FROM QUANTUM DIFFERENCE EQUATION TO DUBRONVIN CONNECTION OF AFFINE TYPE A QUIVER VARIETIES

TIANQING ZHU

ABSTRACT. This is the continuation of the article [Z23]. In this article we will give a detailed analysis of the quantum difference equation of the equivariant K-theory of the affine type A quiver varieties. We will give a good representation of the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$ such that the monodromy operator $\mathbf{B}_{\mathbf{m}}(z)$in the formula can be written in the $U_q(\mathfrak{sl}_2)$-form or in the $U_q(\hat{\mathfrak{gl}}_1)$-form. We also give the detailed analysis of the connection matrix for the quantum difference equation in the nodal limit $p \to 0$. Using these two results, we prove that the degeneration limit of the quantum difference equation is the Dubrovin connection for the quantum cohomology of the affine type A quiver varieties, and the monodromy representation for the Dubrovin connection is generated by the monodromy operators $\mathbf{B}_{\mathbf{m}}$.

CONTENTS

1. Introduction

This paper is a continuation of the paper [Z23]. We do the explicit calculation for the quantum difference equation for the affine type A quiver varieties $M(\mathbf{v},\mathbf{w})$ with stability condition $\boldsymbol{\theta}=(1,1,\cdots,1)$.

Quantum difference equation is the difference equation analog of the quantum differential equation in [MO12]. It is the difference equation of the capping operator $\mathbf{J}(u,z)\in K_T(X\times X)_{loc}[[q^{\mathscr{L}}]]_{\mathscr{L}\in\mathrm{Pic}(X)}$ described in [OS22] and [O15]. Geometrically speaking, capping operator is the operator giving the K-theoretic counting of the stable quasi-map from $\mathbb{P}^1$ to the fixed Nakajima quiver varieties $M(\mathbf{v},\mathbf{w})$. The difference equation can be written as:

$$\Psi(q^{\mathscr{L}}z)=\mathbf{M}_{\mathscr{L}}(z)\Psi(z)\in K_T(X)_{loc}\otimes\mathbb{C}((\mathrm{Pic}(X))) \tag{1.1}$$

The fundamental solution around $z=0$ gives strong connection to the capping operator $J(u,z)$. The solution $\Psi(z)$ around different region also gives rise to the analytic continuation of $J(u,z)$ over the Picard torus $\mathrm{Pic}(X)\otimes\mathbb{C}^{\times}$, which relates the capping operator of quiver varieties of different stability conditions.

To solve the quantum difference equation, we need to first figure out the expression of the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$. Generally it is conjectured that $\mathbf{M}_{\mathscr{L}}(z)$ can be expressed in terms of the generators of the corresponding quantum affine algebra $U_q(\mathfrak{g}_Q)$. The first systematic construction is given by Okounkov and Smirnov [OS22]. They use the quantum algebra defined by the RTT formalism of the K-theoretic stable envelope to express the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$, and the formula can be written as the product of the monodromy operators $\mathbf{B}_w(z)$:

$$\mathbf{M}_{\mathscr{L}}(z)=\mathscr{L}\prod_{w\in Walls}\mathbf{B}_w(z) \tag{1.2}$$

Each monodromy operator $\mathbf{B}_w(z)\in\widehat{U_q(\mathfrak{g}_w)}$ lies in the completion of the slope subalgebra $\widehat{U_q(\mathfrak{g}_w)}$ and can be solved via the ABRR equation in [OS22] and [ABRR97].

It is conjectured that the quantum algebra $U_q(\hat{\mathfrak{g}}_Q)$ constructed via the K-theoretic stable envelope is isomorphic to the quantum affine algebra of the corresponding quiver type $U_q(\hat{\mathfrak{g}}_Q)$. The conjecture implies that the quantum difference operator has the explicit expression in terms of the generators of the quantum affine algebra $U_q(\hat{\mathfrak{g}}_Q)$.

On the algebraic side, one can construct the analog of the quantum difference equation on the quantum affine algebra $U_q(\hat{\mathfrak{g}}_Q)$ [Z23]. For the affine type A case, the construction is based on the slope factorization of the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ given by [N15]. The formula for the slope subalgebra gives the explicit form of the quantum difference operator and the monodormy operator. The algebraic construction of the quantum difference equation is equivalent to the original quantum difference equation once the conjecture about the quantum algebra and K-theoretic stable envelope is true.

1.1. **Quantum difference equation for $M(\mathbf{v},\mathbf{w})$.** In this paper we study the quantum difference equation of the equivariant K-theory of the affine type A quiver varieties $M(\mathbf{v},\mathbf{w})$ from the algebraic construction given in [Z23]. The first main result is that the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$ can have a good representation:

Theorem 1.1. *(See Theorem 6.3) For the generic path $[-s-\mathscr{L},-s)$ between $-s-\mathscr{L}$ and $-s$ of the quantum difference operator:*

$$\mathbf{M}_{\mathscr{L}}(z)=\mathscr{L}\prod_{\mathbf{m}\in Walls}^{\rightarrow}\mathbf{B}_{\mathbf{m}} \tag{1.3}$$

with s being generic. Then each $\mathbf{B}_{\mathbf{m}}$ can be written either in one of the following form:

- *$U_q(\mathfrak{sl}_2)$ type:*

$$\begin{aligned}&\prod_{k=0}^{\stackrel{\rightarrow}{\infty}}\exp_{q^2}(-(q-q^{-1})z^{k(\mathbf{v}_\gamma)}p^{k\mathbf{m}\cdot(\mathbf{v}_\gamma)}q^{-k(\mathbf{v}_\gamma)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}f_\gamma e'_\gamma))\\&=\sum_{n=0}^{\infty}\frac{(q-q^{-1})^n}{[n]_{q^2}!}\frac{(-1)^n}{\prod_{v=1}^n(1-z^{\mathbf{v}_\gamma}p^{\mathbf{m}\cdot\mathbf{v}_\gamma}q^{-v(\mathbf{v}_\gamma)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}})}f_\gamma^n e_\gamma'^n\end{aligned} \tag{1.4}$$

- *$U_q(\hat{\mathfrak{gl}}_1)$ type.*

$$\mathbf{m}(\prod_{h=1}^{g}(\exp(-\sum_{k=1}^{\infty}\frac{n_kq^{-\frac{k|\boldsymbol{\delta}_h|}{2}}}{1-z^{-k|\boldsymbol{\delta}_h|}p^{k\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-\frac{k|\boldsymbol{\delta}_h|}{2}}}\alpha_{-k}^{\mathbf{m},h}\otimes\alpha_k^{\mathbf{m},h}) \tag{1.5}$$

Roughly speaking, the theorem implies that for the generic slope points $\mathbf{m}\in\mathbb{Q}^n$, the corresponding monodromy operator $\mathbf{B}_{\mathbf{m}}(z)$ can be totally written in either $U_q(\mathfrak{sl}_2)$ type or $U_q(\hat{\mathfrak{gl}}_1)$ type. This explains the philosophy that the slope subalgebra $\mathscr{B}_{\mathbf{m}}$ can be thought of as being generated by the wall subalgebra $U_q(\mathfrak{g}_w)$ such that the wall $w\subset\mathbb{Q}^n$ contains the slope point $\mathbf{m}$.

If we do the comparison with the quantum difference operators given by Okounkov-Smirnov in [OS22], the first thing we can check is the comparison between the wall set $\mathrm{Wall}(M(\mathbf{v},\mathbf{w}))$ defined in [Z23] and the set of points on the walls of K-theoretic stable envelope for $M(\mathbf{v},\mathbf{w})$. As expected, these are sets are actually the same:

Proposition 1.2. *(Prop 5.4, Prop 5.5) The wall set $Wall(M(\mathbf{v},\mathbf{w}))$ coincides with the set of points $Wall^{Stab}(M(\mathbf{v},\mathbf{w}))$ in the wall given by the K-theoretic stable envelope of $K_T(M(\mathbf{v},\mathbf{w}))$*

1.2. **Connection matrix of the quantum difference equation.** One important aspects about the quantum difference equation is the connection matrix between the solution expanding from $z=0^{\boldsymbol{\theta}_1}$ to $z=0^{\boldsymbol{\theta}_2}$. It is defined as:

$$\mathbf{Mon}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(z):=\Psi_{\boldsymbol{\theta}_1}(z)^{-1}\Psi_{\boldsymbol{\theta}_2}(z) \tag{1.6}$$

Here $\boldsymbol{\theta}_i \in (\mathbb{Z}-\{0\})^n$ such that $0^\theta = 0$ if $\theta > 0$, $0^\theta = \infty$ if $\theta < 0$. $\Psi_{\boldsymbol{\theta}}(z)$ is the fundamental solution around $z = 0^{\boldsymbol{\theta}}$.

In the settings of the Fuchsian q-difference equation, the connection matrix plays the role as the monodromy operator in the settings of the differential equations with regular singularities.

For simplicity, in this paper we only give the explicit calculation of the connection matrix for $\boldsymbol{\theta}_1 = \boldsymbol{\theta} = (1,1,\cdots,1)$, $\boldsymbol{\theta}_2 = -\boldsymbol{\theta} = (-1,-1,\cdots,-1)$. Part of the reason is that this connection matrix $\mathbf{Mon}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(z)$ can be thought of as the "largest" connection matrix with the factorization:

$$\mathbf{Mon}_{\boldsymbol{\theta},-\boldsymbol{\theta}}(z) = \mathbf{Mon}_{\boldsymbol{\theta},\boldsymbol{\theta}_1}(z)\mathbf{Mon}_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(z)\cdots\mathbf{Mon}_{\boldsymbol{\theta}_n,-\boldsymbol{\theta}}(z) \tag{1.7}$$

If we know the fundamental solutions, the connection matrix can be computed out via the terms in the fundamental solutions. The main result for the computation is the $p \to 0$ limit of the regular part of the connection matrix $\mathbf{Mon}^{reg}(zp^s)$:

Theorem 1.3. *(Theorem 6.6) For generic $s \in \mathbb{Q}^n$ such that $s_i \geq 0$, $s_i < 0$ the connection matrix as the following asymptotic at $p \to 0$:*

$$\lim_{p\to 0}\boldsymbol{Mon}^{reg}(p^s z) = \begin{cases} \prod^{\leftarrow}_{0\leq\mathbf{m}<s}(\mathbf{B}^*_{\mathbf{m}})^{-1}\cdot\mathbf{T}, & s \geq 0 \\ \prod_{s<\mathbf{m}<0}\mathbf{B}^*_{\mathbf{m}}\cdot\mathbf{T}, & s < 0 \end{cases} \tag{1.8}$$

Here $0 \leq \mathbf{m} < s$ means the slope points $\mathbf{m}$ in one generic path from 0 to the point s without intersecting s.

Here $\mathbf{B}_{\mathbf{m}} = \mathbf{m}((1\otimes S_{\mathbf{m}})(R^-_{\mathbf{m}})^{-1})$. In the case of finite A type, the operator $\mathbf{B}_{\mathbf{m}}$ was identified as the generators of the monodromy representation of the trigonometric Casimir connection [GL11].

1.3. **Degeneration of the quantum difference equation.** One important feature about the quiver variety $M(\mathbf{v},\mathbf{w})$ is the Dubrovin connection over its quantum cohomology $H^*_T(M(\mathbf{v},\mathbf{w}))[[q^d]]_{d\in H^2(M(\mathbf{v},\mathbf{w}))_{eff}}$.

For the Dubrovin connection for the affine type A quiver varieties $M(\mathbf{v},\mathbf{w})$, it can be written as:

$$\nabla_\lambda = d_\lambda - Q(\lambda), \qquad \lambda \in H^2(M(\mathbf{v},\mathbf{w}),\mathbb{Z}) \tag{1.9}$$

Here $Q(\lambda) = c_1(\lambda)\cup(-)+\cdots$ is the quantum multiplication operator. It has been proved by Maulik-Okounkov [MO12] that the quantum multiplication operator can be written as:

$$Q(\lambda) = c_1(\lambda) + \sum_{\alpha>0}\frac{(\alpha,\lambda)}{1-q^{-\alpha}}e_\alpha e_{-\alpha} + \text{constant} \tag{1.10}$$

where $e_{\pm\alpha} \in \mathfrak{g}^{MO}$ are the generators of the Maulik-Okounkov Lie algebra of the affine type A, and constant stands for some scaling operators. It has been proved in [BD23] that the Lie algebra $\mathfrak{g}^{MO}$ is isomorphic to the affine Lie algebra $\hat{\mathfrak{sl}}_n$, which means that the quantum multiplication operator can be written as:

$$Q(\lambda) = c_1(\lambda) \cup - + \sum_{i<j} \frac{(\lambda \cdot [i,j))}{1 - q^{-[i,j)}} E_{[i,j)} E_{-[i,j)} + \text{const} \tag{1.11}$$

where $E_{\pm[i,j)} \in \hat{\mathfrak{sl}}_n$ are the generators of the affine Lie algebra $\hat{\mathfrak{sl}}_n$.

The important feature of the Dubrovin connection is that the solution and the monodromy has played the vital role in the Gromov-Witten theory of the quiver varieties, while it can also be expressed in terms of the representation theory of the quantum groups. It is also conjectured that the monodromy representation of the Dubrovin connection for $M(\mathbf{v}, \mathbf{w})$ should be generated by $\mathbf{B}_w := \mathbf{m}((1 \otimes S_{\mathbf{w}})(R_w^-)^{-1})$ with $w \subset \text{Pic}(M(\mathbf{v}, \mathbf{w})) \otimes \mathbb{Q}$ the wall hyperplane in the rational Picard group. R_w^- is the lower triangular R-matrix for the wall subalgebra $U_q(\mathfrak{g}_w)$.

The second main result of the paper is that we can prove the algebraic version of the conjecture:

Theorem 1.4. *(Theorem 7.7) The monodromy representation:*

$$\pi_1(\mathbb{P}^r \backslash \textbf{\textit{Sing}}) \to End(H_T(M(\mathbf{v}, \mathbf{w}))) \tag{1.12}$$

of the Dubrovin connection is generated by $\mathbf{B}_{\mathbf{m}} = \mathbf{m}((1 \otimes S_{\mathbf{m}})(R_{\mathbf{m}}^-)^{-1})$.

The proof of the theorem relies on two facts: We take suitable degeneration limit of the quantum difference equation for $M(\mathbf{v}, \mathbf{w})$, the quantum difference equation would degenerate to the Dubrovin connection of the quantum cohomology of $H_T^*(M(\mathbf{v}, \mathbf{w}))$. This can be stated as the following theorem:

Theorem 1.5. *(Theorem 7.2) The degeneration limit of the quantum difference operator* $\mathbf{M}_{\mathscr{L}}(z)$ *coincides with the quantum multiplication operator* $Q(\mathscr{L})$ *up to a constant operator.*

This theorem tells us that one could describe the monodromy representation of the Dubrovin connection can be described in terms of the degeneration limit of the connection matrix in 7.32:

$$\text{Trans}(s) = \lim_{\tau \to 0} \mathbf{Mon}(z = e^{2\pi i s}, t_1 = e^{2\pi i \hbar_1 \tau}, t_2 = e^{2\pi i \hbar_2 \tau}, q = e^{-2\pi i \tau}) \tag{1.13}$$

It turns out that we could identify $\text{Trans}(s)$ with the $p \to 0$ limit of the regular part of the connection matrix:

Proposition 1.6. *(Proposition 7.4)* $Trans(s) = \lim_{p\to 0} \textbf{\textit{Mon}}^{reg}(p^s, e^{2\pi i \hbar_1}, e^{2\pi i \hbar_2}, p)$ *for* $s \in \mathbb{R}^n \backslash Walls$

The monodromy representation is generated by $\mathrm{Trans}(s')^{-1}\mathrm{Trans}(s)$ for different generic s',s. For s',s close enough, $\mathrm{Trans}(s')^{-1}\mathrm{Trans}(s)$ is equal to $\mathbf{B}_{\mathbf{m}}$. Thus in this way we have finished the proof of Theorem 7.7.

The proof of the degeneration of the connection matrix to the monodromy representation heavily uses the fact on the modular duality for the Riemann theta function. For completeness we will review the Riemann theta function on the abelian variety in the appendix.

It is worth noting that the construction of the quantum difference equations in [Z23] can be generalized to arbitrary quivers and arbitrary weighted modules of the corresponding shuffle algebras $\mathcal{S}$. We will give the construction in [Z24]. For the quiver of finite ADE type, the consturction will be just the copy of the construction given by Etingof and Varchenko in [EV02], and the degeneration to the corresponding trigonometric Casimir connection is given in [BM15].

1.4. **Structure of the paper.** The structure of the paper goes in the following way. In section 2 and section 3, we will introduce some basic ingredients about the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$, affine Yangians $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ and their action on the equivariant K-theory $K_T(M(\mathbf{w}))$ and on the equivariant cohomology $H^*_T(M(\mathbf{w}))$ of the affine type A quiver varieties. We will connect them by the degeneration limit in section 4. Also we will introduce the slope factorization of the quantum toroidal algebra in section 5.

In section 6 we will review the construction of the quantum difference equation for the affine type A quiver varieties, and we show that the wall set of the affine type A quiver varieties coincides with the wall set given by the K-theoretic stable envelopes.

In section 7 we will give the detailed analysis of the quantum difference equation, we will show that the quantum difference operator admits good representation, which is Theorem 6.3 and use this result to prove that the degenration limit of the quantum difference equation is equivalent to the Dubrovin connection (Theorem 7.2) and the monodromy representation is generated by the monodromy operators (Theorem 7.7).

In section 9, we will show some computation for the examples of the equivariant Hilbert scheme $\mathrm{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$.

Acknowledgments. The author would like to thank Andrei Negut, Andrei Okounkov, Andrey Smirnov and Hunter Dinkins for their helpful discussions on quantum groups and stable envelopes. The author is supported by the international collaboration grant BMSTC and ACZSP grant. Part of this work was done while the author was visiting Department of Mathematics at Columbia University and Mathematics Department of University of North Carolina at Chapel Hill. The author thanks for their hospitality and provisio of excellent working environment.

2. Quantum toroidal algebra and quiver varieties

2.1. Quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$. The quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ is a $\mathbb{Q}(q,t)$-algebra defined as:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) = \mathbb{Q}(q,t)\langle \{e^{\pm}_{i,d}\}^{d\in\mathbb{Z}}_{1\leq i\leq n}, \{\varphi^{\pm}_{i,d}\}^{d\in\mathbb{N}_0}_{1\leq i\leq n}\rangle/(\ ) \tag{2.1}$$

The relation between the generators can be described in the generating functions:

$$e^{\pm}_i(z) = \sum_{d\in\mathbb{Z}} e^{\pm}_{i,d} z^{-d} \quad \varphi^{\pm}_i(z) = \sum_{d=0}^{\infty} \varphi^{\pm}_{i,d} z^{\mp d} \tag{2.2}$$

with $\varphi^{\pm}_{i,d}$ commute among themselves and:

$$\begin{aligned} e^{\pm}_i(z)\varphi^{\pm'}_j(w)\cdot\zeta\left(\frac{w^{\pm 1}}{z^{\pm 1}}\right) &= \varphi^{\pm'}_j(w)e^{\pm}_i(z)\cdot\zeta\left(\frac{z^{\pm 1}}{w^{\pm 1}}\right) \\ e^{\pm}_i(z)e^{\pm}_j(w)\cdot\zeta\left(\frac{w^{\pm 1}}{z^{\pm 1}}\right) &= e^{\pm}_j(w)e^{\pm}_i(z)\cdot\zeta\left(\frac{z^{\pm 1}}{w^{\pm 1}}\right) \\ \left[e^{+}_i(z), e^{-}_j(w)\right] &= \delta^j_i\delta\left(\frac{z}{w}\right)\cdot\frac{\varphi^{+}_i(z)-\varphi^{-}_i(w)}{q-q^{-1}} \end{aligned} \tag{2.3}$$

Here $i,j\in\{1,\cdots,n\}$ and z,w are variables of color i and j. Here:

$$\zeta\left(\frac{x_i}{x_j}\right) = \frac{\left[\frac{x_j}{qtx_i}\right]^{\delta^i_{j-1}}\left[\frac{tx_j}{qx_i}\right]^{\delta^i_{j+1}}}{\left[\frac{x_j}{x_i}\right]^{\delta^i_j}\left[\frac{x_j}{q^2x_i}\right]^{\delta^i_j}} \tag{2.4}$$

with the Serre relation:

(2.5)

$$e^{\pm}_i(z_1)\,e^{\pm}_i(z_2)\,e^{\pm}_{i\pm'1}(w) + \left(q+q^{-1}\right)e^{\pm}_i(z_1)\,e^{\pm}_{i\pm'1}(w)e^{\pm}_i(z_2) + e^{\pm}_{i\pm'1}(w)e^{\pm}_i(z_1)\,e^{\pm}_i(z_2) +$$
$$+\,e^{\pm}_i(z_2)\,e^{\pm}_i(z_1)\,e^{\pm}_{i\pm'1}(w) + \left(q+q^{-1}\right)e^{\pm}_i(z_2)\,e^{\pm}_{i\pm'1}(w)e^{\pm}_i(z_1) + e^{\pm}_{i\pm'1}(w)e^{\pm}_i(z_2)\,e^{\pm}_i(z_1) = 0$$

The standard coproduct structure is imposed as:

$$\Delta: U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \longrightarrow U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)\widehat{\otimes}U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \tag{2.6}$$

$$\begin{aligned} \Delta\left(e^{+}_i(z)\right) &= \varphi^{+}_i(z)\otimes e^{+}_i(z) + e^{+}_i(z)\otimes 1 & \Delta\left(\varphi^{+}_i(z)\right) &= \varphi^{+}_i(z)\otimes\varphi^{+}_i(z) \\ \Delta\left(e^{-}_i(z)\right) &= 1\otimes e^{-}_i(z) + e^{-}_i(z)\otimes\varphi^{-}_i(z) & \Delta\left(\varphi^{-}_i(z)\right) &= \varphi^{-}_i(z)\otimes\varphi^{-}_i(z) \end{aligned} \tag{2.7}$$

2.2. **Quantum affine algebra $U_q(\hat{\mathfrak{gl}}_n)$.** The quantum affine algebra $U_q(\hat{\mathfrak{gl}}_n)$ is a $\mathbb{Q}(q)$-algebra generated by:

$$\mathbb{Q}(q)\langle e_{\pm[i;j]},\psi_s^{\pm1},c^{\pm1}\rangle^{s\in\{1,\cdots,n\}}_{(i<j)\in\mathbb{Z}^2/(n,n)\mathbb{Z}} \tag{2.8}$$

The generators $e_{\pm[i;j)}$ satisfy the well-known RTT relation:

$$R(\frac{z}{w})T_1^+(z)T_2^+(w)=T_2^+(w)T_1^+(z)R(\frac{z}{w}) \tag{2.9}$$

$$R(\frac{z}{w})T_1^-(z)T_2^-(w)=T_2^-(w)T_1^-(z)R(\frac{z}{w}) \tag{2.10}$$

$$R(\frac{z}{wc})T_2^-(w)T_1^+(z)=T_1^+(z)T_2^-(w)R(\frac{zc}{w}) \tag{2.11}$$

Here $T^{\pm}(z)$ are the generating function for $e_{\pm[i;j)}$:

$$\begin{aligned} T^+(z)&=\sum_{1\leq i\leq n}^{i\leq j}e_{[i;j)}\psi_i\cdot E_{j\text{ mod }n,i}z^{\lceil\frac{j}{n}\rceil-1}\\ T^-(z)&=\sum_{1\leq i\leq n}^{i\leq j}e_{-[i;j)}\psi_i^{-1}\cdot E_{i,j\text{ mod }n}z^{-\lceil\frac{j}{n}\rceil+1}\end{aligned} \tag{2.12}$$

And $R(z/w)$ is the standard R-matrix for $\mathfrak{gl}_n$:

$$R\left(\frac{z}{w}\right)=\sum_{1\leq i,j\leq n}E_{ii}\otimes E_{jj}\left(\frac{zq-wq^{-1}}{w-z}\right)^{\delta_j^i}+\left(q-q^{-1}\right)\sum_{1\leq i\neq j\leq n}E_{ij}\otimes E_{ji}\frac{w^{\delta_{i>j}}z^{\delta_{i<j}}}{w-z} \tag{2.13}$$

In addition, we have the relations:

$$\psi_k\cdot e_{\pm[i;j)}=q^{\pm\left(\delta_k^i-\delta_k^j\right)}e_{\pm[i;j)}\cdot\psi_k\quad\forall\,\text{arcs}[i;j)\text{ and }k\in\mathbb{Z} \tag{2.14}$$

The algebra $U_q(\hat{\mathfrak{gl}}_n)$ has the coproduct structure given by:

$$\Delta(T^+(z))=T^+(z)\otimes T^+(zc_1),\qquad\Delta(T^-(z))=T^-(zc_2)\otimes T^-(z) \tag{2.15}$$

and the antipode map:

$$S(T^{\pm}(z))=(T^{\pm}(z))^{-1} \tag{2.16}$$

The corresponding antipode map elements can be written as:

$$\begin{aligned} S^+(z)&=\sum_{1\leq i\leq n}^{i\leq j}(-1)^{j-i}f_{[i;j)}\psi_j\cdot E_{j\text{ mod }n,i}z^{\lceil\frac{j}{n}\rceil-1}\\ S^-(z)&=\sum_{1\leq i\leq n}^{i\leq j}(-1)^{j-i}f_{-[i;j)}\psi_j^{-1}\cdot E_{i,j\text{ mod }n}z^{-\lceil\frac{j}{n}\rceil+1}\end{aligned} \tag{2.17}$$

Unwinding the relation we have that:

(2.18)
$$\frac{e_{\pm[a,c)}e_{\pm[b,d)}}{q^{\delta_a^b-\delta_b^d+\delta_a^d}}-\frac{e_{\pm[b,d)}e_{\pm[a,c)}}{q^{\delta_c^b-\delta_b^d+\delta_c^d}}=(q-q^{-1})[\sum_{a\leq x<c}^{x\equiv d}e_{\pm[b,c+d-x)}e_{\pm[a,x)}-\sum_{a<x\leq c}^{x\equiv b}e_{\pm[x,c)}e_{\pm[a+b-x,d)}]$$

$$[e_{[a,c)},e_{-[b,d)}]=(q-q^{-1})[\sum_{a\leq x<c}^{x\equiv b}\frac{e_{[c+b-x,d)}e_{[a,x)}}{q^{\delta_c^b+\delta_c^a-\delta_b^a}}\frac{\psi_x}{\psi_c}-\sum_{a<x\leq c}^{x\equiv d}\frac{e_{[x,c)}e_{-[b,a+d-x)}}{q^{-\delta_a^b+\delta_b^d-\delta_a^d}}\frac{\psi_x}{\psi_a}] \tag{2.19}$$

The isomorphism $U_q(\hat{\mathfrak{gl}}_n)\cong U_q(\hat{\mathfrak{sl}}_n)\otimes U_q(\hat{\mathfrak{gl}}_1)$ is given by the following:

$$e_{[i;i+1)}=x_i^+(q-q^{-1}) \tag{2.20}$$

$$e_{-[i;i+1)}=x_i^-(q^{-2}-1) \tag{2.21}$$

For the generators $\{p_k\}_{k\in\mathbb{Z}}$ of the Heisenberg algebra $U_q(\hat{\mathfrak{gl}}_1)\subset U_q(\hat{\mathfrak{gl}}_n)$, they satisfy the following commutation relations:

$$[p_k,p_{-k}]=\frac{c^{nk}-c^{-nk}}{n_k},\qquad n_k=\frac{q^k-q^{-k}}{k} \tag{2.22}$$

It can be packaged into the following proposition proved by Negut in [N19]:

Proposition 2.1. *There exists constants $\alpha_1,\alpha_2,\cdots\in\mathbb{Q}(q)$ such that:*

$$f_{\pm[i;j)}=\sum_{k=0}^{\left\lfloor\frac{j-i}{n}\right\rfloor}e_{\pm[i;j-nk)}g_{\pm k} \tag{2.23}$$

Here $\sum_{k=0}^{\infty}g_{\pm k}x^k=\exp(\sum_{k=1}^{\infty}\alpha_k p_{\pm k}x^k)$

This formula allows us to obtain the expression for $p_{\pm k}$ iteratively.

2.3. **Shuffle algebra realization of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$.** Here we review the reconstruction of the quantum toroidal algebra via the shuffle algebra and the induced slope factorization of the quantum toroidal algebra. For details see [N15].

Fix a fractional field $\mathbb{F}=\mathbb{Q}(q,t)$ and consider the space of symmetric rational functions:

$$\widehat{\mathrm{Sym}}(V):=\bigoplus_{\mathbf{k}=(k_1,\cdots,k_n)\in\mathbb{N}^n}\mathbb{F}(\cdots,z_{i1},\cdots,z_{ik_i},\cdots)^{\mathrm{Sym}}_{1\leq i\leq n} \tag{2.24}$$

Here "Sym" means to symmetrize the rational function for each $z_{i1},\cdots,z_{ik_i}$. We endow the vector space with the shuffle product:

$$F*G=\mathrm{Sym}[\frac{F(\cdots,z_{ia},\cdots)G(\cdots,z_{jb},\cdots)}{\mathbf{k}!\mathbf{l}!}\prod_{1\leq a\leq k_i}^{1\leq i\leq n}\prod_{k_j+1\leq b\leq k_j+l_j}^{1\leq j\leq n}\zeta(\frac{z_{ia}}{z_{jb}})] \tag{2.25}$$

We define the subspace $\mathcal{S}^{\pm}_{\mathbf{k}}\subset\widehat{\mathrm{Sym}}(V)$ by:

$$\mathcal{S}^{+}:=\{F(\cdots,z_{ia},\cdots)=\frac{r(\cdots,z_{ia},\cdots)}{\prod_{1\leq a\neq b\leq k_i}^{1\leq i\leq n}(qz_{ia}-q^{-1}z_{ib})}\}\tag{2.26}$$

where $r(\cdots,z_{i1},\cdots,z_{ik_i},\cdots)_{1\leq a\leq k_i}^{1\leq i\leq n}$ is any symmetric Laurent polynomial that satisfies the wheel conditions:

$$r(\cdots,q^{-1},t^{\pm},q,\cdots)=0\tag{2.27}$$

for any three variables of colors $i,\cdots,i\pm1,i$.

The shuffle algebra $\mathcal{S}^{+}$ has two natural bigrading given by the number of the variables in each color $\mathbf{k}\in\mathbb{N}^n$ and the homogeneous degree of the rational functions $d\in\mathbb{Z}$:

$$\mathcal{S}^{+}=\bigoplus_{(\mathbf{k},d)\in\mathbb{N}^n\times\mathbb{Z}}\mathcal{S}^{+}_{\mathbf{k},d}\tag{2.28}$$

Similarly we can define the negative shuffle algebra $\mathcal{S}^{-}:=(\mathcal{S}^{+})^{op}$ which is the same as $\mathcal{S}^{+}$ with the opposite shuffle product. Now we slightly enlarge the positive and negative shuffle algebras by the generators $\{\varphi^{\pm}_{i,d}\}_{1\leq i\leq r}^{d\geq0}$:

$$\mathcal{S}^{\geq}=\langle\mathcal{S}^{+},\{(\varphi^{+}_{i,d})_{1\leq i\leq r}^{d\geq0}\}\rangle,\qquad\mathcal{S}^{\leq}=\langle\mathcal{S}^{-},\{(\varphi^{-}_{i,d})_{1\leq i\leq r}^{d\geq0}\}\rangle\tag{2.29}$$

And here $\varphi^{\pm}_{i,d}$ commute with themselves and have the relation with $\mathcal{S}^{\pm}$ as follows:

$$\varphi_i^{+}(w)F=F\varphi_i^{+}(w)\prod_{1\leq a\leq k_j}^{1\leq j\leq n}\frac{\zeta(w/z_{ja})}{\zeta(z_{ja}/w)}\tag{2.30}$$

$$\varphi_i^{-}(w)G=G\varphi_i^{-}(w)\prod_{1\leq a\leq k_j}^{1\leq j\leq n}\frac{\zeta(z_{ja}/w)}{\zeta(w/z_{ja})}\tag{2.31}$$

The Drinfeld pairing $\langle-,-\rangle:\mathcal{S}^{\leq}\otimes\mathcal{S}^{\geq}\to\mathbb{Q}(q,t)$ between the positive and negative shuffle algebras $\mathcal{S}^{\geq,\neq}$ is given by:

$$\langle\varphi_i^{-}(z),\varphi_j^{+}(w)\rangle=\frac{\zeta(w/z)}{\zeta(z/w)}\tag{2.32}$$

$$\langle G,F\rangle=\frac{1}{\mathbf{k}!}\int_{|z_{ia}|=1}^{|q|<1|p|}\frac{G(\cdots,z_{ia},\cdots)F(\cdots,z_{ia},\cdots)}{\prod_{a\leq k_i,b\leq k_j}^{1\leq i,j\leq n}\zeta_p(z_{ia}/z_{jb})}\prod_{1\leq a\leq k_i}^{1\leq i\leq n}\frac{dz_{ia}}{2\pi iz_{ia}}|_{p\mapsto q}\tag{2.33}$$

for $F\in\mathcal{S}^{+}$, $G\in\mathcal{S}^{-}$. Here ζ_p is defined as follows:

$$\zeta_p(\frac{x_i}{x_j})=\zeta(\frac{x_i}{x_j})\frac{[\frac{x_j}{p^2x_i}]^{\delta^i_j}}{[\frac{x_j}{q^2x_i}]^{\delta^i_j}}\tag{2.34}$$

This defines the shuffle algebra $\mathcal{S} := \mathcal{S}^{\leq}\hat{\otimes}\mathcal{S}^{\geq}$. Also the coproduct $\Delta : \mathcal{S} \to \mathcal{S}\hat{\otimes}\mathcal{S}$ is given by:

(2.35)

$$\Delta(\varphi_i^{\pm}(w)) = \varphi_i^{\pm}(w) \otimes \varphi_i^{\pm}(w)$$

(2.36)

$$\Delta(F) = \sum_{0\leq \mathbf{l}\leq \mathbf{k}} \frac{[\prod_{1\leq j\leq n}^{b>l_j} \varphi_j^{+}(z_{jb})]F(\cdots, z_{i1}, \cdots, z_{il_i} \otimes z_{i,l_i+1}, \cdots, z_{ik_i}, \cdots)}{\prod_{1\leq i\leq n}^{a\leq l_i}\prod_{1\leq j\leq n}^{b>l_j} \zeta(z_{jb}/z_{ia})}, \qquad F \in \mathcal{S}^{+}$$

(2.37)

$$\Delta(G) = \sum_{0\leq \mathbf{l}\leq \mathbf{k}} \frac{G(\cdots, z_{i1}, \cdots, z_{il_i} \otimes z_{i,l_i+1}, \cdots, z_{ik_i}, \cdots)[\prod_{1\leq j\leq n}^{b>l_j} \varphi_j^{-}(z_{jb})]}{\prod_{1\leq i\leq n}^{a\leq l_i}\prod_{1\leq j\leq n}^{b>l_j} \zeta(z_{ia}/z_{jb})}, \qquad G \in \mathcal{S}^{-}$$

In human language, the above coproduct formula means the following:For the right hand side of the formula, in the limit $|z_{ia}| << |z_{jb}|$ for all $a \leq l_i$ and $b > l_i$, and then place all monomials in $\{z_{ia}\}_{a\leq l_i}$ to the left of the $\otimes$ symbol and all monomials in $\{z_{jb}\}_{b>l_j}$ to the right of the $\otimes$ symbol. Also we have the antipode map $S : \mathcal{S} \to \mathcal{S}$ which is an anti-homomorphism of both algebras and coalgebras:

$$S(\varphi_i^{+}(z)) = (\varphi_i^{+}(z))^{-1}, \qquad S(F) = [\prod_{1\leq a\leq k_i}^{1\leq i\leq n} (-\varphi_i^{+}(z_{ia}))^{-1}] * F \tag{2.38}$$

$$S(\varphi_i^{-}(z)) = (\varphi_i^{-}(z))^{-1}, \qquad S(G) = G * [\prod_{1\leq a\leq k_i}^{1\leq i\leq n} (-\varphi_i^{-}(z_{ia}))^{-1}] \tag{2.39}$$

The following theorem has been proved by[N15]:

Theorem 2.2. *There is a bigraded isomorphism of bialgebras*

$$\Upsilon : U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \to \mathcal{S} \tag{2.40}$$

given by:

$$\Upsilon(\varphi_{i,d}^{\pm}) = \varphi_{i,d}^{\pm}, \qquad \Upsilon(e_{i,d}^{\pm}) = \frac{z_{i1}^{d}}{[q^{-2}]} \tag{2.41}$$

2.4. **Slope subalgebra of the shuffle algebra.** One of the wonderful property for the shuffle algebra is that it admits the slope factorization.

For the definition of the slope subalgebra see section 5.2 in [N15]

It is known that the slope subalgebra $\mathcal{B}_{\mathbf{m}}$ for the quantum toroidal algebra $U_{q,t}(\widehat{\widehat{\mathfrak{sl}}}_n)$ is isomorphic to:

$$\mathcal{B}_{\mathbf{m}} \cong \bigotimes_{h=1}^{g} U_q(\widehat{\mathfrak{gl}}_{l_h}) \tag{2.42}$$

For the proof of the above isomorphism 2.42 see [N15]. The isomorphism is constructed in the following way. For $\mathbf{m} \in \mathbb{Q}^n$, it is proved in [N15] that as the shuffle algebra, the Drinfeld double of the following:

$$\mathcal{B}_{\mathbf{m}} := \mathcal{B}_{\mathbf{m}}^{+} \otimes \mathbb{C}[\varphi_{i,0}^{\pm}] \otimes \mathcal{B}_{\mathbf{m}}^{-}/(\text{relations}) \tag{2.43}$$

Here $\mathcal{B}_{\mathbf{m}}^{+}$ and $\mathcal{B}_{\mathbf{m}}^{-}$ is defined as (5.29) and (5.30) in [N15], and we omit the details of the definition here. In this section we use the generators of the slope subalgebra $\mathcal{B}_{\mathbf{m}}$ as the definition of the slope subalgebra.

Let us define the generators of the slope algebra $\mathcal{B}_{\mathbf{m}}$. For $\mathbf{m} \cdot [i;j\rangle \in \mathbb{Z}$, we denote the following elements:

$$P_{\pm[i;j)}^{\pm\mathbf{m}} = \mathrm{Sym}\left[\frac{\prod_{a=i}^{j-1} z_a^{\lfloor m_i+\ldots+m_a\rfloor - \lfloor m_i+\ldots+m_{a-1}\rfloor}}{t^{\mathrm{ind}\, d_{[i,j\rangle}^{m}} q^{i-j} \prod_{a=i+1}^{j-1}\left(1-\frac{q_2 z_a}{z_{a-1}}\right)} \prod_{i\leq a<b<j} \zeta\left(\frac{z_b}{z_a}\right)\right] \in \mathcal{S}^{\pm} \tag{2.44}$$

$$Q_{\mp[i;j)}^{\pm\mathbf{m}} = \mathrm{Sym}\left[\frac{\prod_{a=i}^{j-1} z_a^{\lfloor m_i+\ldots+m_{a-1}\rfloor - \lfloor m_i+\ldots+m_a\rfloor}}{t^{-\mathrm{ind}_{[i,j)}^{m}} \prod_{a=i+1}^{j-1}\left(1-\frac{q_1 z_{a-1}}{z_a}\right)} \prod_{i\leq a<b<j} \zeta\left(\frac{z_a}{z_b}\right)\right] \in \mathcal{S}^{\mp} \tag{2.45}$$

Here $\mathrm{ind}_{[i;j)}^{\mathbf{m}}$ is defined as:

$$\mathrm{ind}_{[i;j)}^{\mathbf{m}} = \sum_{a=i}^{j-1} (m_i + \cdots + m_a - \lfloor m_i + \cdots + m_{a-1} \lfloor) \tag{2.46}$$

The positive and negative part of the slope subalgebra $\mathcal{B}_{\mathbf{m}}^{\pm}$ for the shuffle algebra $\mathcal{S}^{\pm}$ can be defined as the algebra generated by $\{P_{\pm[i;j)}^{\mathbf{m}}\}_{i\leq j}$ and $\{Q_{\pm[i;j)}^{\mathbf{m}}\}_{i\leq j}$. And the slope subalgebra $\mathcal{B}_{\mathbf{m}}$ is the similarly the Drinfeld double of $\mathcal{B}_{\mathbf{m}}^{\pm}$ with the neutral elements $\{\varphi_{\pm[i;j)}\}$.

We can check that the antipode map $S_{\mathbf{m}} : \mathcal{B}_{\mathbf{m}} \to \mathcal{B}_{\mathbf{m}}$ has the following relation:

$$S_{\mathbf{m}}(P_{[i;j)}^{\mathbf{m}}) = Q_{[i;j)}^{\mathbf{m}}, \qquad S_{\mathbf{m}}(Q_{-[i;j)}^{\mathbf{m}}) = P_{-[i;j)}^{\mathbf{m}} \tag{2.47}$$

$$\Delta_{\mathbf{m}}\left(P_{[i;j)}^{\mathbf{m}}\right) = \sum_{a=i}^{j} P_{[a;j)}^{\mathbf{m}} \varphi_{[i;a)} \otimes P_{[i;a)}^{\mathbf{m}} \quad \Delta_{\mathbf{m}}\left(Q_{[i;j)}^{\mathrm{m}}\right) = \sum_{a=i}^{j} Q_{[i;a)}^{\mathbf{m}} \varphi_{[a;j)} \otimes Q_{[a;j)}^{\mathbf{m}} \tag{2.48}$$

(2.49)

$$\Delta_{\mathbf{m}}\left(P^{\mathbf{m}}_{-[i;j)}\right)=\sum_{a=i}^{j}P^{\mathbf{m}}_{-[a;j)}\otimes P^{\mathbf{m}}_{-[i;a)}\varphi_{-[a;j)}\quad \Delta_{\mathbf{m}}\left(Q^{\mathbf{m}}_{-[i;j)}\right)=\sum_{a=i}^{j}Q^{\mathbf{m}}_{-[i;a)}\otimes Q^{\mathbf{m}}_{-[a;j)}\varphi_{-[i;a)}$$

Here $\Delta_{\mathbf{m}}$ is defined as (5.27) and (5.28) in [N15]. The isomorphism 2.42 is given by:

$$e_{[i;j)}=P^{\mathbf{m}}_{[i;j)_h},\qquad e_{-[i;j)}=Q^{\mathbf{m}}_{-[i;j)_h},\qquad \varphi_k=\varphi_{[k;v_{\mathbf{m}}(k))} \tag{2.50}$$

2.5. **Geometric action on $K_T(M(\mathbf{w}))$.** We first fix the notation for the Nakajima quiver varieties.

Given a quiver $Q=(I,E)$, consider the following quiver representation space

$$\mathrm{Rep}(\mathbf{v},\mathbf{w}):=\bigoplus_{h\in E}\mathrm{Hom}(V_{i(h)},V_{o(h)})\oplus\bigoplus_{i\in I}\mathrm{Hom}(V_i,W_i) \tag{2.51}$$

here $\mathbf{v}=(\dim(V_1),\cdots,\dim(V_I))$ is the dimension vector for the vector spaces at the vertex, $\mathbf{w}=(\dim(W_1),\cdots,\dim(W_I))$ is the dimension vector for the framing vector spaces.

Denote $G_{\mathbf{v}}:=\prod_{i\in I}GL(V_i)$ and $G_{\mathbf{w}}:=\prod_{i\in I}GL(W_i)$. There is a natural action of $G_{\mathbf{v}}$ and $G_{\mathbf{w}}$ on $\mathrm{Rep}(\mathbf{v},\mathbf{w})$, and thus a natural Hamiltonian action on $T^*\mathrm{Rep}(\mathbf{v},\mathbf{w})$ with respect to the standard symplectic form ω, now we have the moment map

$$\mu:T^*\mathrm{Rep}(\mathbf{v},\mathbf{w})\to\mathfrak{g}_{\mathbf{v}}^* \tag{2.52}$$

$$\mu(X_e,Y_e,A_i,B_i)=\sum_e X_{i(e)}Y_{i(e)}-Y_{o(e)}X_{o(e)}+A_iB_i \tag{2.53}$$

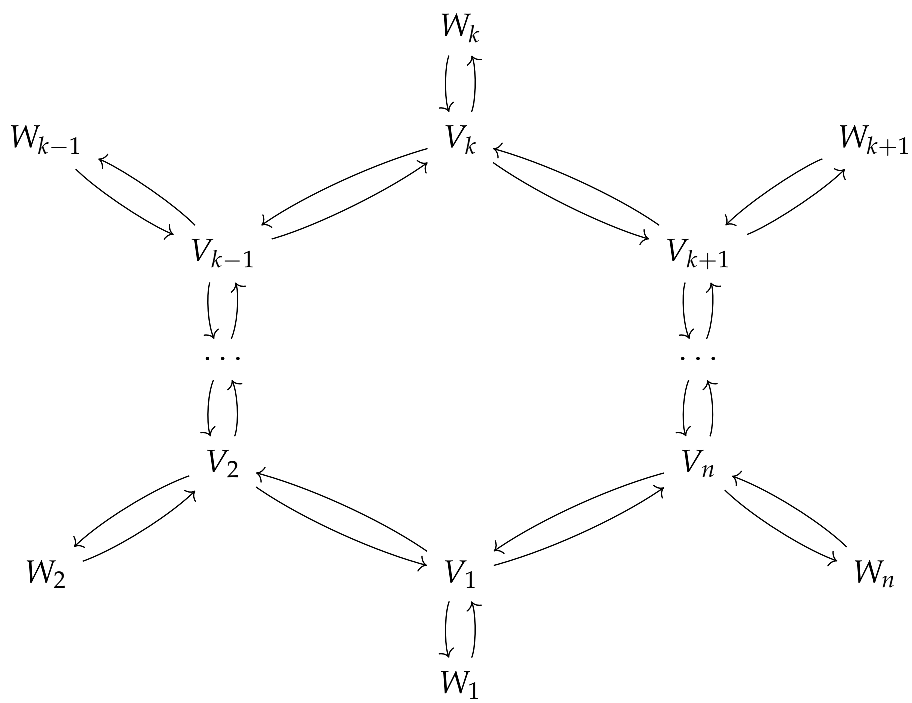
One can define **Nakajima variety**:

$$M_{\theta,0}(\mathbf{v},\mathbf{w}):=\mu^{-1}(0)//_{\theta}G_{\mathbf{v}} \tag{2.54}$$

with all the θ-stable $G_{\mathbf{v}}$-orbits in $\mu^{-1}(0)$. Here we choose the stability condition $\theta=(1,\cdots,1)$.

We will abbreviate $M(\mathbf{v},\mathbf{w})$ as the Nakajima quiver variety defined in 2.54.

In this paper we only consider the affine type A quiver, i.e. the edges are given by $e=(i,i+1)$ with $i=i\mathrm{mod}(n)$. i.e. it is the quiver of the following type:



The action of $\mathbb{C}_q^* \times \mathbb{C}_t^* \times G_{\mathbf{w}}$ on $M(\mathbf{v}, \mathbf{w})$ is given by:

$$(q, t, U_i) \cdot (X_e, Y_e, A_i, B_i)_{e \in E, i \in I} = (qtX_e, qt^{-1}Y_e, qA_iU_i, qU_i^{-1}B_i) \tag{2.55}$$

The fixed points set of the torus action $\mathbb{C}_q^* \times \mathbb{C}_t^*$ on $M_{\theta,0}(\mathbf{v}, \mathbf{w})$ is indexed by the $|\mathbf{w}|$-partitions $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_{\mathbf{w}})$. For each box $\square \in (\lambda_1, \cdots .\lambda_{\mathbf{w}})$. We define the following two character functions:

$$\chi_{\square} = u_i q^{x+y+1} t^{x-y}, \qquad \square \in \lambda_i \tag{2.56}$$

$$\chi_{\square}^{ch} = v_i + \hbar_1(x+1) + \hbar_2(y+1) \tag{2.57}$$

The first one is often used in the equivariant K-theory. The second one is often used in the equivariant cohomology theory.

If we choose the cocharacter $\sigma : \mathbb{C}^* \to G_{\mathbf{w}}$ such that $\mathbf{w} = u_1\mathbf{w}_1 + \cdots + u_k\mathbf{w}_k$, we have that:

$$M(\mathbf{v}, \mathbf{w})^{\sigma} = \bigsqcup_{\mathbf{v}_1 + \cdots + \mathbf{v}_k = \mathbf{v}} M(\mathbf{v}_1, \mathbf{w}_1) \times \cdots \times M(\mathbf{v}_k, \mathbf{w}_k) \tag{2.58}$$

We denote:

$$K_T(M(\mathbf{w})) := \bigoplus_{\mathbf{w}} K_{\mathbb{C}_q^*\times\mathbb{C}_t^*\times G_{\mathbf{w}}}(M(\mathbf{v},\mathbf{w}))_{loc} \tag{2.59}$$

Thus we can choose the fixed point basis $|\boldsymbol{\lambda}\rangle$ of the corresponding partition $\boldsymbol{\lambda}$ to span the vector space $K_T(M(\mathbf{w}))$.

Here we briefly review the geometric action of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ on $K_T(M(\mathbf{w}))$. The construction is based on Nakajima's simple correspondence.

Consider a pair of vectors $(\mathbf{v}_+,\mathbf{v}_-)$ such that $\mathbf{v}_+ = \mathbf{v}_- + \mathbf{e}_i$. There is a simple correspondence

$$Z(\mathbf{v}_+,\mathbf{v}_-,\mathbf{w}) \hookrightarrow M(\mathbf{v}_+,\mathbf{w}) \times M(\mathbf{v}_-,\mathbf{w}) \tag{2.60}$$

parametrises pairs of quadruples $(X^\pm, Y^\pm, A^\pm, B^\pm)$ that respect a fixed collection of quotients $(V^+ \to V^-)$ of codimension δ^i_j with only the semistable and zeros part for the moment map μ for each $M(\mathbf{v}_+,\mathbf{w})$. And the variety $Z(\mathbf{v}_+,\mathbf{v}_-,\mathbf{w})$ is smooth with a tautological line bundle:

$$\mathscr{L}|_{V^+\to V^-} = \mathrm{Ker}(V_i^+ \to V_i^-) \tag{2.61}$$

and the natural projection maps:

$$\begin{tikzcd} & Z(\mathbf{v}_+,\mathbf{v}_-,\mathbf{w}) \arrow[dl, "\pi_+"'] \arrow[dr, "\pi_-"] & \\ M(\mathbf{v}_+,\mathbf{w}) & & M(\mathbf{v}_-,\mathbf{w}) \end{tikzcd} \tag{2.62}$$

Using this we could consturct the operator:

$$e^\pm_{i,d} : K_T(M(\mathbf{v}^\mp,\mathbf{w})) \to K_T(M(\mathbf{v}^\pm,\mathbf{w})), \qquad e^\pm_{i,d}(\alpha) = \pi_{\pm *}(\mathrm{Cone}(d\pi_\pm)\mathscr{L}^d \pi^\pm_\mp(\alpha)) \tag{2.63}$$

and take all $\mathbf{v}$ we have the operator $e^\pm_{i,d} : K_T(M(\mathbf{w})) \to K_T(M(\mathbf{w}))$. Also we have the action of $\varphi^\pm_{i,d}$ given by the multiplication of the tautological class, which means that:

$$\varphi_i^\pm(z) = \overline{\frac{\zeta(\frac{z}{X})}{\zeta(\frac{X}{z})}} \prod_{1\leq j\leq \mathbf{w}}^{u_j\equiv i} \frac{[\frac{u_j}{\hbar z}]}{[\frac{z}{\hbar u_j}]} \tag{2.64}$$

In particular, the element $\varphi_{i,0}$ acts on $K_T(M(\mathbf{v},\mathbf{w}))$ as $q^{\alpha_i^T(\mathbf{w}-C\mathbf{v})}$.

The above construction gives the following well-known result:

Theorem 2.3. *For all $\mathbf{w} \in \mathbb{N}^r$, the operator $e^\pm_{i,d}$ and $\varphi^\pm_{i,d}$ give rise to an action of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ on $K_T(M(\mathbf{w}))$.*

In terms of the shuffle algebra, we can give the explicit formula of the action of the quantum toroial algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ on $K_T(M(\mathbf{w}))$:

Given $F \in \mathcal{S}^+_{\mathbf{k}}$, we have that

$$\langle\boldsymbol{\lambda}|F|\boldsymbol{\mu}\rangle = F(\chi_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}})\prod_{i=1}^{\mathbf{w}}[\frac{u_i}{q\chi_{\blacksquare}}]] \tag{2.65}$$

Similarly, for $G \in \mathcal{S}^-_{-\mathbf{k}}$, we have

$$\langle\boldsymbol{\mu}|G|\boldsymbol{\lambda}\rangle = G(\chi_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\lambda}}\zeta(\frac{\chi_{\square}}{\chi_{\blacksquare}})\prod_{i=1}^{\mathbf{w}}[\frac{\chi_{\blacksquare}}{qu_i}]]^{-1} \tag{2.66}$$

Here $F(\chi_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}})$ and $G(\chi_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}})$ are the rational symmetric function evaluation at the box $\chi_{\square}$ with $\square \in \boldsymbol{\lambda}\backslash\boldsymbol{\mu}$.

3. Affine Yangian and Kac-Moody algebra

In this section we introduce the affine Yangian which is suitable in our settings.

We introduce the following color-dependent rational function:

$$\omega(x_i - x_j) = \frac{(x_j - x_i - \hbar_1)^{\delta^i_{j-1}}(x_j - x_i - \hbar_2)^{\delta^i_{j+1}}}{(x_j - x_i)^{\delta^i_j}(x_j - x_i - \hbar_1 - \hbar_2)^{\delta^i_j}} \tag{3.1}$$

Consider the vector space

$$\mathscr{V}^+ := \bigoplus_{\mathbf{n}\in\mathbb{N}^I}\mathscr{V}_{\mathbf{n}}, \qquad \mathscr{V}_{\mathbf{n}} := \mathbb{F}[z_{i1},\cdots,z_{in_i}]^{sym}_{i\in I} \tag{3.2}$$

we endow $\mathscr{V}^+$ with the following **shuffle product**:

$$\begin{aligned} R(z_{i1},\cdots,z_{in_i})_{i\in I} * R'(z_{i1},\cdots,z_{in'_i})_{i\in I} =& \mathrm{Sym}[\frac{R(z_{i1},\cdots,z_{in_i})_{i\in I}R'(z_{i,n_i+1},\cdots,z_{i,n_i+n'_i})_{i\in I}}{\mathbf{n}!\mathbf{n}'!}\times \\ &\times\prod_{i,j\in I}\prod_{a=1}^{n_i}\prod_{b=n_j+1}^{n_j+n'_j}\zeta_{ij}(z_{ia}-z_{jb})] \end{aligned} \tag{3.3}$$

We consider the subalgebra $\mathscr{A}^+ \subset \mathscr{V}^+$ generated by:

$$\{z^d_{i1}\}_{i\in I,d\geq 0} \tag{3.4}$$

In this setting, we define the positive affine Yangian $Y^+_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ as $\mathscr{A}^+$. The generators z^d_{i1} is denoted as $E^+_{i,d}$.

3.1. **Affine Yangians.** In this subsection we define the affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ by means of the additive shuffle algebra. Most of the details can be found in [BT19].

The affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ is defined as a vector space:

$$Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n) := \mathcal{S}^+ \otimes \mathbb{Q}[\xi_{i,n}]_{n\geq 0} \otimes \mathcal{S}^- \tag{3.5}$$

Here $\mathcal{S}^- := (\mathcal{S}^+)^{op}$. The relation here is defined as follows:

$$p_{ij}(z,\sigma_j^+)\xi_i(z)E_{j,s}^+ = -p_{ji}(\sigma_j^+,z)E_{j,s}^+\xi_i(z) \tag{3.6}$$

$$p_{ji}(\sigma_j^-,z)\xi_i(z)E_{j,s}^- = -p_{ij}(z,\sigma_j^-)E_{j,s}^-\xi_i(z) \tag{3.7}$$

Here $p_{ij}(z,w)$ is defined as follows:

$$p_{ij}(z,w) = \begin{cases} z-w-\hbar_1 & i=j-1 \\ z-w-\hbar_2 & i=j+1 \\ (w-z)(w-z-\hbar_1-\hbar_2) & i=j \end{cases} \tag{3.8}$$

$\sigma_i^\pm : Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)^\pm \to Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)^\pm$ is the homomorphism via sending $\xi_{j,r} \mapsto \xi_{j,r}$, $E_{j,r}^\pm \mapsto E_{j,r+\delta_{ij}}^\pm$

$$[E_i^+(z), E_j^-(w)] = -\hbar_2\delta_{ij}\frac{\xi_i(z)-\xi_j(w)}{z-w} \tag{3.9}$$

As well as the relation in $\mathcal{S}^+$ and $\mathcal{S}^-$ given by the shuffle algebra structure.

3.1.1. *Lie algebra.* The affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ contains the Lie algebra $\hat{\mathfrak{sl}}_n$ such that:

$$\hat{\mathfrak{sl}}_n = (\text{Trace zero } n\times n \text{ matrices with value in } \mathbb{C}[z^{\pm 1}]) \oplus \mathbb{C}\cdot\gamma \tag{3.10}$$

with γ being central in $\hat{\mathfrak{sl}}_n$ and the Lie bracket is written as:

$$[Xz^k, Yz^l] = [X,Y]z^{k+l} + \delta_{k+l}^0 k\cdot \mathrm{Tr}(XY)\gamma \tag{3.11}$$

We use the following way to express the generator of the affine Lie algebra $\hat{\mathfrak{sl}}_n$. Introduce the notation:

$$E_{[i,j)} = E_{i \bmod n, j \bmod n}\cdot z^{[\frac{j-1}{n}]-[\frac{i-1}{n}]} \tag{3.12}$$

$$E_{-[i,j)} = E_{j \bmod n, i \bmod n}\cdot z^{[\frac{j-1}{n}]-[\frac{i-1}{n}]} \tag{3.13}$$

3.2. **Action on the equivariant cohomology of quiver varieties.** The algebra action of $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ on the equivariant cohomology of quiver varieties $H_T^*(M(\mathbf{w}))$ can be described as the degeneration limit of the quantum toroidal algebra action $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ on the equivariant K-theory of quiver varieties $K_T^*(M(\mathbf{w}))$.

$$\langle\boldsymbol{\lambda}|F|\boldsymbol{\mu}\rangle = F(\chi^{ch}_{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\mu}}\omega(\chi_{\blacksquare}-\chi_{\square})\prod_{i=1}^{\mathbf{w}}(v_i-\hbar_1-\hbar_2-\chi_{\blacksquare})] \tag{3.14}$$

Here $|\boldsymbol{\lambda}\rangle$ is the fixed-point basis in the localised equivariant cohomology $H_T^*(\mathbf{M}(\mathbf{w})) := \bigoplus_{\mathbf{v}} H_T^*(\mathbf{M}(\mathbf{v},\mathbf{w}))_{loc}$ of the affine type A quiver varieties. Similarly, the action of the Cartan current $\xi(z)$ is given by the multiplication of the tautological class:

$$\xi(z) = (\overline{\frac{\omega(z-X)}{\omega(X-z)}})^+\prod_{1\leq j\leq\mathbf{w}}^{u_j\equiv i}\frac{a_j-\hbar-z}{z-\hbar-a_j} \tag{3.15}$$

Here $(\overline{\frac{\omega(z-X)}{\omega(X-z)}})^+$ means expanding the rational function of z in terms of the non-negative power of z.

3.3. **Generators of the affine Yangian.** The affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ has a minimal set of generators similar to the situation as that of the quantum toroidal algebra.

Proposition 3.1. *The affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ is generated by*

$$\{E^+_{i,0},E^-_{i,0},\xi_{i,0},\xi_{i,1}\} \tag{3.16}$$

Proof. This is proved by using the formula 3.5 and 3.6. □

From the proposition one can see that the affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\hat{\mathfrak{sl}}}_n)$ contains the affine Lie algebra $\hat{\mathfrak{sl}}_n$.

3.3.1. *Coproducts on the affine Yangians.* The Drinfeld coproduct on $Y_{\hbar_1,\hbar_2}(\hat{\hat{\mathfrak{sl}}}_n)$ is given by:

$$\begin{aligned}&\Delta(E_i(z)) = E_i(z)\otimes 1+\xi_i(z)\otimes E_i(z),\qquad \Delta(F_i(z)) = F_i(z)\otimes\xi_i(z)+1\otimes F_i(z)\\&\Delta(\xi_i^{\pm}(z)) = \xi_i(z)\otimes\xi_i(z)\end{aligned} \tag{3.17}$$

The standard coproduct can be defined as follows. Set the reduced classical r-matrix of $\hat{\mathfrak{sl}}_n$:

$$r = \sum_{\beta\in\Phi_+}x^-_{\beta,0}\otimes x^+_{\beta,0} \tag{3.18}$$

where $x^{\pm}_{\beta,0} \in \mathfrak{g}_{\pm\beta}$ are root vectors in $\hat{\mathfrak{sl}}_n$ such that $(x^{-}_{\beta,0}, x^{-}_{\beta,0}) = 1$.

Here

Denote $t_{i,1} = \xi_{i,1} - \frac{1}{2}\xi_{i,0}^2$

The standard coproduct $\Delta : Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n) \to Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n) \otimes Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$ is defined via the following:

$$\Delta(\xi_{i,0}) = \xi_{i,0} \otimes 1 + 1 \otimes \xi_{i,0} \tag{3.19}$$

$$\Delta(E^{\pm}_{i,0}) = E^{\pm}_{i,0} \otimes 1 + 1 \otimes E^{\pm}_{i,0} \tag{3.20}$$

$$\Delta(t_{i,1}) = t_{i,1} \otimes 1 + 1 \otimes t_{i,1} + \mathrm{ad}(\xi_{i,0} \otimes 1)r \tag{3.21}$$

4. Degeneration of the quantum toroidal algebra to affine Yangians

In this section we describe the degeneration of the quantum toroidal algebra to the affine Yangians.

First let us explain what does it mean when we say the word "degeneration".

We first need to use the following fact:

Proposition 4.1. *There is an algebra embedding:*

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \hookrightarrow \prod_{\mathbf{w}} End(K(\mathbf{w})) \tag{4.1}$$

given by the matrix coefficients of the generators of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$.

Proof. See [N23] and [Z23]. □

Note that $K(\mathbf{w})$ is the $K_{T_{\mathbf{w}}}(pt)_{loc} \cong \mathbb{Q}(q,t,u_1,\cdots,u_{\mathbf{w}})$-module. Using the Chern character map:

$$\begin{aligned} &ch : K_{T_{\mathbf{w}}}(pt) \to \widehat{H_{T_{\mathbf{w}}}(pt)} \\ &q_i \mapsto e^{\kappa\hbar_i}, \qquad u_i \mapsto e^{\kappa z_i} \end{aligned} \tag{4.2}$$

It induces the following map:

$$ch : K_{T_{\mathbf{w}}}(M(\mathbf{w})) \to \widehat{H_{T_{\mathbf{w}}}(M(\mathbf{w}))} \tag{4.3}$$

such that the $\widehat{H_{T_{\mathbf{w}}}(pt)}$ module $\widehat{H_{T_{\mathbf{w}}}(M(\mathbf{w}))}$ is spanned by the fixed point basis with coefficients in $\widehat{H_{T_{\mathbf{w}}}(pt)}$.

Now we turn to the definition of the asymptotic expansion of elements in $\widehat{H_{T_{\mathbf{w}}}(M(\mathbf{w}))}$. Given $f \in Im(ch)$, now we do the Laurent expansion of f with respect to κ. We denote f^{ch} as the lowest order component of f with respect to κ if it exists.

The following fact is easy to prove:

Lemma 4.2. *For arbitrary $f \in Im(ch)$, f^{ch} exists.*

Proof. Given $f \in \mathrm{Im}(ch)$, it means that f can be expressed as:

$$f(e^{\kappa z_i}, e^{\kappa\hbar_i}) = \frac{P(e^{\kappa z_i}, e^{\kappa\hbar_i})}{Q(e^{\kappa z_i}, e^{\kappa\hbar_i})} \tag{4.4}$$

with $P(e^{\kappa z_i}, e^{\kappa\hbar_i})$ and $Q(e^{\kappa z_i}, e^{\kappa\hbar_i})$ are polynomials in the variables $e^{\kappa z_i}$ and $e^{\kappa\hbar_i}$. Thus taking the cohomological limit f^{ch}, it truly exists. □

4.1. **Degeneration of the quantum toroidal algebra to the affine Yangian.** Now we turn to the case of the degeneration of the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ to the affine Yangian $Y_{t_1,t_2}(\hat{\mathfrak{sl}}_n)$.

We use the following fact about the minimal generators for the algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ and $Y_{t_1,t_2}(\hat{\mathfrak{sl}}_n)$. The generators of $Y_{t_1,t_2}(\hat{\mathfrak{sl}}_n)$ has been known in the Proposition 3.1. The following proposition can be found in many references, e.g.[N23].

Proposition 4.3. *The quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ is generated by*

$$\{e^+_{i,0}, e^-_{i,0}, \varphi^\pm_{i,0}, \varphi^\pm_{i,\pm1}\} \tag{4.5}$$

with the central elements.

Similarly, the affine Yangian $Y_{t_1,t_2}(\hat{\mathfrak{sl}}_n)$ is generated by $\{e^+_{i,0}, e^-_{i,0}, H^\pm_{i,0}, H_{i,\pm1}\}$ with the central elements.

Also the relations with the MO quantum affine algebra and the MO affine Yangian in [N23] and [SV17]:

Theorem 4.4. *There are algebra embeddings:*

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n) \hookrightarrow U_q^{MO}(\hat{\mathfrak{g}}_Q), \qquad Y_{t_1,t_2}(\hat{\mathfrak{sl}}_n) \hookrightarrow Y_\hbar^{MO}(\mathfrak{g}_Q) \tag{4.6}$$

which is compatible with the embedding.

The following proposition will be vital in the degeneration of the quantum difference equation to the quantum differential equation:

Proposition 4.5. *The degeneration limit of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ lies in $Y_{t_1,t_2}(\hat{\mathfrak{sl}}_n)$.*

Proof. This is proved by comparing the degeneration limit of the matrix coefficients of the elements of $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ with the matrix coefficienst of the elements of $Y_{t_1,t_2}(\hat{\mathfrak{sl}}_n)$. □

5. REVIEW OF THE QUANTUM DIFFERENCE EQUATION

In this subsection we give a review of the construction of the quantum difference equation in [Z23] for the quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ and quiver varieties of affine type A.

The quantum toroidal algebra $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ admits the slope factorization:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)=\bigotimes_{\mu\in\mathbb{Q}}^{\rightarrow}\mathscr{B}_{\mathbf{m}+\mu\boldsymbol{\theta}} \tag{5.1}$$

For each slope subalgebra $\mathscr{B}_{\mathbf{m}}$, one can associate an element $J^{\pm}_{\mathbf{m}}(\lambda)\in\mathscr{B}_{\mathbf{m}}\hat{\otimes}\mathscr{B}_{\mathbf{m}}$ such that they satisfy the ABRR equation:

$$J^{+}_{\mathbf{m}}(\lambda)q^{-\lambda}_{(1)}q^{\Omega}R^{+}_{\mathbf{m}}=q^{-\lambda}_{(1)}q^{\Omega}J^{+}_{\mathbf{m}}(\lambda),\qquad q^{\Omega}R^{-}_{\mathbf{m}}q^{-\lambda}_{(1)}J^{-}_{\mathbf{m}}(\lambda)=J^{-}_{\mathbf{m}}(\lambda)q^{\Omega}q^{-\lambda}_{(1)} \tag{5.2}$$

5.1. **Asymptotics of $J_{\mathbf{m}}(z)$.** with $R^{-}_{\mathbf{m}}:=(R_{\mathbf{m}})_{21}$. The operator $J^{\pm}_{\mathbf{m}}$ satisfy the following relation:

$$S_{\mathbf{m}}\otimes S_{\mathbf{m}}((J^{+}_{\mathbf{m}}(\lambda))_{21})=J^{-}_{\mathbf{m}}(\lambda) \tag{5.3}$$

Lemma 5.1.

$$\lim_{z\to\infty}J^{+}_{\mathbf{m}}(z^{\theta})=1,\qquad \lim_{z\to 0}J^{+}_{\mathbf{m}}(z^{\theta})=R^{+}_{\mathbf{m}} \tag{5.4}$$

here θ is the parametre above in the root factorization of the quantum toroidal algebra. Here we require that $\boldsymbol{\theta}>0$

Proof. We shall give a proof by direct calcultation, which is useful in the following analysis of the difference equations.

We do the factorization of $J_{\mathbf{m}}$ and $R_{\mathbf{m}}$ with respect to the degree:

$$J_{\mathbf{m}}=1+\sum_{\mathbf{n}>0}J_{\mathbf{m}|\mathbf{n}},\qquad (R_{\mathbf{m}})^{-1}=1+\sum_{\mathbf{n}>0}R_{\mathbf{m}|\mathbf{n}} \tag{5.5}$$

Using the ABRR equation we have the following recursion equations:

$$J_{\mathbf{m}|\mathbf{n}}(z)=\frac{1}{z^{\mathbf{n}}q^{k}-1}\sum_{\substack{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}\\ \mathbf{n}_1<\mathbf{n}}}J_{\mathbf{m}|\mathbf{n}_1}(z)R_{\mathbf{m}|\mathbf{n}_2} \tag{5.6}$$

Now we take $(z_1,\cdots,z_n)=(z^{\theta_1},\cdots,z^{\theta_n})$. As $z\to\infty$, it is obvious that $J_{\mathbf{m}|\mathbf{n}}=0$.

If $z\to 0$, we have the following expression for $J_{\mathbf{m}|\mathbf{n}}(0^{\theta})$:

$$J_{\mathbf{m}|\mathbf{n}}(0^{\theta})=\sum_{k}\sum_{\mathbf{n}_1+\cdots+\mathbf{n}_k=\mathbf{n}}(-1)^{k}R_{\mathbf{m}|\mathbf{n}_1}R_{\mathbf{m}|\mathbf{n}_2}\cdots R_{\mathbf{m}|\mathbf{n}_k} \tag{5.7}$$

Thus we have that:

$$\begin{aligned} J_{\mathbf{m}}(0^{\theta}) =&1+\sum_{k}\sum_{\mathbf{n}_1,\cdots,\mathbf{n}_k}(-1)^k R_{\mathbf{m}|\mathbf{n}_1}R_{\mathbf{m}|\mathbf{n}_2}\cdots R_{\mathbf{m}|\mathbf{n}_k} \\ =&(1+\sum_{\mathbf{n}>0}R_{\mathbf{m}})^{-1} \\ =&R_{\mathbf{m}} \end{aligned} \tag{5.8}$$

□

Now we turn to the analysis of the asymptotics of the following case:

$$\lim_{q\to 0}J_{\mathbf{m}}(zq^s),\qquad zq^s=(z_1q^{s_1},\cdots z_nq^{s_n}) \tag{5.9}$$

We can see that the lemma 5.1 corresponds to the case $s>0$ and $s<0$, which corresponds to the bunch of hyperplanes $s_i=0$ with $i=1,\cdots,n$.

The following proposition reveals that for all $\mathbf{m}\in\mathbb{Q}^n$, $\lim_{p\to 0}J_{\mathbf{m}}(zp^s)$ is a locally constant function on s:

Proposition 5.2. *$\lim_{p\to 0}J_{\mathbf{m}}(zp^s)$ is locally constant on the variable $s\in\mathbb{R}^n$, and it jumps as s crosses the following hyperplanes:*

$$\mathbf{n}\cdot s+k=0,\qquad \mathbf{n}\in(\mathbb{Z}_{\geq 0})^n-\{0\} \tag{5.10}$$

Proof. The ABRR equation now can be re-written as:

$$J_{\mathbf{m}|\mathbf{n}}(zq^s)=\frac{1}{z^{\mathbf{n}}q^{\mathbf{n}\cdot s}q^k-1}\sum_{\substack{\mathbf{n}_1+\mathbf{n}_2=\mathbf{n}\\ \mathbf{n}_1<\mathbf{n}}}J_{\mathbf{m}|\mathbf{n}_1}(zq^s)R_{\mathbf{m}|\mathbf{n}_2} \tag{5.11}$$

So as $q\to 0$, if $\mathbf{n}\cdot s>0$, $\lim_{p\to 0}J_{\mathbf{m}|\mathbf{n}}(zp^s)$ is nonzero, and if $\mathbf{n}\cdot s<0$, $\lim_{p\to 0}J_{\mathbf{m}|\mathbf{n}}(zp^s)=0$. Thus we can see that □

Using this, the monodromy operator is defined as:

$$\mathbf{B}_{\mathbf{m}}(\lambda)=m(1\otimes S_{\mathbf{m}}(\mathbf{J}_{\mathbf{m}}^{-}(\lambda)^{-1}))|_{\lambda\to\lambda+\kappa} \tag{5.12}$$

Here $\kappa=\frac{C\mathbf{v}-\mathbf{w}}{2}$.

Let $\mathscr{L}\in Pic(X)$ be a line bundle. Now we fix a slope $s\in H^2(X,\mathbb{R})$ and choose a path in $H^2(X,\mathbb{R})$ from s to $s-\mathscr{L}$. This path crosses finitely many slope points in some order $\{\mathbf{m}_1,\mathbf{m}_2,\cdots,\mathbf{m}_m\}$. And for this choice of a slope, line bundle and a path we associate the following operator:

$$\mathbf{B}^s_{\mathscr{L}}(\lambda)=\mathscr{L}\mathbf{B}_{\mathbf{m}_m}(\lambda)\cdots\mathbf{B}_{\mathbf{m}_1}(\lambda) \tag{5.13}$$

We define the q-difference operators:

$$\mathscr{A}^s_{\mathscr{L}}=T_{\mathscr{L}}^{-1}\mathbf{B}^s_{\mathscr{L}}(\lambda) \tag{5.14}$$

It has been proved in [Z23] that the q-difference operator $\mathscr{A}^s_{\mathscr{L}}$ is independent of the choice of the path from s to $s-\mathscr{L}$ if we choose the generic path.

The well-definedness and the rigidity of the definition of the q-difference operators $\mathscr{A}^s_{\mathscr{L}}$ lies in the following lemma:

Lemma 5.3. *For arbitrary point s in $Pic(M(\mathbf{v},\mathbf{w}))\otimes\mathbb{R}$, there exists an small arc neighborhood V_s of s such that for every point $s'\in V_s$:*

$$\mathscr{A}^{s'}_{\mathscr{L}}=\mathscr{A}^s_{\mathscr{L}} \tag{5.15}$$

Proof. Note that for arbitrary path from s to $s-\mathscr{L}$, there are only finitely many slope points $\{\mathbf{m}_1,\cdots,\mathbf{m}_n\}$ on the path giving the nontrivial action on $K_T(M(\mathbf{v},\mathbf{w}))$. This means that the small change of s would not affect the q-difference operator $\mathscr{A}^s_{\mathscr{L}}$. □

In our settings, we always choose the slope points s to be generic, i.e. s not lies on the wall. Note that this means that the q-difference operator $\mathscr{A}^s_{\mathscr{L}}$ is locally constant on s, thus we always fix s in a generic position, i.e. $\mathbf{B}_s(\lambda)=1$ on $K_T(M(\mathbf{v},\mathbf{w}))$. We will see that the generic choice of the starting point s would give us a convenient way to compute the quantum difference operator $\mathbf{B}^s_{\mathscr{L}}(\lambda)$.

5.2. **Wall structures over the affine type A quiver varieties.** In this subsection we analyze the wall structure of the affine type A quiver varieties and compare it with the wall structure defined via the K-theoretic stable envelope.

Recall the wall set of a quiver variety $M(\mathbf{v},\mathbf{w})$ is defined as:

(5.16)
$$\text{Walls}(M(\mathbf{v},\mathbf{w}))=\{\mathbf{m}\in\mathbb{Q}^r|\text{Reduced part of } R_{\mathbf{m}} \text{ acts on } K_T(M(\mathbf{v},\mathbf{w}))\otimes K_T(M(\mathbf{v},\mathbf{w})) \text{ trivially}\}$$

Also recall that the wall R-matrix in the K-theoretic stable envelope is defined as:

$$R_w:=\text{Stab}^{-1}_{\mathscr{C},s}\circ\text{Stab}_{\mathscr{C},s'} \tag{5.17}$$

Here the slope $s,s'\in\text{Pic}(X)\otimes\mathbb{Q}$ are separated by the wall hyperplane $w\subset\text{Pic}(X)\otimes\mathbb{Q}$. Or equivalently, the wall w is the set of points in $\text{Pic}(X)\otimes\mathbb{Q}$ such that the K-theoretic stable envelope $\text{Stab}_{\mathscr{C},s}$ would jump when s go across a point in the wall w. Here we determine the wall structure over the case of affine type A quiver varieties.

There are two ways to determine the wall structure, first by the Theorem 2 in [OS22], the wall R-matrices has the following expression:

$$R^+_w|_{F_2\times F_1}=\begin{cases}1 & F_1=F_2\\ O(u^{\mathscr{L}_w|_{F_2}-\mathscr{L}_w|_{F_1}}) & F_1\leq F_2\\ 0 & \text{otherwise}\end{cases} \tag{5.18}$$

and the $O(\cdots)$ part is nonzero if and only if $\mathscr{L}_w|_{F_2}-\mathscr{L}_w|_{F_1}$ is integral. This means that the wall consists of the point $\mathscr{L}\in\text{Pic}(X)\otimes\mathbb{Q}$ such that $\langle\mu(F_\alpha)-\mu(F_\beta),\mathscr{L}\rangle\in\mathbb{Z}$. In this case of the affine type A quiver varieties, these walls are equivalent to the following set of periodic hyperplane arrangements:

$$z_i+\cdots+z_{j-1}+n(z_1+\cdots+z_n)=m, \qquad n\geq 0, m\geq 0 \tag{5.19}$$

In this way we obtain the following proposition:

Proposition 5.4. *The wall set of the K-theoretic stable envelope of $M(\mathbf{v},\mathbf{w})$ coincides with the above set of period hyperplane arranegments defined by the equation 5.19.*

One of the key observation is that the wall set defined in the algebraic setting "coincide" with the one in the K-theoretic stable envelope settings.

Proposition 5.5. $\mathbf{m}\in$ *Walls*$(M(\mathbf{v},\mathbf{w}))$ *if and only if* $\mathbf{m}\in H_{\alpha}$ *where* $H_{\alpha}\in$ *Walls*$^{Stab}(M(\mathbf{v},\mathbf{w}))$.

Proof. If $\mathbf{m}\in\text{Walls}(M(\mathbf{v},\mathbf{w}))$, this means that $\mathbf{m}$ lies in the set of hyperplane arrangements defined in 5.19, and vice versa.

□

In the Proposition 2.13 in [Z23] we proved the following fact of the connection between the slope subalgebra $\mathcal{B}_{\mathbf{m}}$ and the subalgebra $U_q^{MO}(\mathfrak{g}_{\mathbf{m}})$ generated by the matrix coefficients of the wall R-matrix $R^{\pm}_{\mathbf{m},\mathbf{m}+\epsilon\boldsymbol{\theta}}$, i.e. It is the algebra generated by the wall subalgebra $U_q^{MO}(\mathfrak{g}_w)$ such that each wall w contains the slope point $\mathbf{m}$:

Proposition 5.6. *There is a natural Hopf algebra embedding* $\mathcal{B}_{\mathbf{m}}\hookrightarrow U_q^{MO}(\mathfrak{g}_{\mathbf{m}})$.

This implies that the roots of $\mathcal{B}_{\mathbf{m}}$ has the same as those of $U_q^{MO}(\mathfrak{g}_{\mathbf{m}})$.

6. Analysis of the quantum difference equations

Fix the affine type A quiver variety $M(\mathbf{v},\mathbf{w})$. In this section we will fix the slope s of the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z):=\mathbf{B}^s_{\mathscr{L}}(z)$ as defined in 5.13. We assume that s will be really generic, i.e. s will not lie on the wall of $M(\mathbf{v},\mathbf{w})$.

6.1. Possible singularities for the monodromy operators.

The explicit formula for the monodromy operators has been computed in [Z23]:

Theorem 6.1. *For the monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda) \in \mathscr{B}_{\mathbf{m}}$, where $\mathbf{B}_{\mathbf{m}}(\lambda)$ is defined in 5.13 and $\mathscr{B}_{\mathbf{m}}$ is defined as 2.42, its representation in $End(K_T(M(\mathbf{v},\mathbf{w})))$ is given by:*

(6.1)

$$
\begin{aligned}
&\mathbf{B}_{\mathbf{m}}(\lambda)\\
=&\prod_{h=1}^{g} :(\exp(-\sum_{k=1}^{\infty}\frac{n_k q^{-\frac{k|\boldsymbol{\delta}_{\boldsymbol{h}}|}{2}}}{1-z^{-k|\boldsymbol{\delta}_{\boldsymbol{h}}|}p^{k\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-\frac{k|\boldsymbol{\delta}_{\boldsymbol{h}}|}{2}}}\alpha^{\mathbf{m},h}_{-k}\alpha^{\mathbf{m},h}_{k})\prod_{k=0}^{\overrightarrow{\infty}}\\
&\times\prod^{\leftarrow}_{\substack{\gamma\in\Delta(A)\\ m\geq 0}}(\exp_{q^2}(-(q-q^{-1})z^{k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}q^{k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}\\
&f_{(\delta-\gamma)+m\delta}e'_{(\delta-\gamma)+m\delta})\exp(-(q-q^{-1})\sum_{m\in\mathbb{Z}_+}\sum_{i,j=1}^{l_h}z^{km\boldsymbol{\delta}_h}p^{km\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-km\boldsymbol{\delta}_h^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2}u_{m,ij}f_{m\delta,\alpha_i}e'_{m\delta,\alpha_i})\\
&\times\prod^{\rightarrow}_{\substack{\gamma\in\Delta(A)\\ m\geq 0}}\exp_{q^2}(-(q-q^{-1})z^{k(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)}q^{-k(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}f_{\gamma+m\delta}e'_{\gamma+m\delta})):
\end{aligned}
$$

and $e'_v = S(e_v)$, $f'_v = S(f_v)$ are the image of the antipode map.

Though the expression for the quantum difference operator is really complicated, we can still extract out the possible singularity of the quantum difference operator.

Proposition 6.2. *The monodromy operator $\mathbf{B}_{\mathbf{m}}(\lambda)$ has possible singularities at*

$$
\begin{aligned}
&z^{-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h}p^{\mathbf{m}\cdot(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}q^{(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}=1\\
&z^{m\boldsymbol{\delta}_h}p^{m\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-m\boldsymbol{\delta}_h^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2}=1\\
&z^{(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)}p^{\mathbf{m}\cdot(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)}q^{-(\mathbf{v}_\gamma+m\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}=1\\
&z^{-k|\boldsymbol{\delta}_{\boldsymbol{h}}|}p^{k\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-\frac{k|\boldsymbol{\delta}_{\boldsymbol{h}}|}{2}}=1
\end{aligned}
\tag{6.2}
$$

Moreover, these singularities are of the regular singularities.

Before the proof, I shall mention that we can see that the singularities of the monodromy operator gives the toric arrangement in the Picard torus $\mathrm{Pic}(X)\otimes\mathbb{C}^\times$. This means that the singularity structure is more complicated than the case of the hyperplane arrangements.

Proof. For simplicity, we only prove the statement for the first formula. Doing the expansion of the formula and we have:

(6.3)
$$
\begin{aligned}
&:\prod_{k=0}^{\rightarrow\infty}\prod^{\leftarrow}_{\substack{\gamma\in\Delta(A)\\ m\geq 0}}(\exp_{q^2}(-(q-q^{-1})z^{k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}p^{k\mathbf{m}\cdot(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}q^{k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}\\
&\times f_{(\delta-\gamma)+m\delta}e'_{(\delta-\gamma)+m\delta}):\\
=&\prod_{k=0}^{\overrightarrow{\infty}}(\sum_{\substack{l_{\gamma,m}=0\\ \gamma\in\Delta(A)\\ m\geq 0}}\frac{(-1)^{\sum_{\gamma,m}l_{\gamma,m}}(q-q^{-1})^{\sum_{\gamma,m}l_{\gamma,m}}}{\prod_{\gamma,m}[l_{\gamma,m}]_{q^2}!}z^{\sum_{\gamma,m}l_{\gamma,m}k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}p^{\sum_{\gamma,m}l_{\gamma,m}k\mathbf{m}\cdot(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}\\
&\times q^{\sum_{\gamma,m}l_{\gamma,m}k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}\prod_{\gamma,m}f^{l_\gamma}_{(\delta-\gamma)+m\delta}e'^{l_{\gamma,m}}_{(\delta-\gamma)+m\delta})\\
=&\sum_{k=0}^{\infty}(\sum_{\substack{l_{\gamma,m}=0\\ \gamma\in\Delta(A)\\ m\geq 0}}\frac{(-1)^{\sum_{\gamma,m}l_{\gamma,m}}(q-q^{-1})^{\sum_{\gamma,m}l_{\gamma,m}}p(\gamma,m,n)}{\prod_{\gamma,m}[l_{\gamma,m}]_{q^2}!}z^{\sum_{\gamma,m}l_{\gamma,m}k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}p^{\sum_{\gamma,m}l_{\gamma,m}k\mathbf{m}\cdot(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)}\\
&\times q^{\sum_{\gamma,m}l_{\gamma,m}k(-\mathbf{v}_\gamma+(m+1)\boldsymbol{\delta}_h)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}\prod_{\gamma,m}f^{l_\gamma}_{(\delta-\gamma)+m\delta}e'^{l_{\gamma,m}}_{(\delta-\gamma)+m\delta})
\end{aligned}
$$

$p(\gamma,m,n)$ is the number of partition of n, it is easy to check that this gives the convergence radius of 1, and thus we finish the proof. □

6.2. **Good representation for the quantum difference operators.** In this section we shall show that for arbitrary quantum difference operators $\mathbf{M}_{\mathscr{L}}(z)$, it can be expressed as the ordered product of the monodromy operators $\mathbf{B}_{\mathbf{m}}(z)$ such that $\mathbf{B}_{\mathbf{m}}(z)$ can be represented via either $U_q(\mathfrak{sl}_2)$-type or $U_q(\hat{\mathfrak{gl}}_1)$-type.

Theorem 6.3. *For the generic path $[-s-\mathscr{L},-s)$ between $-s-\mathscr{L}$ and $-s$ of the quantum difference operator:*

$$
\mathbf{M}_{\mathscr{L}}(z)=\mathscr{L}\prod^{\rightarrow}_{\mathbf{m}\in Walls}\mathbf{B}_{\mathbf{m}}(z) \tag{6.4}
$$

with s sufficiently small. Then each $\mathbf{B}_{\mathbf{m}}(z)$ can be written either in one of the following form:

- *$U_q(\mathfrak{sl}_2)$ type:*

$$
\begin{aligned}
&\prod_{k=0}^{\overrightarrow{\infty}}\exp_{q^2}(-(q-q^{-1})z^{k(\mathbf{v}_\gamma)}p^{k\mathbf{m}\cdot(\mathbf{v}_\gamma)}q^{-k(\mathbf{v}_\gamma)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}}f_\gamma e'_\gamma))\\
=&\sum_{n=0}^{\infty}\frac{(q-q^{-1})^n}{[n]_{q^2}!}\frac{(-1)^n}{\prod_{\nu=1}^n(1-z^{\mathbf{v}_\gamma}p^{\mathbf{m}\cdot\mathbf{v}_\gamma}q^{-\nu(\mathbf{v}_\gamma)^T((\frac{n^2r-1}{2})\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1\gamma}})}f^n_\gamma e'^n_\gamma
\end{aligned} \tag{6.5}
$$

with

$$f_\gamma = P^{\mathbf{m}}_{[i,j)}, \qquad e'_\gamma = S_{\mathbf{m}}(Q^{\mathbf{m}}_{-[i,j)}) \tag{6.6}$$

- $U_q(\hat{\mathfrak{gl}}_1)$ *type.*

$$\mathbf{m}(\prod_{h=1}^{g}(\exp(-\sum_{k=1}^{\infty}\frac{n_k q^{-\frac{k|\boldsymbol{\delta}_h|}{2}}}{1-z^{-k|\boldsymbol{\delta}_h|}p^{k\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-\frac{k|\boldsymbol{\delta}_h|}{2}}}\alpha^{\mathbf{m},h}_{-k}\otimes\alpha^{\mathbf{m},h}_{k}) \tag{6.7}$$

with

$$\alpha^{\mathbf{m},h}_{-k} = P^{\mathbf{m}}_{[h,h+k\delta_{\mathbf{m}})}, \qquad \alpha^{\mathbf{m},h}_{k} = S_{\mathbf{m}}(Q^{\mathbf{m}}_{-[h,h+k\delta_{\mathbf{m}})}) \tag{6.8}$$

Proof. The construction of the path can be given as follows. Fix the quiver variety $M(\mathbf{v},\mathbf{w})$ and the slope subalgebra $\mathscr{B}_{\mathbf{m}}$. Recall that the generators of $\mathscr{B}_{\mathbf{m}}$ is given by $P^{\mathbf{m}}_{[i;i+1)_h}$. All we need to do is to find a slope $\mathbf{m}\mathbb{Q}^r$ such that $[i;i+1)_h$ exceeds the quiver dimension vector $\mathbf{v}=(n,\cdots,n)$ except for only one generators $P^{\mathbf{m}}_{[j;j+1)_h}$ for some j.

To carry out the procedure, recall that $\mathbf{m}\cdot[i;i+1)_h$ satisfying the above conditions can be written as:

$$(n-1)\mathbf{m}\cdot\boldsymbol{\theta}+m_{i+1}+\cdots+m_k\in\mathbb{Z} \tag{6.9}$$

Thus we define the following set of hyperplanes in $\mathbb{Q}^r$:

$$\mathbf{v}\in\mathbb{Q}^r, \qquad [i;j)\cdot\mathbf{v}\in\mathbb{Z} \tag{6.10}$$

We require that for the integer vector $[i;j)=n_1e_1+n_2e_2+\cdots+n_re_r$, $n_i\leq v_i$. Here $\mathbf{v}=(v_1,\cdots,v_r)$. The following lemma is easy to prove:

Lemma 6.4. *The monodromy operator $\mathbf{B}_{\mathbf{m}}(z)$ acts on $K_T(M(\mathbf{v},\mathbf{w}))$ nontrivially if $[i,j)\cdot\mathbf{m}\in\mathbb{Z}$ for some $[i,j)$ satisfies the condition above. Moreover, if $[i,j)\cdot\mathbf{m}\in\mathbb{Z}$, $P^{\mathbf{m}}_{[i,j)}$ acts on $K_T(M(\mathbf{v},\mathbf{w}))$ non-trivially.*

For these hyperplanes $H_{[i,j),n}$, choose $\mathbf{m}\in\mathbb{Q}^r$ such that $\mathbf{m}$ intersects with only one hyperplane $H_{[i,j),n}$. It is easy to see that the point $\mathbf{m}$ such that intersects with only one hyperplane is dense. Therefore for each hyperplane , one can choose a generic point $\mathbf{m}$ for each hyperplane and connect them together.

Then we determine the type of the monodromy operator $\mathbf{B}_{\mathbf{m}}(z)$ at each slope points $\mathbf{m}$ with the corresponding intersecting hyperplane $H_{[i,j),n}$. If $[i,j)=n\boldsymbol{\theta}$, $\mathscr{B}_{\mathbf{m}}\cong U_q(\hat{\mathfrak{gl}}_1)^{\otimes r}$. In this case the monodromy operator $\mathbf{B}_{\mathbf{m}}$ can be written as:

$$\mathbf{B}_{\mathbf{m}}(z)=(\textbf{Heisenberg type}) \tag{6.11}$$

as in 6.5. Here $[i,j)$ is chosen such that $\mathbf{m}\cdot[i,j)\in\mathbb{Z}$.

Otherwise,

$$\mathbf{B}_{\mathbf{m}}(z)=(U_q(\mathfrak{sl}_2)\ \mathbf{type}) \tag{6.12}$$

as in 6.7. Here $\delta_{\mathbf{m}}$ is the minimal vector $[i,j)$ such that $\delta_{\mathbf{m}}\cdot\mathbf{m}\in\mathbb{Z}$. □

Remark. The theorem implies that for the generic points $\mathbf{m}\in\mathbb{Q}^r$ on the wall $[i,j)\cdot\mathbf{m}\in\mathbb{Z}$, the corresponding monodromy operator $\mathbf{B}_{\mathbf{m}}(z)$ is independent of the choice of the generic points $\mathbf{m}$. This agrees with the monodromy operator $\mathbf{B}_w(\lambda)$ defined by Okounkov and Smirnov in [OS22] where the monodromy operator $\mathbf{B}_w(\lambda)$ is defined only on the wall w for the K-theoretic stable envelope.

6.3. **General solution for the quantum difference equations.** In this section we use the good representation to give the fundamental solution of the quantum difference equation of the affine type A quiver varieties $M(\mathbf{v},\mathbf{w})$.

For simplicity, we denote $X=M(\mathbf{v},\mathbf{w})$. It is well-known that $\mathrm{Pic}(X)$ is generated by its tautological line bundles $\mathscr{L}_i$, and we denote $\mathscr{O}(1)=\mathscr{L}_1\cdots\mathscr{L}_n$. In this way the quantum difference equation at direction $\mathscr{O}(1)$ can be written as:

$$\Psi(p^{\mathscr{O}(1)}z)=\Psi(pz_1,\cdots,pz_n)=\mathbf{M}_{\mathscr{O}(1)}(z)\Psi(z),\qquad z\in\mathrm{Pic}(X)\otimes\mathbb{C}^* \tag{6.13}$$

For simplicity we only concentrate on the quantum difference equation for $\mathscr{O}(1)$. For the quantum difference equation at other directions, i.e. $\mathbf{M}_{\mathscr{L}}(z)$ with $\mathscr{L}$ other than $\mathscr{O}(1)$, the analysis is similar.

6.3.1. *Fundamental solution of QDE near $z=0$.* First note that at $z=0\in\mathrm{Pic}(X)\otimes\mathbb{C}^\times$, i.e. the corresponding chamber in $\mathrm{Pic}(X)\otimes\mathbb{R}$ corresponds to the chamber such that all the parametres $a_1,\cdots,a_n$ goes to $-\infty$. Let $M(0)=M_n(0)M_{n-1}(0)\cdots M_1(0)=\mathscr{L}_n\otimes\cdots\otimes\mathscr{L}_1$ and this matrix is diagonal in $K_T(X)$ with respect to the fixed point basis. And let P be the matrix with columns given by fixed point eigenvectors $[\lambda]$. We denote by E_0 the diagonal matrix of eigenvalues, so that:

$$M(0)P=PE_0 \tag{6.14}$$

And the eigenvalue diagonal matrix E_0 is $\det(\mathscr{V}_1)|_\lambda\cdots\det(\mathscr{V}_n)|_\lambda\in K_T(\mathrm{pt})_{loc}$. In this way we can decompose the solution into the singular part and the regular part. There are two choices of the solution, the first is the elliptic type solutions:

$$\Psi_0^{ell}(z)=\Psi_0^{reg}(z)\prod_{i=1}^n(\Theta(E_0^{(i)},z_i)) \tag{6.15}$$

Here $E_0^{(i)}$ is the diagonal matrix with the value $\det(\mathscr{V}_i)|_\lambda$. And $\Theta(E_0^{(i)},z_i)$ is defined as:

$$\Theta(E_0^{(i)},z_i)=\mathrm{diag}(\frac{\theta(z_i,p)}{\theta(z_i\det(\mathscr{V}_i)^{-1}|_\lambda,p)})_\lambda \tag{6.16}$$

The second is the multiplicative type:

$$\Psi_0(z)=P\Psi_0^{reg}(z)\exp(\sum_i\frac{\ln(E_0^{(i)})\ln(z_i)}{\ln(p)}) \tag{6.17}$$

6.3.2. ***Computation of*** $\Psi_0^{reg}(z)$ ***via*** $\mathcal{O}(1)$. For the second procedure, we can just write down the regular part of the solution as:

$$\Psi_0^{reg}(z)=\overleftarrow{\prod_{k=0}^{\infty}}\mathbf{M}^*(p^kz)^{-1}=\overleftarrow{\prod_{k=0}^{\infty}}\mathbf{M}_k^*(z)^{-1} \tag{6.18}$$

with

$$\mathbf{M}^*(z)=E_0^{-1}P^{-1}\mathbf{M}_{\mathcal{O}(1)}(z)P,\mathbf{M}_k^*(z):=E_0^{-k-1}P^{-1}\mathbf{M}_{\mathcal{O}(1)}(p^kz)PE_0^k \tag{6.19}$$

Recall that the quantum difference operator $\mathbf{M}_{\mathcal{O}(1)}$ can be written as the product of $\mathbf{B}_{\mathbf{m}}$ such that $\mathbf{B}_{\mathbf{m}}$ are either of $U_q(\mathfrak{sl}_2)$-type or $U_q(\hat{\mathfrak{gl}}_1)$-type.

For example, if we take the path from s to $s-\mathcal{O}(1)$ as the straight interval from 0 to $-\mathcal{O}(1)$, the regular part of the fundamental solution around $z=0$ can also be written as:

$$\Psi_0^{reg}(z)=\overleftarrow{\prod_{\mathbf{m}\in(-\infty,s)}}\mathbf{B}_{\mathbf{m}}^*(z)^{-1} \tag{6.20}$$

Here $\mathbf{m}\in(-\infty,0)$ means that we fix the initial point s close to 0 and the end point $s-n\mathcal{O}(1)$, and we require that $n\to\infty$.

6.3.3. *Fundamental solution near* $z=\infty$. For the solution around $z=\infty$, by the Lemma 3.2 in [Z23], the operator $\mathbf{M}_{\mathcal{O}(1)}(\infty)$ consists of the product of the form $\mathbf{m}((1\otimes S_{\mathbf{m}})(R_{\mathbf{m}}^-)^{-1})$. It is a diagonalizable matrix over $\mathbb{Q}(q,t)$ with eigenvalues given by the rational functions of q,t. We shall denote the corresponding matrix of eigenvectors as H, and $\mathbf{E}_\infty$ be the digaonal matrix of eigenvalues:

$$\mathbf{M}_{\mathcal{O}(1)}(\infty)H=H\mathbf{E}_\infty \tag{6.21}$$

Now we do the decomposition of $\mathbf{E}_\infty$ in each direction z_i. Note that for each direction z_i we have the difference equation:

$$\Psi(p^{\mathcal{L}_i}z)=\mathbf{M}_{\mathcal{L}_i}(z)\Psi(z) \tag{6.22}$$

So equivalently, the difference equation on $\mathcal{O}(1)=\mathcal{L}_1\cdots\mathcal{L}_n$ can be written as:

$$\Psi(p^{\mathcal{O}(1)}z)=\mathbf{M}_{\mathcal{L}_1}(p^{\mathcal{O}(1)-\mathcal{L}_1}z)\mathbf{M}_{\mathcal{L}_2}(p^{\mathcal{O}(1)-\mathcal{L}_1-\mathcal{L}_2}z)\cdots\mathbf{M}_{\mathcal{L}_n}(z)\Psi(z) \tag{6.23}$$

In this way we can see that:

$$\mathbf{M}_{\mathcal{O}(1)}(\infty)=\mathbf{M}_{\mathcal{L}_1}(\infty)\cdots\mathbf{M}_{\mathcal{L}_n}(\infty) \tag{6.24}$$

Furthermore $\mathbf{M}_{\mathscr{L}_i}(\infty)\mathbf{M}_{\mathscr{L}_j}(\infty)=\mathbf{M}_{\mathscr{L}_j}(\infty)\mathbf{M}_{\mathscr{L}_i}(\infty)$, and this means that $\mathbf{M}_{\mathscr{L}_i}(\infty)$ share the same eigenvectors H with different eigenvalues $E^{(i)}_{\infty}$.

In this way we can first construct the elliptic solution as:

$$\Psi^{ell}_{\infty}=H\Psi^{reg}_{\infty}(z)\prod_{i}\Theta(E^{(i)}_{\infty},z_i) \tag{6.25}$$

the solution around $z=\infty$ can be written as:

$$\Psi_{\infty}(z)=H\Psi^{reg}_{\infty}(z)\exp(\sum_{i}\frac{\ln(E^{(i)}_{\infty})\ln(z_i)}{\ln(p)}) \tag{6.26}$$

such that

$$\Psi^{reg}_{\infty}(z)\mathbf{E}_{\infty}=H^{-1}\mathbf{M}_{\mathscr{O}(1)}(z)H\Psi^{reg}_{\infty}(z) \tag{6.27}$$

We use the similar procedure to compute the regular part of $\Psi^{reg}_{\infty}(z)$, and it is easy to see that the solution is of the form:

$$\Psi^{reg}_{\infty}(z)=\prod_{k=0}^{\overset{\rightarrow}{\infty}}\mathbf{M}^{*}_{-k}(z) \tag{6.28}$$

with

$$\mathbf{M}^{*}_{-k}(z)=\mathbf{E}^{k-1}_{\infty}\mathbf{M}^{*}(zp^{-k})\mathbf{E}^{-k}_{\infty} \tag{6.29}$$

Also we denote

$$\mathbf{B}^{*}_{\mathbf{m}}(z):=H^{-1}\mathbf{B}_{\mathbf{m}}(z)H,\qquad \mathbf{B}^{*}_{\mathbf{m}}=H^{-1}\mathbf{B}_{\mathbf{m}}H \tag{6.30}$$

Now the fundamental solution around $z=\infty$ can be written as:

$$\Psi^{reg}_{\infty}(z)=\lim_{N\to\infty}\Psi^{reg}_{N}(z)=\lim_{N\to\infty}(\overset{\rightarrow}{\prod_{\mathbf{m}\in[0,N)}}\mathbf{B}^{*}_{\mathbf{m}}(z))(\mathbf{M}_{\mathscr{O}(1)}(0)^{*})^{N}\mathbf{E}^{-N}_{\infty} \tag{6.31}$$

6.4. **Regular fundmental solution of different slopes.** Now given two different generic slopes $s,s'\in\mathbb{Q}^n$. We can connect the regular part of the fundamental solutions $\Psi^{reg,s}_{0}(z)$, $\Psi^{reg,s'}_{0}(z)$ at slopes s,s' by the monodromy operators.

Lemma 6.5. *Suppose that $\Psi^{reg,s}_{0}(z)$, $\Psi^{reg,s'}_{0}(z)$ are the regular fundamental solutions at $z=0$ for the quantum difference operators $\mathscr{A}^{s}_{\mathscr{L}}$, $\mathscr{A}^{s'}_{\mathscr{L}}$. Then,*

$$\Psi^{reg,s'}_{0}(z)=\prod_{\mathbf{m}\in(s',s)}\mathbf{B}_{\mathbf{m}}(z)\Psi^{reg,s}_{0}(z) \tag{6.32}$$

Here (s,s') means that we choose a path from s to s' such that $\mathbf{m}$ are the slope points in the path, which is a finite product.

Proof. The proof is given by induction.

$$\begin{aligned}\prod_{\mathbf{m}\in(s',s)}\mathbf{B}_{\mathbf{m}}(zp^{\mathscr{L}})\Psi_0^{reg,s}(zp^{\mathscr{L}})\mathscr{L} &= \prod_{\mathbf{m}\in(s,s')}\mathbf{B}_{\mathbf{m}}(zp^{\mathscr{L}})\mathbf{B}^s_{\mathscr{L}}(z)\Psi_0^{reg,s}(z)\\ &= \prod_{\mathbf{m}\in(s,s')}\mathbf{B}_{\mathbf{m}}(zp^{\mathscr{L}})\mathscr{L}\prod_{\mathbf{m}\in(s-\mathscr{L},s)}\mathbf{B}_{\mathbf{m}}(z)\Psi_0^{reg,s}(z)\\ &=\mathscr{L}\prod_{\mathbf{m}\in(s'-\mathscr{L},s-\mathscr{L})}\mathbf{B}_{\mathbf{m}}(z)\prod_{\mathbf{m}\in(s-\mathscr{L},s)}\mathbf{B}_{\mathbf{m}}(z)\Psi_0^{reg,s}(z)\\ &=\mathscr{L}\prod_{\mathbf{m}\in(s'-\mathscr{L},s)}\mathbf{B}_{\mathbf{m}}(z)\Psi_0^{reg,s}(z)\\ &=\mathscr{L}\prod_{\mathbf{m}\in(s'-\mathscr{L},s')}\mathbf{B}_{\mathbf{m}}(z)\prod_{\mathbf{m}\in(s',s)}\mathbf{B}_{\mathbf{m}}(z)\Psi_0^{reg,s}(z)\end{aligned} \tag{6.33}$$

By the uniqueness of the solution, the proof is finished. □

6.5. **Connection matrix for the two solutions.** The transition matrix between two fundamental solutions $\Psi_0(z)$ and $\Psi_\infty(z)$ is defined as:

$$\mathbf{Mon}(z) := \Psi_0(z)^{-1}\Psi_\infty(z) \tag{6.34}$$

It is clear that $\mathbf{Mon}(p^{\mathscr{L}}z) = \mathbf{Mon}(z)$, and we expect that $\mathbf{Mon}(z)$ is an abelian function over the abelian variety $\mathrm{Pic}(M(\mathbf{v},\mathbf{w}))\otimes\mathbb{C}^\times/(z\sim zp^{\mathscr{L}})$. However, note that $\mathbf{Mon}(z)$ might be a multivalued function over z, which means that $\mathbf{Mon}(ze^{2\pi i})\neq\mathbf{Mon}(z)$.

Also for the sake of the regular part of the fundamental solution, we can also define the regular part of the transition matrix as:

$$\mathbf{Mon}^{reg}(z) := \exp\Big(\sum_i\frac{\ln(E_0^{(i)})\ln(z_i)}{\ln(p)}\Big)\mathbf{Mon}(z)\exp\Big(-\sum_i\frac{\ln(E_\infty^{(i)})\ln(z_i)}{\ln(p)}\Big) \tag{6.35}$$

And it is easy to check that:

$$\mathbf{Mon}^{reg}(pz) = E_0\mathbf{Mon}^{reg}(z)E_\infty^{-1} \tag{6.36}$$

The reason why we choose the singular part to be of this form is because that under the circumstances, the elliptic solution $\mathbf{Mon}^{ell}(z)$ is single-valued and it lives in $GL(M(\mathbf{E}^r_p))$ the space of invertible matrices valued in the meromorphic function over the abelian variety $M(\mathbf{E}^r_p)$. Using the fact that for the variety $\mathbf{E}^r_p$, the meromorphic function over $\mathbf{E}^r_p$ is generated by the linear combinations of the ratio of the theta function of the form:

$$\prod_{i_1}\frac{\theta(z_1a_{i_1})}{\theta(z_1b_{i_1})}\cdots\prod_{i_r}\frac{\theta(z_ra_{i_r})}{\theta(z_rb_{i_r})}\times\prod_{\mathbf{k}}\frac{\theta(z_1^{k_1}z_2^{k_2}\cdots z_r^{k_r}a_{\mathbf{k}})}{\theta(z_1^{k_1}z_2^{k_2}\cdots z_r^{k_r}b_{\mathbf{k}})} \tag{6.37}$$

such that $\prod_{i_1} a_{i_1} = \prod_{i_1} b_{i_1}, \cdots, \prod_{i_r} a_{i_r} = \prod_{i_r} b_r, \prod_{\mathbf{k}} a_{\mathbf{k}} = \prod_{\mathbf{k}} b_{\mathbf{k}}$. And now via the relation between $\mathbf{Mon}^{ell}(z)$ and $\mathbf{Mon}(z)$ we have that:

(6.38)

$$\exp(-\sum_i \frac{\ln(E_0^{(i)})\ln(z_i)}{\ln(p)})\mathbf{Mon}(z)\exp(\sum_i \frac{\ln(E_\infty^{(i)})\ln(z_i)}{\ln(p)}) = \prod_i(\Theta(E_0^{(i)}, z_i))^{-1}\mathbf{Mon}^{ell}(z)\prod_i(\Theta(E_\infty^{(i)}, z_i))$$

By construction, we can see that $\mathbf{Mon}^{reg}(z)$ can be written as the linear combination of the theta function as the following form:

(6.39)

$$\prod_{i=1}^r(\Theta(E_0^{(i)}, z_i))^{-1}\prod_{i_1}\frac{\theta(z_1 a_{i_1})}{\theta(z_1 b_{i_1})}\cdots\prod_{i_r}\frac{\theta(z_r a_{i_r})}{\theta(z_r b_{i_r})}\times\prod_{\mathbf{k}}\frac{\theta(z_1^{k_1}z_2^{k_2}\cdots z_r^{k_r}a_{\mathbf{k}})}{\theta(z_1^{k_1}z_2^{k_2}\cdots z_r^{k_r}b_{\mathbf{k}})}\prod_{i=1}^r(\Theta(E_\infty^{(i)}, z_i))$$

6.6. $p \to 0$ **limit.** In this subsection we turn to analyze the limit $p \to 0$ for the function of the type $f(p^s z)$ with $s \in \mathbb{R}^r$.

Now we choose our path such that $\mathbf{B}_{\mathbf{m}}(z)$ are all of the generic point $\mathbf{m}$. It remains to analyze $\mathbf{B}_w(p^s z)$, by the result above,

$$\lim_{p\to 0}\mathbf{B}_{\mathbf{m}}(p^s z) = \begin{cases} \mathbf{m}((1\otimes S_{\mathbf{m}})(R^+_{\mathbf{m}})^{-1}_{21}) & s < \mathbf{m}, \mathbf{m}\in U_s \\ \mathbf{B}_{\mathbf{m}}(z)|_{p=1} & s = \mathbf{m} \\ \mathbf{B}_{\mathbf{m}}(z_I, \theta_J) & s_I = \mathbf{m}_I, s_{I^c} \neq \mathbf{m}_{I^c} \\ 1 & \text{Otherwise} \end{cases} \tag{6.40}$$

Here $s < \mathbf{m}$ means the partial order that for $s = (s_1, \cdots, s_n), \mathbf{m} = (m_1, \cdots, m_n)$ $s < \mathbf{m}$ if there is at least one s_i, m_i such that $s_i < m_i$. And $\theta_i = 0$ if $s_i < m_i$ and ∞ if $s_i > m_i$. U_s is a suitable small neighborhood of s.

Denote $\mathbf{B}_{\mathbf{m}} := \mathbf{m}((1\otimes S_{\mathbf{m}})(R^+_{\mathbf{m}})^{-1}_{21})$.

Now we choose a generic representation of the quantum difference operator $\mathbf{M}_{\mathcal{O}(1)}(z) = \mathcal{O}(1)\prod^{\rightarrow}_{\mathbf{m}\in\text{Walls}}\mathbf{B}_{\mathbf{m}}(z)$ such that each monodromy operator $\mathbf{B}_{\mathbf{m}}(z)$ are either of the $U_q(\mathfrak{sl}_2)$-type or of the $U_q(\hat{\mathfrak{gl}}_1)$-type. The following theorem gives the expression for the $p \to 0$ limit of the connection matrix:

Theorem 6.6. *For generic $s \in \mathbb{Q}^n$ the connection matrix as the following asymptotic at $p \to 0$:*

$$\lim_{p\to 0}\boldsymbol{Mon}^{reg}(p^s z) = \begin{cases} \prod^{\leftarrow}_{0\leq\mathbf{m}<s}(\mathbf{B}^*_{\mathbf{m}})^{-1}\cdot\mathbf{T}, & s \geq 0 \\ \prod_{s<\mathbf{m}<0}\mathbf{B}^*_{\mathbf{m}}\cdot\mathbf{T}, & s < 0 \end{cases} \tag{6.41}$$

Here $\mathbf{T} := P^{-1}H$, $0 \leq \mathbf{m} < s$ means the slope points $\mathbf{m}$ in one generic path from 0 to the point s without intersecting s. Here $s \geq 0$ and $s < 0$ stands for $s = (s_1, \cdots, s_n)$ such that $s_i \leq 0$ or $s_i > 0$ for every component s_i. The upper $\mathbf{B}^*_{\mathbf{m}}$ is defined as $P^{-1}\mathbf{B}_{\mathbf{m}}P$, and the lower $\mathbf{B}^*_{\mathbf{m}}$ is defined as $H^{-1}\mathbf{B}_{\mathbf{m}}H$

Roughly speaking, the thoerem tells us that the nodal limit of the regular part of the connection matrix $\mathbf{Mon}^{reg}$ can be described as the ordered product of the monodromy operators. The choice of the monodromy operators on the path would not affect the expression of $\mathbf{Mon}^{reg}$.

For the generic s, we can in principle write down the formula for $\lim_{p\to 0}\mathbf{Mon}^{reg}(p^s z)$ in terms of the monodromy operators $\mathbf{B}^*_{\mathbf{m}}$, but in that case the formula would be extremely complicated and chaotic. But we will see in the proof that there is an explicit formula for $\lim_{p\to 0}\mathbf{Mon}^{reg}(p^s z)$.

To prove the theorem, we first analyze the $p \to 0$ limit of the fundamental solution $\psi_0^{reg} = \overleftarrow{\prod}_{k=0}^{\infty}\mathbf{M}^*_k(z)^{-1}$ around $z=0$. We denote:

$$C(s) = \{\mathbf{x}\in\mathbb{R}^r | x_i \le s_i\} \tag{6.42}$$

and the following set:

$$B_k := \{\mathbf{s}\in\mathbb{R}^r | \exists w \in \mathrm{Wall}_0 \text{ such that } \exists I \subset \{1,\cdots,r\}, s_I + k \le w_I\} \subset \mathbb{R}^r \tag{6.43}$$

For convenience we consider the following stratification of B_k:

$$B_k = B_k^o \cup \partial B_k \tag{6.44}$$

with B_k^o denoting the interior points of B_k, and ∂B_k denotes the points on the boundary of B_k.

It is easy to see that $B_k \subset B_0$ and moreover $B_{k+1}\subset B_k$. Using this fact we can compute the nodal limit of $\psi_0^{reg}(p^s z)$:

Proposition 6.7. *The asymptotic behavior of $\psi_0^{reg}(z)$ for $s \in B_l - B_{l+1}$ with $l \ge 0$ goes in the following way:*

$$\lim_{p\to 0}\psi_0^{reg}(p^s z) = \begin{cases} \overleftarrow{\prod}_{k=0}^{l-1}(\mathbf{M}^*_k)^{-1}\cdot \mathbf{M}^*_{l,asymp}(s)^{-1} & s\in B_l^o \\ \overleftarrow{\prod}_{k=0}^{l-1}(\mathbf{M}^*_k)^{-1}\cdot \mathbf{M}^*_{l,asymp}(z,s)^{-1} & s\in \partial B_l \end{cases} \tag{6.45}$$

where

(6.46)

$$\mathbf{M}^*_{l,asymp}(z,s) := E_0^{-l-1}P^{-1}(\mathcal{O}(1)\prod_{\substack{w\in Walls_0\cap C(s)^c\\ w<s,s_{\sigma(1)}}}\mathbf{B}_w\cdot\mathbf{B}_{w_1,s_{\sigma(1)}}(z_I,\theta_J)|_{p=1}\cdot\prod_{\substack{w\in Walls_0\cap C(s)^c\\ w<s,s_{\sigma(2)}}}\mathbf{B}_w\cdot\mathbf{B}_{w_1,s_{\sigma(2)}}(z_I,\theta_J)$$

$$\cdots\prod_{\substack{w\in Walls_0\cap C(s)^c\\ w<s,s_{\sigma(l)}}}\mathbf{B}_w\cdot\mathbf{B}_{w_1,s_{\sigma(l)}}(z_I,\theta_J)|_{p=1}\cdot\prod_{\substack{w\in Walls_0\cap C(s)^c\\ w<s,s_{\sigma(l+1)}}}\mathbf{B}_w)PE_0^k$$

$$\mathbf{M}^*_{l,asymp}(s) := E_0^{-l-1}P^{-1}(\mathcal{O}(1)\prod_{w\in Walls_l\cap C(s)^c}\mathbf{B}_w)PE_0^k$$

Here Walls_l stands for the set of wall slope points in a small neighborhood of $[l, l+1]^{\times r}$. $(w_1, s_{\sigma(l)})$ stands for the points $(w_1, \cdots, s_{\sigma(l)}, \cdots, w_r)$ and here $\sigma_{\sigma(i)}$ is ordered such that $s_{\sigma(1)} \leq s_{\sigma(2)} \leq \cdots \leq s_{\sigma(r)}$.

Proof. This follows from the formula 6.18 of the fundamental solution $\psi_0^{reg}(z)$. The term $\mathbf{M}^*_{l,asymp}(s)$ and $\mathbf{M}^*_{l,asymp}(z,s)$ comes from computing the precise limit of $\mathbf{M}^*_k(zp^s)$ using the formula 6.40. We leave the computation of the limit as the exercise. □

We use the similar procedure to compute the regular part of $\Psi^{reg}_\infty(z)$, and it is easy to see that the solution is of the form:

$$\Psi^{reg}_\infty(z) = \prod_{k=0}^{\overrightarrow{\infty}} \mathbf{M}^*_{-k}(z) \tag{6.47}$$

with

$$\mathbf{M}^*_{-k}(z) = E^{k-1}_\infty \mathbf{M}^*(zp^{-k}) \mathbf{E}^{-k}_\infty \tag{6.48}$$

Also we denote

$$\mathbf{B}^*_{\mathbf{m}}(z) := H^{-1} \mathbf{B}_{\mathbf{m}}(z) H, \qquad \mathbf{B}^*_{\mathbf{m}} = H^{-1} \mathbf{B}_{\mathbf{m}} H \tag{6.49}$$

For the general case, we set

$$D(s) = \{\mathbf{x} \in \mathbb{R}^r | x_i \geq s_i\} \tag{6.50}$$

And define B_l for $l \leq 0$ in a similar way:

$$B_{-k} := \{\mathbf{s} \in \mathbb{R}^r | \exists \mathbf{m} \in \text{Wall}_0 \text{ such that } \exists I \subset \{1, \cdots, r\}, s_I - k \geq m_I\} \subset \mathbb{R}^r \tag{6.51}$$

Under this situation, doing the similar calculation above, we have that:

Proposition 6.8. *The asymptotic behavior of $\psi^{reg}_\infty(z)$ for $s \in B_{-l} - B_{-l-1}$ with $l \geq 0$ goes in the following way:*

$$\lim_{p \to 0} \psi^{reg}_\infty(p^s z) = \begin{cases} \mathbf{M}^*_{-l,asymp}(s) \cdot \prod_{k=-l+1}^{\overleftarrow{-1}} (\mathbf{M}^*_k)^{-1} & s \in B^o_{-l} \\ \mathbf{M}^*_{-l,asymp}(z,s) \cdot \prod_{k=-l+1}^{\overleftarrow{-1}} (\mathbf{M}^*_k)^{-1} & s \in \partial B_{-l} \end{cases} \tag{6.52}$$

where
(6.53)

$$\mathbf{M}^*_{l,asymp}(z,s) := E_\infty^{l-1}(P^*)^{-1}(\mathcal{O}(1)\prod_{\substack{w\in Walls_0\cap D(s)^c\\ w<s,s_{\sigma(1)}}}\mathbf{B}_w\cdot\mathbf{B}_{w_1,s_{\sigma(1)}}(z_I,\theta_J)|_{p=1}\cdot\prod_{\substack{w\in Walls_0\cap D(s)^c\\ w<s,s_{\sigma(2)}}}\mathbf{B}_w\cdot\mathbf{B}_{w_1,s_{\sigma(2)}}(z_I,\theta$$

$$\cdots\prod_{\substack{w\in Walls_0\cap D(s)^c\\ w<s,s_{\sigma(l)}}}\mathbf{B}_w\cdot\mathbf{B}_{w_1,s_{\sigma(l)}}(z_I,\theta_J)|_{q=1}\cdot\prod_{\substack{w\in Walls_0\cap D(s)^c\\ w<s,s_{\sigma(l+1)}}}\mathbf{B}_w)P^*E_\infty^{-l}$$

$$\mathbf{M}^*_{l,asymp}(s) := E_\infty^{l-1}(P^*)^{-1}(\mathcal{O}(1)\prod_{w\in Walls_l\cap D(s)^c}\mathbf{B}_w)E_\infty^{l-1}(P^*)^{-1}$$

here $Walls_l$ stands for the set of walls in a small neighborhood of $[l,l+1]^{\times n}$. $(w_1,s_{\sigma(l)})$ stands for the points $(w_1,\cdots,s_{\sigma(l)},\cdots,w_r)$ and here $\sigma_{\sigma(i)}$ is ordered such that $s_{\sigma(1)}\le s_{\sigma(2)}\le\cdots\le s_{\sigma(r)}$.

Proof. The proof is similar to proposition 6 in [S21]. We consider a finite approximation of infinite product

$$\Psi_N^{reg}(z)=\mathbf{M}^*_{-1}(z)\mathbf{M}^*_{-2}(z)\cdots\mathbf{M}^*_{-N}(z)=\mathbf{M}^*(\frac{z}{p})\mathbf{M}^*(\frac{z}{p^2})\cdots\mathbf{M}^*(\frac{z}{p^N})E_\infty^{-N} \tag{6.54}$$

Via the translation formula, we can write down:

$$\Psi_N^{reg}(z)=(\overrightarrow{\prod_{[0,N)\boldsymbol{\theta}\in\mathrm{Walls}_0+\mathbb{N}}}\mathbf{B}^*_w(z))(\mathbf{M}(0)^*)^NE_\infty^{-N} \tag{6.55}$$

Now for $s\in B^o_{-l}$, we have that:

$$\lim_{q\to 0}\Psi_N^{reg}=(\overrightarrow{\prod_{(\mathrm{Walls}_0+\mathbb{N})\cap[0,N)\cap C(s)^c}}\mathbf{B}^*_w(z))(\mathbf{M}(0)^*)^NE_\infty^{-N} \tag{6.56}$$

Also note that by taking $z=\infty$, we have that:

$$E_\infty^N=(\overrightarrow{\prod_{[0,N)\boldsymbol{\theta}\in\mathrm{Walls}_0+\mathbb{N}}}\mathbf{B}^*_w(z))(\mathbf{M}(0)^*)^N \tag{6.57}$$

Which means that

$$(\overrightarrow{\prod_{[0,N\boldsymbol{\theta})\in\mathrm{Walls}_0+\mathbb{N}}}\mathbf{B}^*_w(\mathbf{M}(0)^*)^NE_\infty^{-N}=1 \tag{6.58}$$

Thus comparing the result we have the statement for $s\in B^o_l$. For the statement that $s\in\partial B_l$, just mimick the computation as in the $\psi_0^{reg}(z)$, which is complicated but straightforward computation.

□

Though the formula above is really complicated, here we only consider the nodal limit when s is the generic point. Using the proposition we can have the following result for the asymptotics of $\Psi_{\infty}^{reg}(zp^s)$ and $\Psi_0^{reg}(zp^s)$ as $p \to 0$:

Proposition 6.9. *The solution $\Psi_{\infty}^{reg}(z)$ and $\Psi_0^{reg}(zp^s)$ has the following limit for generic s:*

$$\lim_{p\to 0}\Psi_{\infty}^{reg}(zp^s)=\begin{cases}\prod^{\leftarrow}_{\mathbf{m}\in[0,s)}(\mathbf{B}^*_{\mathbf{m}})^{-1} & s\geq 0\\ 1 & s<0\end{cases} \tag{6.59}$$

$$\lim_{p\to 0}\Psi_0^{reg}(zp^s)=\begin{cases}\prod^{\leftarrow}_{\mathbf{m}\in(s,0]}(\mathbf{B}^*_{\mathbf{m}}) & s\leq 0\\ 1 & s\geq 0\end{cases} \tag{6.60}$$

Here $[0,s)$ ($(s,0]$) stands for a path from 0 (s) to s (0) containing 0. Here $s \geq 0$ and $s < 0$ stands for $s = (s_1, \cdots, s_n)$ such that $s_i \leq 0$ or $s_i > 0$ for every component s_i.

Now combining the formula 6.59 and 6.60, we have obtained the Theorem 6.6.

7. Degeneration to the Dubrovin connection

In this section we shall construct the degeneration limit of the quantum difference equations

7.1. Dubrovin connection for quiver varieties.

Fix a quiver variety $M_\theta(\mathbf{v},\mathbf{w})$, it has been proved in [MO12] that the Dubrovin connection for the quantum equivariant cohomology $H_T^*(M_\theta(\mathbf{v},\mathbf{w}))$ is given as:

$$\nabla_\lambda = d_\lambda - Q(\lambda), \qquad \lambda\in H^2(M_\theta(\mathbf{v},\mathbf{w}))\cong \mathrm{Pic}(M_\theta(\mathbf{v},\mathbf{w})) \tag{7.1}$$

Here:

$$Q(\lambda)=c_1(\lambda)+\hbar\sum_{\theta\cdot\alpha>0}\frac{\alpha(\lambda)}{1-q^{-\alpha}}e_\alpha e_{-\alpha}+\text{constant opeartor} \tag{7.2}$$

Here $\{e_\alpha\}$ are the generators of the MO Lie algebra $\mathfrak{g}_Q^{MO}\subset Y_\hbar(\mathfrak{g}_Q^{MO})$. It has been proved [BD23] that this MO Lie algebra $\mathfrak{g}_Q^{MO}$ is isomorphic to the BPS Lie algebra $\mathfrak{g}_Q\subset Y_\hbar(\mathfrak{g}_Q)$. The generators $e_{\pm\alpha}$ in $\mathfrak{g}_Q^{MO}$ were sent to $E_{\pm[i,j)}$ in $\hat{\mathfrak{sl}}_n$.

There are many interesting properties for the Dubrovin connection. The Dubrovin connection has regular singularities $q^{-\alpha}=1$ and $q=0,\infty$, and the corresponding monodromy representation:

$$\pi_1((\mathbb{C}^*)^n-\mathrm{Sing})\to \mathrm{Aut}(H_T^*(M_\theta(\mathbf{v},\mathbf{w}))) \tag{7.3}$$

can be represented by the quantum Weyl group for the corresponding quantum group $U_q(\mathfrak{g}_Q)$.

In our cases, for the affine type A quiver varieties $M(\mathbf{v},\mathbf{w})$ for $\theta=(1,\cdots,1)$. The corresponding BPS Lie algebra $\mathfrak{g}_Q$ is isomorphic to $\hat{\mathfrak{gl}}_n$. The corresponding Dubrovin connection can be written as:

$$\begin{aligned} Q(\lambda) =& c_1(\lambda)+\hbar\sum_{\alpha>0}\frac{\alpha(\lambda)}{1-q^{-\alpha}}e_{\alpha}e_{-\alpha}+\cdots \\ =& c_1(\lambda)+\hbar\sum_{\alpha\in\text{real positive roots}+\{0\}}\sum_{k\geq 0}\frac{(\alpha+k\delta)\lambda}{1-q^{-\alpha-k\delta}}e_{\alpha+k\delta}e_{-\alpha-k\delta}+\cdots \end{aligned} \tag{7.4}$$

For example, if we take $\lambda=\mathscr{O}(1)$, we have that:

$$Q(\mathscr{O}(1))=c_1(\mathscr{O}(1))+\hbar\sum_{\alpha\in\text{real positive roots}+\{0\}}\sum_{k\geq 0}\frac{kn+|\alpha|}{1-q^{-\alpha-k\delta}}e_{\alpha+k\delta}e_{-\alpha-k\delta}+\cdots \tag{7.5}$$

Similarly, if $\lambda=\mathscr{L}_i$, we have:

$$Q(\mathscr{L}_i)=c_1(\mathscr{L}_i)+\hbar\sum_{k\geq 0}\frac{1+k}{1-q^{-\alpha-k\delta}}e_{\alpha+k\delta}e_{-\alpha-k\delta}+\cdots \tag{7.6}$$

Moreover, we can write down the quantum differential equation as:

$$Q(\mathscr{L})=c_1(\mathscr{L})+\hbar\sum_{[i,j)}\frac{\mathscr{L}\cdot[i,j)}{1-z^{-[i,j)}}E_{[i,j)}E_{-[i,j)}+\cdots \tag{7.7}$$

7.2. **Degeneration limit of the slope subalgebra.** In this subsection we describe the degeneration limit of the slope subalgebra $\mathscr{B}_{\mathbf{m}}\subset U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ into the subalgebra of $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$.

The description we shall use is as the following: Using the algebra embedding:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)\hookrightarrow\prod_{\mathbf{w}}\text{End}(K(\mathbf{w})) \tag{7.8}$$

We express the generators of $\mathscr{B}_{\mathbf{m}}\subset U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ into the matrix coefficients in $\prod_{\mathbf{w}}\text{End}(K(\mathbf{w}))$. Then we proceed the degeneration limit of $K(\mathbf{w})$ via the Chern character map $K(\mathbf{w})\to\widehat{H(\mathbf{w})}$.

Then via the algebra embedding $U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)\hookrightarrow U_q(\hat{\mathfrak{g}}_Q^{MO})$ and the fact that the degeneration limit of $U_q(\hat{\mathfrak{g}}_Q^{MO})$ lies in $Y_{\hbar}(\mathfrak{g}_Q^{MO})\cong Y_{\hbar}(\mathfrak{g}_Q)$, we have that the degeneration limit of $\mathscr{B}_{\mathbf{m}}\subset U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)$ lies in $Y_{\hbar}(\mathfrak{g}_Q)$.

To get the formula for the matrix element, recall the formula:

$$\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\mu}\rangle=P^{\mathbf{m}}_{[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}})\prod_{k=1}^{\mathbf{w}}[\frac{u_k}{q\chi_{\blacksquare}}]],\qquad \mathbf{m}\cdot[i,j)\in\mathbb{Z} \tag{7.9}$$

$$\langle\boldsymbol{\mu}|Q^{\mathbf{m}}_{-[i;j)}|\boldsymbol{\lambda}\rangle=Q^{\mathbf{m}}_{-[i;j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\lambda}}\zeta(\frac{\chi_{\square}}{\chi_{\blacksquare}})\prod_{k=1}^{\mathbf{w}}[\frac{\chi_{\blacksquare}}{qu_k}]]^{-1},\qquad \mathbf{m}\cdot[i,j)\in\mathbb{Z} \tag{7.10}$$

Here:

$$P^{\pm\mathbf{m}}_{\pm[i;j)}=\mathrm{Sym}\left[\frac{\prod_{a=i}^{j-1}z_a^{\lfloor m_i+\dots+m_a\rfloor-\lfloor m_i+\dots+m_{a-1}\rfloor}}{t^{\mathrm{ind}\,d^m_{[i,j)}}q^{i-j}\prod_{a=i+1}^{j-1}\left(1-\frac{q_2z_a}{z_{a-1}}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{z_b}{z_a}\right)\right] \tag{7.11}$$

$$Q^{\pm\mathbf{m}}_{\mp[i;j)}=\mathrm{Sym}\left[\frac{\prod_{a=i}^{j-1}z_a^{\lfloor m_i+\dots+m_{a-1}\rfloor-\lfloor m_i+\dots+m_a\rfloor}}{t^{-\mathrm{ind}^m_{[i,j)}}\prod_{a=i+1}^{j-1}\left(1-\frac{q_1z_{a-1}}{z_a}\right)}\prod_{i\leq a<b<j}\zeta\left(\frac{z_a}{z_b}\right)\right] \tag{7.12}$$

Now we shall give the asymptotic behavior of the matrix coefficients $\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\mu}\rangle$ in terms of the order of $\hbar$. In the very first order, we have that:

(7.13)

$$\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\mu}\rangle^{coh}=\mathrm{Sym}[\frac{1}{\prod_{a=i+1}^{j-1}(t_2+\chi_a-\chi_{a-1})}\prod_{i\leq a<b<j}\omega(\chi_b-\chi_a)]\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\mu}}\zeta(\frac{\chi_{\blacksquare}}{\chi_{\square}})\prod_{k=1}^{\mathbf{w}}[\frac{u_k}{q\chi_{\blacksquare}}]]^{coh}$$

In this way one can find that $\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\mu}\rangle^{coh}$ is independent of the choice of $\mathbf{m}$. In this case we choose $P^{\mathbf{0}}_{[i;j)}$.

We want to prove the following thing:

Proposition 7.1. *$\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\mu}\rangle^{coh}$ coincides with $\langle\boldsymbol{\lambda}|E_{[i;j)}|\boldsymbol{\mu}\rangle$, and $\langle\boldsymbol{\lambda}|Q^{\mathbf{m}}_{-[i;j)}|\boldsymbol{\mu}\rangle^{coh}$ coincides with $\langle\boldsymbol{\lambda}|E_{-[i;j)}|\boldsymbol{\mu}\rangle$. Here $E_{[i;j)}\in\hat{\mathfrak{sl}}_n\subset\hat{\mathfrak{gl}}_n$.*

Proof. Before the proof, one thing should be commented that the proof follows the spirit that the degeneration of $U_q(\hat{\mathfrak{gl}}_n)$ to $U(\hat{\mathfrak{gl}})n)$ depends on the fact that the quantum R-matrix would degenerate to the classical r-matrix in the degenerate limit.

Here we only show the proof for $\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\mu}\rangle^{coh}$. The proof for $\langle\boldsymbol{\lambda}|Q^{\mathbf{m}}_{-[i;j)}|\boldsymbol{\mu}\rangle^{coh}$ is similar.

Define the following rational functions:

$$P^{coh}_{[i,j)}:=\mathrm{Sym}[\frac{1}{\prod_{a=i+1}^{j-1}(t_2+z_a-z_{a-1})}\prod_{i\leq a<b<j}\omega(z_b-z_a)] \tag{7.14}$$

It is easy to see that $\langle\boldsymbol{\mu}|P^{\mathbf{m}}_{[i;j)}|\boldsymbol{\lambda}\rangle^{coh}=\langle\boldsymbol{\mu}|P^{coh}_{[i,j)}|\boldsymbol{\lambda}\rangle$. Thus it remains to prove that $P^{coh}_{[i;j)}=E_{[i;j)}\in\hat{\mathfrak{sl}}_n\subset\hat{\mathfrak{gl}}_n$.

We use the induction on $[i,j)$, now suppose that the equality is true for $[i,k)$ and $[k,j)$. We need to prove the following equality:

$$P^{coh}_{[i,j)} = [P^{coh}_{[i,k)}, P^{coh}_{[k,j)}] \tag{7.15}$$

First we have that $E_{[i,i+1)} = (\frac{1}{q-q^{-1}}P^{\mathbf{0}}_{[i,i+1)})^{coh}$, and we define $x^{\pm}_i := \frac{1}{q-q^{-1}}e_{\pm[i,i+1)}$ so that $x^{\pm,coh}_i$ is well-defined as $E_{[i,i+1)}$. Inductively, we also define $x^{\pm}_{[i,j)} := \frac{1}{q-q^{-1}}e_{\pm[i,j)}$, we will prove that they have the well-defined cohomological limit.

To prove this, note that the isomorphism of the slope subalgebra of slope $\mathbf{0}$ $\mathscr{B}_{\mathbf{0}} \cong U_q(\hat{\mathfrak{gl}}_n)$ with the quantum affine algebra sends $e_{[i,j)}$ to $P^{\mathbf{0}}_{[i,j)}$, $e_{-[i,j)}$ to $Q^{\mathbf{0}}_{-[i,j)}$ satisfying the relation 2.19. It is easy to see that for $1 \le i \le j \le k \le n$:

$$e_{[i,k)} = -\frac{1}{q-q^{-1}}(e_{[i,j)}e_{[j,k)} - q^{-1}e_{[j,k)}e_{[i,j)}) \tag{7.16}$$

which is equivalent to

$$x_{[i,k)} = -(x_{[i,j)}x_{[j,k)} - q^{-1}x_{[j,k)}x_{[i,j)}) \tag{7.17}$$

Now we take $q \to 1$ limit we obtain that:

$$e^{coh}_{[i,k]} = -E_{[i,k)} \tag{7.18}$$

Moreover:

(7.19)
$$\frac{1}{(q-q^{-1})}[e_{\pm[i,j)}e_{\pm[j,k)} - q^{-1}e_{\pm[j,k)}e_{\pm[i,j)}] = \sum_{i\le x<j}^{x\equiv k} e_{\pm[j,j+k-x)}e_{\pm[i,x)} - \sum_{i<x\le j}^{x\equiv j} e_{\pm[x,j)}e_{\pm[i+j-x,k)}$$
$$= \sum_{i\le x<j}^{x\equiv k} e_{\pm[j,j+k-x)}e_{\pm[i,x)} - \sum_{i<x<j}^{x\equiv j} e_{\pm[x,j)}e_{\pm[i+j-x,k)} - e_{\pm[i,k)}$$

Using the induction we have that:

(7.20)
$$[x_{\pm[i,j)}x_{\pm[j,k)} - q^{-1}x_{\pm[j,k)}x_{\pm[i,j)}] = (q-q^{-1})\sum_{i\le x<j}^{x\equiv k} x_{\pm[j,j+k-x)}x_{\pm[i,x)} - (q-q^{-1})\sum_{i<x\le j}^{x\equiv j} x_{\pm[x,j)}x_{\pm[i+j-x,k)}$$
$$=(q-q^{-1})\sum_{i\le x<j}^{x\equiv k} x_{\pm[j,j+k-x)}x_{\pm[i,x)} - (q-q^{-1})\sum_{i<x<j}^{x\equiv j} x_{\pm[x,j)}x_{\pm[i+j-x,k)} - x_{\pm[i,k)}$$

As $q \to 1$, the quadratic terms on the right hand side vanishes.

Hence we have proved that:

$$E_{[i,j)} = -e^{coh}_{[i,j)} \tag{7.21}$$

for arbitrary i,j.

∎

Remark. The above proof is really well-known since it just reflects the fact that the $q\to 1$ of the quantum affine algebra $U_q(\hat{\mathfrak{gl}}_n)$ degenerates to the affine Kac-moody algebra $U(\hat{\mathfrak{gl}}_n)$. The point of the proof here is to show the matrix coefficients of $U(\hat{\mathfrak{gl}}_n)$ acting on the equivariant cohomology $H_T^*(M(\mathbf{w}))$ of quiver varieties, which is useful in the precise computation.

7.3. **Main result.** Having established the degeneration property between quantum affine toroidal algebra and the affine Yangians, here we show our main result:

Theorem 7.2. *The degeneration limit of the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$ coincides with the quantum multiplication operator $Q(\mathscr{L})$ up to a constant operator.*

Proof. To prove the theorem, let us first state the following useful result on the degeneration

Proposition 7.3. *In the cohomological limit, the monodromy operator $\mathbf{B_m}(\lambda)$ reads:*

(7.22)
$$\begin{aligned}\mathbf{B}^{coh}_{\mathbf{m}}(\lambda)=&\sum_{h=1}^{g}(\sum_{k=1}^{\infty}\frac{1}{1-z^{-k|\delta_h|}}\alpha_{-k}^{\mathbf{m},h(0)}\alpha_{k}^{\mathbf{m},h(0)}+\sum_{\substack{\gamma\in\Delta(A)\\ m\geq 0}}\frac{1}{1-z^{-\mathbf{v}_r+(m+1)\delta_h}}f_{(\delta-\gamma)+m\delta}e_{(\delta-\gamma)+m\delta}\\ &+\sum_{m>0}\sum_{i,j=1}^{l_h}\frac{1}{1-z^{m\delta_h}}u_{m,ij}f_{m\delta,\alpha_j}e_{m\delta,\alpha_i}+\sum_{\substack{\gamma\in\Delta(A)\\ m\geq 0}}\frac{1}{1-z^{\mathbf{v}_\gamma+m\delta_h}}f_{\gamma+m\delta}e_{\gamma+m\delta})\end{aligned}$$

And here $u_{m,ij}=min\{i,j\}-\frac{ij}{n}$, and the generators (e.g. $f_{\gamma+m\delta}e_{\gamma+m\delta}$)) are the cohomological limit of the generators under the following homomorphism:

$$U_{q,t}(\hat{\hat{\mathfrak{sl}}}_n)\hookrightarrow\prod_{\mathbf{w}}End(K(\mathbf{w}))\xrightarrow{ch}\prod_{\mathbf{w}}End(H(\mathbf{w}))\tag{7.23}$$

Proof. This is from the defintion of the degeneration limit:

$$\mathbf{B}^{coh}_{\mathbf{m}}(\lambda):=\lim_{\hbar\to 0}\frac{1}{\hbar}(\mathbf{B}_{\mathbf{m}}(\lambda)-1)\tag{7.24}$$

Then using the Proposition 4.5 we can show that all the generators $f_{\gamma+m\delta}e_{\gamma+m\delta}$ lies in the affine Yangian $Y_{\hbar_1,\hbar_2}(\hat{\mathfrak{sl}}_n)$.

∎

Now we choose a path such that each $\mathbf{m}$ is at the generic points. In this case the cohomological limit of the monodromy operator can be written as one of the following form:

$$\sum_{h=1}^{n}(\sum_{k=1}^{\infty}\frac{1}{1-z^{-k|\boldsymbol{\delta}_h|}}\alpha_{-k}^{\boldsymbol{m},h(0)}\alpha_{k}^{\boldsymbol{m},h(0)}),\qquad \frac{1}{1-z^{-\alpha_i^{\mathbf{m}}}}e_{-\alpha_i}^{\mathbf{m}}e_{\alpha_i}^{\mathbf{m}} \tag{7.25}$$

Now given the quantum difference operator $\mathbf{M}_{\mathscr{L}}(z)$ being written as:

$$\mathbf{M}_{\mathscr{L}}(z)=\mathscr{L}\mathbf{B}_{w_m}(z)\cdots\mathbf{B}_{w_0}(z) \tag{7.26}$$

Under the degenerate limit, the cohomological limit can be written as:

$$\sum_{h=1}^{n}(\sum_{\frac{a}{b}\in\text{Walls}}\sum_{k=1}^{\infty}\frac{1}{1-z^{-bk\boldsymbol{\theta}}}\alpha_{-k}^{a/b,h}\alpha_{k}^{a/b,h})+\sum_{[i;j)\in\text{Walls}}\frac{1}{1-z^{-[i;j)}}Q_{-[i;j)}^{\mathbf{m}}P_{[i;j)}^{\mathbf{m}} \tag{7.27}$$

Note that each $\alpha_{-k}^{a/b,h}\alpha_{k}^{a/b,h}$ has the degeneration limit as $\alpha_{-bk}^{h}\alpha_{bk}^{h}$, and $Q_{-[i;j)}^{\mathbf{m}}P_{[i;j)}^{\mathbf{m}}$ has the degeneration limit as $e_{-[i,j)}e_{[i,j)}$. With this fact it remains to match the coefficients of the Dubrovin connection.

For the term of the form $e_{-[i,j)}e_{[i,j)}$. Note that $Q_{-[i;j)}^{\mathbf{m}}P_{[i;j)}^{\mathbf{m}}$ exists iff $\mathbf{m}\cdot[i,j)\in\mathbb{Z}$. To make sure that the wall $\mathbf{m}$ is in $(0,1]^r$, it remians to prove that there are $\mathscr{L}\cdot[i,j)$ walls for $e_{-[i,j)}e_{[i,j)}$.

We first prove the claim for $\mathscr{L}=\mathcal{O}(1)=:\boldsymbol{\theta}=(1,\cdots,1)$, note that it is equivalent to count the hyperplanes $\mathbf{m}\cdots[i,j)\in\mathbb{Z}$ with fixed $[i,j)$ such that the corresponding hyperplane intersects the boundary of $(0,1]^r$. While the hyperplane satisfying the above condition is equivalent to the hyperplane satisfying the following equation:

$$\mathbf{m}\cdot[i,j)=l,\qquad l\in[1,\boldsymbol{\theta}\cdot[i,j)] \tag{7.28}$$

The reason is that if $\mathbf{m}\cdot[i,j)=0$, it is easy to see that the hyperplane only intersects $\partial[0,1]^n$ with $\mathbf{0}$, thus does not intersect $\partial(0,1]^n$. The hyperplane $\mathbf{m}\cdot[i,j)=\boldsymbol{\theta}\cdot[i,j)$ intersects $\partial(0,1]^n$ with $\boldsymbol{\theta}$, and the hyperplane $\mathbf{m}\cdot[i,j)=\boldsymbol{\theta}\cdot[i,j)+1$ does not intersect with $\partial(0,1]^n$.

For the general $\mathscr{L}\in\mathbb{N}^n-\mathbf{0}$, denote $\mathscr{L}$ as $(k_1,k_2,\cdots,k_n)$. Now the hyperplane $\mathbf{m}\cdot[i,j)\in\mathbb{Z}$ would be contained in the wall of the quantum difference operator $\mathbf{M}_{\mathscr{L}}$ iff it intersects with $\partial((0,k_1]\times\cdots\times(0,k_n])$. So under the circumstances, the hyperplane with fixed $[i,j)$ satisfying the above condition has the equation written as:

$$\mathbf{m}\cdot[i,j)=a,\qquad a\in[1,\mathscr{L}\cdot[i,j)] \tag{7.29}$$

Which means that the coefficients of $e_{-[i,j)}e_{[i,j)}$ in $\mathbf{M}_{\mathscr{L}}^{coh}$ is equal to $\mathscr{L}\cdot[i,j)$. Thus finish the proof. □

7.4. $p \to 1$ **limit of the solution of the quantum difference equation.** With the result of the degeneration limit of the quantum difference equation. We can start to construct the $p \to 1$ degeneration limit of the solution to the quantum difference equation.

Let $\Psi_{0,\infty}(q_1, q_2, p, z)$ be the solution to the quantum difference equation described above. Then we take $z = e^{2\pi s}$, $q_i = e^{2\pi i\hbar_i\tau}$, $q = e^{-2\pi i\tau}$. By the Theoreom 7.2, for the degeneration limit of the solution $\psi_{0,\infty}(z)$, which is defined as follows:

$$\psi_{0,\infty}(z) = \lim_{\tau\to 0}\Psi_{0,\infty}(e^{2\pi i\hbar_1\tau}, e^{2\pi i\hbar_2\tau}, e^{-2\pi i\tau}, z) \in H_T(\widehat{M(\mathbf{v},\mathbf{w})})_{loc} \tag{7.30}$$

It is the solution to the corresponding Dubrovin connection. The solution lies in the completion of $H_T(M(\mathbf{v},\mathbf{w}))_{loc}$. Using this, we can define the transport of the solution of the Dubrovin connection:

$$\mathrm{Trans}(s) := \psi_0(e^{2\pi is})^{-1}\psi_\infty(e^{2\pi is}) \in H_T(\widehat{M(\mathbf{v},\mathbf{w})})_{loc} \tag{7.31}$$

Similarly as a result of the Theorem 7.2, the transport of the solution is a limit the monodromy:

$$\mathrm{Trans}(s) = \lim_{\tau\to 0}\mathbf{Mon}(z = e^{2\pi is}, t_1 = e^{2\pi i\hbar_1\tau}, t_2 = e^{2\pi i\hbar_2\tau}, q = e^{-2\pi i\tau}) \tag{7.32}$$

in $H_T(\widehat{M(\mathbf{v},\mathbf{w})})_{loc}$. Here $s = (s_1, \cdots, s_n) \in \mathbb{R}^n$ is the generic point in $\mathbb{R}^n$.

The description for the monodromy operator from the monodromy operator generally can only be written as the infinite product of matrices. However, the help of the elliptic stable envelope could help us make the description for the monodromy operator much more elegant.

In this way we find that the transport matrix can be described in terms of $\mathbf{B}_{\mathbf{m}}$.

Proposition 7.4. *Trans*$(s) = \lim_{p\to 0}$ $\boldsymbol{Mon}^{reg}(p^s, e^{2\pi i\hbar_1}, e^{2\pi i\hbar_2}, p)$ *for* $s \in \mathbb{R}^n\backslash$*Walls*

Proof. Note that $\mathbf{Mon}(q, t_1, t_2, z)$ can be written as the following:

$$\exp(-\sum_i \frac{\ln(E_i^{(0)})\ln(z_i)}{\ln(q)})\sum_{\mathbf{a},\mathbf{b}} L_{\mathbf{a},\mathbf{b}}\prod_{j=1}^r\prod_{i_j,l_j}\frac{\theta(z_{i_j}^{l_j}a_{i_j})}{\theta(z_{i_j}^{l_j}b_{i_j})}\exp(\sum_i\frac{\ln(E_i^{(\infty)})\ln(z_i)}{\ln(q)}) \tag{7.33}$$

Using the formula 6.38:

(7.34)

$$\exp(\sum_i\frac{\ln(E_0^{(i)})\ln(z_i)}{\ln(p)})\mathbf{Mon}(z)\exp(-\sum_i\frac{\ln(E_\infty^{(i)})\ln(z_i)}{\ln(p)}) = \prod_i(\Theta(E_0^{(i)}, z_i))\mathbf{Mon}^{ell}(z)\prod_i(\Theta(E_\infty^{(i)}, z_i))^{-1}$$

It is known that $\mathbf{Mon}^{ell}(z)$ is the linear combination of the riemann theta function, and it is also q-periodic. And since $\mathbf{Mon}(q, t_1, t_2, z)$ is q-periodic, it must be the combination of the product of the form $\exp(k\frac{\ln(z)\ln(a/b)}{\ln(q)})\frac{\theta(z^k a)}{\theta(z^k b)}$, and thus the proposition follows from the following identity 9.20 in the appendix:

$$\lim_{\tau\to 0}\exp(2\pi i s(\epsilon_1-\epsilon_2))\frac{\theta(e^{2\pi i(s+\tau\epsilon_1)},e^{2\pi i\tau})}{\theta(e^{2\pi i(s+\tau\epsilon_2)},e^{2\pi i\tau})}=\lim_{q\to 0}\frac{\theta(q^s e^{2\pi i\epsilon_1},q)}{\theta(q^s e^{2\pi i\epsilon_2},q)} \tag{7.35}$$

This identity can be generalized to:

$$\lim_{\tau\to 0}\exp(2\pi i(k\cdot s)\lambda)\frac{\prod_i\theta(e^{2\pi i(k\cdot s+\tau a_i)},e^{2\pi i\tau})}{\prod_j\theta(e^{2\pi i(k\cdot s+\tau b_j)},e^{2\pi i\tau})}=\lim_{q\to 0}\frac{\prod_i\theta(q^{k\cdot s}e^{2\pi i a_i},q)}{\prod_j\theta(q^{k\cdot s}e^{2\pi i b_j},q)},\lambda=\sum_i a_i-\sum_j b_j \tag{7.36}$$

Then using the formula to our setting, we have:

$$\begin{aligned}&\lim_{\tau\to 0}\exp(\sum_i\frac{\ln(E_0^{(i)}/E_\infty^{(i)})\ln(z_i)}{\ln(p)})\prod_{\mathbf{k}}\frac{\theta(z_1^{k_1}\cdots z_n^{k_n}a_{\mathbf{k}})}{\theta(z_1^{k_1}\cdots z_n^{k_n}b_{\mathbf{k}})}\\ =&\lim_{\tau\to 0}\exp(\sum_i\frac{\ln(E_0^{(i)}/E_\infty^{(i)})\ln(z_i)}{\ln(p)})\exp(-\sum_{\mathbf{k}}\frac{k_i\ln(z_i)\ln(a_{\mathbf{k}}/b_{\mathbf{k}})}{\ln(p)})\exp(\sum_{\mathbf{k}}\frac{k_i\ln(z_i)\ln(a_{\mathbf{k}}/b_{\mathbf{k}})}{\ln(p)})\times\\ &\times\prod_{\mathbf{k}}\frac{\theta(z_1^{k_1}\cdots z_n^{k_n}a_{\mathbf{k}})}{\theta(z_1^{k_1}\cdots z_n^{k_n}b_{\mathbf{k}})}=\lim_{q\to 0}\prod_{\mathbf{k}}\frac{\theta(q^{\mathbf{k}\cdot\mathbf{s}}a_{\mathbf{k}})}{\theta(q^{\mathbf{k}\cdot\mathbf{s}}b_{\mathbf{k}})}\end{aligned} \tag{7.37}$$

Here we use the fact that $E_{(0)}^{(i)}/E_\infty^{(i)}=\prod_{\mathbf{k}}(\frac{a_{\mathbf{k}}}{b_{\mathbf{k}}})^{k_i}$. Now we have finished the proof of the theorem. □

Proposition 7.5. *The transport of the quantum connection from $z=0$ to $z=\infty$ intersecting a line γ intersecting $|z_i|=1$ at a nonsingular point $z=e^{2\pi is}$ with $z_i=z_j$ are equal, and s does not meet the wall set we have*

$$Trans(s)=\begin{cases}\prod^{\leftarrow}_{w\in(0,s)}(\mathbf{B}_w^*)^{-1}\cdot\mathbf{T} & s\geq 0\\ \prod^{\rightarrow}_{w\in(s,0)}\mathbf{B}_w^*\cdot\mathbf{T} & s<0\end{cases} \tag{7.38}$$

Moreover, if we release the condition of $(z_1,\cdots,z_r)$ to the polydisk $\{|z_i|=1\}\in\mathbb{C}^n$, one could obtain the following result for the transition operator:

Theorem 7.6. *The transport of the quantum connection from $z=0$ to $z=\infty$ intersecting the polydisk at a nonsingular point $z=e^{2\pi is}$ with $|z_i|=1$ for each $i=1,\cdots,n$, and s does not* *meet the wall set, then it equals:*

(7.39)

$$Trans(s)=\begin{cases}\prod^{\rightarrow}_{w_i\in Walls_i\subset Walls\cap C(s)^c}(\mathbf{B}^*_{w_i})\prod^{\leftarrow}_{w_i\in Walls_i\subset Walls^-\cap D(s)^c}(\mathbf{B}^*_{w_i})^{-1}\\ s\in B_k\cap B_{-l}\\ \prod^{\leftarrow}_{w_i\in Walls_i\subset Walls^-\cap D(s)^c}(\mathbf{B}^*_{w_i})^{-1}, & s\in\bigcup_{w\in Walls_0}\{\mathbf{s}\in\mathbb{R}^n|s_i-l\leq w_i\}\\ \prod^{\rightarrow}_{w_i\in Walls_i\subset Walls\cap C(s)^c}(\mathbf{B}^*_{w_i}), & s\in\bigcup_{w\in Walls_0}\{\mathbf{s}\in\mathbb{R}^n|s_i+k\geq w_i\}\end{cases}$$

Proof. Combining the nodal limit of $\psi^{reg}_0(p^sz)$ and $\psi^{reg}_\infty(p^sz)$ in the Proposition 6.7 and the Proposition 6.8. Then combine them with the Proposition 7.4, we obtain the theorem. $\square$

Using this formula, we can see the following things.

Suppose now that given a solution $\psi_0(z)$ of the qde and extend it to $\psi_\infty(z)$ via the curve γ intersecting the unit polydisk at $e^{2\pi is}$, $s\in\mathbb{R}^n$, we know that the transport formula is given by $\text{Trans}_{DT}(s)$. Now we move the curve γ a little bit such that the new curve γ' intersect the unit polydisk with $e^{2\pi is'}$. Now we require that the move of s and s' is close enough such that:

(7.40)

$$(\text{Walls}\cap C(s)^c\cap C(s')^c)^c_{\text{Walls}}\cap(\text{Walls}^-\cap D(s)^c\cap D(s')^c)^c_{\text{Walls}^-}\text{ has only one element}$$

In human langauge, it means that the corresponding wall elements of s and s' has only one element difference. In this way, inversing the curve γ' and consider the loop $\gamma'^{-1}\gamma$, this corresponds to the element $\text{Trans}_{DT}(s')^{-1}\text{Trans}_{DT}(s)$, and if we denote the wall of the difference by w, we can see that:

$$\text{Trans}(s')^{-1}\text{Trans}(s)=\mathbf{B}^*_w \tag{7.41}$$

While we know that the analytic continutation via the loop $\gamma'^{-1}\gamma$ is unique up to homotopy, thus now we have defined the map:

$$\pi_1(\mathbb{P}^r\backslash\mathbf{Sing})\to\text{End}(H_T(M(\mathbf{v},\mathbf{w}))) \tag{7.42}$$

with each loop γ mapped to an ordered product of $\prod_w\mathbf{B}^*_w$.

To obtain the complete description of the monodromy representation, note that since the other quantum difference operator $T^{-1}_{\mathscr{L}}\mathbf{M}_{\mathscr{L}}(z)$ commutes with $T^{-1}_{\mathscr{O}(1)}\mathbf{M}_{\mathscr{O}(1)}(z)$.

In conclusion, we have the following monodromy representation:

Theorem 7.7. *The monodromy representation:*

$$\pi_1(\mathbb{P}^r\backslash\boldsymbol{Sing})\to End(H_T(M(\mathbf{v},\mathbf{w}))) \tag{7.43}$$

of the Dubrovin connection is generated by $\mathbf{B}_{\mathbf{m}}^{*}$ *with* $q_1 = e^{2\pi i\hbar_1}, q_2 = e^{2\pi i\hbar_2}$*, i.e. the monodromy operators* $\mathbf{B}_{\mathbf{m}}$ *in the fixed point basis.*

8. CASE STUDY: THE EQUIVARIANT HILBERT SCHEME OF A_r SINGULARITY

In this section we study in detail about the Hilbert scheme $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$ of A_r surfaces.

As a quiver variety, the corresponding quiver representation is written as:

$$\text{Rep}(\mathbf{v},\mathbf{w}) = \bigoplus_{i\in\mathbb{Z}/r\mathbb{Z}} \text{Hom}(V_i, W_i) \oplus \text{Hom}(V_0, W_0), \qquad V_i = \mathbb{C}^n, W_0 = \mathbb{C} \tag{8.1}$$

The fixed point of the equivariant Hilbert scheme $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$ is denoted by the partitions λ such that $|\lambda| = nr$ such that:

$$\#\{\square \in \lambda | c_{\square} \text{mod } r = i\} = n \tag{8.2}$$

8.1. Wall structure of the Hilbert scheme $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$.

Now we use the above example to give an explicit calcultaion of the wall structure of the Hilbert scheme $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$.

From the definition it is easy to see that the wall hyperplane in $\text{Pic}(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])) \otimes \mathbb{Q} \cong \mathbb{Q}^r$ for the Hilbert scheme $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$ is given by:

$$\mathbf{m} \cdot [i,j) \in \mathbb{Z}, \qquad 1 \leq i \leq j \leq nr \tag{8.3}$$

The finite hyperplane is given by:

$$\mathbf{m} \cdot [i,j) = 0 \tag{8.4}$$

In this case the quantum difference operator can be written as:

$$\mathbf{M}_{\mathscr{L}}(z) = \mathscr{L} \prod_{\substack{\mathbf{m}\in\text{generic points of walls} \\ \mathbf{z}\cdot[i,j)=n, 1\leq i\leq j\leq nr}} \mathbf{B}_{\mathbf{m}}(z) \tag{8.5}$$

such that:

$$\mathbf{B}_{\mathbf{m}}(z) = \begin{cases} \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_{q^2}!} \frac{(-1)^n}{\prod_{l=1}^{n}(1-z^{n[i,j)}p^{|\mathbf{m}|}q^{-ln\boldsymbol{\theta}\cdot((\frac{n^2r-1}{2}\boldsymbol{\theta}+\mathbf{e}_1)-2-2\delta_{1l})})} (Q^{\mathbf{m}}_{-[i,j)})^n (P^{\mathbf{m}}_{[i,j)})^n & (j-i)\text{mod } r \neq 0 \\ \prod_{h=1}^{r} :\exp(-\sum_{k=1} \frac{n_k q^{-\frac{k|\boldsymbol{\delta}_h|}{2}}}{1-z^{-k|\boldsymbol{\delta}_h|}p^{k\mathbf{m}\cdot\boldsymbol{\delta}_h}q^{-\frac{k|\boldsymbol{\delta}_h|}{2}}} \alpha^{\mathbf{m},h}_{-k}\alpha^{\mathbf{m},h}_{k}): & (j-i)\text{mod } r = 0 \end{cases} \tag{8.6}$$

Here we introduce two ways of computing the matrix coefficients of quantum difference operator in terms of stable basis and fixed point basis.

For the stable basis case, one could write down the matrix coefficients of $\mathbf{B}_{\mathbf{m}}(z)$ in terms of the normalized stable basis:

$$P^{\mathbf{m}}_{[i,j)}\cdot s^{\mathbf{m}}_{\boldsymbol{\mu}}=\sum_{\substack{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}=C\text{ is a type }[i,j)\\ \text{cavalcade of }\mathbf{m}\text{-ribbons}}}s^{\mathbf{m}}_{\boldsymbol{\lambda}}(1-q^2)^{\#_C}q^{\text{ht }C+\text{ind}^{\mathbf{m}}_C+N^+_C}\tag{8.7}$$

$$Q^{\mathbf{m}}_{-[i,j)}\cdot s^{\mathbf{m}}_{\boldsymbol{\mu}}=\sum_{\substack{\boldsymbol{\lambda}\backslash\boldsymbol{\mu}=S\text{ is a type }[i,j)\\ \text{stampede of }\mathbf{m}\text{-ribbons}}}s^{\mathbf{m}}_{\boldsymbol{\mu}}(1-q^2)^{\#_S}q^{\text{wd}(S)-\text{ind}^{\mathbf{m}}_S-j+i+N^-_S}\tag{8.8}$$

Using the formula, we have that:

$$\begin{aligned}(Q^{\mathbf{m}}_{-[i,j)})^n(P^{\mathbf{m}}_{[i,j)})^n\cdot s^{\mathbf{m}}_{\boldsymbol{\lambda}_0}=&\sum_{\substack{\boldsymbol{\lambda}_i\backslash\boldsymbol{\lambda}_{i-1}=C_i\text{ is a type }[i,j)\\ \text{cavalcade of }\mathbf{m}\text{-ribbons}}}(Q^{\mathbf{m}}_{-[i,j)})^n s^{\mathbf{m}}_{\boldsymbol{\lambda}_n}(1-q^2)^{\sum_{i=1}^n\#_{C_i}}q^{\sum_{i=1}^n(\text{ht }C_i+\text{ind}^{\mathbf{m}}_{C_i}+N^+_{C_i})}\\ =&\sum_{\substack{\boldsymbol{\lambda}_{n+i-1}\backslash\boldsymbol{\lambda}_{n+i}=S_i\text{ is a type }[i,j)\\ \text{stampede of }\mathbf{m}\text{-ribbons}}}\sum_{\substack{\boldsymbol{\lambda}_i\backslash\boldsymbol{\lambda}_{i-1}=C_i\text{ is a type }[i,j)\\ \text{cavalcade of }\mathbf{m}\text{-ribbons}}}s^{\mathbf{m}}_{\lambda_{2n}}(1-q^2)^{\sum_{i=1}^n\#_{C_i}+\#_{S_i}}q^{\sum_{i=1}^n(\text{ht }C_i+\text{ind}^{\mathbf{m}}_{C_i}}\\ &\times q^{\sum_{i=1}^n(\text{wd}(S_i)-\text{ind}^{\mathbf{m}}_{S_i}-j+i+N^-_{S_i})}\end{aligned}\tag{8.9}$$

In conclusion, the matrix coefficients of the quantum difference operators $\mathbf{M}_{\mathscr{L}}(z)$ can be written as:

$$\begin{aligned}(s^{\mathbf{m}}_{\boldsymbol{\mu}},\mathbf{M}_{\mathscr{L}}(z)s^{\mathbf{m}}_{\boldsymbol{\lambda}})=&(s^{\mathbf{m}}_{\boldsymbol{\mu}},\mathscr{L}\prod_{\substack{\mathbf{m}\in\text{generic points of walls}\\ \mathbf{z}\cdot[i,j)=n,1\leq i\leq j\leq nr}}\mathbf{B}_{\mathbf{m}}(z)s^{\mathbf{m}}_{\boldsymbol{\lambda}})\\ =&(s^{\mathbf{m}}_{\boldsymbol{\mu}},\mathscr{L}s^{\mathbf{m}}_{\boldsymbol{\lambda}_o})\prod_{\mathbf{m},\boldsymbol{\lambda}_i}(s^{\mathbf{m}}_{\boldsymbol{\lambda}_i},\mathbf{B}_{\mathbf{m}}(z)s^{\mathbf{m}}_{\boldsymbol{\lambda}_{i-1}})R^{\mathbf{m},\mathbf{m}'}_{\boldsymbol{\lambda}_{i+1},\boldsymbol{\lambda}_i}\end{aligned}\tag{8.10}$$

Here $\mathbf{m}'$ is the slope right next to the $\mathbf{m}$ with respect to the order of the product. $R^{\mathbf{m},\mathbf{m}'}_{\boldsymbol{\lambda}_{i+1},\boldsymbol{\lambda}_i}$ is the K-theoretic wall R-matrix of slopes $\mathbf{m}$ and $\mathbf{m}'$ at the fixed point basis $\boldsymbol{\lambda}_{i+1}$ and $\boldsymbol{\lambda}_i$.

In the fixed point basis, recall the formula:

$$\langle\boldsymbol{\lambda}|P^{\mathbf{m}}_{[i,j)}|\boldsymbol{\mu}\rangle=P^{\mathbf{m}}_{[i,j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\mu}}\zeta(\frac{\chi_\blacksquare}{\chi_\square})[\frac{u_1}{q\chi_\blacksquare}]]\tag{8.11}$$

$$\langle\boldsymbol{\mu}|Q^{\mathbf{m}}_{-[i,j)}|\boldsymbol{\lambda}\rangle=Q^{\mathbf{m}}_{-[i,j)}(\boldsymbol{\lambda}\backslash\boldsymbol{\mu})\prod_{\blacksquare\in\boldsymbol{\lambda}\backslash\boldsymbol{\mu}}[\prod_{\square\in\boldsymbol{\lambda}}\zeta(\frac{\chi_\square}{\chi_\blacksquare})[\frac{\chi_\blacksquare}{qu_1}]]^{-1}\tag{8.12}$$

So we can write that:

(8.13)
$$(Q^{\mathbf{m}}_{-[i,j)})^n (P^{\mathbf{m}}_{[i,j)})^n|\boldsymbol{\lambda}_0\rangle = (Q^{\mathbf{m}}_{-[i,j)})^n|\boldsymbol{\lambda}_n\rangle \prod_{l=1}^{n} P^{\mathbf{m}}_{[i,j)}(\boldsymbol{\lambda}_l\backslash\boldsymbol{\lambda}_{l-1}) \prod_{\blacksquare_l\in\boldsymbol{\lambda}_l\backslash\boldsymbol{\lambda}_{l-1}} [\prod_{\square_{l-1}\in\boldsymbol{\lambda}_{l-1}} \zeta(\frac{\chi_{\blacksquare_l}}{\chi_{\square_{l-1}}})[\frac{u_1}{q\chi_{\blacksquare_l}}]]$$
$$=\prod_{l=1}^{n} Q^{\mathbf{m}}_{-[i,j)}(\boldsymbol{\lambda}_{n+l-1}\backslash\boldsymbol{\lambda}_{n+l})P^{\mathbf{m}}_{[i,j)}(\boldsymbol{\lambda}_l\backslash\boldsymbol{\lambda}_{l-1}) \prod_{\substack{\blacksquare_l\in\boldsymbol{\lambda}_l\backslash\boldsymbol{\lambda}_{l-1}\\ \blacksquare_{n+l}\in\boldsymbol{\lambda}_{n+l}\backslash\boldsymbol{\lambda}_{n+l-1}}} [\prod_{\substack{\square_{l-1}\in\boldsymbol{\lambda}_{l-1}\\ \square_{n+l-1}\in\boldsymbol{\lambda}_{n+l-1}}} \frac{\zeta(\frac{\chi_{\blacksquare_l}}{\chi_{\square_{l-1}}})}{\zeta(\frac{\chi_{\square_{n+l-1}}}{\chi_{\blacksquare_{n+l}}})}\frac{[\frac{u_1}{q\chi_{\blacksquare_l}}]}{[\frac{\chi_{\blacksquare_{n+l}}}{qu_1}]}]]|\boldsymbol{\lambda}_{2n}\rangle$$

8.2. **Example: Hilb$_3$([$\mathbb{C}^2/\mathbb{Z}_2$]).** Now we give the example of the equivariant Hilbert scheme $\mathrm{Hilb}_3([\mathbb{C}^2/\mathbb{Z}_2])$. As the quiver variety, the corresponding quiver representation is

$$\mathrm{Hom}(\mathbb{C}^3,\mathbb{C}^3)^{\oplus 2}\oplus\mathrm{Hom}(\mathbb{C}^3,\mathbb{C}^2) \tag{8.14}$$

The T-fixed point of the equivariant Hilbert scheme $\mathrm{Hilb}_3([\mathbb{C}^2/\mathbb{Z}_2])$ is given by the single partition λ such that $|\lambda|=6$ and such that for the boxes $\square\in\lambda$ such that $\#\{\square\in\lambda|c_\square \bmod 2=0\}=\#\{\square\in\lambda|c_\square \bmod 2=1\}3$. Here $c_\square=i-j$ with (i,j) the coordinate of the box. Written in the Young diagram, the fixed point set corresponds to the following Young diagrams:

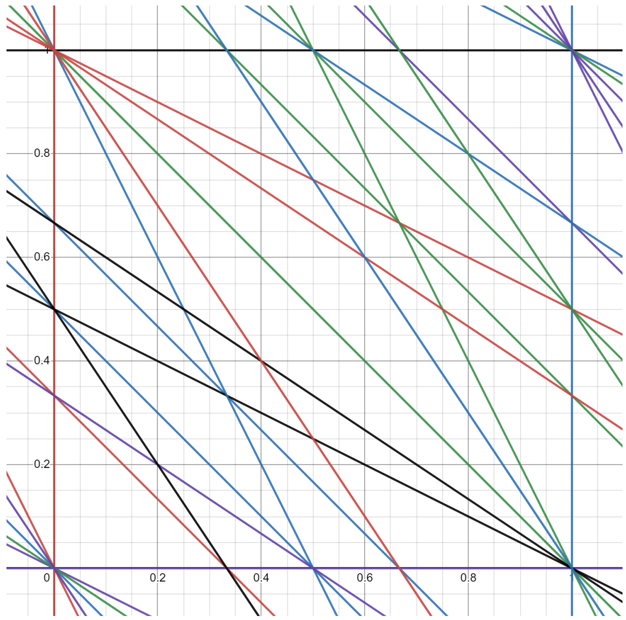
$$\begin{array}{l}0\\1\\0\\1\\0\\1\end{array}\qquad \begin{array}{ll}0&1\\1&\\0&\\1&\\0&\end{array}\qquad \begin{array}{ll}0&1\\1&0\\0&\\1&\end{array}\qquad \begin{array}{ll}0&1\\1&0\\0&1\end{array}\qquad \begin{array}{lll}0&1&0\\1&&\\0&&\\1&&\end{array}\qquad \begin{array}{lll}0&1&0\\1&0&1\end{array} \tag{8.15}$$

$$\begin{array}{llll}0&1&0&1\\1&0&&\end{array}\qquad \begin{array}{lllll}0&1&0&1&0\\1&&&&\end{array}\qquad \begin{array}{llllll}0&1&0&1&0&1\end{array} \tag{8.16}$$

The wall structure on $\mathrm{Hilb}_3([\mathbb{C}^2/\mathbb{Z}_2])$ is given by the following hyperplanes in $\mathbb{Q}^2$:

(8.17)
$$x=n_1,\qquad y=n_2,\qquad x+y=n_3,\qquad x+2y=n_4$$
$$2x+y=n_5,\qquad 2x+2y=n_6,\qquad 3x+2y=n_7,\qquad 2x+3y=n_8,\qquad 3x+3y=n_9$$

The following is the graphs of the wall structure for $\mathrm{Hilb}_3([\mathbb{C}^2/\mathbb{Z}_2])$ in the neighborhood of $[0,1]\times[0,1]$:



By the computation you can see that for the generic slope point $\mathbf{m}$ on the wall $x = n_1$, $y = n_2$, $x + 2y = n_4$, $2x + y = n_5$, $3x + 2y = n_7$, $2x + 3y = n_8$, the monodromy operator $\mathbf{B_m}$ is of the $U_q(\mathfrak{sl}_2)$ type. For the generic slope points $\mathbf{m}$ on the wall $x + y = n_3$, $2x + 2y = n_6$, $3x + 3y = n_9$, the monodromy operator $\mathbf{B_m}$ is of the $U_q(\hat{\mathfrak{gl}}_1)$-type.

8.2.1. *Matrix Coefficients of* $\mathbf{B_m}$. We first write down the monodromy operators $\mathbf{B_m}$ explicitly for $K_T(\mathrm{Hilb}_3([\mathbb{C}^2/\mathbb{Z}_2])$:

$$\mathbf{B_m}(z) = \begin{cases} \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_{q^2}!} \frac{(-1)^n}{\prod_{l=1}^{n}(1-z^{n[i,j)}p^{|\mathbf{m}|})} (Q^{\mathbf{m}}_{-[i,j)})^n (P^{\mathbf{m}}_{[i,j)})^n & (j-i)\mathrm{mod}\ 2 \neq 0 \\ \prod_{h=1}^{2} : \exp(-\sum_{k=1} \frac{n_k}{1-z^{-k|\boldsymbol{\delta}_h|}p^{k\mathbf{m}\cdot\boldsymbol{\delta}_h}} \alpha^{\mathbf{m},h}_{-k} \alpha^{\mathbf{m},h}_{k}) & (j-i)\mathrm{mod}\ 2 = 0 \end{cases} \tag{8.18}$$

This means that

$$\mathbf{B_m} = \begin{cases} \sum_{n=0}^{\infty} \frac{(q-q^{-1})^n}{[n]_{q^2}!} (Q^{\mathbf{m}}_{-[i,j)})^n (P^{\mathbf{m}}_{[i,j)})^n & (j-i)\mathrm{mod}\ 2 \neq 0 \\ \prod_{h=1}^{2} : \exp(-\sum_{k=1} n_k \alpha^{\mathbf{m},h}_{-k} \alpha^{\mathbf{m},h}_{k}) & (j-i)\mathrm{mod}\ 2 = 0 \end{cases} \tag{8.19}$$

Now we do the classification of the monodromy operators on each wall:
(8.20)

$$\mathbf{B}_{\mathbf{m}}(z)=\begin{cases}\sum_{n=0}^{1}\frac{(q-q^{-1})^n}{[n]_{q^2}!}\frac{(-1)^n}{\prod_{l=1}^{n}(1-z_2^n p^{|\mathbf{m}|})}(Q^{\mathbf{m}}_{-[2,3)})^n(P^{\mathbf{m}}_{[2,3)})^n & x=1\\ \sum_{n=0}^{1}\frac{(q-q^{-1})^n}{[n]_{q^2}!}\frac{(-1)^n}{\prod_{l=1}^{n}(1-z_1^n p^{|\mathbf{m}|})}(Q^{\mathbf{m}}_{-[1,2)})^n(P^{\mathbf{m}}_{[1,2)})^n & y=1\\ \sum_{n=0}^{1}\frac{(q-q^{-1})^n}{[n]_{q^2}!}\frac{(-1)^n}{\prod_{l=1}^{n}(1-z_1^{2n}z_2^n p^{|\mathbf{m}|})}(Q^{\mathbf{m}}_{-[1,4)})^n(P^{\mathbf{m}}_{[1,4)})^n & 2x+y=1,2,3\\ \sum_{n=0}^{1}\frac{(q-q^{-1})^n}{[n]_{q^2}!}\frac{(-1)^n}{\prod_{l=1}^{n}(1-z_1^{n}z_2^{2n} p^{|\mathbf{m}|})}(Q^{\mathbf{m}}_{-[2,5)})^n(P^{\mathbf{m}}_{[2,5)})^n & x+2y=1,2,3\\ \sum_{n=0}^{1}\frac{(q-q^{-1})^n}{[n]_{q^2}!}\frac{(-1)^n}{\prod_{l=1}^{n}(1-z_1^{3n}z_2^{2n} p^{|\mathbf{m}|})}(Q^{\mathbf{m}}_{-[1,7)})^n(P^{\mathbf{m}}_{[1,7)})^n & 3x+2y=1,2,3\\ 1 & 2x+3y=1,2,3\\ \prod_{h=1}^{2}(1-\sum_{k=1}^{3}\frac{n_k}{1-z_1^{-k}z_2^{-k}p^k}\alpha^{\mathbf{m},h}_{-k}\alpha^{\mathbf{m},h}_{k}+\frac{n_1n_2}{(1-z_1^{-1}z_2^{-1}p)(1-z_1^{-2}z_2^{-2}p^2)}\times & \\ \times(\alpha^{\mathbf{m},h}_{-1}\alpha^{\mathbf{m},h}_{-2}\alpha^{\mathbf{m},h}_{1}\alpha^{\mathbf{m},h}_{2})-\frac{n_1^3}{6(1-z_1^{-1}z_2^{-1}p)^3}(\alpha^{\mathbf{m},h}_{-1})^3(\alpha^{\mathbf{m},h}_{1})^3) & x+y=1,2\\ \prod_{h=1}^{2}(1-\frac{n_1}{1-z_1^{-2}z_2^{-2}p^2}\alpha^{\mathbf{m},h}_{-2}\alpha^{\mathbf{m},h}_{2}) & 2x+2y=1,3\\ \prod_{h=1}^{2}(1-\frac{n_1}{1-z_1^{-3}z_2^{-3}p^3}\alpha^{\mathbf{m},h}_{-1}\alpha^{\mathbf{m},h}_{1}) & 3x+3y=1,2,4,5\end{cases}$$

Using the formula in the fixed point basis as in 2.65 and 2.66, we can write down the explicit matrix coefficients of the monodromy operators. Otherwise we can use the corresponding stable basis $s^{\mathbf{m}}_{\boldsymbol{\lambda}}$ to compute the matrix coefficients of the monodromy operators. The transition between two stable basis $s^{\mathbf{m}_1}_{\boldsymbol{\lambda}}, s^{\mathbf{m}_2}_{\boldsymbol{\lambda}}$ of different slopes $\mathbf{m}_1$ and $\mathbf{m}_2$ is determined by the wall R-matrix $R^{\mathbf{m}_1,\mathbf{m}_2}_{\mathscr{C}}$ of K-theoretic stable envelope, which has been explained in [Z23].

8.3. **Connection to the Dubrovin connections of Hilbert scheme of A_{r-1}-singularities.** By the main theorem in this paper, the quantum difference equation on $K_T(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r]))$ would degenerate to the Dubrovin connection on the quantum cohomology $H_T^*(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r]))$:

$$\begin{aligned}&Q_{\mathscr{O}(1)}=c_1(\mathscr{O}(1))\cup(-)+(\hbar_1+\hbar_2)\sum_{[i,j)}\frac{|[i,j)|}{1-z^{-[i,j)}}E_{[i,j)}E_{-[i,j)}+\cdots\\ &Q_{\mathscr{L}_l}=c_1(\mathscr{L}_l)\cup(-)+(\hbar_1+\hbar_2)\sum_{[i,j)}\frac{|[i,j)|_l}{1-z^{-[i,j)}}E_{[i,j)}E_{-[i,j)}+\cdots\end{aligned}\tag{8.21}$$

Here $|[i,j)|$ is the sum of the coefficients of $[i,j)$, $|[i,j)|_l$ is the sum of the coefficients of $[i,j)$ in the direction $\mathbf{e}_l$.

In [MO09], Maulik and Oblomkov have proved that for the Dubrovin connection over the quantum cohomology of the Hilbert scheme of A_{r-1}-singularities $H_T^*(\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r}))$, the Dubrovin connection can be written as:

(8.22)
$$
\begin{aligned}
M_{\mathscr{O}(1)} =& c_1(\mathscr{O}(1))\cup(-)+(\hbar_1+\hbar_2)\sum_{1\leq i\leq j\leq r}\sum_{k\in\mathbb{Z}}:e_{ji}(k)e_{ij}(-k):\frac{k(z_1\cdots z_r)^k z_i\cdots z_{j-1}}{1-(z_1\cdots z_r)^k z_i\cdots z_{j-1}}\\
&+\sum_{k\geq 1}[rt_1t_2p_{-k}(1)p_k(1)+\sum_{i=1}^{r-1}p_{-k}(E_i)p_k(\omega_i)](\frac{k(z_1\cdots z_r)^k}{1-(z_1\cdots z_r)^k}-\frac{z_1\cdots z_r}{1-z_1\cdots z_r})
\end{aligned}
$$

(8.23)
$$
M_{\mathscr{L}_i}=c_1(\mathscr{L}_i)\cup(-)-(\hbar_1+\hbar_2)\sum_{1\leq i\leq l\leq j\leq r}\sum_{k\in\mathbb{Z}}:e_{ji}(k)e_{ij}(-k):\frac{(z_1\cdots z_r)^k z_i\cdots z_{j-1}}{1-(z_1\cdots z_r)^k z_i\cdots z_{j-1}}
$$

Here we use the notations in [MO09]. The equivariant Hilbert scheme $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$ and the Hilbert scheme $\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r})$ are the cyclic quiver varieties of the same quiver representation, but with different stability condition. By the flop isomorphism $F: H_T^*(\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r}))\cong H_T^*(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r]))$ defined in [MO12], together with commutative diagram (4.38) in [MO12] and the Theorem 7.2.1 in [MO12], we have that the quantum multiplication by divisors for $\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])$ and $\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r})$ are intertwined by the flop isomorphism up to a scaling operator:

(8.24)
$$
\begin{array}{ccc}
H_T^*(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])) & \xrightarrow{F} & H_T^*(\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r}))\\
\big\downarrow Q_{\mathscr{L}_i} & & \big\downarrow M_{\mathscr{L}_i}\\
H_T^*(\text{Hilb}_n([\mathbb{C}^2/\mathbb{Z}_r])) & \xrightarrow{F} & H_T^*(\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r}))
\end{array}
$$

It is conjectured that the quantum difference operator $\mathbf{M}_{\mathscr{L}_i}(z)$ in the eigenbasis H of $\mathbf{M}_{\mathscr{L}_i}(\infty)$ has the degeneration limit as the quantum multiplication by $M_{\mathscr{L}_i}$ in the quantum cohomology of $H_T^*(\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r}))$.

On the viewpoint from the stable envelope, this means that the regular solution of the quantum difference equation from $z=\infty$ to $z=0$ has the connection matrix identified as the elliptic geometric R-matrix defined in [AO21]. On the algebraic point of view, this means that we need to identify $\mathbf{M}_{\mathscr{L}_i}(z)$ with the line bundle $\mathscr{L}_i$ on $\text{Hilb}_n(\widehat{\mathbb{C}^2/\mathbb{Z}_r})$ up to conjugacy. For now we still don't know how to prove the fact in a straightway, and this will be put into the future study.

9. Appendix: Basics of the abelian functions

In this appendix we shall give a brief introduction on the theory of the function field of meromorphic functions over complex tori. The complex tori we consider is $\mathscr{A}:=\mathbb{C}^n/L$ with $L\subset\mathbb{C}^n$ a periodic full-rank lattice in $\mathbb{C}^n$.

9.1. **Theta Functions.** Given Ω an $n \times n$ symmetric complex matrices with positive definite imaginary part, we define the Jacobi theta function $\theta(z, \Omega)$ associated to $z \in (\mathbb{C}^*)^n$ and Ω as:

$$\theta(z;\Omega) := \sum_{\mathbf{n}\in\mathbb{Z}^n} z^{\mathbf{n}} \exp(\frac{1}{2}\mathbf{n}^t\Omega\mathbf{n}) \tag{9.1}$$

It is easy to check that:

$$\vartheta(e^{2\pi i}z;\Omega) = \theta(z;\Omega), \qquad \theta(\exp(\Omega\mathscr{L})z;\Omega) = \exp(-\frac{1}{2}\mathscr{L}^t\Omega\mathscr{L})z^{-\mathscr{L}}\theta(z;\Omega) \tag{9.2}$$

Moreover if we take $z = e^{2\pi i \boldsymbol{\xi}}$, the Jacobi theta function can be written as:

$$\vartheta(z;\Omega) := \sum_{\mathbf{n}\in\mathbb{Z}^n} \exp(2\pi i \mathbf{n}^t\boldsymbol{\xi}) \exp(\frac{1}{2}\mathbf{n}^t\Omega\mathbf{n}) \tag{9.3}$$

Also given $a, b \in \mathbb{R}^g$, we define the theta function $\vartheta\begin{pmatrix} a \\ b \end{pmatrix}(z;\Omega)$ as:

$$\vartheta\begin{pmatrix} a \\ b \end{pmatrix}(z;\Omega) = \exp(\frac{1}{2}a^t\Omega a + a^t(\xi + b))\vartheta(\xi + \Omega\cdot a + b;\Omega) \tag{9.4}$$

such that:

$$\vartheta\begin{pmatrix} a \\ b \end{pmatrix}(z + \Omega\cdot m + n;\Omega) = \exp(2\pi i(a^t n - b^t m))\exp(-\frac{1}{2}m^t\Omega m - 2\pi i m^t z)\vartheta\begin{pmatrix} a \\ b \end{pmatrix}(z;\Omega), \qquad \forall m, n \in \mathbb{Z} \tag{9.5}$$

As the special example, if we take $n = 1$ and $\Omega = \tau$, and set $q = e^{2\pi i\tau}$, the Jacobi theta function can be written as:

$$\vartheta(z;q) := \sum_{n\in\mathbb{Z}} q^{\frac{n^2}{2}} z^n = \prod_{n=1}^{\infty}(1 - q^{2n})(1 + q^{2n-1}z)(1 + q^{2n-1}z^{-1}) \tag{9.6}$$

It satisfies the following modular property:

$$\vartheta(e^{2\pi i\xi}; e^{2\pi i\tau}) = \frac{i}{\sqrt{\tau}} e^{-\frac{i\pi\xi^2}{\tau}} \vartheta(e^{\frac{2\pi i\xi}{\tau}}; e^{-\frac{2\pi i}{\tau}}) \tag{9.7}$$

In the context of [AO21], they use the odd Jacobi theta function:

$$\theta(z|q) = (z^{1/2} - z^{-1/2}) \prod_{n\geq 1}(1 - q^n z)(1 - q^n z^{-1}) \tag{9.8}$$

The relation between the Jacobi theta function and the odd Jacobi theta function can be written as:

$$\vartheta(-zq^{1/2};q) = -z^{\frac{1}{2}}\varphi(q)\theta(z), \qquad \varphi(q) = \prod_{n=1}^{\infty}(1 - q^{2n}) \tag{9.9}$$

In our settings, We will Jacobi theta function $\vartheta(z;\Omega)$ and the odd Jacobi theta function $\theta(z|q)$ depending on the situation. It does not matter too much which kind of theta function we would use since they are only different up to a scaling transformation.

9.2. **Line bundles over the abelian varieties.** Fix the complex space $\mathbb{C}^n = V$ and a rank $2n$ lattice $\Lambda \subset V$, and we define $X = V/\Lambda$. The line bundle L on X is uniquely determined by the pair (H,χ). the Hermitian form $H : V \times V \to \mathbb{C}$ such that $\mathrm{Im}(\Lambda,\Lambda) \subset \mathbb{Z}$. $\chi : \Lambda \to S^1 \subset \mathbb{C}$ is the semicharacter for H, which is defined as:

$$\chi(\lambda+\mu) = \chi(\lambda)\chi(\mu)\exp(\pi i \mathrm{Im}\, H(\lambda,\mu)), \qquad \forall \lambda,\mu \in \Lambda \tag{9.10}$$

The construction is given by that given (H,χ) we can define the automorphy form $a = a_{(H,\chi)} : \Lambda \times V \to \mathbb{C}^*$ by

$$a(\lambda,v) := \chi(\lambda)\exp(\pi H(v,\lambda) + \frac{\pi}{2}H(\lambda,\lambda)) \tag{9.11}$$

which satisfies the cocycle relation:

$$a(\lambda+\mu,v) = a(\lambda,v+\mu)a(\mu,v) \tag{9.12}$$

In this way, we can define the line bundle $L(H,\chi) := V \times \mathbb{C}/\Lambda$ such that Λ acts on $V \times \mathbb{C}$ by $\lambda \cdot (v,t) = (v+\lambda, a(\lambda,v)t)$.

Now suppose that given a line bundle L with the automorphy form $f(\lambda,v)$, the elements in the global holomorphic section $H^0(L)$ of L can be thought of as the set of holomorphic function $\vartheta : V \to \mathbb{C}$ such that

$$\vartheta(v+\lambda) = f(\lambda,v)\vartheta(v) \tag{9.13}$$

It is known that the global section of the line bundle can be written as $\vartheta(z^{\mathbf{n}}e^{2\pi\Omega\cdot\mathbf{m}};\Omega)$. We can choose vector $\mathbf{n} \in \mathbb{Z}^n$ and $\mathbf{m} \in \mathbb{R}^n$ such that $\vartheta(z^{\mathbf{n}}e^{2\pi\Omega\cdot\mathbf{m}};\Omega)$ satisfy the difference equation 9.13.

Moreover, if we consider the space of meromorphic sections $\Gamma_{rat}(L)$, it is generated by the products of the theta functions of the form:

$$\prod_i \vartheta(z^{\mathbf{n}_i}e^{2\pi\Omega\cdot\mathbf{m}_i};\Omega)^{\lambda_i}, \qquad \lambda_i \in \mathbb{Z} \tag{9.14}$$

For the specific example that we use in this paper, we consider the space of meromorphic sections of the structure sheaf $\Gamma_{rat}(\mathcal{O}_X)$, i.e. the space of abelian functions over X. This means that the section $f(z) \in \Gamma_{rat}(\mathcal{O}_X)$ is meromorphic and Λ-periodic. This implies that $\Gamma_{rat}(\mathcal{O}_X)$ is generated by the theta functions of the form $\prod_i \vartheta(z^{\mathbf{n}_i}e^{2\pi\Omega\cdot\mathbf{m}_i};\Omega)^{\lambda_i}$, $\lambda_i \in \mathbb{Z}$ such that $\sum_i \lambda_i\mathbf{n}_i = \sum_i \Omega\cdot(\lambda_i\mathbf{m}_i) = 0$.

More precisely, if we choose the lattice of the form $\Lambda_{\Omega,D} = \Omega\cdot\mathbb{Z}^g \oplus D\cdot\mathbb{Z}^g$. Given $a,b \in \mathbb{R}^g$, let $\{a_k | 1 \leq k \leq r^g\}$ be a system of representatives of $(a + D^{-1}\mathbb{Z}^g)/\mathbb{Z}^g$, and let $\{b_l | 1 \leq l \leq \prod_{i=1}^g d_i\}$ be a system of representatives of $(r^{-1}b + r^{-1}\mathbb{Z}^g)/\mathbb{Z}^g$.

Then the family of theta functions $\{\theta\begin{pmatrix} a_k \\ b_l \end{pmatrix}(z; r^{-1}\Omega)\}$ defines a projective embedding $\mathbb{C}^g/\Lambda_{\Omega,D} \hookrightarrow \mathbb{P}^{(\prod_{i=1}^g d_i)\cdot r^g - 1}$.

Using the projective embedding, we can show that the abelian function on $\mathbb{C}^g/\Lambda_{\Omega,D}$ is generated by the following family of quotients of theta functions:

$$\frac{\vartheta\begin{pmatrix} a_{k_1} \\ b_{l_1} \end{pmatrix}(z; r^{-1}\Omega)}{\vartheta\begin{pmatrix} a_{k_2} \\ b_{l_2} \end{pmatrix}(z; r^{-1}\Omega)}, \qquad 1 \leq k_1, k_2 \leq r^g, 1 \leq l_1, l_2 \leq \prod_{i=1}^{g} d_i \tag{9.15}$$

9.3. **Nodal Limit of the abelian functions.** In our setting we only consider the special case that $\Omega = \tau \mathrm{Id}$, $D = \mathrm{Id}$ with $\tau \in \mathbb{H}$. In this case:

$$\vartheta\begin{pmatrix} a \\ b \end{pmatrix}(z; \Omega) = \exp\left(\sum_{i=1}^{g} \pi i \tau a_i^2 + 2\pi i a_i (z_i + b_i)\right) \prod_{i=1}^{g} \vartheta(z_i + a_i\tau + b_i | \tau) \tag{9.16}$$

In this case, we can choose $b = 0$, and the abelian function on $\mathbb{C}^g/\Lambda_{\Omega,D}$ is generated by:

$$\exp\left(\sum_{i=1}^{g} \pi i r^{-1}\tau(a_i^2 - b_i^2) + 2\pi i (a_i - b_i) z_i\right) \prod_{i=1}^{g} \frac{\vartheta(z_i + r^{-1}a_i\tau | r^{-1}\tau)}{\vartheta(z_i + r^{-1}b_i\tau | r^{-1}\tau)} \tag{9.17}$$

Using the modular duality for the theta function:

$$\vartheta(e^{2\pi i \xi}, e^{2\pi i \tau}) = \frac{i}{\sqrt{\tau}} e^{-\frac{i\pi\xi^2}{\tau}} \vartheta(e^{\frac{2\pi i \xi}{\tau}}, e^{-\frac{2\pi i}{\tau}}) \tag{9.18}$$

Now we compute the limit $\tau \to 0$ of the above generators:

$$\begin{aligned} &\lim_{\tau \to 0} \exp\left(\sum_{i=1}^{g} \pi i r^{-1}\tau(a_i^2 - b_i^2) + 2\pi i (a_i - b_i) z_i\right) \prod_{i=1}^{g} \frac{\vartheta(z_i + r^{-1}a_i\tau | r^{-1}\tau)}{\vartheta(z_i + r^{-1}b_i\tau | r^{-1}\tau)} \\ =& \lim_{\tau \to 0} \prod_{i=1}^{g} \frac{\vartheta(\frac{rz_i}{\tau} + a_i | -\frac{r}{\tau})}{\vartheta(\frac{rz_i}{\tau} + b_i | -\frac{r}{\tau})} = \lim_{q \to 0} \prod_{i=1}^{g} \frac{\theta(q^{-z_i} e^{2\pi i a_i})}{\theta(q^{-z_i} e^{2\pi i b_i})}, \qquad q = e^{-\frac{2\pi i r}{\tau}} \end{aligned} \tag{9.19}$$

i.e. we have that in terms of the odd Jacobi theta function:

$$\lim_{\tau \to 0} \exp\left(\sum_{i=1}^{g} 2\pi i (a_i - b_i) z_i\right) \prod_{i=1}^{g} \frac{\theta(e^{2\pi i (z_i + r^{-1}a_i\tau)} | e^{2\pi i r^{-1}\tau})}{\theta(e^{2\pi i (z_i + r^{-1}b_i\tau)} | e^{2\pi i r^{-1}\tau})} = \lim_{q \to 0} \prod_{i=1}^{g} \frac{\theta(q^{-z_i} e^{2\pi i a_i})}{\theta(q^{-z_i} e^{2\pi i b_i})}, \qquad q = e^{-\frac{2\pi i r}{\tau}} \tag{9.20}$$

This formula also appears in the Lemma 9 of [S21], which plays an important role in the computation of the cohomological limit of the connection matrix and the elliptic stable envelope of the Hilbert scheme of points $\mathrm{Hilb}_n(\mathbb{C}^2)$.

REFERENCES

[ABRR97] Arnaudon D, Buffenoir E, Ragoucy E, et al. Universal solutions of quantum dynamical Yang-Baxter equations[J]. arXiv preprint q-alg/9712037, 1997.

[AKMMSZ18] Awata H, Kanno H, Mironov A, et al. (q, t)-KZ equations for quantum toroidal algebra and Nekrasov partition functions on ALE spaces[J]. Journal of High Energy Physics, 2018, 2018(3): 1-70.

[AO21] Aganagic M, Okounkov A. Elliptic stable envelopes[J]. Journal of the American Mathematical Society, 2021, 34(1): 79-133.

[BD23] Botta T M, Davison B. Okounkov's conjecture via BPS Lie algebras[J]. arXiv preprint arXiv:2312.14008, 2023.

[ATL15] Appel A, Toledano-Laredo V. Monodromy of the Casimir connection of a symmetrisable Kac-Moody algebra[J]. arXiv preprint arXiv:1512.03041, 2015.

[BM15] Balagović M. Degeneration of trigonometric dynamical difference equations for quantum loop algebras to trigonometric Casimir equations for Yangians[J]. Communications in Mathematical Physics, 2015, 334(2): 629-659.

[BMO11] Braverman A, Maulik D, Okounkov A. Quantum cohomology of the Springer resolution[J]. Advances in Mathematics, 2011, 227(1): 421-458.

[BP08] Bryan J, Pandharipande R. The local Gromov-Witten theory of curves[J]. Journal of the American Mathematical Society, 2008, 21(1): 101-136.

[BT19] Bershtein M, Tsymbaliuk A. Homomorphisms between different quantum toroidal and affine Yangian algebras[J]. Journal of Pure and Applied Algebra, 2019, 223(2): 867-899.

[CK99] Cox D A, Katz S. Mirror symmetry and algebraic geometry[M]. Providence, RI: American Mathematical Society, 1999.

[D21] Dinkins H. Elliptic stable envelopes of affine type A quiver varieties[J]. arXiv preprint arXiv:2107.09569, 2021.

[D22] Danilenko I. Quantum Differential Equation for Slices of the Affine Grassmannian[J]. arXiv preprint arXiv:2210.17061, 2022.

[EV02] Etingof P, Varchenko A. Dynamical Weyl groups and applications[J]. Advances in Mathematics, 2002, 167(1): 74-127.

[GL11] Gautam S, Laredo V T. Monodromy of the trigonometric Casimir connection for sl2[R]. Noncommutative birational geometry, representations and combinatorics, 2011.

[GNW18] Guay N, Nakajima H, Wendlandt C. Coproduct for Yangians of affine Kac–Moody algebras[J]. Advances in Mathematics, 2018, 338: 865-911.

[KR90] Kirillov A N, Reshetikhin N. q-Weyl group and a multiplicative formula for universal R-matrices[J]. 1990.

[KT91] Khoroshkin S M, Tolstoy V N. Universal R-matrix for quantized (super) algebras[J]. Communications in Mathematical Physics, 1991, 141: 599-617.

[KT92] Tolstoy V N, Khoroshkin S M. The universal R-matrix for quantum untwisted affine Lie algebras[J]. Functional analysis and its applications, 1992, 26(1): 69-71.

[KT92B] Khoroshkin S M, Tolstoy V N. The uniqueness theorem for the universal R-matrix[J]. Letters in mathematical physics, 1992, 24: 231-244.

[MNO06A] Maulik D, Nekrasov N, Okounkov A, et al. Gromov–witten theory and donaldson–thomas theory, i[J]. Compositio Mathematica, 2006, 142(5): 1263-1285.

[MNO06B] Maulik D, Nekrasov N, Okounkov A, et al. Gromov–witten theory and donaldson–thomas theory, ii[J]. Compositio Mathematica, 2006, 142(5): 1286-1304.

[MO09] Maulik D, Oblomkov A. Quantum cohomology of the Hilbert scheme of points on A_n-resolutions[J]. Journal of the American Mathematical Society, 2009, 22(4): 1055-1091.

[MO12] Maulik D, Okounkov A. Quantum groups and quantum cohomology[J]. arXiv preprint arXiv:1211.1287, 2012.

[MOO11] Maulik D, Oblomkov A, Okounkov A, et al. Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds[J]. Inventiones mathematicae, 2011, 186: 435-479.

[MV07] Mirković I, Vybornov M. Quiver varieties and Beilinson-Drinfeld Grassmannians of type A[J]. arXiv preprint arXiv:0712.4160, 2007.

[N15] Negut A. Quantum algebras and cyclic quiver varieties[M]. Columbia University, 2015.

[N19] Neguţ A. The PBW basis of $U_{q,\bar{q}}(\ddot{\mathfrak{gl}}_n)$[J]. arXiv preprint arXiv:1905.06277, 2019.

[N20] Neguţ A. Quantum toroidal and shuffle algebras[J]. Advances in Mathematics, 2020, 372: 107288.

[N22] Neguț A. SHUFFLE ALGEBRAS FOR QUIVERS AND R-MATRICES[J]. Journal of the Institute of Mathematics of Jussieu, 2022: 1-36.

[N23] Neguţ A. Quantum loop groups and K-theoretic stable envelopes[J]. arXiv preprint arXiv:2303.12041, 2023.

[NH99] Nakajima H. Lectures on Hilbert schemes of points on surfaces[M]. American Mathematical Soc., 1999.

[O15] Okounkov A. Lectures on K-theoretic computations in enumerative geometry[J]. arXiv preprint arXiv:1512.07363, 2015.

[OP10] Okounkov A, Pandharipande R. The local Donaldson–Thomas theory of curves[J]. Geometry & Topology, 2010, 14(3): 1503-1567.

[OP10B] Okounkov A, Pandharipande R. Quantum cohomology of the Hilbert scheme of points in the plane[J]. Inventiones mathematicae, 2010, 179(3): 523-557.

[O20] Okounkov A. Nonabelian stable envelopes, vertex functions with descendents, and integral solutions of q-difference equations[J]. arXiv preprint arXiv:2010.13217, 2020.

[OS22] Okounkov A, Smirnov A. Quantum difference equation for Nakajima varieties[J]. Inventiones mathematicae, 2022, 229(3): 1203-1299.

[R01] Rosu I. Equivariant elliptic cohomology and rigidity[J]. American Journal of Mathematics, 2001, 123(4): 647-677.

[R11] Rybnikov G L. On the fundamental group of the complement of a complex hyperplane arrangement[J]. Functional Analysis and Its Applications, 2011, 45(2): 137-148.

[S16] Smirnov A. Rationality of capped descendent vertex in K-theory[J]. arXiv preprint arXiv:1612.01048, 2016.

[S20] Smirnov A. Elliptic stable envelope for Hilbert scheme of points in the plane[J]. Selecta Mathematica, 2020, 26(1): 1-57.

[S21] Smirnov A. Quantum differential and difference equations for $\mathrm{Hilb}^n(\mathbb{C}^2)$[J]. arXiv preprint arXiv:2102.10726, 2021.

[S87] Salvetti M. Topology of the complement of real hyperplanes in CN[J]. Invent. math, 1987, 88(3): 603-618.

[SV13] Schiffmann O, Vasserot E. The elliptic Hall algebra and the K-theory of the Hilbert scheme of A2[J]. Duke Mathematical Journal, 2013, 162(2): 279-366.

[SV17] Schiffmann O, Vasserot E. On cohomological Hall algebras of quivers: Yangians (2017)[J]. arXiv preprint arXiv:1705.07491.

[SVC13] Schiffmann O, Vasserot E. Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on A 2[J]. Publications mathématiques de l'IHÉS, 2013, 118(1): 213-342.

[Z23] Zhu T. Quantum Difference equation for the affine type A quiver varieties I: General Construction[J]. arXiv preprint arXiv:2308.00550, 2023.

[Z24] Zhu T. Quantum difference equation from shuffle algebras. In preparation.

Yau Mathematical Sciences Center

Email address: ztq20@tsinghua.edu.cn