# GOTZMANN'S PERSISTENCE THEOREM FOR SMOOTH PROJECTIVE TORIC VARIETIES 

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#### Abstract

Gotzmann's persistence theorem enables us to confirm the Hilbert polynomial of a subscheme of projective space by checking the Hilbert function in just two points, regardless of the dimension of the ambient space. We generalise this result to products of projective spaces, and then extend our result to any smooth projective toric variety. In the case of products of projective spaces, the number of points depends solely on the Picard rank of the ambient space, rather than on the dimension. For a more general smooth projective toric variety this number depends on the number of Hilbert basis elements of the nef cone.


## 1. Introduction

The Hilbert scheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{n}\right)$ is a widely studied object in algebraic geometry. From an algebraic perspective this scheme parameterises homogeneous saturated ideals with a given Hilbert polynomial. In HS04 Haiman and Sturmfels extended these ideas to the multigraded setting, where the Hilbert scheme parameterises homogeneous ideals with a given Hilbert function in a ring $R$ graded by some abelian group. A case of particular interest is when the multigraded ring in question is the Cox ring $R$ of a smooth projective toric variety $X$. In this case the Hilbert function of a homogeneous ideal in $R$ eventually agrees with a polynomial for degrees sufficiently far into the nef cone of $X$. We can therefore define $\operatorname{Hilb}_{P}(X)$ in an analogous manner to $\operatorname{Hilb}_{P}\left(\mathbb{P}^{n}\right)$. We consider the parameter space of homogeneous ideals in $R$ which are saturated with respect to the irrelevant ideal of $X$ and have Hilbert polynomial $P$.

It is natural to ask what properties of standard-graded Hilbert schemes have an extension to the smooth projective toric variety case. Of particular interest are two theorems of Gotzmann, regularity and persistence, which are used in the explicit construction of the Hilbert scheme. Gotzmann's regularity theorem gives a bound on the Castelnuovo-Mumford regularity of a standard-graded ideal as introduced in (Mum66]. Let $I \subset S$ be a homogeneous ideal in a standard-graded polynomial ring. The Hilbert function of $I$ is $H_{I}(b)=\operatorname{dim}_{k}\left((S / I)_{b}\right)$. Similarly, $P_{I}(t)$ will denote the Hilbert polynomial of $I$. Using this language, Gotzmann's persistence theorem can be informally restated in the following way. For a homogeneous ideal $I$ in a standard-graded polynomial ring, a Hilbert polynomial $P(t)$, and sufficiently large $d \in \mathbb{N}$, checking that $H_{I}(d)=P(d)$ and $H_{I}(d+1)=P(d+1)$ guarantees that $P_{I}(t)=P(t)$. For a formal statement, see Theorem 2.3. The surprising aspect here is that by checking the value of $H_{I}(t)$ in just two points we have identified the polynomial $P_{I}(t)$, as opposed to the expected $\operatorname{deg}\left(P_{I}\right)+1$ points. Combining this result with Gotzmann's regularity theorem allows us to obtain explicit equations for the Hilbert scheme $\operatorname{Hilb}_{P}\left(\mathbb{P}^{n}\right)$.

Maclagan and Smith define Castelnuovo-Mumford regularity for the multigraded case MS04 and generalise Gotzmann's regularity theorem MS05 to any smooth projective toric variety. Their generalisation recovers Gotzmann's original result when $X=\mathbb{P}^{n}$. Work of Crona Cro06 gives a persistence theorem for ideals in the Cox ring of $\mathbb{P}^{n} \times \mathbb{P}^{m}$. Further work in this area includes that of Favacchio Fav20 which characterises the Hilbert functions of ideals in the Cox ring of $\mathbb{P}^{n} \times \mathbb{P}^{m}$, and work of Gasharov Gas97] which generalises Gotzmann's persistence theorem to finitely generated modules over a standard-graded polynomial ring.

We are able to extend Crona's theorem to the product of $s$ projective spaces; see Theorem 3.11, we then use this to give a result akin to the earlier restatement of Gotzmann's persistence theorem for products of projective spaces. We introduce some terminology that will allow us to state this theorem. For a homogeneous ideal $J$ lying in the $\mathbb{Z}^{n}$-graded Cox ring $R$ of a smooth
projective toric variety $X$, Definition 4.3 gives a formal description of the associated polynomial $P_{J}\left(t_{1}, \ldots, t_{n}\right)$. We say that $P \in \mathbb{Q}\left[t_{1}, \ldots, t_{s}\right]$ is a Hilbert polynomial on $R$ if there exists some ideal $J \subset R$, homogeneous with respect to the $\mathbb{Z}^{n}$-grading, such that $P_{J}=P$.
Theorem 1.1. Let $S=k\left[x_{1,0}, \ldots, x_{1, n_{1}}, \ldots, x_{s, 0}, \ldots, x_{s, n_{s}}\right]$ be the Cox ring of $\mathbb{P}^{n_{1}} \times \cdots \times \mathbb{P}^{n_{s}}$, with $\operatorname{deg}\left(x_{i, j}\right)=e_{i}$. Let $I \subset S$ be an ideal, homogeneous with respect to the $\mathbb{Z}^{s}$-grading, and let $P \in \mathbb{Q}\left[t_{1}, \ldots, t_{s}\right]$ be a Hilbert polynomial on $S$. Then there exists a point $\left(d_{1}, \ldots, d_{s}\right) \in \mathbb{N}^{s}$ such that if $H_{I}\left(b_{1}, \ldots, b_{s}\right)=P\left(b_{1}, \ldots, b_{s}\right)$ for all points in $\left\{\left(b_{1}, \ldots, b_{s}\right) \in \mathbb{N}^{s} \mid b_{i} \in\left\{d_{i}, d_{i}+1\right\}\right\}$, then $P_{I}=P$.

Theorem 1.1 says that for an ideal $I$ in the Cox ring of the product of $s$ projective spaces we can confirm $P_{I}\left(t_{1}, \ldots, t_{s}\right)=P\left(t_{1}, \ldots, t_{s}\right)$ simply by checking $H_{I}\left(b_{1}, \ldots, b_{s}\right)$ at the vertices of an $s$-dimensional hypercube in $\mathbb{N}^{s}$. This is therefore a generalisation of our earlier informal restatement of Gotzmann's original persistence result.

We then extend Theorem 1.1 to more general smooth projective toric varieties. The situation is particularly nice for Picard-rank-2 varieties, whose Cox rings are described explicitly at the beginning of Section 4 .

Theorem 1.2. Let $J \subset R$, where $R$ is the Cox ring of a smooth Picard-rank-2 toric variety and $J$ is an ideal homogeneous with respect to the $\mathbb{Z}^{2}$-grading. Let $S$ be the Cox ring of $\mathbb{P}^{n} \times \mathbb{P}^{m}$, with $n=\operatorname{dim}_{k}\left(R_{(1,0)}\right)-1, m=\operatorname{dim}_{k}\left(R_{(0,1)}\right)-1$. Let $P\left(t_{1}, t_{2}\right)$ be a Hilbert polynomial on $R$. Then there exists $\left(d_{1}, d_{2}\right) \in \mathbb{N}^{2}$ such that if

$$
\begin{array}{cl}
H_{J}\left(d_{1}, d_{2}\right)=P\left(d_{1}, d_{2}\right), & H_{J}\left(d_{1}+1, d_{2}\right)=P\left(d_{1}+1, d_{2}\right) \\
H_{J}\left(d_{1}, d_{2}+1\right)=P\left(d_{1}, d_{2}+1\right), & H_{J}\left(d_{1}+1, d_{2}+1\right)=P\left(d_{1}+1, d_{2}+1\right),
\end{array}
$$

then $P_{J}=P$.
For a smooth projective toric variety $X$ the semigroup of nef line bundles on $X$ is finitely generated, and the nef cone of $X$ is a convex pointed polyhedral cone with unique minimal Hilbert basis. This fact is useful for varieties with higher Picard rank, where we obtain the following theorem.
Theorem 1.3. Let $X$ be a smooth projective toric variety of Picard rank n. Let $R$ be the Cox ring of $X$, with a $\mathbb{Z}^{n}$-grading resulting from the identification $\operatorname{Pic}(X) \cong \mathbb{Z}^{n}$. Let $J \subset R$ be an ideal, homogeneous with respect to the $\mathbb{Z}^{n}$-grading. Suppose the nef cone of $X$ has a Hilbert basis with $j$ generators. Let $P\left(t_{1}, \ldots, t_{n}\right)$ be a Hilbert polynomial on $R$. Then there exists at most $2^{j+1}$ points $\left(r_{1}, \ldots, r_{n}\right) \in \mathbb{N}^{n}$ such that checking $H_{J}\left(r_{1}, \ldots, r_{n}\right)=P\left(r_{1}, \ldots, r_{n}\right)$ for all of these points guarantees that $P_{J}=P$.
The $2^{j+1}$ points of Theorem 1.3 form a zonotope, see Remark 4.10, generalising the hypercube seen in Theorem 1.1.
The rest of this paper is laid out as follows. In Section 2 we outline necessary background on strongly multistable ideals, allowing us to prove results relating to the product of $s$ projective spaces in Section 3. We conclude Section 3 with a proof of Theorem 1.1 and related remarks. In Section 4 we see how Theorem 1.1 can be applied to smooth projective toric varieties of Picard rank $n \geq 2$, where we prove Theorems 1.2 and 1.3 .

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## 2. Strongly multistable ideals

2.1. Gotzmann's persistence theorem. In this section we outline some necessary background to prove Theorem 1.1. We begin by stating Gotzmann's original persistence theorem. To do so we require the following lemma and definition.

Lemma 2.1 (BH93, Lemma 4.2.6]). Fix $d \in \mathbb{Z}_{>0}$ and $a \in \mathbb{N}$. Then $a$ can be written uniquely in the form

$$
a=\binom{\kappa(d)}{d}+\binom{\kappa(d-1)}{d-1}+\cdots+\binom{\kappa(1)}{1},
$$

where $\kappa(d)>\kappa(d-1)>\cdots>\kappa(1) \geq 0$.
Definition 2.2. Let $a \in \mathbb{N}, d \in \mathbb{Z}_{>0}$. Define

$$
a^{\langle d\rangle}=\binom{\kappa(d)+1}{d+1}+\binom{\kappa(d-1)+1}{d-1+1}+\cdots+\binom{\kappa(1)+1}{1+1}
$$

with $\kappa(d), \ldots, \kappa(1)$ as above. We set $0^{\langle d\rangle}=0$.
A key result of Macaulay Mac27 shows that for $b \in \mathbb{Z}>0$ we have

$$
H_{I}(b+1) \leq H_{I}(b)^{\langle b\rangle} .
$$

We now state Gotzmann's persistence theorem.
Theorem 2.3 ([Got78], see also BH93] Theorem 4.3.3). Let $S=k\left[x_{0}, \ldots, x_{n}\right]$ be a standardgraded polynomial ring, and let $I \subset S$ be a homogeneous ideal generated in degrees $\leq d$. Suppose $H_{I}(d+1)=H_{I}(d)^{\langle d\rangle}$. Then $H_{I}(b+1)=H_{I}(b)^{\langle b\rangle}$ for all $b \geq d$.

As highlighted in the introduction, we can relate this theorem to the Hilbert polynomial of I. In general, any Hilbert polynomial $P(t) \in \mathbb{Q}[t]$ of a standard-graded ideal can be written in the form

$$
P(t)=\sum_{i=1}^{D}\binom{t+a_{i}-i+1}{t-i+1}
$$

for some $D \in \mathbb{Z}_{>0}$ and $a_{1} \geq a_{2} \geq \cdots \geq a_{D} \geq 0$. This number $D$ is called the Gotzmann number of $P(t)$. For $t \geq D-1$, we observe that $P(t+1)=P(t)^{\langle t\rangle}$, which we expect from Theorem 2.3 .
2.2. Strongly multistable ideals. We now outline some necessary background on strongly multistable ideals. Throughout the rest of Sections 2 and 3 let

$$
S=k\left[x_{1,0}, \ldots, x_{1, n_{1}}, \ldots, x_{s, 0}, \ldots, x_{s, n_{s}}\right]
$$

be the Cox ring of the product of $s$ projective spaces $\mathbb{P}^{n_{1}} \times \cdots \times \mathbb{P}^{n_{s}}$, with $\operatorname{deg}\left(x_{i, j}\right)=\boldsymbol{e}_{\boldsymbol{i}} \in \mathbb{Z}^{s}$. We give $S$ the degree lexicographic monomial ordering induced by $x_{1,0} \succ \cdots \succ x_{1, n_{1}} \succ \cdots \succ$ $x_{s, 0} \succ \cdots \succ x_{s, n_{s}}$. More succinctly, we have $x_{i, j} \succ x_{k, l}$ if $i>k$, or $i=k$ and $j>l$. We use $\boldsymbol{x}^{u}$ as shorthand for a monomial in $S$, with $\boldsymbol{x}^{u}=\prod x_{i, j}^{u_{i, j}}$ for some $u_{i, j} \in \mathbb{N}$.

We also set the following piece of notation. An ideal $I$ is generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right) \in$ $\mathbb{N}^{s}$ if there exist homogeneous generators $\left\{f_{1}, \ldots, f_{r}\right\}$ for $I$ such that for each $\left(d_{i 1}, \ldots, d_{i s}\right)=$ $\operatorname{deg}\left(f_{i}\right)$, we have $d_{i j} \leq a_{j}$.
Definition 2.4. A monomial ideal $I \subset S$ is strongly multistable if for all monomials $\boldsymbol{x}^{u} \in I$, $1 \leq i \leq s$ and $0 \leq j \leq n_{i}$, we have that if $x_{i, j} \mid \boldsymbol{x}^{u}$ then $x_{i, k} \frac{x^{u}}{x_{i, j}} \in I$ for all $0 \leq k<j$.
Definition 2.5. For a monomial $\boldsymbol{x}^{\boldsymbol{u}} \in S$ define

$$
\begin{aligned}
m^{i}\left(\boldsymbol{x}^{\boldsymbol{u}}\right) & =\max \left\{j: x_{i, j} \mid \boldsymbol{x}^{u}\right\}, \\
m_{i}\left(\boldsymbol{x}^{u}\right) & =\min \left\{j: x_{i, j} \mid \boldsymbol{x}^{u}\right\} .
\end{aligned}
$$

For every homogeneous ideal of $S$ there exists a strongly multistable ideal with the same Hilbert function. One key example is the multigeneric initial ideal, multigin $(I)$ as outlined in CDNG22. Work of Crona Cro06 focuses on the specific case of strongly bistable ideals, which lie in the Cox ring of $\mathbb{P}^{n} \times \mathbb{P}^{m}$, which we denote $S_{2}$. In this case the bigeneric initial ideal was first defined in ACDN00.
Lemma 2.4 of Cro06 leads to a partial decomposition of the quotient ring $S_{2} / I$ for a strongly bistable ideal $I \subset S_{2}$, with similarities to Stanley decompositions. This partial decomposition allows us to understand the Hilbert function of a strongly bistable ideal. We can generalise this lemma to the strongly multistable case, with the proof being a direct generalisation of Crona's proof.

Lemma 2.6. Let $I \subset S$ be a strongly multistable ideal generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. For a monomial $\boldsymbol{x}^{\boldsymbol{u}} \in S_{\left(a_{1}, \ldots, a_{s}\right)}$ and integers $\left\{v_{i} \geq a_{i}\right\}$ define

$$
H\left(\boldsymbol{x}^{\boldsymbol{u}}\right)=\left\{\boldsymbol{x}^{\boldsymbol{u}} \boldsymbol{x}^{\boldsymbol{v}} \mid \boldsymbol{x}^{\boldsymbol{v}} \in S_{\left(v_{1}-a_{1}, \ldots, v_{s}-a_{s}\right)}, m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}}\right) \geq m^{i}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)\right\} .
$$

Then $H=\bigcup_{\boldsymbol{x}^{u} \in I_{\left(a_{1}, \ldots, a_{s}\right)}} H\left(\boldsymbol{x}^{\boldsymbol{u}}\right)$ generates the vector space $I_{\left(v_{1}, \ldots, v_{s}\right)}$. Moreover $H\left(\boldsymbol{x}^{\boldsymbol{u}}\right) \cap H\left(\boldsymbol{x}^{\boldsymbol{u}^{\prime}}\right)=$ $\emptyset$ for $\boldsymbol{x}^{\boldsymbol{u}} \neq \boldsymbol{x}^{u^{\prime}}$, so $H$ is a monomial basis for $I_{\left(v_{1}, \ldots, v_{s}\right)}$.
Proof. To show that $H$ generates $I_{\left(v_{1}, \ldots, v_{s}\right)}$ as a $k$-vector space, we show that for any monomial

$$
\boldsymbol{x}^{u} \in I_{\left(a_{1}, \ldots, a_{s}\right)}
$$

and monomial $\boldsymbol{x}^{\boldsymbol{v}} \in S_{\left(a_{1}-v_{1}, \ldots, a_{s}-v_{s}\right)}$, then $\boldsymbol{x}^{\boldsymbol{u}} \boldsymbol{x}^{\boldsymbol{v}}$ lies in $H$. Suppose $m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}}\right)<m^{i}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)$ for some $i$. It follows from the strongly bistable property that

$$
\boldsymbol{x}^{\boldsymbol{u}_{\mathbf{2}}}=x_{i, m_{i}\left(\boldsymbol{x}^{v}\right)} \frac{\boldsymbol{x}^{u}}{x_{i, m^{i}\left(\boldsymbol{x}^{u}\right)}} \in I_{\left(a_{1}, \ldots, a_{s}\right)} .
$$

Further if we let

$$
\boldsymbol{x}^{\boldsymbol{v}_{\mathbf{2}}}=x_{i, m^{i}\left(\boldsymbol{x}^{u}\right)} \frac{\boldsymbol{x}^{v}}{x_{i, m_{i}}\left(\boldsymbol{x}^{v}\right)}
$$

then $\boldsymbol{x}^{\boldsymbol{u}} \boldsymbol{x}^{\boldsymbol{v}}=\boldsymbol{x}^{\boldsymbol{u}_{\mathbf{2}}} \boldsymbol{x}^{\boldsymbol{v}_{\mathbf{2}}}, m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}_{\mathbf{2}}}\right) \geq m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}}\right)$ and $m^{i}\left(\boldsymbol{x}^{\boldsymbol{u}_{\mathbf{2}}}\right) \leq m^{i}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)$. We now repeat the above process for $\boldsymbol{x}^{\boldsymbol{u}_{\mathbf{2}}}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\mathbf{2}}}$. If $k$ is the maximum power of $x_{i, m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}}\right)}$ dividing $\boldsymbol{x}^{\boldsymbol{v}}$ then $k$ repeats is enough to ensure that $m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}}\right)>m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}}\right)$. It follows that after a finite number $n$ of repeats $m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{n}}}\right) \geq m^{i}\left(\boldsymbol{x}^{u_{n}}\right)$. Thus we can assume $m_{i}\left(\boldsymbol{x}^{\boldsymbol{v}}\right) \geq m^{i}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)$ for all $i$, completing the first part of the proof. To see that $H\left(\boldsymbol{x}^{u}\right) \cap H\left(\boldsymbol{x}^{u^{\prime}}\right)=\emptyset$ for $\boldsymbol{x}^{\boldsymbol{u}} \neq \boldsymbol{x}^{u^{\prime}}$, let

$$
\boldsymbol{x}^{u}=c_{1} \ldots c_{s}, \quad \boldsymbol{x}^{u^{\prime}}=c_{1}^{\prime} \ldots c_{s}^{\prime}
$$

with $c_{i}, c_{i}^{\prime}$ monomials in $S_{a_{i} e_{i}}$. Suppose $\boldsymbol{x}^{\boldsymbol{u}} \neq \boldsymbol{x}^{u^{\prime}}$. Then without loss of generality there exists a pair $\left\{c_{i}, c_{i}^{\prime}\right\}$ with $c_{i} \succ c_{i}^{\prime}$. Let $j$ be the smallest index such that the power of $x_{i, j}$ is larger for $c_{i}$ than $c_{i}^{\prime}$. Note that $m^{i}\left(\boldsymbol{x}^{\boldsymbol{u}^{\prime}}\right)=m^{i}\left(c_{i}^{\prime}\right)>j$ since $c_{i}$ and $c_{i}^{\prime}$ have equal degree. It therefore follows that for a monomial $\boldsymbol{x}^{\boldsymbol{v}} \in H\left(\boldsymbol{x}^{\boldsymbol{u}}\right)$ the power of $x_{i, j}$ dividing $\boldsymbol{x}^{\boldsymbol{v}}$ is strictly larger than for any monomial lying in $H\left(\boldsymbol{x}^{u^{\prime}}\right)$. Thus $H\left(\boldsymbol{x}^{\boldsymbol{u}}\right) \cap H\left(\boldsymbol{x}^{\boldsymbol{u}^{\prime}}\right)=\emptyset$.

Note that

$$
\bigcup_{x^{u} \in S_{\left(a_{1}, \ldots, a_{s}\right)}} H\left(\boldsymbol{x}^{u}\right)
$$

partitions the monomials of degree $\left(v_{1}, \ldots, v_{s}\right)$. Further,

$$
\bigcup_{x^{u} \in I_{\left(a_{1}, \ldots, a_{s}\right)}} H\left(\boldsymbol{x}^{u}\right)
$$

and

$$
\begin{equation*}
\bigcup_{\boldsymbol{x}^{u} \in S_{\left(a_{1}, \ldots, a_{s}\right) \backslash I_{\left(a_{1}, \ldots, a_{s}\right)}}} H\left(\boldsymbol{x}^{u}\right) \tag{2.1}
\end{equation*}
$$

partition the monomials forming a basis for $I_{\left(v_{1}, \ldots, v_{s}\right)}$ and $(S / I)_{\left(v_{1}, \ldots, v_{s}\right)}$ respectively. For $\sigma=$ $\left\{c_{1}, \ldots, n_{1}, c_{2}, \ldots, n_{2}, \ldots, c_{s}, \ldots, n_{s}\right\}$ let

$$
S_{\sigma}=k\left[x_{i, j} \mid c_{i} \leq j \leq n_{i}\right] .
$$

Then the partition (2.1) shows that for

$$
\sigma_{\boldsymbol{u}}=\left\{m_{1}\left(\boldsymbol{x}^{\boldsymbol{u}}\right), \ldots, n_{1}, m_{2}\left(\boldsymbol{x}^{\boldsymbol{u}}\right), \ldots, n_{2}, \ldots, m_{s}\left(\boldsymbol{x}^{\boldsymbol{u}}\right), \ldots, n_{s}\right\},
$$

is a decomposition of $(S / I)_{\left(\geq a_{1}, \ldots, \geq a_{s}\right)}$. This leads to the following definition.

Definition 2.7. Let $I \subset S$ be a strongly multistable ideal generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. A partial Stanley decomposition of $S / I$ is a set

$$
\mathfrak{S}_{\left(a_{1}, \ldots, a_{s}\right)}=\left\{\left(\boldsymbol{x}^{\boldsymbol{u}}, \sigma_{\boldsymbol{u}}\right) \mid \boldsymbol{x}^{\boldsymbol{u}} \text { is a monomial in } S_{\left(a_{1}, \ldots, a_{s}\right)} \backslash I_{\left(a_{1}, \ldots, a_{s}\right)}\right\}
$$

We have

$$
\operatorname{dim}_{k}\left((S / I)_{\left(t_{1}, \ldots, t_{s}\right)}\right)=\sum_{\mathfrak{S}_{\left(a_{1}, \ldots, a_{s}\right)}} \operatorname{dim}_{k}\left(\left(S_{\sigma_{u}}\right)_{\left(t_{1}-a_{1}, \ldots, t_{s}-a_{s}\right)}\right)
$$

for $\left(t_{1}, \ldots, t_{s}\right)$ with $t_{i} \geq a_{i}$ for all $i$.
This partial Stanley decomposition therefore allows us to calculate the Hilbert function of $S / I$ in sufficiently high degrees. By passing from an ideal $I$ to multigin $(I)$ we can apply this idea to more general ideals.

Example 2.8. Let $S_{2}=k\left[x_{0}, x_{1}, y_{0}, y_{1}\right]$ be the Cox ring of $\mathbb{P}^{1} \times \mathbb{P}^{1}$, and let $I=\left(x_{0}, x_{1} y_{0}\right) \subset S_{2}$ be an ideal. Since $I$ is strongly bistable and generated in degrees $\leq(2,2)$ we can construct a partition using the monomials of degree (2,2). The only monomial in $S_{2_{(2,2)}} \backslash I_{(2,2)}$ is $\boldsymbol{x}^{\boldsymbol{u}}=x_{1}^{2} y_{1}^{2}$. Here $m_{x}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)=1=m_{y}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)$ and $\sigma_{\boldsymbol{u}}=\{1,1\}$. Therefore $\mathfrak{S}_{(2,2)}=\left(x_{1}^{2} y_{1}^{2},\{1,1\}\right)$, and

$$
\operatorname{dim}_{k}\left(\left(S_{2} / I\right)_{\left(t_{1}, t_{2}\right)}\right)=\operatorname{dim}_{k}\left(k\left[x_{1}, y_{1}\right]_{\left(t_{1}-2, t_{2}-2\right)}\right)
$$

for $t_{1} \geq 2, t_{2} \geq 2$.
2.3. Multilex ideals. Aramova, Crona and De Negri ACDN00, Section 4] give a generalisation of lexicographic ideals for the bigraded case, called bilex ideals. We will end this section by extending this definition to the multigraded case, and proving some useful results.

Definition 2.9. Let $M \subset S_{\left(a_{1}, \ldots, a_{s}\right)}$ be a set of monomials. Then:
(i) $M$ is $x_{i}$-lex if for all $\boldsymbol{x}^{\boldsymbol{u}} \in S_{a_{i} e_{i}}, \boldsymbol{x}^{\boldsymbol{v}} \in S_{\left(a_{1}, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_{s}\right)}$ we have that $\boldsymbol{x}^{\boldsymbol{u}} \boldsymbol{x}^{\boldsymbol{v}} \in M$ implies that $\boldsymbol{x}^{\boldsymbol{u}^{\prime}} \boldsymbol{x}^{\boldsymbol{v}} \in M$ for all $\boldsymbol{x}^{\boldsymbol{u}^{\prime}} \in S_{a_{i} \boldsymbol{e}_{\boldsymbol{i}}}$ with $\boldsymbol{x}^{\boldsymbol{u}^{\prime}} \succ \boldsymbol{x}^{\boldsymbol{u}}$.
(ii) $M$ is multilex if it is $x_{i}$-lex for all $1 \leq i \leq s$. It is $x_{1}, \ldots, x_{i}$-lex if it is $x_{j}$-lex for $1 \leq j \leq i$.
(iii) A monomial ideal $I \subset S$ is multilex if $I_{\left(a_{1}, \ldots, a_{s}\right)}$ is generated by a multilex set for all $\left(a_{1}, \ldots, a_{s}\right) \in \mathbb{N}^{s}$.

Observe that a multilex ideal is always strongly multistable. In the bigraded case, where $s=2$, Aramova Crona and De Negri [ACDN00, Lemma 4.13] show that for every strongly bistable ideal $I$, there is a multilex ideal $I^{\text {bilex }}$ with the same Hilbert function. We show that this result can be extended to the strongly multistable case. To do so, we will need the following definitions.

Definition 2.10. For a set of monomials $M \subset S_{\left(a_{1}, \ldots, a_{s}\right)}$ we can decompose

$$
M=\bigcup_{j=1}^{k} M_{j} \boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}}
$$

for some $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}} \in S_{\left(a_{1}, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_{s}\right)}$ and $M_{j} \subset S_{a_{i} \boldsymbol{e}_{\boldsymbol{i}}}$. Define

$$
M^{x_{i} \operatorname{lex}}=\bigcup_{j=1}^{k} M_{j}^{\operatorname{lex}} \boldsymbol{x}^{u_{j}}
$$

Let $M^{x_{1} \ldots x_{i} \operatorname{lex}}=\left(\left(\left(M^{x_{1} \operatorname{lex}}\right)^{x_{2} \operatorname{lex}}\right) \cdots\right)^{x_{i} \operatorname{lex}}$.
Remark 2.11. For any sets of monomials $A \subset B \subset S_{\left(a_{1}, \ldots, a_{s}\right)}$, and any $1 \leq i \leq s, A^{x_{i} l e x} \subset$ $B^{x_{i} \text { lex }}$.

Definition 2.12. Given a monomial $\boldsymbol{x}^{\boldsymbol{u}} \in S_{\left(a_{1}, \ldots, a_{s}\right)}$, write $\boldsymbol{x}^{\boldsymbol{u}}=\boldsymbol{x}_{\mathbf{1}}^{\boldsymbol{u}_{\boldsymbol{1}}} \ldots \boldsymbol{x}_{\boldsymbol{s}}^{\boldsymbol{u}_{\boldsymbol{s}}}$, with each $\boldsymbol{x}_{\boldsymbol{i}}^{\boldsymbol{u}_{\boldsymbol{i}}} \in$ $S_{a_{i} e_{i}}$. Define the multi-lexsegment of $\boldsymbol{x}^{\boldsymbol{u}}$ by

$$
L\left(\boldsymbol{x}^{\boldsymbol{u}}\right)=\left\{\boldsymbol{x}_{\boldsymbol{1}}^{\boldsymbol{u}_{\boldsymbol{1}}^{\prime}} \ldots \boldsymbol{x}_{\boldsymbol{s}}^{\boldsymbol{u}_{\boldsymbol{s}}^{\prime}} \mid \boldsymbol{x}_{\boldsymbol{j}}^{\boldsymbol{u}_{\boldsymbol{j}}^{\prime}} \in S_{a_{j} e_{j}}, \quad \boldsymbol{x}_{\boldsymbol{j}}^{\boldsymbol{u}_{\boldsymbol{j}}^{\prime}} \succeq \boldsymbol{x}_{\boldsymbol{j}}^{\boldsymbol{u}_{\boldsymbol{j}}} \text { for all } j\right\}
$$

Remark 2.13. For a set of monomials $M \subset S_{\left(a_{1}, \ldots, a_{s}\right)}, M$ is $x_{1}, \ldots, x_{i}$-lex if and only if for all $\boldsymbol{x}^{\boldsymbol{u}} \boldsymbol{x}^{\boldsymbol{v}} \in M$, with $\boldsymbol{x}^{\boldsymbol{u}} \in S_{\left(a_{1}, \ldots, a_{i}, 0, \ldots, 0\right)}, \boldsymbol{x}^{\boldsymbol{v}} \in S_{\left(0, \ldots, 0, a_{i+1}, \ldots, a_{s}\right)}$, we have $L\left(\boldsymbol{x}^{\boldsymbol{u}}\right) \boldsymbol{x}^{\boldsymbol{v}} \subset M$.

To generalise the results in Section 4 of ACDN00], we will focus on the case that $M$ is a strongly multistable set of monomials. We begin by generalising [ACDN00, Theorem 4.8].

Lemma 2.14. Suppose the strongly multistable set $M$ is $x_{1}, \ldots, x_{i-1}$-lex for some $2 \leq i \leq s$. Then $M^{x_{i} l e x}$ is $x_{1}, \ldots, x_{i}$-lex.

Proof. The proof is essentially the same as ACDN00, Theorem 4.8]. We decompose $M$ as follows:

$$
M=\bigcup_{j=1}^{l_{1}} \bigcup_{k=1}^{l_{2}} x^{u_{j}} M_{j, k} x^{v_{k}}
$$

where $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}} \in S_{\left(a_{1}, \ldots, a_{i-1}, 0, \ldots, 0\right)}, M_{j, k} \subset S_{a_{i} \boldsymbol{e}_{i}}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}} \in S_{\left(0, \ldots, 0, a_{i+1}, \ldots, a_{s}\right)}$. We will order the $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}}$ such that $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}^{\prime}}} \succ \boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}}$ if and only if $j^{\prime}<j$, and similar for the $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}}$. Observe that

$$
M^{x_{i} \operatorname{lex}}=\bigcup_{j=1}^{l_{1}} \bigcup_{k=1}^{l_{2}} x^{u_{j}} M_{j, k}^{\operatorname{lex}} \boldsymbol{x}^{v_{k}}
$$

An element in $M^{x_{i} \operatorname{lex}}$ can therefore be written as $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}} \boldsymbol{x}^{\boldsymbol{w}} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}}$, with $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}} \in S_{\left(a_{1}, \ldots, a_{i-1}, 0, \ldots, 0\right)}$, $\boldsymbol{x}^{\boldsymbol{w}} \in M_{j, k}^{\mathrm{lex}}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}} \in S_{\left(0, \ldots, 0, a_{i+1}, \ldots, a_{s}\right)}$. We need to check that

$$
L\left(\boldsymbol{x}^{u_{j}} \boldsymbol{x}^{\boldsymbol{w}}\right) \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}} \subset M^{x_{i} \operatorname{lex}} .
$$

To see that this is true, let $\boldsymbol{x}^{\boldsymbol{u}^{\prime}} \in S_{\left(a_{1}, \ldots, a_{i-1}, 0, \ldots, 0\right)}$ and $\boldsymbol{x}^{\boldsymbol{w}^{\boldsymbol{\prime}}} \in S_{a_{i} \boldsymbol{e}_{\boldsymbol{i}}}$ with $\boldsymbol{x}^{\boldsymbol{u}^{\prime}} \in L\left(\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}}\right)$ and $\boldsymbol{x}^{\boldsymbol{w}^{\prime}} \in L\left(\boldsymbol{x}^{\boldsymbol{w}}\right)$. We wish to show that $\boldsymbol{x}^{\boldsymbol{u}^{\prime}} \boldsymbol{x}^{\boldsymbol{w}^{\prime}} \boldsymbol{x}^{\boldsymbol{v}} \in M^{x_{i} \text { lex }}$. Since $M$ is $x_{1}, \ldots, x_{i-1}$-lex it follows that for any $\boldsymbol{x}^{u^{\prime}} \in L\left(\boldsymbol{x}^{u_{j}}\right)$ and any $\boldsymbol{x}^{\boldsymbol{w}^{\prime \prime}} \in M_{j, k}, \boldsymbol{x}^{u^{\prime}} \boldsymbol{x}^{\boldsymbol{w}^{\prime \prime}} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}} \in M$. In particular $\boldsymbol{x}^{u^{\prime}}=\boldsymbol{x}^{\boldsymbol{u}_{j^{\prime}}}$ for some $j^{\prime}<j$ and $\boldsymbol{x}^{\boldsymbol{w}^{\prime \prime}} \in M_{j^{\prime}, k}$. Since this holds for any $\boldsymbol{x}^{\boldsymbol{w}^{\prime \prime}} \in M_{j, k}, M_{j, k} \subset M_{j^{\prime}, k}$ and consequently $M_{j, k}^{\mathrm{lex}} \subset M_{j^{\prime}, k}^{\mathrm{lex}}$. In particular, $\boldsymbol{x}^{\boldsymbol{w}} \in M_{j^{\prime}, k}^{\mathrm{lex}}$ and so $\boldsymbol{x}^{\boldsymbol{u}^{\prime}} \boldsymbol{x}^{\boldsymbol{w}} \boldsymbol{x}^{\boldsymbol{v}} \in M^{x_{i} \operatorname{lex}}$. By definition $M^{x_{i} \text { lex }}$ is $x_{i}$-lex so it then follows that $\boldsymbol{x}^{u^{\prime}} \boldsymbol{x}^{\boldsymbol{w}^{\prime}} \boldsymbol{x}^{\boldsymbol{v}} \in M^{x_{i} \text { lex }}$.

To define $M^{\text {multilex }}=M^{x_{1} \ldots x_{s} \text { lex }}$ we now need only check that if $M$ is strongly multistable and $x_{1}, \ldots, x_{i-1}$-lex, then $M^{x_{i} \text { lex }}$ is also strongly multistable. It then follows that $M^{\text {multilex }}$ is indeed multilex by Lemma 2.14

Lemma 2.15. Let $M$ be a strongly multistable set, and let $1 \leq i \leq s$. If $i \geq 2$, suppose further that $M$ is also $x_{1}, \ldots, x_{i-1}$-lex. Then $M^{x_{i} l e x}$ is strongly multistable.
Proof. We again follow the proof of ACDN00, Theorem 4.8]. As in Lemma 2.14 for $i \geq 2$ we decompose

$$
M=\bigcup_{j=1}^{l_{1}} \bigcup_{k=1}^{l_{2}} x^{u_{j}} M_{j, k} x^{\boldsymbol{v}_{k}}
$$

where $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{j}}} \in S_{\left(a_{1}, \ldots, a_{i-1}, 0, \ldots, 0\right)}, M_{j, k} \subset S_{a_{i} e_{i}}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}} \in S_{\left(0, \ldots, 0, a_{i+1}, \ldots, a_{s}\right)}$. If $i=1$, we simply have

$$
M=\cup_{k=1}^{l_{2}} M_{1, k} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}}
$$

for $M_{1, k} \subset S_{a_{1} e_{1}}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}} \in S_{\left(0, a_{2}, \ldots, a_{s}\right)}$. We need to show that for $\boldsymbol{m} \in M^{x_{i} \operatorname{lex}}$, if $x_{l, p}$ divides $\boldsymbol{m}$, then $x_{l, q} \frac{m}{x_{l, p}} \in M^{x_{i} \operatorname{lex}}$ for all $0 \leq q<p$. If $l \leq i$ then this holds by Lemma 2.14, since $M^{x_{i} \operatorname{lex}}$ is $x_{1}, \ldots, x_{i}$-lex. Otherwise, note that the $X=\left\{\boldsymbol{x}^{v_{1}}, \ldots, \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{l}}}\right\}$ form a strongly multistable set since $M$ is strongly multistable. For $l>i$, if $x_{l, p}$ divides $\boldsymbol{x}^{\boldsymbol{v}_{k}}$ for some $k$ then $x_{l, q} \frac{x^{v_{k}}}{x_{l, p}} \in X$ for all $0 \leq q<p$. We write $x_{l, q} \boldsymbol{x}^{v_{k}}=\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}^{\prime}}}$ for some $k^{\prime}<k$. Since $M$ is strongly multistable, for fixed $j$, we have $M_{j, k} \subset M_{j, k^{\prime}}$ and consequently $M_{j, k}^{\mathrm{lex}} \subset M_{j, k^{\prime}}^{\mathrm{lex}}$. It follows that if $x_{l, p}$ divides $\boldsymbol{m} \in M^{x_{i} \operatorname{lex}}$ for $l>i$ then $x_{l, q} \frac{m}{x_{l, p}} \in M^{x_{i} \operatorname{lex}}$ for all $0 \leq q<p$.

To generalise the result of ACDN00, we prove the following lemma.

Lemma 2.16. Let $I \subset S$ be a strongly multistable ideal. Then

$$
I^{\text {multilex }}=\bigoplus_{\left(a_{1}, \ldots, a_{s}\right) \in \mathbb{N}^{s}}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }}
$$

is an ideal.
Proof. We now follow the proof of ACDN00, Lemma 4.12]. For a set of monomials $M \subset S$, define $X_{p} M=\left\{x_{p, q} \boldsymbol{m} \mid 0 \leq q \leq n_{p}, \boldsymbol{m} \in M\right\}$. To verify that $I^{\text {multilex }}$ is an ideal, we need only check that for any $\left(a_{1}, \ldots, a_{s}\right) \in \mathbb{N}$ and $1 \leq p \leq s$, we have $X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }} \subset$ $\left(I_{\left(a_{1}, \ldots, a_{p}+1, \ldots, a_{s}\right)}\right)^{\text {multilex }}$. Note that $X_{p} I_{\left(a_{1}, \ldots, a_{s}\right)} \subset I_{\left(a_{1}, \ldots, a_{p}+1, \ldots, a_{s}\right)}$, hence $\left(X_{p} I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }} \subset$ $\left(I_{\left(a_{1}, \ldots, a_{p}+1, \ldots, a_{s}\right)}\right)^{\text {multilex }}$. It follows that if we can show that $X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }} \subset\left(X_{p} I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }}$ then we are done. Decompose $I_{\left(a_{1}, \ldots, a_{s}\right)}$ as

$$
I_{\left(a_{1}, \ldots, a_{s}\right)}=\bigcup_{j=1}^{l_{1}} M_{j} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}}
$$

for $M_{j} \subset S_{\left(a_{1}, \ldots, a_{p-1}, 0, \ldots, 0\right)}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}} \in S_{\left(0, \ldots, 0, a_{p}, \ldots, a_{s}\right)}$. Observe that

$$
\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p-1} \operatorname{lex}}=\bigcup_{j=1}^{l_{1}} M_{j}^{x_{1} \ldots x_{p-1} \operatorname{lex}} \boldsymbol{x}^{\boldsymbol{v}_{j}}
$$

Let $X_{p}\left\{\boldsymbol{x}^{\boldsymbol{v}_{1}}, \ldots, \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{l}_{1}}}\right\}=\left\{\boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{1}}}, \ldots, \boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{l}_{\mathbf{2}}}}\right\}$. We similarly decompose

$$
X_{p} I_{\left(a_{1}, \ldots, a_{s}\right)}=\bigcup_{k=1}^{l_{2}} N_{k} \boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{k}}}
$$

for $N_{k} \subset S_{\left(a_{1}, \ldots, a_{p-1}, 0, \ldots, 0\right)}$ and $\boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{k}}} \in S_{\left(0, \ldots, 0, a_{p}, \ldots, a_{s}\right)}$. For all $1 \leq j \leq l_{1}$ and $0 \leq q \leq n_{p}$ we have $x_{p, q} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}}=\boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{k}}}$ for some $1 \leq k \leq l_{2}$, with $M_{j} \subset N_{k}$. If $M_{j} \subset N_{k}$ it follows that $M_{j}^{x_{1} \ldots x_{p-1} \operatorname{lex}} \subset N_{k}^{x_{1} \ldots x_{p-1} \operatorname{lex}}$ by Remark 2.11 . Consequently,

$$
X_{p} \bigcup_{j=1}^{l_{1}} M_{j}^{x_{1} \ldots x_{p-1} \operatorname{lex}} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}} \subset \bigcup_{k=1}^{l_{2}} N_{k}^{x_{1} \ldots x_{p-1} \operatorname{lex}} \boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{k}}}
$$

In other words, $X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p-1} \operatorname{lex}} \subset\left(X_{p} I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p-1} \text { lex }}$. A similar argument shows that $X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }} \subset\left(X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p} \text { lex }}\right)^{x_{p+1} \ldots x_{s} \text { lex }}$. The strategy here is to decompose

$$
\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p} \operatorname{lex}}=\bigcup_{j=1}^{l_{1}} M_{j} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}}
$$

with $M_{j} \subset S_{\left(0, \ldots, 0, a_{p+1}, \ldots, a_{s}\right)}$ and $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{k}}} \in S_{\left(a_{1}, \ldots, a_{p}, 0, \ldots, 0\right)}$, and observe that $\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }}=$ $\bigcup_{j=1}^{l_{1}} M_{j}^{x_{p+1} \ldots x_{s} \operatorname{lex}} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}}$. We then consider the decomposition

$$
X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p} \operatorname{lex}}=\bigcup_{k=1}^{l_{2}} N_{k} \boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{k}}}
$$

with $X_{p}\left\{\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{1}}}, \ldots, \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{l}_{1}}}\right\}=\left\{\boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{1}}}, \ldots, \boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{l}_{\mathbf{2}}}}\right\}$ and $N_{k} \subset S_{\left(0, \ldots, 0, a_{p+1}, \ldots, a_{s}\right)}$. As before, we see that for all $1 \leq j \leq l_{1}$ and $0 \leq q \leq n_{p}$, we have $x_{p, q} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}}=\boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{k}}}$ for some $1 \leq k \leq l_{2}$, and consequently $M_{j} \subset N_{k}$ for this value of $k$. By Remark 2.11 we then have $M_{j}^{x_{p+1} \ldots x_{s} \operatorname{lex}} \subset N_{k}^{x_{p+1} \ldots x_{s} \text { lex }}$. It follows that

$$
X_{p} \bigcup_{j=1}^{l_{1}} M_{j}^{x_{p+1} \ldots x_{s} \operatorname{lex}} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}} \subset \bigcup_{k=1}^{l_{2}} N_{k}^{x_{p+1} \ldots x_{s} \operatorname{lex}} \boldsymbol{x}^{\boldsymbol{w}_{\boldsymbol{k}}}
$$

or in other words, $X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }} \subset\left(X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p} \operatorname{lex}}\right)^{x_{p+1} \ldots x_{s} \operatorname{lex}}$.
We have a chain of inclusions

$$
X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p} \operatorname{lex}} \subset\left(X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p-1} \operatorname{lex}}\right)^{x_{p} \operatorname{lex}} \subset\left(X_{p} I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p} \operatorname{lex}}
$$

The second inclusion follows immediately from the previous part and Remark 2.11. The first inclusion is a direct generalisation of the first part of the proof of ACDN00, Lemma 4.12]. Combining everything, we obtain

$$
X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }} \subset\left(X_{p}\left(I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{x_{1} \ldots x_{p} \operatorname{lex}}\right)^{x_{p+1} \ldots x_{s} \operatorname{lex}} \subset\left(X_{p} I_{\left(a_{1}, \ldots, a_{s}\right)}\right)^{\text {multilex }}
$$

This completes the proof.
Theorem 2.17. For every ideal $I \subset S$ which is homogeneous with respect to the $\mathbb{Z}^{s}$-grading, there exists a multilex ideal with the same Hilbert function.

Proof. As highlighted in Section [2.2, for every $I$ there exists a strongly multistable ideal with the same Hilbert function. The ideal multigin $(I)$ is one such example. Lemmas $2.14,2.15$ and 2.16 then show that for any strongly multistable ideal there exists a multilex ideal with the same Hilbert function.

We will exploit Theorem 2.17 in Section 3 to find the degrees in which persistence occurs.

## 3. Persistence type results for products of projective spaces

We now establish some preliminary results required to prove Theorem 1.1. We begin by understanding the structure of Hilbert polynomials on $S$. We then relate multilex ideals and bilex ideals, and use this to generalise a theorem of Crona [Cro06, Theorem 4.10]. We conclude with the proof of Theorem 1.1 and a comparison to the original result of Gotzmann.
3.1. The structure of Hilbert polynomials on $S$. In this subsection we will apply the results of Section 2 on strongly multistable and multilex ideals to better understand the structure of $P_{I}\left(t_{1}, \ldots, t_{s}\right)$.

Lemma 3.1. Let $I_{1}, \ldots, I_{l} \subset S$ be homogeneous monomial ideals, generated in degrees $\leq$ $\left(a_{1}, \ldots, a_{s}\right)$, with each $a_{i} \geq 2$. Then for $b_{i} \geq 1$,

$$
H_{I_{1}}\left(b_{1}, \ldots, b_{s}\right)+\cdots+H_{I_{l}}\left(b_{1}, \ldots, b_{s}\right)=H_{J}\left(b_{1}, \ldots, b_{s}\right)
$$

for a monomial ideal J generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$ in the Cox ring of $\mathbb{P}^{l\left(n_{1}+1\right)-1} \times \cdots \times$ $\mathbb{P}^{l\left(n_{s}+1\right)-1}$. It follows that

$$
P_{J}\left(t_{1}, \ldots, t_{s}\right)=P_{I_{1}}\left(t_{1}, \ldots, t_{s}\right)+\cdots+P_{I_{l}}\left(t_{1}, \ldots, t_{s}\right) .
$$

Proof. We begin by relabelling the variables of $I_{2}$, replacing $x_{i, j}$ with $y_{i, j}$, to obtain an ideal $I_{2}^{\prime} \subset k\left[y_{1,0}, \ldots, y_{1, n_{1}}, \ldots, y_{s, 0}, \ldots, y_{s, n_{s}}\right]$. Let

$$
S^{\prime}=k\left[x_{1,0}, \ldots, x_{1, n_{1}}, y_{1,0}, \ldots, y_{1, n_{1}}, \ldots, x_{s, 0}, \ldots, x_{s, n_{s}}, y_{s, 0}, \ldots, y_{s, n_{s}}\right],
$$

with $\operatorname{deg}\left(x_{i, j}\right)=\operatorname{deg}\left(y_{i, j}\right)=\boldsymbol{e}_{\boldsymbol{i}}$. Note that $S^{\prime}$ is the Cox ring of

$$
\mathbb{P}^{2 n_{1}+1} \times \cdots \times \mathbb{P}^{2 n_{s}+1} .
$$

Consider the ideal $J_{1}=I_{1}+I_{2}^{\prime}+K \subset S^{\prime}$, where $K$ is generated by all monomials of the form $x_{i, j} y_{p, q}$. The Hilbert function of this ideal agrees with $H_{I_{1}}\left(b_{1}, \ldots, b_{s}\right)+H_{I_{2}}\left(b_{1}, \ldots, b_{s}\right)$ for $b_{i} \geq 1$. Note that each $x_{i, j} y_{p, q}$ has degree $\leq(2, \ldots, 2)$, so $J_{1}$ is generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. We also observe that $J_{1}$ is a monomial ideal. Now applying the same argument to $J_{1}$ and $I_{3}$ allows us to obtain a monomial ideal $J_{2}$ in the Cox ring of $\mathbb{P}^{3\left(n_{1}+1\right)-1} \times \cdots \times \mathbb{P}^{3\left(n_{s}+1\right)-1}$, generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$, and with $H_{J_{1}}\left(b_{1}, \ldots, b_{s}\right)+H_{I_{3}}\left(b_{1}, \ldots, b_{s}\right)=H_{J_{2}}\left(b_{1}, \ldots, b_{s}\right)$. We continue to repeat this argument to obtain the ideal $J$, which is a monomial ideal in the Cox ring of $\mathbb{P}^{l\left(n_{1}+1\right)-1} \times \cdots \times \mathbb{P}^{l\left(n_{s}+1\right)-1}$.

For $P \in \mathbb{Q}\left[t_{1}, \ldots, t_{s}\right]$ set $\operatorname{maxdeg}(P)$ to be the vector $\boldsymbol{v}=\left(v_{i}\right) \in \mathbb{N}^{s}$, where each $v_{i}$ is the maximum power of $t_{i}$ dividing at least one of the terms of $P$. Combining Lemma 2.6 and Lemma 3.1 we prove the following lemma, which allows us to better understand the Hilbert polynomial of a multigraded ideal.

Lemma 3.2. Let $P_{I}\left(t_{1}, \ldots, t_{s}\right)$ be the Hilbert polynomial of an ideal $I \subset S$, which is homogeneous with respect to the $\mathbb{Z}^{s}$-grading. Fix $\left(a_{1}, \ldots, a_{s}\right) \in \mathbb{N}^{s}$ such that $a_{i} \geq 2$ for all $i$ and such that $I^{\text {multilex }}$ is generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. Suppose that $P_{I}\left(t_{1}, \ldots, t_{s}\right)$ has $\operatorname{maxdeg}\left(P_{I}\right)=\left(p_{1}, \ldots, p_{s}\right)$. Then we may write

$$
P_{I}\left(t_{1}, \ldots, t_{s}\right)=\sum_{i_{s}=0}^{p_{s}} \cdots \sum_{i_{2}=0}^{p_{2}} F_{i_{2} \ldots i_{s}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}},
$$

where each $F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$ is the Hilbert polynomial for some standard-graded monomial ideal $J_{i_{2} \ldots i_{s}}$ generated in degrees $\leq a_{1}$. Further, we have $H_{J_{i_{2}} \ldots i_{s}}\left(b_{1}\right)=F_{i_{2} \ldots i_{s}}\left(b_{1}\right)$ for $b_{1} \geq a_{1}$, and consequently $H_{I}\left(b_{1}, \ldots, b_{r}\right)=P_{I}\left(b_{1}, \ldots, b_{r}\right)$ when all $b_{i} \geq a_{i}$.
Proof. We pass to $J=I^{\text {multilex }}$, which has the same Hilbert function as $I$ but is strongly multistable. Consider the set

$$
\mathcal{I}=\left\{\left(i_{2}, \ldots, i_{s}\right) \in \mathbb{N}^{s-1} \mid 0 \leq i_{j} \leq p_{j}\right\} .
$$

For $\boldsymbol{i}=\left(i_{2}, \ldots, i_{s}\right) \in \mathcal{I}$ consider the set of monomials

$$
L^{i}=\left\{\boldsymbol{x}^{\boldsymbol{u}} \in S_{\left(0, a_{2}, \ldots, a_{s}\right)} \mid i_{j}=n_{j}-m^{j}\left(\boldsymbol{x}^{\boldsymbol{u}}\right) \text { for all } 2 \leq j \leq s\right\} .
$$

Fix some $\boldsymbol{x}^{u_{1}} \in L^{i}$. We consider the set of monomials $M_{1}^{i} \subset S_{\left(a_{1}, 0, \ldots, 0\right)}$ with $\boldsymbol{x}^{\boldsymbol{u}_{1}} M_{1}^{i} \subset$ $J_{\left(a_{1}, \ldots, a_{s}\right)}$. The set $M_{1}^{i}$ generates a strongly stable ideal $J_{1}^{i}$ in the standard-graded polynomial ring $k\left[x_{1,0}, \ldots, x_{1, n_{1}}\right]$ with Hilbert function $H_{J_{1}^{i}}\left(b_{1}\right)$. Since $J_{1}^{i}$ is strongly stable and generated in degrees $\leq a_{1}$ we observe that $H_{J_{1}^{i}}\left(b_{1}\right)=P_{J_{1}^{i}}\left(b_{1}\right)$ for $b_{1} \geq a_{1}$. Consider the set of monomials

$$
N_{1}^{i}=S_{\left(a_{1}, 0, \ldots, 0\right)} \backslash M_{1}^{i} .
$$

For each $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}} \in N_{1}^{\boldsymbol{i}}, \boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{1}}} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}}$ corresponds to a monomial in $S_{\left(a_{1}, \ldots, a_{s}\right)} \backslash J_{\left(a_{1}, \ldots, a_{s}\right)}$. For such a monomial $\boldsymbol{x}^{\boldsymbol{u}}=\boldsymbol{x}^{\boldsymbol{u}_{1}} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}}$ the corresponding term in the partial Stanley decomposition of $J$ is

$$
k\left[x_{1, m_{1}\left(\boldsymbol{x}^{v_{j}}\right)}, \ldots, x_{1, n_{1}}, x_{2, m_{2}\left(\boldsymbol{x}^{u_{1}}\right)}, \ldots, x_{2, n_{2}}, \ldots, x_{s, m_{s}\left(\boldsymbol{x}^{u_{1}}\right)}, \ldots, x_{s, n_{s}}\right]\left(-a_{1}, \ldots,-a_{s}\right) .
$$

This contributes

$$
\begin{equation*}
\operatorname{dim}_{k}\left(k\left[x_{1, m_{1}\left(\boldsymbol{x}^{v_{j}}\right)}, \ldots, x_{\left.1, n_{1}\right]}\right]_{t_{1}-a_{1}}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}} \tag{3.1}
\end{equation*}
$$

to the Hilbert polynomial of $I$, where $i_{j}=n_{j}-m^{j}\left(\boldsymbol{x}^{u_{1}}\right)$. Since the monomials in $N_{1}^{i}$ form a partial Stanley decomposition for $J_{1}^{i}$ we have that

$$
\sum_{\boldsymbol{x}^{v_{j} \in N_{1}^{i}}} \operatorname{dim}_{k}\left(k\left[x_{1, m_{1}\left(\boldsymbol{x}^{v_{j}}\right)}, \ldots, x_{1, n_{1}}\right]_{t_{1}-a_{1}}\right)=P_{J_{1}^{i}}\left(t_{1}\right)
$$

for $t_{1} \geq a_{1}$. Consequently, the expression (3.1) becomes

$$
\begin{equation*}
P_{J_{1}^{i}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}} . \tag{3.2}
\end{equation*}
$$

We repeat the above procedure for all possible monomials $\boldsymbol{x}^{u_{l}} \in L^{i}$. Note that all monomials in $L^{i}$ for a fixed $\boldsymbol{i}$ will have the same binomial coefficients appearing in the corresponding piece of the Hilbert polynomial (3.2). It therefore makes sense for us to add the $P_{J_{l}}\left(t_{1}\right)$ together. By Lemma 3.1 the sum of the Hilbert polynomials of the $J_{l}^{i}$ agrees with the Hilbert polynomial for a new monomial ideal $J_{i_{2} \ldots i_{s}}$ generated in degrees $\leq a_{1}$. We will denote the Hilbert polynomial of $J_{i_{2} \ldots i_{s}}$ by $F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$. We have

$$
\begin{equation*}
F_{i_{2} \ldots i_{s}}\left(t_{1}\right)=\sum_{x^{u_{l} \in L^{i}}} P_{J_{l}^{i}}\left(t_{1}\right) . \tag{3.3}
\end{equation*}
$$

Note that since $H_{J_{l}^{i}}\left(b_{1}\right)=P_{J_{l}^{i}}\left(b_{1}\right)$ for all $b_{1} \geq a_{1}$, we have

$$
F_{i_{2} \ldots i_{s}}\left(b_{1}\right)=\sum_{x^{u_{l} \in L^{i}}} P_{J_{l}^{i}}\left(b_{1}\right)=\sum_{\boldsymbol{x}^{u_{l} \in L^{i}}} H_{J_{l}^{i}}\left(b_{1}\right)=H_{J_{i_{2} \ldots i_{s}}}\left(b_{1}\right)
$$

for all $b_{1} \geq a_{1}$. By equation (3.3) the monomials in $L^{i}$ contribute in total

$$
F_{i_{2} \ldots i_{s}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}}
$$

to the Hilbert polynomial of $I$. Varying $\boldsymbol{i} \in \mathcal{I}$ and repeating this procedure we obtain

$$
P_{I}\left(t_{1}, \ldots, t_{s}\right)=\sum_{i_{s}=0}^{p_{s}} \cdots \sum_{i_{2}=0}^{p_{2}} F_{i_{2} \ldots i_{s}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}}
$$

where each $F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$ is the Hilbert polynomial of a monomial ideal $J_{i_{2} \ldots i_{s}}$ generated in degrees $\leq a_{1}$.
Example 3.3. Let

$$
S_{4}=k\left[x_{1,0}, x_{2,0}, x_{2,1}, x_{3,0}, x_{3,1}, x_{3,2}, x_{4,0}, x_{4,1}\right]
$$

be the Cox ring of $\mathbb{P}^{0} \times \mathbb{P}^{1} \times \mathbb{P}^{2} \times \mathbb{P}^{1}$. Let $J=\left(x_{2,0}, x_{3,0}, x_{3,1}\right) \subset S_{4}$, and observe that $J$ is a multilex ideal generated in degrees $\leq(2,2,2,2)$. We have $P_{J}\left(t_{1}, t_{2}, t_{3}, t_{4}\right)=t_{4}+1$, with $\operatorname{maxdeg}\left(P_{J}\right)=(0,0,0,1)$. By Lemma 3.2 we can write

$$
P_{I}\left(t_{1}, t_{2}, t_{3}, t_{4}\right)=F_{000}\left(t_{1}\right)+F_{001}\left(t_{1}\right)\binom{t_{4}-2+1}{1}
$$

for some standard-graded Hilbert polynomials $F_{000}\left(t_{1}\right)$ and $F_{001}\left(t_{1}\right)$. It follows that we must have $F_{001}\left(t_{1}\right)=1, F_{000}\left(t_{1}\right)=2$.

The following terminology will be used throughout the rest of this section.
Definition 3.4. Let $P_{I}\left(t_{1}, \ldots, t_{s}\right)$ be a Hilbert polynomial of an ideal $I \subset S$ with $I^{\text {multilex }}$ generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. By Lemma 3.2 we can write

$$
P_{I}\left(t_{1}, \ldots, t_{s}\right)=\sum_{i_{s}=0}^{p_{s}} \cdots \sum_{i_{2}=0}^{p_{2}} F_{i_{2} \ldots i_{s}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}},
$$

where the $F_{i_{2} \ldots i_{s}}\left(t_{1}\right) \in \mathbb{Q}\left[t_{1}\right]$ are Hilbert polynomials of standard-graded ideals generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. We will extend the definition of the $F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$, setting $F_{i_{2} \ldots i_{s}}\left(t_{1}\right)=0$ for any $\left(i_{2}, \ldots, i_{s}\right) \in \mathbb{N}^{s-1}$ with some $i_{j}>p_{j}$. For $1<r<s$ and $\boldsymbol{b}=\left(b_{1}, \ldots, b_{r-1}, b_{r+1}, \ldots, b_{s}\right) \in$ $\mathbb{N}^{s-1}$, define

$$
P_{\boldsymbol{b}}^{r}\left(t_{r}\right)=\sum_{i_{r}=0}^{p_{r}} \cdots \sum_{i_{2}=0}^{p_{2}} F_{i_{2} \ldots i_{r} b_{r+1} \ldots b_{s}}\left(b_{1}\right) \prod_{j=2}^{r-1}\binom{b_{j}-a_{j}+i_{j}}{i_{j}}\binom{t_{r}-a_{r}+i_{r}}{i_{r}} \in \mathbb{Q}\left[t_{r}\right] .
$$

If $r=1$ then we write $\boldsymbol{b}=\left(b_{2}, \ldots, b_{s}\right) \in \mathbb{N}^{s-1}$ and define

$$
P_{\boldsymbol{b}}^{1}\left(t_{1}\right)=F_{b_{2} \ldots b_{s}}\left(t_{1}\right) .
$$

For $r=s$ write $\boldsymbol{b}=\left(b_{1}, \ldots, b_{s-1}\right)$ and define

$$
P_{\boldsymbol{b}}^{s}\left(t_{s}\right)=P_{I}\left(b_{1}, \ldots, b_{s-1}, t_{s}\right) .
$$

Notice that if $b_{i}>p_{i}$ for some $i>r$ then $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)=0$. For $1 \leq r<s, P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ is the coefficient of $\binom{t_{r+1}-a_{r+1}+b_{r+1}}{b_{r}} \ldots\binom{t_{s}-a_{s}+b_{s}}{b_{s}}$ in $P_{I}\left(b_{1}, \ldots, b_{r-1}, t_{r}, \ldots, t_{s}\right)$.

For $r>1$, fix $\boldsymbol{b}=\left(b_{1}, \ldots, b_{r-1}, b_{r+1}, \ldots, b_{s}\right) \in \mathbb{N}^{s-1}$ and $i_{r} \in \mathbb{N}$. Further, set $\hat{\boldsymbol{b}}=$ $\left(b_{1}, \ldots, b_{r-2}, i_{r}, b_{r+1}, \ldots, b_{s}\right) \in \mathbb{N}^{s-1}$. Then the coefficient of $\binom{t_{r}-a_{r}+i_{r}}{i_{r}}$ in $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ is given by $P_{\hat{b}}^{r-1}\left(b_{r-1}\right)$.
Remark 3.5. Suppose $\operatorname{maxdeg}\left(P_{I}\right)=\left(p_{1}, \ldots, p_{s}\right)$. For fixed $b_{1}, \ldots, b_{r-1}$ with $b_{i} \geq a_{i}$ for all $i$, $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ has degree $\leq p_{r}$ for all choices of $b_{r+1}, \ldots, b_{s}$. Further, we claim that there is at least one choice of $b_{r+1}, \ldots, b_{s}$ such that equality is achieved. For $r=1$ this follows from the definition, so we focus on the case $r>1$. Observe that since $\operatorname{maxdeg}\left(P_{I}\right)=\left(p_{1}, \ldots, p_{s}\right)$ there is at least one choice of $i_{2}, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{s}$ with $0 \leq i_{j} \leq p_{j}$ for all $j$ such that $F_{i_{2} \ldots i_{r-1} p_{r} i_{r+1} \ldots i_{s}}\left(t_{1}\right) \neq 0$. Recall from Lemma 3.2 that $F_{i_{2} \ldots i_{r-1} p_{r} i_{r+1} \ldots i_{s}}\left(b_{1}\right)=H_{J}\left(b_{1}\right)$ for some standard-graded ideal $J$ when $b_{1} \geq a_{1}$. Thus, if $F_{i_{2} \ldots i_{r-1} p_{r} i_{r+1} \ldots i_{s}}\left(b_{1}\right)=0$ then Macaulay's bound for the Hilbert function
of an ideal tells us that $F_{i_{2} \ldots i_{r-1} p_{r} i_{r+1} \ldots i_{s}}(u) \leq 0$ for $u \geq b_{1}$. This forces $F_{i_{2} \ldots i_{r-1} p_{r} i_{r+1} \ldots i_{s}}\left(t_{1}\right)$ to be the zero polynomial, which is a contradiction. Thus $F_{i_{2} \ldots i_{r-1} p_{r} i_{r+1} \ldots i_{s}}\left(b_{1}\right) \neq 0$. We may therefore fix $b_{r+1}=i_{r+1}, \ldots, b_{s}=i_{s}$ and observe that for this choice of $i_{2}, \ldots, i_{r-1}$

$$
F_{i_{2} \ldots i_{r-1} p_{r} b_{r+1} \ldots b_{s}}\left(b_{1}\right)\binom{b_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{b_{r-1}-a_{r-1}+i_{r-1}}{i_{r-1}}>0,
$$

since the binomial coefficients are never zero for $b_{i} \geq a_{i}$. The coefficient of $\binom{t_{r}-a_{r}+p_{r}}{p_{r}}$ in $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ is given by

$$
\sum_{i_{r-1}=0}^{p_{r-1}} \cdots \sum_{i_{2}=0}^{p_{2}} F_{i_{2} \ldots i_{r-1} p_{r} b_{r+1} \ldots b_{s}}\binom{b_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{b_{r-1}-a_{r-1}+i_{r-1}}{i_{r-1}} .
$$

Since $b_{1} \geq a_{1}$, we have $F_{i_{2} \ldots i_{1} p_{1} p_{r+1} \ldots b_{s}}\left(b_{1}\right) \geq 0$ for all choices of $i_{2} \ldots i_{r-1}$. As established there is one choice of $i_{2}, \ldots, i_{r-1}$ such that this polynomial is strictly positive. it follows that the coefficient of $\binom{t_{r}-a_{r}+p_{r}}{p_{r}}$ in $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ is strictly positive, and so $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ has degree $p_{r}$ for this choice of $b_{r+1}, \ldots, b_{s}$.

Example 3.6. Returning to Example 3.3, let $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ denote the polynomials as in Definition 3.4 for $P_{J}\left(t_{1}, t_{2}, t_{3}, t_{4}\right)=t_{4}+1$. We compute $P_{(2,0,0))}^{2}\left(t_{2}\right)$ and $P_{(2,0,1)}^{2}\left(t_{2}\right)$. Recall that $F_{001}\left(t_{1}\right)=1$ and $F_{000}\left(t_{1}\right)=2$, and that maxdeg $\left(P_{J}\right)=(0,0,0,1)$. We obtain

$$
\begin{aligned}
& P_{(2,0,0)}^{2}\left(t_{2}\right)=\sum_{i_{2}=0}^{0} F_{i_{2} 00}(2)\binom{t_{2}-2+i_{2}}{i_{2}}=F_{000}(2)=2, \\
& P_{(2,0,1)}^{2}\left(t_{2}\right)=\sum_{i_{2}=0}^{0} F_{i_{2} 01}(2)\binom{t_{2}-2+i_{2}}{i_{2}}=F_{001}(2)=1 .
\end{aligned}
$$

3.2. Extension of results on bilex ideals. We now extend known results about bilex ideals to multilex ideals, using the following lemmas.

Lemma 3.7. Let $I_{1}, \ldots, I_{l} \subset S_{2}=k\left[x_{0}, \ldots, x_{n}, y_{0}, \ldots, y_{m}\right]$ be bilex ideals generated in degrees $\leq\left(a_{1}^{\prime}, a_{2}\right)$. Fix $a_{1} \in \mathbb{N}$. Then there exists $n^{\prime} \in \mathbb{N}$ and a bilex ideal $J$ in the Cox ring of $\mathbb{P}^{n^{\prime}} \times \mathbb{P}^{m}$ such that for all $b_{2} \geq a_{2}$

$$
H_{I_{1}}\left(a_{1}, b_{2}\right)+\cdots+H_{I_{l}}\left(a_{1}, b_{2}\right)=H_{J}\left(1, b_{2}\right) .
$$

Further we may assume $J$ is generated in degrees $\leq\left(1, a_{2}\right)$. We have

$$
P_{I_{1}}\left(a_{1}, t_{2}\right)+\cdots+P_{I_{l}}\left(a_{1}, t_{2}\right)=P_{J}\left(1, t_{2}\right)
$$

as polynomials in $\mathbb{Q}\left[t_{2}\right]$.
Proof. Fix $d=\binom{a_{1}+n}{a_{1}}$ and let $n^{\prime}=l d-1$. Let $S_{2}^{\prime}=k\left[z_{0}, \ldots, z_{l d-1}, y_{0}, \ldots, y_{m}\right]$, with $\operatorname{deg}\left(z_{i}\right)=$ $(1,0), \operatorname{deg}\left(y_{j}\right)=(0,1)$ Let $\left\{\boldsymbol{x}^{u_{1}}, \ldots, \boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{d}}}\right\}$ be the set of monomials in $S_{2_{\left(a_{1}, 0\right)}}$, and consider the family of maps of vector spaces

$$
\begin{aligned}
\varphi_{i}: S_{2_{\left(a_{1}, 0\right)}} & \rightarrow S_{2_{(1,0)}^{\prime}}^{\prime} \\
\boldsymbol{x}^{u_{j}} & \mapsto x_{j-1+(i-1) d}
\end{aligned}
$$

for $1 \leq i \leq l$. Define the set of monomials

$$
M_{b_{2}}^{i}=\left\{\varphi_{i}\left(\boldsymbol{x}^{\boldsymbol{u}}\right) \boldsymbol{y}^{\boldsymbol{v}} \mid \boldsymbol{x}^{\boldsymbol{u}} \in S_{2_{\left(a_{1}, 0\right)}}, \boldsymbol{y}^{\boldsymbol{v}} \in S_{2_{\left(0, b_{2}\right)}}, \boldsymbol{x}^{\boldsymbol{u}} \boldsymbol{y}^{\boldsymbol{v}} \in I_{i}\right\} \subset S_{2}^{\prime}
$$

for $1 \leq i \leq l, b_{2} \in \mathbb{N}$. Notice that for $i \neq j, M_{b_{2}}^{i}$ is disjoint from $M_{b_{2}}^{j}$, and that each $M_{b_{2}}^{i}$ is bilex when viewed as an ideal in the ring $k\left[z_{(i-1) d}, \ldots z_{i d-1}, y_{0}, \ldots, y_{m}\right]$. Let $M_{b_{2}}=\bigcup_{i=1}^{l} M_{b_{2}}^{i}$. Then
(i) $M_{b_{2}}$ is bilex for the correct relabelling of the $z_{0}, \ldots, z_{l d-1}$,
(ii) $H_{I_{1}}\left(a_{1}, b_{2}\right)+\cdots+H_{I_{l}}\left(a_{1}, b_{2}\right)=\left|S_{2_{\left(1, b_{2}\right)}}^{\prime}\right|-\left|M_{b_{2}}\right|$,
(iii) for $J=\left(M_{a_{2}}\right) \subset S_{2}^{\prime},\left|S_{2_{\left(1, b_{2}\right)}^{\prime}}^{\prime}\right|-\left|M_{b_{2}}\right|=H_{J}\left(1, b_{2}\right)$ for all $b_{2} \geq a_{2}$.

To see (i), note that the monomials $N_{b_{2}}^{i}=\left\{\boldsymbol{y}^{\boldsymbol{v}} \in S_{2_{\left(0, b_{2}\right)}} \mid z_{i} \boldsymbol{y}^{\boldsymbol{v}} \in M_{b_{2}}\right\}$ form a lexsegment for any $z_{i} \in S_{2}^{\prime}$. It follows that $M_{b_{2}}$ is $y$-lex as in Definition 2.9, but may not be $x$-lex. However, we can relabel the $z_{i}$ so that the $N_{b_{2}}^{i}$ are (non strictly) descending in size and thus obtain a bilex set. Part (ii) follows from the definition of $M_{b_{2}}$, and the disjointness of the $M_{b_{2}}^{i}$. Finally, part (iii) holds since each $I_{i}$ is generated in degrees $\leq\left(a_{1}^{\prime}, a_{2}\right)$. The result then follows, since $J$ is a bilex ideal after relabelling variables.
Lemma 3.8. Let $I \subset S$ be an ideal, homogeneous with respect to the $\mathbb{Z}^{n}$-grading, with $I^{\text {multilex }}$ generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. As in Lemma 3.2 we write

$$
P_{I}\left(t_{1}, \ldots, t_{s}\right)=\sum_{i_{s}=0}^{p_{s}} \cdots \sum_{i_{2}=0}^{p_{2}} F_{i_{2} \ldots i_{s}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}},
$$

where each $F_{i_{2} \ldots i_{s}}$ is the Hilbert polynomial of a standard graded ideal. Let $P_{\boldsymbol{b}}^{r}\left(t_{r}\right) \in \mathbb{Q}\left[t_{r}\right]$ denote a polynomial as in definition 3.4 for $P_{I}\left(t_{1}, \ldots, t_{s}\right)$. Then for fixed $2 \leq r \leq s$ and $\boldsymbol{b}=\left(b_{1}, \ldots, b_{r-1}, b_{r+1}, \ldots, b_{s}\right) \in \mathbb{N}^{s-1}$ with $b_{i} \geq a_{i}$ for $i<r$, there exists a bigraded ideal $K$ with $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)=P_{K}\left(1, t_{r}\right)$. Further, we can always find such an ideal $K$ which is a bilex ideal in the Cox ring of $\mathbb{P}^{m} \times \mathbb{P}^{n_{r}}$ for some $m \in \mathbb{N}$, and with $K$ generated in degrees $\leq\left(1, a_{r}\right)$.
Proof. Recall from Definition 3.4 that if $b_{i}>p_{i}$ for any $i>r$ then $P_{b}^{r}\left(t_{r}\right)=0$, so we focus on the case that $b_{i} \leq p_{i}$ for $i>r$. We pass from $I$ to $J=I^{\text {multilex }}$, which is generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. We begin with the case $r=s$. For a given monomial $\boldsymbol{x}^{v_{\boldsymbol{j}}} \in S_{\left(b_{1}, \ldots, b_{s-2}, 0,0\right)}$ set $M_{j} \subset S_{\left(0, \ldots, 0, b_{s-1}, a_{s}\right)}$ to be the set of monomials such that $M_{j} \boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}} \subset J_{\left(b_{1}, \ldots, b_{s-1}, a_{s}\right)}$, and denote by $J_{j} \subset k\left[x_{s-1,0}, \ldots, x_{s-1, n_{s-1}}, x_{s, 0}, \ldots, x_{s, n_{s}}\right]$ the bigraded ideal generated by $M_{j}$. Since $J$ is multilex, $J_{j}$ is bilex, and generated in degrees $\leq\left(b_{s-1}, a_{s}\right)$. Further, we have

$$
P_{J}\left(b_{1}, \ldots, b_{s-2}, t_{s-1}, t_{s}\right)=\sum_{x^{v_{j} \in S_{\left(b_{1}, \ldots, b_{s-2}, 0,0\right)}}} P_{J_{j}}\left(t_{s-1}, t_{s}\right) .
$$

Applying Lemma 3.7, we observe that there exists $m \in \mathbb{N}$ such that

$$
P_{J}\left(b_{1}, \ldots, b_{s-1}, t_{s}\right)=\sum_{x^{v_{\boldsymbol{j}} \in S_{\left(b_{1}, \ldots, b_{s-2}, 0,0\right)}}} P_{J_{j}}\left(b_{s-1}, t_{s}\right)=P_{K}\left(1, t_{s}\right)
$$

for an ideal $K \subset k\left[x_{0}, \ldots, x_{m}, y_{0}, \ldots, y_{n_{s}}\right]$. For the case $2 \leq r<s$ the polynomial $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ is the coefficient of

$$
\binom{t_{r+1}-a_{r+1}+b_{r+1}}{b_{r+1}} \ldots\binom{t_{s}-a_{s}+b_{s}}{b_{s}}
$$

in $P_{I}\left(b_{1}, \ldots, b_{r-1}, t_{r}, \ldots, t_{s}\right)$. Consider the set of monomials

$$
M=\left\{\boldsymbol{x}^{\boldsymbol{u}} \in S_{\left(0, \ldots, 0, a_{r+1}, \ldots, a_{s}\right)} \mid m^{j}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)=n_{j}-b_{j} \text { for all } r<j \leq s\right\} .
$$

A monomial $\boldsymbol{x}^{\boldsymbol{u}} \in S_{\left(a_{1}, \ldots, a_{s}\right)} \backslash J_{\left(a_{1}, \ldots, a_{s}\right)}$ which satisfies $m^{j}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)=n_{j}-b_{j}$ for all $r<j \leq s$ will contribute a term to the partial Stanley decomposition whose associated Hilbert polynomial is of the form

$$
G\left(t_{1}, \ldots, t_{r}\right)\binom{t_{r+1}-a_{r+1}+b_{r+1}}{b_{r+1}} \ldots\binom{t_{s}-a_{s}+b_{s}}{b_{s}}
$$

for some $G\left(t_{1}, \ldots, t_{r}\right)$. Therefore, to find the coefficient of

$$
\binom{t_{r+1}-a_{r+1}+b_{r+1}}{b_{r+1}} \ldots\binom{t_{s}-a_{s}+b_{s}}{b_{s}}
$$

in $P_{I}\left(t_{1}, \ldots, t_{r}\right)$ we focus on monomials whose degree $\left(0, \ldots, 0, a_{r+1}, \ldots, a_{s}\right)$ part is in the set $M$. For a given $\boldsymbol{x}^{u_{i}} \in M$ consider the set of monomials $M_{i} \subset S_{\left(a_{1}, \ldots, a_{r-1}, a_{r}, 0 \ldots, 0\right)}$, such that $\boldsymbol{x}^{u_{i}} M_{i} \subset J_{\left(a_{1}, \ldots, a_{s}\right)}$. Since $J$ is multilex the set $M_{i}$ generates a multilex ideal

$$
J_{i} \subset S_{r}=k\left[x_{1,0}, \ldots, x_{1, n_{1}}, \ldots, x_{r, 0} \ldots, x_{r, n_{r}}\right],
$$

generated in degrees $\leq\left(a_{1}, \ldots, a_{r}\right)$. Varying $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{i}}}$ we get a collection of these multilex ideals $J_{i}$. For a given $\boldsymbol{x}^{u_{i}}$ let $N_{i}$ denote the degree $\left(a_{1}, \ldots, a_{r}, 0, \ldots, 0\right)$ monomials not in $M_{i}$. The
monomials in $N_{i}$ form a partial Stanley decomposition for the strongly multistable ideal $J_{i}$. Further, the monomials $\bigcup_{\boldsymbol{x}^{u_{i} \in M}} \boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{i}}} N_{i}$ are exactly the monomials in $S_{\left(a_{1}, \ldots, a_{s}\right)} \backslash J_{\left(a_{1}, \ldots, a_{s}\right)}$ satisfying $m^{j}\left(\boldsymbol{x}^{\boldsymbol{u}}\right)=n_{j}-b_{j}$ for all $r<j \leq s$. The pieces of the partial Stanley decomposition of $I$ associated to the monomials in $\boldsymbol{x}^{\boldsymbol{u}_{\boldsymbol{i}}} N_{i}$ contribute

$$
P_{J_{i}}\left(t_{1}, \ldots, t_{r}\right)\binom{t_{r+1}-a_{r+1}+b_{r+1}}{b_{r+1}} \ldots\binom{t_{s}-a_{s}+b_{s}}{b_{s}}
$$

to the Hilbert polynomial of $I$. It follows that

$$
P_{\boldsymbol{b}}^{r}\left(t_{r}\right)=\sum_{\boldsymbol{x}^{u_{i} \in M}} P_{J_{i}}\left(b_{1}, \ldots, b_{r-1}, t_{r}\right) .
$$

For a fixed $J_{i}$ and fixed monomial $\boldsymbol{x}^{\boldsymbol{v}_{\boldsymbol{j}}} \in S_{r_{\left(b_{1}, \ldots, b_{r-2}, 0,0\right)}}$ define the bigraded ideal $J_{i, j} \subset$ $k\left[x_{r-1,0}, \ldots, x_{r-1, n_{r-1}}, x_{r, 0}, \ldots, x_{r, n_{r}}\right]$, which is generated by the set of monomials $M_{i, j} \subset S_{r_{\left(0, \ldots, 0, b_{r-1}, a_{r}\right)}}$ such that $M_{i, j} \boldsymbol{x}^{v_{j}} \subset J_{i_{\left(b_{1}, \ldots, b_{r-1}, a_{r}\right)} \text {. }}$ The proof is now similar to the $r=s$ case. The ideals $J_{i, j}$ are bilex since each $J_{i}$ is multilex. Further, we have

$$
P_{J_{i}}\left(b_{1}, \ldots, b_{r-2}, t_{r-1}, t_{r}\right)=\sum_{x^{v_{j} \in S_{r_{\left(b_{1}, \ldots, b_{r-2}, 0,0\right)}}}} P_{J_{i, j}}\left(t_{r-1}, t_{r}\right) .
$$

Now applying Lemma 3.7, there exists $m_{i} \in \mathbb{N}$ such that

$$
P_{J_{i}}\left(b_{1}, \ldots, b_{r-1}, t_{r}\right)=\sum_{x^{v_{j} \in S_{r}} \sum_{\left(b_{1}, \ldots, b_{r-2}, 0,0\right)}} P_{J_{i, j}}\left(b_{r-1}, t_{r}\right)=P_{K_{i}}\left(1, t_{r}\right)
$$

for a bilex ideal $K_{i} \subset k\left[x_{0}, \ldots, x_{m_{i}}, y_{0}, \ldots, y_{n_{r}}\right]$, generated in degrees $\leq\left(1, a_{r}\right)$. Again applying Lemma 3.7, there exists $m \in \mathbb{N}$ such that

$$
P_{\boldsymbol{b}}^{r}\left(t_{r}\right)=\sum_{x^{u_{i} \in M}} P_{J_{i}}\left(b_{1}, \ldots, b_{r-1}, t_{r}\right)=\sum_{x^{u_{i} \in M}} P_{K_{i}}\left(1, t_{r}\right)=P_{K}\left(1, t_{r}\right)
$$

for a bilex ideal $K \subset k\left[x_{0}, \ldots, x_{m}, y_{0}, \ldots, y_{n_{r}}\right]$ generated in degrees $\leq\left(1, a_{r}\right)$.
Remark 3.9. Lemma 3.8 shows that for a multilex ideal $I$ generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$ and a collection $\left\{b_{1}, \ldots, b_{s-1}\right\}$ with $b_{i} \geq a_{i}$ there is a bilex ideal $K$ generated in degrees $\leq\left(1, a_{s}\right)$ such that

$$
P_{I}\left(b_{1}, \ldots, b_{s-1}, t_{s}\right)=P_{K}\left(1, t_{s}\right) .
$$

We may further assume that $K$ lies in the Cox ring of $\mathbb{P}^{m} \times \mathbb{P}^{n_{s}}$ for some $m \in \mathbb{N}$. The choice of fixing $t_{1}, \ldots t_{s-1}$ was arbitrary, and a similar result holds for any $P\left(b_{1}, \ldots, b_{i-1}, t_{i}, b_{i+1}, \ldots, b_{s}\right)$.

We will need the following result of Crona for bigraded ideals.
Theorem 3.10 ([Cro06, Theorem 4.10]). Let $S_{2}=k\left[x_{0}, \ldots, x_{n}, y_{0}, \ldots, y_{m}\right]$ be the Cox ring of $\mathbb{P}^{n} \times \mathbb{P}^{m}$ with the usual $\mathbb{Z}^{2}$-grading. Let $I$ be a bigraded homogeneous ideal of $S_{2}$. Let

$$
\begin{equation*}
H_{I}\left(b_{1}, b_{2}\right)=\binom{n+b_{1}}{n} \mathfrak{q}\left(b_{1}\right)+\mathfrak{r}\left(b_{1}\right) \tag{3.4}
\end{equation*}
$$

be the Euclidean division of $H_{I}\left(b_{1}, b_{2}\right)$ by $\binom{n+b_{1}}{n}$. Fix $t_{2}=b_{2} \in \mathbb{Z}_{>0}$. Suppose there is a strongly bistable ideal with the same Hilbert function as I generated in degrees $\leq\left(a_{1}, a_{2}\right)$. Then
(i)

$$
H_{I}\left(b_{1}+1, b_{2}\right)=\binom{b_{1}+1+n}{n} \mathfrak{q}\left(b_{1}\right)+\mathfrak{r}\left(b_{1}\right)^{\left\langle b_{1}\right\rangle}
$$

for $b_{1} \gg 0$.
(ii) For all $u \geq a_{1}$, if

$$
H_{I}\left(u+1, b_{2}\right)=\binom{u+1+n}{n} \mathfrak{q}(u)+\mathfrak{r}(u)^{\langle u\rangle}
$$

then

$$
H_{I}\left(b_{1}+1, b_{2}\right)=\binom{b_{1}+1+n}{n} \mathfrak{q}\left(b_{1}\right)+\mathfrak{r}\left(b_{1}\right)^{\left\langle b_{1}\right\rangle}
$$

for all $b_{1} \geq u$.
Similar results hold for the Euclidean division by $\binom{b_{2}+m}{m}$, where $b_{1} \in \mathbb{Z}_{\geq 0}$ is now fixed and $b_{2}$ varies.

It is natural to use Remark 3.9 to extend this result to the product of $s$ projective spaces.
Theorem 3.11. Let $I \subset S$ be a homogeneous ideal, whose Hilbert function agrees with that of some multilex ideal generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$. Fix $i \in \mathbb{N}$ with $1 \leq i \leq s$. For fixed $b_{1} \geq a_{1}, \ldots, b_{i-1} \geq a_{i-1}, b_{i+1} \geq a_{i+1}, \ldots b_{s} \geq a_{s}$ consider the Euclidean division of $H_{I}\left(b_{1}, \ldots, b_{s}\right)$ by $\binom{n_{i}+b_{i}}{n_{i}}$, which we write

$$
H_{I}\left(b_{1}, \ldots, b_{s}\right)=\binom{n_{i}+b_{i}}{n_{i}} \mathfrak{q}\left(b_{i}\right)+\mathfrak{r}\left(b_{i}\right) .
$$

Then we have
(i)

$$
H_{I}\left(b_{1}, \ldots, b_{i-1}, b_{i}+1, b_{i+1}, \ldots, b_{s}\right)=\binom{n_{i}+b_{i}+1}{n_{i}} \mathfrak{q}\left(b_{i}\right)+\mathfrak{r}\left(b_{i}\right)^{\left\langle b_{i}\right\rangle}
$$

for $b_{i} \gg 0$.
(ii) For all $u \geq a_{i}$ if

$$
H_{I}\left(b_{1}, \ldots, b_{i-1}, u+1, b_{i+1}, \ldots, b_{s}\right)=\binom{n_{i}+u+1}{n_{i}} \mathfrak{q}(u)+\mathfrak{r}(u)^{\langle u\rangle}
$$

then

$$
H_{I}\left(b_{1}, \ldots, b_{i-1}, b_{i}+1, b_{i+1}, \ldots, b_{s}\right)=\binom{n_{i}+b_{i}+1}{n_{i}} \mathfrak{q}\left(b_{i}\right)+\mathfrak{r}\left(b_{i}\right)^{\left\langle b_{i}\right\rangle}
$$

for all $b_{i} \geq u$.
Proof. Recall that

$$
P_{I}\left(b_{1}, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_{s}\right)=H_{I}\left(b_{1}, \ldots, b_{i-1}, u, b_{i+1}, \ldots, b_{s}\right)
$$

when $u \geq a_{i}$ and $b_{j} \geq a_{j}$ for all possible $j$. By Remark 3.9 there exists $m \in \mathbb{N}$ such that $H_{I}\left(b_{1}, \ldots, b_{i_{1}}, u, b_{i+1}, \ldots, b_{s}\right)=P_{K}(1, u)$ for some bilex ideal $K$ in the Cox ring of $\mathbb{P}^{m} \times \mathbb{P}^{n_{i}}$, generated in degrees $\leq\left(1, a_{i}\right)$. The result then follows by Theorem 3.10.
Remark 3.12. A similar application of Remark 3.9 allows us to extend Aramova, Crona and De Negri's generalisation of Macaulay's bound on the Hilbert function of a bigraded ideal ACDN00, Theorem 4.18] to multigraded ideals.
3.3. Persistence for products of projective spaces. We now have all the tools required to prove the main theorem of this section.
Proof of Theorem 1.1. Suppose $I^{\text {multilex }}$ and $J^{\text {multilex }}$ are generated in degrees $\leq\left(a_{1}, \ldots, a_{s}\right)$, with $a_{i} \geq 2$ for all $i$. Recall that $H_{I}\left(b_{1}, \ldots, b_{s}\right)=P_{I}\left(b_{1}, \ldots, b_{s}\right)$ and $H_{J}\left(b_{1}, \ldots, b_{s}\right)=P_{J}\left(b_{1}, \ldots, b_{s}\right)$ when all $b_{i} \geq a_{i}$. Suppose that $\operatorname{maxdeg}(P)=\left(p_{1}, \ldots, p_{s}\right)$. By Lemma 3.2, $P\left(t_{1}, \ldots, t_{s}\right)$ can be written in the form

$$
P\left(t_{1}, \ldots, t_{s}\right)=\sum_{i_{s}=0}^{p_{s}} \cdots \sum_{i_{2}=0}^{p_{2}} F_{i_{2} \ldots i_{s}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}},
$$

where the $F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$ are Hilbert polynomials of standard-graded ideals generated in degrees $\leq a_{1}$. For a known $P\left(t_{1}, \ldots, t_{s}\right)$ we can explicitly find each of the $F_{i_{2} \ldots i_{s}}$. Similarly, we can write

$$
P_{I}\left(t_{1}, \ldots, t_{s}\right)=\sum_{i_{s}=0}^{q_{s}} \cdots \sum_{i_{2}=0}^{q_{2}} G_{i_{2} \ldots i_{s}}\left(t_{1}\right)\binom{t_{2}-a_{2}+i_{2}}{i_{2}} \ldots\binom{t_{s}-a_{s}+i_{s}}{i_{s}},
$$

for some $q_{2}, \ldots, q_{s} \in \mathbb{N}$, where each $G_{i_{2}, \ldots, i_{s}}\left(t_{1}\right)$ is the Hilbert polynomial of some standardgraded ideal generated in degrees $\leq a_{1}$. The polynomial $P_{I}\left(t_{1}, \ldots, t_{s}\right)$ is therefore uniquely determined by $q_{2}, \ldots, q_{s}$ and the $G_{i_{2} \ldots i_{s}}$.

The first step of the proof is to fix the point $\left(d_{1}, \ldots, d_{s}\right) \in \mathbb{N}^{s}$. We denote by $P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ the polynomials as in definition 3.4 for $P$ and $Q_{b}^{r}\left(t_{r}\right)$ those for $P_{I}$. Recall from Lemma 3.8 that for each $2 \leq r \leq s$ and $\boldsymbol{b} \in \mathbb{N}^{s-1}$ with $b_{i} \geq a_{i}$ for all $i<r$, there exists a bilex ideal $K \subset k\left[x_{0}, \ldots, x_{m}, y_{0}, \ldots, y_{n_{r}}\right]$ such that $P_{b}^{r}\left(t_{r}\right)=P_{K}\left(1, t_{r}\right)$. We may further assume $K$ is generated in degrees $\leq\left(1, a_{r}\right)$. The same holds for the $Q_{\boldsymbol{b}}^{r}\left(t_{r}\right)$. In particular we can apply Theorem 3.10 to both sets of polynomials. Let

$$
P_{\boldsymbol{b}}^{r}(u)=\binom{u+n_{r}}{n_{r}} \mathfrak{q}(u)+\mathfrak{r}(u)
$$

be the Euclidean division of $P_{\boldsymbol{b}}^{r}(u)$ by $\binom{u+n_{r}}{n_{r}}$, and define $c\left(P_{\boldsymbol{b}}^{r}\left(t_{r}\right)\right)$ to be the minimum value of $u \in \mathbb{N}$ such that

$$
P_{\boldsymbol{b}}^{r}(u)=\binom{u+1+n_{r}}{n_{r}} \mathfrak{q}(u)+\mathfrak{r}(u)^{\langle u\rangle}
$$

This minimum must exist by Theorem 3.10 part ( $i$. We now choose $d_{1} \in \mathbb{N}$ with $d_{1} \geq a_{1}$ and such that $d_{1}$ is larger than the maximal Gotzmann number of the Hilbert polynomials $F_{i_{2} \ldots i_{s}}$. Once $d_{1}$ is fixed, set

$$
\mathcal{B}_{2}=\left\{\left(b_{1}, b_{3}, \ldots, b_{s}\right) \in \mathbb{N}^{s-1} \mid b_{1} \in\left\{d_{1}, d_{1}+1\right\}, 0 \leq b_{i} \leq p_{i} \text { for } 2<i \leq s\right\}
$$

and set $c_{2}=\max _{\boldsymbol{b} \in \mathcal{B}_{2}}\left\{c\left(P_{\boldsymbol{b}}^{2}\left(t_{2}\right)\right)\right\}$. We then set $d_{2}=\max \left(a_{2}, c_{2}\right)$. In this way we recursively define

$$
\begin{array}{r}
\mathcal{B}_{r}=\left\{\left(b_{1}, \ldots, b_{r-1}, b_{r+1}, \ldots, b_{s}\right) \in \mathbb{N}^{s-1} \mid b_{i} \in\left\{d_{i}, d_{i}+1\right\} \text { for } i<r\right. \\
\left.0 \leq b_{i} \leq p_{i} \text { for } i>r\right\}
\end{array}
$$

$c_{r}=\max _{\boldsymbol{b} \in \mathcal{B}_{r-1}}\left\{c\left(P_{\boldsymbol{b}}^{r}\left(t_{r}\right)\right)\right\}$, and $d_{r}=\max \left(a_{r}, c_{r}\right)$, up to $r=s$. With $\left(d_{1}, \ldots, d_{s}\right)$ now fixed, we proceed with the proof.

We have defined $\mathcal{B}_{r}$ for $2 \leq r \leq s$. We similarly define

$$
\mathcal{B}_{1}=\left\{\left(b_{2}, \ldots, b_{s}\right) \in \mathbb{N}^{s-1} \mid 0 \leq b_{i} \leq p_{i} \text { for all } i\right\}
$$

We assume as in the statement of the theorem that $H_{I}\left(b_{1}, \ldots, b_{s}\right)=P\left(b_{1}, \ldots, b_{s}\right)$ for all points in $\mathcal{B}_{s}$. Our aim is to show that $q_{i}=p_{i}$ for all $2 \leq i \leq s$, and $G_{i_{2} \ldots i_{s}}\left(t_{1}\right)=F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$ for all $i_{2}, \ldots, i_{s}$ by using induction. We will prove the following statements by induction on decreasing $r$ for all $1 \leq r \leq s$.
(i) $Q_{\boldsymbol{b}}^{r}\left(d_{r}\right)=P_{\boldsymbol{b}}^{r}\left(d_{r}\right)$ and $Q_{\boldsymbol{b}}^{r}\left(d_{r}+1\right)=P_{\boldsymbol{b}}^{r}\left(d_{r}+1\right)$ for all $\boldsymbol{b} \in \mathcal{B}_{r}$.
(ii) $q_{i}=p_{i}$ for all possible $i>r$.

We begin with the base case of $r=s$. For $(i)$, this case is exactly the assumption of Theorem 1.1. that $H_{I}\left(b_{1}, \ldots, b_{s}\right)=P\left(b_{1}, \ldots, b_{s}\right)$ for all points in $\left\{\left(b_{1}, \ldots, b_{s}\right) \in \mathbb{N}^{s} \mid b_{i} \in\left\{d_{i}, d_{i}+1\right\}\right\}$. For $(i i)$, the statement is vacuously true.

For the induction step, we will show that for $2 \leq r \leq s,(i)$ and (ii) holding for $r$ implies (i) and ( $i i$ ) hold for $r-1$. To see this is true, suppose $(i)$ and ( $i i$ ) hold for some $2 \leq r \leq s$. Let

$$
P_{\boldsymbol{b}_{r}}^{r}\left(d_{r}\right)=\binom{d_{r}+n_{r}}{d_{r}} \mathfrak{q}\left(d_{r}\right)+\mathfrak{r}\left(d_{r}\right)
$$

be the Euclidean division of $P_{\boldsymbol{b}_{r}}^{r}\left(d_{r}\right)$ by $\binom{d_{r}+n_{r}}{d_{r}}$. We chose $d_{r}$ such that $d_{r} \geq c\left(P_{\boldsymbol{b}}^{\boldsymbol{r}}\left(t_{r}\right)\right)$ for all possible $\boldsymbol{b}_{\boldsymbol{r}} \in \mathcal{B}_{r}$. It follows that

$$
\begin{equation*}
Q_{\boldsymbol{b}_{r}}^{r}\left(d_{r}+1\right)=P_{\boldsymbol{b}_{r}}^{r}\left(d_{r}+1\right)=\binom{d_{r}+1+n_{r}}{d_{r}+1} \mathfrak{q}\left(d_{r}\right)+\mathfrak{r}\left(d_{r}\right)^{\left\langle d_{r}\right\rangle} \tag{3.5}
\end{equation*}
$$

for all $\boldsymbol{b}_{r} \in \mathcal{B}_{r}$ by $(i)$. As observed earlier, Lemma 3.8 implies that we can apply Crona's Theorem 3.10 to $Q_{b_{r}}^{r}\left(t_{r}\right)$. Since equation (3.5) holds, the assumption of Theorem 3.10 part (ii) is satisfied. We therefore see that $Q_{\boldsymbol{b}_{\boldsymbol{r}}}^{r}(u)=P_{\boldsymbol{b}_{r}}^{r}(u)$ for all $u \geq d_{r}$ and all $\boldsymbol{b}_{\boldsymbol{r}} \in \mathcal{B}_{r}$, implying that $Q_{\boldsymbol{b}}^{r}\left(t_{r}\right)=P_{\boldsymbol{b}}^{r}\left(t_{r}\right)$ in $\mathbb{Q}\left[t_{r}\right]$ for all $\boldsymbol{b}_{\boldsymbol{r}} \in \mathcal{B}_{r}$. Since $q_{i}=p_{i}$ for all $i>r$ by (ii), we can apply

Remark 3.5 to conclude that $q_{r}=p_{r}$. We then match the coefficients of $\binom{t_{r}-i_{r}+a_{r}}{i_{r}}$ in $Q_{\boldsymbol{b}_{r}}^{r}\left(t_{r}\right)$ and $P_{\boldsymbol{b}_{r}}^{r}\left(t_{r}\right)$ for all $0 \leq i_{r} \leq p_{r}$ and all $\boldsymbol{b}_{r} \in \mathcal{B}_{r}$ to see that

$$
Q_{\boldsymbol{b}_{r-1}}^{r-1}\left(d_{r-1}\right)=P_{\boldsymbol{b}_{r-1}}^{r-1}\left(d_{r-1}\right), \quad Q_{\boldsymbol{b}_{r-1}}^{r-1}\left(d_{r-1}+1\right)=P_{\boldsymbol{b}_{r-1}}^{r-1}\left(d_{r-1}+1\right),
$$

for all $\boldsymbol{b}_{r-1} \in \mathcal{B}_{r-1}$. This completes the proof of the induction step.
Since ( $i$ ) and ( $i i$ ) hold for $r=1$, it follows that $q_{i}=p_{i}$ for all $2 \leq i \leq s$, and that

$$
G_{i_{2} \ldots i_{s}}\left(d_{1}\right)=F_{i_{2} \ldots i_{s}}\left(d_{1}\right), \quad G_{i_{2} \ldots i_{s}}\left(d_{1}+1\right)=F_{i_{2} \ldots i_{s}}\left(d_{1}+1\right)
$$

for all $\left(i_{2}, \ldots, i_{s}\right) \in \mathcal{B}_{1}$. Since we chose $d_{1}$ to be larger then the Gotzmann number of every Hilbert polynomial $F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$, we have

$$
G_{i_{2} \ldots i_{s}}\left(d_{1}+1\right)=F_{i_{2} \ldots i_{s}}\left(d_{1}+1\right)=F_{i_{2} \ldots i_{s}}\left(d_{1}\right)^{\left\langle d_{1}\right\rangle}=G_{i_{2} \ldots i_{s}}\left(d_{1}\right)^{\left\langle d_{1}\right\rangle}
$$

for every $\left(i_{2}, \ldots, i_{s}\right) \in \mathcal{B}_{1}$. We therefore apply Gotzmann's persistence theorem to each $G_{i_{2} \ldots . . i_{s}}\left(t_{1}\right)$ and conclude that $G_{i_{2} \ldots i_{s}}\left(t_{1}\right)=F_{i_{2} \ldots i_{s}}\left(t_{1}\right)$ in $\mathbb{Q}\left[t_{1}\right]$ for all $\left(i_{2}, \ldots, i_{s}\right) \in \mathcal{B}_{1}$. It follows that $P_{I}=P$.
Example 3.13. As in Example 3.3 let

$$
S_{4}=k\left[x_{1,0}, x_{2,0}, x_{2,1}, x_{3,0}, x_{3,1}, x_{3,2}, x_{4,0}, x_{4,1}\right],
$$

and consider the ideal $I=\left(x_{3,0}, x_{2,0} x_{3,1}, x_{2,0} x_{3,2}, x_{2,1} x_{3,1}\right) \subset S_{4}$. Let $J$ and $P_{J}\left(t_{1}, t_{2}, t_{3}, t_{4}\right)$ be as in Example 3.3. We will apply Theorem 1.1 to show that $P_{I}\left(t_{1}, t_{2}, t_{3}, t_{4}\right)=P_{J}\left(t_{1}, t_{2}, t_{3}, t_{4}\right)$. Recall that $F_{001}\left(t_{1}\right)=1, F_{000}\left(t_{1}\right)=2$. The maximal Gotzmann number of these polynomials is two, so we set $d_{1}=2$. Calculating the rest of the $P_{\boldsymbol{b}}^{2}\left(t_{2}\right)$ as in Example 3.6, we obtain

$$
P_{(2,0,0)}^{2}\left(t_{2}\right)=P_{(3,0,0)}^{2}\left(t_{2}\right)=2, \quad P_{(2,0,1)}^{2}\left(t_{2}\right)=P_{(3,0,1)}^{2}\left(t_{2}\right)=1,
$$

and we have $c\left(P_{\left(b_{1}, b_{2}, b_{3}\right)}^{2}\left(t_{2}\right)\right) \leq 2$ for these polynomials, so we set $d_{2}=2$. Similarly,

$$
\begin{aligned}
& P_{(2,2,0)}^{3}\left(t_{3}\right)=P_{(2,3,0)}^{3}\left(t_{3}\right)=P_{(3,2,0)}^{3}\left(t_{3}\right)=P_{(3,3,0)}^{3}\left(t_{3}\right)=2, \\
& P_{(2,2,1)}^{3}\left(t_{3}\right)=P_{(2,3,1)}^{3}\left(t_{3}\right)=P_{(3,2,1)}^{3}\left(t_{3}\right)=P_{(3,3,1)}^{3}\left(t_{3}\right)=1,
\end{aligned}
$$

so we set $d_{3}=2$. Finally $P_{\left(b_{1}, b_{2}, b_{3}\right)}^{4}\left(t_{4}\right)=t_{4}+1$ for all $b_{i} \in\{2,3\}$, with $c\left(t_{4}+1\right)=0$, so we can again take $d_{4}=2$. Thus, verifying that $H_{I}\left(b_{1}, b_{2}, b_{3}, b_{4}\right)=P_{J}\left(b_{1}, b_{2}, b_{3}, b_{4}\right)$ for all $\left(b_{1}, b_{2}, b_{3}, b_{4}\right)$ with $2 \leq b_{i} \leq 3$ guarantees that $P_{I}\left(t_{1}, t_{2}, t_{3}, t_{4}\right)=t_{4}+1$.

The key difference between Theorem 1.1 and Gotzmann's original persistence result lies in the choice of the point $\left(d_{1}, \ldots, d_{s}\right)$. In Gotzmann's original result we can choose $d$ to be the Gotzmann number of the prospective Hilbert polynomial $P(t)$. Work of Iarrobino and Kleiman IK99, Appendix C] shows that the Gotzmann number of $P$ is equal to

$$
\inf \left\{m \mid m \in \operatorname{reg}(I) \text { for all } I \subset S \text { with } P_{I}(t)=P(t)\right\} .
$$

Work of Bayer shows that this infimum is achieved as $\min (\operatorname{reg}(L))$ where $L$ is the unique saturated lexicographic ideal with Hilbert polynomial $P$. Thus for saturated lex ideals $I$ and $J$ and $d \in \operatorname{reg}(I) \cap \operatorname{reg}(J)$, we have $H_{I}(d+1)=H_{I}(d)^{\langle d\rangle}$ and $H_{J}(d+1)=H_{J}(d)^{\langle d\rangle}$. It follows that if $H_{I}(d+1)=H_{J}(d+1)$ and $H_{I}(d)=H_{J}(d)$ then $P_{I}=P_{J}$, and in fact $I=J$. In other words, we cannot have different saturated lexicographic ideals whose Hilbert functions agree in degree $d$ and $d+1$. We might hope that a similar result is true for products of projective spaces. However the following example shows that this does not generalise even to the $\mathbb{P}^{n} \times \mathbb{P}^{m}$ case.
Example 3.14. Let $S=k\left[x_{0}, x_{1}, x_{2}, x_{3}, y_{0}, y_{1}, y_{2}, y_{3}\right]$ be the Cox ring of $X=\mathbb{P}^{3} \times \mathbb{P}^{3}$, where $\operatorname{deg}\left(x_{i}\right)=(1,0)$ and $\operatorname{deg}\left(y_{j}\right)=(0,1)$. Let

$$
\begin{gather*}
I=\left(x_{0}, x_{1}, y_{0}, y_{1}^{2}, x_{2} y_{1}, x_{2} y_{2}^{3}\right),  \tag{3.6}\\
J=\left(x_{0}, y_{0}, x_{1} y_{1}, x_{1} y_{2}, x_{2} y_{1}, x_{2} y_{2}, y_{1}^{3}, y_{1}^{2} y_{2}\right) . \tag{3.7}
\end{gather*}
$$

Note that $I$ and $J$ are generated in degrees $\leq(1,3)$, and are both bilex and $B$-saturated, where $B$ is the irrelevant ideal of $X$. Computing the multigraded regularity of $I$ and $J$ with the Macaulay2 [GS] virtual resolutions package ABLS20] we observe $(1,3) \in \operatorname{reg}(I) \cap \operatorname{reg}(J)$,
or equivalently $(0,2) \in \operatorname{reg}(S / I) \cap \operatorname{reg}(S / J)$. Consider the set $\mathcal{D}=\{(1,3),(2,3),(1,4),(2,4)\}$. Then $H_{I}\left(t_{1}, t_{2}\right)=H_{J}\left(t_{1}, t_{2}\right)$ for all $\left(t_{1}, t_{2}\right) \in \mathcal{D}$. However

$$
H_{I}(3,3)=16 \neq H_{J}(3,3)=17
$$

Since the Hilbert functions of both $I$ and $J$ agree with their Hilbert polynomials in degree $(3,3)$ by corollary 2.15 of MS05], $I$ and $J$ necessarily have different Hilbert polynomials. Indeed, we verify in Macaulay2 using the correspondence scrolls package DEFOSAS that $P_{I}\left(t_{1}, t_{2}\right)=$ $3 t_{1}+2 t_{2}+1$ and $P_{J}\left(t_{1}, t_{2}\right)=\frac{1}{2} t_{1}^{2}+\frac{3}{2} t_{1}+2 t_{2}+2$.

We therefore have to use a more complicated method to find appropriate $\boldsymbol{d}=\left(d_{1}, \ldots, d_{s}\right)$ such that checking the Hilbert function of our ideal in a finite number of points around $\boldsymbol{d}$ verifies its Hilbert polynomial. The improvement made by Theorem 1.1 is that the number of points required to check no longer depends on the dimension of our ambient space, as in algorithm 6.3 of MS05.
Remark 3.15. An increased understanding of the degrees of the generators of $I^{\text {multilex }}$ would allow us to combine Theorem 1.1 with Maclagan and Smith's result on multigraded regularity to obtain a supportive set as in HS04, Proposition 3.2], of size $2^{s}$ for $P\left(t_{1}, \ldots, t_{s}\right)$.

## 4. Extension to more general toric varieties

4.1. The Picard rank 2 case. In this section we see how the results of Section 3 apply to Picard-rank-2 toric varieties. Let $d, s \in \mathbb{Z}$ with $d \geq 2,1 \leq s \leq d-1$. Let $\boldsymbol{a}=\left(a_{1}, \ldots, a_{s}\right) \in \mathbb{Z}^{s}$ with $0 \leq a_{1} \leq a_{2} \leq \cdots \leq a_{s}$. Work of Kleinschmidt Kle88 defines the smooth projective toric variety $X_{d}(\boldsymbol{a})$ associated to $d$ and $\boldsymbol{a}$. Its Cox ring is $\bar{R}=k\left[z_{1}, \ldots, z_{d+2}\right]$, with the grading given by the $2 \times(d+2)$ matrix $A$ :

$$
A=\left(\begin{array}{cccccccc}
-a_{1} & \ldots & -a_{s} & 0 & 1 & \ldots & 1 & 1 \\
1 & \ldots & 1 & 1 & 0 & \ldots & 0 & 0
\end{array}\right)
$$

With this choice of grading the semigroup $\mathbb{N} \mathcal{C}$ of nef line bundles on $X_{d}(\boldsymbol{a})$ is identified with $\mathbb{N}^{2}$. Kleinschmidt further showed that every Picard-rank-2 smooth projective toric variety is isomorphic to $X_{d}(\boldsymbol{a})$ for some choice of $d$ and $\boldsymbol{a}$. We therefore restrict our focus to varieties with Cox rings in the above form. We can relate the ring $R$ to the Cox ring of the product of two projective spaces as follows.

Lemma 4.1. Let $R=k\left[z_{1}, \ldots, z_{d+2}\right]$ be the Cox ring of a Picard-rank-2 toric variety and let $J \subset R$ be an ideal homogeneous with respect to the $\mathbb{Z}^{2}$-grading. Let $n=\operatorname{dim}_{k}\left(R_{(1,0)}\right)-1$ and $m=\operatorname{dim}_{k}\left(R_{(0,1)}\right)-1$. Set $S$ to be the Cox ring of $\mathbb{P}^{n} \times \mathbb{P}^{m}$. Then there is a surjective map of $k$-algebras $\psi: S \rightarrow \widetilde{R}$, where $\widetilde{R}=\bigoplus_{\boldsymbol{b} \in \mathbb{N}^{2}} R_{\boldsymbol{b}}$.
Proof. Let $S=k\left[x_{0}, \ldots, x_{n}, y_{0}, \ldots, y_{m}\right]$, with the grading given by $\operatorname{deg}\left(x_{i}\right)=(1,0)$ and $\operatorname{deg}\left(y_{j}\right)=$ $(0,1)$. Note that $n=d-s$, since $R_{(1,0)}$ has a basis given by $\left\{z_{s+2}, \ldots, z_{d+2}\right\}$. We have chosen $S$ such that $R_{(1,0)}$ and $R_{(0,1)}$ are isomorphic to the vector spaces $S_{(1,0)}$ and $S_{(0,1)}$. We define a map of $k$-algebras $\psi: S \rightarrow \widetilde{R}$. Explicitly, set $\psi\left(x_{i}\right)=z_{s+2+i}$ and set $\psi\left(y_{i}\right)=e_{i}$, where $e_{0}, \ldots, e_{m}$ is the monomial basis of $R_{(0,1)}$. We now verify that the map $\psi$ is surjective.

Consider a monomial $\boldsymbol{z}^{\boldsymbol{u}}=z_{1}^{r_{1}} \ldots z_{d+2}^{r_{d+2}} \in R_{\left(b_{1}, b_{2}\right)}$ for some $\left(b_{1}, b_{2}\right) \in \mathbb{N}^{2}$. We have that

$$
\begin{align*}
r_{s+2}+\cdots+r_{d+2} & =r_{1} a_{1}+\cdots+r_{s} a_{s}+b_{1}  \tag{4.1}\\
r_{1}+\cdots+r_{s+1} & =b_{2} \tag{4.2}
\end{align*}
$$

We can rearrange the monomial $\boldsymbol{z}^{\boldsymbol{u}}$ to exhibit it as the product of $b_{1}$ elements of $R_{(1,0)}$ and $b_{2}$ elements of $R_{(0,1)}$ as follows. Consider the multiset

$$
\mathcal{Z}=\bigcup_{i=s+2}^{d+2} \bigcup_{j=1}^{r_{i}}\left\{z_{i}\right\}
$$

In other words,

$$
\mathcal{Z}=\left\{z_{s+2}, \ldots, z_{s+2}, \ldots, z_{d+2}, \ldots, z_{d+2}\right\},
$$

where each $z_{i}$ is repeated $r_{i}$ times. Equation (4.1) implies that $\mathcal{Z}$ has size $r_{1} a_{1}+\cdots+r_{s} a_{s}+b_{1}$. We relabel $r_{1} a_{1}+\cdots+r_{s} a_{s}$ of the elements of $\mathcal{Z}$ to variables of the form $z_{i, j, k}$ for $1 \leq i \leq s$, $1 \leq j \leq r_{i}, 1 \leq k \leq a_{i}$. For a fixed $i$ between 1 and $s$ we have $r_{i} a_{i}$ elements lying in the multiset $\mathcal{Z}$ labelled $z_{i, 1,1}, \ldots, z_{i, r_{i}, a_{i}}$. Let $f$ denote the product of the remaining $b_{1}$ elements of $\mathcal{Z}$, which as previously observed all lie in $R_{(1,0)}$. Then

$$
\boldsymbol{z}^{u}=f z_{s+1}^{r_{s+1}} \prod_{i=1}^{s} \prod_{j=1}^{r_{i}}\left(z_{i} \prod_{k=1}^{a_{i}} z_{i, j, k}\right)
$$

Note that each

$$
z_{i} \prod_{k=1}^{a_{i}} z_{i, j, k}
$$

is an element of $R_{(0,1)}$. This rearrangement of the monomial $\boldsymbol{z}^{u}$ shows that it is the product of elements in $R_{(1,0)}$ and $R_{(0,1)}$, ensuring that $\psi$ is surjective.

Proof of Theorem 1.2. Define $\widetilde{J}=\bigoplus_{b \in \mathbb{N}^{2}} J_{b}$. Composing the surjective map from Lemma 4.1 with the map $\widetilde{R} \rightarrow \widetilde{R} / \widetilde{J}$ we obtain a surjective map $\varphi: S \rightarrow \widetilde{R} / \widetilde{J}$. For $I=\operatorname{ker}(\varphi) \subset S$ we have $H_{I}\left(b_{1}, b_{2}\right)=H_{J}\left(b_{1}, b_{2}\right)$ for $\left(b_{1}, b_{2}\right) \in \mathbb{N}^{2}$. Consequently, we apply Theorem 1.1 to $I \subset S$ to find appropriate $\left(d_{1}, d_{2}\right) \in \mathbb{N}^{2}$ such that if

$$
\begin{array}{cl}
H_{J}\left(d_{1}, d_{2}\right)=P\left(d_{1}, d_{2}\right), & H_{J}\left(d_{1}+1, d_{2}\right)=P\left(d_{1}+1, d_{2}\right), \\
H_{J}\left(d_{1}, d_{2}+1\right)=P\left(d_{1}, d_{2}+1\right), & H_{J}\left(d_{1}+1, d_{2}+1\right)=P\left(d_{1}+1, d_{2}+1\right),
\end{array}
$$

then $P_{I}\left(t_{1}, t_{2}\right)=P\left(t_{1}, t_{2}\right)$. Since $P_{I}\left(t_{1}, t_{2}\right)=P_{J}\left(t_{1}, t_{2}\right)$ this then guarantees that $P_{J}\left(t_{1}, t_{2}\right)=$ $P\left(t_{1}, t_{2}\right)$.

Example 4.2. Consider the Hirzebruch surface $\mathcal{H}_{1}$ with Cox ring $R=k\left[z_{0}, z_{1}, z_{2}, z_{3}\right]$, with $\operatorname{deg}\left(z_{0}\right)=\operatorname{deg}\left(z_{2}\right)=(1,0), \operatorname{deg}\left(z_{1}\right)=(-1,1), \operatorname{deg}\left(z_{3}\right)=(0,1)$. Consider the ideal $J=\left(z_{0}\right)$ in $R$. We will find the Hilbert polynomial $P_{J}\left(t_{1}, t_{2}\right)$ using Theorem 1.2. Observe that $R_{(1,0)}$ has basis $\left\{z_{0}, z_{2}\right\}$ and $R_{(0,1)}$ has basis $\left\{z_{1} z_{0}, z_{1} z_{2}, z_{3}\right\}$. We therefore have a surjective homomorphism from $S_{2}=k\left[x_{0}, x_{1}, y_{0}, y_{1}, y_{2}\right]$, the Cox ring of $\mathbb{P}^{1} \times \mathbb{P}^{2}$, to $\widetilde{R}$ with kernel ( $x_{0} y_{1}-x_{1} y_{0}$ ). Extending the above map we have a surjection from $S_{2}$ to $\widetilde{R} / \widetilde{J}$ with kernel $I=\left(x_{0}, x_{0} y_{1}-x_{1} y_{0}\right)=\left(x_{0}, x_{1} y_{0}\right)$, which is multilex. The Hilbert function of $S_{2} / I$ in degree $\boldsymbol{t} \in \mathbb{N}^{2}$ is equal to that of $R / J$. Let $P\left(t_{1}, t_{2}\right)=t_{2}+1$ and suppose we want to verify that $P_{J}\left(t_{1}, t_{2}\right)=P\left(t_{1}, t_{2}\right)$. As in Lemma 3.2, we write

$$
P\left(t_{1}, t_{2}\right)=F_{0}\left(t_{1}\right)\binom{t_{2}-a_{2}+0}{0}+F_{1}\left(t_{1}\right)\binom{t_{2}-a_{2}+1}{1} .
$$

Since $P(t)$ is the Hilbert polynomial of the multilex ideal $\left(x_{0}, y_{0}\right)$ we can let $\left(a_{1}, a_{2}\right)=(2,2)$ to obtain

$$
P\left(t_{1}, t_{2}\right)=F_{0}\left(t_{1}\right)\binom{t_{2}-2+0}{0}+F_{1}\left(t_{1}\right)\binom{t_{2}-2+1}{1} .
$$

It follows that $F_{1}\left(t_{1}\right)=1, F_{0}\left(t_{1}\right)=2$, with maximal Gotzmann number 2. Fixing $d_{1}=2$, we observe that $c\left(P_{2}^{2}\left(t_{2}\right)\right)=c\left(P_{3}^{2}\left(t_{2}\right)\right)=c\left(t_{2}+1\right)=1$, so we can also fix $d_{2}=2$. Therefore by Theorem 1.1 it is enough to check $H_{J}\left(t_{1}, t_{2}\right)=P\left(t_{1}, t_{2}\right)$ for

$$
\left(t_{1}, t_{2}\right) \in\{(2,2),(2,3),(3,2),(3,3)\}
$$

to verify that $P_{J}\left(t_{1}, t_{2}\right)=P\left(t_{1}, t_{2}\right)$.
4.2. Higher Picard rank toric varieties. The surjective morphism of rings in Lemma 4.1 is actually a stronger condition than is needed to extend results about products of projective spaces to more general smooth projective toric varieties.

For the rest of this section, let $R$ be the Cox ring of a smooth projective toric variety $X$ with Picard rank $n$, with a grading given by fixing an isomorphism of $\operatorname{Pic}(X)$ with $\mathbb{Z}^{n}$. Let $\mathbb{N C}$ be the semigroup of nef line bundles on $X$, with unique minimal Hilbert basis $\mathcal{C}=\left\{\boldsymbol{c}_{\boldsymbol{1}}, \ldots, \boldsymbol{c}_{\boldsymbol{j}}\right\}$.
Definition 4.3. For an ideal $J \subset R$, homogeneous with respect to the $\mathbb{Z}^{n}$-grading, the Hilbert function of $J$ is given by

$$
\begin{aligned}
H_{J}: \mathbb{Z}^{n} & \rightarrow \mathbb{N} \\
\boldsymbol{b}=\left(b_{1}, \ldots, b_{n}\right) & \mapsto \operatorname{dim}_{k}\left((R / J)_{\boldsymbol{b}}\right) .
\end{aligned}
$$

This function agrees with a polynomial $P_{J}\left(t_{1}, \ldots, t_{n}\right)$ for all $\left(b_{1}, \ldots, b_{n}\right)$ sufficiently far from the boundary of $\mathbb{N C}$. This polynomial is the Hilbert polynomial of the ideal $J$. The notion of being sufficiently far from the boundary of $\mathbb{N C}$ can be made rigorous using Castelnuovo-Mumford regularity, see [MS05, Corollary 2.15].

Remark 4.4. It is important to note that the explicit polynomial $P_{J}\left(t_{1}, \ldots, t_{n}\right)$ we obtain for an ideal $J \subset R$ is dependent on our choice of isomorphism $\operatorname{Pic}(X) \cong \mathbb{Z}^{n}$. For example, in the $\mathbb{P}^{r}$ case it is standard to choose the isomorphism $\mathcal{O}(d) \mapsto d \in \mathbb{Z}$, which gives the Cox ring the standard grading. However we could just as easily choose the isomorphism $\mathcal{O}(d) \mapsto-d \in \mathbb{Z}$, in which case each variable in the Cox ring of $\mathbb{P}^{r}$ would have degree -1 , and the Hilbert polynomial of the Cox ring would be $P(b)=\binom{r-b}{-b}$ instead of the usual $P(b)=\binom{r+b}{b}$. In the case of $\mathbb{P}^{r}$ there is a standard choice of isomorphism, but for more general smooth projective toric varieties this can fail to be the case.

Maclagan and Smith MS04] give a generalisation of Castelnuovo-Mumford regularity for an $R$-module M or coherent sheaf $\mathcal{F}$ on $X$, denoted $\operatorname{reg}(M)$ and $\operatorname{reg}(\mathcal{F})$. Note that $\operatorname{reg}(M)$ and $\operatorname{reg}(\mathcal{F})$ are sets contained inside $\mathbb{Z}^{n} \cong \operatorname{Pic}(X)$. This is different to the standard-graded case, where we define $\operatorname{reg}(M)=\min \{m \in \mathbb{Z} \mid M$ is $m$-regular $\}$. In the multigraded case there may not be a clear minimum element so it makes sense to instead consider the set of all $\boldsymbol{m} \in \mathbb{Z}^{n}$ such that $M$ is $\boldsymbol{m}$-regular.

Definition 4.5. For $\boldsymbol{c}_{\mathbf{0}} \in \operatorname{reg}(R)$ denote by $S_{\boldsymbol{c}_{0}, X}$ the Cox ring of

$$
\mathbb{P}^{n_{0}} \times \cdots \times \mathbb{P}^{n_{j}}
$$

where $n_{i}=\operatorname{dim}_{k}\left(R_{\boldsymbol{c}_{\boldsymbol{i}}}\right)-1$. Explicitly, we write

$$
S_{c_{0}, X}=k\left[x_{0,0}, \ldots, x_{0, n_{0}}, \ldots, x_{j, 0}, \ldots, x_{j, n_{j}}\right]
$$

with $\operatorname{deg}\left(x_{i, j}\right)=\boldsymbol{e}_{\boldsymbol{i}}$. Here, $\boldsymbol{e}_{\mathbf{0}}, \ldots, \boldsymbol{e}_{\boldsymbol{j}}$ is the standard basis for $\mathbb{Z}^{j+1}$. Consider the following map

$$
\begin{aligned}
f_{\boldsymbol{c}_{\mathbf{0}}}: \mathbb{Q}^{j+1} & \rightarrow \mathbb{Q}^{n} \\
\left(b_{0}, \ldots, b_{j}\right) & \mapsto b_{0} \boldsymbol{c}_{\mathbf{0}}+b_{1} \boldsymbol{c}_{\mathbf{1}}+\cdots+b_{j} \boldsymbol{c}_{\boldsymbol{j}}
\end{aligned}
$$

If we explicitly let $\boldsymbol{c}_{\boldsymbol{i}}=\left(c_{i 1}, \ldots, c_{i n}\right)$, then we have

$$
f_{c_{0}}\left(b_{0}, \ldots, b_{j}\right)=\left(\begin{array}{lll}
b_{0} & \ldots & b_{j}
\end{array}\right)\left(\begin{array}{ccc}
c_{01} & \ldots & c_{0 n} \\
\vdots & \vdots & \vdots \\
c_{j 1} & \ldots & c_{j n}
\end{array}\right)
$$

We may view $f_{c_{0}}$ as a map on affine spaces $\mathbb{A}_{\mathbb{Q}}^{j+1} \rightarrow \mathbb{A}_{\mathbb{Q}}^{n}$. From this perspective, let $f_{\boldsymbol{c}_{0}}^{\#}: \mathbb{Q}\left[t_{1}, \ldots, t_{n}\right] \rightarrow$ $\mathbb{Q}\left[s_{0}, \ldots, s_{j}\right]$ be the induced map on the corresponding coordinate rings. Again, we give this map explicitly as

$$
f_{c_{0}}^{\#} P\left(t_{1}, \ldots, t_{n}\right)=P\left(\left(\begin{array}{lll}
s_{0} & \ldots & s_{j}
\end{array}\right)\left(\begin{array}{ccc}
c_{01} & \ldots & c_{0 n} \\
\vdots & \vdots & \vdots \\
c_{j 1} & \ldots & c_{j n}
\end{array}\right)\right)
$$

In particular since the nef cone of $X$ has full dimension, $f_{c_{0}}$ is surjective, and so $f_{c_{0}}^{\#}$ is injective.

Definition 4.6. We say that a polynomial $P\left(t_{1}, \ldots, t_{n}\right) \in \mathbb{Q}\left[t_{1}, \ldots, t_{n}\right]$ is admissible on $R$ if there is $\boldsymbol{c}_{\mathbf{0}} \in \operatorname{reg}(R)$ and a Hilbert polynomial $Q\left(s_{0}, \ldots, s_{j}\right)$ on $S_{\boldsymbol{c}_{\mathbf{0}, X}}$ such that $f_{\boldsymbol{c}_{\mathbf{0}}}^{\#} P\left(s_{0}, \ldots, s_{j}\right)=$ $Q\left(s_{0}, \ldots, s_{j}\right)$.

The following lemma allows us to establish that any Hilbert polynomial on $R$ is admissible for any $\boldsymbol{c}_{\mathbf{0}} \in \operatorname{reg}(R)$. We will make use of the following theorem of Maclagan and Smith.

Theorem 4.7 ([MS04, Theorem 1.4]). For a smooth projective toric variety $X$ and coherent sheaf $\mathcal{F}$, and for $\boldsymbol{p}$ in $\operatorname{reg}(\mathcal{F})$ and $\boldsymbol{q}$ in $\mathbb{N C}$, there is a surjective morphism

$$
H^{0}(X, \mathcal{F}(\boldsymbol{p})) \otimes H^{0}\left(X, \mathcal{O}_{X}(\boldsymbol{q})\right) \rightarrow H^{0}(X, \mathcal{F}(\boldsymbol{p}+\boldsymbol{q}))
$$

Lemma 4.8. Let $J \subset R$ be an ideal, homogeneous with respect to the $\mathbb{Z}^{n}$-grading. Let $\boldsymbol{c}_{\mathbf{0}} \in$ $\operatorname{reg}(R)$, and let $S_{c_{0}, X}, f_{c_{0}}$ be as in Definition 4.5. Then there exists an ideal $I \subset S_{c_{0}, X}$ such that there is an isomorphism $\psi_{\boldsymbol{b}}:\left(S_{\boldsymbol{c}_{\mathbf{0}}, X} / I\right)_{\boldsymbol{b}} \rightarrow(R / J)_{f_{\boldsymbol{c}_{\mathbf{0}}}(\boldsymbol{b})}$ for any $\boldsymbol{b}=\left(b_{0}, \ldots, b_{j}\right) \in \mathbb{N}^{j+1}$ with $b_{0} \geq 1$.

Proof. For $\boldsymbol{b}=\left(b_{0}, \ldots, b_{j}\right) \in \mathbb{N}^{j+1}$ with $b_{0} \geq 1$, we will define a map $\tau_{\boldsymbol{b}}:\left(S_{\boldsymbol{c}_{\mathbf{0}}, X}\right)_{\boldsymbol{b}} \rightarrow R_{f_{c_{\mathbf{0}}}(\boldsymbol{b})}$. Consider the morphism of $k$-algebras $\tau$ induced by

$$
\begin{aligned}
\tau: S_{c_{0}, X} & \rightarrow R \\
x_{i, k} & \mapsto \boldsymbol{e}_{\boldsymbol{k + 1}}^{i}
\end{aligned}
$$

where $\boldsymbol{e}_{\boldsymbol{k}}^{\boldsymbol{i}}$ is the $k$ th element of the monomial basis for $R_{\boldsymbol{c}_{\boldsymbol{i}}}$. The map $\tau_{\boldsymbol{b}}$ is defined by restricting $\tau$ to $\left(S_{c_{0}, X}\right)_{\boldsymbol{b}}$. Let $\pi: R \rightarrow R / J$ be the usual quotient map. To prove the lemma, we need to show that $\tau_{\boldsymbol{b}}$ is surjective. We then postcompose with $\left.\pi\right|_{f_{c_{\mathbf{0}}}(\boldsymbol{b})}: R_{f_{c_{\mathbf{0}}(\boldsymbol{b})}} \rightarrow(R / J)_{f_{c_{\mathbf{0}}(\boldsymbol{b})}}$ to obtain a surjective map to $(R / J)_{f_{c_{0}}(\boldsymbol{b})}$. To see that $\tau_{\boldsymbol{b}}$ is surjective for $\boldsymbol{b}=\left(b_{0}, \ldots, b_{j}\right) \in \mathbb{N}^{j+1}$ with $b_{0} \geq 1$, we first apply Theorem 4.7 we observe that there is a surjective map

$$
H^{0}\left(X, \mathcal{O}_{X}\left(b_{0} \boldsymbol{c}_{\mathbf{0}}+\cdots+\left(b_{j}-1\right) \boldsymbol{c}_{\boldsymbol{j}}\right)\right) \otimes H^{0}\left(X, \mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{j}}\right)\right) \rightarrow H^{0}\left(X, \mathcal{O}_{X}\left(f_{\boldsymbol{c}_{\mathbf{0}}}(\boldsymbol{b})\right)\right)
$$

We then repeatedly apply Theorem 4.7 to obtain a surjective map

$$
\begin{equation*}
\bigotimes_{i=0}^{j} H^{0}\left(X, \mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{i}}\right)\right)^{\otimes b_{i}} \rightarrow H^{0}\left(X, \mathcal{O}_{X}\left(f_{\boldsymbol{c}_{\mathbf{0}}}(\boldsymbol{b})\right)\right) \tag{4.3}
\end{equation*}
$$

Note that there is a canonical isomorphism $H^{0}\left(X, \mathcal{O}_{X}\left(f_{c_{\mathbf{0}}}(\boldsymbol{b})\right)\right) \cong R_{f_{c_{\mathbf{0}}}(\boldsymbol{b})}$. Further, since $h^{0}\left(X, \mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{i}}\right)\right)=n_{i}+1$, we can identify each $H^{0}\left(X, \mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{i}}\right)\right)$ with $k\left[x_{i, 0}, \ldots, x_{i, n_{i}}\right]_{1}$. The $\operatorname{map}(4.3)$ to $R_{f_{c_{0}(\boldsymbol{b})}}$ is induced by mapping $x_{i, k} \mapsto e_{\boldsymbol{k}}^{i}$, and then identifying tensor products of such elements with their product in $R$. Since this multiplication in $R$ commutes, this map factors through the vector space obtained by replacing each $H^{0}\left(X, \mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{i}}\right)\right)^{\otimes b_{i}}$ with $\operatorname{Sym}^{b_{i}}\left(H^{0}\left(X, \mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{i}}\right)\right)\right)$, which we can itself identify with $k\left[x_{i, 0}, \ldots, x_{i, n_{i}}\right]_{b_{i}}$. It follows that the map (4.3) factors through the vector space

$$
k\left[x_{0,0}, \ldots, x_{0, n_{0}}\right]_{b_{0}} \otimes k\left[x_{1,0}, \ldots, x_{1, n_{1}}\right]_{b_{1}} \otimes \cdots \otimes k\left[x_{j, 0}, \ldots, x_{j, n_{j}}\right]_{b_{j}}
$$

We now observe that

$$
\left(S_{\boldsymbol{c}_{\mathbf{0}}, X}\right)_{\boldsymbol{b}} \cong k\left[x_{0,0}, \ldots, x_{0, n_{0}}\right]_{b_{0}} \otimes k\left[x_{1,0}, \ldots, x_{1, n_{1}}\right]_{b_{1}} \otimes \cdots \otimes k\left[x_{j, 0}, \ldots, x_{j, n_{j}}\right]_{b_{j}}
$$

and that the map induced from $\left(S_{c_{0}, X}\right)_{\boldsymbol{b}}$ to $R_{f_{c_{0}(\boldsymbol{b}}}$ is exactly the map $\tau_{\boldsymbol{b}}$ defined earlier. We therefore obtain a commutative diagram


In particular since equation (4.3) is a surjection it follows that $\tau_{b}$ is also a surjection. Consequently the induced map $\widetilde{\tau_{\boldsymbol{b}}}:\left(S_{c_{0}, X}\right)_{\boldsymbol{b}} \rightarrow(R / J)_{f_{c_{0}}(\boldsymbol{b})}$ is also a surjection. We define an ideal $I=\bigcup_{b \in \mathbb{N}^{j+1}, b_{0} \geq 1}\left(\operatorname{ker}\left(\widetilde{\tau_{\boldsymbol{b}}}\right)\right)$. Observe that for $f \in \operatorname{ker}\left(\widetilde{\tau_{\boldsymbol{b}}}\right), g \in S_{c_{0}, X}$ with $\operatorname{deg}(g)=\boldsymbol{b}^{\prime}$, we have $f g \in \operatorname{ker}\left(\widetilde{\tau_{b+b^{\prime}}}\right)$. It follows that $I_{b}=\operatorname{ker}\left(\widetilde{\tau_{b}}\right)$. We conclude that we have an isomorphism of vector spaces $\left(S_{c_{0}, X} / I\right)_{\boldsymbol{b}} \cong(R / J)_{f_{c_{0}}(\boldsymbol{b})}$ for all $\boldsymbol{b} \in \mathbb{N}^{j+1}$ with $b_{0} \geq 1$. We will denote these isomorphisms by $\psi_{\boldsymbol{b}}$.

Remark 4.9. It follows from Lemma 4.8 that for any homogeneous $J \subset R$ and $\boldsymbol{c}_{\mathbf{0}} \in \operatorname{reg}(R)$, there exists a homogeneous ideal $I \subset S_{c_{0}, X}$ such that $H_{I}\left(b_{0}, \ldots, b_{j}\right)=H_{J}\left(f_{c_{0}}\left(b_{0}, \ldots, b_{j}\right)\right)$ for $\left(b_{0}, \ldots, b_{j}\right) \in \mathbb{N}^{j+1}$ with $b_{0} \geq 1$. It follows that $P_{J}\left(t_{1}, \ldots, t_{n}\right)$ is admissible, with $f_{c_{0}}^{\#} P_{J}\left(s_{0}, \ldots, s_{j}\right)=$ $P_{I}\left(s_{0}, \ldots, s_{j}\right)$.

Although the situation is a little more complicated than the Picard rank 2 case, we are now able to prove Theorem 1.3.

Proof of Theorem 1.3. Fixing some $\boldsymbol{c}_{\mathbf{0}} \in \operatorname{reg}(R)$ we consider the ideal $I$ from Lemma 4.8 with $\left(S_{c_{0}, X} / I\right)_{\boldsymbol{b}} \cong(R / J)_{f_{c_{0}}(\boldsymbol{b})}$. As established in Remark 4.9, $P\left(t_{1}, \ldots, t_{n}\right)$ is an admissible polynomial for any $\boldsymbol{c}_{\mathbf{0}} \in \operatorname{reg}(R)$, with $f_{\boldsymbol{c}_{0}}^{\#} P\left(s_{0}, \ldots, s_{j}\right)=Q\left(s_{0}, \ldots, s_{j}\right)$ for a Hilbert polynomial $Q\left(s_{0}, \ldots, s_{j}\right)$ on $S_{\boldsymbol{c}_{0}, X}$. We observe that for $\boldsymbol{b}=\left(b_{0}, \ldots, b_{j}\right) \in \mathbb{N}^{j+1}$, with $b_{0} \geq 1$ we have $H_{I}(\boldsymbol{b})=H_{J}\left(f_{c_{0}}(\boldsymbol{b})\right)$. In particular we therefore have an equality of polynomials

$$
P_{I}\left(s_{0}, \ldots, s_{j}\right)=f_{c_{0}}^{\#} P_{J}\left(s_{0}, \ldots, s_{j}\right) .
$$

We apply Theorem 1.1 to $I$ to find appropriate $\left\{d_{0}, \ldots, d_{j}\right\}$ such that verifying

$$
H_{I}\left(b_{0}, \ldots, b_{j}\right)=Q\left(b_{0}, \ldots, b_{j}\right)
$$

for $b_{i} \in\left\{d_{i}, d_{i}+1\right\}$ guarantees that $P_{I}\left(s_{0}, \ldots, s_{j}\right)=Q\left(s_{0}, \ldots, s_{j}\right)$. Equivalently, verifying

$$
H_{J}\left(f_{c_{0}}\left(b_{0}, \ldots, b_{j}\right)\right)=P\left(f_{c_{0}}\left(b_{0}, \ldots, b_{j}\right)\right)
$$

for $b_{i} \in\left\{d_{i}, d_{i}+1\right\}$ guarantees that $f_{\mathbf{c o}_{0}}^{\#} P_{J}\left(s_{0}, \ldots, s_{j}\right)=f_{c_{0}}^{\#} P\left(s_{0}, \ldots, s_{j}\right)$, so we need to check exactly the points $\left(r_{1}, \ldots, r_{n}\right) \in\left\{f_{c_{0}}(\boldsymbol{b}) \in \mathbb{N}^{n} \mid b_{i} \in\left\{d_{i}, d_{i}+1\right\}\right\}$. As highlighted in Definition 4.5, $f_{c_{0}}^{\#}$ is injective, and so we conclude that $P_{J}\left(t_{1}, \ldots, t_{n}\right)=P\left(t_{1}, \ldots, t_{n}\right)$.

Remark 4.10. The points

$$
\mathcal{D}=\left\{f_{\boldsymbol{p}}\left(b_{0}, \ldots, b_{j}\right) \mid b_{i} \in\left\{d_{i}, d_{i}+1\right\}\right\}
$$

form the vertices of a zonotope in $\mathbb{R}^{n}$. We have an upper bound on the size of $\mathcal{D}$ given by $|\mathcal{D}| \leq 2^{j+1}$. However, by combining [Buc43, A geometrical lemma] and [Zas75, Section 6B] for $n<j+1$ we get an improved bound of

$$
|\mathcal{D}| \leq \sum_{i=0}^{n}\binom{j+1}{i}
$$

Further, when $\mathbf{0} \in \operatorname{reg}(X)$, we can take $\boldsymbol{c}_{\mathbf{0}}=\mathbf{0}$ in which case $|\mathcal{D}|=2^{j}$, which is the minimum possible.

Remark 4.11. The proof of Lemma 4.1 relies on the known grading of the Cox ring of $X_{d}(\boldsymbol{a})$. However this result can be extended to other smooth projective toric varieties $X$ where the map

$$
\begin{equation*}
H^{0}\left(X, \mathcal{O}_{X}(\alpha)\right) \otimes H^{0}\left(X, \mathcal{O}_{X}(\beta)\right) \rightarrow H^{0}\left(X, \mathcal{O}_{X}(\alpha+\beta)\right) \tag{4.4}
\end{equation*}
$$

is surjective for all nef line bundles $\mathcal{O}_{X}(\alpha), \mathcal{O}_{Y}(\beta)$. The idea is to take a Hilbert basis $\mathcal{C}=$ $\left\{\boldsymbol{c}_{\boldsymbol{1}}, \ldots, \boldsymbol{c}_{\boldsymbol{j}}\right\}$ of the nef cone $\mathcal{K}$ of $X$, and set $n_{i}=h^{0}\left(X, \mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{i}}\right)\right)-1$. Let $S$ be the Cox ring of $\mathbb{P}^{n_{1}} \times \cdots \times \mathbb{P}^{n_{j}}$, and identify generators $x_{0}, \ldots, x_{n_{i}}$ of the $\mathbb{P}^{n_{i}}$ piece with the monomial basis for $R_{\boldsymbol{c}_{\boldsymbol{i}}}$. Using the surjectivity of (4.4) for all $\mathcal{O}_{X}\left(\boldsymbol{c}_{\boldsymbol{i}}\right)$, we obtain a surjective morphism of rings from $S$ to $\bigoplus_{t \in \mathbb{N C}} R_{t}$. It follows that we can extend this to a surjective morphism of rings from $S$ to $\bigoplus_{t \in \mathbb{N C}}(R / J)_{t}$ for an ideal $J \subset R$. Oda Oda08 conjectured that for all smooth projective toric varieties the map (4.4) is surjective for all ample line bundles $\mathcal{O}_{X}(\alpha)$ and nef line bundles $\mathcal{O}_{X}(\beta)$. If this conjecture holds then a similar result to Theorem 1.2 would hold for all smooth projective toric varieties, and the technique of introducing an extra copy of $\mathbb{P}^{n}$ used in Theorem 1.3 would no longer be required.

Remark 4.12. The Hilbert polynomial of a subscheme of a d-dimensional smooth projective toric variety $X$ of Picard rank $n$ can have up to $\binom{n+d-1}{d}$ coefficients. This means that naively we have to check $\binom{n+d-1}{d}$ points to find this polynomial. Theorems 1.1 and 1.2 ensure that we only have to check $2^{n}$ points to find the Hilbert polynomial in the case of products of projective spaces or a Picard-rank-2 toric variety. This is an improvement for $d \gg 0$. Theorem 1.3 ensures we need only check $2^{j+1}$ points, where $j$ is the size of the Hilbert basis of the nef cone. The key contribution is that the complexity of finding the Hilbert polynomial no longer depends on the dimension of $X$, leading to a computational advantage for varieties of larger dimensions.
Example 4.13. Consider the fan $\Sigma \subset \mathbb{R}^{3}$ with rays

$$
\begin{aligned}
\rho_{0}=(1,0,0), & \rho_{1}=(0,1,0),
\end{aligned} \quad \rho_{2}=(-1,1,0), ~(0,0,1), \quad \rho_{5}=(0,0,-1), ~ \$ \rho_{3}=(0,-1,0), \quad \rho_{4}=(0,0)
$$

and maximal cones generated by the subsets

$$
\begin{aligned}
& \left\{\rho_{0}, \rho_{1}, \rho_{4}\right\},\left\{\rho_{0}, \rho_{1}, \rho_{5}\right\},\left\{\rho_{1}, \rho_{2}, \rho_{4}\right\},\left\{\rho_{1}, \rho_{2}, \rho_{5}\right\} \\
& \left\{\rho_{2}, \rho_{3}, \rho_{4}\right\},\left\{\rho_{2}, \rho_{3}, \rho_{5}\right\},\left\{\rho_{0}, \rho_{3}, \rho_{4}\right\},\left\{\rho_{0}, \rho_{3}, \rho_{5}\right\}
\end{aligned}
$$

The associated normal toric variety $X$ is smooth and projective with Picard rank 3. After fixing an isomorphism $\operatorname{Pic}(X) \cong \mathbb{Z}^{3}$ we write the Cox ring of $X$ as $R=k\left[y_{0}, \ldots, y_{5}\right]$, where $\operatorname{deg}\left(y_{0}\right)=\operatorname{deg}\left(y_{2}\right)=(-1,1,0), \operatorname{deg}\left(y_{1}\right)=(1,0,0), \operatorname{deg}\left(y_{3}\right)=(0,1,0)$ and $\operatorname{deg}\left(y_{4}\right)=\operatorname{deg}\left(y_{5}\right)=$ $(0,0,1)$. The nef cone of $X$ is generated by $\boldsymbol{c}_{1}=(-1,1,0), \boldsymbol{c}_{\boldsymbol{2}}=(0,1,0)$ and $\boldsymbol{c}_{3}=(0,0,1)$, and $\operatorname{reg}(R)=\mathbb{N} \mathcal{C}$. In particular $(0,0,0) \in \operatorname{reg}(R)$. This makes the application of Theorem 1.3 especially nice. Set $\boldsymbol{c}_{\mathbf{0}}=(0,0,0)$. We have

$$
R_{\boldsymbol{c}_{\mathbf{0}}}=\langle 1\rangle, \quad R_{\boldsymbol{c}_{\mathbf{1}}}=\left\langle y_{0}, y_{2}\right\rangle, \quad R_{\boldsymbol{c}_{\mathbf{2}}}=\left\langle y_{3}, y_{0} y_{1}, y_{1} y_{2}\right\rangle, \quad R_{\boldsymbol{c}_{3}}=\left\langle y_{4}, y_{5}\right\rangle
$$

so we set

$$
S_{c_{0}, X}=k\left[x_{1,0}, x_{2,0}, x_{2,1}, x_{3,0}, x_{3,1}, x_{3,2}, x_{4,0}, x_{4,1}\right]
$$

which is the Cox ring of $\mathbb{P}^{0} \times \mathbb{P}^{1} \times \mathbb{P}^{2} \times \mathbb{P}^{1}$. Consider the ideal $J=\left(y_{3}, y_{0}^{2} y_{1}, y_{0} y_{1} y_{2}\right) \subset R$. The ideal $I$ in Lemma 4.8 is given by $I=\left(x_{3,0}, x_{2,0} x_{3,1}, x_{2,0} x_{3,2}, x_{2,1} x_{3,1}\right)$, which is multilex. We have $\left(S_{\boldsymbol{c}_{\mathbf{0}}, X} / I\right)_{\boldsymbol{b}} \cong(R / J)_{f_{c_{\mathbf{0}}}(\boldsymbol{b})}$ for $\boldsymbol{b} \in \mathbb{N}^{4}$. Consider the polynomial $P\left(t_{1}, t_{2}, t_{3}\right)=t_{3}+1$, with $Q\left(s_{0}, s_{1}, s_{2}, s_{3}\right)=f_{c_{0}}^{\#} P\left(s_{0}, s_{1}, s_{2}, s_{3}\right)=s_{3}+1$. To check that $P_{J}\left(t_{1}, t_{2}, t_{3}\right)=P\left(t_{1}, t_{2}, t_{3}\right)$ we need only apply Theorem 1.1 to $I \subset S_{c_{0}, X}$ to find appropriate $\left\{d_{0}, \ldots, d_{3}\right\}$ such that checking $H_{I}\left(b_{0}, b_{2}, b_{3}, b_{3}\right)=Q\left(b_{0}, b_{1}, b_{2}, b_{3}\right)$ for $b_{i} \in\left\{d_{i}, d_{i}+1\right\}$ confirms that $P_{I}=Q$. We now observe that modulo a change of notation this is identical to the case of Example 3.13, and set $\left(d_{0}, d_{1}, d_{2}, d_{3}\right)=(2,2,2,2)$. We calculate the values of $f_{\boldsymbol{c}_{\mathbf{0}}}(\boldsymbol{b})$ for $\boldsymbol{b}=\left(b_{0}, b_{1}, b_{2}, b_{3}\right) \in \mathbb{N}^{4}$ with $2 \leq b_{i} \leq 3$. We obtain eight vertices for our zonotope, as expected by Remark 4.10.

$$
\begin{aligned}
\mathcal{D}= & \{(-2,4,2),(-2,4,3),(-2,5,2),(-2,5,3), \\
& (-3,5,2),(-3,5,3),(-3,6,2),(-3,6,3)\} .
\end{aligned}
$$

Checking that $H_{I}\left(r_{1}, r_{2}, r_{3}\right)=P\left(r_{1}, r_{2}, r_{3}\right)$ for all $\left(r_{1}, r_{2}, r_{3}\right) \in \mathcal{D}$ guarantees that $P_{J}=P$.

## References

[ABLS20] Ayah Almousa, Juliette Bruce, Michael Loper, and Mahrud Sayrafi, The virtual resolutions package for Macaulay2, Journal of Software for Algebra and Geometry 10 (2020), no. 1, 51-60.
[ACDN00] Annetta Aramova, Kristina Crona, and Emanuela De Negri, Bigeneric initial ideals, diagonal subalgebras and bigraded Hilbert functions, J. Pure Appl. Algebra 150 (2000), no. 3, 215-235.
[BH93] Winfried Bruns and Jürgen Herzog, Cohen-Macaulay rings, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.
[Buc43] R. C. Buck, Partition of space, Amer. Math. Monthly 50 (1943), 541-544.
[CDNG22] Aldo Conca, Emanuela De Negri, and Elisa Gorla, Radical generic initial ideals, Vietnam J. Math. 50 (2022), no. 3, 807-827.
[Cro06] Kristina Crona, Standard bigraded Hilbert functions, Comm. Algebra 34 (2006), no. 2, 425-462.
[DEFOSAS] David Eisenbud, Frank-Olaf Schreyer, and Alessio Sammartano, CorrespondenceScrolls: A Macaulay2 package. Version 0.6.
[Fav20] Giuseppe Favacchio, A numerical property of Hilbert functions and lex segment ideals, J. Korean Math. Soc. 57 (2020), no. 3, 777-792.
[Gas97] Vesselin Gasharov, Extremal properties of Hilbert functions, Illinois J. Math. 41 (1997), no. 4, 612629.
[Got78] Gerd Gotzmann, Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes, Math. Z. 158 (1978), no. 1, 61-70 (German).
[GS] Daniel R. Grayson and Michael E. Stillman, Macaulay2, a software system for research in algebraic geometry.
[HS04] Mark Haiman and Bernd Sturmfels, Multigraded Hilbert schemes, J. Algebraic Geom. 13 (2004), no. 4, 725-769.
[IK99] Anthony Iarrobino and Vassil Kanev, Power sums, Gorenstein algebras, and determinantal loci, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999. Appendix C by Iarrobino and Steven L. Kleiman.
[Kle88] Peter Kleinschmidt, A classification of toric varieties with few generators, Aequationes Math. $\mathbf{3 5}$ (1988), no. 2-3, 254-266.
[Mac27] F. S. Macaulay, Some Properties of Enumeration in the Theory of Modular Systems, Proc. London Math. Soc. (2) 26 (1927), 531-555.
[MS04] Diane Maclagan and Gregory G. Smith, Multigraded Castelnuovo-Mumford regularity, J. Reine Angew. Math. 571 (2004), 179-212.
[MS05] $\qquad$ , Uniform bounds on multigraded regularity, J. Algebraic Geom. 14 (2005), no. 1, 137-164.
[Mum66] David Mumford, Lectures on curves on an algebraic surface, Annals of Mathematics Studies, vol. No. 59, Princeton University Press, Princeton, NJ, 1966. With a section by G. M. Bergman.
[Oda08] Tadao Oda, Problems on Minkowski sums of convex lattice polytopes (2008), arXiv 0812.1418, primary class math.AG
[Zas75] Thomas Zaslavsky, Facing up to arrangements: face-count formulas for partitions of space by hyperplanes, Mem. Amer. Math. Soc. 1 (1975), vii+102.

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