COMPARING VARIETIES GENERATED BY CERTAIN WREATH PRODUCTS OF GROUPS

V. H. MIKAELIAN

An exhaustive version of the thesis to a talk presented at the:
Groups & Algebras in Bicocca Conference (GABY)
University of Milano-Bicocca (Milan, Italy), June 17 to June 21, 2024.
https://staff.matapp.unimib.it/~/gaby/gaby2024/index.html

1. Introduction

In the current talk we would like to present the main results of recent work [22] in which we classify cases when *the wreath products of distinct pairs of groups generate the same variety*. This classification allows us to study the subvarieties of some nilpotent-by-abelian product varieties $\mathfrak{U}\mathfrak{V}$ with the help of wreath products of groups. For background information on varieties of groups, on generating groups of varieties, on products of varieties, or on wreath products we refer to Hanna Neumann's monograph [25], to the related articles [26, 8, 2, 3, 27, 6], and to literature cited therein.

In particular, using wreath products we discover such subvarieties in nilpotent-by-abelian products $\mathfrak{U}\mathfrak{V}$ which *have the same nilpotency class, the same length of solubility, and the same exponent, but which still are distinct subvarieties*. Obtained classification strengthens results on varieties generated by wreath products in the mentioned articles and elsewhere in the literature, see [22] for references.

Wreath products are among the main tools to study products $\mathfrak{U}\mathfrak{V}$ of any varieties $\mathfrak{U}$ and $\mathfrak{V}$ of groups. Under wreath products we by default suppose *Cartesian* wreath products, but all the results we bring are true for *direct* wreath products also. In the literature the wreath product methods most typically consider certain groups A and B generating the varieties $\mathfrak{U}$ and $\mathfrak{V}$ respectively, and then they find extra conditions, under which the wreath product $A \,\mathrm{Wr}\, B$ generates $\mathfrak{U}\mathfrak{V}$, i.e., conditions, under which the equality

$$(*) \qquad \operatorname{var}(A \,\mathrm{Wr}\, B) = \operatorname{var}(A) \operatorname{var}(B)$$

holds for the given A and B.

The advantage of such an approach is that having the equality $(*)$ we using Birkhoff's Theorem can get *all* the groups in $\operatorname{var}(A)\operatorname{var}(B) = \mathfrak{U}\mathfrak{V}$ by applying the operations of taking the homomorphic images, subgroups, Cartesian products to the *single* group $A \,\mathrm{Wr}\, B$ only, see [25, 15.23].

Generalizing some known results in the cited literature, we in [11], [18], [19], [20], [21], [23] were able to suggest criteria classifying all the cases when $(*)$

holds for groups from certain particular classes of groups: abelian groups, p-groups, nilpotent groups of finite exponent, etc. See the very brief outline of results presented in Section 5 of [23].

In [22] we turned to a sharper problem of comparison of two varieties, both generated by wreath products. Namely, take A_1, B_1 and A_2, B_2 to be pairs of non-trivial groups such that $\operatorname{var}(A_1) = \operatorname{var}(A_2)$, $\operatorname{var}(B_1) = \operatorname{var}(B_2)$, and then distinguish the cases when:

$$\operatorname{var}(A_1 \operatorname{Wr} B_1) = \operatorname{var}(A_2 \operatorname{Wr} B_2). \tag{$**$}$$

2. The main criterion and examples

To write down the main classification criterion from [22] we need some very simple notation. Namely, by Prüper's Theorem any abelian group B of finite exponent is a direct product of its p-primary components $B(p)$, and each of such components is a direct product of certain cyclic p-groups $C_{p^{u_1}}, C_{p^{u_2}}, \dots$ If among the latters the cardinality of copies isomorphic to $C_{p^{u_k}}$ is $m_{p^{u_k}}$, we can write their direct product as $C_{p^u}^{m_{p^u}}$. Then the component $B(p)$ can be rewritten as a direct product of such factors:

$$B(p) = C_{p^{u_1}}^{m_{p^{u_1}}} \times \cdots \times C_{p^{u_r}}^{m_{p^{u_r}}}, \tag{2.1}$$

where we may suppose $u_1 > \cdots > u_r$, see [5, Section 35]. If $B(p)$ is finite, then all the cardinals $m_{p^{u_1}}, \dots, m_{p^{u_r}}$ together with all the factors $C_{p^{u_k}}^{m_{p^{u_k}}}$ are finite. Otherwise, at least one of those factors has to be infinite, and we can denote $C_{p^{u_k}}^{m_{p^{u_k}}}$ to be the *first one* of them, i.e., $m_{p^{u_k}}$ is an infinite (at least countable) cardinal, and all the preceding cardinals $m_{p^{u_1}}, \dots, m_{p^{u_{k-1}}}$ are finite.

Let B_1 and B_2 be abelian groups of finite exponents divisible by some prime p. Call their p-primary components $B_1(p)$ and $B_2(p)$ *equivalent*, if in (2.1) their first *infinite* direct factors have the same exponent, and all their preceding *finite* direct factors coincide. More precisely, if $B_1(p) = C_{p^{u_1}}^{m_{p^{u_1}}} \times \cdots \times C_{p^{u_r}}^{m_{p^{u_r}}}$ and $B_2(p) = C_{p^{v_1}}^{m_{p^{v_1}}} \times \cdots \times C_{p^{v_s}}^{m_{p^{v_s}}}$, then $B_1(p) \equiv B_2(p)$ if and only if:

(1) when $B_1(p), B_2(p)$ are *finite*, then $B_1(p) \equiv B_2(p)$ iff $B_1(p) \cong B_2(p)$;
(2) when $B_1(p), B_2(p)$ are *infinite*, then $B_1(p) \equiv B_2(p)$ iff there is a k so that:
   i) $u_i = v_i$ and $m_{p^{u_i}} = m_{p^{v_i}}$ for each $i = 1, \dots, k-1$;
   ii) $C_{p^{u_k}}^{m_{p^{u_k}}}$ is the first infinite factor for $B_1(p)$; $C_{p^{v_k}}^{m_{p^{v_k}}}$ is the first infinite factor for $B_2(p)$, and $u_k = v_k$;
(3) else $B_1(p), B_2(p)$ are *not* equivalent.

The above definition is not short, but it is very intuitive to understand:

Example 2.1. $C_{3^5}^6 \times C_{3^4}^8 \times C_{3^3}^{\aleph_0} \times C_{3^2}^5 \times C_3^4$ is equivalent to $C_{3^5}^6 \times C_{3^4}^8 \times C_{3^3}^{\mathfrak{c}} \times C_3^{50}$, but it is *not* equivalent to $C_{3^5}^6 \times C_{3^4}^8 \times C_{3^2}^{\aleph_0} \times C_3^4$. Here $\aleph_0$ and $\mathfrak{c}$ stand for countable and continuum cardinals. In the first two of the above groups the first *infinite* factors $C_{3^3}^{\aleph_0}$ and $C_{3^3}^{\mathfrak{c}}$ are of the same exponent 3^3 (without being isomorphic), and they both have the same two initial *finite* factors $C_{3^5}^6 \times C_{3^4}^8$. Whereas in the third group the first *infinite* factor $C_{3^2}^{\aleph_0}$ is of another exponent $3^2 \neq 3^3$.

In these terms our main criterion reads:

Theorem 2.2. *Let A_1, A_2 be any non-trivial nilpotent groups of exponent m generating the same variety, and let B_1, B_2 be any non-trivial abelian groups of exponent n generating the same variety, where any prime divisor p of n also divides m.*
Then $(**)$ holds for A_1, A_2, B_1, B_2 if and only if $B_1(p) \equiv B_2(p)$ for each p.

Notice how the roles of the passive and active groups of these wreath products are different: for A_1, A_2 we just require that $\operatorname{var}(A_1) = \operatorname{var}(A_2)$, whereas for B_1, B_2 we put extra conditions on structures of their decompositions. And when B_1, B_2 are finite, the extra conditions simply mean $B_1 \cong B_2$.

Example 2.3. To see an application of Theorem 2.2 take Q_8 to be the quaternion group, and take M_{27} to be the semidirect product of C_9 and of C_3, acting on it by nontrivial automorphisms. Q_8 is of order 8, of exponent 4, and of nilpotency class 2, while M_{27} is of order 27, of exponent 9, and of nilpotency class 2. Then pick $A_1 = Q_8 \times M_{27} \times C_{25}$, $A_2 = Q_8 \times Q_8 \times M_{27}^{\aleph_0} \times C_{25} \times C_5 \times C_5$, $B_1 = C_{3^5}^6 \times C_{3^4}^8 \times C_{3^3}^{\aleph_0} \times C_{3^2}^5 \times C_3^4 \times C_5$, $B_2 = C_{3^5}^6 \times C_{3^4}^8 \times C_{3^3}^{\mathfrak{c}} \times C_3^{50} \times C_5$. Three conditions $\operatorname{var}(A_1) = \operatorname{var}(A_2)$, $B_1(3) \equiv B_2(3)$, $B_1(5) \equiv B_2(5)$ are easy to verify, see Example 2.1. Hence $(**)$ holds for this choice of A_1, A_2, B_1, B_2. On the other hand, we will no longer have an equality choosing either $B_2 = C_{3^5}^6 \times C_{3^4}^8 \times C_{3^2}^{\aleph_0} \times C_3^4 \times C_5$ (because $3^3 \neq 3^2$), or $B_2 = C_{3^5}^6 \times C_{3^4}^8 \times C_{3^3}^{\mathfrak{c}} \times C_3^{50} \times C_5 \times C_5$ (because $(C_5 \times C_5) \not\equiv C_5$).

3. Applications to subvariety structures

Theorem 2.2 covers the cases of nilpotent A_1, A_2 and abelian B_1, B_2, with some restrictions on exponents. Besides getting a generalization of $(*)$ our study of equality $(**)$ is motivated by some applications one of which we would like to outline here.

Classification of subvariety structures of $\mathfrak{UV}$ is incomplete even when $\mathfrak{U}$ and $\mathfrak{V}$ are such "small" varieties as the abelian varieties $\mathfrak{A}_m$ and $\mathfrak{A}_n$ respectively. Here are some of the results in this direction: $\mathfrak{A}_p$ (for prime numbers p) are the simplest non-trivial varieties, as they consist of the Cartesian powers of the cycle C_p only. L.G. Kovács and M.F. Newman in [9] fully described the subvariety structure in the product $\mathfrak{A}_p^2 = \mathfrak{A}_p \mathfrak{A}_p$ for $p > 2$. Later they continued this classification for the varieties $\mathfrak{A}_{p^u} \mathfrak{A}_p$. Their research was unpublished for many years, and it appeared in 1994 only [10] (parts of their proof are present in [4]). Another direction is description of subvarieties in the product $\mathfrak{A}_m \mathfrak{A}_n$ where m and n are *coprime*. This is done by C. Houghton (mentioned by Hanna Neumann in [25, 54.42]), by P.J. Cossey (Ph.D. thesis [7], mentioned by R.A. Bryce in [4]). A more general result of R.A. Bryce classifies the subvarieties of $\mathfrak{A}_m \mathfrak{A}_n$, where m and n are *nearly prime* in the sense that, if a prime p divides m, then p^2 does not divide n [4].

In 1967 Hanna Neumann wrote that classification of subvarieties of $\mathfrak{A}_m\mathfrak{A}_n$ for arbitrary m and n *"seems within reach"* [25]. And R.A. Bryce in 1970 mentioned that *"classifying all metabelian varieties is at present slight"* [4]. However, nearly half a century later this task is not yet accomplished: Yu.A. Bakhturin and A.Yu. Olshanskii remarked in the survey [1] of 1988 (appeared in English in 1991) that *"classification of all nilpotent metabelian group varieties has not been completed yet"*.

As this brief summary shows, one of the cases, when the subvariety structure of $\mathfrak{U}\mathfrak{V}$ is *less known*, is the case when $\mathfrak{U}$ and $\mathfrak{V}$ have non-coprime exponents divisible by *high* powers p^u for *many* prime numbers p. Thus, even if we cannot classify all the subvarieties in some product varieties $\mathfrak{U}\mathfrak{V}$, it may be interesting to find those subvarieties in $\mathfrak{U}\mathfrak{V}$, which are generated by wreath products. We, surely, can take any groups $A \in \mathfrak{U}$ and $B \in \mathfrak{V}$, and then $A \,\mathrm{Wr}\, B$ will generate some subvariety in $\mathfrak{U}\mathfrak{V}$. But in order to make this approach reasonable, we yet have to detect *if or not two wreath products of that type generate the same subvariety*, i.e, if or not the equality (∗∗) holds for the given pairs of groups.

Yet another outcome of this research may be stressed. In the literature the different subvarieties are often distinguished by their different nilpotency classes, different lengths of solubility, or different exponents (see, for example, classification of subvarieties of $\mathfrak{A}_p^2$ in [9]). Using wreath products technique, we in [22] construct such subvarieties of $\mathfrak{U}\mathfrak{V}$, which have the same nilpotency class, the same length of solubility, the same exponent, but which still are distinct subvarieties, see examples in [22]. Other related research can be foind in [13], [14], [15], [16], [24].

References

[1] Yu.A. Bakhturin, A.Yu. Olshanskii, *Identities*, Algebra II: Noncommutative rings. Identities, Encycl. Math. Sci. 18, 107–221 (1991); translation from Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Fundam. Napravleniya 18, 117–240 (1988).

[2] G. Baumslag, *Wreath products and p-groups*, Proc. Camb. Philos. Soc. 55 (1959), 224–231.

[3] G. Baumslag, B. H. Neumann, Hanna Neumann, P. M. Neumann *On varieties generated by finitely generated group*, Math. Z., 86 (1964), 93–122.

[4] R.A. Bryce, *Metabelian groups and varieties*, Phil. Trans. Roy. Soc. 266 (1970), 281–355.

[5] L.Fuchs, *Infinite Abelian Groups, volume I*, Academic Press, N-Y and London (1970).

[6] R.G. Burns, *Verbal wreath products and certain product varieties of groups*, J. Austral. Math. Soc. 7 (1967), 356–374.

[7] P. J. Cossey, *On varieties of A-groups*, Ph.D. thesis, A.N.U., (1966).

[8] G. Higman, *Some remarks on varieties of groups*, Quart. J. Math. Oxford, (2) 10 (1959), 165–178.

[9] L.G. Kovács, M.F. Newman, *On non-Cross varieties of groups*, J. Austral. Math. Soc., 12 (1971), 2, 129–144.

[10] L.G. Kovács, M.F. Newman, *Torsionfree varieties of metabelian groups*, de Giovanni, Francesco (ed.) et al., Infinite groups 1994. Proceedings of the international conference, Ravello, Italy, May 23-27, 1994. Berlin: Walter de Gruyter. 125–128 (1996).

[11] V.H. Mikaelian, *On varieties of groups generated by wreath products of abelian groups*, Abelian groups, rings and modules (Perth, Australia, 2000), Contemp. Math., 273, Amer. Math. Soc., Providence, RI (2001), 223–238, DOI: dx.doi.org/10.1090/conm/273.

[12] V. H. Mikaelian, *On wreath products of finitely generated Abelian groups*, Advances in Group Theory, (2002), 13–24.
[13] V. H. Mikaelian, *Two problems on varieties of groups generated by wreath products*, International Journal of Mathematics and Mathematical Sciences, 31 (2002), 65–75, DOI: doi.org/10.1155/S0161171202012528.
[14] V. H. Mikaelian, *Infinitely many not locally soluble SI*-groups*, Ricerche di Matematica, 52 (2003), 1, 1–19.
[15] V. H. Mikaelian, *On embedding properties of SD-groups*, International Journal of Mathematics and Mathematical Sciences (2004), 65–76, DOI: doi.org/10.1155/S0161171204211280.
[16] V. H. Mikaelian, *On a problem on explicit embeddings of the group* $\mathbb{Q}$, International Journal of Mathematics and Mathematical Sciences (2005), 2119–2123, DOI: doi.org/10.1155/IJMMS.2005.2119.
[17] V. H. Mikaelian, *On finitely generated soluble non-Hopfian groups, an application to a problem of Neumann*, International Journal of Algebra and Computation, 17 (2007) 05/06, 1107–1113, DOI: doi.org/10.1142/S0218196707004086.
[18] V. H. Mikaelian, *Metabelian varieties of groups and wreath products of abelian groups*, Journal of Algebra, 313 (2007), 2, 455–485, DOI: doi.org/10.1016/j.jalgebra.2004.02.040.
[19] V.H. Mikaelian, *Varieties Generated by Wreath Products of Abelian and Nilpotent Groups*, Algebra and Logic, 54 (2015), 1, 70–73, DOI: doi.org/10.17377/alglog.2015.54.109.
[20] V. H. Mikaelian, *The criterion of Shmel'kin and varieties generated by wreath products of finite groups*, Algebra and logic 56 (2017), 2, 108–115, DOI: doi.org/10.17377/alglog.2017.56.203.
[21] V.H. Mikaelian, *On K_p-series and varieties generated by wreath products of p-groups*, International Journal of Algebra and Computation, 28 (2018), 8, 1693—1703, DOI: doi.org/10.1142/S0218196718400143.
[22] V. H. Mikaelian, *Subvariety structures in certain product varieties of groups*, Journal of Group Theory, 21 (2018), 5, 865–884, DOI: doi.org/10.1515/jgth-2018-0017.
[23] V.H. Mikaelian, *A classification theorem for varieties generated by wreath products of groups*, Izvestiya: Mathematics (Izvestiya RAN, Ser. Math.), 82:5 (2018), DOI: doi.org/10.4213/im8694.
[24] V. H. Mikaelian, *The Higman operations and embeddings of recursive groups* Journal of Group Theory, 26 (2023), 1067–1093, DOI: doi.org/10.1515/jgth-2021-0095.
[25] Hanna Neumann, *Varieties of Groups*, Varieties of groups (Ergebn. Math. Grenzg., 37), Berlin-Heidelberg-New York, Springer-Verlag 1967.
[26] B.H. Neumann, Hanna Neumann, Peter M.Neumann, *Wreath products and varieties of groups*, Math. Zeitschrift 80 (1962), 44–62
[27] A.L. Shmel'kin, *Wreath products and varieties of groups*, Izv. AN SSSR, ser. matem., 29 (1965), 149–170 (Russian). Summary in English: Soviet Mathematics. Vol. 5. No. 4 (1964).

E-mail: v.mikaelian@gmail.com, vmikaelian@ysu.am
Web: researchgate.net/profile/Vahagn-Mikaelian