of S_n and by the inequalities (4.1), there exists $\varsigma \in]0,+\infty[$ such that

$$|D^\eta S_n(x-y)| \leq \varsigma^{|\eta|}|\eta|!|x-y|^{-|\eta|-(n-2)} \qquad \forall x \in \mathbb{R}^n \setminus K_1\,,\ y \in K_1\,, \tag{4.15}$$

for all $\eta \in \mathbb{N}^n \setminus \{0\}$. If $n \geq 3$ as in statement (ii), then we also have

$$|S_n(x-y)| \leq \frac{1}{(n-2)s_n}|x-y|^{-(n-2)} \qquad \forall x \in \mathbb{R}^n \setminus K_1\,,\ y \in K_1$$

and accordingly

$$\lim_{x\to\infty} \sup_{|\gamma|\leq m} \sup_{y\in K_1} |D_y^\gamma(S_n(x-y))| = 0\,.$$

Hence, the above inequality (4.14) implies that statement (ii) holds true.

We now consider case $n = 2$ as in (iii). If $\operatorname{supp}\tau = \emptyset$, then the statement is obvious. Let $\operatorname{supp}\tau \neq \emptyset$, $x_0 \in \operatorname{supp}\tau$. Then we have

$$\begin{aligned} \theta(x) =&< \tau(y), S_2(x-y) > \\ &- < \tau, 1 > S_2(x-x_0)+ < \tau, 1 > S_2(x-x_0) \qquad \forall x \in \mathbb{R}^2 \setminus K_1 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} &| < \tau(y), S_2(x-y) > - < \tau, 1 > S_2(x-x_0)| \\ &\quad = | < \tau(y), \omega_{x,\tau}(y)S_2(x-y) > - < \tau(y), S_2(x-x_0) > | \\ &\quad = | < \tau(y), \omega_{x,\tau}(y)S_2(x-y) - S_2(x-x_0) > | \\ &\quad \leq c_{\tau,K_1} \sup_{|\gamma|\leq m} \sup_{y\in K_1} |D_y^\gamma\left(\omega_{x,\tau}(y)S_2(x-y) - S_2(x-x_0)\right)| \\ &\quad = c_{\tau,K_1} \sup_{|\gamma|\leq m} \sup_{y\in K_1} |D_y^\gamma\left(S_2(x-y) - S_2(x-x_0)\right)| \quad \forall x \in \mathbb{R}^2 \setminus K_1\,. \end{aligned} \tag{4.17}$$

Since

$$S_2(x-y) - S_2(x-x_0) = \frac{1}{2\pi}\log\left(1 + \left(\frac{|x-y|}{|x-x_0|} - 1\right)\right)$$

for all $x \in \mathbb{R}^2 \setminus K_1$, $y \in K_1$ and

$$\left|\frac{|x-y|}{|x-x_0|} - 1\right| = \left|\frac{|x-y| - |x-x_0|}{|x-x_0|}\right| \leq \frac{|y-x_0|}{|x-x_0|} \qquad \forall x \in \mathbb{R}^2 \setminus K_1, y \in K_1\,,$$

inequalities (4.15) imply that

$$\lim_{x\to\infty} \sup_{|\gamma|\leq m} \sup_{y\in K_1} |D_y^\gamma\left(S_2(x-y) - S_2(x-x_0)\right)| = 0$$

and accordingly inequality (4.17) implies that the harmonic function

$$< \tau(y), S_2(x-y) > - < \tau, 1 > S_2(x-x_0)$$

of the variable $x \in \mathbb{R}^2 \setminus \operatorname{supp} \tau$ is harmonic at infinity. Hence, equality (4.16) implies that the function θ is harmonic at infinity if and only if the harmonic function $< \tau, 1 > S_2(x - x_0)$ is harmonic at infinity in the variable $x \in \mathbb{R}^2 \setminus \operatorname{supp} \tau$. Since $< \tau, 1 > S_2(x - x_0)$ is harmonic at infinity in the variable $x \in \mathbb{R}^2 \setminus \operatorname{supp} \tau$ if and only if $< \tau, 1 >= 0$, the proof of (iii) is complete. □

Then we can apply Proposition 4.12 to $\tau = (r^t_{|\overline{\Omega}} \mu)$ and obtain information on $\mathcal{P}^-_\Omega[\mu]$ as in (4.11). Then we introduce the following definition.

Definition 4.18 *Let $\alpha \in]0, 1]$, $m \in \mathbb{N}$. Let Ω be a bounded open subset of $\mathbb{R}^n$. If $\mu \in (C^{m,\alpha}(\overline{\Omega}))'$ and if $\mathcal{P}^+_\Omega[\mu]$ is represented by a continuous function in Ω that admits a continuous extension to $\overline{\Omega}$ (that we denote by the same symbol) and if the harmonic function that represents $\mathcal{P}^-_\Omega[\mu]$ admits a continuous extension to $\overline{\Omega^-}$ (that we denote by the same symbol), and if*

$$\mathcal{P}^+_\Omega[\mu](x) = \mathcal{P}^-_\Omega[\mu](x) \qquad \forall x \in \partial\Omega\,, \tag{4.19}$$

then we set

$$P_\Omega[\mu](x) \equiv \mathcal{P}^+_\Omega[\mu](x) = \mathcal{P}^-_\Omega[\mu](x) \qquad \forall x \in \partial\Omega\,. \tag{4.20}$$

In the specific case $m = 2$, we are interested in distributions μ having α-Hölder continuous volume potentials $\mathcal{P}^\pm_\Omega[\mu]$. Thus we introduce the following definition.

Definition 4.21 *Let $\alpha \in]0, 1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let*

$$\begin{aligned} P^{-2,\alpha}(\overline{\Omega}) = \Big\{ & \mu \in (C^{2,\alpha}(\overline{\Omega}))' : \mathcal{P}^+_\Omega[\mu] \in C^{0,\alpha}(\overline{\Omega}), \qquad (4.22) \\ & \mathcal{P}^-_\Omega[\mu] \in C^{0,\alpha}_{\mathrm{loc}}(\overline{\Omega^-}),\ \mu \textit{ satisfies condition } (4.19) \Big\}. \end{aligned}$$

Next we note that the restriction of an element of $(C^{1,\alpha}(\overline{\Omega}))'$ to $C^{2,\alpha}(\overline{\Omega})$ belongs to $(C^{2,\alpha}(\overline{\Omega}))'$ and we turn to compute the distributional volume potential for the specific form of μ's in $\left(C^{1,\alpha}(\overline{\Omega})\right)'$ that are extensions of elements of $C^{-1,\alpha}(\overline{\Omega})$ in the sense of Proposition 3.1.

Proposition 4.23 *Let $n \in \mathbb{N} \setminus \{0, 1\}$. Let $\alpha \in]0, 1[$. Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^n$. If $f = f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j \in C^{-1,\alpha}(\overline{\Omega})$, then $\mathcal{P}_\Omega[E^\sharp[f]]$ is the distribution that is associated to the function*

$$\begin{aligned} & \int_\Omega S_n(x-y) f_0(y)\, dy \qquad (4.24) \\ & \quad + \sum_{j=1}^n \int_{\partial\Omega} S_n(x-y)(\nu_\Omega)_j(y) f_j(y)\, d\sigma_y + \sum_{j=1}^n \frac{\partial}{\partial x_j} \int_\Omega S_n(x-y) f_j(y)\, dy \end{aligned}$$

for almost all $x \in \mathbb{R}^n$.

Proof. If $v \in \mathcal{D}(\mathbb{R}^n)$, then

$$\frac{\partial}{\partial y_j}\int_{\mathbb{R}^n} S_n(x-y)v(x)dx = \int_{\mathbb{R}^n}\frac{\partial}{\partial y_j}(S_n(x-y))v(x)dx = -\int_{\mathbb{R}^n}\frac{\partial}{\partial x_j}S_n(x-y)v(x)dx$$

for all $x \in \mathbb{R}^n$ (cf. *e.g.*, [5, Prop. 7.6]). Hence, Proposition 3.1 and the Fubini Theorem imply that

$$\begin{aligned}
< \mathcal{P}_\Omega[E^\sharp[f]], v > &=< (r^t_{|\overline{\Omega}}E^\sharp[f]) * S_n, v > \\
&=< E^\sharp[f](y), r_{|\overline{\Omega}} < S_n(\eta), v(y+\eta) > \\
&= \int_\Omega f_0(y)\int_{\mathbb{R}^n} S_n(\eta)v(y+\eta)\, d\eta dy \\
&\quad + \sum_{j=1}^n \int_{\partial\Omega} f_j(y)(\nu_\Omega)_j(y)\int_{\mathbb{R}^n} S_n(\eta)v(y+\eta)\, d\eta d\sigma_y \\
&\quad - \sum_{j=1}^n \int_\Omega f_j(y)\frac{\partial}{\partial y_j}\int_{\mathbb{R}^n} S_n(\eta)v(y+\eta)\, d\eta dy \\
&= \int_\Omega f_0(y)\int_{\mathbb{R}^n} S_n(x-y)v(x)\, dxdy \\
&\quad + \sum_{j=1}^n \int_{\partial\Omega} f_j(y)(\nu_\Omega)_j(y)\int_{\mathbb{R}^n} S_n(x-y)v(x)\, dxd\sigma_y \\
&\quad - \sum_{j=1}^n \int_\Omega f_j(y)\frac{\partial}{\partial y_j}\int_{\mathbb{R}^n} S_n(x-y)v(x)dx\, dy \\
&= \int_{\mathbb{R}^n}\int_\Omega S_n(x-y)f_0(y)\, dy\, v(x)dx \\
&\quad + \sum_{j=1}^n \int_{\mathbb{R}^n}\int_{\partial\Omega} f_j(y)(\nu_\Omega)_j(y)S_n(x-y)d\sigma_y\, v(x)dx \\
&\quad - \sum_{j=1}^n \int_\Omega f_j(y)\int_{\mathbb{R}^n}\frac{\partial}{\partial y_j}\left(S_n(x-y)\right)v(x)dx\, dy \\
&= \int_{\mathbb{R}^n}\int_\Omega S_n(x-y)f_0(y)dy\, v(x)dx \\
&\quad + \sum_{j=1}^n \int_{\mathbb{R}^n}\int_{\partial\Omega} f_j(y)(\nu_\Omega)_j(y)S_n(x-y)d\sigma_y\, v(x)dx \\
&\quad + \sum_{j=1}^n \int_{\mathbb{R}^n}\int_\Omega \frac{\partial}{\partial x_j}S_n(x-y)f_j(y)dy\, v(x)dx
\end{aligned}$$

and accordingly, $\mathcal{P}_\Omega[E^\sharp[f]]$ is the distribution that is associated to the function in (4.24). $\square$

Next we introduce the following (known) definition that we need below.

Definition 4.25 *Let Ω be a bounded open subset of $\mathbb{R}^n$ of class C^1. If $\phi \in C^0(\partial\Omega)$, then we denote by $v_\Omega[\phi]$ the single (or simple) layer potential with moment (or density) ϕ, i.e., the function from $\mathbb{R}^n$ to $\mathbb{R}$ defined by*

$$v_\Omega[\phi](x) \equiv \int_{\partial\Omega} S_n(x-y)\phi(y)\, d\sigma_y \qquad \forall x \in \mathbb{R}^n . \tag{4.26}$$

Under the assumptions of Definition 4.25, it is known that $v_\Omega[\phi]$ is continuous in $\mathbb{R}^n$ and we set

$$v_\Omega^+[\phi] = v_\Omega[\phi]_{|\Omega} , \qquad v_\Omega^-[\phi] = v_\Omega[\phi]_{|\Omega^-} , \tag{4.27}$$

(cf. *e.g.*, [5, Thm. 4.22]). Then we have the following variant of a known result (cf. [10, Thm. 3.6 (ii)], [5, Thm. 7.19]), which shows that if $f \in C^{-1,\alpha}(\overline{\Omega})$, then the extension $E^\sharp[f]$ in the sense of Proposition 3.1 determines an element of $P^{-2,\alpha}(\overline{\Omega})$.

Proposition 4.28 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the following statements hold.*

(i) If $f = f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j \in C^{-1,\alpha}(\overline{\Omega})$, then

$$\mathcal{P}_\Omega^+[E^\sharp[f]] \in C^{1,\alpha}(\overline{\Omega}),\ \mathcal{P}_\Omega^-[E^\sharp[f]] \in C^{1,\alpha}_{\mathrm{loc}}(\overline{\Omega^-}) \tag{4.29}$$

and equality (4.20) holds true. Moreover,

$$\Delta \mathcal{P}_\Omega^+[E^\sharp[f]] = f \qquad \text{in } \mathcal{D}'(\Omega) . \tag{4.30}$$

(ii) The map $\mathcal{P}_\Omega^+[E^\sharp[\cdot]]$ is linear and continuous from $C^{-1,\alpha}(\overline{\Omega})$ to $C^{1,\alpha}(\overline{\Omega})$.

(iii) Let $r \in]0,+\infty[$ be such that $\overline{\Omega} \subseteq \mathbb{B}_n(0,r)$. The map $\mathcal{P}_\Omega^-[E^\sharp[\cdot]]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ is linear and continuous from $C^{-1,\alpha}(\overline{\Omega})$ to $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)} \setminus \Omega)$.

Proof. For a proof of the first membership in (4.29) and of statement (ii), we refer to [5, Thm. 7.19]. Equality (4.30) follows by equalities (3.3), (4.4). Then equality (4.20) follows by formula (4.24), by the continuity in $\mathbb{R}^n$ of the single layer potential with density in $C^{0,\alpha}(\partial\Omega)$ (cf. *e.g.*, [5, Thm. 4.22]) and by the continuous differentiability in $\mathbb{R}^n$ of volume potentials with density in $C^{0,\alpha}(\overline{\Omega})$ (cf. Theorem A.1 of the Appendix).

We now prove (iii) by exploiting Lemma 2.9 and thus by following a variant of the proof of [5, Thm. 7.19].

We first prove that if $(f_0, f_1, \ldots, f_n) \in (C^{0,\alpha}(\overline{\Omega}))^{n+1}$, then the restriction to $\overline{\mathbb{B}_n(0,r)} \setminus \Omega$ of (4.24) defines an element of $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)} \setminus \Omega)$ and that the map B_- from $(C^{0,\alpha}(\overline{\Omega}))^{n+1}$ to $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)}\setminus\Omega)$ that takes $(f_0, f_1, \ldots, f_n)$ to the restriction to $\overline{\mathbb{B}_n(0,r)}\setminus\Omega$ of the term $B_-[f_0, f_1, \ldots, f_n]$ of (4.24) is linear and continuous. Here we note that

$$B_-[f_0, f_1, \ldots, f_n] = \mathcal{P}_\Omega^-[E^\sharp[\Xi[f_0, f_1, \ldots, f_n]]] \quad \forall (f_0, f_1, \ldots, f_n) \in (C^{0,\alpha}(\overline{\Omega}))^{n+1} .$$

For the continuity of the first and third addendum of (4.24) from the space $(C^{0,\alpha}(\overline{\Omega}))^{n+1}$ to $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)}\setminus\Omega)$, we refer to the classical result Theorem (ii) A.5 of the Appendix with $m=0$. Since $v_\Omega[\cdot]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega}$ is known to be continuous from $C^{0,\alpha}(\partial\Omega)$ to $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)}\setminus\Omega)$ (cf. *e.g.*, (4.27), [6, Thm. 7.1 (i)]), the membership of ν_Ω in $\left(C^{0,\alpha}(\partial\Omega)\right)^n$ and the continuity of the pointwise product in $C^{0,\alpha}(\partial\Omega)$ imply that also the second addendum that defines $B_-[f_0, f_1, \ldots, f_n]$ is linear and continuous from $(C^{0,\alpha}(\overline{\Omega}))^{n+1}$ to $C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)}\setminus\Omega)$. In particular, if $f \in C^{-1,\alpha}(\overline{\Omega})$ and $f_0 + \sum_{j=1}^n \frac{\partial}{\partial x_j} f_j$, then

$$\mathcal{P}_\Omega^-[E^\sharp[f]]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega} = \mathcal{P}_\Omega^-[E^\sharp[\Xi[f_0, f_1, \ldots, f_n]]]_{|\overline{\mathbb{B}_n(0,r)}\setminus\Omega} \in C^{1,\alpha}(\overline{\mathbb{B}_n(0,r)}\setminus\Omega)\,.$$

Then Lemma 2.9 implies that statement (iii) holds true. The last membership in (4.29) follows by statement (iii). □

Remark 4.31 We note that if $f \in C^{-1,\alpha}(\overline{\Omega})$, then Proposition 3.1 implies that $E^\sharp[f]$ belongs to $(C^{1,\alpha}(\overline{\Omega}))'$ and accordingly to $(C^{2,\alpha}(\overline{\Omega}))'$. Then Proposition 4.28 implies that $E^\sharp[f]$ belongs to $P^{-2,\alpha}(\overline{\Omega})$ (cf. Definition 4.21).

5 A distributional form of normal derivative for Hölder continuous solutions of the Poisson equation

Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\tilde{f} \in (C^{2,\alpha}(\overline{\Omega}))'$. We plan to define a normal derivative for a function $u \in C^0(\overline{\Omega})$ that satisfies the equality

$$\Delta u = \tilde{f}_{|\Omega} \qquad \text{in } \mathcal{D}'(\Omega)\,. \tag{5.1}$$

Actually a form of the normal derivative that depends on $\tilde{f}$ too. If u were to belong to the Sobolev space $H^1(\Omega)$ of functions in $L^2(\Omega)$ which have first order distributional derivatives in $L^2(\Omega)$ and $\tilde{f} \in \left(H^1(\Omega)\right)'$, then one could classically define the distributional normal derivative $\partial_{\nu,\tilde{f}}u$ to be the only element of the dual $H^{-1/2}(\partial\Omega)$ of the space $H^{1/2}(\partial\Omega)$ of traces on $\partial\Omega$ of $H^1(\Omega)$ that is defined by the equality

$$< \partial_{\nu,\tilde{f}}u, v >\equiv \int_\Omega DuD(Ev)\,dx + < \tilde{f}, Ev > \qquad \forall v \in H^{1/2}(\partial\Omega)\,, \tag{5.2}$$

where E is any bounded extension operator from $H^{1/2}(\partial\Omega)$ to $H^1(\Omega)$ (cf. *e.g.*, Lions and Magenes [15], Nečas [22, Chapt. 5], Nedelec and Planchard [23, p. 109], Costabel [4], McLean [18, Chapt. 4], Mikhailov [19], Mitrea, Mitrea and Mitrea [21, §4.2]). Then it is known that $\partial_{\nu,\tilde{f}}u$ may well depend on the specific choice of $\tilde{f}$ such that $\tilde{f}_{|\Omega} = \Delta u$ and it is also known that if we formulate further assumptions on u such as $\Delta u \in L^2(\Omega)$, then one could write $\int_\Omega (\Delta u) Ev\, dx$ instead of $< \tilde{f}, Ev >$ in (5.2) and thus one could define a canonical form $\partial_\nu u$ of the normal derivative of u on $\partial\Omega$ (with no need of some extra $\tilde{f}$). For a discussion on this issue, we refer to Costabel [4], Mikhailov [19].

In all cases, definition (5.2) implies that $\partial_{\nu,\tilde{f}} u$ is required to satisfy a generalized form of the classical first Green Identity as in (5.2).

However functions in $C^0(\overline{\Omega})$ or even in $C^{0,\alpha}(\overline{\Omega})$ are not necessarily in $H^1(\Omega)$ (for a discussion on this point we refer to Bramati, Dalla Riva and Luczak [3]). Thus we now develop the scheme of [12] for case $\tilde{f} = 0$ and introduce a different notion of distributional normal derivative $\partial_{\nu,\tilde{f}} u$ that requires that $\partial_{\nu,\tilde{f}} u$ satisfies a generalized form of the classical second Green Identity. To do so, we introduce the following classical result on the Green operator for the interior Dirichlet problem. For a proof, we refer for example to [13, §4].

Theorem 5.3 *Let $m \in \mathbb{N}$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{\max\{m,1\},\alpha}$. Then the map $\mathcal{G}_{d,+}$ from $C^{m,\alpha}(\partial\Omega)$ to the closed subspace*

$$C_h^{m,\alpha}(\overline{\Omega}) \equiv \{u \in C^{m,\alpha}(\overline{\Omega}), u \text{ is harmonic in } \Omega\} \tag{5.4}$$

of $C^{m,\alpha}(\overline{\Omega})$ that takes v to the only solution $v^\sharp$ of the Dirichlet problem

$$\begin{cases} \Delta v^\sharp = 0 & \text{in } \Omega\,, \\ v^\sharp_{|\partial\Omega} = v & \text{on } \partial\Omega \end{cases} \tag{5.5}$$

is a linear homeomorphism.

Next we introduce the (classical) interior Steklov-Poincaré operator (or interior Dirichlet-to-Neumann map).

Definition 5.6 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. The classical interior Steklov-Poincaré operator is defined to be the operator S_+ from*

$$C^{1,\alpha}(\partial\Omega) \quad \text{to} \quad C^{0,\alpha}(\partial\Omega) \tag{5.7}$$

takes $v \in C^{1,\alpha}(\partial\Omega)$ to the function

$$S_+[v](x) \equiv \frac{\partial}{\partial\nu}\mathcal{G}_{d,+}[v](x) \qquad \forall x \in \partial\Omega\,. \tag{5.8}$$

Since the classical normal derivative is continuous from $C^{1,\alpha}(\overline{\Omega})$ to $C^{0,\alpha}(\partial\Omega)$, the continuity of $\mathcal{G}_{d,+}$ implies that $S_+[\cdot]$ is linear and continuous from $C^{1,\alpha}(\partial\Omega)$ to $C^{0,\alpha}(\partial\Omega)$. We are now ready to introduce the following definition.

Definition 5.9 *Let $\alpha \in]0,1[$, $m \in \{1,2\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\tilde{f} \in (C^{m,\alpha}(\overline{\Omega}))'$. If $u \in C^0(\overline{\Omega})$ satisfies equation (5.1) in the sense of distributions in Ω, then we define the distributional normal derivative $\partial_{\nu,\tilde{f}} u$ to be the only element of the dual $(C^{m,\alpha}(\partial\Omega))'$ that satisfies the following equality*

$$< \partial_{\nu,\tilde{f}} u, v > \equiv \int_{\partial\Omega} u S_+[v]\, d\sigma + < \tilde{f}, \mathcal{G}_{d,+}[v] > \qquad \forall v \in C^{m,\alpha}(\partial\Omega)\,. \tag{5.10}$$

Here we have introduced the Definition 5.9 for functions u of $C^0(\overline{\Omega})$ that solve the Poisson equation (5.1), but one could do the same also for functions that solve the Poisson equation (5.1) in other function spaces that have a trace operator on $\partial\Omega$ and in cases in which we do not have information on the integrability of the first order partial derivatives of u in Ω.

In case $m = 1$ and if we further we require that Δu belongs to $C^{-1,\alpha}(\overline{\Omega})$, then Proposition 3.1 implies the existence of a 'canonical' extension $E^\sharp[\Delta u] \in \left(C^{1,\alpha}(\overline{\Omega})\right)'$ of Δu (that is an element of $C^{-1,\alpha}(\overline{\Omega})$ and accordingly of $\mathcal{D}'(\Omega)$) and thus we can define a 'canonical' normal derivative of u just by taking $\tilde{f} = E^\sharp[\Delta u]$ in Definition 5.9. Namely, we can introduce the following definition.

Definition 5.11 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $u \in C^0(\overline{\Omega})$ and $\Delta u \in C^{-1,\alpha}(\overline{\Omega})$, then we define the distributional normal derivative of u by the equality*

$$\partial_\nu u \equiv \partial_{\nu, E^\sharp[\Delta u]} u \,, \tag{5.12}$$

i.e., $\partial_\nu u$ is the only element of the dual $(C^{1,\alpha}(\partial\Omega))'$ that satisfies the following equality

$$< \partial_\nu u, v > \equiv \int_{\partial\Omega} u S_+[v] \, d\sigma + < E^\sharp[\Delta u], \mathcal{G}_{d,+}[v] > \qquad \forall v \in C^{1,\alpha}(\partial\Omega) \,. \tag{5.13}$$

Remark 5.14 If $\Delta u = 0$, then we have precisely the definition of [13, §5].

Next we show that if $u \in C^{1,\alpha}(\overline{\Omega})$, then the canonical normal derivative of u coincides with the distribution that is associated to the classical normal derivative of u. Namely, we prove the following statement.

Lemma 5.15 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $u \in C^{1,\alpha}(\overline{\Omega})$, then*

$$< \partial_\nu u, v > \equiv < \partial_{\nu, E^\sharp[\Delta u]} u, v > = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma \quad \forall v \in C^{1,\alpha}(\partial\Omega) \,, \tag{5.16}$$

where $\frac{\partial u}{\partial \nu}$ in the right hand side denotes the classical normal derivative of u on $\partial\Omega$.

Proof. Since $\Delta u = \sum_{j=1}^n \frac{\partial}{\partial x_j} \frac{\partial u}{\partial x_j}$ and $\frac{\partial u}{\partial x_j} \in C^{0,\alpha}(\overline{\Omega})$ for all $j \in \{1, \ldots, n\}$, we have $\Delta u \in C^{-1,\alpha}(\overline{\Omega})$ and the definition of $\partial_\nu u$ and Proposition 3.1 implies that

$$\begin{aligned} < \partial_\nu u, v > &= \int_{\partial\Omega} u S_+[v] \, d\sigma + < E^\sharp[\Delta u], \mathcal{G}_{d,+}[v] > \qquad (5.17) \\ &= \int_{\partial\Omega} u \frac{\partial}{\partial \nu} \mathcal{G}_{d,+}[v] \, d\sigma + \int_{\partial\Omega} \sum_{j=1}^n (\nu_\Omega)_j \frac{\partial u}{\partial x_j} v \, d\sigma \end{aligned}$$

$$-\sum_{j=1}^{n}\int_{\Omega}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\mathcal{G}_{d,+}[v]\,dx \quad \forall v\in C^{1,\alpha}(\partial\Omega)\,.$$

Since

$$\mathrm{div}\,(uD\mathcal{G}_{d,+}[v]) = Du(D\mathcal{G}_{d,+}[v])^t + u\Delta\mathcal{G}_{d,+}[v] = Du(D\mathcal{G}_{d,+}[v])^t \in C^{0,\alpha}(\overline{\Omega})\,,$$

then the Divergence Theorem implies that

$$\int_{\partial\Omega} u\frac{\partial}{\partial\nu}\mathcal{G}_{d,+}[v]\,d\sigma = \int_{\Omega}\mathrm{div}\,(uD\mathcal{G}_{d,+}[v])\;dx = \int_{\Omega} Du(D\mathcal{G}_{d,+}[v])^t\,dx$$

for all $v\in C^{1,\alpha}(\overline{\Omega})$ (cf., *e.g.*, [5, Thm. 4.1]). Hence, equality (5.17) implies the validity of equality (5.16). □

In the sequel, we use the classical symbol $\frac{\partial u}{\partial\nu}$ also for $\partial_\nu u = \partial_{\nu,E^\sharp[\Delta u]}u$ when no ambiguity can arise.

We now introduce an appropriate space of functions for which we can define the canonical normal derivative as in Definition 5.11 and that we later exploit to solve the interior Neumann problem.

Definition 5.18 *Let $\alpha\in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let*

$$C^{0,\alpha}(\overline{\Omega})_\Delta \equiv \left\{u\in C^{0,\alpha}(\overline{\Omega})\,:\,\Delta u\in C^{-1,\alpha}(\overline{\Omega})\right\}, \tag{5.19}$$

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})_\Delta} \equiv \|u\|_{C^{0,\alpha}(\overline{\Omega})} + \|\Delta u\|_{C^{-1,\alpha}(\overline{\Omega})} \qquad \forall u\in C^{0,\alpha}(\overline{\Omega})_\Delta\,.$$

Since $C^{0,\alpha}(\overline{\Omega})$ and $C^{-1,\alpha}(\overline{\Omega})$ are Banach spaces, $\left(\|u\|_{C^{0,\alpha}(\overline{\Omega})_\Delta}, \|\cdot\|_{C^{0,\alpha}(\overline{\Omega})_\Delta}\right)$ is a Banach space. We also note that $C^{0,\alpha}_h(\overline{\Omega})\subseteq C^{0,\alpha}(\overline{\Omega})_\Delta$ and that the inclusion is continuous.

Next we introduce a function space on the boundary of Ω for the normal derivatives of the functions of $C^{0,\alpha}(\overline{\Omega})_\Delta$. To do so, we resort to the following definition of [13, Defn. 13.2, 15.10, Thm. 18.1].

Definition 5.20 *Let $\alpha\in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let*

$$V^{-1,\alpha}(\partial\Omega)\equiv\left\{\mu_0+S_+^t[\mu_1]\,:\,\mu_0,\mu_1\in C^{0,\alpha}(\partial\Omega)\right\}, \tag{5.21}$$

$$\|\tau\|_{V^{-1,\alpha}(\partial\Omega)}\equiv\inf\left\{\|\mu_0\|_{C^{0,\alpha}(\partial\Omega)}+\|\mu_1\|_{C^{0,\alpha}(\partial\Omega)}\,:\,\tau=\mu_0+S_+^t[\mu_1]\right\},$$
$$\forall\tau\in V^{-1,\alpha,\pm}(\partial\Omega)\,,$$

where S_+^t is the transpose map of S_+.

As shown in [13, §13], $(V^{-1,\alpha}(\partial\Omega), \|\cdot\|_{V^{-1,\alpha}(\partial\Omega)})$ is a Banach space. By definition of the norm, $C^{0,\alpha}(\partial\Omega)$ is continuously embedded into $V^{-1,\alpha}(\partial\Omega)$. Moreover, the following statement holds (cf. [13, §18]).

Theorem 5.22 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the map $\frac{\partial}{\partial\nu}$ from the closed subspace $C_h^{0,\alpha}(\overline{\Omega})$ of $C^{0,\alpha}(\overline{\Omega})$ to $V^{-1,\alpha}(\partial\Omega)$ is linear and continuous. Moreover,*

$$\frac{\partial u}{\partial\nu} = S_+^t[u_{|\partial\Omega}] \qquad \forall u \in C_h^{0,\alpha}(\overline{\Omega}) \,. \tag{5.23}$$

(see (5.4) for the definition of $C_h^{0,\alpha}(\overline{\Omega})$).

We are now ready to prove the following statement on the continuity of the normal derivative on $C^{0,\alpha}(\overline{\Omega})_\Delta$.

Theorem 5.24 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Then the canonical normal derivative ∂_ν from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to $V^{-1,\alpha}(\partial\Omega)$ is linear and continuous.*

Proof. By definition, the canonical normal derivative ∂_ν is linear from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to $\left(C^{1,\alpha}(\partial\Omega)\right)'$. Next we note that

$$u = u - \mathcal{P}_\Omega^+[E^\sharp[\Delta u]] + \mathcal{P}_\Omega^+[E^\sharp[\Delta u]] \qquad \forall u \in C^{0,\alpha}(\overline{\Omega})_\Delta \,. \tag{5.25}$$

By Proposition 4.28 (i), (ii) and by the definition of norm in $C^{0,\alpha}(\overline{\Omega})_\Delta$, the map A_1 from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to the closed subspace $C_h^{0,\alpha}(\overline{\Omega})$ of $C^{0,\alpha}(\overline{\Omega})$ (see (5.4)) that takes u to

$$A_1[u] \equiv u - \mathcal{P}_\Omega^+[E^\sharp[\Delta u]]$$

is linear and continuous. By Theorem 5.22, ∂_ν is linear and continuous from $C_h^{0,\alpha}(\overline{\Omega})$ to $V^{-1,\alpha}(\partial\Omega)$. Hence, $\partial_\nu A_1[\cdot]$ is linear and continuous from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to $V^{-1,\alpha}(\partial\Omega)$.

By Proposition 4.28 (ii) and by the definition of norm in $C^{0,\alpha}(\overline{\Omega})_\Delta$, the operator $\mathcal{P}_\Omega^+[E^\sharp[\Delta\cdot]]$ is linear and continuous from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to $C^{1,\alpha}(\overline{\Omega})$. Then Lemma 5.15 implies that ∂_ν is linear and continuous from $C^{1,\alpha}(\overline{\Omega})$ to $C^{0,\alpha}(\partial\Omega)$. Since $C^{0,\alpha}(\partial\Omega)$ is continuously embedded into $V^{-1,\alpha}(\partial\Omega)$, we conclude that $\partial_\nu\mathcal{P}_\Omega^+[E^\sharp[\Delta\cdot]]$ is linear and continuous from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to $V^{-1,\alpha}(\partial\Omega)$. Hence, the map from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to $V^{-1,\alpha}(\partial\Omega)$ that takes u to $\partial_\nu u = \partial_\nu A_1[u] + \partial_\nu\left(\mathcal{P}_\Omega^+[E^\sharp[\Delta u]]\right)$ is linear and continuous. □

6 A nonvariational form of the interior Neumann problem for the Poisson equation

Let $\alpha \in]0,1[$, $m \in \{1,2\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. By exploiting the Definition 5.9 of distributional normal derivative, we can state the following Neumann problem. Given $\tilde{f} \in (C^{m,\alpha}(\overline{\Omega}))'$, $g \in (C^{m,\alpha}(\partial\Omega))'$, find all $u \in C^0(\overline{\Omega})$ such that the following interior Neumann problem is satisfied.

$$\begin{cases} \Delta u = \tilde{f}_{|\Omega} & \text{in } \mathcal{D}'(\Omega)\,, \\ \partial_{\nu,\tilde{f}} u = g & \text{in } (C^{m,\alpha}(\partial\Omega))'\,, \end{cases} \tag{6.1}$$

where $\partial_{\nu,\tilde{f}} u$ is as in Definition 5.9. Since the solutions of the Neumann problem (6.1) may well have infinite Dirichlet integral, we address to problem (6.1) as 'nonvariational interior Neumann problem for the Poisson equation'.

For the interior nonvariational Neumann problem to have solutions, the data $\tilde{f}$ and g have to satisfy certain compatibility conditions that are akin to the corresponding compatibility conditions for the variational Neumann problem for the Poisson equation, as we show in the following Lemma (see Section 2 for the notation on the connected components Ω_j of Ω).

Lemma 6.2 *Let $\alpha \in]0,1[$, $m \in \{1,2\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\tilde{f} \in (C^{m,\alpha}(\overline{\Omega}))'$, $g \in (C^{m,\alpha}(\partial\Omega))'$. If the interior Neumann problem (6.1) has a solution $u \in C^0(\overline{\Omega})$, then*

$$< g, \chi_{\partial\Omega_j} >=< \tilde{f}, \chi_{\overline{\Omega_j}} > \qquad \forall j \in \{1,\ldots,\kappa^+\}\,. \tag{6.3}$$

Proof. First we note that $\chi_{\partial\Omega_j}$ is locally constant on $\partial\Omega$ and that accordingly $\chi_{\partial\Omega_j} \in C^{m,\alpha}(\partial\Omega)$ for all $j \in \{1,\ldots,\kappa^+\}$. Next we note that $\chi_{\overline{\Omega_j}}$ solves the Dirichlet problem (5.5) with $v = \chi_{\partial\Omega_j}$ and that accordingly $\mathcal{G}_{d,+}[\chi_{\partial\Omega_j}] = \chi_{\overline{\Omega_j}}$. Hence, the validity of the interior Neumann problem (6.1) implies that

$$\begin{aligned} < g, \chi_{\partial\Omega_j} > &=< \partial_{\nu,\tilde{f}} u, \chi_{\partial\Omega_j} > \\ &\equiv \int_{\partial\Omega} u \frac{\partial \mathcal{G}_{d,+}[\chi_{\partial\Omega_j}]}{\partial\nu}\, d\sigma + < \tilde{f}, \mathcal{G}_{d,+}[\chi_{\partial\Omega_j}] > \\ &= \int_{\partial\Omega} u \frac{\partial \chi_{\overline{\Omega_j}}}{\partial\nu}\, d\sigma + < \tilde{f}, \chi_{\overline{\Omega_j}} >=< \tilde{f}, \chi_{\overline{\Omega_j}} > \qquad \forall j \in \{1,\ldots,\kappa^+\}\,. \end{aligned}$$

□

Then by Remark 5.14 and by [13, §7], we have the following statement that shows that the possible continuous solutions of the nonvariational interior Neumann problem (6.1) are unique up to locally constant functions, exactly as in the classical case.

Theorem 6.4 *Let $\alpha \in]0,1[$, $m \in \{1,2\}$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Let $\tilde{f} \in (C^{m,\alpha}(\overline{\Omega}))'$, $g \in (C^{m,\alpha}(\partial\Omega))'$.*

If u_1, $u_2 \in C^0(\overline{\Omega})$ solve the interior Neumann problem (6.1), then $u_1 - u_2$ is constant in each connected component of Ω. In particular, all solutions of the nonvariational interior Neumann problem in $C^0(\overline{\Omega})$ can be obtained by adding to u_1 an arbitrary function which is constant on the closure of each connected component of Ω.

In this paper, we solve the nonvariational interior Neumann problem (6.1) in the case in which $m = 1$ and the datum $\tilde{f}$ in the interior is of the form $\tilde{f} = E^\sharp[f]$ for some $f \in C^{-1,\alpha}(\overline{\Omega})$ and in case solutions admit a canonical normal derivative as in Definition 5.11. To do so, we reformulate problem (6.1) as in the following elementary statement.

Proposition 6.5 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $f \in C^{-1,\alpha}(\overline{\Omega})$. Then a function $u \in C^0(\overline{\Omega})$ such that $\Delta u \in C^{-1,\alpha}(\overline{\Omega})$ satisfies the nonvariational Neumann problem (6.1) with $\tilde{f} = E^\sharp[f]$, $m = 1$ if and only if u satisfies the following interior nonvariational Neumann problem*

$$\begin{cases} \Delta u = f & \text{in } \mathcal{D}'(\Omega)\,, \\ \partial_\nu u = g & \text{in } (C^{1,\alpha}(\partial\Omega))'\,, \end{cases} \tag{6.6}$$

where $\partial_\nu u$ is the canonical normal derivative as in Definition 5.11.

Proof. By Proposition 3.1, we have $E^\sharp[f]_{|\Omega} = f$. Then the nonvariational Neumann problem (6.1) with $\tilde{f} = E^\sharp[f]$, $m = 1$ holds if and only if

$$\begin{cases} \Delta u = f & \text{in } \mathcal{D}'(\Omega)\,, \\ \partial_{\nu,E^\sharp[f]} u = g & \text{in } (C^{1,\alpha}(\partial\Omega))'\,. \end{cases} \tag{6.7}$$

Next we note that equality $\Delta u = f$ in $\mathcal{D}'(\Omega)$, the membership of Δu in $C^{-1,\alpha}(\overline{\Omega})$ and Proposition 3.1 imply that $E^\sharp[\Delta u] = E^\sharp[f]$. Then problem (6.7) is equivalent to problem

$$\begin{cases} \Delta u = f & \text{in } \mathcal{D}'(\Omega)\,, \\ \partial_{\nu,E^\sharp[\Delta u]} u = g & \text{in } (C^{1,\alpha}(\partial\Omega))'\,, \end{cases}$$

i.e., to problem (6.6). □

We are now ready to prove the following existence theorem.

Theorem 6.8 *Let $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. If $(f,g) \in C^{-1,\alpha}(\overline{\Omega}) \times V^{-1,\alpha}(\partial\Omega)$ and if*

$$< g, \chi_{\partial\Omega_j} > = \mathcal{I}_{\Omega_j}[f] \qquad \forall j \in \{1,\ldots,\kappa^+\}\,, \tag{6.9}$$

then the interior nonvariational Neumann problem (6.6) has at least a solution $u \in C^{0,\alpha}(\overline{\Omega})_\Delta$ (for the definition of $\mathcal{I}_{\Omega_j}[\cdot]$, see Proposition 2.10). All other solutions in $C^0(\overline{\Omega})$ can be obtained by adding to u a function that is constant on the closure of each connected component of Ω. Moreover, the operator (Δ, ∂_ν) from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to the closed subspace

$$\left\{(f,g) \in C^{-1,\alpha}(\overline{\Omega}) \times V^{-1,\alpha}(\partial\Omega) : < g, \chi_{\partial\Omega_j} > = \mathcal{I}_{\Omega_j}[f]\ \forall j \in \{1,\ldots,\kappa^+\}\right\} \tag{6.10}$$

of $C^{-1,\alpha}(\overline{\Omega}) \times V^{-1,\alpha}(\partial\Omega)$ that takes u to $(\Delta u, \partial_\nu u)$ is a linear and continuous surjection and the null space $\mathrm{Ker}\,(\Delta, \partial_\nu)$ consists of the functions which are constant on the closure of each connected component of Ω.

Proof. If $f = f_0 + \sum_{l=1}^{n} \frac{\partial}{\partial x_l} f_l \in C^{-1,\alpha}(\overline{\Omega})$, then $\mathcal{P}^+_\Omega[E^\sharp[f]] \in C^{1,\alpha}(\overline{\Omega})$ and $\Delta \mathcal{P}^+_\Omega[E^\sharp[f]] = f$ in Ω (see Proposition 4.28). We now show that the interior Neumann problem

$$\begin{cases} \Delta h = 0 & \text{in } \Omega\,, \\ \partial_\nu h = g - \frac{\partial}{\partial \nu}\mathcal{P}^+_\Omega[E^\sharp[f]] & \text{on } \partial\Omega \end{cases} \tag{6.11}$$

has a solution $h \in C^{0,\alpha}(\overline{\Omega})$. By Proposition 4.28, we have $\mathcal{P}^+_\Omega[E^\sharp[f]] \in C^{1,\alpha}(\overline{\Omega})$ and accordingly

$$\frac{\partial}{\partial \nu}\mathcal{P}^+_\Omega[E^\sharp[f]] \in C^{0,\alpha}(\partial\Omega) \subseteq V^{-1,\alpha}(\partial\Omega)\,.$$

Thus it suffices to show that g satisfies the compatibility conditions

$$< g - \frac{\partial}{\partial \nu_\Omega}\mathcal{P}^+_\Omega[E^\sharp[f]], \chi_{\partial\Omega_j} >= 0 \qquad \forall j \in \{1, \dots, \kappa^+\}$$

for the data of the nonvariational interior Neumann problem for the Laplace operator (cf. [13, §20]).

By (3.4) and the classical Theorem A.5 of the Appendix with $m = 0$ we have $\mathcal{P}^+_\Omega[E^\sharp[f_l]] \in C^{2,\alpha}(\overline{\Omega})$ for all $l \in \{1, \dots, n\}$. Then (4.30) implies that $\Delta \mathcal{P}^+_\Omega[E^\sharp[f_l]] = f_l$ in Ω for all $l \in \{1, \dots, n\}$. By known results on the single layer potential, $v^+_\Omega[f_l(\nu_\Omega)_l]$ belongs to $C^{1,\alpha}(\overline{\Omega})$ and is harmonic in Ω for all $l \in \{1, \dots, n\}$ (cf. *e.g.*, (4.27), [5, Thm. 4.25]). Then formula (4.24) for $\mathcal{P}^+_\Omega[E^\sharp[f]]$, the Divergence Theorem (cf. *e.g.*, [5, Thm. 4.1]), the first Green Identity (cf. *e.g.*, [5, Thm. 4.2]), and Proposition 2.10 on the definition of $\mathcal{I}_{\Omega_j}$ imply that

$$\begin{aligned}
< g - \frac{\partial}{\partial \nu_\Omega}\mathcal{P}^+_\Omega[f], \chi_{\partial\Omega_j} >=< g, \chi_{\partial\Omega_j} > - \int_{\partial\Omega_j} \frac{\partial}{\partial \nu_\Omega}\mathcal{P}^+_\Omega[E^\sharp[f]]\, d\sigma =< g, \chi_{\partial\Omega_j} > \\
- \int_{\partial\Omega_j} \frac{\partial}{\partial \nu_\Omega}\mathcal{P}^+_\Omega[f_0]\, d\sigma - \int_{\partial\Omega_j} \sum_{s=1}^{n} (\nu_\Omega)_s \frac{\partial}{\partial x_s}\left(\sum_{l=1}^{n} \frac{\partial}{\partial x_l}\mathcal{P}^+_\Omega[f_l]\right) d\sigma \\
- \sum_{l=1}^{n} \int_{\partial\Omega_j} \frac{\partial}{\partial \nu_\Omega} v^+_\Omega[f_l(\nu_\Omega)_l]\, d\sigma \\
=< g, \chi_{\partial\Omega_j} > - \int_{\partial\Omega_j} f_0\, d\sigma - \int_{\partial\Omega_j} \sum_{s=1}^{n} (\nu_\Omega)_s \frac{\partial}{\partial x_s}\left(\sum_{l=1}^{n} \frac{\partial}{\partial x_l}\mathcal{P}^+_\Omega[f_l]\right) d\sigma \\
=< g, \chi_{\partial\Omega_j} > -\mathcal{I}_{\Omega_j}\left[f_0 + \sum_{s=1}^{n} \frac{\partial}{\partial x_s}\frac{\partial}{\partial x_s}\left(\sum_{l=1}^{n} \frac{\partial}{\partial x_l}\mathcal{P}^+_\Omega[f_l]\right)\right] \\
=< g, \chi_{\partial\Omega_j} > -\mathcal{I}_{\Omega_j}\left[f_0 + \sum_{l=1}^{n} \frac{\partial}{\partial x_l}\Delta\mathcal{P}^+_\Omega[f_l]\right] \\
=< g, \chi_{\partial\Omega_j} > -\mathcal{I}_{\Omega_j}\left[f_0 + \sum_{l=1}^{n} \frac{\partial}{\partial x_l} f_l\right]
\end{aligned}$$

$$
=< g,\chi_{\partial\Omega_j} > -\mathcal{I}_{\Omega_j}[f] = 0 \qquad \forall j\in\{1,\dots,\kappa^+\}\,.
$$

Hence, the nonvariational interior Neumann problem for the Laplace operator (6.11) has a solution $h\in C^{0,\alpha}(\overline{\Omega})$ (cf. [13, §20]).

Then $u\equiv h+\mathcal{P}^+_\Omega[E^\sharp[f]]$ belongs to $C^{0,\alpha}(\overline{\Omega})$ and solves the Neumann problem of the statement, i.e., $(\Delta u,\partial_\nu u)=(f,g)$.

Since the components of (Δ,∂_ν) are linear and continuous, (Δ,∂_ν) is linear and continuous from $C^{0,\alpha}(\overline{\Omega})_\Delta$ to $C^{-1,\alpha}(\overline{\Omega})\times V^{-1,\alpha}(\partial\Omega)$ (see the Definition 5.18 of nom in $C^{0,\alpha}(\overline{\Omega})_\Delta$ and Theorem 5.24).

Nex we show that if $u\in C^{0,\alpha}(\overline{\Omega})_\Delta$, then the pair $(\Delta u,\partial_\nu u)$ belongs to the space in (6.10). If $u\in C^{0,\alpha}(\overline{\Omega})_\Delta$, then the compatibility conditions of Lemma 6.2 and Proposition 6.5 on the formulation of the Neumann problem with the canonical normal derivative imply that

$$
< \partial_\nu u,\chi_{\partial\Omega_j} >=< E^\sharp[\Delta u],\chi_{\overline{\Omega_j}} > \qquad \forall j\in\{1,\dots,\kappa^+\}\,. \tag{6.12}
$$

Since $\Delta u\in C^{-1,\alpha}(\overline{\Omega})$, there exists $(f_0,\dots,f_n)\in C^{0,\alpha}(\overline{\Omega})^{n+1}$ such that

$$
\Delta u = f_0+\sum_{l=1}^n \frac{\partial}{\partial x_l} f_l
$$

and thus Proposition 2.10 on $\mathcal{I}_{\Omega_j}$ and Proposition 3.1 (ii) imply that

$$
\begin{aligned}
< E^\sharp[\Delta u],\chi_{\overline{\Omega_j}} >&= \int_{\Omega_j} f_0\,dx+\int_{\partial\Omega_j}\sum_{l=1}^n(\nu_\Omega)_l f_l\,d\sigma \\
&\quad -\sum_{l=1}^n\int_{\Omega_j} f_l\frac{\partial\chi_{\overline{\Omega_j}}}{\partial x_l}\,dx=\mathcal{I}_{\Omega_j}[\Delta u] \qquad \forall j\in\{1,\dots,\kappa^+\}\,.
\end{aligned} \tag{6.13}
$$

Hence, equalities (6.12) and (6.13) imply that the pair $(\Delta u,\partial_\nu u)$ belongs to the space in (6.10).

By the above argument, the operator $(\Delta,\frac{\partial}{\partial\nu_\Omega})$ is surjective onto the space in (6.10). By Theorem 6.4 and Proposition 6.5, we know that all other solutions in $C^0(\overline{\Omega})$ can be obtained by adding to u a locally constant function and that $\mathrm{Ker}\,(\Delta,\partial_\nu)$ consists of the functions which are locally constant in $\overline{\Omega}$.

By Proposition 2.10, the operator $\mathcal{I}_{\Omega_j}$ from $C^{-1,\alpha}(\overline{\Omega})$ to $\mathbb{R}$ is linear and continuous. Since the operator from $V^{-1,\alpha}(\partial\Omega)$ to $\mathbb{R}$ that takes g to $< g,\chi_{\partial\Omega_j} >$ is linear and continuous, then the map from $C^{-1,\alpha}(\overline{\Omega})_\Delta\times V^{-1,\alpha}(\partial\Omega)$ to $\mathbb{R}$ that takes (f,g) to $< g,\chi_{\partial\Omega_j} > -\mathcal{I}_{\Omega_j}[f]$ is linear and continuous for all $j\in\{1,\dots,\kappa^+\}$. Thus the space in (6.10) is closed in $C^{-1,\alpha}(\overline{\Omega})\times V^{-1,\alpha}(\partial\Omega)$. □

A Appendix: Classical properties of the harmonic volume potential

We now present some classical results on the harmonic volume potential in the specific form that we need in the paper.

Theorem A.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$. Then the following statements hold.*

(i) If $n \geq 3$, then the volume potential $\mathcal{P}_\Omega[\cdot]$ is linear and continuous from $L^\infty(\Omega)$ to $C^1_b(\mathbb{R}^n)$ and

$$\frac{\partial}{\partial x_j}\mathcal{P}_\Omega[f](x) = \int_\Omega \frac{\partial S_n}{\partial x_j}(x-y)f(y)\,dy \qquad \forall x \in \mathbb{R}^n \tag{A.2}$$

for all $f \in L^\infty(\Omega)$ and $j \in \{1,\dots,n\}$.

(ii) If $n = 2$, then the restriction $\mathcal{P}_\Omega[\cdot]_{|\overline{\mathbb{B}_n(0,r)}}$ of the volume potential is linear and continuous from $L^\infty(\Omega)$ to $C^1_b(\overline{\mathbb{B}_n(0,r)})$ for all $r \in]0,+\infty[$ such that $\overline{\Omega} \subseteq \mathbb{B}_n(0,r)$ and formula (A.2) holds true.

Proof. Let $\varphi \in L^\infty(\Omega)$. By Gilbarg and Trudinger [8, Lem. 4.1] (see also [5, Prop. 7.6]) and by the classical differentiability theorem for integrals depending on a parameter (for $x \in \mathbb{R}^n \setminus \overline{\Omega}$), we have $\mathcal{P}_\Omega[f] \in C^1(\mathbb{R}^n)$ and formula (A.2) holds true for $n \geq 2$. The elementary inequality

$$m_n(\Omega \cap \mathbb{B}_n(x,r)) \leq \omega_n r^n \qquad \forall r \in]0,+\infty[$$

implies that Ω is upper $(n-1)$-Ahlfors regular with respect to $\mathbb{R}^n$ (cf. [11, (1.4)]). Then Lemma 3.4 of [11] implies that

$$c'_s \equiv \sup_{x\in\mathbb{R}^n} \int_\Omega \frac{dy}{|x-y|^s} < +\infty \tag{A.3}$$

for all $s \in]0,n[$ (an inequality that one could also prove directly by elementary calculus). In case $n = 2$, we also note that

$$\sup\left\{|x-y|^{1/2}\log|x-y| : \, x \in \mathbb{B}_n(0,r),\ y \in \Omega\,, x \neq y\right\} < +\infty \tag{A.4}$$

for all $r \in]0,+\infty[$ as in (ii). Then the Hölder inequality, formula (A.2) and inequalities (A.3), (A.4) imply the validity of statements (i), (ii). □

By Proposition A.1 and by a classical result, we can state the following theorem (cf. *e.g.*, Miranda [20, Thm. 3.I, p. 320]).

Theorem A.5 *Let $m \in \mathbb{N}$, $\alpha \in]0,1[$. Let Ω be a bounded open subset of $\mathbb{R}^n$ of class $C^{m+1,\alpha}$. Then the following statements hold.*

(i) $\mathcal{P}^+_\Omega[\cdot]$ *is linear and continuous from* $C^{m,\alpha}(\overline{\Omega})$ *to* $C^{m+2,\alpha}(\overline{\Omega})$.

(ii) $\mathcal{P}^-_\Omega[\cdot]$ *is linear and continuous from* $C^{m,\alpha}(\overline{\Omega})$ *to* $C^{m+2,\alpha}(\overline{\mathbb{B}_n(0,r)} \setminus \Omega)$ *for all* $r \in]0,+\infty[$ *such that* $\overline{\Omega} \subseteq \mathbb{B}_n(0,r)$.

Statements and Declarations

Competing interests: This paper does not have any conflict of interest or competing interest.

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