DEFORMATION MAPS OF QUASI-TWILLED LIE ALGEBRAS

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ABSTRACT. In this paper, we provide a unified approach to study the cohomology theories and deformation theories of various types of operators in the category of Lie algebras, including modified r-matrices, crossed homomorphisms, derivations, homomorphisms, relative Rota-Baxter operators, twisted Rota-Baxter operators, Reynolds operators and deformation maps of matched pairs of Lie algebras. The main ingredients are quasi-twilled Lie algebras. We introduce two types of deformation maps of a quasi-twilled Lie algebra. Deformation maps of type I unify modified r-matrices, crossed homomorphisms, derivations and homomorphisms between Lie algebras, while deformation maps of type II unify relative Rota-Baxter operators, twisted Rota-Baxter operators, Reynolds operators and deformation maps of matched pairs of Lie algebras. We further give the controlling algebras and cohomologies of these two types of deformation maps, which not only recover the existing results for crossed homomorphisms, derivations, homomorphisms, relative Rota-Baxter operators, twisted Rota-Baxter operators and Reynolds operators, but also leads to some new results which are unable to solve before, e.g. the controlling algebras and cohomologies of modified r-matrices and deformation maps of matched pairs of Lie algebras.

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1. Introduction

A classical approach to study a mathematical structure is to associate to it invariants. Among these, cohomology theories occupy a central position as they enable for example to control deformation or extension problems. The concept of a formal deformation of an algebraic structure began with the seminal work of Gerstenhaber [21, 22] for associative algebras. Nijenhuis and

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Richardson extended this study to Lie algebras [41, 43]. Deformations of other algebraic structures such as pre-Lie algebras have also been developed [8]. More generally, deformation theory for algebras over quadratic operads was developed by Balavoine [3]. For more general operads we refer the reader to [28, 35, 39], and the references therein. There is a well known slogan, often attributed to Deligne, Drinfeld and Kontsevich: *every reasonable deformation theory is controlled by a differential graded (dg) Lie algebra, determined up to quasi-isomorphism.* This slogan has been made into a rigorous theorem by Lurie and Pridham, cf. [37, 45].

It is also meaningful to deform *maps* compatible with given algebraic structures. Recently, the cohomology and deformation theories of various operators were established with fruitful applications, e.g morphisms [5, 7, 14, 18, 19, 20], derivations [53], *O*-operators (also called relative Rota-Baxter operators) [9, 15, 50, 54], crossed homomorphisms [12, 25, 44], twisted Rota-Baxter operators and Reynolds operators [10, 11]. The key step in most of the above studies is to construct the controlling algebra, namely an algebra whose Maurer-Cartan elements are the given structures, using the method of derived brackets [30, 38, 55]. Then twisting the controlling algebra by a Maurer-Cartan element, one can obtain the algebra that governs deformations of the given operator, as well as the coboundary operator in the deformation complex.

Modified *r*-matrices, namely solutions of the modified Yang-Baxter equation, are important operators that have deep applications in mathematical physics, e.g. the Lax equation and the factorization problem [46, 48, 49]. Even though there are very fruitful results for various kinds of operators as aforementioned, but the controlling algebra for modified *r*-matrices is still unknown. On the other hand, the notion of deformation maps of matched pairs of Lie algebras was introduced in [1, 2] in the study of classifying compliments. It is also useful to develop the cohomology and deformation theories for deformation maps of matched pairs of Lie algebras.

We propose a unified approach to study all aforementioned operators. On the one hand, we recover all the existing theories. On the other hand, we obtain some new results. We give the controlling algebra of modified r-matrices, which is a curved L_{∞} -algebra, and establish the cohomology and deformation theories for deformation maps of matched pairs of Lie algebras. We study all the operators under the general framework of quasi-twilled Lie algebras, which are generalization of quasi-Lie bialgebras [29, 31], and include direct sum of Lie algebras, semidirect products, action Lie algebras and matched pairs of Lie algebras as particular cases. We introduce two types of deformation maps of a quasi-twilled Lie algebra. Deformation maps of type I unify modified r-matrices, crossed homomorphisms, derivations and homomorphisms between Lie algebras, and deformation maps of type II unify relative Rota-Baxter operators, twisted Rota-Baxter operators, Reynolds operators and deformation maps of matched pairs of Lie algebras. We further give the controlling algebras and cohomologies of these two types of deformation maps, and realize the above purposes.

Note that simultaneous deformations of parts of aforementioned operators and algebras were studied in [5, 13, 19, 33, 58], and it would be helpful to develop a unified approach to study simultaneous deformations, which will be considered in the future.

The paper is organized as follows. In Section 2, we introduce the notion of a quasi-twilled Lie algebra and give various examples. In Section 3, we introduce the notion of a deformation map of type I of a quasi-twilled Lie algebra, and give its controlling algebra and cohomology. In particular, we obtain the curved L_{∞} -algebra, whose Maurer-Cartan elements are modified r-matrices. In Section 4, we introduce the notion of a deformation map of type II of a quasi-twilled Lie algebra, and give its controlling algebra and cohomology. As a byproduct, we give the controlling algebra and cohomology of a deformation map of a matched pair of Lie algebras.

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2. Quasi-twilled Lie algebras

In this section, we introduce the notion of a quasi-twilled Lie algebra, and give various examples.

Let g be a vector space. Define the graded vector space $\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}g, \mathfrak{g})$ with the degree of elements in $\operatorname{Hom}(\wedge^n g, \mathfrak{g})$ being n-1. For $f \in \operatorname{Hom}(\wedge^m g, \mathfrak{g}), g \in \operatorname{Hom}(\wedge^n g, \mathfrak{g})$, the Nijenhuis-Richardson bracket $[\cdot, \cdot]_{NR}$ is defined by

$$[f,g]_{NR} := f \circ g - (-1)^{(m-1)(n-1)}g \circ f,$$

with $f \circ g \in \text{Hom}(\wedge^{m+n-1}\mathfrak{g}, \mathfrak{g})$ being defined by

$$(1) (f \circ g)(x_1, \cdots, x_{m+n-1}) := \sum_{\sigma \in S(n, m-1)} (-1)^{\sigma} f(g(x_{\sigma(1)}, \cdots, x_{\sigma(n)}), x_{\sigma(n+1)}, \cdots, x_{\sigma(m+n-1)}),$$

where the sum is over (n, m-1)-shuffles. Recall that a permutation $\tau \in S_n$ is called an (i, n-i)-shuffle if $\tau(1) < \cdots < \tau(i)$ and $\tau(i+1) < \cdots < \tau(n)$. Then $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{g}), [\cdot, \cdot]_{NR})$ is a graded Lie algebra [41, 43]. With this setup, a Lie algebra structure on \mathfrak{g} is precisely a solution $\pi \in \operatorname{Hom}(\wedge^2\mathfrak{g}, \mathfrak{g})$ of the Maurer-Cartan equation

$$[\pi, \pi]_{NR} = 0.$$

Let g and h be vector spaces. The elements in g are denoted by x and the elements in h are denoted by u. For a multilinear map $\kappa : \wedge^k g \otimes \wedge^l h \to g$, we define $\hat{\kappa} \in \text{Hom}(\wedge^{k+l}(g \oplus h), g \oplus h)$ by

$$\hat{\kappa}\Big((x_1,u_1),\cdots,(x_{k+l},u_{k+l})\Big) = \sum_{\tau \in S(k,l)} (-1)^{\tau} \Big(\kappa(x_{\tau(1)},\cdots,x_{\tau(k)},u_{\tau(k+1)},\cdots,u_{\tau(k+l)}),0\Big).$$

Similarly, for a multilinear map $\kappa : \wedge^k \mathfrak{g} \otimes \wedge^l \mathfrak{h} \to \mathfrak{h}$, we define $\hat{\kappa} \in \operatorname{Hom}(\wedge^{k+l}(\mathfrak{g} \oplus \mathfrak{h}), \mathfrak{g} \oplus \mathfrak{h})$ by

$$\hat{\kappa}\Big((x_1,u_1),\cdots,(x_{k+l},u_{k+l})\Big) = \sum_{\tau \in S(k,l)} (-1)^{\tau}\Big(0,\kappa(x_{\tau(1)},\cdots,x_{\tau(k)},u_{\tau(k+1)},\cdots,u_{\tau(k+l)})\Big).$$

The linear map $\hat{\kappa}$ is called a **lift** of κ . We define $g^{k,l} = \wedge^k g \otimes \wedge^l \mathfrak{h}$. Then $\wedge^n (g \oplus \mathfrak{h}) \cong \bigoplus_{k+l=n} g^{k,l}$ and $\operatorname{Hom}(\wedge^n (g \oplus \mathfrak{h}), g \oplus \mathfrak{h}) \cong (\bigoplus_{k+l=n} \operatorname{Hom}(g^{k,l}, g)) \oplus (\bigoplus_{k+l=n} \operatorname{Hom}(g^{k,l}, \mathfrak{h}))$, where the isomorphism is given by the lift. See [33, 54] for more details. In the sequel, we will omit the notation $\hat{\cdot}$.

Definition 2.1. Let $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}})$ be a Lie algebra with a decomposition into two subspaces $\mathcal{G} = \mathfrak{g} \oplus \mathfrak{h}$. The triple $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ is called a **quasi-twilled Lie algebra** if \mathfrak{h} is a Lie subalgebra of $(\mathcal{G}, [\cdot, \cdot]_{\mathcal{G}})$.

Let $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ be a quasi-twilled Lie algebra. Denote the Lie bracket of \mathcal{G} by Ω . Then there exists $\pi \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}), \ \rho \in \text{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{h}), \ \mu \in \text{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{h}), \ \eta \in \text{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{g}) \text{ and } \theta \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h}),$ such that

(2)
$$\Omega = \pi + \rho + \mu + \eta + \theta.$$

More precisely, for all $x, y \in \mathfrak{g}$, $u, v \in \mathfrak{h}$, we have

$$\Omega((x, u), (y, v)) = (\pi(x, y) + \eta(x, v) - \eta(y, u), \mu(u, v) + \rho(x, v) - \rho(y, u) + \theta(x, v)).$$

In fact, $[\Omega, \Omega]_{NR} = 0$ if and only if the following equations hold

(3)
$$\begin{cases} [\mu,\mu]_{NR} &= 0, \\ [\pi,\pi]_{NR} + 2\eta \circ \theta &= 0, \\ 2[\mu,\eta]_{NR} + [\eta,\eta]_{NR} &= 0, \\ [\rho,\mu]_{NR} + \rho \circ \eta &= 0, \\ [\rho,\theta]_{NR} + [\pi,\theta]_{NR} &= 0, \\ [\pi,\eta]_{NR} + \eta \circ \rho &= 0, \\ [\mu,\theta]_{NR} + [\pi,\rho]_{NR} + \theta \circ \eta + \frac{1}{2}[\rho,\rho]_{NR} &= 0. \end{cases}$$

Proposition 2.2. With the above notations, $\sigma: \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ is a representation of the Lie algebra (\mathfrak{h}, μ) on the vector space \mathfrak{g} , where

$$\sigma(v)x = -\eta(x, v), \quad \forall x \in \mathfrak{g}, v \in \mathfrak{h}.$$

Proof. By (3), we have $2[\mu, \eta]_{NR} + [\eta, \eta]_{NR} = 0$ which implies that $\sigma(v)x = -\eta(x, v)$ is a representation of the Lie algebra (\mathfrak{h}, μ) on the vector space \mathfrak{g} .

Quasi-twilled Lie algebras can be viewed as natural generalizations of quasi-Lie bialgebras. Let g be a vector space and $\pi \in \wedge^2 g^* \otimes g$, $\mu \in g^* \otimes \wedge^2 g$ and $\theta \in \wedge^3 g^*$. Recall that the quadruple (g, π, μ, θ) is called a **quasi-Lie bialgebra** [6, 29, 31] if

$$\frac{1}{2}\{\pi,\pi\}+\{\mu,\theta\}=0,\quad \{\pi,\mu\}=0,\quad \{\mu,\mu\}=0,\quad \{\pi,\theta\}=0,$$

where $\{\cdot,\cdot\}$ is the big bracket, which is the canonical Poisson bracket on $T^*[2]g[1]$. For convenience and to be consistence with the notations for quasi-twilled Lie algebras, we will also view $\pi \in \wedge^2 g^* \otimes g$, $\mu \in g^* \otimes \wedge^2 g$ and $\theta \in \wedge^3 g^*$ as maps in $\operatorname{Hom}(\wedge^2 g, g)$, $\operatorname{Hom}(\wedge^2 g^*, g^*)$ and $\operatorname{Hom}(\wedge^2 g, g^*)$ respectively. Then it is well known that the double $g \oplus g^*$ is a Lie algebra and it follows that $(g \oplus g^*, g, g^*)$ is a quasi-twilled Lie algebra.

In the sequel, we give various examples of quasi-twilled Lie algebras.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra. For $\lambda \in \mathbb{K}$, define a bracket operation $[\cdot, \cdot]_M$ on $\mathfrak{g} \oplus \mathfrak{g}$ by

$$[(x, u), (y, v)]_M = ([x, v]_{\mathfrak{q}} - [y, u]_{\mathfrak{q}}, \lambda [x, y]_{\mathfrak{q}} + [u, v]_{\mathfrak{q}}), \quad \forall x, y, u, v \in \mathfrak{g}.$$

That is $\pi = \rho = 0$, $\eta = \mu = [\cdot, \cdot]_g$ and $\theta = \lambda[\cdot, \cdot]_g$ in (2). Then we have that $[\cdot, \cdot]_M$ is a Lie algebra structure on $g \oplus g$. Denote this Lie algebra by $g \oplus_M g$.

Example 2.3. Let $(g, [\cdot, \cdot]_g)$ be a Lie algebra. Then $(g \oplus_M g, g, g)$ is a quasi-twilled Lie algebra.

Let $\rho: \mathfrak{g} \to \operatorname{Der}(\mathfrak{h})$ be an action of a Lie algebra \mathfrak{g} on a Lie algebra \mathfrak{h} . For any $\lambda \in \mathbb{K}$, then $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{\rho})$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_{\rho}$ is given by

$$(4) \qquad [(x,u),(y,v)]_{\varrho} = ([x,y]_{\mathfrak{g}},\rho(x)v - \rho(y)u + \lambda[u,v]_{\mathfrak{h}}), \quad \forall x,y \in \mathfrak{g}, u,v \in \mathfrak{h}.$$

This Lie algebra is denoted by $\mathfrak{g} \ltimes_{\rho} \mathfrak{h}$ and called the **action Lie algebra**.

Example 2.4. Let $\rho : \mathfrak{g} \to \operatorname{Der}(\mathfrak{h})$ be an action of a Lie algebra \mathfrak{g} on a Lie algebra \mathfrak{h} . Then $(\mathfrak{g} \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ is a quasi-twilled Lie algebra.

Example 2.5. Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra \mathfrak{g} on a vector space V. Then $(\mathfrak{g} \ltimes_{\rho} V, \mathfrak{g}, V)$ is a quasi-twilled Lie algebra, where $\mathfrak{g} \ltimes_{\rho} V$ is the **semidirect product Lie algebra** with the Lie bracket

$$[(x, u), (y, v)]_{\rho} = ([x, y]_{\mathfrak{g}}, \rho(x)v - \rho(y)u), \quad \forall u, v \in V, x, y \in \mathfrak{g}.$$

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$ be Lie algebras. Then there is the direct product Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, [\cdot, \cdot]_{\oplus})$, where

$$[(x, u), (y, v)]_{\oplus} = ([x, y]_{\mathfrak{g}}, [u, v]_{\mathfrak{h}}), \quad \forall x, y \in \mathfrak{g}, u, v \in \mathfrak{h}.$$

Example 2.6. The direct product Lie algebra $(g \oplus h, g, h)$ is a quasi-twilled Lie algebra.

Example 2.7. Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of a Lie algebra \mathfrak{g} on a vector space V and $\omega \in \operatorname{Hom}(\wedge^2 \mathfrak{g}, V)$ be a 2-cocycle. Then $(\mathfrak{g} \oplus V, [\cdot, \cdot]_{\rho,\omega})$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_{\rho,\omega}$ is given by

(5)
$$[(x, u), (y, v)]_{\rho,\omega} = ([x, y]_{\mathfrak{g}}, \rho(x)v - \rho(y)u + \omega(x, y)), \quad \forall x, y \in \mathfrak{g}, u, v \in V.$$

Denote this Lie algebra by $\mathfrak{g} \ltimes_{\rho,\omega} V$. Moreover, $(\mathfrak{g} \ltimes_{\rho,\omega} V, \mathfrak{g}, V)$ is a quasi-twilled Lie algebra.

Example 2.8. As a special case of Example 2.7, consider $V = \mathfrak{g}$, $\rho = \text{ad}$ and $\omega(x, y) = [x, y]_{\mathfrak{g}}$. Then we obtain a quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\mathrm{ad},\omega} \mathfrak{g}, \mathfrak{g}, \mathfrak{g})$.

Remark 2.9. In fact, the above examples can be unified via extensions of Lie algebras. Recall that a Lie algebra \mathcal{G} is an extension of a Lie algebra \mathfrak{g} by a Lie algebra \mathfrak{h} if we have the following exact sequence:

$$0 \longrightarrow \mathfrak{h} \stackrel{i}{\longrightarrow} \mathcal{G} \stackrel{\pi}{\longrightarrow} \mathfrak{g} \longrightarrow 0.$$

By choosing a section $s: \mathfrak{g} \to \mathcal{G}$ of the extension (6), \mathcal{G} is equal to $s(\mathfrak{g}) \oplus i(\mathfrak{h})$, and $i(\mathfrak{h})$ is a subalgebra. Thus, $(\mathcal{G}, s(\mathfrak{g}), i(\mathfrak{h}))$ is a quasi-twilled Lie algebra.

A matched pair of Lie algebras consists of a pair of Lie algebras (g, h), a representation $\rho: g \to gl(h)$ of g on h and a representation $\eta: h \to gl(g)$ of h on g such that

(7)
$$\rho(x)[u,v]_{\mathfrak{h}} = [\rho(x)u,v]_{\mathfrak{h}} + [u,\rho(x)v]_{\mathfrak{h}} + \rho((\eta(v)x)u - \rho(\eta(u)x)v,$$

(8)
$$\eta(u)[x,y]_{g} = [\eta(u)x,y]_{g} + [x,\eta(u)y]_{g} + \eta(\rho(y)u)x - \eta(\rho(x)u)y,$$

for all $x, y \in \mathfrak{g}$ and $u, v \in \mathfrak{h}$. We will denote a matched pair of Lie algebras by $(\mathfrak{g}, \mathfrak{h}; \rho, \eta)$, or simply by $(\mathfrak{g}, \mathfrak{h})$. Let $(\mathfrak{g}, \mathfrak{h}; \rho, \eta)$ be a matched pair of Lie algebras. Then there is a Lie algebra structure on the direct sum space $\mathfrak{g} \oplus \mathfrak{h}$ with the Lie bracket $[\cdot, \cdot]_{\bowtie}$ given by

$$[(x, u), (y, v)]_{\bowtie} = ([x, y]_{\mathfrak{q}} + \eta(u)y - \eta(v)x, [u, v]_{\mathfrak{h}} + \rho(x)v - \rho(y)u).$$

Denote this Lie algebra by $\mathfrak{g} \bowtie \mathfrak{h}$.

Example 2.10. Let $(g, h; \rho, \eta)$ be a matched pair of Lie algebras. Then $(g \bowtie h, g, h)$ is a quasitwilled Lie algebra. Note that the representation η in the definition of a matched pair and the η in (2) are related as follows: $\rho(x)v = \rho(x, v)$, $\eta(u)(y) = -\eta(y, u)$.

Remark 2.11. In Section 3, we will introduce the notion of a deformation map of type I of a quasi-twilled Lie algebra, and will see that deformation maps of type I of quasi-twilled Lie algebras given in Examples 2.3, 2.4, 2.5 and 2.6 are exactly modified *r*-matrices (solutions of the modified classical Yang-Baxter equation), crossed homomorphisms, derivations and Lie algebra homomorphisms.

Remark 2.12. In Section 4, we will introduce the notion of a deformation map of type II of a quasi-twilled Lie algebra, and will see that deformation maps of type II of quasi-twilled Lie algebras given in Examples 2.4, 2.5, 2.7 2.8 and 2.10 are exactly relative Rota-Baxter operators of weight λ , relative Rota-Baxter operators of weight 0, twisted Rota-Baxter operators, Reynolds operators and deformation maps of a matched pair of Lie algebras. So deformation maps of type

II of quasi-twilled Lie algebras provide a unified approach to study relative Rota-Baxter operators, twisted Rota-Baxter operators, Reynolds operators and deformation maps of a matched pair of Lie algebras.

3. The controlling algebras and cohomologies of deformation maps of type I

In this section, $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ is always a quasi-twilled Lie algebra, and the Lie bracket on \mathcal{G} is denoted by

$$\Omega = \pi + \rho + \mu + \eta + \theta,$$

where $\pi \in \operatorname{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}), \ \rho \in \operatorname{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{h}), \ \mu \in \operatorname{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{h}), \ \eta \in \operatorname{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{g}) \ \text{and} \ \theta \in \operatorname{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h}).$

3.1. **Deformation maps of type I of a quasi-twilled Lie algebra.** In this subsection, we introduce the notion of deformation maps of type I of a quasi-twilled Lie algebra, which unify modified r-matrices, crossed homomorphisms, derivations and homomorphisms between Lie algebras

Definition 3.1. Let $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ be a quasi-twilled Lie algebra. A **deformation map of type I** $(\mathfrak{D}$ -map for short) of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ is a linear map $D : \mathfrak{g} \to \mathfrak{h}$ such that

$$D(\eta(x, D(y)) - \eta(y, D(x)) + \pi(x, y)) = \mu(D(x), D(y)) + \rho(x, D(y)) - \rho(y, D(x)) + \theta(x, y).$$

Remark 3.2. \mathfrak{D} -maps may not exists. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho,\omega} V, \mathfrak{g}, V)$ given in Example 2.7 obtained from a representation ρ of \mathfrak{g} on V and a 2-cocycle ω . A linear map $D:\mathfrak{g}\to\mathfrak{h}$ is a \mathfrak{D} -map if and only if

$$\omega(x,y) = -\left(\rho(x)D(y) - \rho(y)D(x) - D([x,y]_g)\right) = d_{\mathsf{CE}}(-D)(x,y), \quad \forall x, y \in \mathfrak{g},$$

where d_{CE} is the corresponding Chevalley-Eilenberg coboundary operator of the Lie algebra g with coefficients in the representation $(V; \rho)$. Thus, the quasi-twilled Lie algebra $(g \ltimes_{\rho,\omega} V, g, V)$ given in Example 2.7 admits a \mathfrak{D} -map if and only if ω is an exact 2-cocycle.

Let $D: \mathfrak{g} \to \mathfrak{h}$ be a linear map. Denote the graph of D by

$$Gr(D) = \{(x, D(x)) | x \in \mathfrak{g}\}.$$

Proposition 3.3. A linear map $D : \mathfrak{g} \to \mathfrak{h}$ is a \mathfrak{D} -map if and only if Gr(D) is a subalgebra. In this case $(\mathfrak{h}, Gr(D))$ is also a matched pair of Lie algebras.

Proof. For all $(x, D(x)), (y, D(y)) \in Gr(D)$, we have

$$\Omega((x, D(x)), (y, D(y)))$$

$$= \left(\pi(x, y) + \eta(x, D(y)) - \eta(y, D(x)), \mu(D(x), D(y)) + \rho(x, D(y)) - \rho(y, D(x)) + \theta(x, y)\right).$$

Thus, Gr(D) is a Lie subalgebra of \mathcal{G} , i.e. $\Omega((x, D(x)), (y, D(y))) \in Gr(D)$, if and only if

$$\mu(D(x), D(y)) + \rho(x, D(y)) - \rho(y, D(x)) + \theta(x, y) = D(\pi(x, y) + \eta(x, D(y)) - \eta(y, D(x))),$$

namely D is a \mathfrak{D} -map of the quasi-twilled Lie algebra ($\mathcal{G}, \mathfrak{g}, \mathfrak{h}$).

Since $\mathcal{G} = Gr(D) \oplus \mathfrak{h}$, it follows that $(\mathfrak{h}, Gr(D))$ is a matched pair of Lie algebras.

Remark 3.4. Let $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ be a quasi-twilled Lie algebra. Then we have $\mathcal{G} = \mathfrak{g} \oplus \mathfrak{h}$. Since any compliment of \mathfrak{h} in \mathcal{G} is isomorphic to a graph of a linear map D from \mathfrak{g} to \mathfrak{h} . Then by Proposition 3.3, to find a space V which is a compliment of \mathfrak{h} in \mathcal{G} such that (\mathfrak{h}, V) is a matched pair of Lie algebras is equivalent to find a \mathfrak{D} -map of the quasi-twilled Lie algebra $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$.

Example 3.5. Consider the quasi-twilled Lie algebra $(g \oplus_M g, g, g)$ given in Example 2.3. In this case, a \mathfrak{D} -map of $(g \oplus_M g, g, g)$ is a linear map $D : g \to g$ such that

$$[D(x), D(y)]_{g} - D([D(x), y]_{g} + [x, D(y)]_{g}) = -\lambda [x, y]_{g}.$$

Note that this equation is called the modified Yang-Baxter equation by Semenov-Tian-Shansky in the seminal work [48], whose solutions are called **modified** *r***-matrices** on the Lie algebra $(g, [\cdot, \cdot]_q)$.

Remark 3.6. Modified r-matrices play an important role in studying solutions of Lax equations [46, 48, 49]. Furthermore, modified r-matrices are intimately related to particular factorization problems in the corresponding Lie algebras and Lie groups. This factorization problem was considered by Reshetikhin and Semenov-Tian-Shansky in the framework of the enveloping algebra of a Lie algebra with a modified r-matrix to study quantum integrable systems [47]. Moreover, any modified r-matrix induces a post-Lie algebra [4].

Example 3.7. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ given in Example 2.4 obtained from the action Lie algebra $g \ltimes_{\rho} \mathfrak{h}$. In this case, a \mathfrak{D} -map of $(g \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ is a linear map $D : \mathfrak{g} \to \mathfrak{h}$ such that

$$D([x, y]_{\mathfrak{g}}) = \rho(x)D(y) - \rho(y)D(x) + \lambda[D(x), D(y)]_{\mathfrak{h}},$$

which is exactly a **crossed homomorphism of weight** λ from the Lie algebra g to the Lie algebra h [36].

Remark 3.8. Note that crossed homomorphisms of weight -1 are ε -derivations on the Lie algebras, which play crucial roles in the Jacobian conjecture [57] and the Mathieu-Zhao subspace theory [56]. On the other hand, crossed homomorphisms of weight 1 are deeply related to the representation theory of Cartan type Lie algebras [44] and post-Lie algebras [40].

Example 3.9. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho} V, \mathfrak{g}, V)$ given in Example 2.5 obtained from the semidirect product Lie algebra $\mathfrak{g} \ltimes_{\rho} V$. In this case, a \mathfrak{D} -map is a linear map $D : \mathfrak{g} \to V$ such that

$$D([x, y]_{\mathfrak{g}}) = \rho(x)D(y) - \rho(y)D(x),$$

which implies that D is a **derivation** from $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ to V. In particular, if ρ is the adjoint representation of \mathfrak{g} on itself, then we obtain the usual derivation.

Example 3.10. Consider the quasi-twilled Lie algebra $(g \oplus h, g, h)$ given in Example 2.6 obtained from the direct product Lie algebra. In this case, a \mathfrak{D} -map of $(g \oplus h, g, h)$ is a linear map $D : g \to h$ such that

$$D([x, y]_{\mathfrak{g}}) = [D(x), D(y)]_{\mathfrak{h}},$$

which is exactly a **Lie algebra homomorphism** from $(g, [\cdot, \cdot]_a)$ to $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})$.

At the end of this subsection, we illustrate the roles that \mathfrak{D} -maps play in the twisting theory. Let $D:\mathfrak{g}\to\mathfrak{h}$ be a linear map. It follows that $D^2=0$ and $[\cdot,D]_{NR}$ is a derivation of the graded Lie algebra $(\oplus_{n=0}^{+\infty}\operatorname{Hom}(\wedge^{n+1}(\mathfrak{g}\oplus\mathfrak{h}),\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{NR})$. Then we gain that $e^{[\cdot,D]_{NR}}$ is an automorphism of the graded Lie algebra $(\oplus_{n=0}^{+\infty}\operatorname{Hom}(\wedge^{n+1}(\mathfrak{g}\oplus\mathfrak{h}),\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{NR})$.

Definition 3.11. Let $D: \mathfrak{g} \to \mathfrak{h}$ be a linear map. The transformation $\Omega^D \triangleq e^{[\cdot,D]_{NR}}\Omega$ is called the **twisting** of Ω by D.

Proposition 3.12. With the above notations, we gain that

(9)
$$\Omega^D = e^{-D} \circ \Omega \circ (e^D \otimes e^D)$$

is a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$ and $e^D : (\mathcal{G}, \Omega^D) \to (\mathcal{G}, \Omega)$ is an isomorphism between Lie algebras.

Proof. Since $e^{[\cdot,D]_{NR}}$ is an automorphism of the graded Lie algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}(\mathfrak{h} \oplus \mathfrak{g}), \mathfrak{h} \oplus \mathfrak{g}), [\cdot,\cdot]_{NR})$, we deduce that Ω^D is a Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{h}$. Moreover, by the similar computation in [52, 54], we obtain that $\Omega^D = e^{-D} \circ \Omega \circ (e^D \otimes e^D)$ and $e^D : (\mathcal{G}, \Omega^D) \to (\mathcal{G}, \Omega)$ is an isomorphism between Lie algebras.

Theorem 3.13. Let $(\mathcal{G}, \mathfrak{h}, \mathfrak{g})$ be a quasi-twilled Lie algebra and $D: \mathfrak{g} \to \mathfrak{h}$ a linear map. Then $((\mathcal{G}, \Omega^D), \mathfrak{g}, \mathfrak{h})$ is a quasi-twilled Lie algebra. Moreover, write $\Omega^D = \pi^D + \rho^D + \mu^D + \eta^D + \theta^D, \pi^D \in \operatorname{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}), \ \rho^D \in \operatorname{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{h}), \ \mu^D \in \operatorname{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{h}), \ \eta^D \in \operatorname{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{g}) \ and \ \theta^D \in \operatorname{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h}).$ We have

$$\pi^{D}(x,y) = \pi(x,y) + \eta(x,D(y)) - \eta(y,D(x)),
\rho^{D}(x,v) = \rho(x,v) + \mu(D(x),v) - D(\eta(x,v)),
\mu^{D}(u,v) = \mu(u,v),
\eta^{D}(x,v) = \eta(x,v),
\theta^{D}(x,y) = \theta(x,y) + \rho(x,D(y)) - \rho(y,D(x)) - D(\pi(x,y))
+ \mu(D(x),D(y)) - D(\eta(x,D(y))) + D(\eta(y,D(x))),$$

for all $x, y \in \mathfrak{g}, u, v \in \mathfrak{h}$.

Consequently, $D: \mathfrak{g} \to \mathfrak{h}$ is a \mathfrak{D} -map if and only if the Lie algebras (\mathfrak{g}, π^D) and (\mathfrak{h}, μ) form a matched pair of Lie algebras.

Proof. By (9), we have

$$\Omega^D((0,u),(0,v)) = e^{-D}\Omega(e^D(0,u),e^D(0,v)) = \Omega((0,u),(0,v)) \in \mathfrak{h}.$$

Thus \mathfrak{h} is a Lie subalgebra of (\mathcal{G}, Ω^D) , which implies that $((\mathcal{G}, \Omega^D), \mathfrak{g}, \mathfrak{h})$ is a quasi-twilled Lie algebra.

For all $x, y \in \mathfrak{g}$, $u, v \in \mathfrak{h}$, by (9), we have

$$\begin{split} &(\pi^D(x,y),\theta^D(x,y)) = \Omega^D\Big((x,0),(y,0)\Big) \\ &= \Big(\pi(x,y) + \eta(x,D(y)) - \eta(y,D(x)), \theta(x,y) + \rho(x,D(y)) - \rho(y,D(x)) - D(\pi(x,y)) \\ &+ \mu(D(x),D(y)) - D(\eta(x,D(y))) + D(\eta(y,D(x)))\Big), \end{split}$$

and

$$\begin{split} (\eta^D(x,v), \rho^D(x,v)) &= & \Omega^D\Big((x,0), (0,v)\Big) \\ &= & (\eta(x,v), \rho(x,v) + \mu(D(x),v) - D(\eta(x,v))), \end{split}$$

which completes the proof.

Apply the above result to the quasi-Lie bialgebra (g, π, μ, θ) , a \mathfrak{D} -map gives rise to a Lie bialgebra.

Proposition 3.14. Let $D: \mathfrak{g} \to \mathfrak{g}^*$ be a \mathfrak{D} -map of the quasi-twilled Lie algebra $(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ obtained from the quasi-Lie bialgebra $(\mathfrak{g}, \pi, \mu, \theta)$ such that $D = -D^*$. Then $((\mathfrak{g}, \pi^D), (\mathfrak{g}^*, \mu^D))$ is a Lie bialgebra.

Proof. By Theorem 3.13, $\theta^D = 0$. Thus $((g, \pi^D), (g^*, \mu^D))$ is a matched pair of Lie algebras. Moreover, by $D = -D^*$, we can deduce that $((g, \pi^D), (g^*, \mu^D))$ is a Lie bialgebra.

3.2. The controlling algebra of \mathfrak{D} -maps. In this subsection, we give the controlling algebra of deformation maps of type I, which is a curved L_{∞} -algebra. An important byproduct is the controlling algebra of modified r-matrices, which is totally unknown before.

Definition 3.15. ([27]) Let $g = \bigoplus_{k \in \mathbb{Z}} g^k$ be a \mathbb{Z} -graded vector space. A **curved** L_{∞} -algebra is a \mathbb{Z} -graded vector space g equipped with a collection $(k \ge 0)$ of linear maps $l_k : \bigotimes^k g \to g$ of degree 1 with the property that, for any homogeneous elements $x_1, \dots, x_n \in g$, we have

(i) (graded symmetry) for every $\sigma \in S_n$,

$$l_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = \varepsilon(\sigma)l_n(x_1, \cdots, x_n),$$

(ii) (generalized Jacobi identity) for all $n \ge 0$,

$$\sum_{i=0}^n \sum_{\sigma \in S(i,n-i)} \varepsilon(\sigma) l_{n-i+1}(l_i(x_{\sigma(1)},\cdots,x_{\sigma(i)}),x_{\sigma(i+1)},\cdots,x_{\sigma(n)}) = 0,$$

where $\varepsilon(\sigma) = \varepsilon(\sigma; x_1, \dots, x_n)$ is the Koszul sign for a permutation $\sigma \in S_n$ and $x_1, \dots, x_n \in \mathfrak{g}$.

We denote a curved L_{∞} -algebra by $(\mathfrak{g}, \{l_k\}_{k=0}^{+\infty})$. A curved L_{∞} -algebra $(\mathfrak{g}, \{l_k\}_{k=0}^{+\infty})$ with $l_0 = 0$ is exactly an L_{∞} -algebra [34].

Definition 3.16. Let $(g, \{l_k\}_{k=0}^{+\infty})$ be a curved L_{∞} -algebra. A **Maurer-Cartan element** is a degree 0 element x satisfying

$$l_0 + \sum_{k=1}^{+\infty} \frac{1}{k!} l_k(x, \dots, x) = 0.$$

Let x be a Maurer-Cartan element of a curved L_{∞} -algebra $(g, \{l_k\}_{k=0}^{+\infty})$. Define $l_k^x : \otimes^k g \to g$ $(k \ge 1)$ by

$$l_k^x(x_1,\dots,x_k) = \sum_{n=0}^{+\infty} \frac{1}{n!} l_{k+n}(\underbrace{x,\dots,x}_{n},x_1,\dots,x_k).$$

Theorem 3.17. ([17, 23]) With the above notation, $(g, \{l_k^x\}_{k=1}^{+\infty})$ is an L_{∞} -algebra which is called the twisted L_{∞} -algebra by x.

Remark 3.18. To ensure the convergence of the series appearing in the definition of Maurer-Cartan elements and Maurer-Cartan twistings above, one need the L_{∞} -algebra being filtered given by Dolgushev and Rogers in [16], or weakly filtered given in [33]. Since all the L_{∞} -algebras under consideration in the sequel satisfy the weakly filtered condition, so we will not mention this point anymore.

We recall Voronov's derived bracket construction [55], which is a powerful method for constructing a curved L_{∞} -algebra.

Definition 3.19. ([55]) A curved V-data consists of a quadruple (L, F, P, Δ) , where

- $(L = \oplus L^i, [\cdot, \cdot])$ is a graded Lie algebra,
- F is an abelian graded Lie subalgebra of $(L, [\cdot, \cdot])$,
- $P: L \to L$ is a projection, that is $P \circ P = P$, whose image is F and kernel is a graded Lie subalgebra of $(L, [\cdot, \cdot])$,
- Δ is an element in L^1 such that $[\Delta, \Delta] = 0$.

When $\Delta \in \ker(P)^1$ such that $[\Delta, \Delta] = 0$, we refer to (L, F, P, Δ) as a *V*-data.

Theorem 3.20. ([55]) Let (L, F, P, Δ) be a curved V-data. Then $(F, \{l_k\}_{k=0}^{+\infty})$ is a curved L_{∞} -algebra, where l_k are given by

$$l_0 = P(\Delta), \quad l_k(x_1, \dots, x_n) = P([\dots [[\Delta, x_1], x_2], \dots, x_n]).$$

Now we are ready to give the controlling algebra of \mathfrak{D} -maps of a quasi-twilled Lie algebra.

Theorem 3.21. Let $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ be a quasi-twilled Lie algebra. Then there is a curved V-data (L, F, P, Δ) as follows:

- the graded Lie algebra $(L, [\cdot, \cdot])$ is given by $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1} \mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g} \oplus \mathfrak{h}), [\cdot, \cdot]_{\operatorname{NR}})$,
- the abelian graded Lie subalgebra F is given by $\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g},\mathfrak{h})$,
- $P: L \to L$ is the projection onto the subspace F,
- $\Delta = \pi + \rho + \mu + \eta + \theta$.

Consequently, we obtain a curved L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} Hom(\wedge^{n+1}\mathfrak{g}, \mathfrak{h}), l_0, l_1, l_2)$, where l_0, l_1, l_2 are given by

$$l_0 = \theta$$

 $l_1(f) = [\pi + \rho, f]_{NR}$
 $l_2(f, g) = [[\mu + \eta, f]_{NR}, g]_{NR}$.

Furthermore, a linear map $D: \mathfrak{g} \to \mathfrak{h}$ is a \mathfrak{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ if and only if D is a Maurer-Cartan element of the above curved L_{∞} -algebra.

Proof. It is obvious that F is an abelian graded Lie subalgebra of L. Since P is the projection onto F, it is obvious that $P^2 = P$. Moreover, the kernel of P is a graded Lie subalgebra of L. Thus (L, F, P, Δ) is a curved V-data. By Theorem 3.20, we obtain a curved L_{∞} -algebra $(F, \{l_k\}_{k=1}^{+\infty})$, where l_k are given by

$$l_k(f_1, \dots, f_n) = P([\dots [[\Delta, f_1]_{NR}, f_2]_{NR}, \dots, f_n]_{NR}).$$

By Theorem 3.20, we have $l_0 = P(\Delta) = \theta$ and

$$l_1(f) = P([\pi + \rho + \mu + \eta + \theta, f]_{NR}) = [\pi + \rho, f]_{NR},$$

$$l_2(f, g) = P([[\pi + \rho + \mu + \eta + \theta, f]_{NR}, g]_{NR}) = [[\mu + \eta, f]_{NR}, g]_{NR},$$

where $f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}), g \in \text{Hom}(\wedge^m \mathfrak{g}, \mathfrak{h})$. Since F is abelian and

$$[[\pi + \rho + \mu + \eta + \theta, f]_{NR}, g]_{NR} \in \text{Hom}(\wedge^{n+m}\mathfrak{g}, \mathfrak{h}),$$

we have $l_k = 0$ for all $k \ge 3$. Moreover, we have

$$\begin{split} &l_0(x,y) + l_1(D)(x,y) + \frac{1}{2}l_2(D,D)(x,y) \\ &= \theta(x,y) + [\pi + \rho, D]_{NR}(x,y) + [[\mu + \eta, D]_{NR}, D]_{NR}(x,y) \\ &= \theta(x,y) + \rho(x,D(y)) - \rho(y,D(x)) - D(\pi(x,y)) + \mu(D(x),D(y)) - D(\eta(x,D(y))) + D(\eta(y,D(x))). \end{split}$$

Thus, D is a Maurer-Cartan element of the curved L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{h}), l_0, l_1, l_2)$ if and only if D is a \mathfrak{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. The proof is finished.

As an immediate application of the above theorem, we obtain the **controlling algebra of modified** r-matrices, namely a curved L_{∞} -algebra whose Maurer-Cartan elements are modified r-matrices.

Corollary 3.22. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \oplus_M \mathfrak{g}, \mathfrak{g}, \mathfrak{g})$ given in Example 2.3. Then $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{g}), l_0, l_1, l_2)$ is a curved L_{∞} -algebra, where l_0, l_1 and l_2 are given by $l_0 = \lambda[\cdot, \cdot]_{\mathfrak{g}}, l_1 = 0$, and

$$l_{2}(f,g)(x_{1},\cdots,x_{p+q})$$

$$= \sum_{\sigma \in S(q,1,p-1)} (-1)^{p}(-1)^{\sigma} f([x_{\sigma(q+1)},g(x_{\sigma(1)},\cdots,x_{\sigma(q)})]_{\mathfrak{g}},x_{\sigma(q+2)},\cdots,x_{\sigma(p+q)})$$

$$- \sum_{\sigma \in S(p,1,q-1)} (-1)^{p(q+1)}(-1)^{\sigma} g([x_{\sigma(p+1)},f(x_{\sigma(1)},\cdots,x_{\sigma(p)})]_{\mathfrak{g}},x_{\sigma(p+2)},\cdots,x_{\sigma(p+q)}))$$

$$+ \sum_{\sigma \in S(p,q)} (-1)^{p(q+1)}(-1)^{\sigma} [f(x_{\sigma(1)},\cdots,x_{\sigma(p)}),g(x_{\sigma(p+1)},\cdots,x_{\sigma(p+q)})]_{\mathfrak{g}}$$

for all $f \in \text{Hom}(\wedge^p \mathfrak{g}, \mathfrak{g}), g \in \text{Hom}(\wedge^q \mathfrak{g}, \mathfrak{g}).$

Moreover, Maurer-Cartan elements of this curved L_{∞} -algebra are exactly modified r-matrices on the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$.

Through Theorem 3.21, one can also recover the controlling algebras for crossed homomorphisms, derivations and Lie algebra homomorphisms after the suspension.

Corollary 3.23. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ given in Example 2.4 obtained from the action Lie algebra $\mathfrak{g} \ltimes_{\rho} \mathfrak{h}$. Then $(\bigoplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}), d, \llbracket \cdot, \cdot \rrbracket)$ is a differential graded Lie algebra, where the differential d is given by

(10)
$$d(f)(x_1, \dots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{n+i} \rho(x_i) f(x_1, \dots, \hat{x}_i, \dots, x_{n+1}) + \sum_{i < j} (-1)^{n+i+j-1} f([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}),$$

for all $f \in \text{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h})$, and the graded Lie bracket $[\![\cdot, \cdot]\!]$ is given by

$$(11) \ \llbracket f,g \rrbracket (x_1,\cdots,x_{p+q}) = \sum_{\sigma \in S(p,q)} (-1)^{pq+1} (-1)^{\sigma} \lambda [f(x_{\sigma(1)},\cdots,x_{\sigma(p)}),g(x_{\sigma(p+1)},\cdots,x_{\sigma(p+q)})]_{\mathfrak{h}},$$

for all $f \in \text{Hom}(\wedge^p \mathfrak{g}, \mathfrak{h}), g \in \text{Hom}(\wedge^q \mathfrak{g}, \mathfrak{h})$. This differential graded Lie algebra is exactly the **controlling algebra for crossed homomorphisms of weight** λ given in [44].

Corollary 3.24. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho} V, \mathfrak{g}, V)$ given in Example 2.5 obtained from the semidirect product Lie algebra $\mathfrak{g} \ltimes_{\rho} V$. Then $(\bigoplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n \mathfrak{g}, V), d)$ is a cochain complex, where d is given by (10). A linear map $D \in \operatorname{Hom}(\mathfrak{g}, V)$ is a derivation if and only if d(D) = 0. Therefore, the cochain complex $(\bigoplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n \mathfrak{g}, V), d)$ can be viewed as the **controlling algebra for derivations** from the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ to V. See [53] for more details.

Corollary 3.25. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ given in Example 2.6 obtained from the direct product Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$. Then $(\bigoplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n \mathfrak{g}, \mathfrak{h}), d, \llbracket \cdot, \cdot \rrbracket)$ is a differential graded Lie algebra, where the graded Lie bracket $\llbracket \cdot, \cdot \rrbracket$ is given by (11) and the differential d is given by

$$d(f)(x_1, \dots, x_{p+1}) = \sum_{i < j} (-1)^{p+i+j-1} f([x_i, x_j]_g, x_1 \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{p+1}),$$

for all $f \in \text{Hom}(\wedge^p \mathfrak{g}, \mathfrak{h})$. This differential graded Lie algebra is exactly the **controlling algebra** for Lie algebra homomorphisms. See [14, 18, 20, 42] for more details.

Let $D: \mathfrak{g} \to \mathfrak{h}$ is a \mathfrak{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. By Theorem 3.21, we obtain that D is a Maurer-Cartan element of the curved L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{h}), l_0, l_1, l_2)$. By Theorem 3.17, we have the twisted L_{∞} -algebra structure on $\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{h})$ as following:

(12)
$$l_1^D(f) = l_1(f) + l_2(D, f),$$

(13)
$$l_2^D(f,g) = l_2(f,g),$$

$$(14) l_k^D = 0, \quad k \ge 3,$$

where $f \in \text{Hom}(\wedge^p \mathfrak{g}, \mathfrak{h}), g \in \text{Hom}(\wedge^q \mathfrak{g}, \mathfrak{h})$. Now we are ready to give the L_{∞} -algebra that controls deformations of the \mathfrak{D} -map D.

Theorem 3.26. Let $D: \mathfrak{g} \to \mathfrak{h}$ be a \mathfrak{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. Then for a linear map $D': \mathfrak{g} \to \mathfrak{h}$, D+D' is a \mathfrak{D} -map if and only if D' is a Maurer-Cartan element of the twisted L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{h}), l_1^D, l_2^D)$, that is D' satisfies the Maurer-Cartan equation:

$$l_1^D(D') + \frac{1}{2}l_2^D(D', D') = 0.$$

Proof. By Theorem 3.21, D + D' is a \mathfrak{D} -map if and only if

$$l_0 + l_1(D + D') + \frac{1}{2}l_2(D + D', D + D') = 0.$$

Since D is a Maurer-Cartan element of the curved L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g},\mathfrak{h}), l_0, l_1, l_2)$, we deduce that the above condition is equivalent to

$$l_1(D') + l_2(D, D') + \frac{1}{2}l_2(D', D') = 0.$$

That is, $l_1^D(D') + \frac{1}{2}l_2^D(D', D') = 0$, which implies that D' is a Maurer-Cartan element of the twisted L_{∞} -algebra $\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{h}), l_1^D, l_2^D$.

Remark 3.27. Apply Theorem 3.26 to Corollary 3.22, one obtains the differential graded Lie algebra governing deformations of a modified r-matrix. Note that this differential graded Lie algebra was first given in [24] via a different approach. Apply Theorem 3.26 to Corollaries 3.23-3.25, one can also obtain the algebras governing deformations of crossed homomorphisms, derivations and Lie algebra homomorphisms.

3.3. **Cohomology of** \mathfrak{D} **-maps.** In this subsection, we introduce a cohomology theory of \mathfrak{D} -maps which will unify the cohomologies of various operators.

Lemma 3.28. Let $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ be a quasi-twilled Lie algebra and $D : \mathfrak{g} \to \mathfrak{h}$ be a \mathfrak{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. Then

$$\pi^D(x,y) = \pi(x,y) + \eta(x,D(y)) - \eta(y,D(x))$$

is a Lie algebra structure on g and

$$\rho^{D}(x)v = \rho(x, v) + \mu(D(x), v) - D(\eta(x, v))$$

is a representation of (\mathfrak{g}, π^D) on \mathfrak{h} .

Proof. Since $((\mathfrak{g}, \pi^D), (\mathfrak{h}, \mu); \rho^D, \eta)$ is a matched pair of Lie algebras, we gain that (\mathfrak{g}, π^D) is a Lie algebra and $(\mathfrak{h}; \rho^D)$ is a representation of (\mathfrak{g}, π^D) .

Let d_{CE}^D : $\operatorname{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h}) \to \operatorname{Hom}(\wedge^{k+1} \mathfrak{g}, \mathfrak{h})$ be the corresponding Chevalley-Eilenberg coboundary operator of the Lie algebra (\mathfrak{g}, π^D) with coefficients in the representation (\mathfrak{h}, ρ^D) . More precisely, for all $f \in \operatorname{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$ and $x_1, \dots, x_{k+1} \in \mathfrak{g}$, we have

(15)
$$d_{CE}^{D}f(x_{1}, \dots, x_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i+1} \rho^{D}(x_{i}) f(x_{1}, \dots, \hat{x}_{i}, \dots, x_{k+1})$$

$$+ \sum_{i < j} (-1)^{i+j} f(\pi^{D}(x_{i}, x_{j}), x_{1}, \dots, \hat{x}_{i}, \dots, \hat{x}_{j}, \dots, x_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i+1} \mu(D(x_{i}), f(x_{1}, \dots, \hat{x}_{i}, \dots, x_{k+1})) + \sum_{i=1}^{k+1} (-1)^{i+1} \rho(x_{i}, f(x_{1}, \dots, \hat{x}_{i}, \dots, x_{k+1}))$$

$$- \sum_{i=1}^{k+1} (-1)^{i+1} D(\eta(x_{i}, f(x_{1}, \dots, \hat{x}_{i}, \dots, x_{k+1})))$$

$$+ \sum_{i < j} (-1)^{i+j} f(\pi(x_{i}, x_{j}), x_{1}, \dots, \hat{x}_{i}, \dots, \hat{x}_{j}, \dots, x_{k+1})$$

$$+ \sum_{i < j} (-1)^{i+j} f(\eta(x_{i}, D(x_{j})) - \eta(x_{j}, D(x_{i})), x_{1}, \dots, \hat{x}_{i}, \dots, \hat{x}_{j}, \dots, x_{k+1}).$$

Now, we define the cohomology of a \mathfrak{D} -map $D : \mathfrak{g} \to \mathfrak{h}$. Define the space of 0-cochains $C^0(D)$ to be 0 and define the space of 1-cochains $C^1(D)$ to be \mathfrak{h} . For $n \geq 2$, define the space of n-cochains $C^n(D)$ by $C^n(D) = \operatorname{Hom}(\wedge^{n-1}\mathfrak{g}, \mathfrak{h})$.

Definition 3.29. Let $D: \mathfrak{g} \to \mathfrak{h}$ be a \mathfrak{D} -map of a quasi-twilled Lie algebra $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. The cohomology of the cochain complex $(\oplus_{i=0}^{+\infty} C^i(D), \mathrm{d}_{\mathsf{CE}}^D)$ is defined to be the **cohomology for the** \mathfrak{D} -map D.

Denote the set of *n*-cocycles by $Z^n(D)$, the set of *n*-coboundaries by $B^n(D)$ and the *n*-th cohomology group by

$$H^n(D) = Z^n(D)/B^n(D), \quad n \ge 0.$$

It is obvious that $u \in \mathfrak{h}$ is closed if and only if

$$\mu(D(x), u) + \rho(x, u) - D(\eta(x, u)) = 0, \quad \forall x \in \mathfrak{g},$$

and $f \in \text{Hom}(\mathfrak{g}, \mathfrak{h})$ is closed if and only if

$$\begin{split} &\mu(D(x),f(y)) - \mu(D(y),f(x)) + \rho(x,f(y)) - \rho(y,f(x)) + D(\eta(y,f(x))) - D(\eta(x,f(y))) \\ &= & f(\pi(x,y)) + f(\eta(x,D(y)) - \eta(y,D(x))), \quad \forall x,y \in \mathfrak{g}. \end{split}$$

Here we provide another intrinsic interpretation of the above coboundary operator.

Let D be a \mathfrak{D} -map of a quasi-twilled Lie algebra $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. Recall the twisted L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{h}), l_1^D, l_2^D)$ from Theorem 3.26, we have

Proposition 3.30. With the above notations, for any $f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$, one has

$$l_1^D(f) = (-1)^{k-1} d_{CE}^D f.$$

Proof. For $f \in \text{Hom}(\wedge^k \mathfrak{g}, \mathfrak{h})$, by (12) and Theorem 3.21, we have

$$l_1^D = l_1(f) + l_2(D, f)$$

= $[\pi + \rho, f]_{NR} + [[\mu + \eta, D]_{NR}, f]_{NR}$

=
$$[\pi^D + \rho^D, f]_{NR}$$

= $(-1)^{k-1} d_{CE}^D(f)$,

which finishes the proof.

Example 3.31. Consider the quasi-twilled Lie algebra $(g \oplus_M g, g, g)$ given in Example 2.3. Let $D : g \to g$ be a modified r-matrix. Then $(g, [\cdot, \cdot]_D)$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_D$ is given by

$$[x, y]_D = [x, D(y)]_g - [y, D(x)]_g$$
.

Moreover, the Lie algebra $(\mathfrak{g},[\cdot,\cdot]_D)$ represents on the vector space \mathfrak{g} via $\rho^D:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ given by

$$\rho^D(x)u = [D(x), u]_{\mathfrak{h}} - D([x, u]_{\mathfrak{g}}), \quad \forall u \in \mathfrak{h}, x \in \mathfrak{g}.$$

The corresponding Chevalley-Eilenberg cohomology is taken to be the **cohomology for a modified** r-matrix [24].

Example 3.32. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ given in Example 2.4 obtained from the action Lie algebra $\mathfrak{g} \ltimes_{\rho} \mathfrak{h}$. Let $D : \mathfrak{g} \to \mathfrak{h}$ be a crossed homomorphism of weight λ . Then the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ represents on the vector space \mathfrak{h} via $\rho^D : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{h})$ given by

$$\rho^{D}(x)u = \rho(x)u + \lambda[D(x), u]_{h}, \quad \forall u \in h, x \in g.$$

The corresponding Chevalley-Eilenberg cohomology is taken to be the **cohomology for a crossed** homomorphism of weight λ [25, 44].

Example 3.33. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho} V, g, V)$ given in Example 2.5 obtained from the action Lie algebra $g \ltimes_{\rho} V$. Let $D : g \to V$ be a derivation from Lie algebra $(g, [\cdot, \cdot]_g)$ to the g-module V. Then the Chevalley-Eilenberg cohomology of the Lie algebra $(g, [\cdot, \cdot]_g)$ with coefficients in the representation $(V; \rho)$ is taken to be the **cohomology for a derivation** from the Lie algebra $(g, [\cdot, \cdot]_g)$ to the g-module V [53].

Example 3.34. Consider the quasi-twilled Lie algebra $(g \oplus h, g, h)$ given in Example 2.6 obtained from the direct product Lie algebra $g \oplus h$. Let $D : g \to h$ be a Lie algebra homomorphism. Then the Lie algebra $(g, [\cdot, \cdot]_g)$ represents on the vector space h via $\rho^D : g \to gl(h)$ given by

$$\rho^{D}(x)u = [D(x), u]_{\mathfrak{h}}, \quad \forall u \in \mathfrak{h}, x \in \mathfrak{g}.$$

The corresponding Chevalley-Eilenberg cohomology is taken to be the **cohomology for a Lie algebra homomorphism**. See [14, 18, 20, 42] for more details.

4. The controlling algebras and cohomologies of deformation maps of type II

In this section, $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ is always a quasi-twilled Lie algebra, and the Lie bracket on \mathcal{G} is denoted by

$$\Omega = \pi + \rho + \mu + \eta + \theta,$$

where $\pi \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}), \ \rho \in \text{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{h}), \ \mu \in \text{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{h}), \ \eta \in \text{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{g}) \text{ and } \theta \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h}).$

4.1. **Deformation maps of type II of a quasi-twilled Lie algebra.** In this subsection, we introduce the notion of deformation maps of type II of a quasi-twilled Lie algebra, which unify relative Rota-Baxter operators, twisted Rota-Baxter operators, Reynolds operators and deformation maps of matched pairs of Lie algebras.

Definition 4.1. Let $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ be a quasi-twilled Lie algebra. A **deformation map of type II** $(\mathcal{D}$ -map for short) of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ is a linear map $B : \mathfrak{h} \to \mathfrak{g}$ such that

$$\pi(B(u), B(v)) + \eta(B(u), v) - \eta(B(v), u) = B(\mu(u, v) + \rho(B(u), v) - \rho(B(v), u) + \theta(B(u), B(v))).$$

These two types of deformation maps are related as follows.

Proposition 4.2. Let $D: \mathfrak{g} \to \mathfrak{h}$ be an invertible linear map. Then D is a \mathfrak{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ if and only if $D^{-1}: \mathfrak{h} \to \mathfrak{g}$ is a \mathcal{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$.

Proof. It is straightforward.

Example 4.3. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ given in Example 2.4 obtained from the action Lie algebra $g \ltimes_{\rho} \mathfrak{h}$. In this case, a \mathcal{D} -map of $(g \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ is a linear map $B : \mathfrak{h} \to \mathfrak{g}$ such that

$$[B(u), B(v)]_{\mathfrak{g}} = B(\rho(B(u))v - \rho(B(v))u + \lambda[u, v]_{\mathfrak{h}}), \quad \forall u, v \in \mathfrak{h},$$

which is exactly is a **relative Rota-Baxter operator of weight** λ on g with respect to the action $(\mathfrak{h}; \rho)$ [4].

Example 4.4. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho} V, g, V)$ given in Example 2.5 obtained from the semidirect product Lie algebra $g \ltimes_{\rho} V$. In this case, a \mathcal{D} -map is a linear map $B: V \to g$ such that

$$[B(u), B(v)]_{\mathfrak{a}} = B(\rho(B(u))v - \rho(B(v))u), \quad \forall u, v \in V,$$

which is exactly is a **relative Rota-Baxter operator of weight** 0 (also called an *O*-operator) on \mathfrak{g} with respect to the representation $(V; \rho)$ [32, 48].

Example 4.5. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho,\omega} V, g, V)$ given in Example 2.7 obtained from a representation ρ of g on V and a 2-cocycle ω . In this case, a \mathcal{D} -map of $(g \ltimes_{\rho,\omega} V, g, V)$ is a linear map $B: V \to g$ such that

$$[B(u), B(v)]_{\mathfrak{a}} = B(\rho(B(u))v - \rho(B(v))u + \omega(B(u), B(v))), \quad \forall u, v \in V,$$

which implies that *B* is a **twisted Rota-Baxter operator** [10].

Example 4.6. Consider the quasi-twilled Lie algebra $(g \ltimes_{ad,\omega} g, g, g)$ given in Example 2.8. In this case, a \mathcal{D} -map of $(g \ltimes_{ad,\omega} g, g, g)$ is a linear map $B : g \to g$ such that

$$[B(u), B(v)]_{\mathfrak{g}} = B([B(u), v]_{\mathfrak{g}} - [B(v), u]_{\mathfrak{g}} + [B(u), B(v)]_{\mathfrak{g}}), \quad \forall u, v \in \mathfrak{g},$$

which implies that B is a **Reynolds operator** [10].

Example 4.7. Consider the quasi-twilled Lie algebra $(g \bowtie \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ given in Example 2.10 obtained from a matched pair of Lie algebras. In this case, a \mathcal{D} -map of $(g \bowtie \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ is a linear map $B: \mathfrak{h} \to \mathfrak{g}$ such that

$$[B(u), B(v)]_{\mathfrak{q}} - \eta(v)B(u) + \eta(u)B(v) = B([u, v]_{\mathfrak{h}} + \rho(B(u))v - \rho(B(v))u), \quad \forall u, v \in \mathfrak{h},$$

which is exactly a **deformation map** of a matched pair of Lie algebras introduced in [1].

At the end of this subsection, we illustrate the roles that \mathcal{D} -maps play in the twisting theory.

Let $B:\mathfrak{h}\to\mathfrak{g}$ be a linear map. It follows that $B^2=0$ and $[\cdot,B]_{NR}$ is a derivation of the graded Lie algebra $(\oplus_{n=0}^{+\infty}\operatorname{Hom}(\wedge^{n+1}(\mathfrak{g}\oplus\mathfrak{h}),\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{NR})$. Then we gain that $e^{[\cdot,B]_{NR}}$ is an automorphism of the graded Lie algebra $(\oplus_{n=0}^{+\infty}\operatorname{Hom}(\wedge^{n+1}(\mathfrak{g}\oplus\mathfrak{h}),\mathfrak{g}\oplus\mathfrak{h}),[\cdot,\cdot]_{NR})$.

Definition 4.8. The transformation $\Omega^B \triangleq e^{[\cdot,B]_{NR}}\Omega$ is called the **twisting** of Ω by B.

Parallel to Lemma 3.12, we have the following lemma.

Lemma 4.9. With the above notations, $\Omega^B = e^{-B} \circ \Omega \circ (e^B \otimes e^B)$ is a Lie algebra structure on \mathcal{G} .

Obviously, Ω^B is decomposed into the unique six substructures $\pi^B \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}), \ \rho^B \in \text{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{h}), \ \mu^B \in \text{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{h}), \ \eta^B \in \text{Hom}(\mathfrak{g} \otimes \mathfrak{h}, \mathfrak{g}), \ \theta^B \in \text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{h}), \ \xi^B \in \text{Hom}(\wedge^2 \mathfrak{h}, \mathfrak{g}).$ We have the following result.

Theorem 4.10. Write $\Omega = \pi + \rho + \mu + \eta + \theta$ and $\Omega^B = \pi^B + \rho^B + \mu^B + \eta^B + \theta^B + \xi^B$. Then we have

$$\begin{split} \pi^B(x,y) &= \pi(x,y) - B(\theta(x,y)), \\ \rho^B(x,v) &= \rho(x,v) - \theta(B(v),x), \\ \mu^B(u,v) &= \mu(u,v) + \rho(B(u),v) - \rho(B(v),u) + \theta(B(u),B(v)), \\ \eta^B(x,v) &= \eta(x,v) - \pi(B(v),x) - B(\rho(x,v)) + B(\theta(B(v),x)), \\ \theta^B(x,y) &= \theta(x,y), \\ \xi^B(u,v) &= \eta(B(u),v) - \eta(B(v),u) + \pi(B(u),B(v)) \\ &- B(\mu(u,v)) - B(\rho(B(u),v) + B(\rho(B(v),u)) + B(\theta(B(v),B(u))), \end{split}$$

for all $x, y \in \mathfrak{g}, u, v \in \mathfrak{h}$.

Consequently, $B: \mathfrak{h} \to \mathfrak{g}$ is a \mathcal{D} -map if and only if $((\mathcal{G}, \Omega^B), \mathfrak{g}, \mathfrak{h})$ is a quasi-twilled Lie algebra.

Proof. It follows from a direct but tedious computation. We omit details.

4.2. The controlling algebra of \mathcal{D} -maps. In this subsection, we give the controlling algebra of deformation maps of type II, which is an L_{∞} -algebra. An important byproduct is the controlling algebra of deformation maps of matched pairs of Lie algebras introduced in [1].

Theorem 4.11. Let $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ be a quasi-twilled Lie algebra. Then there is a V-data (L, F, P, Δ) as following:

- $\bullet \ \ the \ graded \ Lie \ algebra \ (L,[\cdot,\cdot]) \ is \ given \ by \ (\oplus_{n=0}^{+\infty} Hom(\wedge^{n+1}\mathfrak{g} \oplus \mathfrak{h},\mathfrak{g} \oplus \mathfrak{h}),[\cdot,\cdot]_{NR}),$
- the abelian graded Lie subalgebra F is given by $\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{h},\mathfrak{g})$,
- $P: L \to L$ is the projection onto the subspace F,
- $\Delta = \pi + \rho + \mu + \eta + \theta$.

Consequently, we obtain an L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{h},\mathfrak{g}), l_1, l_2, l_3)$, where l_1, l_2, l_3 are given by

$$\begin{split} l_1(f) &= [\mu + \eta, f]_{\text{NR}}, \\ l_2(f_1, f_2) &= [[\pi + \rho, f_1]_{\text{NR}}, f_2]_{\text{NR}}, \\ l_3(f_1, f_2, f_3) &= [[[\theta, f_1]_{\text{NR}}, f_2]_{\text{NR}}, f_3]_{\text{NR}}, \\ l_k &= 0, \quad k \ge 4. \end{split}$$

Furthermore, a linear map $B: \mathfrak{h} \to \mathfrak{g}$ is a \mathcal{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ if and only if B is a Maurer-Cartan element of the above L_{∞} -algebra.

Proof. It is obvious that F is an abelian graded Lie subalgebra of L and $[\Delta, \Delta]_{NR} = 0, \Delta \in \ker(P)^1$. Since P is the projection onto F, it is obvious that $P^2 = P$. Thus (L, F, P, Δ) is a V-data.

By Theorem 3.20, we obtain an L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{h},\mathfrak{g}),\{l_k\}_{k=1}^{+\infty})$, where l_k are given by

$$l_k(f_1, \dots, f_n) = P([\dots [[\Delta, f_1]_{NR}, f_2]_{NR}, \dots, f_n]_{NR}).$$

For $f \in \text{Hom}(\wedge^k \mathfrak{h}, \mathfrak{g})$, we have

$$l_1(f) = P([\pi + \rho + \mu + \eta + \theta, f]_{NR}) = [\mu + \eta, f]_{NR}.$$

Moreover, for $f_1 \in \text{Hom}(\wedge^p \mathfrak{h}, \mathfrak{g}), f_2 \in \text{Hom}(\wedge^q \mathfrak{h}, \mathfrak{g})$ and $f_3 \in \text{Hom}(\wedge^r \mathfrak{h}, \mathfrak{g})$, we obtain

$$l_2(f_1, f_2) = P([[\pi + \rho + \mu + \eta + \theta, f_1]_{NR}, f_2]_{NR}) = [[\pi + \rho, f_1]_{NR}, f_2]_{NR}$$

and

$$l_3(f_1, f_2, f_3) = P([[[\pi + \rho + \mu + \eta + \theta, f_1]_{NR}, f_2]_{NR}, f_3]_{NR}) = [[[\theta, f_1]_{NR}, f_2]_{NR}, f_3]_{NR}.$$

Since F is abelian and $[[[\pi + \rho + \mu + \eta + \theta, f_1]_{NR}, f_2]_{NR}, f_3]_{NR} \in \text{Hom}(\wedge^{p+q+r-1}\mathfrak{h}, \mathfrak{g})$, we have $l_k = 0$ for all $k \ge 4$.

It is straightforward to obtain

$$l_{1}(B)(u,v) + \frac{1}{2}l_{2}(B,B)(u,v) + \frac{1}{6}l_{3}(B,B,B)(u,v)$$

$$= [\mu + \eta, B]_{NR}(u,v) + \frac{1}{2}[[\pi + \rho, B]_{NR}, B]_{NR}(u,v) + \frac{1}{6}[[[\theta, B]_{NR}, B]_{NR}, B]_{NR}(u,v)$$

$$= -B(\mu(u,v)) + \eta(B(u),v) - \eta(B(v),u) + \pi(B(u),B(v)) + B(\rho(B(v),u) - \rho(B(u),v))$$

$$+B(\theta(B(v),B(u))).$$

Thus, $B \in \text{Hom}(\mathfrak{h}, \mathfrak{g})$ is a Maurer-Cartan element of $(\bigoplus_{n=0}^{+\infty} \text{Hom}(\wedge^{n+1}\mathfrak{h}, \mathfrak{g}), l_1, l_2, l_3)$ if and only if $B : \mathfrak{h} \to \mathfrak{g}$ is a \mathcal{D} -map of $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. The proof is finished.

Corollary 4.12. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho} \mathfrak{h}, \mathfrak{g}, \mathfrak{h})$ given in Example 2.4 obtained from the action Lie algebra $\mathfrak{g} \ltimes_{\rho} \mathfrak{h}$. Then $(\bigoplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g}), d, \llbracket \cdot, \cdot \rrbracket)$ is a differential graded Lie algebra, where the differential $d : \operatorname{Hom}(\wedge^p \mathfrak{h}, \mathfrak{g}) \to \operatorname{Hom}(\wedge^{p+1} \mathfrak{h}, \mathfrak{g})$ is given by

$$df(u_1, \dots, u_{p+1}) = \sum_{i < j} (-1)^{p+i+j-1} \lambda f([u_i, u_j]_{\mathfrak{h}}, u_1, \dots, \hat{u_i}, \dots, \hat{u_j}, \dots, u_{p+1}),$$

and $[\cdot,\cdot]$ is given by

for all $f_1 \in \text{Hom}(\wedge^p \mathfrak{h}, \mathfrak{g})$, $f_2 \in \text{Hom}(\wedge^q \mathfrak{h}, \mathfrak{g})$. This differential graded Lie algebra is exactly the **controlling algebra for relative Rota-Baxter operators of weight** λ initially given in [51, Corollary 2.17].

Corollary 4.13. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho} V, \mathfrak{g}, V)$ given in Example 2.5 obtained from the semidirect product Lie algebra $\mathfrak{g} \ltimes_{\rho} V$. Then $(\bigoplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n V, \mathfrak{g}), \llbracket \cdot, \cdot \rrbracket)$ is a graded Lie algebra, where the graded Lie bracket $\llbracket \cdot, \cdot \rrbracket$ is given by (16). This graded Lie algebra is exactly the **controlling algebra for** *O***-operators** on \mathfrak{g} with respect to the representation $(V; \rho)$ given in [50, Proposition 2.3].

Corollary 4.14. Consider the quasi-twilled Lie algebra $(\mathfrak{g} \ltimes_{\rho,\omega} V, \mathfrak{g}, V)$ given in Example 2.7 obtained from a representation ρ of \mathfrak{g} on V and a 2-cocycle ω . Then $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}V, \mathfrak{g}), l_2, l_3)$ is an L_{∞} -algebra, where l_2 and l_3 are given by

$$l_{2}(f_{1}, f_{2})(u_{1}, \cdots, u_{p+q})$$

$$= \sum_{\sigma \in S(q, 1, p-1)} (-1)^{p} (-1)^{\sigma} f_{1}(\rho(f_{2}(u_{\sigma(1)}, \cdots, u_{\sigma(q)}))(u_{\sigma(q+1)}), u_{\sigma(q+2)}, \cdots, u_{\sigma(q+p)})$$

$$-(-1)^{p(q+1)} \sum_{\sigma \in S(p, 1, q-1)} (-1)^{\sigma} f_{2}(\rho(f_{1}(u_{\sigma(1)}, \cdots, u_{\sigma(p)}))(u_{\sigma(p+1)}), u_{\sigma(p+2)}, \cdots, u_{\sigma(q+p)})$$

$$+(-1)^{p(q+1)} \sum_{\sigma \in S(p, q)} (-1)^{\sigma} [f_{1}(u_{\sigma(1)}, \cdots, u_{\sigma(p)}), f_{2}(u_{\sigma(p+1)}, \cdots, u_{\sigma(p+q)})]_{g},$$

and

$$\begin{split} &l_{3}(f_{1},f_{2},f_{3})(u_{1},\cdots,u_{p+q+r-1})\\ &=\sum_{\sigma\in S(q,r,p-1)}(-1)^{p+q+qr}(-1)^{\sigma}f_{1}(\Phi(f_{2}(u_{\sigma(1)},\cdots,u_{\sigma(q)}),f_{3}(u_{\sigma(q+1)},\cdots,u_{\sigma(q+r)})),\\ &u_{\sigma(q+r+1)},\cdots,u_{\sigma(p+q+r-1)})\\ &-\sum_{\sigma\in S(p,r,q-1)}(-1)^{pq+pr}(-1)^{\sigma}f_{2}(\Phi(f_{1}(u_{\sigma(1)},\cdots,u_{\sigma(p)}),f_{3}(u_{\sigma(p+1)},\cdots,u_{\sigma(p+r)})),\\ &u_{\sigma(p+r+1)},\cdots,u_{\sigma(p+q+r-1)})\\ &+\sum_{\sigma\in S(p,q,r-1)}(-1)^{pq+pr+q+r+q+r}(-1)^{\sigma}f_{3}(\Phi(f_{1}(u_{\sigma(1)},\cdots,u_{\sigma(p)}),f_{2}(u_{\sigma(p+1)},\cdots,u_{\sigma(p+q)})),\\ &u_{\sigma(p+q+1)},\cdots,u_{\sigma(p+q+r-1)}). \end{split}$$

for all $f_1 \in \text{Hom}(\wedge^p \mathfrak{h}, \mathfrak{g})$, $f_2 \in \text{Hom}(\wedge^q \mathfrak{h}, \mathfrak{g})$ and $f_3 \in \text{Hom}(\wedge^r \mathfrak{h}, \mathfrak{g})$. This L_{∞} -algebra is exactly the **controlling algebra for twisted Rota-Baxter operators** given in [10, Theorem 3.2].

Corollary 4.15. Consider the quasi-twilled Lie algebra ($\mathfrak{g} \ltimes_{\operatorname{ad},\Phi} \mathfrak{g}, \mathfrak{g}, \mathfrak{g}$) given in Example 2.8. Parallel to Corollary 4.14, ($\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{g}, \mathfrak{g}), l_2, l_3$) is an L_{∞} -algebra. This L_{∞} -algebra is exactly the **controlling algebra for Reynolds operators**. See [10] for more details.

Theorem 4.11 can not only recover some known results, but also gives rise to some new results, e.g. it gives rise to the **controlling algebra for deformation maps of a matched pair of Lie algebras**.

Corollary 4.16. Consider the quasi-twilled Lie algebra ($\mathfrak{g} \bowtie \mathfrak{h}, \mathfrak{g}, \mathfrak{h}$) given in Example 2.10 obtained from a matched pair of Lie algebras. Then $(\bigoplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g}), d, \llbracket \cdot, \cdot \rrbracket)$ is a differential graded Lie algebra, where $d : \operatorname{Hom}(\wedge^p \mathfrak{h}, \mathfrak{g}) \to \operatorname{Hom}(\wedge^{p+1} \mathfrak{h}, \mathfrak{g})$ is given by

$$df(u_1, \dots, u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{p+i} \eta(u_i) f(u_1, \dots, \hat{u_i}, \dots, u_{p+1}))$$

+
$$\sum_{i \le j} (-1)^{p+i+j-1} f([u_i, u_j]_{\mathfrak{h}}, u_1, \cdots, \hat{u_i}, \cdots, \hat{u_j}, \cdots, u_{p+1})$$

and the graded Lie bracket $[\cdot, \cdot]$ is given by (16). Maurer-Cartan elements of this differential graded Lie algebra are exactly deformation maps of a matched pair of Lie algebras.

At the end of this subsection, we give the L_{∞} -algebra governing deformations of \mathcal{D} -maps of a quasi-twilled Lie algebra.

Let *B* be a \mathcal{D} -map of the quasi-twilled Lie algebra (\mathcal{G} , \mathfrak{g} , \mathfrak{h}). By Theorem 3.17, we have the twisted L_{∞} -algebra structure on $\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{h}, \mathfrak{g})$ as following:

(17)
$$l_1^B(f) = l_1(f) + l_2(B, f) + \frac{1}{2}l_3(B, B, f),$$

(18)
$$l_2^B(f_1, f_2) = l_2(f_1, f_2) + l_3(B, f_1, f_2),$$

(19)
$$l_3^B(f_1, f_2, f_3) = l_3(f_1, f_2, f_3),$$

$$(20) l_k^B = 0, \quad k \ge 4,$$

Theorem 4.17. Let B be a \mathcal{D} -map of the quasi-twilled Lie algebra $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. Then for a linear map $B': \mathfrak{h} \to \mathfrak{g}$, $B+B': \mathfrak{h} \to \mathfrak{g}$ is a \mathcal{D} -map of the quasi-twilled Lie algebra $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$ if and only if B' is a Maurer-Cartan element of the twisted L_{∞} -algebra $(\oplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{h}, \mathfrak{g}), l_1^B, l_2^B, l_3^B)$.

Proof. By Theorem 4.11, the linear map B + B' is a \mathcal{D} -map if and only if

$$l_1(B+B') + \frac{1}{2!}l_2(B+B',B+B') + \frac{1}{3!}l_3(B+B',B+B',B+B') = 0.$$

Moreover, by the fact B is a \mathcal{D} -map, we gain that the above condition is equivalent to

$$l_1(B') + l_2(B, B') + \frac{1}{2}l_3(B, B, B') + \frac{1}{2}l_2(B', B') + \frac{1}{2}l_3(B, B', B') + \frac{1}{3!}l_3(B', B', B') = 0.$$

Thus $B+B':\mathfrak{h}\to\mathfrak{g}$ is a \mathcal{D} -map of the quasi-twilled Lie algebra $(\mathcal{G},\mathfrak{g},\mathfrak{h})$ if and only if B' is a Maurer-Cartan element of the twisted L_{∞} -algebra $(\oplus_{n=0}^{+\infty}\mathrm{Hom}(\wedge^{n+1}\mathfrak{h},\mathfrak{g}),l_1^B,l_2^B,l_3^B)$.

Apply the above theorem to Corollary 4.16, we obtain the differential graded Lie algebra governing deformations of deformation maps of a matched pair of Lie algebras.

Corollary 4.18. Let $B: \mathfrak{h} \to \mathfrak{g}$ be a deformation map of a matched pair $(\mathfrak{g}, \mathfrak{h}; \rho, \eta)$ of Lie algebras. Then $(s(\mathfrak{g}, \mathfrak{h}; \rho, \eta), \mathfrak{g}), \mathfrak{g}^B, [[\cdot, \cdot]])$ is a differential graded Lie algebra, where $[[\cdot, \cdot]]$ is given by (16), and \mathfrak{d}^B is given by

$$d^{B} f(u_{1}, \dots, u_{p+1})$$

$$= \sum_{i=1}^{p+1} (-1)^{p+i} \eta(u_{i}) f(u_{1}, \dots, \hat{u}_{i}, \dots, u_{p+1}) - \sum_{i < j} (-1)^{p+i+j} f([u_{i}, u_{j}]_{\mathfrak{h}}, u_{1}, \dots, \hat{u}_{i}, \dots, \hat{u}_{j}, \dots, u_{p+1})$$

$$+ \sum_{i=1}^{p+1} (-1)^{p+i} B(\rho(f(u_{1}, \dots, \hat{u}_{p}, \dots, u_{p+1})) u_{i}) - \sum_{i=1}^{p+1} (-1)^{i+p} f(\rho(B(u_{i})), u_{1}, \dots, \hat{u}_{i}, \dots, u_{p+1})$$

$$+ \sum_{i=1}^{p+1} (-1)^{i+p} [B(u_{i}), f(u_{1}, \dots, \hat{u}_{i}, \dots, u_{p+1})]_{\mathfrak{g}}.$$

Moreover, for a linear map $B': \mathfrak{h} \to \mathfrak{g}$, B+B' is a deformation map if and only if B' is a Maurer-Cartan element of the differential graded Lie algebra $(\oplus_{n=1}^{+\infty} \operatorname{Hom}(\wedge^n \mathfrak{h}, \mathfrak{g}), d^B, \llbracket \cdot, \cdot \rrbracket)$.

Remark 4.19. Apply Theorem 4.17 to Corollaries 4.12-4.15, one can also obtain the algebras governing deformations of relative Rota-Baxter operators, twisted Rota-Baxter operators and Reynolds operators. See [10, 15, 50] for more details.

4.3. **Cohomologies of** \mathcal{D} -maps. In this subsection, we introduce a cohomology theory of a \mathcal{D} -map using the Chevalley-Eilenberg cohomology of a Lie algebra, and illustrate that it unifies the cohomologies of relative Rota-Baxter operators, twisted Rota-Baxter operators and Reynolds operators. It also helps us to define a cohomology theory of a deformation map of a matched pair of Lie algebras.

Lemma 4.20. Let $B:\mathfrak{h}\to\mathfrak{g}$ be a \mathcal{D} -map of a quasi-twilled Lie algebra $(\mathcal{G},\mathfrak{g},\mathfrak{h})$. Then

$$\mu^{B}(u, v) = \mu(u, v) + \rho(B(u), v) - \rho(B(v), u) + \theta(B(u), B(v))$$

is a Lie algebra structure on h and

$$\sigma(v)x = -\eta^{B}(x, v) = -\eta(x, v) + \pi(B(v), x) + B(\rho(x, v)) - B(\theta(B(v), x))$$

is a representation of the Lie algebra (\mathfrak{h}, μ^B) on the vector space \mathfrak{g} .

Proof. It follows from Theorem 4.10 and Proposition 2.2 directly.

Let $d_{CE}^B: \operatorname{Hom}(\wedge^k \mathfrak{h}, \mathfrak{g}) \to \operatorname{Hom}(\wedge^{k+1} \mathfrak{h}, \mathfrak{g})$ be the corresponding Chevalley-Eilenberg coboundary operator of the Lie algebra (\mathfrak{h}, μ^B) with coefficients in the representation (\mathfrak{g}, σ) . More precisely, for all $f \in \operatorname{Hom}(\wedge^k \mathfrak{h}, \mathfrak{g})$ and $u_1, \dots, u_{k+1} \in \mathfrak{h}$, we have

$$(21) \ d_{CE}^{B} f(u_{1}, \dots, u_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i+1} \sigma(u_{i}) f(u_{1}, \dots, \hat{u}_{i}, \dots, u_{k+1}) + \sum_{i < j} (-1)^{i+j} f(\mu^{B}(u_{i}, u_{j}), u_{1}, \dots, \hat{u}_{i}, \dots, \hat{u}_{j}, \dots, u_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i} \eta(f(u_{1}, \dots, \hat{u}_{i}, \dots, u_{k+1}), u_{i}) + \sum_{i=1}^{k+1} (-1)^{i+1} \pi(B(u_{i}), f(u_{1}, \dots, \hat{u}_{i}, \dots, u_{k+1}))$$

$$+ \sum_{i=1}^{k+1} (-1)^{i+1} B(\rho(f(u_{1}, \dots, \hat{u}_{i}, \dots, u_{k+1})), u_{i}) - \sum_{i=1}^{k+1} (-1)^{i+1} B(\theta(B(u_{i}), f(u_{1}, \dots, \hat{u}_{i}, \dots, u_{k+1})))$$

$$+ \sum_{i < j} (-1)^{i+j} f(\mu(u_{i}, u_{j}), u_{1}, \dots, \hat{u}_{i}, \dots, \hat{u}_{j}, \dots, u_{k+1})$$

$$+ \sum_{i < j} (-1)^{i+j} f(\theta(B(u_{i}), B(u_{j})), u_{1}, \dots, \hat{u}_{i}, \dots, \hat{u}_{j}, \dots, u_{k+1})$$

$$+ \sum_{i < j} (-1)^{i+j} f(\rho(B(u_{i}), u_{j}) - \rho(B(u_{j}), u_{i}), u_{1}, \dots, \hat{u}_{i}, \dots, \hat{u}_{j}, \dots, u_{k+1}) .$$

Now, we define the cohomology of a \mathcal{D} -map $B: \mathfrak{h} \to \mathfrak{g}$. Define the space of 0-cochains $C^0(B)$ to be 0 and define the space of 1-cochains $C^1(B)$ to be \mathfrak{g} . For $n \geq 2$, define the space of n-cochains $C^n(B)$ by $C^n(B) = \operatorname{Hom}(\wedge^{n-1}\mathfrak{h}, \mathfrak{g})$.

Definition 4.21. Let $(\mathcal{G}, \mathfrak{h}, \mathfrak{g})$ be a quasi-twilled Lie algebra and $B : \mathfrak{h} \to \mathfrak{g}$ be a \mathcal{D} -map of $(\mathcal{G}, \mathfrak{h}, \mathfrak{g})$. The cohomology of the cochain complex $(\bigoplus_{i=0}^{+\infty} C^i(B), \mathrm{d}_{\mathsf{CE}}^B)$ is defined to be the **cohomology for the** \mathcal{D} -map B.

Denote the set of *n*-cocycles by $Z^n(B)$, the set of *n*-coboundaries by $B^n(B)$ and the *n*-th cohomology group by

$$H^{n}(B) = Z^{n}(B)/B^{n}(B), \quad n \ge 0.$$

It is obvious that $x \in \mathfrak{g}$ is closed if and only if

$$-\eta(x,u) + \pi(B(u),x) + B(\rho(x,u)) - B(\theta(B(u),x)) = 0, \quad \forall u \in \mathfrak{h},$$

and $f \in \text{Hom}(\mathfrak{h}, \mathfrak{g})$ is closed if and only if

$$-\eta(f(v), u) + \eta(f(u), v) + \pi(B(u), f(v)) - \pi(B(v), f(u)) + B(\rho(f(v), u)) - B(\rho(f(u), v))$$

= $B(\theta(B(u), f(v))) - B(\theta(B(v), f(u))) + f(\mu(u, v) + \theta(B(u), B(v))) + f(\rho(B(u), v) - \rho(B(v), u)),$ for all $u, v \in h$.

Here we provide an intrinsic interpretation of the above coboundary operator.

Let $B: \mathfrak{h} \to \mathfrak{g}$ be a \mathcal{D} -map of a quasi-twilled Lie algebra $(\mathcal{G}, \mathfrak{g}, \mathfrak{h})$. The twisted L_{∞} -algebra $(\bigoplus_{n=0}^{+\infty} \operatorname{Hom}(\wedge^{n+1}\mathfrak{h}, \mathfrak{g}), l_1^B, l_2^B, l_3^B)$ controls deformations of the \mathcal{D} -map B. Parallel to Proposition 3.30, we have the following proposition.

Proposition 4.22. With the above notations, for any $f \in \text{Hom}(\wedge^k \mathfrak{h}, \mathfrak{g})$, one has

$$l_1^B(f) = (-1)^{k-1} d_{CF}^B f.$$

Definition 4.21 also recover the existing cohomology theories of relative Rota-Baxter operators, twisted Rota-Baxter operators and Reynolds operators.

Example 4.23. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho} \mathfrak{h}, g, \mathfrak{h})$ given in Example 2.4 obtained from the action Lie algebra $g \ltimes_{\rho} \mathfrak{h}$. Let $B : \mathfrak{h} \to g$ be a relative Rota-Baxter operator of weight λ on g with respect to the action $(\mathfrak{h}; \rho)$. Then $(\mathfrak{h}, [\cdot, \cdot]_B)$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_B$ is given by

$$[u, v]_B = \lambda [u, v]_h + \rho(B(u))v - \rho(B(v))u, \quad \forall u, v \in \mathfrak{h}.$$

Moreover, the Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_R)$ represents on the vector space \mathfrak{g} via $\sigma : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ given by

$$\sigma(v)x = [B(v), x]_{\mathfrak{g}} + B(\rho(x)v), \quad \forall v \in \mathfrak{h}, x \in \mathfrak{g}.$$

The corresponding Chevalley-Eilenberg cohomology is taken to be the **cohomology for the relative Rota-Baxter operator of weight** λ . See [15, 26] for more details.

Example 4.24. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho} V, g, V)$ given in Example 2.5 obtained from the semidirect product Lie algebra $g \ltimes_{\rho} V$. Let $B : V \to g$ be a relative Rota-Baxter operator of weight 0 or an O-operator on g with respect to the representation $(V; \rho)$. Then $(V, [\cdot, \cdot]_B)$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_B$ is given by

$$[u, v]_B = \rho(B(u))v - \rho(B(v))u, \quad \forall u, v \in V.$$

Moreover, the Lie algebra $(V, [\cdot, \cdot]_B)$ represents on the vector space \mathfrak{g} via $\sigma : V \to \mathfrak{gl}(\mathfrak{g})$ given by

$$\sigma(u)x = [B(u), x]_{\mathfrak{q}} + B(\rho(x)u), \quad \forall u \in V, x \in \mathfrak{g}.$$

The corresponding Chevalley-Eilenberg cohomology is taken to be the **cohomology for the** O-**operator** B. See [50] for more details.

Example 4.25. Consider the quasi-twilled Lie algebra $(g \ltimes_{\rho,\omega} V, g, V)$ given in Example 2.7 obtained from a representation ρ of g on V and a 2-cocycle ω . Let $B : \mathfrak{h} \to \mathfrak{g}$ be a twisted Rota Baxter operator. Then $(V, [\cdot, \cdot]_B)$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_B$ is given by

$$[u, v]_B = \rho(B(u))v - \rho(B(v))u + \omega(B(u), B(v)), \quad \forall u, v \in V.$$

Moreover, the Lie algebra $(V, [\cdot, \cdot]_B)$ represents on the vector space g via $\sigma : V \to \mathfrak{gl}(\mathfrak{g})$ given by

$$\sigma(v)x = [B(v), x]_{\mathfrak{g}} + B(\rho(x)v) - B(\omega(B(v), x)), \quad \forall v \in V, x \in \mathfrak{g}.$$

The corresponding Chevalley-Eilenberg cohomology is taken to be the **cohomology for the twisted Rota-Baxter operator** B. See [10] for more details.

Example 4.26. Consider the quasi-twilled Lie algebra $(g \ltimes_{ad, [\cdot, \cdot]_g} g, g, g)$ given in Example 2.8. Let $B : g \to g$ be a Reynolds operator. Then $(g, [\cdot, \cdot]_B)$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_B$ is given by

$$[x, y]_B = [B(x), y]_g - [B(y), x]_g + [B(x), B(y)]_g, \quad \forall x, y \in g.$$

Moreover, the Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_B)$ represents on the vector space \mathfrak{g} via $\sigma : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ given by

$$\sigma(x)y = [B(x), y]_{\mathfrak{q}} + B([y, x]_{\mathfrak{q}}) - B([B(x), y]_{\mathfrak{q}}), \quad \forall x, y \in \mathfrak{g}.$$

The corresponding Chevalley-Eilenberg cohomology is taken to be the **cohomology for the Reynolds operator** B. See [10] for more details.

Definition 4.21 leads the following definition of cohomologies of deformation maps of a matched pair of Lie algebras.

Consider the quasi-twilled Lie algebra ($g \bowtie \mathfrak{h}, \mathfrak{g}, \mathfrak{h}$) given in Example 2.10 obtained from a matched pair of Lie algebras. Let $B : \mathfrak{h} \to \mathfrak{g}$ be a deformation map of a matched pair of Lie algebras. Then $(\mathfrak{h}, [\cdot, \cdot]_B)$ is a Lie algebra, where the Lie bracket $[\cdot, \cdot]_B$ is given by

$$[u, v]_B = [u, v]_{\mathfrak{h}} + \rho(B(u))v - \rho(B(v))u, \quad \forall u, v \in \mathfrak{h}.$$

Moreover, the Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_B)$ represents on the vector space g via $\sigma : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ given by

$$\sigma(v)x = \eta(v)x + [B(v),x]_{\mathfrak{g}} + B(\rho(x)v), \quad \forall v \in \mathfrak{h}, x \in \mathfrak{g}.$$

Definition 4.27. The corresponding Chevalley-Eilenberg cohomology of the Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_B)$ with coefficients in the representation (\mathfrak{g}, σ) is taken to be the **cohomology for the deformation** map B of a matched pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras.

Remark 4.28. In [1], authors constructed the Lie algebra structure $[\cdot, \cdot]_B$ on \mathfrak{h} via another approach, namely transfer the Lie algebra structure on Gr(B) to \mathfrak{h} .

Remark 4.29. In [10, 15, 26, 50], it was showed that one can use the established cohomology theory to classify infinitesimal deformations of relative Rota-Baxter operators, twisted Rota-Baxter operators and Reynolds operators. Similarly, one can also study infinitesimal deformations of a deformation map of a matched pair and show that they are classified by the second cohomology group of a deformation map of a matched pair given in Definition 4.27. We leave the details to readers.

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