Convergence of a Finite Volume Scheme for Compactly Heterogeneous Scalar Conservation Laws

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Abstract

We build a finite volume scheme for the scalar conservation law $\partial_t u + \partial_x(H(x,u)) = 0$ with initial condition $u_o \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R})$ for a wide class of flux function H, convex with respect to the second variable. The main idea for the construction of the scheme is to use the theory of discontinuous flux. We prove that the resulting approximating sequence converges in $\mathbf{L}^p_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$, $p \in [1,+\infty[$, to the entropy solution.

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Contents

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1 Introduction

Consider the following Cauchy problem for x-dependent scalar conservation law:

$$\begin{cases} \partial_t u(t,x) + \partial_x \left(H(x,u(t,x))\right) = 0 & (t,x) \in]0,+\infty[\times\mathbb{R} \\ u(0,x) = u_o(x) & x \in \mathbb{R}. \end{cases} \tag{CL}$$

Equations of this type often occur, for instance, in the modeling of physical phenomena related to traffic flow [43, 33, 22], porous media [21, 23] or sedimentation problems [15, 16].
It is known that solutions of (**CL**) are discontinuous, regardless of the regularity of the data. They are to be understood in the entropy sense of [29]. The following quantity often recurs below,

$$\forall x,u,k \in \mathbb{R}, \quad \Phi(x,u,k) := \operatorname{sgn}(u-k)\,\left(H(x,u) - H(x,k)\right). \tag{1.1}$$

Definition 1.1. Let $H \in \mathbf{C}^1(\mathbb{R}^2,\mathbb{R})$ and $u_o \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R})$. We say that $u \in \mathbf{L}^\infty(]0,+\infty[\times\mathbb{R},\mathbb{R})$ is an entropy solution to (**CL**) if for all $\varphi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^+\times\mathbb{R},\mathbb{R}^+)$ and $k\in\mathbb{R}$,

$$\begin{aligned} \int_0^{+\infty}\int_{\mathbb{R}} |u-k|\partial_t\varphi + \Phi(x,u,k)\partial_x\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^{+\infty}\int_{\mathbb{R}} \operatorname{sgn}(u-k)\partial_x H(x,k)\varphi \,\mathrm{d}x\,\mathrm{d}t \\ + \int_{\mathbb{R}} |u_o(x)-k|\varphi(0,x)\,\mathrm{d}x \geq 0. \end{aligned} \tag{1.2}$$

Remark 1.1. In appearance, Definition 1.1 is weaker than the classical [29, Definition 1] since it does not require the existence of a strong trace at the initial time. It is in particular more manageable to limit process. Nevertheless, it can be shown that if $H \in \mathbf{C}^3(\mathbb{R}^2,\mathbb{R})$, then Definition 1.1 ensures that each entropy solution admits a representative belonging to $\mathbf{C}([0,+\infty[,\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R},\mathbb{R}))$, see [13, Theorem 2.6].

In [29], the author proved existence, uniqueness and stability with the respect to the initial datum for (**CL**) in the framework of entropy solutions under, among others, the growth assumptions

$$\partial_u H \in \mathbf{L}^\infty(\mathbb{R}^2,\mathbb{R}) \quad \text{and} \quad \sup_{(x,u)\in\mathbb{R}^2} \left(-\partial^2_{xu}H(x,u)\right) < +\infty, \tag{1.3}$$

highlighting the fact that the author treated the space dependency in (**CL**) as a source term. In his paper, Kruzhkov proved existence using the vanishing viscosity method, that is through a parabolic regularization of (**CL**). The following example motivates the necessity to abandon it.

Example 1.1. Fix positive constants X, V_1, V_2, R_1, R_2, and let $\theta,\rho \in \mathbf{C}^3(\mathbb{R},]0,+\infty[)$ be such that $\theta(x) = V_1$, resp. $\rho(x) = R_1$, for $x < -X$ and $\theta(x) = V_2$, resp. $\rho(x) = R_2$, for $x > X$. Define

$$H(x,u) := \theta(x)\,u\left(1 - \frac{u}{\rho(x)}\right).$$

Then with this flow, (**CL**) is the so-called "LWR" (Lightill-Whitham, Richards) model [30, 35] for a flow of vehicles described by their density u along a rectilinear road with maximal speed, resp. density, smoothly varying from V_1 to V_2, resp. from R_1 to R_2. Clearly, H does not satisfy (1.3).

Remark 1.2. *A priori*, Kruzhkov results do not apply in the case of Example 1.1. For completeness, let us however mention that a truncation argument could be used to extend them if the initial datum takes values between the stationary solutions $u(t,x) = 0$ and $u(t,x) = \rho(x)$.

Recently, the authors of [13, 37] developed an alternative framework to tackle (**CL**), one not requiring (1.3) and inspired by Example 1.1:

$$\textbf{Smoothness}: \quad H \in \mathbf{C}^3(\mathbb{R}^2, \mathbb{R}). \tag{C3}$$

$$\textbf{Compact Heterogeneity}: \quad \exists X > 0,\ \forall (x,u) \in \mathbb{R}^2, \quad |x| \geq X \implies \partial_u H(x,u) = 0. \tag{CH}$$

$$\textbf{Strong Convexity}: \quad \forall x \in \mathbb{R},\ u \mapsto \partial_u H(x,u) \text{ is an increasing } \mathbf{C}^1\text{-diffeomorphism of } \mathbb{R} \text{ onto itself.} \tag{CVX}$$

Assumption (**CVX**) ensures that for all $x \in \mathbb{R}$, the mapping $u \mapsto H(x,u)$ is strictly convex. Naturally, it can be replaced by the strong concavity assumption: for all $x \in \mathbb{R}$, $u \mapsto \partial_u H(x,u)$ is a decreasing $\mathbf{C}^1$-diffeomorphism of $\mathbb{R}$ onto itself.
The condition (**CH**) expresses the compact spatial heterogeneity of H and is not, apparently, common in the context of conservation laws. We expect that it might be relaxed.
Like we previously mentioned, assumptions (**C3**)-(**CH**)-(**CVX**) comprise flows that do not fit in the classical Kruzhkov framework and that are relevant from the modeling point of view, for instance, the flow of Example 1.1 (in the concave case). On the other hand, Kruzhkov results apply to general balance laws in several space dimensions. Let us recall the following.

Theorem 1.2. *[13, Theorem 2.6 and Theorem 2.18] Assume that H satisfies (**C3**)–(**CVX**). Then for all $u_o \in \mathbf{L}^\infty(\mathbb{R}, \mathbb{R})$, the Cauchy problem (**CL**) admits a unique entropy solution $u \in \mathbf{L}^\infty(]0, +\infty[\times\mathbb{R}, \mathbb{R})$. Moreover, if $v_o \in \mathbf{L}^\infty(\mathbb{R}, \mathbb{R})$ and if v is the associated entropy solution, then there exists $L > 0$ such that for all $R > 0$ and $t > 0$,*

$$\int_{|x|\leq R} |u(t,x) - v(t,x)| \,\mathrm{d}x \leq \int_{|x|\leq R+Lt} |u_o(x) - v_o(x)| \,\mathrm{d}x$$

$$\int_{|x|\leq R} (u(t,x) - v(t,x))^+ \,\mathrm{d}x \leq \int_{|x|\leq R+Lt} (u_o(x) - v_o(x))^+ \,\mathrm{d}x\,.$$

The proof of existence in [13], like in [29], relies on the vanishing viscosity method, but there, the authors exploit the correspondence with the Hamilton-Jacobi equation (and its parabolic approximation), going back and forth between the two frameworks, and gathering information for both equations at each step.

Remark 1.3. To be precise, let us mention that in [13], the convexity assumption (**CVX**) is relaxed to a uniform coercivity assumption coupled to a genuine nonlinearity assumption:

$$\textbf{Uniform Coercivity}: \quad \forall h \in \mathbb{R},\ \exists\, \mathcal{U}_h \in \mathbb{R},\ \forall (x,u) \in \mathbb{R}^2, \quad |H(x,u)| \leq h \implies |u| \leq \mathcal{U}_h. \tag{UC}$$

$$\textbf{Weak Genuine NonLinearity}: \quad \text{for a.e. } x \in \mathbb{R}, \text{ the set } \{p \in \mathbb{R} \ :\ \partial^2_{uu} H(x,p) = 0\} \text{ has empty interior.} \tag{WGNL}$$

The strong convexity implies (**WGNL**), while (**UC**) follows from the fact that H admits a Nagumo function. Indeed, thanks to (**CH**) and (**CVX**), we can use [37, Lemma 8.1.3 and Corollary 8.1.4] which ensure that there exists a function $\phi \in \mathbf{C}(\mathbb{R}^+, \mathbb{R})$ that verifies:

$$\forall x, u \in \mathbb{R},\ H(x,u) \geq \phi(|u|) \quad \text{and} \quad \frac{\phi(r)}{r} \underset{r\to+\infty}{\longrightarrow} +\infty.$$

The convexity was used in [14] to characterize, for (**CL**), the attainable set and the set of initial data evolving at a prescribed time into a prescribed profile.

An alternative to the vanishing viscosity method to construct solutions to (**CL**), and this is the main focus of the paper, is to build and prove the convergence of finite volume schemes. The most recent results include the works of [20, 19, 11] (in the case $H(x,u) = \theta(x)f(u)$), [9] (convergence and error analysis), [10] (with a source term) or [42] (on a bounded domain). In all these works, the flux function enters the framework of Kruzhkov by satisfying whether (1.3), or the stronger requirement $\partial_x H \equiv 0$. To the author's knowledge, no convergence result is available for flux functions that do not verify (1.3).
The aim of this paper is precisely to build a finite volume scheme for (**CL**) and prove its convergence to the entropy solution under the assumptions (**C3**)–(**CVX**). The main result is Theorem 3.12, establishing the convergence of the sequence generated by the scheme described in Section 3. As a byproduct, we provide an alternate existence result for the Cauchy problem (**CL**), one that does rely on the vanishing viscosity method. The main difficulty to overcome in the convergence analysis of the scheme is the obtaining of *a priori* $\mathbf{L}^\infty$ bounds for the approximating sequence. Indeed, flux functions verifying (**C3**)–(**CVX**) are not globally Lipschitz therefore, assigning an *a priori* CFL condition is not straightforward. That is the main reason why we rely on the theory of discontinuous flux, see [1, 2, 3, 4, 15, 21, 26, 28, 32, 39] and the references therein, which, for completeness, we adapt in Section 2. In Section 3, we discretize the space dependency of H and the idea behind the construction of the scheme is to treat each interface as a discontinuous flux problem. Outside the compact $[-X, X]$, the scheme reduces to a standard three point monotone finite volume scheme. The contribution of the discontinuous flux theory is that we can build by hand discrete steady states of the scheme, see Lemma 2.7 and Lemma 3.3, making him, in a way, well-balanced. This is how we obtain $\mathbf{L}^\infty$ bounds, Lemma 3.4. For the convergence analysis, since global **BV** are not expected in the context of discontinuous flux, we rely on the compensated compactness method [31, 36], taking advantage of the genuine nonlinearity of H under assumption (**CVX**).

2 Discontinuous Flux Theory

We start by recalling results regarding conservation laws with discontinuous flux. For the purpose of this section, let us fix $f_l, f_r \in \mathbf{C}^3(\mathbb{R}, \mathbb{R})$ two convex functions satisfying (**CVX**). Consider:

$$\partial_t u + \partial_x (F(x,u)) = 0, \quad F(x,u) := \begin{cases} f_l(u) & \text{if } x \leq 0 \\ f_r(u) & \text{if } x > 0. \end{cases} \tag{2.1}$$

Since the works of [1, 2, 3, 4, 15, 21, 26, 28, 39], it is now well known that additionally to the conservation of mass (Rankine-Hugoniot condition)

$$\text{for a.e. } t > 0, \quad f_l(u(t, 0-)) = f_r(u(t, 0+)),$$

an entropy criterion must be imposed at the interface to select one solution. This choice is often guided by physical consideration. Among the most common criteria, we can cite: the minimal jump condition [21, 25, 24], the vanishing viscosity criterion [5, 6], or the flux maximization [3].
If Φ_l, resp. Φ_r, denotes the Kruzhkov entropy flux associated with f_l, resp. with f_r, then in this section, define $\Phi = \Phi(x,u)$ as the Kruzhkov entropy associated with F:

$$\forall x \in \mathbb{R},\ \forall a, b \in \mathbb{R}, \quad \Phi(x,a,b) := \operatorname{sgn}(a-b)(F(x,a) - F(x,b)) = \begin{cases} \Phi_l(a,b) & \text{if } x \leq 0 \\ \Phi_r(a,b) & \text{if } x > 0. \end{cases}$$

2.1 Dissipative germs

For the study of (2.1), we follow [4], where the traces of the solution at the interface $\{x = 0\}$ are explicitly treated. It will be useful to have a name for the critical points of $f_{l,r}$, say $\alpha_{l,r}$. Notice that for all

$y \in]\min f_r, +\infty[$, the equation $f_r(u) = y$ admits exactly two solutions, $S_r^-(y) < \alpha_r < S_r^+(y)$. When $y = \min f_r$, then $S_r^-(y) = S_r^+(y) = \alpha_r$. This motivates the following definition.

Definition 2.1. The admissibility germ for (2.1) is the subset $\mathcal{G}$ defined as the union of:

$$\begin{aligned} \mathcal{G}_1 &:= \left\{(k_l, k_r) \in \mathbb{R}^2 \ : \ k_l \geq \alpha_l, \ k_r = S_r^+(f_l(k_l))\right\} \\ \mathcal{G}_2 &:= \left\{(k_l, k_r) \in \mathbb{R}^2 \ : \ k_l \leq \alpha_l, \ k_r = S_r^-(f_l(k_l))\right\} \\ \mathcal{G}_3 &:= \left\{(k_l, k_r) \in \mathbb{R}^2 \ : \ k_l > \alpha_l, \ k_r = S_r^-(f_l(k_l))\right\}. \end{aligned} \tag{2.2}$$

The germ contains all the possible traces along $\{x = 0\}$ of the solutions to (2.1). Notice that by construction, any couple in the germ satisfies the Rankine-Hugoniot condition. Conversely, some couples verifying the Rankine-Hugoniot condition have been excluded from the germ, more precisely the ones belonging to

$$\left\{(k_l, k_r) \in \mathbb{R}^2 \ : \ k_l < \alpha_l, \ k_r = S_r^+(f_l(k_l))\right\}.$$

The reason lies in the following proposition, see in particular (2.4).

Proposition 2.2. *$\mathcal{G}$ defined in Definition 2.1 is a maximal $\mathbf{L}^1$-dissipative germ, meaning that hold:*
(i) For all $(u_l, u_r) \in \mathcal{G}$, $f_l(u_l) = f_r(u_r)$.
(ii) Dissipative inequality:

$$\forall (u_l, u_r), (k_l, k_r) \in \mathcal{G}, \quad \Phi_l(u_l, k_l) - \Phi_r(u_r, k_r) \geq 0. \tag{2.3}$$

(iii) Maximality condition: let $(u_l, u_r) \in \mathbb{R}^2$ such that $f_l(u_l) = f_r(u_r)$. Then

$$\forall (k_l, k_r) \in \mathcal{G}, \ \Phi_l(u_l, k_l) - \Phi_r(u_r, k_r) \geq 0 \implies (u_l, u_r) \in \mathcal{G}. \tag{2.4}$$

Proof. Point *(i)* is clear, by construction, while *(ii)* follows from a straightforward case by case study. We now prove *(iii)* by way of contradiction: let $(u_l, u_r) \in \mathbb{R}^2$ such that $f_l(u_l) = f_r(u_r)$ and assume that

$$\forall (k_l, k_r) \in \mathcal{G}, \ \Phi_l(u_l, k_l) - \Phi_r(u_r, k_r) \geq 0 \quad \text{and} \quad (u_l, u_r) \notin \mathcal{G}.$$

From the Rankine-Hugoniot condition, we deduce that $u_r = S_r^+(f_l(u_l))$, with $u_l < \alpha_l$. Take $k_r = S_r^-(f_(k_l))$ with $k_l \in]u_l, \alpha_l[$. Clearly, $(k_l, k_r) \in \mathcal{G}_2 \subset \mathcal{G}$, therefore by assumption, $\Phi_l(u_l, k_l) - \Phi_r(u_r, k_r) \geq 0$. But the choice of k_l raises the contradiction:

$$\Phi_l(u_l, k_l) - \Phi_r(u_r, k_r) = (f_l(k_l) - f_l(u_l)) - (f_r(u_r) - f_r(k_r)) = 2(f_l(k_l) - f_l(u_l)) < 0.$$

□

The theory of [4] proposes an abstract framework for the study of (2.1). For each germ satisfying the requirements of Proposition 2.2 holds, one can define a notion of solution for (2.1), see Theorem 2.4 *2.(i)-(ii)*, below. On the other hand, the authors of [3, 2] give a formula for the flux at the interface for Riemann problems of (2.1). In the following proposition, we link the two points of view. Let us adopt the notations:

$$\forall a, b \in \mathbb{R}, \quad a \wedge b := \min\{a, b\} \quad \text{and} \quad a \vee b := \max\{a, b\}. \tag{2.5}$$

Proposition 2.3. *For all $(u_l, u_r) \in \mathbb{R}^2$, define the interface flux:*

$$F_{\text{int}}(u_l, u_r) := \max\{f_l(u_l \vee \alpha_l), f_r(\alpha_r \wedge u_r)\}, \tag{2.6}$$

and the remainder term:

$$\mathcal{R}(u_l, u_r) := |F_{\text{int}}(u_l, u_r) - f_l(u_l)| + |F_{\text{int}}(u_l, u_r) - f_r(u_r)|. \tag{2.7}$$

Then the following points hold.
(i) For all $(u_l, u_r) \in \mathbb{R}^2$, $(u_l, u_r) \in \mathcal{G} \iff \mathcal{R}(u_l, u_r) = 0$.
(ii) For all $(u_l, u_r) \in \mathbb{R}^2$ and for all $(k_l, k_r) \in \mathcal{G}$, $\Phi_r(u_r, k_r) - \Phi_l(u_l, k_l) \leq \mathcal{R}(u_l, u_r)$.

Proof. The fact that we have an "explicit" description of $\mathcal{G}$ facilitates the proof.
(i) If $(u_l, u_r) \in \mathcal{G}_1 \cup \mathcal{G}_2$, then $u_l - \alpha_l$ and $u_r - \alpha_r$ have the same sign, therefore

$$F_{\text{int}}(u_l, u_r) = \begin{cases} f_l(u_l) & \text{if } u_l \geq \alpha_l \\ f_r(u_r) & \text{if } u_l < \alpha_l \end{cases} = f_l(u_l) = f_r(u_r).$$

On the other hand, if $(u_l, u_r) \in \mathcal{G}_3$, then

$$F_{\text{int}}(u_l, u_r) = \max\{f_l(u_l), f_r(u_r)\} = f_l(u_l) = f_r(u_r).$$

This ensures that $\mathcal{R}(u_l, u_r) = 0$.
Conversely, assume that $\mathcal{R}(u_l, u_r) = 0$. In particular, the Rankine-Hugoniot holds. We deduce that

$$\mathcal{R}(u_l, u_r) = 0 \implies (u_l, u_r) \in \mathcal{G} \text{ or } u_r = S_r^+(f_l(u_l)), \; u_l < \alpha_l.$$

But in the latter case, $F_{\text{int}}(u_l, u_r) = \max\{\min f_l, \min f_r\} \neq f_l(u_l)$. Therefore, $(u_l, u_r) \in \mathcal{G}$.
(ii) For clarity, set $\mu := \max\{\min f_l, \min f_r\}$ and notice that

$$\mathcal{R}(u_l, u_r) = \begin{cases} |f_l(u_l) - \mu| + |f_r(u_r) - \mu| & \text{if } u_l < \alpha_l \text{ and } u_r \geq \alpha_r \\ |f_l(u_l) - f_r(u_r)| & \text{otherwise.} \end{cases}$$

Let us call $q := \Phi_r(u_r, k_r) - \Phi_l(u_l, k_l)$. We proceed with a case by case study. We use the fact that $f_{l,r}$ is decreasing on $]-\infty, \alpha_{l,r}]$ and increasing on $[\alpha_{l,r}, +\infty[$.
Case 1: $u_l \leq k_l$ and $u_r \leq k_r$. Then $q = f_l(u_l) - f_r(u_r) \leq \mathcal{R}(u_l, u_r)$.
Case 2: $u_l > k_l$ and $u_r \leq k_r$. Then $q = (f_r(k_r) - f_r(u_r)) + (f_l(k_l) - f_l(u_l))$. Notice that because of the germ structure, the case $k_l < u_l < \alpha_l$ and $\alpha_r \leq u_r \leq k_r$ cannot happen. Therefore, in case 2, $\mathcal{R}(u_l, u_r) = |f_l(u_l) - f_r(u_r)|$. We have

- $k_l < u_l \leq \alpha_l$ and $u_r \leq k_r \leq \alpha_r$
  $\implies q \leq f_l(k_l) - f_l(u_l) = f_r(u_r) - f_l(u_l) + (f_r(k_r) - f_r(u_r)) \leq f_r(u_r) - f_l(u_l) \leq \mathcal{R}(u_l, u_r)$

  $k_l < u_l \leq \alpha_l$ and $u_r \leq \alpha_r < k_r$ does not happen
- $k_l < u_l \leq \alpha_l$ and $\alpha_r < u_r \leq k_r$ does not happen
- $\alpha_l < k_l < u_l$ and $u_r \leq k_r \leq \alpha_r \implies q \leq 0$
  $k_l < \alpha_l < u_l$ and $u_r \leq k_r \leq \alpha_r$
  $\implies q \leq f_l(k_l) - f_l(u_l) = f_r(u_r) - f_l(u_l) + (f_r(k_r) - f_r(u_r)) \leq f_r(u_r) - f_l(u_l) \leq \mathcal{R}(u_l, u_r)$
  $\alpha_l < k_l < u_l$ and $u_r \leq \alpha_r \leq k_r$
  $\implies q \leq f_r(k_r) - f_r(u_r) = f_l(u_l) - f_r(u_r) + (f_l(k_l) - f_l(u_l)) \leq f_l(u_l) - f_r(u_r) \leq \mathcal{R}(u_l, u_r)$
  $\alpha_l < k_l < u_l$ and $u_r \leq \alpha_r \leq k_r$ does not happen
- $\alpha_l < k_l < u_l$ and $\alpha_r < u_r \leq k_r$
  $\implies q \leq f_r(k_r) - f_r(u_r) = f_l(u_l) - f_r(u_r) + (f_l(k_l) - f_l(u_l)) \leq f_l(u_l) - f_r(u_r) \leq \mathcal{R}(u_l, u_r)$
  $k_l < \alpha_l < u_l$ and $\alpha_r < u_r \leq k_r$ does not happen.

When we say "does not happen", we mean that germ structure prevents the situation from occurring.
Case 3: $u_l \leq k_l$ and $u_r > k_r$. Similar to case 2.
Case 4: $u_l > k_l$ and $u_r > k_r$. Similar to case 1. $\square$

Remark 2.1. Notice that for all $k \in \mathbb{R}$,

$$\mathcal{R}(k,k) = \begin{cases} |\max\{\min f_l, \min f_r\} - f_l(k)| + |\max\{\min f_l, \min f_r\} - f_r(k)| & \text{if } \alpha_r \leq k < \alpha_l \\ |f_l(k) - f_r(k)| & \text{otherwise.} \end{cases}$$

2.2 Definitions of solution and uniqueness

We are now in position to properly give the definition of solutions for (2.1).

Theorem 2.4. *Define $\mathcal{G}$ as (2.2) and $\mathcal{R}$ as (2.7). Let $u_o \in \mathbf{L}^\infty(\mathbb{R}, \mathbb{R})$ and $u \in \mathbf{L}^\infty(]0,+\infty[\times\mathbb{R}, \mathbb{R})$. Then the following statements are equivalent.*
1. For all test functions $\varphi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ and for all $(k_l, k_r) \in \mathbb{R}^2$,

$$\int_0^{+\infty} \int_{\mathbb{R}} |u - \kappa(x)| \partial_t \varphi + \Phi(x, u, \kappa(x)) \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} |u_o(x) - \kappa(x)| \varphi(0, x) \, \mathrm{d}x \\ + \int_0^{+\infty} \mathcal{R}(k_l, k_r) \varphi(t, 0) \, \mathrm{d}t \geq 0. \tag{2.8}$$

2.(i) For all test functions $\varphi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^+ \times \mathbb{R}^*, \mathbb{R}^+)$ and for all $(k_l, k_r) \in \mathbb{R}^2$,

$$\int_0^{+\infty} \int_{\mathbb{R}} |u - \kappa(x)| \partial_t \varphi + \Phi(x, u, \kappa(x)) \partial_x \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_{\mathbb{R}} |u_o(x) - \kappa(x)| \varphi(0, x) \, \mathrm{d}x \geq 0. \tag{2.9}$$

2.(ii) For a.e. $t \in]0, +\infty[$, $(u(t, 0-), u(t, 0+)) \in \mathcal{G}$.
When one of these statements holds, we say that u is an entropy solution to (2.1) with initial datum u_o.

Proof. See the proof of [4, Theorem 3.18]. □

Remark 2.2. In both (2.8) and (2.9), κ denotes the piecewise constant function

$$\kappa(x) = \begin{cases} k_l & \text{if } x \leq 0 \\ k_r & \text{if } x > 0. \end{cases}$$

We see that in the second definition, the traces of the solution along $\{x = 0\}$ explicitly appear. The condition

$$\text{for a.e. } t > 0, \quad (u(t, 0-), u(t, 0+)) \in \mathcal{G}$$

is to be understood in the sense of strong traces. An important fact we stress is that it is not restrictive to assume that general entropy solutions, that is to say bounded functions verifying (2.9), admit strong traces. Usually, it is ensured provided a nondegeneracy assumption on the flux function. Here, since f_l and f_r are genuinely nonlinear in the sense that

$$\forall s \in \mathbb{R}, \quad \text{meas}\Big(\{p \in \mathbb{R} \ : \ f'_{l,r}(p) = s\}\Big) = 0,$$

existence of strong traces is well known since [41].
In Theorem 2.4, Definition 1 is well suited for passage to the limit of a.e. convergent sequences of exact or approximate solutions. This is the one we use to prove existence in the next section. On the other hand, Definition 2 is well-adapted to prove stability with respect to the initial datum.

Theorem 2.5. *Fix $u_o, v_o \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R})$. We denote by u, resp. v, an entropy solution to (2.1) with initial datum u_o, resp. v_o. Set*

$$U := \max\{\|u\|_{\mathbf{L}^\infty(]0,+\infty[\times\mathbb{R})}, \|v\|_{\mathbf{L}^\infty(]0,+\infty[\times\mathbb{R})}\}, \quad L := \max\left\{\sup_{|p|\le U}|f_l'(p)|, \sup_{|p|\le U}|f_r'(p)|\right\}.$$

Then for all $R>0$ and for all $t>0$,

$$\begin{aligned}\int_{|x|\le R}|u(t,x)-v(t,x)|\,\mathrm{d}x &\le \int_{|x|\le R+Lt}|u_o(x)-v_o(x)|\,\mathrm{d}x\\ \int_{|x|\le R}(u(t,x)-v(t,x))^+\,\mathrm{d}x &\le \int_{|x|\le R+Lt}(u_o(x)-v_o(x))^+\,\mathrm{d}x\,.\end{aligned} \tag{2.10}$$

Proof. See the proofs of [4, Theorems 3.11-3.19]. □

2.3 Numerical scheme

We now produce and prove the convergence of a simple finite volume scheme toward an entropy solution of (2.1) in the sense of (2.8).
Let $u_o \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R})$. Without loss of generality, we can assume that there exists $(m, M) \in \mathbb{R}^2$ with $M \ge \max\{\alpha_l, \alpha_r\}$ and $m \le \min\{\alpha_l, \alpha_r\}$, such that for a.e. $x \in \mathbb{R}$, $m \le u_o(x) \le M$.
Fix a spatial mesh size $\Delta x > 0$ and time step $\Delta t > 0$. We assume that the ratio $\lambda := \Delta t/\Delta x$ satisfies the CFL condition

$$2\lambda L \le 1, \quad L := \max\left\{\sup_{\underline{u}\le p\le\overline{u}}|f_l'(p)|, \sup_{\underline{u}\le p\le\overline{u}}|f_r'(p)|\right\}, \tag{2.11}$$

with

$$\underline{u} := \min\left\{S_l^-(f_r(m)), S_r^-(f_l(m))\right\} \quad \text{and} \quad \overline{u} := \max\left\{S_l^+(f_r(M)), S_r^+(f_l(M))\right\}. \tag{2.12}$$

These two values come from the choice of two particular steady states of the scheme, see Lemma 2.7 below.
For all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, set the notations $t^n = n\Delta t$, $x_j = j\Delta x$ and discretize the initial datum:

$$\forall j \in \mathbb{Z}, \quad u_j^o := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} u_o(x)\,\mathrm{d}x\,.$$

With reference to (2.6), define the numerical flux:

$$\forall j \in \mathbb{Z},\ \forall u, v \in \mathbb{R}, \quad F_j(u,v) := \begin{cases} \mathrm{God}_l(u,v) & \text{if } j \le -1\\ F_{\text{int}}(u,v) & \text{if } j = 0\\ \mathrm{God}_r(u,v) & \text{if } j \ge 1,\end{cases} \tag{2.13}$$

where $\mathrm{God}_{l,r}$ are the Godunov fluxes associated with $f_{l,r}$. The marching formula of the scheme reads

$$u_j^{n+1} = u_j^n - \lambda\Big(F_{j+1}(u_j^n, u_{j+1}^n) - F_j(u_{j-1}^n, u_j^n)\Big). \tag{2.14}$$

It is worth mentioning that by itself, F_{int} is a monotone, locally Lipschitz, numerical flux, see [2, Section 4]. Also, let us precise that in general, F_{int} is not consistent, meaning that for all $k \in \mathbb{R}$, $F_{\text{int}}(k,k)$ is not equal to $f_l(k)$ or to $f_r(k)$. For instance, if $\min f_l \ge \min f_r$, then we have:

$$F_{\text{int}}(k,k) = \begin{cases} \max\{\min f_l, f_r(k)\} & \text{if } \ k \le \alpha_l, \alpha_r\\ \min f_l & \text{if } \ \alpha_r \le k \le \alpha_l\\ \max\{f_l(k), f_r(k)\} & \text{if } \ \alpha_l < k \le \alpha_r\\ f_l(k) & \text{if } \ k > \alpha_l, \alpha_r.\end{cases} \tag{2.15}$$

Finally, define the following piecewise constant function:

$$u_\Delta := \sum_{n=0}^{+\infty} \sum_{j\in\mathbb{Z}} u_j^n \mathbf{1}_{[t^n,t^{n+1}[} \mathbf{1}_{[x_j,x_{j+1}[}.$$

It will be convenient to write the scheme under the form:

$$u_j^{n+1} = H_j(u_{j-1}^n, u_j^n, u_{j+1}^n), \tag{2.16}$$

where $H_j = H_j(a, b, c)$ is given by the right-hand side of (2.14).

2.3.1 Stability, entropy inequalities

Definition 2.6. We say that a sequence $(v_j)_j$ is a steady state of (2.16) if

$$\forall j \in \mathbb{Z}, \quad H_j(v_{j-1}, v_j, v_{j+1}) = v_j.$$

Lemma 2.7. *There exist two steady states of* (2.16), $(m_j)_j$ *and* $(M_j)_j$ *such that:*
(i) $\sup_j M_j = \max\left\{S_l^+(f_r(M)), S_r^+(f_l(M))\right\}$ *;*
(ii) $\inf_j m_j = \min\left\{S_l^-(f_r(m)), S_r^-(f_l(m))\right\}$ *;*
(iii) for all $j \in \mathbb{Z}$, $m_j \le u_j^o \le M_j$.

Proof. We only give the details for the construction of $(M_j)_j$. Assume that $f_l(M) \le f_r(M)$. Then, define

$$\forall j \in \mathbb{Z}, \quad M_j := \begin{cases} S_l^+(f_r(M)) & \text{if} \quad j \le -1 \\ M & \text{if} \quad j \ge 0. \end{cases}$$

It is readily checked that $(S_l^+(f_r(M)), M) \in \mathcal{G}$. This ensures that $(M_j)_j$ is a steady state of the scheme. Exploiting the monotonicity of f_l:

$$f_l\left(S_l^+(f_r(M))\right) = f_r(M) \ge f_l(M) \implies S_l^+(f_r(M)) \ge M.$$

On the other hand, if $f_l(M) < f_r(M)$, set instead:

$$\forall j \in \mathbb{Z}, \quad M_j := \begin{cases} M & \text{if} \quad j \le -1 \\ S_r^+(f_l(M)) & \text{if} \quad j \ge 0. \end{cases}$$

□

Steady states are now used to derive $\mathbf{L}^\infty$ bounds.

Theorem 2.8. *Consider the two steady states from Lemma 2.7. Then under the CFL condition* (2.11)-(2.12)*, the scheme* (2.16) *is monotone, meaning that for all* $j \in \mathbb{Z}$, H_j *is nondecreasing with respect to its three variables. Consequently, for all* $n \in \mathbb{N}$,

$$\underline{u} \le m_j \le u_j^n \le M_j \le \overline{u}. \tag{2.17}$$

Proof. We proceed by induction. By construction of the steady states, (2.17) holds for $n = 0$. Assume that it holds for some $n \in \mathbb{N}$. The CFL condition ensures that for all $j \in \mathbb{Z}$, H_j is nondecreasing with respect to its three variables. To see it, one simply has to differentiate H_j with respect to its three variables. By monotonicity, for all $j \in \mathbb{Z}$,

$$\begin{aligned} m_j \le u_j^n \le M_j &\implies H_j(m_{j-1}, m_j, m_{j+1}) \le H_j(u_{j-1}^n, u_j^n, u_{j+1}^n) \le H_j(M_{j-1}, M_j, M_{j+1}) \\ &\implies m_j \le u_j^{n+1} \le M_j, \end{aligned}$$

concluding the induction. □

Lemma 2.9. *Let $k_l, k_r \in [\underline{u}, \overline{u}]$. Define the sequence*

$$\forall j \in \mathbb{Z}, \quad \kappa_j := \begin{cases} k_l & \text{if} \quad j \leq -1 \\ k_r & \text{if} \quad j \geq 0. \end{cases}$$

Then, under (2.11)-(2.12), the scheme (2.16) satisfies the following discrete entropy inequalities:

$$\begin{aligned} \forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad & \Big(|u_j^{n+1} - \kappa_j| - |u_j^n - \kappa_j|\Big)\,\Delta x + (\Phi_{j+1/2}^n - \Phi_{j-1/2}^n)\Delta t \\ & \leq \Big\{ |F_{\text{int}}(k_l, k_r) - f_l(k_l)|\delta_{j=-1} + |F_{\text{int}}(k_l, k_r) - f_r(k_r)|\delta_{j=0} \Big\} \Delta t, \end{aligned} \tag{2.18}$$

where

$$\Phi_{j+1/2}^n := F_{j+1}(u_j^n \vee \kappa_j, u_{j+1}^n \vee \kappa_{j+1}) - F_{j+1}(u_j^n \wedge \kappa_j, u_{j+1}^n \wedge \kappa_{j+1}).$$

Proof. If $j \notin \{-1, 0\}$, (2.18) is a standard consequence of the monotonicity of the scheme and of the fact that constants are preserved away from the interface.
Assume now for instance that $j = -1$. First, notice that

$$k_l = H_{-1}(k_l, k_l, k_r) + \lambda\left(F_{\text{int}}(k_l, k_r) - f_l(k_l)\right),$$

implying, by monotonicity, that

$$\begin{aligned} & H_{-1}(u_{j-1}^n \wedge \kappa_{j-1}, u_j^n \wedge \kappa_j, u_{j+1}^n \wedge \kappa_{j+1}) - \lambda\left(F_{\text{int}}(k_l, k_r) - f_l(k_l)\right)^- \\ & \leq k_l \leq H_{-1}(u_{j-1}^n \vee \kappa_{j-1}, u_j^n \vee \kappa_j, u_{j+1}^n \vee \kappa_{j+1}) + \lambda\left(F_{\text{int}}(k_l, k_r) - f_l(k_l)\right)^+. \end{aligned}$$

Consequently,

$$\begin{aligned} |u_j^{n+1} - \kappa_j| &= \max\{u_j^{n+1}, \kappa_j\} - \min\{u_j^{n+1}, \kappa_j\} \\ &\leq H_{-1}(u_{j-1}^n \vee \kappa_{j-1}, u_j^n \vee \kappa_j, u_{j+1}^n \vee \kappa_{j+1}) + \lambda\left(F_{\text{int}}(k_l, k_r) - f_l(k_l)\right)^+ \\ &\quad - H_{-1}(u_{j-1}^n \wedge \kappa_{j-1}, u_j^n \wedge \kappa_j, u_{j+1}^n \wedge \kappa_{j+1}) + \lambda\left(F_{\text{int}}(k_l, k_r) - f_l(k_l)\right)^- \\ &= |u_j^n - \kappa_j| - \lambda\left(\Phi_{j+1/2}^n - \Phi_{j-1/2}^n\right) + \lambda|F_{\text{int}}(k_l, k_r) - f_l(k_l)|, \end{aligned}$$

which is exactly (2.18) in the case $j = -1$. The proof for the case $j = 0$ is similar, so we omit its details.
□

Theorem 2.10. *Assume that (2.11)-(2.12) hold. Let $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ and $k_l, k_r \in [\underline{u}, \overline{u}]$. Let $T > 0$ such that $\varphi(t, x) = 0$ if $t \geq T$ and $x \in \mathbb{R}$. Set $\kappa = k_l \mathbf{1}_{\mathbb{R}^-} + k_r \mathbf{1}_{\mathbb{R}^+}$. Then, there exist $c_1, c_2 > 0$ depending only on $\underline{u}$, $\overline{u}$, T, φ, f_l and f_r such that*

$$\begin{aligned} & \int_0^{+\infty} \int_{\mathbb{R}} |u_\Delta - \kappa(x)|\partial_t \varphi + \Phi(x, u_\Delta, \kappa(x))\partial_x \varphi \,\, \mathrm{d}x\, \mathrm{d}t \\ & + \int_{\mathbb{R}} |u_\Delta(0, x) - \kappa(x)|\varphi(0, x)\, \mathrm{d}x + \int_0^{+\infty} \mathcal{R}(k_l, k_r)\varphi(t, 0)\, \mathrm{d}t \\ \geq & -c_1(\Delta t + \Delta x) - c_2 \int_0^{+\infty} \int_{\mathbb{R}} |(u_\Delta(t, x + \Delta x) - u_\Delta(t, x))\partial_x \varphi(t, x)|\, \mathrm{d}x\, \mathrm{d}t. \end{aligned} \tag{2.19}$$

Proof. Define for all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, $\varphi_j^n := \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} \varphi(t^n, x)\,\mathrm{d}x$. Multiply (2.18) by φ_j^n, take the double sum and apply Abel summation by parts. We obtain $A + B + C \geq 0$, where

$$
\begin{aligned}
A &= \sum_{n=1}^{+\infty}\sum_{j\in\mathbb{Z}} |u_j^n - \kappa_j|(\varphi_j^n - \varphi_j^{n-1})\Delta x + \sum_{j\in\mathbb{Z}} |u_j^o - \kappa_j|\varphi_j^o \Delta x\\
B &= \sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}} \Phi_{j+1/2}^n(\varphi_{j+1}^n - \varphi_j^n)\Delta t\\
C &= \sum_{n=0}^{+\infty}\Big\{|F_{\text{int}}(k_l, k_r) - f_l(k_l)|\varphi_{-1}^n + |F_{\text{int}}(k_l, k_r) - f_r(k_r)|\varphi_o^n\Big\}\Delta t.
\end{aligned}
$$

Term A**.** Clearly,

$$
\begin{aligned}
A &= \int_{\Delta t}^{+\infty}\int_{\mathbb{R}} |u_\Delta - \kappa(x)|\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |u_\Delta(0,x) - \kappa(x)|\varphi(0,x)\,\mathrm{d}x\\
&\leq \int_0^{+\infty}\int_{\mathbb{R}} |u_\Delta - \kappa(x)|\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |u_\Delta(0,x) - \kappa(x)|\varphi(0,x)\,\mathrm{d}x + (\overline{u} - \underline{u})\|\partial_t\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+,\mathbf{L}^1)}\Delta t.
\end{aligned}
$$

Term B**.** For all $j \in \mathbb{N}$,

$$
\begin{aligned}
\Phi_{j+1/2}^n = &\underbrace{\text{God}_r(u_j^n \vee k_r, u_{j+1}^n \vee k_r) - \text{God}_r(u_j^n \vee k_r, u_j^n \vee k_r)}_{B_1} + \underbrace{\Phi_r(u_j^n, k_r)}_{B_2}\\
&+ \underbrace{\text{God}_r(u_j^n \wedge k_r, u_j^n \wedge k_r) - \text{God}_r(u_j^n \wedge k_r, u_{j+1}^n \wedge k_r)}_{B_3}.
\end{aligned}
$$

We estimate $B_1 + B_3$ as

$$
\begin{aligned}
\left|\sum_{n=0}^{+\infty}\sum_{j\geq 0}(B_1 + B_3)(\varphi_{j+1}^n - \varphi_j^n)\Delta t\right| &\leq 2L\Big(T\|\partial_{xx}^2\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+,\mathbf{L}^1)} + \|\partial_{tx}^2\varphi\|_{\mathbf{L}^1}\Big)(\Delta x + \Delta t)\\
&\quad + 2L\sum_{n=0}^{+\infty}\sum_{j\geq 0}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} |u_{j+1}^n - u_j^n| \cdot |\partial_x\varphi(t,x)|\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
$$

For B_2, we have

$$
\begin{aligned}
&\sum_{n=0}^{+\infty}\sum_{j\geq 0} B_2(\varphi_{j+1}^n - \varphi_j^n)\Delta t\\
&= \Delta t\sum_{n=0}^{+\infty}\sum_{j\geq 0}\int_{x_j}^{x_{j+1}} \Phi_r(u_j^n, k_r)\left(\frac{\varphi(t^n, x + \Delta x) - \varphi(t^n, x)}{\Delta x} - \partial_x\varphi(t^n, x)\right)\mathrm{d}x\\
&+ \sum_{n=0}^{+\infty}\sum_{j\geq 1}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} \Phi_r(u_j^n, k_r)(\partial_x\varphi(t^n, x) - \partial_x\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t\\
&+ \int_0^{+\infty}\int_0^{+\infty} \Phi(x, u_\Delta, \kappa(x))\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\leq T\sup_{\underline{u}\leq p\leq\overline{u}} |\Phi_r(p, k_r)| \cdot \|\partial_{tx}^2\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+,\mathbf{L}^1)}\Delta x + \sup_{\underline{u}\leq p\leq\overline{u}} |\Phi_r(p, k_r)| \cdot \|\partial_{tx}^2\varphi\|_{\mathbf{L}^1}\Delta t\\
&+ \int_0^{+\infty}\int_0^{+\infty} \Phi(x, u_\Delta, \kappa(x))\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}
$$

We do similar computations for $j \leq -2$. For $j=-1$, write

$$\left|\sum_{n=0}^{+\infty} \Phi^n_{-1/2}(\varphi^n_{j+1}-\varphi^n_j)\Delta t\right| \leq 2T \left\{\sup_{\underline{u}\leq p\leq \overline{u}} |f_l(p)| + |f_r(p)|\right\} \|\partial_x \varphi\|_{\mathbf{L}^\infty}\Delta x.$$

Term C. Finally, write

$$\left|C - \int_0^{+\infty} \mathcal{R}(k_l,k_r)\varphi(t,0)\,\mathrm{d}t\right| \leq 4T \left\{\sup_{\underline{u}\leq p\leq \overline{u}} |f_l(p)| + |f_r(p)|\right\} \left(\|\partial_x\varphi\|_{\mathbf{L}^\infty}\Delta x + \|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t\right).$$

□

2.3.2 Compactness and convergence

The final step is to pass to the limit in (2.19). For that, we prove the strong compactness of $(u_\Delta)_\Delta$.

Theorem 2.11. *Let $u_o \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R})$ and let $(u_\Delta)_\Delta$ be the sequence generated by the scheme described in Section 2.3. Then there exists a subsequence of $(u_\Delta)_\Delta$ that converges in $\mathbf{L}^p_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$ for all $p\in[1,+\infty[$ and a.e. on $]0,+\infty[\times\mathbb{R}$ to some function $u\in\mathbf{L}^\infty(]0,+\infty[\times\mathbb{R},\mathbb{R})$.*

Proof. Let us mention that if $u_o \in \mathbf{BV}_{\mathbf{loc}}(\mathbb{R},\mathbb{R})$, since (2.1) is invariant by time translation, one can derive localized **BV** estimates away from the interface following [7, Lemma 4.2] or [8, Lemmas 5.3, 5.4]. The strict convexity of f_l and f_r is essential.
If $u_o \notin \mathbf{BV}(\mathbb{R},\mathbb{R})$, one can derive one-sided Lipschitz bounds on $(u_\Delta)_\Delta$, provided that f_l and f_r are strictly convex, following [40, Section 4] or [38, Section 2.4]. This approach also requires to choose either the Godunov flux or the Engquist-Osher flux away from the interface, that is when $j\neq 0$ in (2.13). □

Theorem 2.12. *Let $u_o\in\mathbf{L}^\infty(\mathbb{R},\mathbb{R})$ and fix $(m,M)\in\mathbb{R}^2$ with $M\geq\max\{\alpha_l,\alpha_r\}$ and $m\leq\min\{\alpha_l,\alpha_r\}$, such that for a.e. $x\in\mathbb{R}$, $m\leq u_o(x)\leq M$. Let $(u_\Delta)_\Delta$ be the sequence generated by the scheme described in Section 2.3 and let u be the limit function from Theorem 2.11. Then u is an entropy solution to (2.1) with initial datum u_o in the sense of (2.8). Moreover, for a.e. $(t,x)\in]0,+\infty[\times\mathbb{R}$,*

$$\min\left\{S_l^-(f_r(m)), S_r^-(f_l(m))\right\} \leq u(t,x) \leq \max\left\{S_l^+(f_r(M)), S_r^+(f_l(M))\right\}. \tag{2.20}$$

Proof. It suffices to pass to the limit in (2.19).
Let us just say that since $(u_\Delta)_\Delta$ converges a.e. on $]0,+\infty[\times\mathbb{R}$, for all test functions φ, the sequence $(u_\Delta\partial_x\varphi)_\Delta$ is strongly compact in $\mathbf{L}^1(]0,+\infty[\times\mathbb{R},\mathbb{R})$. As a consequence of the Riesz-Fréchet-Kolmogorov compactness characterization, for all $\varphi\in\mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}^+\times\mathbb{R},\mathbb{R})$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}} |(u_\Delta(t,x+\Delta x)-u_\Delta(t,x))\partial_x\varphi(t,x)|\,\mathrm{d}x\,\mathrm{d}t \underset{\Delta\to 0}{\longrightarrow} 0.$$

□

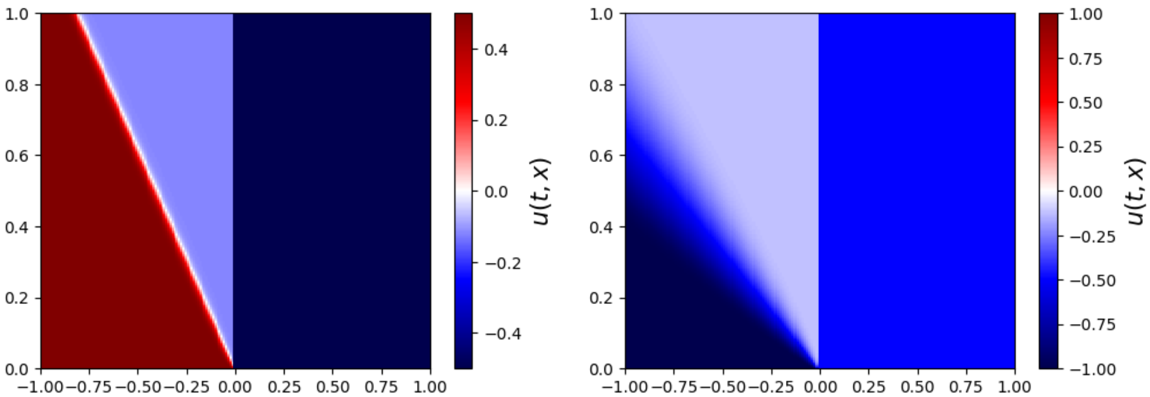
2.4 The Riemann problem

Let us explain how to solve the Riemann problem for (2.1). To fix the ideas, assume that $\min f_l \le \min f_r$. Given $(u_l, u_r) \in \mathbb{R}^2$, first compute the interface flux given by (2.6). We have:

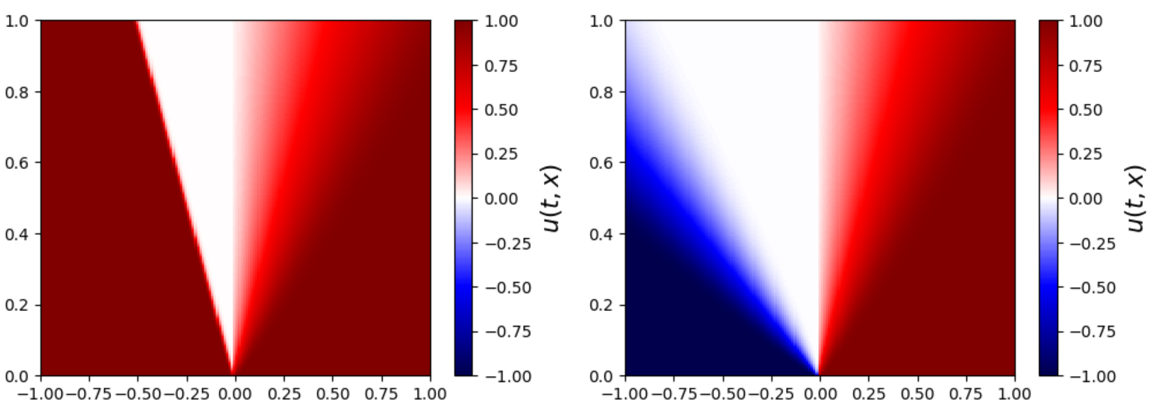
$$F_{\mathrm{int}}(u_l, u_r) = \begin{cases} f_r(u_r) & \text{if} \quad u_l \le \alpha_l \text{ and } u_r \le \alpha_r \quad \text{(I)} \\ \min f_r & \text{if} \quad u_l \le \alpha_l \text{ and } u_r > \alpha_r \quad \text{(II)} \\ \max\{f_l(u_l), f_r(u_r)\} & \text{if} \quad u_l > \alpha_l \text{ and } u_r \le \alpha_r \quad \text{(III)} \\ \max\{f_l(u_l), \min f_r\} & \text{if} \quad u_l > \alpha_l \text{ and } u_r > \alpha_r \quad \text{(IV)} \end{cases} \tag{2.21}$$

The value of $F_{\mathrm{int}}(u_l, u_r)$ imposes the value of one the traces at the interface. For instance, if $F_{\mathrm{int}}(u_l, u_r) = f_r(u_r)$, then the right trace is equal to u_r. Then to determine the left trace, one solves $f_l(\gamma) = f_r(u_r)$, with the requirement that the solution to the classical Riemann problem with flux f_l and states (γ, u_r) only displays waves of negative speeds. We list below the different structure of the solution.

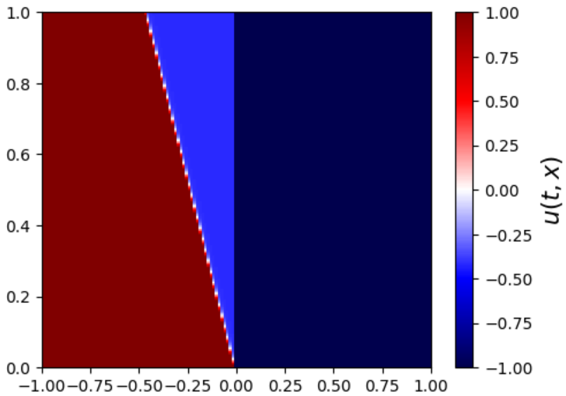
- If $(u_l, u_r) \in \mathcal{G}$, then the solution is a stationary non-classical shock (SNS).
- (I) in (2.21): gluing of shock/rarefaction-SNS-constant u_r. In the case $\min f_l \ge \min f_r$, the solution is instead the gluing of rarefaction-SNS-shock, or the gluing of shock/rarefaction-SNS-constant u_r.



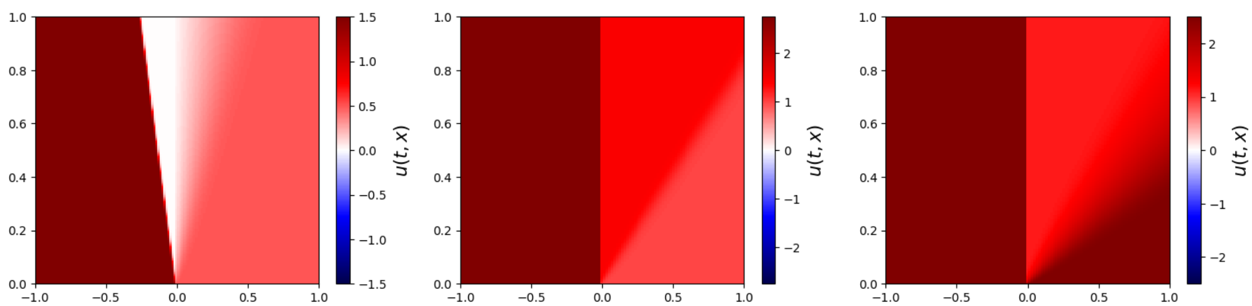
- (II) in (2.21): gluing of shock/rarefaction-SNS-rarefaction. In the case $\min f_l \ge \min f_r$, the solution is instead the gluing of rarefaction-SNS-shock/rarefaction.



• (III) in (2.21): gluing of shock-SNS-constant u_r, or gluing of constant u_l-SNS-shock.



• (IV) in (2.21): gluing of shock-SNS-rarefaction, or gluing of constant u_l-SNS-shock/rarefaction. In the case $\min f_l \geq \min f_r$, the solution is instead the gluing of constant u_l-SNS-shock/rarefaction.



2.5 Extension

We now extend the results of Sections 2.1-2.2 to the case where F in (2.1) presents a finite number of space discontinuities. More precisely, let us fix $\mathcal{P} \in \mathbb{N}^*$, $(y_p)_{p\in[\![1;\mathcal{P}]\!]}$ an increasing finite sequence of numbers, and $(h_p)_{p\in[\![0;\mathcal{P}]\!]}$ a family of $\mathbf{C}^3(\mathbb{R},\mathbb{R})$ convex functions satisfying (**CVX**). For all $p \in [\![0;\mathcal{P}]\!]$, we call α_p the critical point of h_p. Consider

$$\partial_t u + \partial_x (F(x,u)) = 0, \quad F(x,u) := \sum_{p=0}^{\mathcal{P}} h_p(u)\mathbf{1}_{]y_p,y_{p+1}[}(x), \quad y_o := -\infty,\ y_{\mathcal{P}+1} := +\infty. \tag{2.22}$$

For all $p \in [\![1;\mathcal{P}]\!]$, we can define a germ, following Section 2.1.

Definition 2.13. The admissibility family of germs for (2.22) is the family $(\mathcal{G}_p)_{p\in[\![1;\mathcal{P}]\!]}$, where for all $p \in [\![1;\mathcal{P}]\!]$, $\mathcal{G}_p$ is the germ defined as the union of the following subsets:

$$\begin{aligned}
\mathcal{G}_{1,p} &:= \left\{(k_l,k_r) \in \mathbb{R}^2 \ :\ k_l \geq \alpha_p,\ k_r = S^{+}_{p+1}(h_p(k_l))\right\} \\
\mathcal{G}_{2,p} &:= \left\{(k_l,k_r) \in \mathbb{R}^2 \ :\ k_l \leq \alpha_p,\ k_r = S^{-}_{p+1}(h_p(k_l))\right\} \\
\mathcal{G}_{3,p} &:= \left\{(k_l,k_r) \in \mathbb{R}^2 \ :\ k_l > \alpha_p,\ k_r = S^{-}_{p+1}(h_p(k_l))\right\}.
\end{aligned} \tag{2.23}$$

For all $p \in [\![1;\mathcal{P}]\!]$, $\mathcal{G}_p$ is a maximal $\mathbf{L}^1$-dissipative germ in the sense of Proposition 2.2. Then, at each interface, for all $(u_l, u_r) \in \mathbb{R}^2$, we define a flux

$$F^p_{\rm int}(u_l, u_r) := \max\{h_p(u_l \vee \alpha_p),\ h_{p+1}(\alpha_{p+1} \wedge u_r)\} \tag{2.24}$$

and the remainder term

$$\mathcal{R}_{\mathcal{G}_p}(u_l, u_r) := |F^p_{\rm int}(u_l, u_r) - h_p(u_l)| + |F^p_{\rm int}(u_l, u_r) - h_{p+1}(u_r)|. \tag{2.25}$$

Following Section 2.2, we give the definition of solutions.

Theorem 2.14. *Let $u_o \in \mathbf{L}^\infty(\mathbb{R}, \mathbb{R})$ and $u \in \mathbf{L}^\infty(]0, +\infty[\times\mathbb{R}, \mathbb{R})$. Define $\Gamma := \bigcup_{p=1}^{\mathcal{P}} \{(t, y_p)\ :\ t \geq 0\}$. Then the following statements are equivalent.*
1. For all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ and for all $(k_p)_{p\in[\![1;\mathcal{P}+1]\!]} \in \mathbb{R}^{\mathcal{P}+1}$,

$$\begin{aligned}\int_0^{+\infty}\int_{\mathbb{R}} |u - \kappa(x)|\ \partial_t\varphi + \Phi(x, u, \kappa(x))\partial_x\varphi\ \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |u_o(x) - \kappa(x)|\varphi(0, x)\,\mathrm{d}x& \\ + \sum_{p=1}^{\mathcal{P}} \int_0^{+\infty} \mathcal{R}_{\mathcal{G}_p}(k_p, k_{p+1})\varphi(t, y_p)\,\mathrm{d}t \geq 0.&\end{aligned} \tag{2.26}$$

2.(i) For all test functions $\varphi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}^+ \times (\mathbb{R}\backslash\Gamma), \mathbb{R}^+)$ and for all $(k_p)_{p\in[\![1;\mathcal{P}+1]\!]} \in \mathbb{R}^{\mathcal{P}+1}$,

$$\int_0^{+\infty}\int_{\mathbb{R}} |u - \kappa(x)|\ \partial_t\varphi + \Phi(x, u, \kappa(x))\partial_x\varphi\ \,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |u_o(x) - \kappa(x)|\varphi(0, x)\,\mathrm{d}x \geq 0. \tag{2.27}$$

2.(ii) For all $p \in [\![1;\mathcal{P}]\!]$, for a.e. $t \in]0, +\infty[$, $(u(t, y_p-), u(t, y_p+)) \in \mathcal{G}_p$.
When one these statements holds, we say that u is an entropy solution to (2.22) with initial datum u_o.

Remark 2.3. In both (2.26) and (2.27), κ denotes the piecewise constant function

$$\kappa(x) = \sum_{p=0}^{\mathcal{P}} k_p \mathbf{1}_{]y_p, y_{p+1}[}(x).$$

The proof of Theorem 2.14 follows from an obvious adaptation of the proof of Theorem 2.4.

Theorem 2.15. *Fix $u_o, v_o \in \mathbf{L}^\infty(\mathbb{R}, \mathbb{R})$. We denote by u, resp. v, an entropy solution to (2.1) with initial datum u_o, resp. v_o. Set*

$$U := \max\{\|u\|_{\mathbf{L}^\infty(]0,+\infty[\times\mathbb{R})}, \|v\|_{\mathbf{L}^\infty(]0,+\infty[\times\mathbb{R})}\}, \quad L := \sup_{\substack{p\in[\![0;\mathcal{P}]\!] \\ |k|\leq U}} |h_p'(k)|.$$

Then for all $R > 0$ and for all $t > 0$,

$$\begin{aligned}\int_{|x|\leq R} |u(t, x) - v(t, x)|\,\mathrm{d}x &\leq \int_{|x|\leq R+Lt} |u_o(x) - v_o(x)|\,\mathrm{d}x \\ \int_{|x|\leq R} (u(t, x) - v(t, x))^+\,\mathrm{d}x &\leq \int_{|x|\leq R+Lt} (u_o(x) - v_o(x))^+\,\mathrm{d}x\,.\end{aligned} \tag{2.28}$$

Proof. Straightforward adaptation of the proof of Theorem 2.5. □

Regarding the existence, let us only stress that for small times (so that the emerging waves from the discontinuity points of F do not interact), the Riemann problem for (2.22) is solved by patching together p solutions to "simple" Riemann problems as described in Section 2.4.

3 From Discontinuous to Continuous

We now take advantage of Section 2 to develop a scheme for (**CL**). After discretizing the space dependency of H, we treat each interface as a discontinuous flux problem. Outside the compact $[-X, X]$, the scheme reduces to a standard three point monotone finite volume scheme. The contribution of the discontinuous flux theory is in Lemmas 3.3-3.4, where we constructed steady states of the scheme to derive *a priori* $\mathbf{L}^\infty$ bounds.

3.1 Definition of the scheme

We describe a semi-Godunov scheme for (**CL**) under Assumptions (**C3**)–(**CVX**).

Lemma 3.1. *Assume that H satisfies (**C3**)–(**CVX**). Then, there exists a unique function $\alpha \in \mathbf{C}^2(\mathbb{R}, \mathbb{R})$ such that for all $x \in \mathbb{R}$, $\partial_u H(x, \alpha(x)) = 0$. Moreover,*

$$\forall x \in \mathbb{R}, \quad |x| \geq X \implies \alpha'(x) = 0.$$

Proof. Straightforward application of the Implicit Function Theorem. □

Fix a spatial mesh size $\Delta x > 0$ and time step $\Delta t > 0$. For all $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, set the notations

$$t^n = n\Delta t, \quad x_j = j\Delta x, \ x_{j+1/2} = (j+1/2)\Delta x, \ I_j :=]x_{j-1/2}, x_{j+1/2}[, \quad X \in I_J, \ J \in \mathbb{N}^*.$$

Discretize the initial datum, α and "the real line":

$$\forall j \in \mathbb{Z}, \quad u_j^o := \frac{1}{\Delta x}\int_{I_j} u_o(x)\,\mathrm{d}x\,, \quad \alpha_j := \alpha(x_j), \quad w_\Delta(x) := \sum_{j\in\mathbb{Z}} x_j \mathbf{1}_{I_j}(x).$$

Notice that for all $j \in \mathbb{Z}$,

$$j \geq J+1 \implies \alpha_j = \alpha(X) \quad \text{and} \quad j \leq -J-1 \implies \alpha_j = \alpha(-X).$$

It will be convenient to adopt the notation

$$\forall j \in \mathbb{Z}, \quad h_j(u) := H(x_j, u).$$

Fix $n \in \mathbb{N}$. Let us explain how, given $(u_j^n)_{j\in\mathbb{Z}}$, we determine $(u_j^{n+1})_{j\in\mathbb{Z}}$. Set $\rho^n = \sum_{j\in\mathbb{Z}} u_j^n \mathbf{1}_{I_j}$. Let us call $\mathcal{U}^n = \mathcal{U}^n(t, x)$ the unique entropy solution to:

$$\begin{cases} \partial_t \mathcal{U}(t,x) + \partial_x \left(H(w_\Delta(x), \mathcal{U}(t,x))\right) &= 0 \qquad (t,x) \in]t^n, t^{n+1}[\times\mathbb{R} \\ \mathcal{U}(t^n, x) &= \rho^n(x) \quad x \in \mathbb{R}. \end{cases}$$

Following Section 2.5, by entropy solution, we mean that for all test functions $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ and for all $(k_j)_{j\in[\![-J-1;J+1]\!]} \in \mathbb{R}^{2J+3}$ used tho defined the piecewise constant function κ, which discontinuities are the edges $x_{j+1/2}$, $j \in [\![-J-1; J]\!]$, we have:

$$\begin{aligned} &\int_{t^n}^{t^{n+1}} \int_{\mathbb{R}} |\mathcal{U}^n - \kappa(x)|\partial_t\varphi + \Phi(w_\Delta(x), \mathcal{U}^n, \kappa(x))\partial_x\varphi \ \mathrm{d}x\,\mathrm{d}t \\ &+ \int_{\mathbb{R}} |\rho^n(x) - \kappa(x)|\varphi(t^n, x)\,\mathrm{d}x - \int_{\mathbb{R}} |\mathcal{U}(t^{n+1}-, x) - \kappa(x)|\varphi(t^{n+1}, x)\,\mathrm{d}x \\ &+ \sum_{j=-J-1}^{J} \int_{t^n}^{t^{n+1}} \mathcal{R}_{\mathcal{G}_{j+1/2}}(k_j, k_{j+1})\varphi(t, x_{j+1/2})\,\mathrm{d}t \geq 0, \end{aligned}$$

where, following Proposition 2.3,

$$\mathcal{R}_{j+1/2}(a,b) := |F_{\text{int}}^{j+1/2}(a,b) - h_j(a)| + |F_{\text{int}}^{j+1/2}(a,b) - h_{j+1}(b)|$$

and

$$F_{\text{int}}^{j+1/2}(a,b) := \max\Big\{\text{God}_j(a,\alpha_j),\ \text{God}_{j+1}(\alpha_{j+1},b)\Big\},$$

is the flux across the interface $x = x_{j+1/2}$, with for all $j \in \mathbb{Z}$, God_j^n denoting the Godunov numerical flux associated with h_j.
Define u_j^{n+1} as

$$\forall j \in \mathbb{Z}, \quad u_j^{n+1} := \frac{1}{\Delta x}\int_{I_j} \mathcal{U}^n(t^{n+1}-, x)\,\mathrm{d}x\,. \tag{3.1}$$

The CFL condition (3.3) will ensure that waves emanating at each interface $x = x_{j+1/2}$, $j \in \mathbb{Z}$ do not interact. Consequently, (3.1) rewrites precisely as

$$\forall j \in \mathbb{Z}, \quad u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}(F_{\text{int}}^{j+1/2}(u_j^n, u_{j+1}^n) - F_{\text{int}}^{j-1/2}(u_{j-1}^n, u_j^n)). \tag{3.2}$$

For the numerical analysis, we defined the approximate solution u_Δ the following way:

$$u_\Delta := \sum_{n=0}^{+\infty} \mathcal{U}^n \mathbf{1}_{[t^n,t^{n+1}[}\mathbf{1}_{I_j}.$$

3.2 Stability

Following Section 2.3, we explicitly construct steady states of the scheme (3.2) to derive $\mathbf{L}^\infty$ bounds.

Lemma 3.2. *Let $c \in \mathbb{R}$ such that $c \geq \sup_{\mathbb{R}} \alpha$. Define for all $j \in \mathbb{Z}$,*

$$v_{j+1} := \left\{ \begin{array}{cl} c & \textit{if} \quad j \leq -J-2 \\ S_{j+1}^+(h_j(v_j)) & \textit{if} \quad j \geq -J-1, \end{array} \right. \qquad w_j := \left\{ \begin{array}{cl} S_j^+(h_{j+1}(w_{j+1})) & \textit{if} \quad j \leq J \\ c & \textit{if} \quad j \geq J+1. \end{array} \right.$$

Then $(v_j)_j$ and $(w_j)_j$ are steady states of (3.2) in the sense of Definition 2.6.

Proof. The key point is that for all $j \in \mathbb{Z}$, $(v_j, v_{j+1}) \in \mathcal{G}_{j+1/2}$, which ensures that

$$h_j(v_j) = F_{\text{int}}^{j+1/2}(v_j, v_{j+1}) = h_{j+1}(v_{j+1}).$$

□

Remark 3.1. Keep the notations of Lemma 3.2.
(i) Notice that $(v_j)_j$ and $(w_j)_j$ are stationary. For instance, for all integers $j \geq J+2$,

$$h_{j-1}(v_{j-1}) = h_j(v_j) = h_{j-1}(v_j) \implies v_{j-1} = v_j.$$

(ii) For all $c \in \mathbb{R}$ such that $c \leq \inf_{\mathbb{R}} \alpha$, the two following sequences are steady states as well:

$$v_{j+1} = \left\{ \begin{array}{cl} c & \text{if} \quad j \leq -J-2 \\ S_{j+1}^-(h_j(v_j)) & \text{if} \quad j \geq -J-1, \end{array} \right. \qquad w_j = \left\{ \begin{array}{cl} S_j^-(h_{j+1}(w_j)) & \text{if} \quad j \leq J \\ c & \text{if} \quad j \geq J+1. \end{array} \right.$$

Lemma 3.3. *Assume that* (**C3**)–(**CVX**) *hold. Let* $(m, M) \in \mathbb{R}^2$ *such that* $m \leq \inf_{\mathbb{R}} \alpha$ *and* $M \geq \sup_{\mathbb{R}} \alpha$. *Then there exist two steady states of* (3.2), $(m_j)_j$ *and* $(M_j)_j$, *such that:*
(i) $(m_j)_j$ *is bounded by below ;*
(ii) $(M_j)_j$ *is bounded by above ;*
(iii) for all $j \in \mathbb{Z}$, $m_j \leq m$ *and* $M_j \geq M$.

Proof. We only give the details for the construction of $(M_j)_j$.
Set $\mathcal{U}_h := \max_{x\in\mathbb{R}} H(x, M)$. By monotonicity, for all $j \in \mathbb{Z}$, $S_j^+(\mathcal{U}_h) \geq M$. Therefore, if $\overline{M} := \max_{j\in\mathbb{Z}} S_j^+(\mathcal{U}_h)$, then $\overline{M} \geq M \geq \max_{x\in\mathbb{R}} \alpha(x)$. Define

$$\forall j \in \mathbb{Z}, \quad M_{j+1} = \begin{cases} \overline{M} & \text{if} \quad j \leq -J-2 \\ S_{j+1}^+(h_j(M_j)) & \text{if} \quad j \geq -J-1. \end{cases}$$

Clearly, $(M_j)_j$ is a steady state of the scheme.

Claim 1: for all $j \in \mathbb{Z}$, $M_j \geq M$. Indeed, for all $j \in \mathbb{Z}$,

$$h_j(M_j) = h_{j+1}(M_{j+1}) = h_{-J-1}(\max_{i\in\mathbb{Z}} S_i^+(\mathcal{U}_h)) \geq h_{-J-1}(S_{-J-1}^+(\mathcal{U}_h)) = \mathcal{U}_h \geq h_j(M),$$

from which we deduce, by monotonicity, that $M_j \geq M$. The claim is proved.

Claim 2: $(M_j)_j$ is bounded by above. By definition, for all $j \in \mathbb{Z}$,

$$M_j = S_j^+\left(H(-X, \overline{M})\right) \leq \max_{x\in\mathbb{R}} h_+^{-1}\left(x, H(-X, \overline{M})\right) := \overline{u},$$

where for all $x \in \mathbb{R}$, we denote by $u \mapsto h_+^{-1}(x, u)$ the reciprocal of $u \mapsto h(x, u)$ on $[\alpha(x), +\infty[$. We see that $\overline{u}$ only depends on H and M. □

We can now prove that the scheme (3.2) is stable.

Lemma 3.4. *Assume that* (**C3**)–(**CVX**) *hold. Let* $u_o \in \mathbf{L}^\infty(\mathbb{R}, \mathbb{R})$. *With reference to Lemma 3.1, fix* $(m, M) \in \mathbb{R}^2$ *satisfying* $m \leq \inf_{\mathbb{R}} \alpha$ *and* $M \geq \sup_{\mathbb{R}} \alpha$, *such that for a.e.* $x \in \mathbb{R}$, $m \leq u_o(x) \leq M$. *Consider* $(m_j)_j$ *and* $(M_j)_j$, *steady states given by Lemma 3.3. Assume that the ratio* $\lambda := \frac{\Delta t}{\Delta x}$ *satisfies the CFL condition:*

$$2\lambda L \leq 1, \quad L := \sup_{\substack{x\in\mathbb{R} \\ \underline{u}\leq p\leq\overline{u}}} |\partial_u H(x, p)|, \quad \underline{u} := \inf_{j\in\mathbb{Z}} m_j, \ \overline{u} := \sup_{j\in\mathbb{Z}} M_j. \tag{3.3}$$

Then the scheme (3.2) *is stable: for all* $n \in \mathbb{N}$,

$$\forall j \in \mathbb{Z}, \quad \underline{u} \leq m_j \leq u_j^n \leq M_j \leq \overline{u}. \tag{3.4}$$

Proof. Let us define the piecewise functions

$$(m_\Delta, M_\Delta) := \sum_{j\in\mathbb{Z}} (m_j, M_j)\mathbf{1}_{I_j}.$$

The key point is that m_Δ and M_Δ are both stationary solutions to

$$\partial_t \mathcal{U}(t, x) + \partial_x \left(H(w_\Delta(x), \mathcal{U}(t, x))\right) = 0.$$

We prove (3.4) by induction on n. It holds for $n = 0$ by construction of the sequences $(m_j)_{j\in\mathbb{Z}}, (M_j)_{j\in\mathbb{Z}}$. Assume now that for some $n \in \mathbb{N}$, (3.4) holds, implying that $m_\Delta \leq \rho^n \leq M_\Delta$. Therefore, Theorem 2.15 ensures that

$$m_\Delta \leq \rho^n \leq M_\Delta \implies \forall t \in [t^n, t^{n+1}], \ m_\Delta \leq \mathcal{U}^n(t) \leq M_\Delta \implies \forall (t, x) \in [t^n, t^{n+1}] \times \mathbb{R}, \ \underline{u} \leq \mathcal{U}^n(t, x) \leq \overline{u}.$$

Since u_j^{n+1} is obtained as the average of $\mathcal{U}^n(t^{n+1}-)$ over I_j, we deduce that for all $j \in \mathbb{Z}$, $\underline{u} \leq u_j^{n+1} \leq \overline{u}$. The proof is concluded. □

Example 3.1. Assume that H takes the form $H(x,u) = \theta(x)\frac{(u-\alpha(x))^2}{2}$, for a suitable choice of functions θ and α. Then for all $j \in \mathbb{Z}$ and $h \in \mathbb{R}^+$, $S_j^+(h) = \alpha_j + \sqrt{\frac{2h}{\theta_j}}$. Applying the procedure of Lemmas 3.3-3.4, we find the bound

$$\overline{u} \leq \frac{\sup\theta}{\inf\theta}\left(M + 2(\sup\alpha - \inf\alpha)\right).$$

3.3 Compensated compactness

Throughout the section, we assume that the hypotheses of Lemma 3.4 hold.
The compensated compactness method and its applications to systems of conservation laws is for instance reviewed in [12, 31]. Modified for our use, the compensated compactness lemma reads as follows.

Lemma 3.5. *Assume that H satisfies* (**CH**)–(**CVX**)*, ensuring its genuine nonlinearity in the sense of*

$$\forall (x,s) \in \mathbb{R}^2, \quad meas\Big(\{u \in \mathbb{R} \ : \ \partial_u H(x,u) = s\}\Big) = 0. \tag{3.5}$$

Let $(u_\varepsilon)_\varepsilon$ be a bounded sequence of $\mathbf{L}^\infty(]0,+\infty[\times\mathbb{R},\mathbb{R})$ such that for all $k \in \mathbb{R}$ and for any $i \in \{1,2\}$, the sequence $(\partial_t S_i(u_\varepsilon) + \partial_x Q_i(x,u_\varepsilon))_\varepsilon$ belongs to a compact subset of $\mathbf{H}^{-1}_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$, where

$$\begin{aligned} S_1(u) &= u - k & Q_1(x,u) &= H(x,u) - H(x,k) \\ S_2(u) &= H(x,u) - H(x,k) & Q_2(x,u) &= \int_k^u \partial_u H(x,\xi)^2 \, \mathrm{d}\xi. \end{aligned} \tag{3.6}$$

Then there exists a subsequence of $(u_\varepsilon)_\varepsilon$ that converges in $\mathbf{L}^p_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$ for all $p \in [1,+\infty[$ and a.e. on $]0,+\infty[\times\mathbb{R}$ to some function $u \in \mathbf{L}^\infty(]0,+\infty[\times\mathbb{R},\mathbb{R})$.

To prove the $\mathbf{H}^{-1}_{\mathbf{loc}}$ compactness required in Lemma 3.5 we will use the following technical result, see [34, 17, 36].

Lemma 3.6. *Let $q, r \in \mathbb{R}$ such that $1 < q < 2 < r$. Let $(\mu_\varepsilon)_\varepsilon$ be a sequence of distributions such that:*
(i) $(\mu_\varepsilon)_\varepsilon$ belongs to a compact subset of $\mathbf{W}^{-1,q}_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$.
(ii) $(\mu_\varepsilon)_\varepsilon$ is bounded in $\mathbf{W}^{-1,r}_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$.
Then $(\mu_\varepsilon)_\varepsilon$ belongs to a compact subset of $\mathbf{H}^{-1}_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$.

For the compactness analysis, we take inspiration from [24, 25].
Let (S,Q) be a regular entropy/entropy flux pair, which we recall, means that

$$\forall x, u \in \mathbb{R}, \quad \partial_u Q(x,u) = S'(u)\, \partial_u H(x,u).$$

Setting Q_Δ as the entropy flux of S associated with the flux $u \mapsto H(w_\Delta(x),u)$, the entropy dissipation of u_Δ is defined as

$$E_\Delta(\phi) := \int_0^{+\infty}\int_{\mathbb{R}} S(u_\Delta)\partial_t\phi + Q_\Delta(x,u_\Delta)\partial_x\phi \ \mathrm{d}x\,\mathrm{d}t\,, \quad \phi \in \mathbf{C}^\infty_{\mathbf{c}}(\mathbb{R}^+\times\mathbb{R},\mathbb{R}). \tag{3.7}$$

Fix $n \in \mathbb{N}$ and $j \in \mathbb{Z}$. Let us consider the entropy dissipation in $P^n_{j+1/2} := [t^n, t^{n+1}[\times]x_j, x_{j+1}[$. By integration by parts and the fact that u_Δ is the exact solution of a Riemann problem in $P^n_{j+1/2}$, we write:

$$
\begin{aligned}
\iint_{P^n_{j+1/2}} & S(u_\Delta)\partial_t\phi + Q_\Delta(x,u_\Delta)\partial_x\phi \ \mathrm{d}x\,\mathrm{d}t \\
&= \int_{x_j}^{x_{j+1}} S(u_\Delta(t^{n+1}-,x))\phi(t^{n+1},x) - S(u_\Delta(t^n,x))\phi(t^n,x)\,\mathrm{d}x \\
&+ \int_{t^n}^{t^{n+1}} Q(x_{j+1},u^n_{j+1})\phi(t,x_{j+1}) - Q(x_j,u^n_j)\phi(t,x_j)\,\mathrm{d}t \\
&+ \int_{t^n}^{t^{n+1}} \Big(Q(x_j,u_\Delta(t,x_{j+1/2}-)) - Q(x_{j+1},u_\Delta(t,x_{j+1/2}+))\Big)\phi(t,x_{j+1/2})\,\mathrm{d}t \\
&+ \int_{t^n}^{t^{n+1}} [\![\sigma S(u_\Delta) - Q_\Delta(x,u_\Delta)]\!]_y \ \phi(t,y(t))\,\mathrm{d}t\,.
\end{aligned}
$$

With reference to Section 2.4, in the right-hand side of the previous equality:
• the third integral is the contribution of the (eventual) stationary non-classical shock in the solution.
• The last integral is the contribution of the (eventual) classical shock in the solution. Hence, σ is the shock velocity, given by the Rankine-Hugoniot condition, $y(t) = x_{j+1/2} + \sigma(t-t^n)$ is the shock curve and

$$
[\![\sigma S(u_\Delta) - Q_\Delta(x,u_\Delta)]\!]_y := (\sigma S(u_\Delta) - Q_\Delta(x,u_\Delta))(t,y(t)+) - (\sigma S(u_\Delta) - Q_\Delta(x,u_\Delta))(t,y(t)-).
$$

It is relevant to say that w_Δ is continuous along $\{x = y(t)\}$, equal to either x_j or x_{j+1}, depending on the sign of σ.
Taking the sum for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, we see that the entropy dissipation rewrites as

$$
\begin{aligned}
E_\Delta(\phi) \quad &= -\int_{\mathbb{R}} S(u_\Delta(0,x))\phi(0,x)\,\mathrm{d}x && \longleftarrow I_1(\phi) \\
&+ \sum_{n=1}^{+\infty} \int_{\mathbb{R}} (S(u_\Delta(t^n-,x)) - S(u_\Delta(t^n,x)))\phi(t^n,x)\,\mathrm{d}x && \longleftarrow I_2(\phi) \\
&+ \sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}} \int_{t^n}^{t^{n+1}} \Big(Q(x_j,u_\Delta(t,x_{j+1/2}-)) - Q(x_{j+1},u_\Delta(t,x_{j+1/2}+))\Big)\phi(t,x_{j+1/2})\ \mathrm{d}t && \longleftarrow I_4(\phi) \\
&+ \sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}\sum_y \int_{t^n}^{t^{n+1}} [\![\sigma S(u_\Delta) - Q_\Delta(x,u_\Delta)]\!]_y \ \phi(t,y(t))\,\mathrm{d}t && \longleftarrow I_3(\phi)
\end{aligned}
\tag{3.8}
$$

As we previously explained, for all $j \in \mathbb{Z}$, the sum $\sum_y$ in (3.8) is either over an empty set, or over a singleton.

First, let us give a bound on the variation of the approximate solution across the discrete time levels.

Lemma 3.7. *Let $T > 0$ and $R > 0$. Fix $N, K \in \mathbb{N}^*$ such that $T \in [t^N, t^{N+1}[$ and $R \in I_K$. Then, there exists a constant $c_1 > 0$ depending on T, R, u_o, and H such that*

$$
\begin{aligned}
& \sum_{n=1}^{N}\sum_{|j|\le K} \int_{I_j} |u_\Delta(t^n,x) - u_\Delta(t^n-,x)|^2\,\mathrm{d}x \le c_1 \\
\textit{and} \quad & \sum_{n=0}^{N}\sum_{|j|\le K}\sum_y \int_{t^n}^{t^{n+1}} [\![\sigma S(u_\Delta) - Q_\Delta(x,u_\Delta)]\!]_y \ \mathrm{d}t \le c_1.
\end{aligned}
\tag{3.9}
$$

Proof. For the purpose of this proof, let us choose $S(u) = \frac{u^2}{2}$ and $Q = Q(x,u)$ its entropy flux. Let $(\phi_k)_k$ be a sequence of nonnegative test functions that converges to $\phi := \mathbf{1}_{[0,T]\times[-R,R]}$. At the limit $k \to +\infty$ in (3.8), we obtain:

$$I_2(\phi) + I_3(\phi) = \int_{|x|\le R} |u_\Delta(0,x)|^2 \,\mathrm{d}x - I_4(\phi) \le 2R\|u_o\|^2_{\mathbf{L}^\infty} - I_4(\phi).$$

Note that $I_3(\phi)$ is nonnegative because the (eventual) shock is classical and therefore produces entropy. Then, notice that by definition of $(u_j^n)_{j,n}$,

$$\begin{aligned}
I_2(\phi) &= \frac{1}{2}\sum_{n=1}^{N}\sum_{|j|\le K}\int_{I_j} u_\Delta(t^n-,x)^2 - u_\Delta(t^n,x)^2 \;\mathrm{d}x\\
&= \frac{1}{2}\sum_{n=1}^{N}\sum_{|j|\le K}\left\{\int_{I_j} (u_\Delta(t^n-,x) - u_\Delta(t^n,x))^2 + 2u_\Delta(t^n,x)(u_\Delta(t^n-,x) - u_\Delta(t^n,x))\right\}\mathrm{d}x\\
&= \frac{1}{2}\sum_{n=1}^{N}\sum_{|j|\le K}\int_{I_j} (u_\Delta(t^n-,x) - u_\Delta(t^n,x))^2\,\mathrm{d}x + \sum_{n=1}^{N}\sum_{|j|\le K} u_j \underbrace{\int_{I_j}(u_\Delta(t^n-,x) - u_j^n)\,\mathrm{d}x}_{=0}.
\end{aligned}$$

Finally, as a consequence of Proposition 2.3 *(ii)*, for all $k \in [\underline{u}, \overline{u}]$,

$$\Phi(x_{j+1/2}-, u_\Delta(t, x_{j+1/2}-), k) - \Phi(x_{j+1/2}+, u_\Delta(t, x_{j+1/2}+), k) \ge -\mathcal{R}_{\mathcal{G}_{j+1/2}}(k,k).$$

Using Remark 2.1, we can bound this remainder term as

$$\mathcal{R}_{\mathcal{G}_{j+1/2}}(k,k) \le \left(2\|\alpha'\|_{\mathbf{L}^\infty} \sup_{\substack{x\in\mathbb{R}\\ p\in[\underline{u},\overline{u}]}} |\partial_u H(x,p)| + \sup_{\substack{x\in\mathbb{R}\\ p\in[\underline{u},\overline{u}]}} |\partial_x H(x,p)|\right)\Delta x.$$

Using an approximation argument to pass from Kruzhkov entropies to any entropy, see [24], we obtain

$$I_4(\phi) \ge -2TR\left(2\|\alpha'\|_{\mathbf{L}^\infty} \sup_{\substack{x\in\mathbb{R}\\ p\in[\underline{u},\overline{u}]}} |\partial_u H(x,p)| + \sup_{\substack{x\in\mathbb{R}\\ p\in[\underline{u},\overline{u}]}} |\partial_x H(x,p)|\right).$$

Estimate (3.9) follows by putting the bounds on $I_1(\phi)$ and $I_4(\phi)$ together. □

We can convert (3.9) into an estimate on the spatial variation of the approximate solutions, following the arguments of [18, Page 67]. For the sake of clarity, we set $\rho^{n,\pm}_{j+1/2} := u_\Delta(t^n, x_{j+1/2}\pm)$.

Lemma 3.8. *Let $T > 0$ and $R > 0$. Fix $N, K \in \mathbb{N}^*$ such that $T \in [t^N, t^{N+1}[$ and $R \in I_K$. Then, there exists a constant $c_2 > 0$ depending on T, R, u_o and H such that*

$$\sum_{n=0}^{N}\sum_{|j|\le K}\left\{(\rho^{n,-}_{j+1/2} - \rho^{n,+}_{j+1/2})^2 + (u_j^n - \rho^{n,-}_{j+1/2})^2 + (u_{j+1}^n - \rho^{n,+}_{j+1/2})^2\right\}\Delta x \le c_2. \tag{3.10}$$

We are now in position to prove the $\mathbf{H}^{-1}_{\mathbf{loc}}$ compactness of the sequence of measures that defines the entropy dissipation.

Lemma 3.9. *Let $(S_i, Q_i)_{i\in\{1,2\}}$ be the entropy/entropy flux pairs defined in Lemma 3.5. Then for any $i \in \{1,2\}$, the sequence of distributions μ_i defined by*

$$\mu_i(\phi) := \int_0^{+\infty} \int_{\mathbb{R}} S_i(u_\Delta)\partial_t\phi + Q_i(x, u_\Delta)\partial_x\phi \, \mathrm{d}x\, \mathrm{d}t\,, \quad \phi \in \mathbf{C}^\infty_\mathbf{c}(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$$

belongs to a compact subset of $\mathbf{H}^{-1}_{\mathbf{loc}}(]0, +\infty[\times\mathbb{R}, \mathbb{R})$.

Proof. We follow the proofs of [24, Lemma 5.5] or [25, Lemma 4.5]. To start, let us fix (S, Q) a smooth entropy/entropy flux pair.
Let us fix a bounded open subset $\Omega \subset]0, +\infty[\times\mathbb{R}$, say $\Omega \subset [0, T] \times [-R, R]$ for some $T > 0$ and $R > 0$. Call $N, K \in \mathbb{N}^*$ such that $T \in [t^N, t^{N+1}[$ and $R \in I_K$, and finally, let $\phi \in \mathbf{C}^\infty_\mathbf{c}(\Omega, \mathbb{R})$. We split μ as

$$\mu(\phi) = \int_0^{+\infty} \int_{\mathbb{R}} (Q(x, u_\Delta) - Q_\Delta(x, u_\Delta))\partial_x\phi \, \mathrm{d}x\, \mathrm{d}t + E_\Delta(\phi),$$

where $E_\Delta(\phi)$ is given by (3.7).
The definition of w_Δ and regularity of H ensures that for any $q_1 \in]1, 2]$, the first term of the right-hand side is strongly compact in $\mathbf{W}^{-1,q_1}(\Omega)$. We now estimate $E_\Delta(\phi)$.
First, observe that since $(u_\Delta)_\Delta$ is bounded in $\mathbf{L}^\infty(\Omega)$, we have

$$|E_\Delta(\phi)| \leq \left(\sup_{\underline{u}\leq p\leq \overline{u}} |S(p)| + \sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq \overline{u}}} |Q(x, p)| \right) \cdot \|\phi\|_{\mathbf{W}^{1,\infty}},$$

implying that $(E_\Delta(\phi))_\Delta$ is bounded in $\mathbf{W}^{-1,r}_{\mathbf{loc}}(\Omega)$ for any $r \in]2, +\infty[$.
Keeping the notations of (3.8), we bound $I_1(\phi)$ and $I_3(\phi)$ as

$$|I_1(\phi)| \leq \|S(u_o)\|_{\mathbf{L}^1([-R,R])} \, \|\phi\|_{\mathbf{L}^\infty(\Omega)}, \quad I_3(\phi) \leq c_1\|\phi\|_{\mathbf{L}^\infty(\Omega)},$$

where we used Lemma 3.7 for $I_3(\phi)$. Consequently, $I_1(\phi)$ and $I_3(\phi)$ are bounded in the space $\mathcal{M}(\Omega)$ of bounded Radon measures, which is compactly embedded in $\mathbf{W}^{-1,q_2}(\Omega)$ if $q_2 \in]1, 2[$. Therefore, $I_1(\phi)$ and $I_3(\phi)$ belong to a compact subset of $\mathbf{W}^{-1,q_2}(\Omega)$.
Now, to estimate $I_2(\phi)$, split it as $I_2(\phi) = I_{2,1}(\phi) + I_{2,2}(\phi)$ where

$$I_{2,1}(\phi) = \sum_{n=1}^{N} \sum_{|j|\leq K} \int_{I_j} (S(u_\Delta(t^n-, x)) - S(u^n_j))\phi(t^n, x_j) \, \mathrm{d}x$$

$$I_{2,2}(\phi) = \sum_{n=1}^{N} \sum_{|j|\leq K} \int_{I_j} (S(u_\Delta(t^n-, x)) - S(u^n_j))(\phi(t^n, x) - \phi(t^n, x_j)) \, \mathrm{d}x\,.$$

We handle $I_{2,1}(\phi)$ by introducing an intermediary point ω^n_j such that

$$S(u_\Delta(t^n-, x)) - S(u^n_j) = S'(u^n_j)(u_\Delta(t^n-, x) - u^n_j) + \frac{S''(\omega^n_j)}{2}(u_\Delta(t^n-, x) - u^n_j)^2.$$

Taking into account the definition of u^n_j and Lemma 3.8, we obtain

$$|I_{2,1}(\phi)| \leq \frac{c_1}{2} \sup_{\underline{u}\leq p\leq \overline{u}} S''(p) \cdot \|\phi\|_{\mathbf{L}^\infty(\Omega)},$$

ensuring that $I_{2,1}(\phi)$ belong to a compact subset of $\mathbf{W}^{-1,q_2}(\Omega)$.

We continue by choosing $\alpha \in]\frac{1}{2}, 1[$ and then writing:

$$
\begin{aligned}
|I_{2,2}(\phi)| &\leq \|\phi\|_{\mathbf{C}^{0,\alpha}} \Delta x^{\alpha} \sum_{n=1}^{N} \sum_{|j|\leq K} \int_{I_j} |S(u_\Delta(t^n-, x)) - S(u_j^n)| \, \mathrm{d}x \\
&\leq \|\phi\|_{\mathbf{C}^{0,\alpha}} \Delta x^{\alpha-1} \left\{ \sum_{n=1}^{N} \sum_{|j|\leq K} \left(\int_{I_j} |S(u_\Delta(t^n-, x)) - S(u_j^n)| \, \mathrm{d}x \right)^2 \right\}^{1/2} \left\{ \sum_{n=1}^{N} \sum_{|j|\leq K} \Delta x^2 \right\}^{1/2} \\
&\leq \sqrt{\frac{2TR}{\lambda}} \|\phi\|_{\mathbf{C}^{0,\alpha}} \Delta x^{\alpha-1/2} \left\{ \sum_{n=1}^{N} \sum_{|j|\leq K} \int_{I_j} |S(u_\Delta(t^n-, x)) - S(u_j^n)|^2 \, \mathrm{d}x \right\}^{1/2} \\
&\leq \sqrt{\frac{2TRc_1}{\lambda}} \sup_{\underline{u}\leq p\leq \overline{u}} |S'(p)| \cdot \|\phi\|_{\mathbf{C}^{0,\alpha}} \ \Delta x^{\alpha-1/2} \qquad \text{Lemma 3.8}
\end{aligned}
$$

We now take advantage of the compact embedding $\mathbf{C}^{0,\alpha}(\Omega) \subset \mathbf{W}^{1,p}(\Omega)$, $p = 2/(1-\alpha)$ to be sure that $I_{2,2}(\phi)$ belongs to a compact subset of $\mathbf{W}^{-1,q_3}(\Omega)$, $q_3 := 2/(1+\alpha) \in]1, \frac{4}{3}[$.
Finally, we estimate $I_4(\phi)$. Let $k \in [\underline{u}, \overline{u}]$ and consider $(S_i, Q_i)_{i\in\{1,2\}}$ the entropy/entropy flux pairs defined in Lemma 3.5.
Because of the Rankine-Hugoniot condition,

$$
\left| [\![Q_{\Delta,1}(x, u_\Delta)]\!]^n_{x_{j+1/2}} \right| = |H(x_j, k) - H(x_{j+1}, k)|,
$$

from which we deduce

$$
|I_{4,1}(\phi)| \leq 2TR \sup_{x\in\mathbb{R}} |\partial_x H(x,k)| \cdot \|\phi\|_{\mathbf{L}^\infty(\Omega)},
$$

ensuring that $I_{4,1}(\phi)$ belongs to a compact subset of $\mathbf{W}^{-1,q_2}(\Omega)$. On the other hand,

$$
\begin{aligned}
\left| [\![Q_{\Delta,2}(x, u_\Delta)]\!]^n_{x_{j+1/2}} \right| &= \int_{\rho^{n,+}_{j+1/2}}^{\rho^{n,-}_{j+1/2}} \partial_u H(x_j, \xi)^2 \, \mathrm{d}\xi + \int_{k}^{\rho^{n,+}_{j+1/2}} \partial_u H(x_j, \xi)^2 - \partial_u H(x_{j+1}, \xi)^2 \, \mathrm{d}\xi \\
&\leq \int_{\rho^{n,+}_{j+1/2}}^{\rho^{n,-}_{j+1/2}} \partial_u H(x_j, \xi)^2 \, \mathrm{d}\xi + 2L(\overline{u} - \underline{u}) \sup_{\substack{x\in\mathbb{R} \\ \underline{u}\leq p\leq\overline{u}}} |\partial^2_{xu} H(x,p)| \Delta x \\
&\leq L \underbrace{\int_{\rho^-}^{\rho^+} |\partial_u H(x_j, \xi)| \, \mathrm{d}\xi}_{Q_{2,1}} + 2L(\overline{u} - \underline{u}) \sup_{\substack{x\in\mathbb{R} \\ \underline{u}\leq p\leq\overline{u}}} |\partial^2_{xu} H(x,p)| \Delta x,
\end{aligned}
$$

where $\rho^- := \min\{\rho^{n,-}_{j+1/2}, \rho^{n,+}_{j+1/2}\}$ and $\rho^+ := \max\{\rho^{n,-}_{j+1/2}, \rho^{n,+}_{j+1/2}\}$. If $u \mapsto \partial_u H(x_j, u)$ does not change sign in $]\rho^-, \rho^+[$, then

$$
\begin{aligned}
|Q_{2,1}| &= |H(x_j, \rho^{n,-}_{j+1/2}) - H(x_j, \rho^{n,+}_{j+1/2})| \\
&\leq \underbrace{|H(x_j, \rho^{n,-}_{j+1/2}) - H(x_{j+1}, \rho^{n,+}_{j+1/2})|}_{=0} + |H(x_{j+1}, \rho^{n,+}_{j+1/2}) - H(x_j, \rho^{n,+}_{j+1/2})| \\
&\leq \sup_{\substack{x\in\mathbb{R} \\ \underline{u}\leq p\leq\overline{u}}} |\partial_x H(x,p)| \ \Delta x.
\end{aligned}
$$

Otherwise, assume to fix the ideas that $\rho^- \leq \alpha_j \leq \rho^+$. Then, setting $L_H := \sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}} |\partial_{uu}H(x,p)|$ we write

$$\begin{aligned}
Q_{2,1} &= \int_{\rho^-}^{\rho^+} |\partial_u H(x_j,\xi) - \partial_u H(x_j,\alpha_j)| \,\mathrm{d}\xi \\
&\leq L_H \int_{\rho^-}^{\rho^+} |\xi - \alpha_j| \,\mathrm{d}\xi \\
&= \frac{L_H}{2}\left((\rho^+ - \alpha_j)^2 + (\rho^- - \alpha_j)^2\right) \\
&\leq L_H(\rho^+ - \rho^-)^2 = L_H\,(\rho^{n,-}_{j+1/2} - \rho^{n,+}_{j+1/2})^2.
\end{aligned}$$

Taking advantage of Lemma 3.8, we conclude that

$$\begin{aligned}
&\sum_{n=0}^{N} \Delta t \sum_{|j|\leq J} \left| [\![Q_{\Delta,2}(x,u_\Delta)]\!]^n_{x_{j+1/2}} \right| \\
&\leq L \sum_{n=0}^{N} \Delta t \sum_{|j|\leq J} \left\{ L_H\,(\rho^{n,-}_{j+1/2} - \rho^{n,+}_{j+1/2})^2 + \sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}} |\partial_x H(x,p)|\,\Delta x + 2(\overline{u}-\underline{u}) \sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}} |\partial^2_{xu}H(x,p)|\Delta x \right\} \\
&\leq L \left\{ L_H + \sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}} |\partial_x H(x,p)| + 2(\overline{u}-\underline{u}) \sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}} |\partial^2_{xu}H(x,p)| \right\} (\lambda c_2 + 2TR),
\end{aligned}$$

which ensures $I_{4,2}(\phi)$ belongs to a compact subset of $\mathbf{W}^{-1,q_2}(\Omega)$.
To summarize, for any $i \in \{1,2\}$, $(\mu_i)_\Delta$ belongs to a compact subset of $\mathbf{W}^{-1,q}_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$, $q = \min\{q_1,q_2,q_3\} < 2$. Additionally, $(\mu_i)_\Delta$ is bounded in $\mathbf{W}^{-1,r}_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$ for any $r \in]2,+\infty[$. Lemma 3.6 applies to ensure that any $i \in \{1,2\}$, $(\mu_i)_\Delta$ belongs to a compact subset of $\mathbf{H}^{-1}_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$. □

Corollary 3.10. *Assume that H satisfies* (**CH**)–(**CVX**). *Fix $u_o \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R})$ and let $(u_\Delta)_\Delta$ be the sequence generated by the scheme described in Section 3.1.*
Then, there exists a limit function $u \in \mathbf{L}^\infty(]0,+\infty[\times\mathbb{R},\mathbb{R})$ such that along a subsequence as $\Delta \to 0$, $(u_\Delta)_\Delta$ converges in $\mathbf{L}^p_{\mathbf{loc}}(]0,+\infty[\times\mathbb{R},\mathbb{R})$ for all $p \in [1,+\infty[$ and a.e. on $]0,+\infty[\times\mathbb{R}$ to u.

Proof. Follows from a combination of Lemma 3.7 and Lemma 3.9. □

3.4 Convergence

To prove that the limit function u of Corollary 3.10 is the entropy solution, we will need the following technical result.

Lemma 3.11. *[25, Lemma 4.8] Let $\Omega \subset]0,+\infty[\times\mathbb{R}$ be a bounded open set, $g \in \mathbf{L}^1(\Omega,\mathbb{R})$ and suppose that $(g_\varepsilon)_\varepsilon$ converges a.e. on Ω to g. Then, there exists a set $\mathcal{L}$, at most countable, such that for any $k \in \mathbb{R}\backslash\mathcal{L}$*

$$\mathrm{sgn}(g_\varepsilon - k) \underset{\varepsilon\to 0}{\longrightarrow} \mathrm{sgn}(g-k) \quad a.e.\ in\ \Omega.$$

We can now state the convergence result.

Theorem 3.12. *Assume that H satisfies* (**CH**)–(**CVX**). *Fix $u_o \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R})$, let $(u_\Delta)_\Delta$ be the sequence generated by the scheme described in Section 3.1 and let u be the limit function from Corollary 3.10.*
Then u is the entropy solution to (**CL**) *with initial datum u_o.*

Proof. We prove that (1.2) holds.

Step 1. Let us introduce the piecewise constant function: $\rho_\Delta = \displaystyle\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}} u_j^n \mathbf{1}_{t^n,t^{n+1}[}\mathbf{1}_{I_j}$.

We claim that $\|u_\Delta - \rho_\Delta\|_{\mathbf{L}^2_{\mathbf{loc}}} \underset{\Delta\to 0}{\longrightarrow} 0$. Indeed, for all $(t,x) \in [t^n, t^{n+1}[\times]x_{j+1/2}, x_{j+1}[$,

$$|u_\Delta(t,x) - \rho_\Delta(t,x)| = |\mathcal{U}^n(t,x) - u_{j+1}^n| \leq |\rho_{j+1/2}^{n,+} - u_{j+1}^n|,$$

since $\mathcal{U}^n$ is the solution of a Riemann problem with left state u_j^n and right state u_{j+1}^n at $x = x_{j+1/2}$. Likewise, for all $(t,x) \in [t^n, t^{n+1}[\times]x_j, x_{j+1/2}[$,

$$|u_\Delta(t,x) - \rho_\Delta(t,x)| = |\mathcal{U}^n(t,x) - u_j^n| \leq |\rho_{j+1/2}^{n,-} - u_j^n|.$$

Therefore, by Lemma 3.8, for all $N, K \in \mathbb{N}^*$,

$$\begin{aligned}
&\sum_{n=0}^{N}\sum_{|j|\leq K}\int_{t^n}^{t^{n+1}}\int_{x_j}^{x_{j+1}} |u_\Delta(t,x) - \rho_\Delta(t,x)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{1}{2}\sum_{n=0}^{N}\sum_{|j|\leq K}\left\{|\rho_{j+1/2}^{n,+} - u_{j+1}^n|^2 + |\rho_{j+1/2}^{n,-} - u_j^n|^2\right\}\Delta x\Delta t \leq \frac{c_1}{2}\Delta t,
\end{aligned}$$

proving the claim. The claim ensures that $(\rho_\Delta)_\Delta$ converges a.e. on $]0,+\infty[\times\mathbb{R}$ to u.

Step 2: Discrete entropy inequalities. Let $k \in [\underline{u}, \overline{u}]$. Under the CFL condition (3.3), we derive the following discrete entropy inequalities, consequence of the monotonicity of the scheme:

$$\begin{aligned}
&\Big(|u_j^{n+1} - k| - |u_j^n - k|\Big)\Delta x + (\Phi_{j+1/2}^n - \Phi_{j-1/2}^n)\Delta t \\
&\leq -\operatorname{sgn}(u_j^{n+1} - k)(F_{\text{int}}^{j+1/2}(k,k) - F_{\text{int}}^{j-1/2}(k,k))\Delta t,
\end{aligned} \tag{3.11}$$

where

$$\Phi_{j+1/2}^n := F_{\text{int}}^{j+1/2}(u_j^n \vee k, u_{j+1}^n \vee k) - F_{\text{int}}^{j+1/2}(u_j^n \wedge k, u_{j+1}^n \wedge k).$$

Indeed, on the one hand, by convexity,

$$\begin{aligned}
|u_j^{n+1} - H_j(k,k,k)| &\geq |u_j^{n+1} - k + \lambda(F_{\text{int}}^{j+1/2}(k,k) - F_{\text{int}}^{j+1/2}(k,k))| \\
&= |u_j^{n+1} - k| + \lambda\operatorname{sgn}(u_j^{n+1} - k)(F_{\text{int}}^{j+1/2}(k,k) - F_{\text{int}}^{j-1/2}(k,k)).
\end{aligned}$$

On the other hand, by monotonicity of the scheme,

$$\begin{aligned}
|u_j^{n+1} - H_j(k,k,k)| &\leq H_j(u_{j-1}^n \vee k, u_j^n \vee k, u_{j+1}^n \vee k) - H_j(u_{j-1}^n \wedge k, u_j^n \wedge k, u_{j+1}^n \wedge k) \\
&= |u_j^n - k| - \lambda(\Phi_{j+1/2}^n - \Phi_{j-1/2}^n).
\end{aligned}$$

Inequality (3.11) follows by combining these two estimates.

Step 3: Convergence. Now let $\varphi \in \mathbf{C}_\mathbf{c}^\infty(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}^+)$ and fix $T > 0$, R such that the support of φ is included in $[0,T] \times [-R,R]$. Define

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z}, \quad \varphi_j^n := \frac{1}{\Delta x}\int_{I_j} \varphi(t^n, x)\,\mathrm{d}x\,.$$

Multiply (3.11) by φ_j^n and take the double sum. A summation by parts provides $A+B+C\geq 0$ with

$$A=\sum_{n=1}^{+\infty}\sum_{j\in\mathbb{Z}}|u_j^n-k|(\varphi_j^n-\varphi_j^{n-1})\Delta x+\sum_{j\in\mathbb{Z}}|u_j^o-k|\varphi_j^o\Delta x$$

$$B=\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}\Phi_{j+1/2}^n(\varphi_{j+1}^n-\varphi_j^n)\Delta t$$

$$C=-\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}\mathrm{sgn}(u_j^{n+1}-k)(F_{\mathrm{int}}^{j+1/2}(k,k)-F_{\mathrm{int}}^{j-1/2}(k,k))\varphi_j^n\Delta t.$$

Clearly,

$$A\underset{\Delta\to 0}{\longrightarrow}\int_0^{+\infty}\int_{\mathbb{R}}|u-k|\partial_t\varphi\ \mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}u_o(x)\varphi(0,x)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Regarding C, we write:

$$\begin{aligned}
C&=-\sum_{n=1}^{+\infty}\sum_{j\in\mathbb{Z}}\mathrm{sgn}(u_j^n-k)(H(x_j,k)-H(x_{j-1},k))\varphi_j^{n-1}\Delta t\\
&=-\sum_{n=1}^{+\infty}\sum_{j\in\mathbb{Z}}\mathrm{sgn}(u_j^n-k)(H(x_j,k)-H(x_{j-1},k))(\varphi_j^{n-1}-\varphi_j^n)\Delta t\\
&+\sum_{n=1}^{+\infty}\sum_{j\in\mathbb{Z}}\mathrm{sgn}(u_j^n-k)(H(x_{j-1},k)-2H(x_j,k)+H(x_{j+1},k))\varphi_j^n\Delta t\\
&-\sum_{n=1}^{+\infty}\sum_{j\in\mathbb{Z}}\mathrm{sgn}(u_j^n-k)\left\{(H(x_{j+1},k)-H(x_j,k))\varphi_j^n\Delta t-\int_{t^n}^{t^{n+1}}\int_{I_j}\partial_xH(x,k)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\right\}\\
&-\int_{\Delta t}^{+\infty}\int_{\mathbb{R}}\mathrm{sgn}(u_\Delta-k)\partial_xH(x,k)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\leq\sup_{x\in\mathbb{R}}|\partial_xH(x,k)|\Big(\|\partial_x\varphi\|_{\mathbf{L}^1}\Delta x+2TR\|\partial_t\varphi\|_{\mathbf{L}^\infty}\Delta t\Big)+T\sup_{x\in\mathbb{R}}|\partial_{xx}^2H(x,k)|\cdot\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+;\mathbf{L}^1)}\Delta x\\
&+4TR\sup_{x\in\mathbb{R}}|\partial_{xx}^2H(x,k)|\cdot\|\varphi\|_{\mathbf{L}^\infty}\Delta x-\int_0^{+\infty}\int_{\mathbb{R}}\mathrm{sgn}(\rho_\Delta-k)\partial_xH(x,k)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned}$$

Hence, using Lemma 3.11,

$$\lim_{\Delta\to 0}C=-\int_0^{+\infty}\int_{\mathbb{R}}\mathrm{sgn}(u-k)\ \partial_xH(x,k)\varphi(t,x)\ \mathrm{d}x\,\mathrm{d}t\,,$$

for any $k\in[\underline{u},\overline{u}]\backslash\mathcal{L}$. Then, repeat the argument from [27] to extend it for all $k\in[\underline{u},\overline{u}]$.
We finally estimate B. Write:

$$\begin{aligned}
\Phi_{j+1/2}^n&=\underbrace{F_{\mathrm{int}}^{j+1/2}(u_j^n\vee k,u_{j+1}^n\vee k)-F_{\mathrm{int}}^{j+1/2}(u_j^n\vee k,u_j^n\vee k)}_{B_1}\\
&+\underbrace{F_{\mathrm{int}}^{j+1/2}(u_j^n\vee k,u_j^n\vee k)-H(x_j,u_j^n\vee k)}_{B_2}+\underbrace{\Phi(x_j,u_j^n,k)}_{B_3}\\
&+\underbrace{H(x_j,u_j^n\wedge k)-F_{\mathrm{int}}^{j+1/2}(u_j^n\wedge k,u_j^n\wedge k)}_{B_4}+\underbrace{F_{\mathrm{int}}^{j+1/2}(u_j^n\wedge k,u_j^n\wedge k)-F_{\mathrm{int}}^{j+1/2}(u_j^n\wedge k,u_{j+1}^n\wedge k)}_{B_5}.
\end{aligned}$$

We see that

$$\left|\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}(B_1+B_5)(\varphi_{j+1}^n-\varphi_j^n)\Delta t\right| \leq 2L\Big(T\|\partial_{xx}^2\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+,\mathbf{L}^1)}+\|\partial_{tx}^2\varphi\|_{\mathbf{L}^1}\Big)(\Delta x+\Delta t)$$
$$+\,2L\int_0^{+\infty}\int_{\mathbb{R}}|\rho_\Delta(t,x+\Delta x)-\rho_\Delta(t,x)|\cdot|\partial_x\varphi(t,x)|\,\mathrm{d}x\,\mathrm{d}t\,.$$

Since $(\rho_\Delta)_\Delta$ converges a.e. on $]0,+\infty[\times\mathbb{R}$, $(\rho_\Delta\partial_x\varphi)_\Delta$ is strongly compact in $\mathbf{L}^1(]0,+\infty[\times\mathbb{R},\mathbb{R})$. As a consequence of the Riesz-Fréchet-Kolmogorov compactness characterization,

$$\int_0^{+\infty}\int_{\mathbb{R}}|\rho_\Delta(t,x+\Delta x)-\rho_\Delta(t,x)|\cdot|\partial_x\varphi(t,x)|\,\mathrm{d}x\,\mathrm{d}t \underset{\Delta\to 0}{\longrightarrow} 0.$$

We now handle B_3. We have

$$\begin{aligned}
&\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}B_3(\varphi_{j+1}^n-\varphi_j^n)\Delta t\\
&=\Delta t\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}\int_{I_j}\Phi(x_j,u_j^n,k)\left(\frac{\varphi(t^n,x+\Delta x)-\varphi(t^n,x)}{\Delta x}-\partial_x\varphi(t^n,x)\right)\mathrm{d}x\\
&+\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}\int_{t^n}^{t^{n+1}}\int_{I_j}\Phi(x_j,u_j^n,k)(\partial_x\varphi(t^n,x)-\partial_x\varphi(t,x))\,\mathrm{d}x\,\mathrm{d}t\\
&+\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}\int_{t^n}^{t^{n+1}}\int_{I_j}(\Phi(x_ju_j^n,k)-\Phi(x,u_j^n,k))\,\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t+\int_0^{+\infty}\int_{\mathbb{R}}\Phi(x,\rho_\Delta,k)\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\\
&\leq T\sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}}|\Phi(x,p,k)|\cdot\|\partial_{tx}^2\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+;\mathbf{L}^1)}\Delta x+\sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}}|\Phi(x,p,k)|\cdot\|\partial_{tx}^2\varphi\|_{\mathbf{L}^1}\Delta t\\
&+\sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}}|\partial_x\Phi(x,p,k)|\cdot\|\partial_x\varphi\|_{\mathbf{L}^1}\Delta x+\int_0^{+\infty}\int_{\mathbb{R}}\Phi(x,\rho_\Delta,k)\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}$$

ensuring that

$$\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}B_3(\varphi_{j+1}^n-\varphi_j^n)\Delta t \underset{\Delta\to 0}{\longrightarrow}\int_0^{+\infty}\int_{\mathbb{R}}\Phi(x,u(t,x),k)\partial_x\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Now, going back to the definition of the interface flux, see Proposition 2.3, we can estimate B_2 and B_4 as

$$\begin{aligned}
\left|\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}(B_2+B_4)(\varphi_{j+1}^n-\varphi_j^n)\Delta t\right| &\leq 2(\sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}}|\partial_xH(x,p)|+L\|\alpha'\|_{\mathbf{L}^\infty})\sum_{n=0}^{+\infty}\sum_{j\in\mathbb{Z}}|\varphi_{j+1}^n-\varphi_j^n|\Delta x\Delta t\\
&\leq 2T\,(\sup_{\substack{x\in\mathbb{R}\\ \underline{u}\leq p\leq\overline{u}}}|\partial_xH(x,p)|+L\|\alpha'\|_{\mathbf{L}^\infty})\cdot\|\partial_x\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^+,\mathbf{L}^1)}\Delta x.
\end{aligned}$$

By passing to the limit $\Delta\to 0$ in $A+B+C\geq 0$, we proved that (1.2) holds, concluding the proof that u is the entropy solution to (**CL**). □

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