AFDM Chirp-Permutation-Index Modulation with Quantum-Accelerated Codebook Design

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***Abstract*—We describe a novel index modulation (IM) scheme exploiting a unique feature of the recently proposed affine frequency division multiplexing (AFDM) in doubly-dispersive (DD) channels. Dubbed AFDM chirp-permutation-index modulation (CPIM), the proposed method encodes additional information via the permutation of the discrete affine Fourier transform (DAFT) chirp sequence, without any sacrifice of the various beneficial properties of the AFDM waveform in DD channels. The effectiveness of the proposed method is validated via simulation results leveraging a novel reduced-complexity minimum mean-squared-error (MMSE)-based maximum likelihood (ML) detector, highlighting the gains over the classical AFDM. As part of the work two interesting problems related to optimizing AFDM-CPIM are identified: the optimal codebook design problem, over a discrete solution space of dimension $\binom{N!}{K}$, where N is the number of subcarriers and K is the number of codewords; and the ML detection problem whose solution space is of dimension KM^N, where M is the constellation size. In order to alleviate the computational complexity of these problems and enable large-scale variations of AFDM-CPIM, the two problems are reformulated as a higher-order binary optimization problem and mapped to the well-known quantum Grover adaptive search (GAS) algorithm for their solution.**

I. Introduction

Recently, affine frequency division multiplexing (AFDM) has emerged as a strong candidate waveform for sixth-generation (6G) wireless systems [1], [2], as it offers a number of advantages such as robustness to doubly-dispersive (DD) channels, attractive features for integrated sensing and communications (ISAC), the ability to minimize self-interference via de-chirping and more [3]. On the other hand, index modulation (IM) is also an attractive technology for 6G systems, as it has other desirable (and to some extent complementary) characteristics such as reduction of radio frequency (RF) chains [4], and ability to easily incorporate physical-layer security [5].

In most IM schemes, a subset of a larger number of available resources are selected in order to encode information, such that the utilization of available resources is "sparsified", which we shall therefore refer to such approaches as sparse IM schemes. As shall be shown later, in Subsection III-A, in the case of AFDM a very peculiar type of IM can also be designed, in which no resource is excluded at any transmission instance[1].

In particular, we refer to the possibility – unique to AFDM in comparison to alternative waveforms such as orthogonal frequency division multiplexing (OFDM) and orthogonal time frequency space (OTFS) [16] – that the chips employed in the construction of the AFDM chirps be permuted in accordance with the information to be transmitted. The resulting technique, hereafter refered to as chirp-permutation-index modulation (CPIM), is one of the main contribution of this article.

As shall be clarified, the consequence of such pseudo-random permutation of AFDM chirp chips, is a substantial difference in that the state of the channel between the transmitter and receiver. To elaborate, it is found that different channels matrices corresponding to a distinct chirp permutations, are very distinguishable, under various definitions of matrix distances. By designing distinct sets of chip permutations that maximize the distinction among the resulting AFDM channels, an optimum CPIM codebook design can be achieved. Denoting the number of subcarriers by N, and the number of codewords ($i.e.$ selected permutations) by K, it follows that the optimal codebook design problem requires a search over a discrete solution space of dimension $\binom{N!}{K}$. In turn, at the receiver, maximum likelihood (ML) detection requires that all possible K permutations be tested, each over all possible N-tuple of symbols taking from a constellation of size M, such that again the search space for ML detection is of dimension KM^N.

Clearly, such combinatorial search spaces can quickly grow well beyond the feasibility of modern digital computers even for relatively small numbers of K, N and M, which motivates the second major contribution of the article, namely, a pair of quantum-computing algorithms for both the ML detection and codebook design of AFDM-CPIM.

We remark that although techniques to reduce decoding complexity exist, for instance via sphere-detection [6], compressive sensing approaches [7], or vectorized message-passing algorithms [8], none of such techniques can actually achieve the same performance of the brute-force ML search. In turn, the codebook design is an even harder problem, related to the well-known max-min dispersion problem [9]. We demonstrate, however, that the codebook design can be first mapped to a binary optimization problem, allowing the query complexity to be reduced to approximately the square root via quantum computing techniques. Such representative solution is the Grover adaptive search (GAS) algorithm, which was shown to be efficiently implemented via quantum oracle in the case of objective functions with integer coefficients in [10], extended to accommodate real-valued coefficients [11].

[1]We emphasize that, of course, sparse IM can also be considered in conjunction to the method here proposed. For example, the set of chirp-domain subcarriers employed during a given transmission instance may be taken from a larger subset. Similarly, spatial-domain variations IM can also be incorporated. Such approaches are transparent to the proposed method.

II. System Model

A. Conventional AFDM Signal Model

The AFDM waveform [1] leverages the inverse discrete affine Fourier transform (IDAFT) [12] to modulate a sequence of information symbols unto the chirp-domain subcarriers.

The N-point forward discrete affine Fourier transform (DAFT) matrix $\mathbf{A} \in \mathbb{C}^{N\times N}$ and its inverse $\mathbf{A}^{-1} \in \mathbb{C}^{N\times N}$ are efficiently described by a discrete Fourier transform (DFT) matrix twisted by two chirp sequences, as

$$\mathbf{A} \triangleq \mathbf{\Lambda}_{c_2}\mathbf{F}_N\mathbf{\Lambda}_{c_1} \in \mathbb{C}^{N\times N}, \tag{1a}$$

$$\mathbf{A}^{-1} \triangleq (\mathbf{\Lambda}_{c_2}\mathbf{F}_N\mathbf{\Lambda}_{c_1})^{-1} = \mathbf{\Lambda}_{c_1}^{\mathsf{H}}\mathbf{F}_N^{\mathsf{H}}\mathbf{\Lambda}_{c_2}^{\mathsf{H}} \in \mathbb{C}^{N\times N}, \tag{1b}$$

where $\mathbf{F}_N \times \mathbb{C}^{N\times N}$ is the normalized N-point DFT matrix, and $\mathbf{\Lambda}_{c_1} \triangleq \mathrm{diag}(\boldsymbol{\lambda}_{c_1}) \in \mathbb{C}^{N\times N}$, $\mathbf{\Lambda}_{c_2} \triangleq \mathrm{diag}(\boldsymbol{\lambda}_{c_2}) \in \mathbb{C}^{N\times N}$ are diagonal chirp matrices whose diagonals are described by the chirp vector $\boldsymbol{\lambda}_{c_i} \triangleq [e^{-j2\pi c_i(0)^2}, \cdots, e^{-j2\pi c_i(N-1)^2}] \in \mathbb{C}^{N\times 1}$ with a central digital frequency of c_i.

The first chirp frequency c_1 is a critical parameter which is optimally configured to the statistics of the doubly-dispersive channel to obtain the notable orthogonality of the AFDM subcarriers over the delay-Doppler domain [1], [2]. On the other hand, the second chirp frequency c_2 is a relatively flexible parameter which does affect the orthogonality of the subcarriers, but changes certain waveform properties such as the ambiguity function for ISAC techniques [13]. The optimal criteria of c_1 and c_2 will be discussed in the following section.

In light of the above, the AFDM modulated transmit signal $\mathbf{s} \in \mathbb{C}^{N\times 1}$ is given by an N-point IDAFT of the symbol vector $\mathbf{x} \in \mathcal{X}^{N\times 1}$ whose elements are drawn from an M-ary complex digital constellation $\mathcal{X} \subset \mathbb{C}$, *i.e.,*

$$\mathbf{s} \triangleq \mathbf{A}^{-1}\mathbf{x} = \mathbf{\Lambda}_{c_1}^{\mathsf{H}}\mathbf{F}_N^{\mathsf{H}}\mathbf{\Lambda}_{c_2}^{\mathsf{H}}\mathbf{x} \in \mathbb{C}^{N\times 1}. \tag{2}$$

B. Received Signal Model over Doubly-Dispersive Channels

Consider a doubly-dispersive channel between a transmitter and a receiver with P significant resolvable propagation paths, where each p-th path induces a unique path delay $\tau_p \in [0, \tau^{\max}]$ and Doppler shift $\nu_p \in [-\nu^{\max}, +\nu^{\max}]$ to the received signal, with the channel statistics described by the delay spread $\tau^{\max}$ and Doppler spread $\nu^{\max}$. The corresponding received signal is obtained by a linear convolution of the transmit signal and the time-varying impulse response function (TVIRF) of the doubly-dispersive channel [2], [14].

Then, by leveraging a cyclic prefix (CP) to the transmit signal, and sampling the signals at a sampling frequency of $f_{\mathrm{s}} \triangleq \frac{1}{T_{\mathrm{s}}}$, the received signal can be expressed in terms of a discrete circular convolutional channel given by

$$\mathbf{r} \triangleq \mathbf{H}\mathbf{s} + \mathbf{w} = \Big(\sum_{p=1}^{P} h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{Z}^{f_p} \cdot \mathbf{\Pi}^{\ell_p}\Big)\mathbf{s} + \mathbf{w} \in \mathbb{C}^{N\times 1}, \tag{5}$$

where $\mathbf{r} \in \mathbb{C}^{N\times 1}$, $\mathbf{s} \in \mathbb{C}^{N\times 1}$, and $\mathbf{w} \in \mathbb{C}^{N\times 1}$ are the discrete vectors of the received signal, transmit signal, and additive white Gaussian noise (AWGN) signal; $\ell_p \triangleq \lfloor \frac{\tau_p}{T_{\mathrm{s}}} \rceil \in \mathbb{N}_0$ and $f_p \triangleq \frac{N\nu_p}{f_s} \in \mathbb{R}$ are the normalized integer[2] path delay and normalized digital Doppler shift of the p-th propagation path; and $\mathbf{H} \triangleq \sum_{p=1}^{P} h_p \cdot \mathbf{\Phi}_p \cdot \mathbf{Z}^{f_p} \cdot \mathbf{\Pi}^{\ell_p} \in \mathbb{C}^{N\times N}$ is the circular convolutional matrix of the doubly-dispersive channel, whose p-th path is fully parametrized of four statistical components: *a)* a complex channel fading coefficient $h_p \in \mathbb{C}$, *b)* a diagonal phase matrix $\mathbf{\Phi}_p \in \mathbb{C}^{N\times N}$ given by eq. (3) corresponding to the chirp-periodic phase of the AFDM CP [1], *c)* a diagonal roots-of-unity matrix $\mathbf{Z} \in \mathbb{C}^{N\times N}$ given by eq. (4) which is taken to the f_p-th power, and *d)* a right-multiplying circular left-shift matrix $\mathbf{\Pi} \in \mathbb{C}^{N\times N}$ taken to the ℓ_p-th integer power, to denote a circular left-shift operation by ℓ_p indices.

In light of the above, the effective input-output relationship of the AFDM system is given by

$$\mathbf{y} \triangleq \mathbf{G}\mathbf{x} + \tilde{\mathbf{w}} = \mathbf{A}\mathbf{r} \in \mathbb{C}^{N\times 1}, \tag{6}$$

where $\mathbf{y} \in \mathbb{C}^{N\times 1}$ is the demodulated signal, $\mathbf{G} \triangleq \mathbf{A}\mathbf{H}\mathbf{A}^{-1} \in \mathbb{C}^{N\times N}$ is the effective AFDM channel matrix, and $\tilde{\mathbf{w}} \triangleq \mathbf{A}\mathbf{w} \in \mathbb{C}^{N\times 1}$ is the effective noise which has the same statistical properties as $\mathbf{w}$ due to unitary transform property of the DAFT.

The unique property of the AFDM effective channel $\mathbf{G}$ is the distinct separability of the channel paths with different integer delay and Doppler shifts [1], [2], leading to diversity optimality and double-dispersion robustness in linear time-variant (LTV) channels [1], given that the DAFT chirp frequencies in eq. (1) are selected to satisfy

$$c_1^{\mathrm{opt}} = \frac{2(f^{\max} + \xi) + 1}{2N} \quad \text{and} \quad c_2^{\mathrm{opt}} \in \{\mathbb{R}\backslash\mathbb{Q}\}, \tag{7}$$

where $\xi \in \mathbb{N}_0$ is the guard width of the AFDM which improves the robustness against fractional Doppler shifts [1], [2], and $\{\mathbb{R}\backslash\mathbb{Q}\}$ denotes the set of real irrational numbers.

III. Proposed AFDM-based CPIM

In light of eq. (7), it can be seen that while the first chirp matrix $\mathbf{\Lambda}_{c_1}$ of the DAFT is of critical value in determining the performance over the doubly-dispersive channel, the second chirp matrix $\mathbf{\Lambda}_{c_2}$ provides a flexible parameter while still attaining the desired properties of the AFDM. This degree of freedom is unique to the AFDM modulation scheme, as other waveforms which achieve full diversity and robustness over doubly-dispersive channels, such as the OTFS waveform [16], do not have such flexibility in the core transform used in the modulation.

[2]It is assumed that the sampling frequency f_{s} is sufficiently high, such that the normalized path delays can be assumed to be integers with negligble error [1], [2], [14].

$$\mathbf{\Phi}_p \triangleq \mathrm{diag}\Big(\big[\overbrace{e^{-j2\pi\cdot c_1(N^2-2N(\ell_p))}, e^{-j2\pi\cdot c_1(N^2-2N(\ell_p-1))}, \cdots, e^{-j2\pi\cdot c_1(N^2-2N(1))}}^{\ell_p \text{ terms}}, \overbrace{1, 1, \cdots, 1, 1}^{N-\ell_p \text{ ones}}\big]\Big) \in \mathbb{C}^{N\times N}. \tag{3}$$

$$\mathbf{Z} \triangleq \mathrm{diag}\Big(\big[e^{-j2\pi(0)/N}, e^{-j2\pi(1)/N}, \cdots, e^{-j2\pi(N-2)/N}, e^{-j2\pi(N-1)/N}\big]\Big) \in \mathbb{C}^{N\times N}. \tag{4}$$

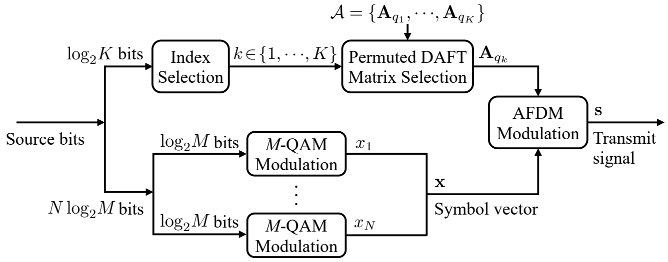


Fig. 1: A schematic diagram of the modulation process of the proposed AFDM-CPIM transmitter.

Therefore, we propose a novel IM scheme based on the second chirp matrix $\mathbf{\Lambda}_{c_2}$ of the AFDM, specifically by exploiting the permutations of the chirp sequence $\boldsymbol{\lambda}_{c_2}$.

A. The AFDM-CPIM Transmitter

Let the permutation operation of an arbitrary vector $\mathbf{x} \in \mathbb{C}^{N\times 1}$ be denoted by $\mathbf{x}_i \triangleq \mathrm{perm}(\mathbf{x}, i)$, where $i \in \{1, \cdots, N!\}$ is the permutation *index* in ascending order. Then, a modified DAFT matrix is described by the i-th permutation the second chirp sequence, *i.e.,*

$$\mathbf{A}_i \triangleq \mathbf{\Lambda}_{c_2,i} \cdot \mathbf{F}_N \cdot \mathbf{\Lambda}_{c_1} \in \mathbb{C}^{N\times N}, \tag{8}$$

where $\mathbf{\Lambda}_{c_2,i} \triangleq \mathrm{diag}(\boldsymbol{\lambda}_{c_2,i})$, and $\boldsymbol{\lambda}_{c_2,i} \triangleq \mathrm{perm}(\boldsymbol{\lambda}_{c_2}, i)$.

Trivially, there exist $N!$ number of permuted DAFT matrices, which can be used in place of the original DAFT matrix $\mathbf{A} = \mathbf{A}_1$ of the AFDM modulator, without changing any of the delay-Doppler and diversity properties of the AFDM waveform. Therefore, in the proposed AFDM-CPIM scheme, additional information is encoded by selecting a specific permuted DAFT matrix to modulate the symbols from a given codebook of K unique permuted DAFT matrices. Namely, the permuted DAFT matrix codebook is described by $\mathcal{A} \triangleq \{\mathbf{A}_{q_k}\}_{k=1}^{K}$ whose selected index of K permutations is described by the index vector $\mathbf{q} \triangleq \{q_1, \cdots, q_k, \cdots, q_K\}$, where $k^* \in \mathcal{K} \triangleq \{1, \cdots, K\}$ denotes the specifically selected index for transmission. Clearly, K can be any value between $K = 2$ (binary CPIM) and $K = 2^{\log_2(N!)}$, which for even moderate values of N becomes immensely large, highlighting the potential of the proposed method.

In summary, the AFDM-CPIM efficiently encodes a source bit sequence of length $B \triangleq N \log_2 M + \log_2(K)$ into the transmit signal by using the first $B_1 \triangleq N \log_2 M$ bits to map N complex symbols from an M-ary constellation, then using the remaining $B_2 \triangleq \log_2(K)$ bits to select the index k^* of the permuted DAFT matrix from the codebok $\mathcal{A}$ used to modulate the symbols, as illustrated in a schematic diagram in Fig. 1.

B. MMSE-based Reduced-Complexity Detector

The complementary AFDM-CPIM detector with perfect channel state information (CSI) $\mathbf{H}$, consequently must detect the correct DAFT matrix from a codebook of K valid matrices, in addition to the N complex symbols from an M-ary constellation $\mathcal{X}$. The corresponding ML problem is given by

$$\tilde{\mathbf{x}}, \tilde{k}^* = \underset{\substack{k\in\mathcal{K}\\ \mathbf{x}\in\mathcal{X}^N}}{\mathrm{argmin}} \big\| \mathbf{y}_k - \mathbf{A}_k \mathbf{H} \mathbf{A}_k^{-1} \mathbf{x} \big\|_2^2 = \underset{\substack{k\in\mathcal{K}\\ \mathbf{x}\in\mathcal{X}^N}}{\mathrm{argmin}} \big\| \mathbf{r} - \mathbf{H} \mathbf{A}_k^{-1} \mathbf{x} \big\|_2^2, \tag{9}$$

where $\mathbf{y}_k \triangleq \mathbf{A}_k \mathbf{r} \in \mathbb{C}^{N\times 1}$ is the demodulated AFDM signal with the k-th permutated DAFT matrix.

However, the above ML detection requires a discrete search space KM^N, such that the detection complexity order is $\mathcal{O}(KM^N)$, which is evident to quickly become infeasible for practical communications systems with large number of subcarriers N. Therefore in the following, we propose a novel reduced-complexity ML detector leveraging a matched minimum mean-squared-error (MMSE) filter bank.

The MMSE filter of the system for $\mathbf{x}$ in eq. (6) with the k-th permuted DAFT matrix $\mathbf{A}_k$, is given by

$$\begin{aligned}\mathbf{M}_k &\triangleq (\mathbf{H}\mathbf{A}_k^{-1})^{\mathsf{H}} \big((\mathbf{H}\mathbf{A}_k^{-1})(\mathbf{H}\mathbf{A}_k^{-1})^{\mathsf{H}} + N_0 \mathbf{I}_N \big)^{-1} \\ &= \mathbf{A}_k \mathbf{H}^{\mathsf{H}} \big(\mathbf{H}\mathbf{H}^{\mathsf{H}} + N_0 \mathbf{I}_N \big)^{-1} \in \mathbb{C}^{N\times N}\end{aligned} \tag{10}$$

where N_0 is the spectral density of the AWGN vector $\mathbf{w}$.

In turn, the estimated symbol vector with the k-th MMSE filter is given by

$$\tilde{\mathbf{x}}_k^{\mathrm{MMSE}} \triangleq \mathcal{P}_{\mathcal{X}}(\mathbf{M}_k \mathbf{r}), \tag{11}$$

where $\mathbf{M}_k \mathbf{r} \in \mathbb{C}^{N\times 1}$ is the raw MMSE-equalized symbol vector, *i.e.,* the soft-estimate of the symbol vector, and $\mathcal{P}_{\mathcal{X}}(\cdot)$ is the projection operator to the discrete set $\mathcal{X}$, *i.e.,* $\tilde{\mathbf{x}}_k^{\mathrm{MMSE}}$ is the hard-decided symbol vector.

In light of the above, the optimal k-th permutation is determined by first applying all K MMSE filters to the received signal, then solving the reduced ML problem as

$$\tilde{k}^* = \underset{k\in\mathcal{K}}{\mathrm{argmin}} \big\| \mathbf{r} - \mathbf{H}\mathbf{A}_k \tilde{\mathbf{x}}_i^{\mathrm{MMSE}} \big\|_2^2, \tag{12}$$

which only requires K evaluations of the ML metric, albeit the K applications of the MMSE filter such that the detection complexity is of order $\mathcal{O}(KN^3)$.

The proposed reduced-complexity MMSE-ML detector is fully described in Algorithm 1, which can achieve a complexity order reduction from $\mathcal{O}(KM^N)$ of the full ML problem in eq. (9) to $\mathcal{O}(KN^3)$, with a trade-off in optimality of the solution in relaxing the ML symbol estimation of via the MMSE estimator. Consequently, in Section IV-B, a quantum algorithm is designed to address the full ML problem of eq. (9) to alleviate the challenging computational complexity and enable the full potential of the proposed AFDM-CPIM scheme.

Algorithm 1: Proposed MMSE-ML AFDM-CPIM Detector

Inputs: Received signal $\mathbf{r}$, permuted DAFT Codebook $\mathcal{A}$, constellation $\mathcal{X}$, channel matrix $\mathbf{H}$, noise variance N_0.
Outputs: Estimated symbol vector $\tilde{\mathbf{x}}$ and codeword index $\tilde{k}^*$

Compute the MMSE filters $\mathbf{M}_k$ $\forall k$ via eq (10);
Obtain the MMSE-equalized vector $\tilde{\mathbf{x}}_k$ $\forall k$ via eq. (11);
Compute the ML metrics $\forall k$ in eq. (12);
Determine the index $k_{\min}$ with the smallest ML metric;
Output $\tilde{k}^* \leftarrow k_{\min}$ as the estimated codeword;
Output $\tilde{\mathbf{x}} \leftarrow \tilde{\mathbf{x}}_{k_{\min}}^{\mathrm{MMSE}}$ as the estimated symbol vector;

IV. Quantum-Acceleration for AFDM-CPIM

In this section, two quantum algorithms are formulated to address the optimal design problem of the permuted DAFT matrix codebook of AFDM-CPIM systems, and the full ML detection problem as outlined in eq. (9), which are highly challenging due to the inherently large discrete solution space. Specifically, we reformulate the two problems as a polynomial binary optimization problem, which enable its solution to be obtained via the GAS quantum algorithm, thus surpassing the computational complexity barrier of classical optimization.

In particular, GAS [10] can solve binary optimization problems with a query complexity $\mathcal{O}(\sqrt{N})$, where $N = 2^n$ is the size of the search space and n the number of binary variables. The procedure of GAS is described in Algorithm 2 below, assuming a fault-tolerant quantum computer equipped with $n+m$ qubits, where m qubits are used for encoding a general objective function value $E(\mathbf{b})$. The method for constructing a quantum circuit of GAS for polynomial binary optimization problems, consisting of a state preparation operator $\mathbf{S}_{y_i}$ and the Grover rotation $\mathbf{R}$, was introduced in [10], extended to real-value variables in [11] and tested on a real quantum computer.

A. Quantum-Accelerated Optimized Codebook Design

In the proposed AFDM-CPIM scheme, K permuted DAFT matrices out of $N!$ candidates are selected as valid codewords, which can convey additional $\log_2(K)$-bit of information. Here, the system performance can be improved by carefully selecting the codewords according to an appropriate distance metric.

It is well known that the minimum distance between two distinct codewords $d_{\min}$, generally becomes the dominant performance factor of wireless communication systems, especially in middle to high SNR region. Therefore, the problem of designing an optimized codebook can be defined as follows.

Given a distance $d_{i,j}$ between a pair of codewords $\mathbf{A}_i$ and $\mathbf{A}_j$, it is required to find a codebook $\mathcal{A}$ such that $|\mathcal{A}| = K$ and $d_{\min} = \min_{i \neq j} \{d_{i,j} \mid \{\mathbf{A}_i, \mathbf{A}_j\} \subseteq \mathcal{A}\}$ is maximized. This problem can be formulated as a polynomial binary optimization problem [10] with the objective function

$$E(\mathbf{b}) = \sum_{i,j} b_i b_j \big((d_{i,j})^{-1}\big)^{\lambda_1} + \lambda_2 \Big(\sum_i b_i - K\Big)^2, \tag{13}$$

Algorithm 2: GAS Algorithm [10], [11]

Inputs: Polynomial binary objective function $E : \mathbb{F}_2^n \to \mathbb{Z}$ w/ # of binary variables n, scaling factor $\lambda = 8/7$.
Outputs: Binary vector solution $\mathbf{b}$.

Sample $\mathbf{b}_0 \in \mathbb{F}_2^n$, set $y_0 = E(\mathbf{b}_0)$, $k = 1$, and $i = 1$;
repeat
  Randomly select the rotation count L_i from the set $\{0, 1, \cdots, \lceil k-1 \rceil\}$;
  Evaluate $\mathbf{R}^{L_i} \mathbf{S}_{y_i} |0\rangle_{n+m}$ to obtain $\mathbf{b}$ and $y = E(\mathbf{b})$;
  if $y < y_i$ **then**
    $\mathbf{b}_{i+1} = \mathbf{b}$, $y_{i+1} = y$, and $k = 1$;
  else
    $\mathbf{b}_{i+1} = \mathbf{b}_i$, $y_{i+1} = y_i$, and $k = \min\{\lambda k, \sqrt{2^n}\}$;
  end if
  $i = i + 1$;
until a termination condition is met

where $\mathbf{b} = (b_1, \cdots, b_{N!})$ are the binary variables representing whether the i-th DAFT matrix codeword $\mathbf{A}_i$ is included in the codebook or not, and λ_1 and λ_2 are penalty coefficients.

The first term in eq. (13) corresponds to the objective of maximizing $d_{\min}$, while the second term corresponds to the constraint of the codebook size K. Then, assuming $d_{i,j} > 1$ for arbitrary indices (i, j), and $\lambda_1 \gg 1$, the minimum distance $d_{\min}$ of the optimal codebook solution of eq. (13) is maximized. This can be easily justified as follows.

Let $d_{\min,1}$ be the maximum possible $d_{\min}$, and $d_{\min,2}$ be the second maximum possible $d_{\min}$. By ignoring the second term in eq. (13) without the loss of generality, the upper bound of eq. (13) for the codebook with $d_{\min} = d_{\min,1}$ is given by $E(\mathbf{b}) \leq \bar{E}(\mathbf{b}) = K\big((d_{\min,1})^{-1}\big)^{\lambda_1}$, while the lower bound of eq. (13) for the codebook with $d_{\min} \leq d_{\min,2}$ is given by $E(\mathbf{b}) \geq \underline{E}(\mathbf{b}) = \big((d_{\min,2})^{-1}\big)^{\lambda_1}$.

When λ_1 is sufficiently large, $\bar{E}(\mathbf{b})$ is always less than $\underline{E}(\mathbf{b})$, and the optimal codebook obtained from eq. (13) with the minimum objective function value satisfies $d_{\min} = d_{\min,1}$.

Finally, by applying the aforementioned GAS in Alg. 2 to the above formulation, the query complexity is expressed as $\mathcal{O}\big(\sqrt{2^{N!}}\big)$, while the query complexity of the classical exhaustive search is given by $\mathcal{O}\big(\binom{N!}{K}\big)$.

B. Quantum-Accelerated ML Detection

In this section, the full ML detection problem of AFDM-CPIM described in eq. (9) is reformulated as a binary optimization problem using the approach proposed in [11], and a novel method to utilize multiple quantum devices in parallel in order to solve the intractable ML problem is proposed.

Following the AFDM-CPIM transmitter structure of Sec. III-A, the symbol vector $\mathbf{x} \in \mathbb{C}^{N \times 1}$ can be described as being mapped from a bit sequence $\mathbf{b} = [b_1, \cdots, b_{B_1}] \in \mathbb{B}^{B_1}$ of length $B_1 = N \log_2 M$, with the mapping function given by

$$\mathbf{x} = g(\mathbf{b}) = g(b_1, \cdots, b_{B_1}), \tag{14}$$

as specified in the 5G New Radio (5GNR) standard [17].

To provide one example, the binary phase shift keying (BPSK) mapping function $\mathbf{x} = g_{\mathrm{BPSK}}(\mathbf{b})$ is given by

$$x_t = \tfrac{1}{\sqrt{2}}[(1 - 2b_t) + j(1 - 2b_t)], \tag{15}$$

for $t = \{0, 1, \cdots, N-1\}$.

Next, we assume that K quantum devices can be used in parallel to solve the ML detection problem of eq. (9), where the binary objective function for the k-th device is denoted by

$$E^{(k)}(\mathbf{b}) \triangleq \big\|\mathbf{r} - \mathbf{H}\mathbf{A}_k^{-1} g(\mathbf{b})\big\|_2^2. \tag{16}$$

By leveraging GAS at each k-th quantum device, the binary solution $\mathbf{b}^{(k)}$ that minimizes the objective function $E^{(k)}$ is obtained. Then, the optimal index k^* is evaluated via comparing the optimal value of the objective function across the K quantum devices and determining the minimum, as

$$k^* = \underset{k \in \mathcal{K}}{\operatorname{argmin}}\ E^{(k)}(\mathbf{b}^{(k)}), \tag{17}$$

from which the optimal solution of the binary vector is consequently determined as $\mathbf{b}^{(k^*)}$.

The ML detection problem of eq. (16) can be formulated as a quadratic function for the BPSK or quadrature phase shift keying (QPSK) case, but must be formulated as a higher-order function for higher-order modulations, which cannot be solved by mathematical programming solvers such as CPLEX[3]or by quantum annealing. By contrast, the GAS supports binary objective functions of any order [11] and achieves a query complexity of $\mathcal{O}(K\sqrt{M^N})$, which is significantly lower than the classical computational complexity of $\mathcal{O}(KM^N)$.

V. Performance Assessment

In Fig. 2, the performance of the proposed AFDM-CPIM is compared against the classical AFDM in terms of the bit-error-rate (BER) with respect to the energy-per-bit-to-noise-spectral-density ratio (E_b/N_0), where the results have been obtained via the proposed MMSE-ML detector in Alg. 1.

It is observed that the proposed AFDM-CPIM outperforms the classical AFDM for sufficient E_b/N_0, benefiting from the additional chirp-permutation information bits embedded without the loss of subcarrier resources. Specifically, the theoretical E_b/N_0 gain given K, can be obtained as $(1 + \frac{\log_2 K}{N \log_2 M})$ dB, where K can be maximally upto $N!$, yielding a maximum attainable gain of 6.7 dB for the system in Fig. 2.

However, a performance degredation is also visible for low E_b/N_0, owing to the imperfection of the MMSE, although its effect is expected to be reduced for increasing system size N.

Next, in Fig. 3a, the BER comparsion of two optimized codebooks under different exemplary distance metrics is provided, namely the Frobenius distance $d_{i,j} \triangleq ||\mathbf{A}_i - \mathbf{A}_j||_F$ and the angular distance $d_{i,j} \triangleq |\mathrm{tr}(\mathbf{A}_i^{\mathsf{H}}\mathbf{A}_j)|^{-1}$, where it is found that the latter provides a superior performance under the MMSE-ML detector. However, the optimal codebook design should also consider the effective distance of the effective channels $\mathbf{G}_i$ and $\mathbf{G}_j$, and more elaborate distance metrics.

In addition, Fig. 3b presents the pairwise angular distances of a small system ($N = 4$) employing binary AFDM-CPIM, *i.e.,* $K = 2$, which clearly illustrates certain codeword pairs with maximal distance. The figure helps to anticipate the potential scale of the optimization problem for large K and N, requiring a search of K elements over the $N! \times N!$ grid.

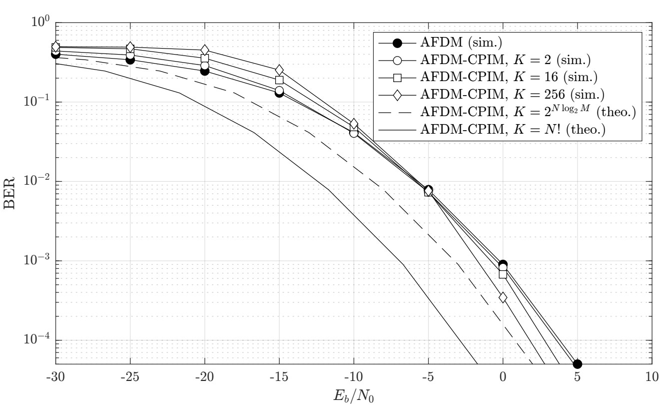
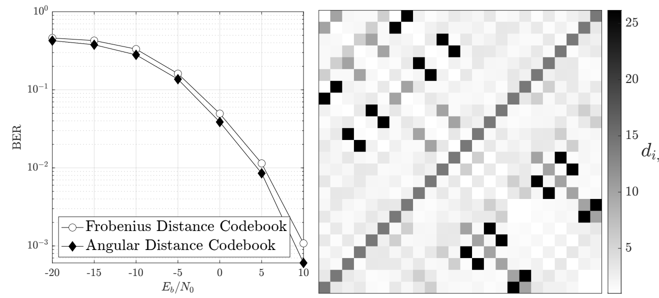


Fig. 2: BER performance of AFDM-CPIM for different simulated and theoretical values of K, over a doubly-dispersive system with $P = 3, \ell^{\max} = 3, f^{\max} = 3, N = 32, M = 2$.



(a) Codebook BER comparison with $N = 8$, $K = 2$, $M = 4$.

(b) Pairwise angular distances $d_{i,j}$, with $N = 4$, $N! = 24$, $K = 2$.

Fig. 3: BER performance and pairwise distance analysis of the optimized AFDM-CPIM codebook.

VI. Conclusion

We proposed a novel IM scheme based on a unique feature inherent to AFDM. The proposed AFDM-CPIM scheme is shown to provide a noticeable performance gain over conventional AFDM, except for the computational complexity at large scales. To address this issue, quantum-accelerated solutions are proposed both for the codebook design and ML detection. An analysis of the pairwise codeword distances suggests that the proposed CPIM scheme can be exploited to realize efficient multi-user systems. More details on the methods here proposed will be provided in the camera-ready, and the extension to multi-user scenarios will be exploited in a follow-up article.

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