SECOND RADIAL EIGENFUNCTIONS TO A FRACTIONAL DIRICHLET PROBLEM AND UNIQUENESS FOR A SEMILINEAR EQUATION

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ABSTRACT. We analyze the shape of radial second Dirichlet eigenfunctions of fractional Schrödinger type operators of the form $(-\Delta)^s + V$ in the unit ball B in $\mathbb{R}^N$ with a nondecreasing radial potential V. Specifically, we show that the eigenspace corresponding to the second radial eigenvalue is simple and spanned by an eigenfunction u which changes sign precisely once in the radial variable and does not have zeroes anywhere else in B. Moreover, by a new Hopf type lemma for supersolutions to a class of degenerate mixed boundary value problems, we show that u has a nonvanishing fractional boundary derivative on ∂B. We apply this result to prove uniqueness and nondegeneracy of positive ground state solutions to the problem $(-\Delta)^s u + \lambda u = u^p$ on B, $u = 0$ on $\mathbb{R}^N \setminus B$. Here $s \in (0,1)$, $\lambda \geq 0$ and $p > 1$ is strictly smaller than the critical Sobolev exponent.

1. INTRODUCTION

Let $s \in (0,1)$ and $B := \{x \in \mathbb{R}^N : |x| < 1\}$ denote the unit ball in $\mathbb{R}^N$. The present paper is devoted to oscillation estimates of the radial second eigenfunctions in the eigenvalue problem

$$(-\Delta)^s w + Vw = \sigma w \quad \text{in } B, \qquad w \equiv 0 \quad \text{in } \mathbb{R}^N \setminus B. \tag{1.1}$$

Here $(-\Delta)^s$ denotes the fractional Laplacian of order s, which, under appropriate smoothness and integrability assumptions on the function w, is pointwisely given by

$$(-\Delta)^s w(x) = c_{N,s} \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{w(x) - w(y)}{|x-y|^{N+2s}} dy \tag{1.2}$$

with $c_{N,s} = 2^{2s} \pi^{-\frac{N}{2}} s \frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}$. Moreover, we consider (1.1) in weak sense. So, by definition, an eigenfunction u of (1.1) is contained in the Sobolev space

$$\mathcal{H}^s(B) := \{w \in H^s(\mathbb{R}^N) : w \equiv 0 \text{ in } \mathbb{R}^N \setminus B\},$$

and it satisfies

$$[w,v]_s + \int_B Vwv \, dx = \sigma \int_B wv \, dx \qquad \text{for all } v \in \mathcal{H}^s(B).$$

Here

$$(v_1, v_2) \mapsto [v_1, v_2]_s = c_{N,s} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(v_1(x) - v_1(y))(v_2(x) - v_2(y))}{|x-y|^{N+2s}} dxdy \tag{1.3}$$

denotes the bilinear form associated with the fractional Laplacian, and we shall also write $[v]_s^2 := [v,v]_s$ in the following. Moreover, $H^s(\mathbb{R}^N)$ is the usual fractional Sobolev space of functions $w \in L^2(\mathbb{R}^N)$ with $[w]_s^2 < \infty$. Here we note that the bilinear form $[\cdot,\cdot]_s$ can also be represented via Fourier transform by

$$[v_1, v_2]_s = \int_{\mathbb{R}^N} |\xi|^{2s} \widehat{v_1}(\xi) \widehat{v_2}(\xi) \, d\xi. \tag{1.4}$$

If $V \in L^q(B)$ for some $q > \max(\frac{N}{2s}, 1)$, then

$$\text{the embedding } \mathcal{H}^s(B) \hookrightarrow L^2(B; V dx) \text{ is compact.} \tag{1.5}$$

This follows since $2q' < 2^*_s$ in this case, where 2^*_s is the critical fractional Sobolev exponent given by

$$2^*_s = \frac{2N}{N-2s} \ \text{ if } 2s < N \qquad \text{and} \qquad 2^*_s = +\infty \ \text{ if } 2s \geq 1 = N.$$

Indeed, we then have a compact Sobolev embedding $\mathcal{H}^s(B) \hookrightarrow L^{2q'}(B)$ and a continuous embedding $L^{2q'}(B) \hookrightarrow L^2(B; V dx)$, the latter being a consequence of Hölder's inequality.

If, in addition, V is a radially symmetric function, then it follows from (1.5) and a classical argument that there exists a sequence of discrete eigenvalues of (1.1) corresponding to radial eigenfunctions. These eigenvalues are given through the min-max characterization

$$\sigma_k(V) = \inf_{\substack{\mathcal{S} \subset \mathcal{H}^s_{rad}(B) \\ \dim(\mathcal{S}) = k}} \sup_{w \in \mathcal{S} \setminus \{0\}} \frac{[w]^2_s + \|w\|^2_{L^2(B; V dx)}}{\|w\|^2_{L^2(B)}}, \qquad k \geq 1, \tag{1.6}$$

where $\mathcal{H}^s_{rad}(B)$ is the closed subspace of radial functions in $\mathcal{H}^s(B)$. It is well-known that the first eigenvalue $\sigma_1(V)$ is simple, and the corresponding eigenspace is spanned by a positive eigenfunction w_1. Moreover, from (1.6) one may, by a standard argument, obtain the alternative useful representation

$$\sigma_2(V) = \inf_{\substack{w \in \mathcal{H}^s_{rad}(B) \\ \langle w, w_1 \rangle_{L^2(B)} = 0}} \frac{[w]^2_s + \|w\|^2_{L^2(B; V dx)}}{\|w\|^2_{L^2(B)}}. \tag{1.7}$$

In the following, we wish to derive qualitative properties of eigenfunctions of (1.1) corresponding to the eigenvalue $\sigma_2(V)$. Up to now, few results about simplicity of Dirichlet eigenvalues and oscillation estimates of Dirichlet eigenfunction of the operator $(-\Delta)^s + V$ in B are available, even in the simple case $V \equiv 0$. Indeed, for $N = 1$, the papers [19, 20] first proved simplicity of $\sigma_k(0)$ for $s \in [1/2, 1)$. This result is recently extended to all $s \in (0, 1)$ in [11], where also generic simplicity of Dirichlet eigenvalues in smooth domains was proven. Finally, the simplicity of $\sigma_k(0)$, for all $k \geq 1$, has been recently proven in [8].

The first main result of the paper is the following. For simplicity, we write $B_r := B_r(0)$ for $r > 0$ from now on.

Theorem 1.1. *Suppose that, for some $q > \max(\frac{N}{2s}, 1)$ and $\beta > 0$,*

$$V \in L^q(B) \cap C^\beta_{loc}(B) \textit{ is radial and radially nondecreasing.} \tag{1.8}$$

Then $\sigma_2(V)$ is simple, and the associated eigenspace is spanned by an eigenfunction w_2 which changes sign exactly once in the radial variable. More precisely, there exists $r_0 \in (0, 1)$ with the property that $w_2 > 0$ on B_{r_0} and $w_2 < 0$ on $B \setminus \overline{B_{r_0}}$. Moreover, the function $w_2\big|_{B_{r_0}}$ is decreasing in the radial variable. In addition, we have

$$\psi_{w_2}(1) < 0, \tag{1.9}$$

where $\psi_{w_2}(1) := \liminf_{|x| \nearrow 1} \dfrac{w_2(x)}{(1 - |x|)^s}$.

Theorem 1.1 should be compared with Theorems 1 and 2 in the paper [16] of Frank, Lenzmann and Silvestre. These theorems are concerned with radial second eigenfunctions of the operator $(-\Delta)^s + V$ in the entire space, see also [17] for the case $N = 1$. Assuming that

$$V \in C^\beta(\mathbb{R}^N) \text{ for some } \beta > \max\{0, 1-2s\},\ V \text{ is radial and radially nondecreasing,} \tag{1.10}$$

it is shown in [16, Theorem 1] that the equation

$$(-\Delta)^s w + Vw = 0 \qquad \text{in } \mathbb{R}^N \tag{1.11}$$

has at most one bounded radial solution with $w(x) \to 0$ as $|x| \to \infty$ which satisfies $w(0) \neq 0$. Moreover, assuming in addition that $(-\Delta)^s + V$ has at least two radial eigenvalues below the essential spectrum, it is shown in [16, Theorem 2] that the second radial eigenvalue is simple and eigenfunctions change sign precisely once.

The proof of [16, Theorem 1] strongly relies on a Hamiltonian identity involving the s-harmonic extensions of solutions w of (1.11). Here, instead, we use a rearrangement argument to show that $w_2(0) \neq 0$ for every nontrivial second eigenfunction of (1.1), which then shows the simplicity of $\sigma_2(V)$ under assumption (1.8). It is interesting to note that this rearrangement argument can also be used for second eigenfunctions of the full space problem and applies under weaker regularity assumptions than (1.10).

Once we have established the property $w_2(0) \neq 0$, we will then use a continuation argument in two steps, starting from second radial eigenfunctions of the classical Dirichlet Laplacian, to show that w_2 changes sign precisely once. A key property used in this continuation argument is the equivalence

$$\text{(I)} \ \ w_2 \text{ changes sign precisely once} \qquad \Longleftrightarrow \qquad \text{(II)} \quad w_2(0)\int_B w_2\,dx < 0.$$

This equivalence is highly useful for the continuation argument as (I) is a closed condition while (II) is an open condition in an appropriate norm. A further open condition is given by (1.9), but (II) is easier to use when considering continuous dependence on parameters. Therefore we will not use (1.9) in the continuation argument. In fact, (1.9) will be established independently as a consequence of a more general Hopf type lemma, see Theorem 5.2 below. We point out the use of a continuation argument is inspired by the proof of [16, Theorem 2], but the argument itself is quite different. For a more detailed comparison, see Remark 3.6 below.

We also mention that Frank, Lenzmann and Silvestre used their analysis in [16] on second radial eigenfunctions to prove uniqueness and nondegeneracy of ground state solutions up to translations of the semilinear equation

$$(-\Delta)^s u + \lambda u = u^p \quad \text{in } \mathbb{R}^N, \qquad u > 0 \quad \text{in } \mathbb{R}^N, \qquad u \in H^s(\mathbb{R}^N), \tag{1.12}$$

where $\lambda > 0$ and $p \in (1, 2_s^* - 1)$, see Theorems 3 and Theorem 4 in [16] and also [17] for earlier work on the case $N = 1$. In the present paper, we shall use Theorem 1.1 to derive the nondegeneracy and uniqueness of ground state solutions to the problem

$$(-\Delta)^s u + \lambda u = u^p \quad \text{in } B, \qquad u > 0 \quad \text{in } B, \qquad u = 0 \quad \text{in } \mathbb{R}^N \setminus B, \tag{1.13}$$

for $\lambda \geq 0$ and $p \in (1, 2_s^* - 1)$. Here, by a ground state solution, we mean solutions u to (1.13) satisfying

$$[w]_s^2 + \lambda \|w\|_{L^2(B)}^2 - p\int_B u^{p-1}w^2\,dx \geq 0 \quad \text{for all } w \in \mathcal{H}^s(B) \text{ with } \int_B u^p w dx = 0. \tag{1.14}$$

We note that this class of solutions include least energy solutions to (1.13). Moreover, using the variational characterization (1.6), it is easy to see that (1.14) is equivalent to

$$\sigma_2(-pu^{p-1}) \geq -\lambda \tag{1.15}$$

Here we note that $V = -pu^{p-1}$ satisfies (1.8) if $u \in \mathcal{H}^s(B)$ solves (1.13), see Section 4 below. We have the following result, which provides an analogue of [16, Theorems 3 and 4] for the fractional Dirichlet problem (1.13) in the unit ball.

Theorem 1.2. *Let $s \in (0,1)$, $\lambda \geq 0$ and $1 < p < 2_s^* - 1$. Then (1.13) possesses a unique ground state solution $u \in \mathcal{H}^s(B)$. Moreover u is nondegenerate, i.e., the linearized problem*

$$(-\Delta)^s w + \lambda w - pu^{p-1}w = 0 \quad \text{in } B, \qquad u \equiv 0 \quad \text{on } \mathbb{R}^N \setminus B \tag{1.16}$$

only has the trivial solution $w \equiv 0$.

We wish to mention some further results related to this theorem. For the full space problem (1.12), uniqueness up to translation in the class of all positive solutions is, up to now, only known for $N = 1$, $s = 1/2$ and $p = 2$, see Amick and Toland [2]. This stands in striking contrast to the local case $s = 1$, in which Kwong has proved uniqueness of positive solutions for the corresponding versions of (1.12) and (1.13) with the help of an ODE analysis. We point out that ODE methods are not applicable in the present nonlocal setting.

For the Dirichlet problem (1.13) in a ball, only very recent results are available. In particular, it has been proved in [7] that (1.13) admits a unique solution which is nondegenerate if s and p belongs to a borderline range of parameters. More precisely, it is assumed in [7] that s is close to 1 or p is close to 1 or $2_s^* - 1$. Moreover, very recently in [6], it is shown, by a compactness argument based on the uniqueness result of [16] for (1.12), that (1.13) with $\lambda > 0$ admits a unique ground state solution if B is replaced with a sufficiently large ball. In our very recent paper [9], we have proved Theorem 1.2 in the special case $N = 1$. Moreover, also in [9], we have shown unique solvability of the fractional one-dimensional Lane-Emden equation, i.e., of (1.13) in the special case $N = 1$ and $\lambda = 0$, within the class of *all* positive solutions. Also very recently and independently, the assertion of Theorem 1.2 was shown in [5] in the special case $\lambda = 0$.

We point out that our argument to derive Theorem 1.2 from Theorem 1.1 is different from the one in [16] since we need to deal with boundary terms arising when applying a fractional integration by parts formula. A useful tool is the nonradial nondegeneracy of positive solutions of (1.13) which we establish in [9] for the full range of parameters $s \in (0,1)$, $1 < p < 2^* - 1$, see also [6] for a different and independent proof. The remaining part of the proof then uses Theorem 1.1 and a fractional integration by parts formula. The key new information needed in the case $\lambda > 0$ is the fact that second radial eigenfunctions w associated with the potential function $V = -pu^{p-1}$ and eigenvalues $\sigma \leq 0$ change sign precisely once in the radial variable. In the case $N = 1$, this property can be deduced from the nonradial nondegeneracy result mentioned above. In fact, in the case $N = 1$, this property can be used to show that the s-harmonic extension W of w, as defined in Section 2 below, has the same number of nodal domains as w when regarded as a function of the radial variable, see [9] for details. A similar result is not available in the case $N > 1$, therefore we rely on Theorem 1.1. The case $\lambda = 0$ in (1.2) is different. In this case, fractional integration by parts shows that nonzero radial solutions of the linearized equation (1.16) must have a vanishing fractional normal derivative at the boundary ∂B. Therefore, the existence of such solutions can be ruled out by a fractional Hopf boundary point lemma for second radial eigenfunctions. We shall derive such a result in Proposition 3.7 below as a consequence of a more general new Hopf type lemma for supersolutions of an extended problem (in a nonradial setting). This new Hopf type lemma is given in Theorem 5.2 in the appendix, and its proof is partly inspired by the proof of [9, Lemma 5.10]. We also note that, independently and differently, a fractional Hopf boundary point lemma for second radial eigenfunctions associated with the potential function $V = -pu^{p-1}$ has been proved in [5].

The paper is organized as follows. In Section 2 we collect some useful information concerning convergence of eigenvalues and some nodal domain estimates. In Section 3, we prove simplicity of second eigenfunction and their precise nodal domain estimates. The proof of Theorem 1.2 is given in Section 4. In Section 5 we state and prove the new Hopf-type lemma mentioned above. We finally collect some topological results on curve intersection in Section 6 which are useful to estimate the number of sign changes of radial second eigenfunctions.

2. Preliminaries

Let Ω be an open bounded set of class $C^{1,1}$, and let

$$\mathcal{H}^s(\Omega) := \{v \in H^s(\mathbb{R}^N) \,:\, v \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\}.$$

We need the following uniform regularity result.

Lemma 2.1. *Let Ω be as above, let $V, F \in L^q(\Omega)$ with $q > \max(N/(2s), 1)$, and let $u \in \mathcal{H}^s(\Omega)$ satisfy $(-\Delta)^s u + Vu = F$ in Ω in weak sense, i.e.,*

$$[u, v]_s + \int_\Omega Vuv\, dx = \int_\Omega Fv\, dx \qquad \text{for all } v \in \mathcal{H}^s(\Omega).$$

Moreover, let $c_0 > 0$.

(i) If $\|V\|_{L^q(\Omega)} \le c_0$, then there exist $\alpha = \alpha(N, s, q, c_0) > 0$ and $C = C(N, s, q, c_0) > 0$ with

$$\|u\|_{C^\alpha(\mathbb{R}^N)} \le C(\|u\|_{L^2(\Omega)} + \|F\|_{L^q(\Omega)}). \tag{2.1}$$

(ii) If $F, V \in L^p(\Omega)$, with $p > \frac{N}{s}$ and $\|V\|_{L^p(\Omega)} \le c_0$, then there exists $C = C(N, s, p, c_0) > 0$ with

$$\|u\|_{C^s(\mathbb{R}^N)} + \|u/d^s\|_{C^{s-\frac{N}{p}}(\overline{\Omega})} \le C(\|u\|_{L^2(\Omega)} + \|F\|_{L^p(\Omega)}), \tag{2.2}$$

where $d(x) := dist(x, \mathbb{R}^N \setminus \Omega)$.

(iii) If $F, V \in C^\beta_{loc}(\Omega)$ with $\beta > 0$ and $2s + \beta \notin \mathbb{N}$, then $u \in C^{2s+\beta}_{loc}(\Omega)$.

Proof. By [12], we have (2.1) and (2.2). Now by interior regularity from [24] (and a bootstrap argument only necessary for $2s < 1$), we obtain (iii). □

The following is also a consequence of Lemma 2.1.

Lemma 2.2. *Let $q > \max(\frac{N}{2s}, 1)$, and let $V, V_n \in L^q(B)$, $n \in \mathbb{N}$ be radial functions satisfying $V_n \to V$ in $L^q(B)$ as $n \to \infty$. Then $\sigma_k(V_n) \to \sigma_k(V)$. Suppose moreover that $\sigma_k(V)$ is simple, and let v be an eigenfunction associated to $\sigma_k(V)$. Then any sequence $(v_n)_n$ of eigenfunctions v_n associated to $\sigma_k(V_n; B)$, normalized such that $\|v_n\|_{L^2(B)} = 1$, possesses a subsequence that converges in $C(\overline{B}) \cap \mathcal{H}^s(B)$ to κv, for some $\kappa \in \mathbb{R} \setminus \{0\}$.*

Proof. Let $b \in L^q(B)$. Since $q > \max(\frac{N}{2s}, 1)$, we have $2 < 2q' < \left[\frac{2N}{N-2s}\right]_+$, with $q' = \frac{q}{q-1}$ and therefore, by Hölder and Sobolev inequalities, we have

$$\int_B bu^2\, dx \le \|b\|_{L^q(B)} \|u\|^2_{L^{2q'}(B)} \le C\|b\|_{L^q(B)}[u]_s^2 \qquad \text{for all } u \in \mathcal{H}^s(B) \tag{2.3}$$

with a constant $C = C(N, s, q) > 0$. Since $\|v_n\|_{L^2(B)} = 1$ for all $n \in \mathbb{N}$, we deduce from (1.6) and (2.3) that

$$\sigma_k(V_n) \le \sigma_k(V) + C\|V_n - V\|_{L^q(B)}\sigma_k(0), \quad \sigma_k(V) \le \sigma_k(V_n) + C\|V_n - V\|_{L^q(B)}\sigma_k(0).$$

As a consequence, $\sigma_k(V_n) \to \sigma_k(V)$ as $n \to \infty$. In particular, this implies that $(v_n)_n$ is bounded in $\mathcal{H}^s(B)$. Therefore, by (1.5), $(v_n)_n$ converges, up to a subsequence, weakly in $\mathcal{H}^s(B)$ and strongly in $L^{2q'}(B)$, hence also strongly in $L^2(B) \cap L^2(B; V dx)$. Moreover, by weak convergence, the limit w satisfies

$$[w, \phi]_s + \int_B V w\phi \, dx = \sigma_k(V) \int_B w\phi \, dx \qquad \text{for all } \phi \in \mathcal{H}^s(B),$$

so w is an eigenfunction of (1.1) corresponding to the eigenvalue $\sigma_k(V)$. Hence, since $\sigma_k(V)$ is simple by assumption, we have $w = \kappa v$ for some $\kappa \in \mathbb{R} \setminus \{0\}$. In particular, this implies that

$$[\kappa v]_s^2 = [w]_s^2 = \int_B (\sigma_k(V) - V)w^2 \, dx = \lim_{n\to\infty} \int_B (\sigma_k(V_n) - V_n)v_n^2 \, dx = \lim_{n\to\infty} [v_n]_s^2,$$

and from this and the weak convergence we deduce that $v_n \to \kappa v$ strongly in $\mathcal{H}^s(B)$. Applying Lemma 2.1 we deduce that $v_n \to \kappa v$ in $C(\overline{B})$. □

In the following, we need to consider the *s-harmonic extension* W of a function $w \in \mathcal{H}^s(B)$, which has been introduced in [4] and is sometimes called the Caffarelli-Silvestre extension. We define $\mathbb{R}^{N+1}_+ = \{(x, t) \in \mathbb{R}^N \times \mathbb{R} : t > 0\}$. For $w \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, we define

$$W(x, t) = p_{N,s} t^{2s} \int_{\mathbb{R}^N} \frac{w(y)dy}{(t^2 + |x - y|^2)^{\frac{N+2s}{2}}} \qquad \text{with} \qquad \frac{1}{p_{N,s}} = \int_{\mathbb{R}^N} \frac{dy}{(1 + |y|^2)^{\frac{N+2s}{2}}}, \tag{2.4}$$

Then we have

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla W) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ W \in C(\overline{\mathbb{R}^{N+1}_+}), \\ \lim_{t\to 0} W(x, t) = w(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \tag{2.5}$$

In this case, we call W the *s-harmonic extension* of w. If moreover Ω is an open subset of $\mathbb{R}^N$ and $w \in C^{2s+\alpha}(\Omega)$ for some $\alpha > 0$, then $(x, t) \mapsto t^{1-2s}\partial_t W(x, t) \in C(\Omega \times [0, \infty))$ and

$$-\lim_{t\to 0} t^{1-2s}\partial_t W(x, t) = a_s(-\Delta)^s w(x) \qquad \text{for all } x \in \Omega \tag{2.6}$$

with some (explicit) positive constant a_s, where $(-\Delta)^s w(x)$ is defined pointwisely by (1.2).

Remark 2.3. *Let $D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ be the completion of $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm*

$$U \mapsto \|U\|^2_{D^{1,2}(\mathbb{R}^{N+1}_+;t^{1-2s})} = \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 t^{1-2s} \, dxdt. \tag{2.7}$$

If $w \in H^s(\mathbb{R}^N)$ is fixed, then the functional in (2.7) admits a unique minimizer in the affine subspace of functions $U \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ satisfying $U = w$ on $\mathbb{R}^N = \partial\mathbb{R}^{N+1}_+$ in trace sense. This minimizer $W \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s})$ is also called the s-harmonic extension of w, and it satisfies

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s}\nabla W \cdot \nabla\varphi \, dtdx = a_s[w, \varphi(\cdot, 0)]_s \quad \textit{for all } \varphi \in D^{1,2}(\mathbb{R}^{N+1}_+; t^{1-2s}). \tag{2.8}$$

Moreover, if, in addition, $w \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, then W coincides with the s-harmonic extension defined pointwisely by (2.4) above.

If $w \in \mathcal{H}^s(B)$ is an eigenfunction of (1.1), then, by Lemma 2.1 and the remarks above, (2.5), (2.6) and (2.8) are true for the s-harmonic extension W of w, which then is contained in the space

$$D_B^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}) = \{U \in D^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}) \,:\, U(\cdot, 0) = 0 \text{ on } \mathbb{R}^N \setminus B\}.$$

Moreover, if w is radial, then the function W is radial in the x-variable. In the following, we need some information on the nodal structure of W in the case where $w = w_2$ is a second eigenfunction of (1.1).

Definition 2.4. *Let $W \in C(\overline{\mathbb{R}_+^{N+1}})$. We call a subset $\mathcal{O} \subset \overline{\mathbb{R}_+^{N+1}}$ a nodal domain of W if $\mathcal{O}$ is a connected component of the set $\{(x,t) \in \overline{\mathbb{R}_+^{N+1}} \,:\, W(x,t) \neq 0\}$.*

We first note the following result which is essentially contained in [16].

Lemma 2.5. *Let $V \in L^q(B)$, with $q > \max(\frac{N}{2s}, 1)$, be a radial function, let $w_2 \in \mathcal{H}^s_{rad}(B)$ be an eigenfunction of (1.1) corresponding to the eigenvalue $\sigma_2 = \sigma_2(V)$, and let W_2 be its s-harmonic extension. Then W_2 has precisely two nodal domains. More precisely, the sets $\{(x,t) \in \overline{\mathbb{R}_+^{N+1}} \,:\, \pm W_2 > 0\}$ are connected, nonempty and intersect the set $B \times \{0\}$.*

Proof. Recalling Remark 2.3 and (2.8), we have the variational characterization

$$\sigma_2(g) a_s = \frac{\int_{\mathbb{R}_+^{N+1}} |\nabla W_2|^2 t^{1-2s} dt dx - a_s \int_B W_2^2 V dx}{\int_B W_2^2 dx} = \inf_{U \in M} \frac{\int_{\mathbb{R}_+^{N+1}} |\nabla U|^2 t^{1-2s} dt dx - a_s \int_B U^2 V dx}{\int_B U^2 dx},$$

where

$$M = \left\{ U \in D_B^{1,2}(\mathbb{R}_+^{N+1}; t^{1-2s}) \setminus \{0\} \quad : \quad \int_B U W_1 dx = 0, \quad U(\cdot, t) \text{ is radial} \right\}$$

and W_1 is the s-harmonic extension of w_1, which achieves the infinimum

$$\inf_{U \in D_B^{1,2}(\mathbb{R}^{N+1}; t^{1-2s})} \frac{\int_{\mathbb{R}_+^{N+1}} |\nabla U|^2 t^{1-2s} dt dx - a_s \int_B U^2 V dx}{\int_B U^2 dx}.$$

By the same argument as in [16, Prop. 5.2], it then follows that W_2 has at most two nodal domains in $\mathbb{R}_+^{N+1}$. Since $w_2 = W_2(\cdot, 0)$ changes sign and $W_2 \in C(\overline{\mathbb{R}_+^{N+1}})$, we see that W_2 has precisely two nodal domains $\{W_2 > 0\}$ and $\{W_2 < 0\}$ in $\overline{\mathbb{R}_+^{N+1}}$. To see that these nodal domains intersect the set $B \times \{0\}$, we argue by contradiction and suppose that $\{W_2 > 0\} \cap B \times \{0\} = \varnothing$. Then $\varphi = W_2 1_{\{W_2 > 0\}} \equiv 0$ on $\mathbb{R}^N \times \{0\}$, and by Remark 2.3, we may use (2.8) with $\varphi = W_2 1_{\{W_2 > 0\}}$ to obtain that

$$\int_{\{W_2 > 0\}} t^{1-2s} |\nabla W_2|^2 d(x,t) = 0.$$

This in turn implies that W_2 is constant in $\mathcal{O}$. Hence, by continuity, $W_2 \equiv 0$ in $\{W_2 > 0\}$ which is not possible. Hence $\{W_2 > 0\} \cap B \times \{0\} \neq \varnothing$, and in the same way we see that $\{W_2 < 0\} \cap B \times \{0\} \neq \varnothing$. □

We also recall the following definition.

Definition 2.6. *Let $L \geq 1$ be an integer and let $w \in C(B)$ be radial, i.e., $w(x) = \widetilde{w}(|x|)$ with some $\widetilde{w} \in C(0,1)$. We say that w changes sign at least L times in the radial variable if there exists $y_i \in (0,1)$, for $i = 0, \dots, L$ with $y_0 < y_1 < \cdots < y_L$ and such that $\widetilde{w}(y_i)\widetilde{w}(y_{i+1}) < 0$ for $i = 0 \dots, L-1$. We also say L changes sign precisely L times in the radial variable if L is the largest number with this property.*

The following is a rather direct consequence of Lemma 2.5 and Lemma 6.1 below, see also [16, 17].

Corollary 2.7. *Let $V \in L^q(B)$, with $q > \max(\frac{N}{2s}, 1)$, be a radial function, and let $w_2 \in \mathcal{H}^s_{rad}(B)$ be an eigenfunction of (1.1) corresponding to the eigenvalue $\sigma_2 = \sigma_2(V)$. Then w_2 changes sign at most twice in the radial variable.*

3. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. We start with the following uniqueness result for radial solutions to (1.1). We start with the following simple lemma, which we shall use multiple times in the following.

Lemma 3.1. *Let V satisfy* (1.8)*, let $w \in \mathcal{H}^s(B)$ be an eigenfunction of (1.1) corresponding to the eigenvalue σ and let W be its s-harmonic extension. If $w(x_0) = 0$ for some $x_0 \in B$, then W changes sign in every relative neighborhood N of* $(x_0, 0)$ *in* $\overline{\mathbb{R}^{N+1}_+}$.

We point out that neither V nor w needs to be radial here.

Proof. We first claim that

$$W \text{ takes negative values in any relative neighborhood } N \text{ of } (x_0, 0) \text{ in } \overline{\mathbb{R}^{N+1}_+}. \tag{3.1}$$

To show this, we suppose by contradiction that there exists a relative neighborhood N of $(x_0, 0)$ in $\overline{\mathbb{R}^{N+1}_+}$ with $W \geq 0$ in N. We have $W \not\equiv 0$ in N since otherwise $W \equiv 0$ in $\overline{\mathbb{R}^{N+1}_+}$ by unique continuation (see e.g. [14]) and therefore $w \equiv 0$, which is impossible. Hence the strong maximum principle implies that $W > 0$ in $N \cap \mathbb{R}^{N+1}_+$. Consequently, since we assume that $w(x_0) = W(x_0, 0) = 0$, it follows from [3, Proposition 4.11] that $-\lim_{t \to 0} t^{1-2s} \partial_t W(x_0, 0) < 0$. On the other hand, by Lemma 2.1(iii) we have $w \in C^{2s+\alpha}_{loc}(B)$, and (2.6) yields

$$-\lim_{t \to 0} t^{1-2s} \partial_t W(x_0, 0) = a_s(-\Delta)^s w(x_0) = (-V(x_0) + \sigma) w(x_0) = 0.$$

This contradiction proves (3.1). Moreover,replacing w with $-w$ and W with $-W$ shows that W also takes positive values in any relative neighborhood N of $(x_0, 0)$ in $\overline{\mathbb{R}^{N+1}_+}$. The claim thus follows. □

Theorem 3.2. *Let V satisfy (1.8) and let $w_2 \in \mathcal{H}^s_{rad}(B)$ be a radial solution to* (1.1) *with $\sigma = \sigma_2(V)$. If $w_2(0) = 0$, then $w_2 \equiv 0$ in B. As a consequence, the eigenvalue $\sigma_2(V)$ is simple.*

Proof. Suppose by contradiction that $w_2(0) = 0$ but $w_2 \not\equiv 0$. We already know that w_2 changes sign at least once and in the radial variable, since it is L^2-orthogonal to the (up to normalization unique) positive first eigenfunction.

Claim 1: w_2 changes sign only once.

To see this, we argue by contradiction and assume, without loss of generality, that there exists $0 < r_1 < r_2 < r_3 < 1$ such that $w_2(r_i) > 0$ for $i = 1, 3$ and $w_2(r_2) < 0$.
Let W_2 be the s-harmonic extension of w_2, and define $\widetilde{W} : \overline{\mathbb{R}_+ \times \mathbb{R}_+} \to \mathbb{R}$ by $\widetilde{W}(|x|, t) =$

$W_2(x,t)$. Then by Lemma 2.5, the sets $\mathcal{O}_+ := \{\widetilde{W} > 0\}$ and $\mathcal{O}_- := \{\widetilde{W} < 0\}$ are (relatively) open in $\overline{\mathbb{R}_+ \times \mathbb{R}_+}$ and connected in $\overline{\mathbb{R}_+ \times \mathbb{R}_+}$, so they are also path connected. In particular, there exists a continuous curve $\gamma : [0,1] \to \mathcal{O}_+$ joining the points $(r_1, 0)$ and $(r_3, 0)$.

Moreover, since we assume that $w_2(0) = \widetilde{W}(0,0) = 0$ and therefore $(0,0) \notin \gamma([0,1])$, we have $d := \mathrm{dist}(\gamma([0,1]), (0,0)) > 0$, and we may use Lemma 3.1 to find $z \in \overline{\mathbb{R}_+ \times \mathbb{R}_+}$ with $|z| < d$ and $\widetilde{W}(z) < 0$. By path connectedness of $\mathcal{O}_-$, we then find a continuous curve $\eta : [0,1] \to \mathcal{O}_-$ joining z and $(r_2, 0)$. By Lemma 6.2 below applied to the points $0 < r_1 < r_2 < r_3$, this curve must intersect γ. This, however, is impossible since $\mathcal{O}_+ \cap \mathcal{O}_- = \varnothing$. From this contradiction, Claim 1 follows.

Next, we write v in place of w_2 to simplify the notation. As a consequence, from (1.1), we have

$$\int_{\mathbb{R}^N} (\sigma_2 - V)(v^+)^2 dx = [v^+]_s^2 - [v^+, v^-]_s \tag{3.2}$$

and

$$\int_{\mathbb{R}^N} (\sigma_2 - V)(v^-)^2 dx = [v^-]_s^2 - [v^+, v^-]_s. \tag{3.3}$$

By Claim 1, we may assume that there exists $r_0 \in (0,1)$ with $v \geq 0$, $v \not\equiv 0$ on $B_{r_0}(0)$ and $v \leq 0$, $v \not\equiv 0$ on $B_1(0) \setminus B_{r_0}(0)$. Let v_* denote the Schwarz symmetrization of the function $v^+ \in \mathcal{H}^s(B)$. Then

$$\mathrm{Supp} v_* \subset B_{r_0} \tag{3.4}$$

and by a classical result of Almgren and Lieb [1, Theorem 9.2 (i)], we have $v_* \in \mathcal{H}^s(B)$ and

$$[v_*]_s^2 \leq [v^+]_s^2. \tag{3.5}$$

We note also that[1]

$$\int_{\mathbb{R}^N} (\sigma_2 - V) v_*^2 dx \geq \int_{\mathbb{R}^N} (\sigma_2 - V)(v^+)^2 dx, \tag{3.6}$$

by the classical Hardy-Littlewood inequality (see e.g. [22, Theorem 3.4]), since the function $\sigma_2 - V$ is nonincreasing by assumption and since v_*^2 equals the Schwarz symmetization of $(v^+)^2$.

In the following, we wish to prove that

$$-[v_*, v^-]_s \leq -[v^+, v^-]_s \tag{3.7}$$

Since $v_* \equiv 0$ on $B_1(0) \setminus B_{r_0}(0)$ and $v^- \equiv 0$ on $B_{r_0}(0)$, we have, using polar coordinates,

$$\begin{aligned} -[v_*, v^-]_s &= 2c_{N,s} \int_{B_1 \setminus B_{r_0}} \int_{B_{r_0}} |x-y|^{-N-2s} v_*(x) v^-(y)\, dxdy \\ &= 2c_{N,s} \int_{r_0}^1 \rho^{N-1} v^-(\rho) \Big(\int_{B_{r_0}} v_*(x) h_\rho(x) dx \Big) d\rho, \end{aligned} \tag{3.8}$$

where, for $\rho \in (r_0, 1)$,

$$h_\rho(x) = \int_{S^{N-1}} |x - \rho y|^{-N-2s} d\sigma(y) = \Theta(|x|, \rho)$$

with

$$\Theta(r, \rho) = \int_{S^{N-1}} |re_1 - \rho y|^{-N-2s} d\sigma(y) = \frac{\alpha_N}{\rho^{N+2s}} {}_2F_1\Big(\frac{N+2s}{2}; s+1; \frac{N}{2}, \frac{r^2}{\rho^2}\Big)$$

[1] We note that if V is unbounded, then the inequality holds with $V 1_{B_{1-\frac{1}{n}}} \in L^\infty(B)$, for $\in \mathbb{N}$. Therefore, by the dominated convergence theorem, we can let $n \to \infty$ to get (3.6).

for $0 \le r < \rho \le 1$, see e.g. [15, Section 5]. Here $\alpha_N = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma(\frac{N-1}{2})}$, and ${}_2F_1$ denotes the hypergeometric function given by

$$x \mapsto {}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} x^n$$

with the Pochhammer symbols $(a)_n, (b)_n$ and $(c_n)_n$. Since, for fixed $a, b, c > 0$, the function $x \mapsto {}_2F_1(a,b;c;x)$ is positive and increasing on $(0,1)$ as $(a)_n, (b)_n$ and $(c_n)_n$ are positive for all n, the function $r \mapsto \Theta(r,\rho)$ is positive and increasing in $r \in (0,\rho)$. Consequently, for $\rho \in (r_0, 1)$ we have, by applying again the Hardy-Littlewood inequality,

$$\begin{aligned}\int_{B_{r_0}} v_*(x)h_\rho(x)dx &= \rho^{1+2s}\int_{B_{r_0}} v_*(x)\Theta(|x|,\rho)dx \le \rho^{1+2s}\int_{B_{r_0}} v^+(x)\tilde{h}(\rho|x|)dx \\ &= \int_{B_{r_0}} v_+(x)h_\rho(x)dx,\end{aligned}$$

which by (3.8) implies that

$$-[v_*, v^-]_s \le 2c_{N,s}\int_{r_0}^1 \rho^{N-1}v^-(\rho)\Big(\int_{B_{r_0}} v_+(x)h_\rho(x)dx\Big)d\rho = -[v_+, v^-]_s,$$

as claimed in (3.7).
In view of (1.7), here exists $\kappa > 0$ such that $\int_B (v_* - \kappa v^-)w_1 dx = 0$ and

$$\int_B (\sigma_2 - V)(v_* - \kappa v^-)^2 dx \le [v_* - \kappa v^-]_s^2 = [v_*]_s^2 + \kappa^2[v^-]_s^2 - 2\kappa[v_*, v^-]_s. \tag{3.9}$$

From this, (3.5) and (3.7), we obtain

$$\int_B (\sigma_2 - V)(v_* - \kappa v^-)^2 dx \le [v^+]_s^2 + \kappa^2[v^-]_s^2 - 2\kappa[v^+, v^-]_s.$$

Combining this with (3.2), (3.3) and (3.4), we get

$$\begin{aligned}\int_B (\sigma_2 - V)(v^+ - \kappa v^-)^2 dx &\le \int_B (\sigma_2 - V)v_*^2 dx + \kappa^2\int_B (\sigma_2 - V)(v^-)^2 dx \\ &= \int_B (\sigma_2 - V)(v_* - \kappa v^-)^2 dx \le [v^+]_s^2 + \kappa^2[v^-]_s^2 - 2\kappa[v^+, v^-]_s \\ &= \int_B (\sigma_2 - V)(v^+ - \kappa v^-)^2 dx + (1 + \kappa^2 - 2\kappa)[v^+, v^-]_s.\end{aligned} \tag{3.10}$$

Since $[v^+, v^-]_s < 0$ and $1 + \kappa^2 - 2\kappa = (1-\kappa)^2$, we then deduce that $\kappa = 1$ and all the inequalities in the display (3.10) become equalities. A a consequence equality holds in (3.9) with $\kappa = 1$ and thus $\omega = v_* - v^- = v_* - w_2^-$ is an eigenfunction of (1.1) corresponding to $\sigma = \sigma_2(V)$. Since, by (3.4), $w_2 \equiv \omega$ on $B_1 \setminus B_{r_0}$, we conclude that $w_2 \equiv \omega$ by fractional unique continuation (see [14]). However, since $w_2^+ \not\equiv 0$, it now follows from the properties of Schwarz symmetrization that

$$v_*(0) = w_2(0) = \|v_*\|_{L^\infty(B)} > 0,$$

which gives a contradiction. The proof is thus finished. □

Next, we need the following key equivalence statement.

Proposition 3.3. *Let V satisfy (1.8), and let $w_2 \in \mathcal{H}^s_{rad}(B) \setminus \{0\}$ be a radial solution to (1.1) with $\sigma = \sigma_2(V)$. Then the following assertions are equivalent:*

(i) w_2 *changes sign precisely once in the radial variable.*
(ii) We have

$$w_2(0) \int_B w_2 \, dx < 0. \tag{3.11}$$

Proof. By Theorem 3.2 we may, replacing w_2 by $-w_2$ if necessary, assume that

$$w_2(0) > 0. \tag{3.12}$$

We first prove that (i) implies (ii). Let $w_1 \in \mathcal{H}^s(B) \cap C(B)$ be the unique L^2-normalized positive eigenfunction corresponding to the first eigenvalue $\sigma_1(V)$. From [18, Corollary 1.2], we may deduce that w_1 is strictly decreasing in its radial variable $|x|$. Let $r_0 \in B$ be such that that $w_2 \gneqq 0$, $w_2 \not\equiv 0$ in B_{r_0} and $w_2 \lneqq 0$, $w_2 \not\equiv 0$ in $B \setminus B_{r_0}$. Since $w_1 = w_1(|x|)$ is strictly decreasing in the radial variable, we then get

$$0 = \int_B w_2 w_1 dx = \int_{B_{r_0}} w_2 w_1 dx + \int_{B \setminus B_{r_0}} w_2 w_1 dx > w_1(r_0) \int_B w_2 dx$$

and hence

$$\int_B w_2 \, dx < 0. \tag{3.13}$$

Combining (3.12) with (3.13), we get (ii).
Next we prove that (ii) implies (i). For this we argue by contradiction and assume that $w_2 = w_2(|x|)$ changes twice in the radial variable, i.e. there exists $0 < r_1 < r_2 < r_3$ with $w(r_1) > 0$, $w(r_2) < 0$ and $w(r_3) > 0$ after replacing w with $-w$ if necessary. We then argue similarly as in the proof of Theorem 3.2. For this we let W_2 be the s-harmonic extension of w_2, and we claim that

$$W_2(0,t) \geq 0 \qquad \text{for all } t > 0. \tag{3.14}$$

To see this, we define $\widetilde{W} : \overline{\mathbb{R}_+ \times \mathbb{R}_+} \to \mathbb{R}$ by $\widetilde{W}(|x|, t) = W_2(x,t)$. By Lemma 2.5, the sets $\mathcal{O}_\pm := \{(x,t) \in \overline{\mathbb{R}_+ \times \mathbb{R}_+} \,:\, \pm W_2 > 0\}$ are (relatively) open in $\overline{\mathbb{R}_+ \times \mathbb{R}_+}$ and connected, hence they are also path connected. In particular, there exists a continuous path $\gamma : [0,1] \to \mathcal{O}_+$ with $\gamma(0) = (r_1, 0)$ and $\gamma(1) = (r_3, 0)$. Arguing by contradiction, we now assume that there exists a point $(0, t_0)$ with $t_0 > 0$ and $W_2(0, t_0) < 0$. Then there exists another continuous path $\eta : [0,1] \to \mathcal{O}_-$ with $\eta(0) = (0, t_0)$ and $\eta(1) = (r_2, 0)$. By Lemma 6.3 below, this curve must intersect γ, but this is impossible since $\mathcal{O}_+ \cap \mathcal{O}_- = \varnothing$. The contradiction shows that (3.14) holds.

Noticing that

$$t^N W_2(0,t) = t^N \widetilde{W}(0,t) = p_{N,s} \int_{\mathbb{R}^N} \frac{w_2(y) dy}{(1 + |y|^2/t^2)^{\frac{N+2s}{2}}}$$

and that $w_2 \in L^1(\mathbb{R}^N)$, we then conclude that

$$\lim_{t \to \infty} t^N \widetilde{W}(0,t) = p_{N,s} \int_{\mathbb{R}^N} w_2(y) dy \tag{3.15}$$

and thus $\int_{\mathbb{R}^N} w_2(y) dy \geq 0$. Together with (3.12), this contradicts our assumption (ii). The contradiction shows that w_2 changes sign only once in the radial variable, as claimed. □

Next, we first consider the case $V \equiv 0$, i.e., eigenfunctions corresponding the second radial eigenvalues of the Dirichlet fractional Laplacian.

Proposition 3.4. *For $s \in (0,1]$, let $\lambda_{2,s} = \sigma_2(0)$ be the second radial eigenvalue of the Dirichlet fractional Laplacian and $\varphi_{s,2}$ be a family of corresponding eigenfunctions. Then $\varphi_{s,2}$ changes sign only once in the radial variable. In particular $\varphi_{s,2}(0) \int_B \varphi_{s,2} \, dx < 0$.*

Proof. We start with the preliminary remark that (3.11) holds in the case $s = 1$, $V \equiv 0$, i.e., we have

$$\varphi_{1,2}(0) \int_B \varphi_{1,2}\, dx < 0. \tag{3.16}$$

Indeed, it is well known that $\varphi_{1,2}$ changes sign precisely once in the radial variable. Moreover, we have

$$\int_B \varphi_{1,2}\, dx = \frac{1}{\lambda_{2,1}} \int_B (-\Delta\varphi_{1,2})\, dx = -\frac{1}{\lambda_{2,1}} \int_{\partial B} \partial_\nu \varphi_{1,2}\, d\sigma, \tag{3.17}$$

where ∂_ν denotes the outer normal derivative on $\partial\Omega$. After replacing $\varphi_{1,2}$ with $-\varphi_{1,2}$ if necessary, we may now assume that $\varphi_{1,2}(0) > 0$. Moreover, since $\varphi_{1,2}$ changes sign precisely once in the radial variable, the classical Hopf Lemma implies that $\partial_\nu\varphi_{1,2} > 0$ on ∂B. Hence (3.17) implies (3.16).

Next, we let $\lambda_{1,s}$ be the first radial eigenvalue of the Dirichlet fractional Laplacian and $\varphi_{s,1} \in \mathcal{H}^s(B)$ be the corresponding positive eigenfunctions, normalized such that

$$\|\varphi_{s,1}\|_{L^\infty(B)} = 1 \qquad \text{for } s \in (0, 1]. \tag{3.18}$$

From the variational characterization we see that $\lambda_{1,s} \le C(N)$. This follows from the fact that for for all $\varphi \in C_c^\infty(B)$ with $\|\varphi\|_{L^2(B)} = 1$, we have, by (1.4),

$$\lambda_{1,s} \le [\varphi]_s^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{\varphi}|^2 d\xi \le \int_{\mathbb{R}^N} (1 + |\xi|)^2 |\widehat{\varphi}|^2 d\xi = \|\varphi\|^2_{H^1(B)}.$$

Recall that $(-\Delta)^s\varphi_{s,1} = \lambda_{1,s}\varphi_{s,1}$ in B. Applying [25, Theorem 1.3], we find that for all $s_0 \in (0, 1)$, there exits $C = C(N, s_0) > 0$ such that

$$\|\varphi_{s,1}\|_{C^s(\overline{B})} \le C \qquad \text{for all } s \in [s_0, 1). \tag{3.19}$$

Hence, if $\overline{s} \in (0, 1]$ and $(s_n)_n \subset (0, 1)$ is a sequence with $s_n \to \overline{s}$, then, up to passing to a subsequence, we have $\varphi_{s_n,1} \to \overline{v}$ in $C(\overline{B})$ for some function $\overline{v} : \overline{B} \to \mathbb{R}$ satisfying $|\overline{v}(x)| \le C(1 - |x|)_+^s$ for all $x \in \mathbb{R}^N$. Moreover $(-\Delta)^{\overline{s}}\overline{v} = \lambda_{1,\overline{s}}\overline{v}$ in $\mathcal{D}'(B)$. If $\overline{s} < 1$, then it follows from Fatou's lemma that

$$[\overline{v}]^2_{\overline{s}} \le \liminf_{n\to\infty} [\varphi_{s_n,1}]^2_{s_n} \le \liminf_{n\to\infty} \lambda_{1,s_n} \|\varphi_{s_n,1}\|^2_{L^2(B)} \le C(N)|B|$$

and therefore $\overline{v} \in \mathcal{H}^{\overline{s}}(B)$. In the case where $\overline{s} = 1$, passing to the limit in (3.19), we deduce that $\nabla\overline{v} \in L^\infty(B)$. Since $\overline{v}$ is nonnegative and $\|\overline{v}\|_{L^\infty(B)} = 1$, we then obtain that $\lambda_{1,\overline{s}}$ is the first Dirichlet radial eigenvalue of $(-\Delta)^{\overline{s}}$ in B and $\overline{v} = \varphi_{\overline{s},1}$. Since it is simple, then for any $\overline{s} \in (0, 1]$,

$$\varphi_{s,1} \to \varphi_{\overline{s},1} \quad \text{ in } C(\overline{B}) \cap \mathcal{H}^s(B) \text{ as } s \to \overline{s}. \tag{3.20}$$

From the variational characterization of $\lambda_{2,s}$, as given in (1.6) with $V = 0$, and from (3.20), it follows that $(\lambda_{2,s})_{s\in(0,1]}$ is bounded. In the following, we may, by normalization and Theorem 3.2, assume that

$$\|\varphi_{s,2}\|_{L^\infty(B)} = 1 \quad \text{and} \quad \varphi_{s,2}(0) > 0 \qquad \text{for all } s \in (0, 1]. \tag{3.21}$$

Arguing as above by using Theorem 3.2 and (3.21) together with the regularity estimate given in [25, Theorem 1.3], we find that

$$\varphi_{s,2} \to \varphi_{\overline{s},2} \text{ in } C(\overline{B}) \qquad \text{as } s \to \overline{s}. \tag{3.22}$$

We now recall that $\varphi_{s,2}$ changes at most sign twice in the radial variable for all $s \in (0, 1)$ by Corollary 2.7. By (3.16), we have $\varphi_{1,2}(0) \int_B \varphi_{1,2}\, dx < 0$. Hence by (3.22) and Proposition 3.3, there exists $s_0 \in (0,1)$ such that $\varphi_{s,2}$ changes sign precisely once in the radial variable for all $s \in (s_0, 1)$. We define

$$s_* := \inf\{s_0 \in (0,1] : \ \varphi_{s,2} \text{ changes sign only once in the radial variable for all } s \in (s_0,1)\}.$$

The proof finishes once we show that $s_* = 0$. Assume on the contrary that $s_* > 0$. Then by (3.22) and the definition of s_*, $\varphi_{s_*,2}$ changes sign only once in the radial variable and thus by Proposition 3.3

$$\varphi_{s_*,2}(0) \int_B \varphi_{s_*,2}\, dx < 0.$$

On the other hand Proposition 3.3 implies that $\varphi_{\tau,2}(0) \int_B \varphi_{\tau,2}\, dx \geq 0$ for all $\tau \in (0, s_*)$. Hence letting $\tau \nearrow s_*$ and using (3.22), we find that $\varphi_{s_*,2}(0) \int_B \varphi_{s_*,2}\, dx \geq 0$. This leads to a contradiction and thus $s_* = 0$, as desired. □

Theorem 3.5. *Let V satisfy (1.8), and let $w_2 \in \mathcal{H}^s_{rad}(B)$ be a nontrivial solution to* (1.1) *with $\sigma = \sigma_2(V)$. Then w_2 changes sign precisely once in the radial variable.*

Proof. For $\tau \in [0,1]$, we define

$$V_\tau : B \to \mathbb{R}, \qquad V_\tau(x) = \tau V(x),$$

and we let $w_{2,\tau}$ be an eigenfunction associated to $\sigma_2(V_\tau)$. By Theorem 3.2, we may normalize $w_{2,\tau}$ such that

$$\|w_{2,\tau}\|_{L^2(B)} = 1 \quad \text{and} \quad w_{2,\tau}(0) > 0 \qquad \text{for all } \tau \in [0,1]. \tag{3.23}$$

Applying Lemma 2.2, Theorem 3.2 and (3.23), we find that, for every $\overline{\tau} \in [0,1]$,

$$w_{2,\tau} \to w_{2,\overline{\tau}} \quad \text{in } C(\overline{B}) \qquad \text{as } \tau \to \overline{\tau}. \tag{3.24}$$

Moreover, for all $\tau \in [0,1]$, the function $w_{2,\tau}$ changes sign at most twice in the radial variable by Corollary 2.7. In addition $w_{2,0}(0) \int_B w_{2,0} dx < 0$ by Proposition 3.4. Therefore from (3.24) and Proposition 3.3, there exists $\varepsilon \in (0,1]$ such that $w_{2,\tau}$ changes sign only once in the radial variable for all $\tau \in [0,\varepsilon]$. We define

$$\tau_* := \sup\{\varepsilon \in [0,1] : \ w_{2,\tau} \text{ changes sign only once for all } \tau \in [0,\varepsilon]\}.$$

By definition of τ_*, (3.24) and Proposition 3.3, we see that w_{2,τ_*} changes sign only once. In particular

$$w_{2,\tau_*}(0) \int_B w_{2,\tau_*} dx < 0. \tag{3.25}$$

We claim that $\tau_* = 1$. Indeed, if we had $\tau_* < 1$, then Proposition 3.3 would yield

$$w_{2,\tau}(0) \int_B w_{2,\tau} dx \geq 0 \quad \text{ for all } \tau \in (\tau_*, 1).$$

Letting $\tau \searrow \tau_*$ in the above inequality and using (3.24), we get $w_{2,\tau_*}(0) \int_B w_{2,\tau_*} dx \geq 0$ which contradicts (3.25). As consequence (3.25) holds with $\tau_* = 1$. Combining this with Proposition 3.3 and Theorem 3.2, we conclude that $w_{2,1} \in \left\{ \frac{w_2}{\|w_2\|_{L^2(B)}}; \frac{-w_2}{\|w_2\|_{L^2(B)}} \right\}$ changes sign precisely once in the radial variable, as claimed. □

Remark 3.6. *The continuity argument we use in the proof of Theorem 1.1 is inspired by the work of Frank, Lenzmann and Silvestre [16]. However our arguments here provide some simplifications in some proofs in [16], because Proposition 3.3 can be extended also to the case where B is replaced with $\mathbb{R}^N$. In fact the main idea behind the proofs is that if w_{2,τ_*} is a simple second eigenfunction of $(-\Delta)^s + V$ in B_R with $R \in (0, +\infty]$ and $s \in (0,1]$, changes sign only once on B_R. Then there cannot exists a 1-parameter family of second eigenfunction*

$w_{2,\tau}$ of $(-\Delta)^{s_\tau}+V_\tau$ in B_R with the property that $w_{2,\tau}\to\pm w_{2,\tau_*}$ in $C(B_R)\cap L^1(B_R)$ and $w_{2,\tau}$ changes sign twice in B_R for all $\tau\neq\tau_*$.
While in [16] for $R=+\infty$ they use the expansion of the Green function of the operator $(-\Delta)^s+1$ on $\mathbb{R}^N$ to derive an open condition given by the sign of $\int_{\mathbb{R}^N} w_{2,\tau_*}dx$, here we simply observe in Proposition 3.3 that this sign is given by $\lim_{t\to\infty} t^N W_{2,\tau_*}(0,t)$ where W_{τ_*} is the s_{τ_*}-harmonic extension of w_{2,τ_*}. Now the sign of this limit is in turn given by the monotonicity of the first eigenfunction and the fact that w_{2,τ_*} changes sign only once in the radial variable.
Comparing the proofs, we see that the role played by the radial Dirichlet eigenvalues $\sigma_i(0)$ is the the one that $\sigma_i(\varepsilon e^{-|x|^2})$ plays in [16], for some $\varepsilon<0$.

With the help of a new local Hopf-type Lemma for the s-harmonic extension given in Theorem 5.2 below, we shall now prove that the fractional normal derivative of a radial second eigenfunction of (1.1) is nontrivial.

Proposition 3.7. *Let V satisfy (1.8), and let $w_2\in\mathcal{H}^s_{rad}(B)\setminus\{0\}$ be a radial solution to (1.1) with $\sigma=\sigma_2(V)$. Then we have*

$$w_2(0)\psi_{w_2}(1)<0, \tag{3.26}$$

where $\psi_{w_2}(1):=\liminf\limits_{|x|\nearrow 1}\dfrac{w_2(x)}{(1-|x|)^s}$.

Proof. By Theorem 3.2 we may, replacing w_2 by $-w_2$ if necessary, again assume that

$$w_2(0)>0. \tag{3.27}$$

By Theorem 3.5, the equivalent properties of Proposition 3.3 are satisfied. Let W_2 be the s-harmonic extension of w_2, and let $\widetilde{W}:\overline{\mathbb{R}_+\times\mathbb{R}_+}\to\mathbb{R}$ be defined as in the proof of Proposition 3.3, i.e., $\widetilde{W}(|x|,t)=W_2(x,t)$. Moreover, we consider again the path connected sets $\mathcal{O}_\pm:=\{(x,t)\in\overline{\mathbb{R}_+\times\mathbb{R}_+}\,:\,\pm W_2>0\}$

By (3.11), (3.15) and (3.27), there must exist $t_*>0$ such that $\widetilde{W}(0,t_*)<0$. Moreover, there exists $r_*\in(0,1)$ such that $\widetilde{W}(r_*,0)<0$. By the path connectedness of $\mathcal{O}_-$, there exists a continuous curve $\eta:[0,1]\to\mathcal{O}_-$ with $\eta(0)=(0,t_*)$ and $\eta(1)=(r_*,0)$. Since w_2 changes sign precisely once in the radial variable, we have that $\psi_{w_2}(1)\le 0$.

Let us now assume by contradiction that $\psi_{w_2}(1)=0$.

We claim that W_2 takes positive values in every relative neighborhood of the point $(e_1,0)$ in $\overline{\mathbb{R}^{N+1}_+}$ in this case, where e_1 denotes the first coordinate vector in $\mathbb{R}^N$. Indeed, suppose by contradiction that $W_2\le 0$ in some relative neighborhood N of $(e_1,0)$ in $\overline{\mathbb{R}^{N+1}_+}$. We note that $W_2\not\equiv 0$ in N, since otherwise $W_2\equiv 0$ in $\overline{\mathbb{R}^{N+1}_+}$ by unique continuation (see e.g. [14]) and therefore $w_2\equiv 0$, which is impossible. Hence $W_2\le 0$, $W_2\not\equiv 0$ in N, and therefore Theorem 5.2 below implies that

$$\psi_{w_2}(1)=\liminf_{r\to 0^+}\frac{W_2((1-r)e_1,0)}{r^s}<0,$$

contrary to our current assumption. The contradiction shows that the function W_2 takes positive values in every relative neighborhood of the point $(e_1,0)$ in $\overline{\mathbb{R}^{N+1}_+}$, as claimed. As a consequence, $\widetilde{W}$ takes positive values in $Q^+_\rho:=\{(r,t)\in\mathbb{R}_+\times\mathbb{R}_+\,:\,|(r,t)-(1,0)|<\rho\}$ for every $\rho>0$. Therefore letting $d=\operatorname{dist}((1,0),\eta([0,1]))>0$, there exists

$$z\in Q^+_{d/2}\qquad\text{with } \widetilde{W}(z)>0. \tag{3.28}$$

On the other hand, by (3.27) there exists $\varepsilon \in (0, r_*)$ such that $\widetilde{W}(\varepsilon, 0) > 0$, and by the path connectedness of $\mathcal{O}_+$ there exists a continuous curve $\gamma : [0,1] \to \mathcal{O}_+$ joining the points $(\varepsilon, 0)$ and z. By Lemma 6.4 below applied to the points $t_*, \varepsilon, r_*, 1$, the curves η and γ intersect. This however is impossible since $\mathcal{O}_+ \cap \mathcal{O}_- = \varnothing$. The contradiction yields $\psi_{w_2}(1) < 0$, and together with (3.27) the claim follows. □

Proof of Theorem 1.1 (completed). Let V satisfy (1.8). By Theorem 3.2, the eigenvalue $\sigma_2(V)$ is simple, and every associated eigenfunction $w_2 = w_2(|x|)$ satisfies $w_2(0) \neq 0$. Moreover, by Theorem 3.5 we may assume, after replacing w_2 by $-w_2$ if necessary, that there exists $r_0 \in (0,1)$ with the property that $w_2 \geq 0$, $w_2 \not\equiv 0$ on B_{r_0} and $w_2 \leq 0$, $w_2 \not\equiv 0$ on $B \setminus \overline{B_{r_0}}$. Then we may follow the second part of the proof of Theorem 3.2 to see that w_2^+ coincides with its Schwarz symmetrization, which implies that $w_2\big|_{B_{r_0}}$ is decreasing in the radial variable. In addition, the property (1.9) follows from Proposition 3.7.

Finally, fractional unique continuation (see [14]) implies that

$$w_2 \text{ is nonzero on a dense (open) subset of } B. \tag{3.29}$$

Since $w_2\big|_{B_{r_0}}$ is decreasing in the radial variable, we thus conclude that $w_2 > 0$ on B_{r_0}. It thus remains to show that $w_2 < 0$ in $B \setminus \overline{B_{r_0}}$, i.e. $w_2 < 0$ in $(r_0, 1)$ as a function of the radial variable. Suppose by contradiction that there exists $r_3 \in (r_0, 1)$ with $w_2(r_3) = 0$. By (3.29), there exist points $r_1 \in (0, r_0)$, $r_2 \in (r_0, r_3)$ and $r_4 \in (r_3, 1)$ with $w_2(r_1) > 0$, $w_2(r_2) < 0$ and $w_2(r_4) < 0$. Let W_2 be the s-harmonic extension of w_2, and let again $\widetilde{W} : \overline{\mathbb{R}_+ \times \mathbb{R}_+} \to \mathbb{R}$ be defined by $\widetilde{W}(|x|, t) = W_2(x, t)$. Moreover, we consider again the path connected sets $\mathcal{O}_\pm := \{(x,t) \in \overline{\mathbb{R}_+ \times \mathbb{R}_+} \,:\, \pm W_2 > 0\}$. We now fix a continuous curve $\gamma : [0,1] \to \mathcal{O}_-$ joining the points $(r_2, 0)$ and $(r_4, 0)$. Since $w_2(r_3) = \widetilde{W}(r_3, 0) = 0$, we have $(r_3, 0) \notin \gamma([0,1])$ and therefore we may, by Lemma 3.1, choose a point $z \in \overline{\mathbb{R}_+ \times \mathbb{R}_+}$ with $\widetilde{W}(z) > 0$ and $|z - (r_3, 0)| < \operatorname{dist}((1,0), \gamma([0,1]))$. By the path connectedness of $\mathcal{O}_+$, there exists a continuous curve $\eta : [0,1] \to \mathcal{O}_+$ joining the points $(r_1, 0)$ and z. Now Lemma 6.3, applied to the points $r_1 < r_2 < r_3 < r_4$, shows that γ and η must intersect, which is impossible as $\mathcal{O}_+ \cap \mathcal{O}_- = \varnothing$. The contradiction shows that $w_2 < 0$ in $B \setminus \overline{B_{r_0}}$, as required. □

4. Nondegeneracy and uniqueness of ground state solutions

In this section we complete the proof of Theorem 1.2. For a radial function $v \in C^s(\mathbb{R}^N)$ with $v \equiv 0$ on $\mathbb{R}^N \setminus B$, we define

$$\psi_v \in L^\infty(0,1), \qquad \psi_v(|x|) := \frac{v(x)}{\operatorname{dist}(x, \mathbb{R}^N \setminus B)^s} = \frac{v(x)}{(1-|x|)^s} \qquad \text{for } x \in B.$$

and, as before, we define $\psi_v(1) := \liminf_{\rho \nearrow 1} \psi_v(\rho)$.

We start by collecting some properties of solutions to (1.13). Throughout this section, we let $p \in (1, 2_s^* - 1)$ and $\lambda \geq 0$ be fixed, and we let $u \in \mathcal{H}^s(B)$ denote a fixed solution of (1.13).

We recall the following well-known properties of u.

Lemma 4.1. *The following statements hold:*

(i) $u \in C^\infty(B) \cap C^s(\mathbb{R}^N)$ and u is radially symmetric and strictly decreasing.

(ii) ψ_u extends to a continuous function on $[0,1]$, and $\psi_u(1) > 0$.

Proof. As noted in [9], we can apply [27, Proposition 3.1] to get $u \in L^\infty(B)$. Then, by a classical bootstrap argument using interior and boundary regularity (see [28] and [24]), we find that $u \in C^s(\mathbb{R}^N) \cap C^\infty(B)$, and that ψ_u extends to a continuous function on $[0,1]$.

From [18, Corollary 1.2] we deduce that u is radially symmetric and strictly decreasing in the radial variable. Finally, $\psi_u(1) > 0$ follows from the fractional Hopf lemma, see e.g. [13, Proposition 3.3]. □

As a consequence, we note that $V = -pu^{p-1}$ satisfies assumption (1.8), with $q = +\infty$. The following lemma has been proved in [9] in the case $N = 1$. The proof is almost the same in the multidimensional case, but we prefer to give the details for the convenience of the reader.

Lemma 4.2. *Let $u \in \mathcal{H}^s(B)$ be a solution to* (1.13), *and let $w \in \mathcal{H}^s_{rad}(B)$ be a radial solution of*

$$(-\Delta)^s w - pu^{p-1} w = -\lambda w \qquad \text{in } B. \tag{4.1}$$

Then $\psi_w \in C([0,1])$ and

$$\int_B u^p w dx = 0 \qquad \text{and} \qquad [u,w]_s = -\lambda \int_B wu\, dx. \tag{4.2}$$

Moreover, the fractional normal derivatives $\psi_u(1)$ and $\psi_w(1)$ of u and w satisfy the identity

$$2s\lambda \int_B uw dx = -\Gamma^2(1+s)|\partial B|\psi_u(1)\psi_w(1). \tag{4.3}$$

Proof. We first note that it follows from Lemma 2.1 that $w \in C^s(\mathbb{R}^N) \cap C^{2s+\alpha}_{loc}(B)$ and $x \mapsto \frac{w(x)}{(1-|x|)^s} \in C^\alpha(\overline{B})$ for some $\alpha > 0$. Next we note that the weak formulations of (1.12) and (4.1) yield that

$$\int_B u^p w\, dx = [u,w]_s + \lambda \int_B wu\, dx = p \int_B u^p w\, dx$$

and therefore (4.2) follows. Moreover, the fractional integration by parts formula given in [26, Theorem 1.9] now yields

$$\begin{aligned} &\int_B \nabla u \cdot x(-\Delta)^s w\, dx + \int_B \nabla w \cdot x(-\Delta)^s u\, dx \\ &= -\Gamma^2(1+s) \int_{\partial B} \psi_u \psi_w\, d\sigma - (N-2s)[u,w]_s \end{aligned} \tag{4.4}$$

By integration by parts and (4.2), we have

$$\begin{aligned} &\int_B \nabla w \cdot x(-\Delta)^s u\, dx = \int_B \nabla w \cdot x(-\lambda u + u^p)\, dx \\ &= -N \int_B w(-\lambda u + u^p)\, dx - \int_B \nabla u \cdot x(-\lambda w + u^{p-1} w)\, dx \\ &= -N[u,w]_s - \int_B \nabla u \cdot x(-\Delta)^s w\, dx. \end{aligned}$$

Combing this with (4.4), we deduce that

$$-N[u,w]_s = -\Gamma^2(1+s) \int_{\partial B} \psi_u \psi_w\, d\sigma - (N-2s)[u,w]_s.$$

This and (4.2) gives (4.3). □

Corollary 4.3. *Let $V = -pu^{p-1}$. Then we have $\sigma_2(V) \neq -\lambda$ for the second radial eigenvalue $\sigma_2(V)$ of (1.1).*

Proof. Suppose by contradiction that $\sigma_2(V) = -\lambda$, and let $w \in \mathcal{H}^s(B)$ be a corresponding eigenfunction, so w satisfies (4.2) and (4.1). Moreover, by Theorem 1.1 we have, after replacing w by $-w$ if necessary, that

$$\psi_{w_2}(1) < 0, \tag{4.5}$$

and there exists $r \in (0,1)$ with the property that

$$w_2 > 0 \text{ on } B_r \qquad \text{and} \qquad w_2 < 0 \text{ on } B \setminus \overline{B_r}. \tag{4.6}$$

Since $\psi_u(1) > 0$, it follows from (4.3) and (4.5) that

$$\lambda \int_B uw\,dx > 0,$$

which, since $\lambda \geq 0$, is only possible if $\lambda > 0$ and $\int_B uw\,dx > 0$. However, from (4.2) and the fact that u is radially symmetric, positive and strictly decreasing in the radial variable, for $e \in \partial B$, we obtain

$$0 = \int_B u^p w dx = \int_{B_r} u^p w dx + \int_{B\setminus B_r} u^p w dx > u^{p-1}(re) \int_B uw dx \qquad \text{with} \quad u^{p-1}(re) > 0,$$

which yields a contradiction. The claim thus follows. □

Theorem 4.4. *Suppose that u is a ground state solution of (1.13). Then u is nondegenerate, i.e., the equation* (4.1) *does not admit nontrivial solutions* $w \in \mathcal{H}^s(B)$.

Proof. We first note that the nonradial nondegeneracy result given in [9, Theorem 1.1] shows that (4.1) does not admit nontrivial solutions $w \in \mathcal{H}^s(B) \setminus \mathcal{H}^s_{rad}(B)$. Moreover, (1.15) and Corollary 4.3 imply that $\sigma_2(V) > -\lambda$ for $V := -pu^{p-1}$. In addition, by (1.13) we have

$$[u]_s^2 + \|u\|^2_{L^2(B;V\,dx)} = -\lambda \|u\|^2_{L^2(B)} - (p-1)\int_B u^p\,dx < -\lambda\|u\|^2_{L^2(B)}$$

and therefore $\sigma_1(-pu^{p-1}) < -\lambda$ by (1.6). Hence (1.1) does not admit nontrivial solutions $w \in \mathcal{H}^s_{rad}(B)$ for $\sigma = -\lambda$, and therefore (4.1) does not admit nontrivial solutions in $\mathcal{H}^s_{rad}(B)$. The claim thus follows. □

Proof of Theorem 1.2. Since ground state solutions solutions of (1.13) are nondegenerate for all $p \in (1, 2^*_s - 1)$, $\lambda \geq 0$ and since uniqueness of solutions to (1.13) holds for p close to 1 by the results in [7], we can use the same continuity argument as in [9] to deduce that uniqueness holds for all allowed values of p. □

5. Appendix I: A Hopf-type boundary point lemma

The aim of this section is to establish a new Hopf-type boundary point lemma, which was used in a special case with radial data in the proof of Theorem 1.1. We believe that this new lemma could be of independent interest. For $x_0 \in \mathbb{R}^N$, we let, as before, $B_r(x_0)$ denote the ball in $\mathbb{R}^N$ of radius r centered at x_0. Moreover, we define $\mathbf{B}_r(x_0)$ as the ball in $\mathbb{R}^{N+1}$ centered at the point $(x_0, 0)$ with radius r and $\mathbf{B}_r^+(x_0) := \mathbb{R}^{N+1}_+ \cap \mathbf{B}_r(x_0)$. If there is no confusion, we will identify, as before, subsets B of $\mathbb{R}^N$ with $B \times \{0\} \subset \overline{\mathbb{R}^{N+1}_+}$. Note that, with this identification, $B_r(x_0)$ coincides with $\partial \mathbf{B}_r^+(x_0) \setminus \mathbb{R}^{N+1}_+$ up to a set of set of zero N-dimensional Lebesgue measure. Moreover, for an relatively open set $A \subset \overline{\mathbb{R}^{N+1}_+}$, we define the weighted Sobolev space $H^1(A; t^{1-2s})$ as space of all functions $U \in L^2(A)$ with

$$\|U\|^2_{H^1(A;t^{1-2s})} := \int_A (|\nabla U|^2 + U^2) t^{1-2s}\,dxdt < \infty,$$

where ∇U denotes the distributional gradient of U. Moreover, we let $H^1_{0,+}(A;t^{1-2s})$ denote the closure of $C_c^\infty(A) \subset H^1(A;t^{1-2s})$ with respect to $\|\cdot\|_{H^1(A;t^{1-2s})}$. We note that, as in the case of classical first order Sobolev spaces without weights, we have $U \in H^1_{0,+}(A;t^{1-2s})$ for every function $U \in H^1(A;t^{1-2s}) \cap C(\overline{A})$ with $U \equiv 0$ on $\partial A \cap \mathbb{R}^{N+1}$.

We recall the following Sobolev trace inequality.

Lemma 5.1. *Let $m = 2_s^* = \frac{2N}{N-2s}$ if $N > 2s$, and let $2 < m < \infty$ in the case $1 = N \leq 2s$. Then there exists $C = C(N,s,m) > 0$ with the property that*

$$\|w\|^2_{L^m(B_1(0))} \leq C \left(\int_{\mathbf{B}_1^+(0)} |\nabla w|^2 t^{1-2s} dxdt + \|w\|^2_{L^2(\partial \mathbf{B}_1^+(0) \cap \{t>0\};t^{1-2s})} \right),$$

for any $w \in H^1(\mathbf{B}_1^+(0);t^{1-2s})$ with $w(\cdot,0) = 0$ on $B_1(0) \setminus B_{1/2}(0)$.

This inequality is classical in the unweighted case $1 = 2s$, and it is proved in [14, Lemma 2.6] in the case $N > 2s$. To deal with the remaining case $1 = N < 2s$, we merely note that

$$\int_{\mathbf{B}_1^+(0)} |\nabla w|^2 dxdt \leq \int_{\mathbf{B}_1^+(0)} |\nabla w|^2 t^{1-2s} dxdt \qquad \text{for any } w \in H^1(\mathbf{B}_1^+(0);t^{1-2s}) \text{ if } s \geq \frac{1}{2}.$$

Therefore, if $m \in (2,\infty)$ is fixed, the inequality for $s = \frac{1}{2}$ implies the one for $s > \frac{1}{2}$.

We are now in the position to formulate our new Hopf-type lemma.

Theorem 5.2. *Let Ω be a $C^{1,1}$ open set with $0 \in \partial\Omega$. Let $V \in L^q(B_r(0)) \cap C^\alpha_{loc}(B_r(0) \cap \Omega)$, for some $q > \max(\frac{N}{2s}, 1)$ and $\alpha > 0$. Let $U \in H^1(\mathbf{B}_r^+(0);t^{1-2s}) \cap C(\Omega \times [0,r))$ be nonnegative and satisfy*

$$\begin{cases} -div(t^{1-2s}\nabla U) \geq 0 & \text{in } \mathbf{B}_r^+(0), \\ -\lim\limits_{t\to 0} t^{1-2s}\partial_t U + VU \geq 0 & \text{in } B_r(0) \cap \Omega. \end{cases} \tag{5.1}$$

in weak sense, i.e.,

$$\int_{\mathbf{B}_r^+(0)} t^{1-2s} \nabla U \cdot \nabla \Phi \, dxdt + \int_{B_r(0)} VU\Phi dx \geq 0 \quad \text{for all } \Phi \in H^1_{0,+}(\mathbf{B}_r^+(0);t^{1-2s}), \Phi \geq 0. \tag{5.2}$$

Then either $U \equiv 0$ in $\mathbf{B}_r^+(0)$ or $\liminf\limits_{\rho \searrow 0} \frac{U(\rho\nu,0)}{\rho^s} > 0$, where ν is the unit interior normal of $\partial\Omega$ at 0.

Proof. Let us assume that $U \not\equiv 0$ in $\mathbf{B}_r^+(0)$. Then by the strong maximum principle $U > 0$ in $\mathbf{B}_r^+(0)$. By assumption, Ω satisfies the interior sphere condition at 0. Therefore, there exists $\tau_0 \in (0, r/2)$ such that for all $\tau \in (0,\tau_0)$, there exists $e_\tau \in \Omega$ such that $B_\tau(e_\tau) \subset \Omega \cap B_r(0)$ and $0 \in \partial B_\tau(e_\tau)$. We claim that the problem

$$\begin{cases} -\text{div}(t^{1-2s}\nabla W) = 0 & \text{in } \mathbf{B}_{r/2}^+(0), \\ -\lim_{t\to 0} t^{1-2s}\partial_t W + VW = 0 & \text{in } B_\tau(e_\tau), \\ W(\cdot,0) = 0 & \text{in } B_{r/2}(0) \setminus B_\tau(e_\tau), \\ W = U & \text{on } \partial \mathbf{B}_{r/2}^+(0) \cap \{t > 0\}. \end{cases} \tag{5.3}$$

admits a (weak) solution if $\tau > 0$ is chosen sufficiently small. By this we mean that W is contained in the affine subspace $\mathcal{H} \subset H^1(\mathbf{B}_{r/2}^+(0);t^{1-2s})$ of functions $W \in H^1(\mathbf{B}_{r/2}^+(0);t^{1-2s})$

which satisfy the last two equations in (5.3) in trace sense, and that

$$\begin{cases} \int_{\mathbf{B}^+_{r/2}(0)} t^{1-2s}\nabla W\cdot\nabla\Phi\,dxdt + \int_{B_\tau(e_\tau)} VW\Phi dx = 0 \\ \text{for all } \Phi\in H^1_{0,+}(\mathbf{B}^+_{r/2}(0);t^{1-2s}) \text{ with } \Phi\equiv 0 \text{ on } B_{r/2}(0)\setminus B_\tau(e_\tau). \end{cases} \tag{5.4}$$

To see this, we minimize the energy functional

$$W\mapsto J(W) := \int_{\mathbf{B}^+_{r/2}(0)} |\nabla W|^2 t^{1-2s}dxdt + \int_{B_\tau(e_\tau)} VW^2dx$$

in $\mathcal{H}$. By Lemma 5.1 and Hölder's inequality, we have

$$\begin{aligned} \Big|\int_{B_\tau(e_\tau)} VW^2dx\Big| &\le \|V\|_{L^q(B_\tau(e_\tau))}\|W\|^2_{L^{2q'}(B^+_{r/2}(0))} \\ &\le C\|V\|_{L^q(B_\tau(e_\tau))}\left(\int_{\mathbf{B}^+_{r/2}(0)} |\nabla W|^2 t^{1-2s}dxdt + \|W\|^2_{L^2(\partial\mathbf{B}^+_{r/2}(0)\cap\mathbb{R}^{N+1}_+;t^{1-2s})}\right) \end{aligned} \tag{5.5}$$

for $W\in\mathcal{H}$ with a constant $C=C(r,N,s)>0$. Hence, if $\tau>0$ is chosen small enough to guarantee that $C\|V\|_{L^q(B_\tau(e_\tau))}<1$, then the function J is coercive on $\mathcal{H}$. Since, furthermore, the trace map $H^1(\mathbf{B}^+_{r/2}(0);t^{1-2s})\hookrightarrow L^2(B_\tau(e_\tau);Vdx)$ is compact, standard weak lower continuity arguments show that J admits a minimizer in $\mathcal{H}$, which then satisfies (5.4).

Next we note that, by the regularity theory in [3], we have $W\in C(\overline{\mathbf{B}^+_{r/2}(0)})$ and $t^{1-2s}\partial_t W\in C(\mathbf{B}^+_\tau(e_\tau))$. In addition, we deduce from (5.2) and (5.4) that

$$U\ge W\ge 0 \qquad \text{in } \overline{\mathbf{B}^+_{r/2}(0)}. \tag{5.6}$$

Indeed, applying (5.4) with $\Phi=W_-=\max(-W,0)\in H^1_{0,+}(\mathbf{B}^+_{r/2}(0);t^{1-2s})$ gives

$$\int_{\mathbf{B}^+_{r/2}(0)} t^{1-2s}|\nabla W_-|^2\,dxdt = -\int_{B_\tau(e_\tau)} V|W_-|^2dx.$$

Estimating as in (5.5), using $C\|V\|_{L^q(B_\tau(e_\tau))}<1$ and the fact that $W_-\equiv U_-\equiv 0$ on $\partial\mathbf{B}^+_{r/2}(0)\cap\mathbb{R}^{N+1}_+$, shows that $\int_{\mathbf{B}^+_{r/2}(0)} t^{1-2s}|\nabla W_-|^2\,dxdt=0$ and therefore $W_-\equiv 0$ in $\mathbf{B}^+_{r/2}(0)$, which gives the second inequality in (5.6). The first inequality in (5.6) follows in a similar way from (5.2) and (5.4).

Moreover, by (5.6) and the strong maximum principle, we have

$$W>0 \qquad \text{in } \mathbf{B}^+_\tau(e_\tau).$$

For fixed τ as above, we let $\tau_2\in(\tau,r/2)$. Since $W\equiv 0$ on $B_{\tau_2}(e_\tau)\setminus B_\tau(e_\tau)$ and $t^{1-2s}\partial_t W\in C(\overline{\mathbf{B}^+_{\tau_2}(e_\tau)}\setminus\overline{\mathbf{B}^+_\tau(e_\tau)})$, by applying [3, Proposition 4.11], we can find a constant $c>0$ such that

$$W(x,t)\ge ct^{2s} \qquad \text{for } (x,t)\in\partial\mathbf{B}^+_{\tau_2}(e_\tau)\setminus B_{\tau_2}(e_\tau). \tag{5.7}$$

We note that for $e\in B_\tau(e_\tau)$, we have $W(e,0)>0$ because otherwise if $W(e,0)=0$ then by [3, Proposition 4.11], we would have

$$0>-\lim_{t\to0} t^{1-2s}\partial_t W(e,t) = -V(e)W(e,0)=0$$

which is not possible. Therefore, fixing $\tau_1\in(0,\tau)$ from now on, we deduce, by compactness and the continuity of W, that

$$W(z)\ge c \qquad \text{for all } z\in\overline{\mathbf{B}^+_{\tau_1}(e_\tau)} \tag{5.8}$$

after making $c > 0$ smaller if necessary. Next, we choose a nonnegative and nontrivial function $f \in C_c^\infty(B_{\tau_1}(e_\tau))$, and we let $\phi \in \mathcal{H}^s(B_\tau(e_\tau))$ be the unique solution to $(-\Delta)^s\phi = f$ in $B_\tau(e_\tau)$. By the fractional Hopf lemma (see e.g. [13, Proposition 3.3]), we have that

$$\lim_{x\to 0} \frac{\phi(x)}{(\tau - |x - e_\tau|)_+^s} > 0. \tag{5.9}$$

Let $\widetilde{\Phi} \in D^{1,2}_{B_\tau(e_\tau)}(\mathbb{R}^{N+1}_+; t^{1-2s}) \cap C(\overline{\mathbb{R}^{N+1}_+})$ denote the s-harmonic extension of ϕ. It then follows from the Poisson kernel representation and the fact that $\phi = 0$ in $\mathbb{R}^N \setminus B_\tau(e_\tau)$ that

$$\widetilde{\Phi}(x,t) \le c' t^{2s} \qquad \text{for all } (x,t) \in \partial\mathbf{B}^+_{\tau_2}(e_\tau) \setminus B_{\tau_2}(e_\tau).$$

and that

$$\widetilde{\Phi}(z) \le c' \qquad \text{for all } z \in \overline{\mathbf{B}^+_{\tau_1}(e_\tau)},$$

for some constant $c' > 0$. We then fix $\eta > 0$ with $c > \eta c'$. By (5.3), (5.7) and (5.8), the function $\Psi := W - \eta\widetilde{\Phi}$ satisfies

$$\begin{cases} -\mathrm{div}(t^{1-2s}\nabla\Psi) \ge 0 & \text{in } A \\ -\lim_{t\to 0} t^{1-2s}\Psi(x,t) \ge -VW(x,0) & \text{for } x \in B_\tau(e_\tau) \setminus \overline{B_{\tau_1}(e_\tau)}, \\ \Psi(x,0) = 0 & \text{for } x \in B_{\tau_2}(e_\tau) \setminus \overline{B_\tau(e_\tau)}, \\ \Psi \ge 0 & \text{on } \partial A \cap \{t > 0\}, \end{cases}$$

where $A := \mathbf{B}^+_{\tau_2}(e_\tau) \setminus \overline{\mathbf{B}^+_{\tau_1}(e_\tau)}$. Here we used that $f = 0$ on $B_\tau(e_\tau) \setminus B_{\tau_1}(e_\tau)$. Multiplying the above equation with $\Psi_- = \max(-\Psi, 0)$ and integrating by parts, we get

$$-\int_A |\nabla\Psi_-|^2 t^{1-2s}\,dxdt \ge -\int_{B_\tau(e_\tau)\setminus B_{\tau_1}(e_\tau)} VW\Psi_-\,dx.$$

Since $\Psi_- = 0$ on $(\partial A \cap \{t > 0\}) \cup \partial B_{\tau_2}(e_\tau) \times \{0\}$, using the Hölder's and Sobolev trace inequalities as in (5.5), we obtain

$$\begin{aligned} -\int_A |\nabla\Psi_-|^2 t^{1-2s}\,dxdt &\ge -\|W\|_{L^2(B_\tau(e_\tau);|V|dx)} \|\Psi_-\|_{L^2(B_\tau(e_\tau);|V|dx)} \\ &\ge -C\|W\|_{L^\infty(B_\tau(e_\tau))} \|V\|^{\frac12}_{L^1(B_\tau(e_\tau))} \|V\|^{\frac12}_{L^q(B_\tau(e_\tau))} \left(\int_A |\nabla\Psi_-|^2 t^{1-2s}\,dxdt\right)^{\frac12}, \end{aligned}$$

with $C = C(N,s,r)$. From this and (5.6), provided $\tau > 0$ is small enough, we get $|\nabla\Psi_-| = 0$ on A. Hence $\Psi_- \equiv 0$ in $\overline{A}$, which in particular implies that $W(x,0) \ge \eta\Phi(x,0) = \eta\phi(x)$ for all $x \in B_\tau(e_\tau)$. By (5.9) and (5.6), we therefore get $\liminf_{\rho\searrow 0} \frac{U(\rho\nu,0)}{\rho^s} > 0$, as claimed. $\square$

6. Appendix II: Some topological lemmas on curve intersection

In this appendix, we collect curve intersection properties which we have used in the previous sections. We start by citing the following lemma from [9, Lemma 7.4].

Lemma 6.1. *Let $x_1 < x_2 < x_3 < x_4$ be real numbers. Suppose that $\gamma, \eta : [0,1] \to \overline{\mathbb{R}^2_+}$ are continuous curves such that $\gamma(0) = (x_1, 0)$, $\gamma(1) = (x_3, 0)$, $\eta(0) = (x_2, 0)$, $\eta(1) = (x_4, 0)$. Then the curves γ and η intersect, i.e. there exists $t, \tilde{t} \in (0,1)$ with $\gamma(t) = \eta(\tilde{t})$.*

We have also used the following slight generalization.

Lemma 6.2. *Let $x_1 < x_2 < x_3 < x_4$ be real numbers, and let $\gamma, \eta : [0,1] \to \overline{\mathbb{R}^2_+}$ be continuous curves. Moreover, suppose that one of the following is satisfied.*

(i) We have $\gamma(0) = (x_1, 0)$, $\gamma(1) = (x_3, 0)$, $|\eta(0) - (x_2, 0)| < \mathrm{dist}((x_2, 0), \gamma([0, 1]))$ *and* $|\eta(1) - (x_4, 0)| < \mathrm{dist}((x_4, 0), \gamma([0, 1]))$

(ii) We have $\eta(0) = (x_2, 0)$, $\eta(1) = (x_4, 0)$, $|\gamma(0) - (x_1, 0)| < \mathrm{dist}((x_1, 0), \eta([0, 1]))$ *and* $|\gamma(1) - (x_3, 0)| < \mathrm{dist}((x_3, 0), \eta([0, 1]))$

Then the curves γ *and* η *intersect.*

Proof. We only prove (i), the proof of (ii) is very similar. For two points $a, b \in \mathbb{R}^2$, we let $[a, b] := \{ta + (1 - t)b \,:\, t \in [0, 1]\}$ denote the closed line segment joining a and b. Then assumption (i) implies that the line segments $[(x_2, 0), \eta(0)]$ and $[\eta(1), (x_4, 0)]$ do not intersect the curve γ. On the other hand, adding these line segments to the curve η, we obtain a curve $\widetilde{\eta} : [0, 1] \to \overline{\mathbb{R}^2_+}$ joining the points $(x_2, 0)$ and $(x_4, 0)$, so by Lemma 6.1 the curve $\widetilde{\eta}$ does intersect γ. It therefore follows that also the original curve η must intersect γ, as claimed. □

Lemma 6.3. *Let* $t_0 > 0$*, and let* $0 \leq x_2 < x_3 < x_4$*. Suppose that* $\gamma, \eta : [0, 1] \to \overline{\mathbb{R}_+ \times \mathbb{R}_+}$ *are continuous curves such that* $\gamma(0) = (0, t_0)$, $\gamma(1) = (x_3, 0)$, $\eta(0) = (x_2, 0)$, $\eta(1) = (x_4, 0)$*. Then the curves* γ *and* η *intersect.*

Proof. We define the continuous curve

$$\widetilde{\gamma} : [-1, 1] \to \overline{\mathbb{R}^2_+}, \qquad \widetilde{\gamma}(t) = \begin{cases} (-\gamma_1(|t|), \gamma_2(|t|)) & \text{if } t < 0, \\ (\gamma_1(t), \gamma_2(t)) & \text{if } t \geq 0. \end{cases}$$

This curve joins the points $(-x_3, 0)$ and $(x_3, 0)$. Since $-x_3 < x_2 < x_3 < x_4$, the curve $\widetilde{\gamma}$ must intersect η by Lemma 6.1. Since $\eta([0, 1]) \subset \overline{\mathbb{R}_+ \times \mathbb{R}_+}$, this implies that η intersects γ, as claimed. □

By the same argument as for Corollary 6.2, we can weaken the assumptions slightly to obtain the following statement.

Lemma 6.4. *Let* $t_0 > 0$*, and let* $0 \leq x_2 < x_3 < x_4$*. Suppose that* $\gamma, \eta : [0, 1] \to \overline{\mathbb{R}_+ \times \mathbb{R}_+}$ *are continuous curves such that* $\gamma(0) = (0, t_0)$, $\gamma(1) = (x_3, 0)$ *and*

$$|\eta(0) - (x_2, 0)| < \mathrm{dist}((x_2, 0), \gamma([0, 1])), \qquad |\eta(1) - (x_4, 0)| < dist((x_4, 0), \gamma([0, 1])).$$

Then the curves γ *and* η *intersect.*

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