# Higher-order topology protected by latent crystalline symmetries 

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#### Abstract

We demonstrate that rotation symmetry is not a necessary requirement for the existence of fractional corner charges in $C_{n}$-symmetric higher-order topological crystalline insulators. Instead, it is sufficient to have a latent rotation symmetry, which may be revealed upon performing an isospectral reduction on the system. We introduce the concept of a filling anomaly for latent crystalline symmetric systems, and propose modified topological invariants. The notion of higherorder topology in two dimensions protected by $C_{n}$ symmetry is thus generalized to a protection by latent symmetry. Our claims are corroborated by concrete examples of models that show non-trivial corner charge in the absence of $C_{n}$-symmetry. This work extends the classification of topological crystalline insulators to include latent symmetries.


## I. INTRODUCTION

Topological phases of matter have been extensively studied and form a cornerstone of condensed matter physics. These phases were originally understood in the framework of the celebrated Altland-Zirnbauer classification for non-interacting systems [1]. It considers the presence of time-reversal, particle-hole, and chiral symmetry to obtain 10 symmetry classes for topological insulators (TIs) and superconductors. It was also realized that there are various relationships between the classes in different dimensions, leading to what is called the ten-fold way [2]. Later, there have been various extensions that lie outside the original Altland-Zirnbauer classification. Examples are Floquet topological materials [3], disordered materials/topological Anderson insulators 4], non-Hermitian systems [5], and topological crystalline insulators (TCI's) [6], to name a few. The latter are known to host quantized electronic boundary states protected by the geometric symmetries of the crystalline system [6•9].

Of particular interest are higher-order topological (crystalline) insulators (HOTIs). While 'conventional' TIs in $D$ dimension host states with codimension one on their $D-1$ dimensional boundary, HOTIs host boundary states with a codimension $d$ on their $D-d$ dimensional boundary. One way of realizing HOTIs is by considering materials with quantized higher multipole moments in the bulk. Ref. [10] proposed a classification of materials with higher multipole moments as an extension of the modern theory of electric polarization [11, 12]. These systems are now known to be topological and characterized in terms of nested Wilson loops. However, systems without a bulk multipole moment may still give rise to quantized edge or corner states. Such systems are in a so-called obstructed atomic limit, a phase with a weaker topological protection than in the conventional topological materials. This was further investigated in Ref. [13], where the presence of rotation symmetry lead to the definition of rotation invariants to characterize the
band topology. Energy spectra with different rotation invariants cannot be deformed into each other without closing the bulk gap and are therefore topologically distinct. Nevertheless, the protection and classification of these novel topological phases of matter requires the presence of a geometric rotation symmetry.

Recently, a new type of symmetry, named latent symmetry, has been proposed [14, 15]. A latent symmetry only becomes apparent at the reduced level upon performing a suitable dimensional reduction, the so-called isospectral reduction (ISR) [16. In particular, seemingly asymmetric Hamiltonians may feature latent geometric symmetries which have a strong impact on the system's eigenstates and eigenenergies [15, 17]. This connection has been used, for instance, in the design of flat-band lattices 18, and to explain "accidental degeneracies" in band structures [19]. Very recently, latent symmetries were also used to construct latent versions of the Su-Schrieffer-Heeger (SSH) model [20] and of the non-Hermitian SSH model [21.

In this work, we explore the implications of the existence of latent symmetries on the classification of topological phases. Our main result is that the presence of a geometric rotation symmetry is not a necessary constraint for obtaining the topological phases introduced in Ref. 13. Indeed, the requirement of a rotation symmetry can be relaxed to the less stringent condition of a latent rotation symmetry.

The outline of this paper is the following. In Section II, we introduce the concept of filling anomaly in the context of the SSH model. We then relate this concept to the latent SSH model, proposed in Ref. [20, discuss the ISR in more detail, and formalise the concept of latent crystalline symmetries. In Section III, we review the topological classification of $C_{n}$-symmetric TCI's introduced in Ref. 13 and argue how the topological invariants should be modified for latent symmetries. In Section IV, the notion of primitive generators is introduced, and the generators proposed in Ref. [13] are generalized. We propose latent primitive generators
that have no direct rotation symmetry, but are latently rotation symmetric. The topological behaviour of these generators is characterised in phase diagrams using topological rotation invariants. Using the generators, we construct examples of crystalline latent HOTIs in Section V. Finally, we conclude our work in Section VI.

## II. FILLING ANOMALY BY LATENT SYMMETRY

Crystalline symmetries impose restrictions on the distribution of electrons in a crystal. As a consequence, it may not be possible to maintain charge neutrality everywhere in the lattice. The simplest example of a model that exhibits this behaviour is the SSH model. Its Bloch-Hamiltonian is given by

$$
h_{\mathrm{SSH}}(k)=\left(\begin{array}{cc}
0 & v+e^{i k}  \tag{1}\\
v+e^{-i k} & 0
\end{array}\right)
$$

where the intracell hopping is denoted by $v$, the intercell hopping is fixed to 1 , and $k$ is the dimensionless crystal momentum (we set the lattice constant $a=1$ ). The SSH model exhibits a mirror symmetry $M$, that is

$$
M h_{\mathrm{SSH}}(k) M^{-1}=h_{\mathrm{SSH}}(-k), \quad M=\left(\begin{array}{ll}
0 & 1  \tag{2}\\
1 & 0
\end{array}\right)
$$

The presence of this mirror symmetry gives rise to two gapped phases, separated by a band closing at $|v|=1$. At half filling, the system is an insulator, and for periodic boundary conditions (PBC), each unit cell hosts one electron. Overall, charge neutrality dictates that each unit cell is composed of an ion with charge $Q=|e|$. For open boundary conditions (OBC), without cutting through unit cells, the distribution of electrons depends on the phase. As a consequence of mirror symmetry, the Wannier centers of the electrons can only be located at either the center or the edge of a unit cell. For $|v|>1$ (trivial phase), the Wannier centers are located at the center of the unit cell and the system can be adiabatically connected to the trivial atomic limit. On the other hand, when $|v|<1$ (topological phase), the Wannier centers sit at the edge of the unit cell and the system can be connected to its obstructed atomic limit. The Wannier centers in both phases of the SSH model are shown in Fig. 1(a). In the trivial phase, $N$ electrons can be distributed symmetrically over $N$ unit cells to yield charge neutrality. However, in the topological phase, this is no longer possible. $N$ unit cells can only be filled in a symmetric manner using $N-1$ or $N+1$ electrons. Since mirror symmetry requires the energies of the edge states to be degenerate [see Fig. 1(b)], increasing the Fermi energy yields an increase of $N-1$ electrons to $N+1$ electrons, skipping $N$. This leads us to the notion of a filling anomaly $\eta$ [13], associated with a crystalline symmetry dividing the lattice in $n$ sectors, each spanning an angle of $2 \pi / n \mathrm{rad}$, with $n \in \mathbb{Z}$,

$$
\begin{equation*}
\eta=\# \text { ions }-\# \text { electrons } \quad \bmod n \tag{3}
\end{equation*}
$$

For the SSH model in the topological phase, we obtain $\eta=N-(N+1) \bmod 2=1$. Mirror symmetry distributes this charge imbalance over the two sides of the system, such that there will be a fractional boundary charge of $e / 2$ at the ends of the system. Consequently, the dipole moment in the topological phase equals $p=e / 2$, while it vanishes in the trivial phase.

Recently, Ref. [20] showed that although mirror symmetry leads to topological edge states in the SSH model, its not a necessary constraint. Indeed, the requirement of preserved mirror symmetry can be relaxed to preserved latent mirror symmetry. A latent symmetry is hidden and only becomes apparent upon performing what is called an isospectral reduction (ISR) -akin to an effective Hamiltonian - on the system [14]. Take a Hamiltonian $H$ as a starting point. The Hilbert space on which it acts may be partitioned in a set $S$ and its complement $\bar{S}$. The $\operatorname{ISR} \mathcal{R}_{S}(H, E)$ of $H$ is then defined through

$$
\begin{equation*}
\mathcal{R}_{S}(H, E)=H_{S S}-H_{S \bar{S}}\left(H_{\overline{S S}}-E \mathbb{I}\right)^{-1} H_{\bar{S} S} \tag{4}
\end{equation*}
$$

where $\mathbb{I}$ is the identity matrix [16]. The ISR converts the linear eigenvalue problem $H \boldsymbol{\psi}=E \boldsymbol{\psi}$ into a reduced, albeit non-linear, eigenvalue problem $\mathcal{R}_{S}(H, E) \psi_{S}=E \psi_{S}$. For example, consider the Bloch


FIG. 1. (a) Sketch of the SSH model in its trivial $(|v|>1)$ and topological $(|v|<1)$ phase. In the trivial (topological) phase, Wannier centers localize in the center (boundary) of the unit cell. (b) Spectrum of the SSH model for PBC (left) and OBC (right), the OBC spectrum shows in-gap edge localized modes. (c) Latent SSH model consisting of four sites with intra cell hoppings $w$ (thin solid line) and $\sqrt{2} w$ (thick solid line). Upon performing an ISR, an energy dependent SSH model is obtained. (d) Spectrum of the latent SSH model for PBC (left) and OBC (right). The OBC spectrum shows in-gap edge states for filling $1 / 4$ and $3 / 4$ in the topological phase.

Hamiltonian

$$
h_{\mathrm{LSSH}}(k)=\left(\begin{array}{cccc}
0 & e^{i k} & \sqrt{2} w & 0  \tag{5}\\
e^{-i k} & 0 & w & w \\
\sqrt{2} w & w & 0 & 0 \\
0 & w & 0 & 0
\end{array}\right)
$$

which is also depicted in Fig. 1(c) for OBC. By inspection, one can conclude that this Hamiltonian does not have a mirror symmetry. Nevertheless, upon performing an ISR over the red sites, we obtain

$$
\begin{align*}
\mathcal{R}_{S}\left[h_{\mathrm{LSSH}}(k), E\right] & =\left(\begin{array}{cc}
a(E) & s(E)+e^{i k} \\
s(E)+e^{-i k} & a(E)
\end{array}\right) \\
& \equiv \mathfrak{h}_{\mathrm{LSSH}}(k), \tag{6}
\end{align*}
$$

which bears a strong resemblance with the SSH model given in Eq. 11, though with energy-dependent on-site potential $a(E)=2 w^{2} / E$ and coupling $s(E)=\sqrt{2} w^{2} / E$. At the level of the ISR, there is a mirror symmetry: $M \mathfrak{h}_{\mathrm{LSSH}}(k) M^{-1}=\mathfrak{h}_{\mathrm{LSSH}}(-k){ }^{1}$. In other words, the system given by Eq. (5) hosts a latent mirror symmetry, which is revealed through the ISR.

The main idea of this work is to employ latent symmetries to construct HOTIs. Before we do so, let us discuss two important properties of latent symmetries and the ISR. Firstly, it has been recently shown that the presence of latent symmetry also implies a certain (though in general non-geometric) symmetry on the level of the original Hamiltonian [22. Let us define the eigenvalue problem $A \mathbf{v}=\lambda \mathbf{v}$ and let $\mathcal{R}_{S}(A, \lambda)$ be the ISR of $A$. If there exist a symmetry $T$ that becomes apparent after the ISR, that is

$$
\begin{equation*}
T \mathcal{R}_{S}(A, \lambda) T^{-1}=\mathcal{R}_{S}(A, \lambda) \tag{7}
\end{equation*}
$$

then there also exists a symmetry on the level of $A$ of the form $Q A Q^{-1}=A$, with $Q=T \oplus \bar{Q}$, where $\bar{Q}$ is a normal matrix that acts on $\bar{S}$. Equivalently,

$$
\begin{equation*}
\left[T, \mathcal{R}_{S}(A, \lambda)\right]_{-}=0 \Rightarrow \exists Q=T \oplus \bar{Q}:[Q, A]_{-}=0 \tag{8}
\end{equation*}
$$

where $[A, B]_{-}=A B-B A$ denotes the commutator. It is important to note that the full operation $Q$ is usually not a geometric symmetry. We will see examples for this statement throughout the manuscript.

With some additional technical preface, a similar reasoning can be applied to crystalline symmetries. Let us assume that our unit cell features a latent $T$ symmetry (for instance, $C_{n}$ ) that becomes apparent after performing an ISR to the set $S$ of its sites, i.e. $\left[\mathcal{R}_{S}(H, E), T\right]_{-}=0$. We then build a lattice such that (i) the ISR performed on the union of sites $S$ in all unit cells

[^0]| n | $\|v\|>1$ |  | $\|v\|<1$ |  |
| :---: | :---: | :---: | :---: | :---: |
|  | $\nu_{\alpha}$ | $\varphi$ |  | $\nu_{\alpha}$ |
|  | 0 | 0 | $\pi$ | $\varphi$ |
| 2 | $\{-\nu, \nu\}$ | 0 | $\{-\nu, \nu\}$ | 0 |
| 3 | $0,\{-\nu, \nu\}$ | 0 | $\pi,\{-\nu, \nu\}$ | $\pi$ |

TABLE I. Wilson loop eigenvalues $\nu_{\alpha}$ and Zak phase $\varphi$ for different fillings $n$ of $h_{\text {LSSH }}(k)$. For $|v|>1$ the model is always trivial, while for $|v|<1$ the model is topological for fillings $n=1$ and $n=3$.
is $T$-symmetric, and (ii) intercell-coupling only exists between sites $S$. It is now easy to show the following: If we denote the reduced Bloch-Hamiltonian of the lattice by $\mathfrak{h}(k)$, we find, analogous to Ref. [10],

$$
\begin{equation*}
T \mathfrak{h}(\mathbf{k}) T^{-1}=\mathfrak{h}\left(D_{T} \mathbf{k}\right) \tag{9}
\end{equation*}
$$

where $D_{T}$ is the representation of $T$ on the vector space of reciprocal lattice vectors. For example, mirror symmetry $(T \equiv M)$ flips spatial coordinates, such that $D_{M} k=-k$. The analogue of Eq. (7) is then

$$
\begin{equation*}
(T \oplus \bar{Q}) h(\mathbf{k})(T \oplus \bar{Q})^{-1}=h\left(D_{T} \mathbf{k}\right) \tag{10}
\end{equation*}
$$

To make things more specific, let us now investigate the latent SSH model introduced in Eq. (5). This model features a (1D) latent mirror symmetry: $T=M=\sigma_{x}$ and $D_{M} k=-k$. The symmetry $Q$ of $h_{\mathrm{LSSH}}(k)$ is given by (details on the derivation of the matrix $Q$ are given in Appendix A

$$
Q=M \oplus \bar{Q}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{11}\\
1 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}
\end{array}\right)
$$

The existence of $Q$ is central to the analysis considered in this work. In this case, it acts as a permutation on the red sites, but as a general orthogonal transformation on the other sites. Importantly, since $Q$ acts similarly on $h_{\text {LSSH }}(k)$ as an ordinary mirror symmetry would, that is $Q h_{\mathrm{LSSH}}(k) Q^{-1}=h_{\mathrm{LSSH}}(-k)$, it poses the exact same restrictions as mirror symmetry would. For example, the Wilson loop eigenvalues $\nu_{\alpha}$ of $h_{\mathrm{LSSH}}(k)$ are restricted to be $0, \pi$ or come in pairs $\{-\nu, \nu\}$. Moreover, its Zak phase $\varphi$ is quantized to 0 or $\pi$. Both the Wilson loop eigenvalues and the Zak phase at different filling factors are shown in Table

Secondly, besides revealing latent symmetries, the ISR also allows for a simpler topological characterisation of Bloch Hamiltonians. More concretely, consider two Hamiltonians $h_{1}\left(\mathbf{k}, \mathbf{t}=\left\{t_{1}, t_{2}, \ldots, t_{n}\right\}\right)$ and $h_{2}(\mathbf{k}, \mathbf{g}=$ $\left.\left\{g_{1}, g_{2}, \ldots, g_{m}\right\}\right)$. The first one, $h_{1}(\mathbf{k}, \mathbf{t})$, represents a known model which exhibits a gap closing at $E=E_{0}$ captured through a condition on the parameters $\mathbf{t}$ that could equivalently be written as $f(\mathbf{t})=0$. For the

SSH-model, for instance, we have $\mathbf{t}=(v)$ and the gap closes at $f(v)=1-|v|=0$; see the example below. Let us then assume that the ISR of the second model has the same form as $h_{1}(\mathbf{k})$, but with energy-dependent parameters $\left\{\mathfrak{t}_{1}(E, \mathbf{g}), \mathfrak{t}_{2}(E, \mathbf{g}), \ldots, \mathfrak{t}_{n}(E, \mathbf{g})\right\} \equiv \mathfrak{t}(E, \mathbf{g})$, up to a constant energy-dependent shift $a(E, \mathbf{g}){ }^{2}$ i.e.,

$$
\begin{equation*}
\mathfrak{h}_{2}(\mathbf{k})=h_{1}[\mathbf{k}, \mathfrak{t}(E, \mathbf{g})]+a(E, \mathbf{g}) \mathbb{I} . \tag{12}
\end{equation*}
$$

Then, we can extract the gap closing energies, $E^{*}$, of $h_{2}(\mathbf{k}, \mathbf{g})$ from

$$
\begin{equation*}
E^{*}-a\left(E^{*}, \mathbf{g}\right)=E_{0} \tag{13}
\end{equation*}
$$

The corresponding values of $\mathbf{g}$ can be obtained by solving $f\left[\mathfrak{t}\left(E^{*}, \mathbf{g}\right)\right]=0$. When the ISR reduces a system to a known model, the topological characterisation of the system is reduced from calculating multiband topological invariants to solving an algebraic problem [20, 21]. As an example, consider now the latent SSH model described by Eq. (5). The 'ordinary' SSH model has a gap closing at $E_{0}=0$ for $f(v)=1-|v|=0$. The gap closing energies $E^{*}$ for the latent model are thus determined through

$$
\begin{equation*}
E^{*}-a\left(E^{*}, w\right)=E^{*}-\frac{2 w^{2}}{E^{*}}=0 \tag{14}
\end{equation*}
$$

Equation 14 has two solutions $E^{*}= \pm \sqrt{2} w$, which is corroborated by Fig. 1(d) showing two sets of in-gap modes in red. The energies $E^{*}$ predict the topological phase transition through $f\left[s\left(E^{*}, w\right)\right]=1-\left|s\left(E^{*}, w\right)\right|=$ 0 , which in this case reduces to $|w|=1$, akin to the SSH model. One can also conclude that in order to respect the symmetries of the system at bulk insulating filling, the system with OBC must feature a filling-anomaly.

## III. TOPOLOGICAL INDICES FROM (LATENT) ROTATIONAL SYMMETRY

In the above, we have discussed a simple latent SSH model, which has been previously investigated both in its Hermitian [20] and its non-Hermitian [21] version. In the remainder of this work, we apply a similar reasoning to develop the concept of latent HOTIs. To this end, we start by reviewing the existing methods for the classification of usual HOTIs. We focus on two-dimensional insulators that preserve timereversal symmetry (class AI in the Altland-Zirnbauer classification [1, 2]). The introduction of crystalline symmetries allows for a further classification of these materials [6, 23]. Here, we will follow Ref. [13].

[^1]
## A. Recap: Topological indices through conventional geometric rotation symmetries

The presence of a rotation symmetry $\hat{C}_{n}$, which rotates sites in a lattice by $2 \pi / n \mathrm{rad}$ around some point, is represented on the level of the Bloch Hamiltonian by

$$
\begin{equation*}
\hat{C}_{n} h(\mathbf{k}) \hat{C}_{n}^{-1}=h\left(D_{C_{n}} \mathbf{k}\right), \tag{15}
\end{equation*}
$$

where, similar to Eq. (9), $D_{C_{n}}$ is a linear transformation on $\mathbf{k}$ that depends on $C_{n}$. At high symmetry points (HSPs) in the Brillouin zone, i.e. points that are mapped to themselves (modulo a reciprocal lattice vector), $D_{C_{n}} \Pi^{(n)}=\Pi^{(n)}$, we have

$$
\begin{equation*}
\left[\hat{C}_{n}, h\left(\boldsymbol{\Pi}^{(n)}\right)\right]_{-}=0 \tag{16}
\end{equation*}
$$

Because $\hat{C}_{n}$ and $h(k)$ commute at a HSP $\boldsymbol{\Pi}^{(n)}$, they share an eigenbasis. Thus, the Bloch states $\left|u\left(\mathbf{k}=\boldsymbol{\Pi}^{(\boldsymbol{n})}\right)\right\rangle$ can be chosen as eigenstates of $\hat{C}_{n}$, such that

$$
\begin{equation*}
\hat{C}_{n}\left|u\left(\Pi^{(n)}\right)\right\rangle=\Pi_{p}^{(n)}\left|u\left(\Pi^{(n)}\right)\right\rangle . \tag{17}
\end{equation*}
$$

Since $\left(\hat{C}_{n}\right)^{n}=\mathbb{I}$, its eigenvalues are the $n$-th roots of unity:

$$
\begin{equation*}
\Pi_{p}^{(n)}=e^{\frac{2 \pi i}{n}(p-1)}, \quad \text { with } \quad p \in\{1,2, \ldots n\} \tag{18}
\end{equation*}
$$

From these eigenvalues, we can construct rotation topological invariants of the form

$$
\begin{equation*}
\left[\Pi_{p}^{(n)}\right] \equiv \# \Pi_{p}^{(n)}-\# \Gamma_{p}^{(n)} \tag{19}
\end{equation*}
$$

where $\# \Pi_{p}^{(n)}$ denotes the number of bands below the energy gap with eigenvalue $\Pi_{p}^{(n)}$ and $\boldsymbol{\Gamma}=\mathbf{0}$ is the gamma point in the Brillouin zone, which is a natural reference point to calculate the rotational invariants ${ }^{3}$, $\left[\Pi_{p}^{(n)}\right]$ characterises the topology of $C_{n}$-symmetric insulators in a similar way to inversion eigenvalues: A difference of inversion eigenvalues between two HSPs indicates band inversion, i.e. non-trivial topology. In a similar manner, the difference of rotation eigenvalues between HSPs, captured by $\left[\Pi_{p}^{(n)}\right]$, allows for a comparison of the representations of rotation symmetry. If different representations exist, the energy bands exhibit nontrivial topology. As a consequence of TRS and the fact that the number of bands is constant through the Brillouin zone, one obtains a set of independent

[^2]

FIG. 2. Latently $C_{n}$ symmetric unit cells. (a-d) Minimal examples of unit cells that show a latent $C_{2}, C_{3}, C_{4}$, and $C_{6}$ symmetry without a direct geometric symmetry. Relevant hopping parameters are indicated. (e-h) Cells obtained upon performing an ISR to the red sites of the unit cells in (a-d), the reduced systems show geometric $C_{n}$ symmetries, revealing the latent symmetries in (a-d). Relevant (energy-dependent) parameters are indicated.
topological indices $\chi^{(n)}$ for $C_{n}$-symmetric materials [13],

$$
\begin{align*}
\chi^{(2)} & =\left(\left[X_{1}^{(2)}\right],\left[Y_{1}^{(2)}\right],\left[M_{1}^{(2)}\right]\right) \\
\chi^{(4)} & =\left(\left[X_{1}^{(2)}\right],\left[M_{1}^{(4)}\right],\left[M_{2}^{(4)}\right]\right) \\
\chi^{(3)} & =\left(\left[K_{1}^{(3)}\right],\left[K_{2}^{(3)}\right]\right) \\
\chi^{(6)} & =\left(\left[M_{1}^{(2)}\right],\left[K_{1}^{(3)}\right]\right) . \tag{20}
\end{align*}
$$

## B. Topological indices through latent rotation symmetries

Having reviewed the theory regarding geometric symmetries, let us now discuss how latent symmetries fit into the scheme. Rather surprisingly, it can be shown that the invariants $\chi^{(n)}$ are still applicable. The key for this insight is Eq. (8). Equipped with this equation, one only needs to replace the symmetry operator $\hat{C}_{n}$ in Eq. 17) by $\hat{C}_{n} \oplus \bar{Q}$, as defined in Eq. (10). Upon doing so, the rotation invariants of latent symmetric models can be evaluated to characterize the topology of the system. This leads to the definition of latent rotation symmetric HOTIs through

$$
\begin{equation*}
\left[\hat{C}_{n} \oplus \bar{Q}, h\left(\boldsymbol{\Pi}^{(n)}\right)\right]_{-}=0 \tag{21}
\end{equation*}
$$

where the total symmetry $\hat{C}_{n} \oplus \bar{Q}$ is not necessarily geometric. Equivalently, we have

$$
\begin{equation*}
\left[\hat{C}_{n}, \mathfrak{h}\left(\boldsymbol{\Pi}^{(n)}\right)\right]_{-}=0 \tag{22}
\end{equation*}
$$

## C. Classification of HOTIs: Dipole moment and corner charges

The rotation invariants $\chi^{(n)}$ can be related to physical properties of the systems, such as the dipole moment $\mathbf{P}^{(n)}$ and corner charge $Q^{(n)}$. These quantities are derived in Ref. [13] for geometric symmetries and can be easily shown to hold also for latent symmetries.

Below, we will directly state the results for the dipole moment,

$$
\begin{align*}
& \mathbf{P}^{(2)}= \frac{e}{2}\left(\left[Y_{1}^{(2)}\right]+\left[M_{1}^{(2)}\right]\right) \mathbf{a}_{\mathbf{1}} \\
& \quad+\frac{e}{2}\left(\left[X_{1}^{(2)}\right]+\left[M_{1}^{(2)}\right]\right) \mathbf{a}_{\mathbf{2}} \\
& \mathbf{P}^{(4)}= \frac{e}{2}\left[X_{1}^{(2)}\right]\left(\mathbf{a}_{\mathbf{1}}+\mathbf{a}_{\mathbf{2}}\right), \\
& \mathbf{P}^{(3)}= \frac{2 e}{3}\left(\left[K_{1}^{(3)}\right]+2\left[K_{2}^{(3)}\right]\right)\left(\mathbf{a}_{\mathbf{1}}+\mathbf{a}_{\mathbf{2}}\right), \\
& \mathbf{P}^{(6)}= \mathbf{0} \tag{23}
\end{align*}
$$

and for the corner charge,

$$
\begin{align*}
Q^{(2)} & =\frac{e}{4}\left(-\left[X_{1}^{(2)}\right]-\left[Y_{1}^{(2)}\right]+\left[M_{1}^{(2)}\right]\right) \\
Q^{(4)} & =\frac{e}{4}\left(\left[X_{1}^{(2)}\right]+2\left[M_{1}^{(4)}\right]+3\left[M_{2}^{(4)}\right]\right) \\
Q^{(3)} & =\frac{e}{3}\left[K_{2}^{(3)}\right] \\
Q^{(6)} & =\frac{e}{4}\left[M_{1}^{(2)}\right]+\frac{e}{6}\left[K_{1}^{(3)}\right] \tag{24}
\end{align*}
$$

The dipole moments are defined modulo $e / A_{\text {cell }}$ times a lattice vector and the corner charges are defined modulo $e$. Here $A_{\text {cell }}$ denotes the area of the unit cell. Notice also that we group the dipole moments $\mathbf{P}^{(2)}$ and $\mathbf{P}^{(4)}$ $\left(\mathbf{P}^{(3)}\right.$ and $\left.\mathbf{P}^{(6)}\right)$ because they share at least one common topological index (the same is done for the corner charges as well).

Before we continue, let us briefly remind the reader about the connection between these two quantities and HOTIs. To this end, we consider a system under OBC, terminated respecting its (latent or geometric) $C_{n}$ symmetry. The system is a HOTI if it is (higher order) topological and insulating. The former requirement translates in a nonzero corner charge. If the dipole moment $\mathbf{P}^{(n)}$ does not vanish, there will be in-gap states when OBC are imposed. This poses two constraints on a HOTI, be it conventional or latent:

- The corner charge $Q^{(n)}$, as given by Eq. 24, is nonzero.
- The dipole moment $\mathbf{P}^{(n)}$, as given by Eq. 23), vanishes.

In the remainder of this manuscript, we will show how one can construct a latent HOTI, that is, a latently $C_{n^{-}}$ symmetric system fulfilling these two criteria.

## IV. BUILDING BLOCKS FOR LATENT HOTIS

A necessary first step to construct a latent HOTI is a sufficiently large set of latently $C_{n}$ symmetric lattices. In this section, we show how this task can be achieved. In particular, in the next Section IV A we will introduce unit cells that feature a latent $C_{2}-, C_{3}-, C_{4}-$, and $C_{6}$-symmetry, as depicted in Fig. 2. Afterwards, in Section IV B we shall use these unit cells to construct lattices, such that the total system retains the latent $C_{n}$ rotation symmetry. Finally, in Section IV C we present a systematic way of constructing latently $C_{n}$-symmetric setups with a pre-defined topological index. Equipped with all these tools, we will then finally construct latent HOTIs in the next Section V .

## A. Latently symmetric unit cells

In the following, we will present unit cells featuring a latent $C_{2}, C_{3}, C_{4}$, or $C_{6}$ symmetry. Before we discuss
them in detail, let us briefly remark how we designed these cells. The $C_{2}$-symmetric cell was found analytically by starting with a four-site long chain with couplings $a, b, c$ and all on-site energies equal to zero. Then, demanding that two sites $u, v$ in this chain are latently symmetric, analytical expressions for the three couplings were found. We remark that there are other techniques for the design of a latently $C_{2}$-symmetric system, with examples ranging from exhaustive search [24] to more sophisticated graph-theoretical results [25]. The interested reader is referred to the literature on graphs $\$^{4}$ with cospectral vertices [25, 26]; every graph with this property has recently been shown to have a latent $C_{2^{-}}$ symmetry [17. For the unit cells with a latent $C_{3}, C_{4}$, and $C_{6}$-symmetry, we used the complement multiplet technique explained in the Supplemental Material of Ref. [19. In particular, we started with a $C_{n}$-symmetric Hamiltonian, which trivially hosts also a latent $C_{n^{-}}$ symmetry ${ }^{5}$ and computed its complement multiplets. Then, we added a new site to the system and connected it to the complement multiplets such that the geometric $C_{n}$-symmetry is broken while its latent $C_{n}$-symmetry is maintained.

## 1. Latent $C_{2}$ symmetry

In Sec. II, the notion of a latently mirror symmetric system was introduced in the context of the latent SSH model. In 1D, $C_{2}$ and mirror symmetry are the same, hence we take the latent SSH unit cell as a building block for latent $C_{2}$-symmetric HOTIs. The unit cell of the latent SSH model, and its ISR to the red sites, are once more displayed in Figs. 2(a) and 2(e), respectively. The Hamiltonian for the single cell in Fig. 2(a) is given by

$$
H_{L}^{(2)}=t_{0}\left(\begin{array}{cccc}
0 & 0 & \sqrt{2} & 0  \tag{25}\\
0 & 0 & 1 & 1 \\
\sqrt{2} & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right)
$$

The parameters in the isospectrally reduced model in Fig. 2(e) are given by $a^{(2)}=2 t_{0}^{2} / E$ and $v_{0}^{(2)}=\sqrt{2} t_{0}^{2} / E$. The reduced model obeys a $C_{2}$-symmetry,

$$
\hat{C}_{2}=\left(\begin{array}{ll}
0 & 1  \tag{26}\\
1 & 0
\end{array}\right)
$$

[^3]which corresponds to the symmetry
\[

Q^{(2)} \equiv \hat{C}_{2} \oplus \bar{Q}^{(2)}=\left($$
\begin{array}{ll}
0 & 1  \tag{27}\\
1 & 0
\end{array}
$$\right) \oplus\left($$
\begin{array}{cc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}
\end{array}
$$\right)
\]

of the full Hamiltonian.

## 2. Latent $C_{3}$ symmetry

An example of a unit cell exhibiting latent $C_{3}$ symmetry is depicted in Fig. 2(b). It has 9 sites, with hopping given by $t_{0}$ (thin black lines) and $2 t_{0}$ (thick black lines). Its Hamiltonian is given by

$$
H_{L}^{(3)}=t_{0}\left(\begin{array}{ccccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 2 & 0 & 0  \tag{28}\\
0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

Upon performing an ISR to the red sites, the cell shown in Fig. 2 (f) is obtained, displaying a $C_{3}$-symmetry. The reduced model has a single energy-dependent hopping parameter $v_{0}^{(3)}=5 t_{0}^{2} / E$ and onsite potential $\left.a^{(3)}=7 t_{0}^{2} / E\right]^{6}$ The corresponding symmetry matrix of the full Hamiltonian $Q^{(3)}$ [cf. Eq. (8)] of the full system depicted in Fig. 2(b) is given by

$$
\begin{aligned}
Q^{(3)} & \equiv \hat{C}_{3} \oplus \bar{Q}^{(3)} \\
& =\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right) \oplus\left(\begin{array}{cccccc}
\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{1}{2} \\
0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
-\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}\right)
\end{aligned}
$$

## 3. Latent $C_{4}$ symmetry

Next, let us investigate the setup shown in Fig. 2(c), which has 13 -sites and features four different hopping parameters: $t_{0}$ (black), $t_{1}$ (green), $t_{2}$ (red), and $t_{3}$ (blue). A double-coloured line implies a sum of the two hopping strengths, i.e. the blue-and-red line has hopping parameter $t_{2}+t_{3}$. The matrix form, $H_{L}^{(4)}$, of Fig. 2(c) is given in Appendix B1 by Eq. B1). Depending on the

[^4]choice of couplings, this unit cell has different geometric symmetries. Firstly, if $t_{1}=t_{2}$ and $t_{3}=0$, it enjoys a $C_{4^{-}}$ symmetry. Keeping $t_{3}=0$ but breaking the equality of the first two couplings such that $t_{1} \neq t_{2}$, this symmetry is partly broken and only a geometric $C_{2}$ symmetry is left. The situation becomes much easier, though, when performing an ISR on the red sites. The resulting reduced model is depicted in Fig. 2 (g); it has an energy-dependent on-site potential $a^{(4)}$ and hopping parameters $v_{0}^{(4)}$ (black) and $v_{1}^{(4)}$ (grey). As can be easily checked, it has a $C_{4}$ symmetry given by
\[

\hat{C}_{4}=\left($$
\begin{array}{llll}
0 & 0 & 0 & 1  \tag{30}\\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}
$$\right)
\]

Thus, regardless of the choice of couplings $t_{i}$, the unit cell is latently $C_{4}$-symmetric. Once again, this demonstrates that the ISR gives a wider, more comprehensive viewpoint on a system than merely checking its geometric symmetries. Before continuing, we remark that-again, by Eq. (8)-this latent $C_{4}$ symmetry corresponds to a non-geometric symmetry $Q^{(4)} \equiv \hat{C}_{4} \oplus \bar{Q}^{(4)}$ of the full unit cell. Expressions for $\bar{Q}^{(4)}$ and the parameters in Fig. $2(\mathrm{~g})$ are given in Appendix B 1.

## 4. Latent $C_{6}$ symmetry

Finally, we come to the unit cell depicted in Fig. 2(d). It has 19 sites and features three different hopping parameters $t_{0}$ (black), $t_{1}$ (green), $t_{2}$ (red). A more complex latent $C_{6}$-symmetric unit cell is given in Appendix B 2. This model is $C_{6}$-symmetric only when $t_{1}=t_{2}=0$. Furthermore, it is $C_{3}$ symmetric if only $t_{2}=0$. Once again, the picture becomes clearer when performing an ISR to the red sites. The resulting reduced model is depicted in Fig. 2(h); it is described in terms of the on-site potential $a^{(6)}$ and hopping parameters $v_{0}^{(6)}$ (black), $v_{1}^{(6)}$ (grey), and $v_{2}^{(6)}$ (red). It can be promptly seen that the reduced model is $C_{6}$-symmetric. Thus, irrespective of the choice of coupling parameters $t_{i}$, the full unit cell is latently $C_{6}$-symmetric. This latent symmetry corresponds to a non-geometric symmetry of the full unit cell, given by $Q^{(6)} \equiv \hat{C}_{6} \oplus \bar{Q}^{(6)}$, with full expressions for $\bar{Q}^{(6)}$ and the parameters in Fig. 2 (h) given in Appendix B 2 .

## B. Lattice structures preserving latent $C_{n}$-symmetries

Equipped with a set of latently $C_{n}$-symmetric unit cells, the next step is to embed these into lattices, such that the total setup keeps this symmetry. As we now show, this task is rather simple. Let us assume that a
given lattice structure is composed of a unit cell with a geometric $C_{n}$-symmetry, such that the lattice as a whole keeps the symmetry. Next, let us replace the geometrically $C_{n}$-symmetric unit cell by one whose ISR on a set $S$ of sites has the same symmetry, i.e. a latently $C_{n}$-symmetric unit cell. It is then a trivial task to show that the lattice's ISR-for clarity, we mean the simultaneous reduction on the union of sites $S$ in each unit cell-is $C_{n}$-symmetric. In other words, the lattice is latently $C_{n}$-symmetric. Examples of systems with these features are discussed in more detail in the following section.

## C. Primitive generators and their topological classification

With the material presented so far, one could easily construct a large number of latently $C_{n}$-symmetric lattices. However, to be HOTIs, they need a vanishing dipole moment and a non-vanishing corner charge. Via Eqs. (23) and (24), both quantities are connected to the topological indices in Eq. 20). In principle, one could then find latent HOTIs by a brute-force search, that is, by computing these quantities for a large number of latently $C_{n}$-symmetric setups and filtering out the ones that are HOTIs. However, there is a more elegant and systematic way that is based on the concept of primitive generators [13]. Essentially, these are building blocks with certain properties, which can be connected to each other in a specific manner, such that the resulting setup features a well-defined topological index.

To introduce these generators, let us start with an interesting fact on topological indices. As pointed out in Ref. [13], two models with the same $C_{n^{-}}$ symmetry, described by Bloch Hamiltonians $h_{1}$ and $h_{2}$, characterized by $\chi_{1}^{(n)}$ and $\chi_{2}^{(n)}$, respectively, may be combined to form a third model

$$
h_{3}=\left(\begin{array}{cc}
h_{1} & \gamma  \tag{31}\\
\gamma^{\dagger} & h_{2}
\end{array}\right)
$$

Here, $\gamma$ connects the two different Hamiltonians in a way that does not close any gaps and that preserves $C_{n^{-}}$ symmetry. The rotation invariant of the new model is then given by $\chi_{3}^{(n)}=\chi_{1}^{(n)}+\chi_{2}^{(n)}$. Consequently, if for a given symmetry $\chi^{(n)}$ has $N$ components, it is sufficient to have $N$ models with linearly independent $\chi^{(n)}$ to span the whole topological phase space. This sets the basis to the notion of primitive generators. The primitive generators form a minimal set from which a setup with an arbitrary topological index can be constructed. From there, the construction of an actual (latent) HOTI is only a minor step.

In the following, we will discuss and classify (latent) primitive generators for every one of the four classes $C_{2}$, $C_{3}, C_{4}$, and $C_{6}$ that are compatible with translational invariance of a crystal. The key results are graphically
depicted in Figs. 3 to 6 and are discussed in more detail in the text. For each of the four classes, we first treat the conventional case of geometric symmetry, and then treat the new case of latent symmetry. This dual treatment might seem redundant, but it serves two purposes. Firstly, the characteristics (for instance, the band structure) of the geometric and latent setups have some striking similarity that would otherwise be unnoticed. Secondly, our treatment of the geometrically symmetric primitive generators represents minor generalisations of the primitive generators introduced in Ref. [13].

In each of the following examples, the primitive generators are the Bloch-Hamiltonians of a crystal obtained by inserting a specific unit cell (with either a geometric or latent $C_{n}$ symmetry) into a specific lattice structure. We remark that a given $C_{n}$ might support different lattice structures.

## 1. $C_{2}$-symmetry

For a $C_{2}$ symmetry, we investigate only one lattice structure, which we call the "stacked SSH model".
a. Geometric stacked SSH. We consider the primitive generator given by the Hamiltonian

$$
h_{1}^{(2)}(\mathbf{k})=\left(\begin{array}{cc}
2 w \cos k_{y} & t+e^{i k_{x}}  \tag{32}\\
t+e^{-i k_{x}} & 2 w \cos k_{y}
\end{array}\right)
$$

It is the Bloch-Hamiltonian of a system obtained by inserting the unit cell depicted in Fig. 3(a) into the stacked SSH lattice structure of Fig. 3(d). The system corresponds to multiple SSH chains stacked in the $y$ direction. Hopping within a chain occurs with an intracell hopping given by $t$ (solid black line) and a horizontal intercell hopping of 1 (dashed black line). The chains are connected with a vertical intercell hopping of $w$ (dot-dashed black line). The lattice vectors are given by $\mathbf{a}_{1}=(1,0)$ and $\mathbf{a}_{2}=(0,1)$. Eq. 32 admits a $C_{2^{-}}$ symmetry given by

$$
\begin{equation*}
\hat{C}_{2} h_{1}^{(2)}\left(k_{x}, k_{y}\right) \hat{C}_{2}^{-1}=h_{1}^{(2)}\left(-k_{x},-k_{y}\right) \tag{33}
\end{equation*}
$$

where $\hat{C}_{2}$ is given by Eq. 26. Consequently, the topology of the system may be characterized using $\chi^{(2)}$. Since the value of $w$ does not affect the gap structure, we set $w=0$. For general values of $t$, the spectrum is gapped at half filling, as shown in Fig. 3(b), giving rise to two distinct phases. These phases are separated by a gap closing at $t=+1(-1)$, taking place along the $\mathbf{X M}(\mathbf{Y} \boldsymbol{\Gamma})$ path in the Brillouin zone. For $|t|<1$, the system is in its topological phase, corresponding to $\chi^{(2)}=(1,0,1)$ and $\mathbf{P}^{(2)}=(e / 2) \mathbf{a}_{\mathbf{1}}$. For $|t|>1$, the system is trivial with $\chi^{(2)}=(0,0,0)$ and $\mathbf{P}^{(2)}=\mathbf{0}$. The different topological phases are depicted in Fig. 3(c). We note that a second, independent, generator for $C_{2}$ can be obtained by rotating Fig. 3 (d) by 90 degrees. This would correspond by letting $k_{x} \rightarrow k_{y}$ and $k_{y} \rightarrow-k_{x}$ in Eq. (32). As a result, the topological phase would


FIG. 3. (a) Unit cell of the geometric $C_{2}$-symmetric primitive generator $h_{1}^{(2)}(\mathbf{k})$, with intracell hopping $t$. Horizontal intercell hopping is indicated by dashed lines and is fixed to 1 ; vertical intercell hopping is represented by a dot-dashed line and is given by $w$. (b) Energy spectrum of $h_{1}^{(2)}(\mathbf{k})$ for $t=1 / 2$ and $w=0$. (c) Phase diagram, in which the rotation invariants $\chi^{(2)}$ are shown for gapped phases. (d) Lattice structure of the $C_{2}$-symmetric primitive generators. The lattice represents stacked 1D SSH chains. (e) Unit cell of the Latent $C_{2}$-symmetric primitive generator $h_{L, 1}^{(2)}(\mathbf{k})$. Values of the hopping parameters are indicated in Fig. 2 (a) and horizontal intercell hoppings (dashed lines) are fixed to 1 while vertical intercell hopping (dot-dashed lines) are given by $w$. (f) Spectrum of $h_{L, 1}^{(2)}(\mathbf{k})$ for $t_{0}=1 / 2$ and $w=0$. (g) Phase diagram at one filled band. Rotation invariants $\chi^{(2)}$ are displayed for gapped phases.
now have $\chi^{(2)}=(0,1,1)$ and $\mathbf{P}^{(2)}=(e / 2) \mathbf{a}_{2}$. This would yield 2 generators for $C_{2}$-symmetric systems. A third generator can be obtained by taking one of the $C_{4}$ generators in the next section and making the hopping in the $x-$ and $y$-direction different.
b. Latent stacked $S S H$. If we insert the unit cell depicted in Fig. 3(e) into the lattice structure of Fig. 3(d),
we obtain a system with a Bloch-Hamiltonian given by

$$
h_{L, 1}^{(2)}(\mathbf{k})=H_{L}^{(2)}+\left(\begin{array}{cccc}
2 w \cos k_{y} & e^{i k_{x}} & 0 & 0  \tag{34}\\
e^{-i k_{x}} & 2 w \cos k_{y} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

In Fig. 3(e), the horizontal intercell hopping (dashed black line) is fixed to 1 and the vertical intercell hopping (dot-dashed black line) is given by $w$. Figure 3(f) depicts the spectrum of Eq. (34) for $t_{0}=1 / 2$ and $w=0$. Notice that the spectrum resembles two copies of the one in Fig. 3(b). Since this primitive generator represents stacked (latent) SSH chains, its phase diagram is the same as that of the SSH chain. At the end of Sec. II, we showed that for this specific latent SSH model, phase transitions occur at $|t|=1$, just like for the SSH model (again, the value of $w$ does not affect the gap structure). Following this reasoning, $h_{L, 1}^{(2)}(\mathbf{k})$ has a gap closing at $\left|t_{0}\right|=1$, separating the trivial phase $\chi^{(2)}=(0,0,0)$ $\left(\left|t_{0}\right|>1\right)$ from the topological phase $\chi^{(2)}=(1,0,1)$ $\left(\left|t_{0}\right|<1\right)$, as shown in the phase diagram in Fig. 3(g).

## 2. $C_{4}$-symmetry

For a $C_{4}$-symmetry, we consider three lattice structures: a "2D SSH", a "breathing square-octagon", and a "stacked breathing square-octagon".
a. Geometric 2D SSH. Inserting the unit cell of Fig. 4(a) into a lattice structure, we obtain Fig. 4(j), which corresponds to a system described by the BlochHamiltonian
$h_{1}^{(4)}(\mathbf{k})=\left(\begin{array}{cccc}0 & t_{1}+e^{i k_{x}} & t_{2} & t_{1}+e^{i k_{y}} \\ t_{1}+e^{-i k_{x}} & 0 & t_{1}+e^{i k_{y}} & t_{2} \\ t_{2} & t_{1}+e^{-i k_{y}} & 0 & t_{1}+e^{-i k_{x}} \\ t_{1}+e^{-i k_{y}} & t_{2} & t_{1}+e^{i k_{x}} & 0\end{array}\right)$.
The system has 4 sites per unit cell, as depicted in Fig. 4(a), and is also known as the 2D SSH model [2729. The sites are connected with hopping parameters $t_{1}$ and $t_{2}$ for nearest-neighbour and next-nearest-neighbour hopping, respectively. The intercell hopping (dashed lines) is fixed to unity. The lattice vectors are given by $\mathbf{a}_{\mathbf{1}}=(1,0)$ and $\mathbf{a}_{\mathbf{2}}=(0,1)$. Equation 35 has a $C_{4^{-}}$ symmetry, that is:

$$
\begin{equation*}
\hat{C}_{4} h_{1}^{(4)}\left(k_{x}, k_{y}\right) \hat{C}_{4}^{-1}=h_{1}^{(4)}\left(k_{y},-k_{x}\right) \tag{36}
\end{equation*}
$$

where $\hat{C}_{4}$ is given by Eq. 30. Figure 4 (b) shows the spectrum of $h_{1}^{(4)}(\mathbf{k})$ for $t_{1}=1 / 2$ and $t_{2}=0$, which is gapped at $1 / 4$ - and $3 / 4$-filling. For this parameter choice, we have $\left[X_{1}^{(2)}\right]=-1,\left[M_{1}^{(4)}\right]=1,\left[M_{2}^{(4)}\right]=0$, hence $\chi^{(4)}=(-1,1,0)$, which corresponds to $\mathbf{P}^{(4)}=$ $e\left(\mathbf{a}_{1}+\mathbf{a}_{\mathbf{2}}\right) / 2$. The spectrum closes its gap for $\left|t_{1}\right|+\left|t_{2}\right|=$ 1, which occurs simultaneously at the $\mathbf{X}$ and $\mathbf{Y}$ point, owing to the $C_{4}$-symmetry. For finite $t_{2}$, the system


FIG. 4. (a,d,g) Unit cells of the geometric $C_{4}$-symmetric primitive generators $h_{1}^{(4)}(\mathbf{k}), h_{2}^{(4)}(\mathbf{k})$, and $h_{3}^{(4)}(\mathbf{k})$, respectively. The nearest-neighbour hoppings are represented by $t_{1}$; the next-nearest-neighbour hoppings are represented by $t_{2}$; the intercell hoppings are indicated by dashed lines and are fixed to be equal to 1 . (b,e,h) Energy spectra of $h_{1}^{(4)}(\mathbf{k})$ at $t_{1}=1 / 2$ and $t_{2}=0$, $h_{2}^{(4)}(\mathbf{k})$ at $t_{1}=1 / 2$ and $t_{2}=0$, and $h_{3}^{(4)}(\mathbf{k})$ at $t_{1}=0$ and $t_{2}=1 / 2$. (c,f,i) Phase diagrams of the geometric primitive generators. For gapped phases, the rotation invariants $\chi^{(4)}$ are shown. ( $\left.\mathrm{j}, \mathrm{k}, \mathrm{l}\right)$ Lattice structures of the $C_{4}-$ symmetric primitive generators. The lattices represent a 2 D SSH , a breathing square-octagon, and a stacked breathing square octagon lattice, respectively. $(\mathrm{m}, \mathrm{p}, \mathrm{s})$ Unit cells of the latent $C_{4}$-symmetric primitive generators $h_{L, 1}^{(4)}(\mathbf{k}), h_{L, 2}^{(4)}(\mathbf{k})$, and $h_{L, 3}^{(4)}(\mathbf{k})$, respectively. Values of the hopping parameters are indicated in Fig. 2 (c) and intercell hoppings (dashed lines) are fixed to 1 . ( $\mathrm{n}, \mathrm{q}, \mathrm{t}$ ) Energy spectra of $h_{L, 1}^{(4)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=1, t_{3}=1 / 2, h_{L, 2}^{(4)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=1 / 2, t_{3}=1$, and $h_{L, 3}^{(4)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=-1, t_{3}=0$. $(\mathrm{o}, \mathrm{r}, \mathrm{u})$ Phase diagrams of $h_{L, 1}^{(4)}(\mathbf{k})$ at 1 filled band, $h_{L, 2}^{(4)}(\mathbf{k})$ at 2 filled bands, and $h_{L, 3}^{(4)}(\mathbf{k})$ at 3 filled bands, as a function of $t_{0}$ and $t_{3}$, $t_{1}=t_{2}=t_{0}$. Rotation invariants $\chi^{(4)}$ are shown for gapped phases.
remains gapless till $\left|t_{1}\right|=1+\left|t_{2}\right|$. At this point, the system becomes gapped once again with $\chi^{(4)}=(0,0,0)$, corresponding to $\mathbf{P}^{(4)}=\mathbf{0}$. This behavior, together with the corresponding symmetry indicators $\chi^{(4)}$, is depicted in Fig. 4(c). A third primitive generator for $C_{2}$ is obtained by letting the hoppings in the $x-$ and $y$-dirrection be different. In that case, the $C_{4}$-symmetry
in Fig. 4(a) gets broken into a $C_{2}$ symmetry given by

$$
\begin{equation*}
\mathcal{I} h_{1}^{(4)}\left(k_{x}, k_{y}\right) \mathcal{I}^{-1}=h_{1}^{(4)}\left(-k_{x},-k_{y}\right) \tag{37}
\end{equation*}
$$

where $\mathcal{I} \equiv \hat{C}_{4}^{2}$.
b. Geometric breathing square-octagon. The second generator for $C_{4}$ is given by the Bloch-Hamiltonian

$$
h_{2}^{(4)}(\mathbf{k})=\left(\begin{array}{cccc}
0 & t_{1} & t_{2}+e^{i k_{x}} & t_{1}  \tag{38}\\
t_{1} & 0 & t_{1} & t_{2}+e^{i k_{y}} \\
t_{2}+e^{-i k_{x}} & t_{1} & 0 & t_{1} \\
t_{1} & t_{2}+e^{-i k_{y}} & t_{1} & 0
\end{array}\right)
$$

The underlying system is obtained by inserting the unit cell of Fig. 4 (d) into a lattice to form the structure shown in Fig. $4(\mathrm{k})$. The internal structure of the unit cell is rotated with regard to $h_{1}^{(4)}(\mathbf{k})$. If the next-nearestneighbour hopping is set to zero, this lattice represents (breathing) T-Graphene; otherwise, we call it the squareoctagon lattice. The lattice has attracted much attention recently, with results ranging from topological phases to flat-band superconductivity 3032 . The spectrum is gapped for $\left|t_{1}\right|+\left|t_{2}\right|<1$ at half filling, accompanied with rotation invariants $\chi^{(4)}=(1,1,-1)$ and $\mathbf{P}^{(4)}=$ $e\left(\mathbf{a}_{\mathbf{1}}+\mathbf{a}_{\mathbf{2}}\right) / 2$. An example of the spectrum for $t_{1}=0$ and $t_{2}=1 / 2$ is shown in Fig. 4 (e). When $\left|t_{1}\right|+\left|t_{2}\right|=1$, the gap generally closes at the $\mathbf{M}$ point. However, when $t_{1}=0$ (and therefore $\left|t_{2}\right|=1$ ), it closes at both the $\mathbf{M}$ and $\boldsymbol{\Gamma}$ points. Moreover, when $t_{2}=0\left(\left|t_{1}\right|=1\right)$, the gap closes over the full $\mathbf{X M}$ and $\mathbf{Y M}$ lines in the Brillouin zone. For $\left|t_{2}\right|>1+\left|t_{1}\right|$, a new gap opens at the $\mathbf{M}$ point (if $\left|t_{1}\right|=0$, this gap opens along the whole $\mathbf{X M}$ and YM lines). This gap is trivial and characterized by $\chi^{(4)}=(0,0,0), \mathbf{P}^{(4)}=\mathbf{0}$. For other parameter choices, the system is gapless, as shown in Fig. 4(f).
c. Geometric stacked breathing square-octagon. Finally, the third $C_{4}$-symmetric generator is governed by the Hamiltonian

$$
\begin{align*}
h_{3}^{(4)}(\mathbf{k})= & \left(\begin{array}{cccc}
0 & t_{1} & t_{2} & t_{1} \\
t_{1} & 0 & t_{1} & t_{2} \\
t_{2} & t_{1} & 0 & t_{1} \\
t_{1} & t_{2} & t_{1} & 0
\end{array}\right)  \tag{39}\\
& +\left(\begin{array}{cccc}
0 & 0 & e^{i\left(k_{x}+k_{y}\right)} & 0 \\
0 & 0 & 0 & e^{-i\left(k_{x}-k_{y}\right)} \\
e^{-i\left(k_{x}+k_{y}\right)} & 0 & 0 & 0 \\
0 & e^{i\left(k_{x}-k_{y}\right)} & 0 & 0
\end{array}\right) .
\end{align*}
$$

The underlying system is obtained by inserting the unit cell of Fig. $4(\mathrm{~g})$ into a lattice to form the structure in Fig. 4(1). The system has the same internal structure as $h_{1}^{(4)}(\mathbf{k})$, as can be seen in Fig. 4 (a), but has different intercell hopping. The lattice is formed by overlapping two breathing square-octagon lattices. The spectrum of $h_{3}^{(4)}(\mathbf{k})$ [ Fig. $\left.4(\mathrm{~h})\right]$ is gapped at half filling for $\left|t_{1}\right|+\left|t_{2}\right|<$ 1 , characterized by the rotation invariants $\chi^{(4)}=(2,0,0)$. This corresponds to $\mathbf{P}^{(4)}=\mathbf{0}$. For $\left|t_{1}\right|+\left|t_{2}\right|=1$, the gap closes at the $\mathbf{X}$ and $\mathbf{Y}$ points (if $t_{1}=0$, it also closes at the $\boldsymbol{\Gamma}$ and $\mathbf{M}$ points). For $\left|t_{2}\right|>1+\left|t_{1}\right|$, a trivial gap opens at the $\boldsymbol{\Gamma}$ and $\mathbf{M}$ points with $\chi^{(4)}=(0,0,0)$, $\mathbf{P}^{(4)}=\mathbf{0}$. For other values of $t_{1}$ and $t_{2}$, the system is gapless, as shown in Fig. 4(i).
d. Latent $C_{4}$-symmetric structures. The latent $C_{4}{ }^{-}$ symmetric cell in Fig. 2(c) can be inserted in the three different lattice structures of Fig. $4(\mathrm{j})$, Fig. $4(\mathrm{k})$, or Fig. 4(l). This yields systems with Bloch-Hamiltonians of the form

$$
h_{L, i}^{(4)}(\mathbf{k})=H_{L}^{(4)}+\left(\begin{array}{cc}
\tilde{h}_{i}^{(4)}(\mathbf{k}) & \emptyset_{4 \times 9}  \tag{40}\\
\emptyset_{9 \times 4} & \emptyset_{9 \times 9}
\end{array}\right)
$$

Here $\tilde{h}_{i}^{(4)}(\mathbf{k})$ are the Hamiltonians given in Eq. 35, Eq. (38), and Eq. (39) with all intracell hoppings set to zero, i.e. only intercell hopping. This is because the second term is only there to connect the latently symmetric cells on a lattice.

For all $h_{L, i}^{(4)}(\mathbf{k})$, we set $t_{1}=t_{2}=t_{0}$. Consequently, the Hamiltonians have a geometric $C_{4}$-symmetry for $t_{3}=0$. For finite values of $t_{3}$, the geometric $C_{4}$ symmetry of $h_{L, i}^{(4)}(\mathbf{k})$ gets broken. Nevertheless, the latent symmetry of the unit cell is inherited, which becomes clear upon taking the ISR to the red sites

$$
\begin{equation*}
\mathfrak{h}_{L, i}^{(4)}(\mathbf{k})=h_{i}^{(4)}\left(\mathbf{k}, t_{1}=v_{0}^{(4)}, t_{2}=v_{1}^{(4)}\right)+a^{(4)} \mathbb{I} \tag{41}
\end{equation*}
$$

which commutes with $\hat{C}_{4}$ given in Eq. 30). The parameters $a^{(4)}, v_{0}^{(4)}$ and $v_{1}^{(4)}$ are polynomials of a degree larger than 5 [See Appendix B 1]. Consequently, it is not possible to analytically resolve the gap closing conditions as outlined at the end of Sec. II. Figure 4(n) shows the spectrum of $h_{L, 1}^{(4)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=1$ and $t_{3}=1 / 2$. The spectrum is gapped at multiple fillings. For simplicity we consider a single filled band. Figure 4 (o) shows the numerically obtained phase diagram of $h_{L, 1}^{(4)}(\mathbf{k})$ for a single filled band for $t_{1}=t_{2}=t_{0}$. The rotation invariants $\chi^{(4)}$ are depicted for the gapped phases. Similarly, the spectrum of $h_{L, 2}^{(4)}(\mathbf{k})$ for $t_{0}=$ $t_{1}=t_{2}=-1$ and $t_{3}=0$ is shown in Fig. 4(q). The corresponding phase diagram for 2 filled bands is shown in Fig. 4(r). Finally, Fig. 4(t) shows the spectrum of $h_{L, 3}^{(4)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=1 / 2$ and $t_{3}=1$. The phase diagram shown in Fig. 4(u) is (numerically) obtained for 3 filled bands. The rotation invariants $\chi^{(4)}$ are depicted for the gapped phases. The topological and trivial phases are represented by the red and blue regions, respectively. The white regions denote metallic gapless states.

## 3. $C_{3}$-symmetry

For $C_{3}$-symmetry, we consider two lattice strucures, namely a "breathing kagome" and a "bearded breathing kagome".
a. Geometric $C_{3}$-symmetric structures. The internal structure of our geometrically $C_{3}$-symmetric unit cells are shown in Figs. 5(a) and (c). They consist of 3 sites connected with an intracell hopping $t_{0}$ and an intercell hopping fixed to 1 . The two lattice structures for


FIG. 5. (a,c) Unit cells of the geometric $C_{3}$-symmetric primitive generators $h_{1}^{(3)}(\mathbf{k})$ and $h_{2}^{(3)}(\mathbf{k})$, respectively. Intracell hopping (black line) has a strength of $t_{0}$, while intercell hopping (dashed lines) is fixed to 1. (b) Spectrum of $h_{1}^{(3)}(\mathbf{k})$ and $h_{2}^{(3)}(\mathbf{k})$ at $t_{0}=1 / 2$. (d,e) Phase diagrams of $h_{1}^{(3)}(\mathbf{k})$ and $h_{2}^{(3)}(\mathbf{k})$ at $2 / 3$-filling. Rotation invariants $\chi^{(3)}$ are indicated for the gapped phases. $(\mathrm{f}, \mathrm{g})$ Lattice structures of the $C_{3}$--symmetric primitive generators. The lattices represent a breathing kagome and a bearded breathing kagome lattice, respectively. (h, j$)$ Unit cells of the latent $C_{3}$-symmetric primitive generators $h_{L, 1}^{(3)}(\mathbf{k})$ and $h_{L, 2}^{(3)}(\mathbf{k})$, respectively. Values of the hopping parameters are indicated in Fig. 2(b) and intercell hoppings (dashed lines) are fixed to 1. (i) Spectrum of $h_{L, 1}^{(3)}(\mathbf{k})$ and $h_{L, 2}^{(3)}(\mathbf{k})$ for $t_{0}=0.6$. (k,l) Phase diagrams at filling $n=2$ and $n=8$ for $h_{L, 1}^{(3)}$ and $h_{L, 2}^{(3)}$, respectively. The rotation invariants $\chi^{(3)}$ are indicated in the different gapped phases.
$C_{3}$ symmetry are presented in Figs. 5(f)-(g). Since the resulting systems correspond to different terminations of the same Kagome lattice, we will treat them together. The Hamiltonian corresponding to the unit cell shown in Fig. 5(a) inserted into a lattice to form the structure
in Fig. 5 (f) is given by

$$
h_{1}^{(3)}(\mathbf{k})=\left(\begin{array}{ccc}
0 & t_{0}+e^{i \mathbf{k} \cdot \mathbf{a}_{1}} & t_{0}+e^{i \mathbf{k} \cdot \mathbf{a}_{\mathbf{2}}}  \tag{42}\\
t_{0}+e^{-i \mathbf{k} \cdot \mathbf{a}_{1}} & 0 & t_{0}+e^{-i \mathbf{k} \cdot \mathbf{a}_{3}} \\
t_{0}+e^{-i \mathbf{k} \cdot \mathbf{a}_{2}} & t_{0}+e^{i \mathbf{k} \cdot \mathbf{a}_{3}} & 0
\end{array}\right)
$$

while the Bloch-Hamiltonian corresponding to the unit cell in Fig. 5(c), which forms the lattice structure in Fig. 5 (g), is given by

$$
h_{2}^{(3)}(\mathbf{k})=\left(\begin{array}{ccc}
0 & t_{0}+e^{i \mathbf{k} \cdot \mathbf{a}_{\mathbf{2}}} & t_{0}+e^{-i \mathbf{k} \cdot \mathbf{a}_{\mathbf{3}}}  \tag{43}\\
t_{0}+e^{-i \mathbf{k} \cdot \mathbf{a}_{\mathbf{2}}} & 0 & t_{0}+e^{-i \mathbf{k} \cdot \mathbf{a}_{1}} \\
t_{0}+e^{i \mathbf{k} \cdot \mathbf{a}_{3}} & t_{0}+e^{i \mathbf{k} \cdot \mathbf{a}_{1}} & 0
\end{array}\right) .
$$

The lattice vectors are defined through $\mathbf{a}_{\mathbf{1}}=(1,0)$, $\mathbf{a}_{\mathbf{2}}=(1 / 2, \sqrt{3} / 2)$, and $\mathbf{a}_{\mathbf{3}}=\mathbf{a}_{\mathbf{1}}-\mathbf{a}_{\mathbf{2}} . h_{1}^{(3)}(\mathbf{k})$ and $h_{2}^{(3)}(\mathbf{k})$ represent a (bearded) breathing Kagome lattice 33 35] which exhibits a $C_{3}$ symmetry of the form

$$
\begin{equation*}
\hat{C}_{3} h_{i}^{(3)}\left(k_{x}, k_{y}\right) \hat{C}_{3}^{-1}=h_{i}^{(3)}\left(D_{C_{3}} \mathbf{k}\right), \tag{44}
\end{equation*}
$$

with $D_{C_{3}} \mathbf{k}=\left(-k_{x}-\sqrt{3} k_{y},-k_{y}+\sqrt{3} k_{x}\right) / 2$ and

$$
\hat{C}_{3}=\left(\begin{array}{lll}
0 & 0 & 1  \tag{45}\\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right)
$$

The spectrum for both Hamiltonians is shown in Fig. 5(b) for $t_{0}=1 / 2$. For $\left|t_{0}\right|<1$, the spectrum of $h_{1}^{(3)}(\mathbf{k})$ $\left[h_{2}^{(3)}(\mathbf{k})\right]$ is gapped at $2 / 3$ filling and is described by a topological invariant $\chi^{(3)}=(1,0)\left[\chi^{(3)}=(1,-1)\right]$ with $\mathbf{P}^{(3)}=(2 e / 3)\left(\mathbf{a}_{\mathbf{1}}+\mathbf{a}_{\mathbf{2}}\right)\left[\mathbf{P}^{(3)}=(e / 3)\left(\mathbf{a}_{\mathbf{1}}+\mathbf{a}_{\mathbf{2}}\right)\right]$. At $t_{0}=1$, the gap closes at $\mathbf{K}$ and $\mathbf{K}^{\prime}$ and opens again in a trivial phase with $\chi^{(3)}=(0,0)$ and $\mathbf{P}^{(3)}=\mathbf{0}$ for $t_{0}>1$. At $t_{0}=-1$, the gap closes at $\boldsymbol{\Gamma}$ and the spectrum remains gapless for $t_{0}<-1$. The phase diagrams of $h_{1}^{(3)}(\mathbf{k})$ and $h_{2}^{(3)}(\mathbf{k})$ are shown in Figs. 5 (d) and (e), respectively.
b. Latent $C_{3}$-symmetric structures. Inserting the latently $C_{3}$-symmetric unit cell Fig. 2(b) into a lattice to form the structures in Figs. 5(f) and (g) yields two latent $C_{3}$-symmetric primitive generators with BlochHamiltonians

$$
h_{L, i}^{(3)}(\mathbf{k})=H_{L}^{(3)}+\left(\begin{array}{cc}
\tilde{h}_{i}^{(3)}(\mathbf{k}) & \emptyset_{3 \times 6}  \tag{46}\\
\emptyset_{6 \times 3} & \emptyset_{6 \times 6}
\end{array}\right),
$$

with $i=1,2 . \quad \tilde{h}_{i}^{(3)}(\mathbf{k})$ are the Hamiltonians given in Eq. (42) and Eq. (43) with all intracell hoppings set to zero, i.e. only intercell hopping (once again, just to connect the larger cells on a lattice). Figures $5(\mathrm{f})$ and (g) show lattices corresponding to $h_{L, 1}^{(3)}$ and $h_{L, 2}^{(3)}$, respectively. Analogous to the non-latent primitive generators for $C_{3}$ symmetry, $h_{L, 1}^{(3)}$ and $h_{L, 2}^{(3)}$ share the same spectrum, as depicted in Fig. 5(i).

The ISR of $h_{L, i}^{(3)}(\mathbf{k})$ to the red sites is given by

$$
\begin{equation*}
\mathfrak{h}_{L, i}^{(3)}(\mathbf{k})=h_{i}^{(3)}\left(\mathbf{k}, t_{0}=v_{0}^{(3)}\right)+a^{(3)} \mathbb{I}, \tag{47}
\end{equation*}
$$

Owing to the simplicity of the parameters $v_{0}^{(3)}\left(=5 t_{0}^{2} / E\right)$ and $a^{(3)}\left(=7 t_{0}^{2} / E\right)$, it is possible to analytically derive the phase diagram of this model. In Sec. IV we showed that $h_{i}^{(3)}(\mathbf{k})$ has a gap closing at $E=1$ for $t_{0}=1$, and at $E=0$ for $t_{0}=-1$. Therefore, we obtain the energies $E_{1}^{*}$ and $E_{2}^{*}$ at which the latent models have a gap closing

$$
\begin{equation*}
E_{1}^{*}-a^{(3)}\left(E_{1}^{*}\right)=1 \quad \text { and } \quad E_{2}^{*}-a^{(3)}\left(E_{2}^{*}\right)=0 \tag{48}
\end{equation*}
$$

Solving the above equations yields

$$
\begin{equation*}
E_{1}^{*}=\frac{1}{2}\left(1 \pm \sqrt{1+28 t_{0}^{2}}\right) \quad \text { and } \quad E_{2}^{*}= \pm \sqrt{7} t_{0} \tag{49}
\end{equation*}
$$

From Eq. 49), we extract the phase transitions through $v_{0}^{(3)}\left(E_{1}^{*}\right)=1$ and $v_{0}^{(3)}\left(E_{2}^{*}\right)=-1$, resulting in

$$
\begin{equation*}
t_{0}= \pm \frac{2 \sqrt{3}}{5}, \pm \frac{\sqrt{7}}{5} \tag{50}
\end{equation*}
$$

The gap closing energies correspond to a filling of $n=2$ and $n=8$ bands out of 9 . Using the above derived constraints, we obtain the phase diagrams depicted in Figs. $5(\mathrm{k})$ and (l) for $h_{L, 1}^{(3)}(\mathbf{k})$ and $h_{L, 2}^{(3)}(\mathbf{k})$, respectively. The rotation invariants $\chi^{(3)}$ are shown for the different phases. The invariants are the same as those obtained for $h_{i}^{(3)}(\mathbf{k})$.

## 4. $C_{6}$-symmetry

For $C_{6}$-symmetry, we consider two lattice structures, namely, the "breathing ruby lattice", and the "Kekule" structure.
a. Geometric breathing ruby lattice The Hamiltonian for the first $C_{6}$-symmetric primitive generator is given by

$$
\begin{aligned}
h_{1}^{(6)}(\mathbf{k}) & =\left(\begin{array}{cccccc}
0 & t_{0} & t_{1} & t_{2} & t_{1} & t_{0} \\
t_{0} & 0 & t_{0} & t_{1} & t_{2} & t_{1} \\
t_{1} & t_{0} & 0 & t_{0} & t_{1} & t_{2} \\
t_{2} & t_{1} & t_{0} & 0 & t_{0} & t_{1} \\
t_{1} & t_{2} & t_{1} & t_{0} & 0 & t_{0} \\
t_{0} & t_{1} & t_{2} & t_{1} & t_{0} & 0
\end{array}\right) \\
& +\left(\begin{array}{cccccc}
0 & 0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{2}} & 0 & e^{i \mathbf{k} \cdot \mathbf{a}_{3}} & 0 \\
0 & 0 & 0 & e^{i \mathbf{k} \cdot \mathbf{a}_{3}} & 0 & e^{i \mathbf{k} \cdot \mathbf{a}_{1}} \\
e^{i \mathbf{k} \cdot \mathbf{a}_{\mathbf{2}}} & 0 & 0 & 0 & e^{i \mathbf{k} \cdot \mathbf{a}_{1}} & 0 \\
0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{3}} & 0 & 0 & 0 & e^{i \mathbf{k} \cdot \mathbf{a}_{2}} \\
e^{-i \mathbf{k} \cdot \mathbf{a}_{3}} & 0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{1}} & 0 & 0 & 0 \\
0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{1}} & 0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{2}} & 0 & 0
\end{array}\right) .
\end{aligned}
$$

It is obtained by inserting the unit cell from Fig. 6(a) into a lattice to obtain the structure in Fig. 6(g). This lattice is a breathing version of a Ruby lattice. The lattice vectors are given by $\mathbf{a}_{\mathbf{1}}=(1,0), \mathbf{a}_{\mathbf{2}}=(1 / 2, \sqrt{3} / 2)$ and $\mathbf{a}_{\mathbf{3}}=\mathbf{a}_{\mathbf{1}}-\mathbf{a}_{\mathbf{2}} . h_{1}^{(6)}(\mathbf{k})$ exhibits a $C_{6}$ symmetry given by

$$
\begin{equation*}
\hat{C}_{6} h_{1}^{(6)}\left(k_{x}, k_{y}\right) \hat{C}_{6}^{-1}=h_{1}^{(6)}\left(D_{C_{6}} \mathbf{k}\right) \tag{52}
\end{equation*}
$$

with $D_{C_{6}} \mathbf{k}=\left(k_{x}-\sqrt{3} k_{y},+\sqrt{3} k_{x}+k y\right) / 2$ and

$$
\hat{C}_{6}=\left(\begin{array}{llllll}
0 & 0 & 0 & 0 & 0 & 1  \tag{53}\\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

First, consider the case $t_{1}=t_{2}=0$. The spectrum is gapped at $2 / 3$-filling if $\left|t_{0}\right|<1$. The gap is characterised by $\chi^{(6)}=(0,2)$, which corresponds to $\mathbf{P}^{(\mathbf{6})}=\mathbf{0}$ and $Q^{(6)}=e / 3$. This can be seen in the spectrum in Fig. 6(b), which is calculated for $t_{0}=1 / 2$. At $t_{0}=1$, the gap closes at the $\boldsymbol{\Gamma}$ point and upon further increase of $t_{0}$, the spectrum remains gapless. If we only impose $t_{2}=0$, the phase diagram in Fig. 6(c) is obtained. For values of $t_{0}$ and $t_{1}$ in the red region, the system is in a topological phase with $\chi^{(6)}=(0,2)$. Leaving the topological phase (red) by increasing $t_{1}$, the gap closes at the $\mathbf{K}$ and $\mathbf{K}^{\prime}$ points and the spectrum is gapless. Upon further increasing $t_{1}$, a trivial gap $\left[\chi^{(6)}=(0,0)\right]$ reopens at the $\mathbf{K}$ and $\mathbf{K}^{\prime}$ points.
b. Geometric Kekulé. The Hamiltonian for the second $C_{6}$-symmetric primitive generator is given by

$$
\begin{align*}
h_{2}^{(6)}(\mathbf{k}) & =\left(\begin{array}{cccccc}
0 & t_{0} & t_{1} & t_{2} & t_{1} & t_{0} \\
t_{0} & 0 & t_{0} & t_{1} & t_{2} & t_{1} \\
t_{1} & t_{0} & 0 & t_{0} & t_{1} & t_{2} \\
t_{2} & t_{1} & t_{0} & 0 & t_{0} & t_{1} \\
t_{1} & t_{2} & t_{1} & t_{0} & 0 & t_{0} \\
t_{0} & t_{1} & t_{2} & t_{1} & t_{0} & 0
\end{array}\right)  \tag{54}\\
& +\left(\begin{array}{cccccc}
0 & 0 & 0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & e^{i \mathbf{k} \cdot \mathbf{a}_{3}} & 0 \\
0 & 0 & 0 & 0 & 0 & e^{i \mathbf{k} \cdot \mathbf{a}_{1}} \\
e^{i \mathbf{k} \cdot \mathbf{a}_{\mathbf{2}}} & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{3}} & 0 & 0 & 0 & 0 \\
0 & 0 & e^{-i \mathbf{k} \cdot \mathbf{a}_{1}} & 0 & 0 & 0
\end{array}\right) .
\end{align*}
$$

The corresponding system is obtained by using the unit cell of Fig. 6(d) to form the lattice structure in Fig. 6(h). Note that it has the same unit cell as in the Breathing Ruby Lattice case, but rotated and with different intercell hopping. The lattice vectors $\mathbf{a}_{1}, \mathbf{a}_{2}$, and $\mathbf{a}_{3}$ are the same as in the breathing ruby lattice case. We note that the different intercell hopping structure forms a breathing honeycomb or Kekulé lattice [36]. Just as Eq. (51], the primitive generator Eq. (54) has a $C_{6}$ symmetry given by Eq. (52). Again, first consider $t_{1}=t_{2}=0$. The spectrum is gapped at $1 / 2$-filling for $-1 / 2<t_{0}<1$, as shown in Fig. 6(e), where $t_{0}=1 / 2$. This gap is characterised by $\chi^{(6)}=(2,0)$, which corresponds to $\mathbf{P}^{(6)}=\mathbf{0}$ and $Q^{(6)}=$ $e / 2$. For fixed $t_{2}=0$, the phase diagram in Fig. 6(f) is obtained. The system is in its topological phase $\chi^{(6)}=$ $(2,0)$ ] for $-1 / 2<t_{0}<1$ and $\left|t_{1}\right|<\left(t_{0}+2\right) / 3$. At $t_{0}=1$ or $\left|t_{1}\right|=\left(t_{0}+2\right) / 3$, the gap closes at the $\boldsymbol{\Gamma}$ point. For $t_{0}>1$ and $\left|t_{1}\right|<\left|t_{0}\right|$, the gap reopens in a trivial phase with $\chi^{(6)}=(0,0)$.

Latent $C_{6}$-symmetric structures. The latent primitive


FIG. 6. (a,d) Unit cells of the geometric $C_{6}$-symmetric primitive generators $h_{1}^{(6)}(\mathbf{k})$ and $h_{2}^{(6)}(\mathbf{k})$. Three $C_{6}$ preserving hoppings $t_{0}$ (black), $t_{1}$ (blue) and $t_{2}$ (red) are indicated. Intercell hopping (dashed) is fixed to 1. (b,e) Spectra of $h_{1}^{(6)}(\mathbf{k})$ and $h_{2}^{(6)}(\mathbf{k})$, for $t_{0}=1 / 2, t_{1}=t_{2}=0$. (c,f) Phase diagrams of $h_{1}^{(6)}(\mathbf{k})$, and $h_{2}^{(6)}(\mathbf{k})$ in the $t_{0}-t_{1}$ plane with $t_{2}=0$. Rotation invariants $\chi^{(6)}$ are indicated for gapped phases. (g,h) Lattice structures of the $C_{6}-$ symmetric primitive generators. The lattices represent a breathing ruby lattice and a kekulé lattice, respectively. (i,l) Unit cells of the latent $C_{6}$-symmetric primitive generators $h_{L, 1}^{(6)}(\mathbf{k})$ and $h_{L, 2}^{(6)}(\mathbf{k})$. Values of the hopping parameters are indicated in Fig. 2 (d) and intercell hoppings (dashed lines) are fixed to 1. $(\mathrm{j}, \mathrm{m})$ Spectra of $h_{L, 1}^{(6)}(\mathbf{k})$ and $h_{L, 2}^{(6)}(\mathbf{k})$, for $t_{0}=3 / 4, t_{1}=7 / 8, t_{2}=1 / 8$. (k,n) Phase diagrams of $h_{L, 1}^{(6)}(\mathbf{k})$ for $n=4$ filled bands and $h_{L, 2}^{(6)}(\mathbf{k})$ for $n=3$ filled bands, with $t_{1}=1 / 4$. Rotation invariants $\chi^{(6)}$ are indicated in the gapped phases.
generators for $C_{6}$-symmetry are given by

$$
h_{L, i}^{(6)}(\mathbf{k})=H_{L}^{(6)}+\left(\begin{array}{cc}
\tilde{h}_{i}^{(6)}(\mathbf{k}) & \emptyset_{6 \times 13}  \tag{55}\\
\emptyset_{13 \times 6} & \emptyset_{13 \times 13}
\end{array}\right)
$$

Here, $\tilde{h}_{i}^{(6)}(\mathbf{k})$ are the Hamiltonians given in Eq. 51, Eq. (54) with all intracell hoppings set to zero, i.e. only intercell hopping (to connect the large unit cells on a lattice). The two systems corresponding to $h_{L, 1}^{(6)}(\mathbf{k})$
and $h_{L, 2}^{(6)}(\mathbf{k})$ are obtained by inserting the latently $C_{6^{-}}$ symmetric unit cell shown in Fig. 2(d) into a lattice structure to form Fig. 6(g) and Fig. 6(h), respectively. The intercell hopping (dashed) is fixed to 1 . The ISR of $\mathfrak{h}_{L, i}^{(6)}(\mathbf{k})$ is given by

$$
\begin{align*}
\mathfrak{h}_{L, i}^{(6)}(\mathbf{k})=h_{i}^{(6)}\left(\mathbf{k}, t_{0}=v_{0}^{(6)}, t_{1}=v_{1}^{(6)}, t_{2}\right. & \left.=v_{2}^{(6)}\right) \\
& +a^{(6)} \mathbb{I} \tag{56}
\end{align*}
$$

which is symmetric under the action of $\hat{C}_{6}$, i.e. Eq. 52 . Since the parameters $a^{(6)}$ and $v_{i}^{(6)}$ are ratios of large order polynomials in $E$, it is not possible to analytically obtain the phase diagrams of $h_{L, i}^{(6)}(\mathbf{k})$. Figure 6(j) [Fig. 6(m)] show the spectrum of $h_{L, 1}^{(6)}(\mathbf{k})\left[h_{L, 2}^{(6)}(\mathbf{k})\right]$ for $t_{0}=t_{1}=$ $t_{2}=1 / 4$. A phase diagram of $h_{L, 1}^{(6)}(\mathbf{k})$ for 4 filled bands is shown in Fig. 6(k), revealing a topological phase characterised by $\chi^{(6)}=(0,2)$ separated by a gapless phase from the trivial phase $\chi^{(6)}=(0,0)$. Moreover, Fig. 6(n) shows a phase diagram of $h_{L, 2}^{(6)}(\mathbf{k})$ for 3 filled bands. Both phase diagrams reveal the presence of the same topological phases as in $h_{i}^{(6)}(\mathbf{k})$.

## V. CONSTRUCTION OF LATENT HOTIS

We are now finally equipped with all necessary tools to allow an efficient design of latent HOTIs. As mentioned above, any HOTI (geometric or latent) must fulfil the following two constraints:

- The corner charge $Q^{(n)}$, as given by Eq. 24), is nonzero.
- The dipole moment $\mathbf{P}^{(n)}$, as given by Eq. 23., vanishes.

In the following, we will construct latent HOTIs featuring a $C_{n}$ symmetry. Although there are many possibilities, for brevity we only show a single example of $C_{3}-, C_{4}-$, and $C_{6}$-symmetry each. $C_{2}$-symmetry is left out as constructing a lattice with only two 'corners' would correspond to a 1D chain, which is already treated in Ref. [20]. Alternatively, a square latent $C_{2}-$ symmetric lattice could be considered which only shows pairwise equal corner charges.
a. Latent $C_{3}$-symmetric HOTI. The latent $C_{3}-$ symmetric generators $h_{L, i}^{(3)}(\mathbf{k})$ have topological phases with $\chi^{(3)}=(0,2)$ and $\chi^{(3)}=(2,0)$, which corresponds to $\mathbf{P}^{(3)}=(e / 3)\left(\mathbf{a}_{\mathbf{1}}+\mathbf{a}_{\mathbf{2}}\right)$ and $\mathbf{P}^{(3)}=(2 e / 3)\left(\mathbf{a}_{\mathbf{1}}+\right.$ $\mathbf{a}_{2}$ ), respectively. For neither of these models the dipole moment vanishes. In Sec. IV C we argued that the generators may be stacked to obtain models with arbitrary rotational invariant, and, thus arbitrary $\mathbf{P}^{(n)}$ and $Q^{(n)}$. By 'stacking' $h_{L, 1}^{(3)}(\mathbf{k})$ and $h_{L, 2}^{(3)}(\mathbf{k})$, we obtain

$$
h_{L}^{(3)}(\mathbf{k})=\left(\begin{array}{cc}
h_{L, 1}^{(3)}(\mathbf{k}) & \mathcal{T}  \tag{57}\\
\mathcal{T}^{\dagger} & h_{L, 2}^{(3)}(\mathbf{k})
\end{array}\right)
$$

where $\mathcal{T}$ couples the two models without breaking $C_{3^{-}}$ symmetry and without closing the gap. Here, we choose

$$
\mathcal{T}=\left(\begin{array}{cc}
T & \emptyset_{3 \times 6}  \tag{58}\\
\emptyset_{6 \times 3} & \emptyset_{6 \times 6}
\end{array}\right), \quad T=\left(\begin{array}{lll}
g & 0 & g \\
g & g & 0 \\
0 & g & g
\end{array}\right)
$$

From Fig. 5. we observe that both $h_{L, i}^{(3)}(\mathbf{k})$ are topological for $t_{0}=0.2$ at 8 filled bands. Taking $g=0.1$ does not
close the gap, such that $h_{L}^{(3)}(\mathbf{k})$ is characterised by $\chi^{(3)}=$ $(0,2)+(2,0)=(2,2)$. This translates to $\mathbf{P}^{(3)}=\mathbf{0}$ and $Q^{(3)}=2 e / 3$. Figure 7(a) shows a triangular flake with OBC corresponding to $h_{L}^{(3)}(\mathbf{k})$, in which every hexagon represents a (stacked) unit cell. The colour of the cells represents $\rho\left(\mathbf{x}_{\text {cell }}\right)-\rho_{0}$ where

$$
\begin{equation*}
\rho\left(\mathbf{x}_{\mathrm{cell}}\right)=e \sum_{\mathbf{x} \in \mathbf{x}_{\mathrm{cell}}} \sum_{i}^{N_{f}}\left|\psi_{i}(\mathbf{x})\right|^{2} \tag{59}
\end{equation*}
$$

is the electronic charge density per unit cell and $N_{f}$ is the amount of filled states. $\psi_{i}(\mathbf{x})$ is the wavefunction of the $i^{\text {th }}$ energy eigenstate of the electron. $\rho_{0}$ is the (ionic) background charge density of the unit cells ( $\rho_{0}=$ $e \times \#$ filled bands $\times \#$ cells). In Fig. 7(a), we take a filling of 16 bands ( 8 filled bands of each model). There is a clear localisation of excess charge in the three corners of the flake. Adding up the excess charge within a single sector (indicated in green), reveals that the corner charge is quantized and equal to $Q_{\text {corner }}=2 e / 3$.
b. Latent $C_{4}$-symmetric HOTI. As an example of $C_{4}$-symmetric latent HOTI, we may take $h_{L, 2}^{(4)}(\mathbf{k})$, which in its topological phase $\left[\chi^{(4)}=(2,0,0)\right]$ at 3 filled bands has $\mathbf{P}^{(4)}=\mathbf{0}$ and $Q^{(4)}=e / 2$, i.e., it is not necessary to stack two models. Figure 7 (b) shows the charge density in a square flake with OBC , described by $h_{L, 2}^{(4)}(\mathbf{k})$ for $t_{0}=$ $t_{1}=t_{2}=1 / 2$ and $t_{3}=1$. Summing over a single sector (green) reveals a total corner charge of $Q_{\text {corner }}=e / 2$, predominantly localized at the corners.
c. Latent $C_{6}$-symmetric HOTI. Finally, from Eq. (23), it follows that the dipole moment always vanishes for any $C_{6}$-symmetric system. Consequently, both primitive generators $h_{L, i}^{(6)}(\mathbf{k})$ will represent a HOTI in their topological phase. Here, we consider $h_{L, 1}^{(6)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=1 / 4$ for 17 filled bands. For these parameters, the system is gapped and characterised by $\chi^{(6)}=(0,2)$, corresponding to $\mathbf{P}^{(6)}=\mathbf{0}$ and $Q^{(6)}=e / 3$. Figure 7(c) shows a hexagonal flake of this system, with OBC. It displays corner charges $Q_{\text {corner }}=e / 3$ in each of the six corners of the flake.

## VI. CONCLUSION

In this paper, we have shown that the existence of non-trivial fractional corner charges in two dimensional systems does not require the preservation of a $C_{n^{-}}$ symmetry. Instead, a latent symmetry is sufficient. This may be understood from the behaviour of the latent symmetry $\hat{C}_{n} \oplus \bar{Q}^{(n)}$ at the HSPs $\boldsymbol{\Pi}$, which is exactly the same as for a non-latent symmetry, i.e. $\left[h(\boldsymbol{\Pi}), \hat{C}_{n} \oplus \bar{Q}^{(n)}\right]_{-}=0$. Owing to this property, any two dimensional system with a latent symmetry of the form $\hat{C}_{n} \oplus \bar{Q}^{(n)}$ may host non-trivial corner modes and can be characterised in terms of rotation invariants. In


FIG. 7. Latent higher-order topological insulators. (a) Triangular OBC flake of $h_{L}^{(3)}(\mathbf{k})$ for $t_{0}=0.2$ and $g=0.1$ at filling $N_{f}=18 N_{\text {cells }}+2$. Each of the corners shows a quantized excess corner charge of $Q_{\text {corner }}=2 e / 3$. (b) Square OBC flake of $h_{L, 2}^{(4)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=1 / 2$ and $t_{3}=1$ at filling $N_{f}=3 N_{\text {cells }}+2$. Each of the corners shows a quantized excess corner charge of $Q_{\text {corner }}=e / 2$. (c) Hexagonal OBC flake of $h_{L, 1}^{(6)}(\mathbf{k})$ for $t_{0}=t_{1}=t_{2}=1 / 4$ at filling $N_{f}=17 N_{\text {cells }}+2$. Each of the corners shows a quantized excess corner charge of $Q_{\text {corner }}=e / 3$.

Section IV we gave examples of primitive generators that have this property. We showed that for some models, a full analytical treatment is possible, even though the model had up to 9 bands. This reveals another strength of our method: if, under an isospectral reduction, a Hamiltonian reduces to an energy-dependent version of a known model, then properties of the known model can be used to characterize the full Hamiltonian. A formal outline of this procedure was given at the end of Section II.

We have given examples of unit cells that show a latent $C_{n}$-symmetry for the four rotation symmetries that tile the two-dimensional space and have provided an outline for how to construct latent HOTIs based on these. Nevertheless, our work does not restrict itself to these unit cells. On the contrary, any cell that obeys a latent $C_{n}$-symmetry may be used to construct a latent HOTI. These unit cells were used merely to give three specific examples of a latent HOTI. We emphasize that these examples do not capture the full range of possible latent HOTIs. Any two primitive generators based on a unit cell - geometric- or latent symmetric - can be combined to form a new generator. One could obtain HOTIs that are composed of one geometric generator and one latent generator. The examples we have provided - namely, the 2D SSH model, a breathing square-octagon lattice, T-Graphene, a breathing kagome lattice, a breathing ruby lattice, and a Kekulé lattice [27-36] - have all been previously used to study a variety of phenomena, which is why we believe that our results might therefore be directly relevant to them.

In addition, we discussed the possibility of a unit cell showing a geometric $C_{3}\left(C_{2}\right)$ symmetry while its ISR showed a latent $C_{6}\left(C_{4}\right)$ symmetry, i.e. an effective symmetry doubling. Consequentially, the corner charges will follow the doubled symmetry. This behaviour can neatly be explained by performing an ISR.

Finally, TCIs protected by $C_{n}$-symmetry only form a subset of all crystalline topological phases. Similar methods to those outlined in our work can be applied to systems with different crystalline symmetries. This leads us to believe that our work opens doors for the latent generalisation of topological phases protected by other crystalline symmetries. Interesting cases for future investigations would be latent versions of topological systems lacking translational symmetry, but still displaying global space-group symmetries. Examples of these are possibly a generalization of a recent work on 1D topological quasicrystalline insulators [37], or a crystalline generalization of the topological states observed in finite fractals 38].

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## Appendix A: Derivation of the $\bar{Q}$ matrix

Reference 19 gives a derivation for the $Q$ (and therefore also $\bar{Q}$ ) matrix. Here, we will briefly outline how to construct $Q$ in the specific case of a latent $C_{n}$-symmetry. Let $H$ be a real symmetric matrix with a latent $C_{n^{-}}$ symmetry, i.e.

$$
\begin{equation*}
\left[\mathcal{R}_{S}(H, E), \hat{C}_{n}\right]_{-}=0 \tag{A1}
\end{equation*}
$$

Then the eigenstates $\{|\phi\rangle\}$ of $H$ can be chosen to satisfy

$$
\begin{equation*}
\hat{C}_{n}|\phi\rangle_{S}=e^{\frac{2 \pi i}{n}(p-1)}|\phi\rangle_{S}, \quad \text { with } \quad p \in\{1,2, \ldots, n\} \quad \text { OR } \quad|\phi\rangle_{S}=\mathbf{0} \tag{A2}
\end{equation*}
$$

where $\mathbf{0}$ is a zero vector in the $S$ subspace. We label these states by $\left|\phi^{(p)}\right\rangle$ and $\left|\phi^{(0)}\right\rangle$, respectively. One can then define the projectors

$$
\begin{equation*}
P_{p}=\sum_{i}\left|\phi_{i}^{(p)}\right\rangle\left\langle\phi_{i}^{(p)}\right|, \quad P_{0}=\sum_{i}\left|\phi_{i}^{(0)}\right\rangle\left\langle\phi_{i}^{(0)}\right| \tag{A3}
\end{equation*}
$$

From these projectors, $Q$ is defined through

$$
\begin{equation*}
Q=\hat{C}_{n} \oplus \bar{Q}=P_{0}+\sum_{p=1}^{n} e^{\frac{2 \pi i}{n}(p-1)} P_{p} \tag{A4}
\end{equation*}
$$

## Appendix B: Expressions

1. $C_{4}$

The Hamiltonian for the unit cell in Fig. 2(c) of the main text is given by

$$
H_{L}^{(4)}=\left(\begin{array}{ccccccccccccc}
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0  \tag{B1}\\
1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & t_{1}+t_{3} \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & t_{2} \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & t_{1} \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & t_{2} \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & t_{3} \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t_{1}+t_{3} & t_{2} & t_{1} & t_{2} & 0 & 0 & t_{3} & 0 & 0
\end{array}\right) .
$$

The symmetry on the level of the full Hamiltonian $H_{L}^{(4)}$ is given by

$$
Q^{(4)}=\hat{C}_{4} \oplus \bar{Q}^{(4)}=\left(\begin{array}{ccccc}
0 & 0 & 0 & 1  \tag{B2}\\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right) \oplus\left(\begin{array}{ccccccccc}
\frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\
\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & 0 \\
-\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\
\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & \frac{1}{4} & 0 \\
\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & 0 \\
-\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{3}{4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)
$$

The reduced parameters that enter Fig. 2 (g) and Eq. 41) are given by

$$
\begin{equation*}
a^{(4)}=\frac{t_{0}^{2}\left(4 E\left(E^{2}-t_{0}^{2}\right)+t_{1}^{2}\left(t_{0}-7 E\right)+2\left(t_{2}-3 t_{3}\right) t_{1}\left(E+t_{0}\right)+t_{2}^{2}\left(t_{0}-7 E\right)+t_{3}^{2}\left(t_{0}-7 E\right)+2 t_{2} t_{3}\left(E+t_{0}\right)\right)}{\left(t_{0}-E\right)\left(E\left(t_{0}-E\right)\left(E+t_{0}\right)+2\left(E t_{1}^{2}+t_{3} t_{1}\left(E+t_{0}\right)+E\left(t_{2}^{2}+t_{3}^{2}\right)\right)\right)} \tag{B3}
\end{equation*}
$$

$$
\begin{equation*}
v_{0}^{(4)}=-\frac{t_{0}\left(E+t_{0}\right)\left(t_{1}^{2}\left(-\left(t_{0}-2 E\right)\right)+t_{1}\left(2 E t_{3}-2 t_{0} t_{2}\right)-t_{2}^{2}\left(t_{0}-2 E\right)-t_{3}^{2}\left(t_{0}-2 E\right)+E\left(t_{0}-E\right)\left(E+t_{0}\right)-2 t_{0} t_{2} t_{3}\right)}{\left(t_{0}-E\right)\left(E\left(t_{0}-E\right)\left(E+t_{0}\right)+2\left(E t_{1}^{2}+t_{3} t_{1}\left(E+t_{0}\right)+E\left(t_{2}^{2}+t_{3}^{2}\right)\right)\right)} \tag{B4}
\end{equation*}
$$

$$
\begin{equation*}
v_{1}^{(4)}=\frac{t_{0}^{2}\left(t_{1}+t_{2}+t_{3}\right)^{2}\left(E+t_{0}\right)}{\left(t_{0}-E\right)\left(E\left(t_{0}-E\right)\left(E+t_{0}\right)+2\left(E t_{1}^{2}+t_{3} t_{1}\left(E+t_{0}\right)+E\left(t_{2}^{2}+t_{3}^{2}\right)\right)\right)} \tag{B5}
\end{equation*}
$$

## 2. $C_{6}$

The Hamiltonian for the unit cell in Fig. 22(d) of the main text is a simplified version of the one denoted in Fig. 8 . To obtain the former, we set $t_{2}=t_{3}=t_{4}=0$ and let $t_{5} \rightarrow t_{2}$. The full Hamiltonian is given by


FIG. 8. A more complex latent $C_{6}-$ symmetric unit cell. The hopping parameters are indicated by different colors.

$$
H_{L}^{(6)}=\left(\begin{array}{ccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & t_{0} & 0 & 0  \tag{B6}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & t_{0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & t_{0} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & t_{0} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & t_{0} & 0 & 0 & 0 \\
0 & 0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & t_{3}+t_{4}+t_{5} \\
0 & 0 & 0 & t_{0} & 0 & 0 & t_{0} & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{3} \\
0 & 0 & 0 & 0 & t_{0} & 0 & 0 & t_{0} & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & t_{0} & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{4} \\
t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & t_{0} & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{5} \\
0 & 0 & t_{0} & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & t_{0} & t_{1} \\
0 & 0 & 0 & t_{0} & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & t_{0} & 0 & 0 & 0 & t_{2}+t_{4}+t_{5} \\
0 & 0 & 0 & 0 & t_{0} & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & t_{0} & 0 & 0 & t_{1}+t_{3} \\
t_{0} & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & t_{0} & 0 & t_{2}+t_{5} \\
t_{0} & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & t_{0} & t_{1}+t_{3}+t_{4} \\
0 & t_{0} & t_{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{0} & 0 & 0 & 0 & t_{0} & 0 & t_{2} \\
0 & 0 & 0 & 0 & 0 & 0 & t_{3}+t_{4}+t_{5} & t_{3} & 0 & t_{4} & 0 & t_{5} & t_{1} & t_{2}+t_{4}+t_{5} & t_{1}+t_{3} & t_{2}+t_{5} & t_{1}+t_{3}+t_{4} & t_{2} & 0
\end{array}\right) .
$$

The symmetry on the level of the full Hamiltonian $H_{L}^{(6)}$ is given by

$$
Q^{(6)}=\hat{C}_{6} \oplus \bar{Q}^{(6)}=\left(\begin{array}{lllllll}
0 & 0 & 0 & 0 & 0 & 1  \tag{B7}\\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{array}\right) \oplus\left(\begin{array}{ccccccccccccc}
\frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} & -\frac{3}{8} & \frac{1}{8} & 0 & -\frac{1}{8} & \frac{3}{8} & 0 & 0 \\
\frac{5}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & 0 & -\frac{3}{8} & \frac{1}{8} & 0 & -\frac{1}{8} & \frac{3}{8} & 0 \\
-\frac{1}{4} & \frac{5}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & \frac{3}{8} & 0 & -\frac{3}{8} & \frac{1}{8} & 0 & -\frac{1}{8} & 0 \\
\frac{1}{8} & -\frac{1}{4} & \frac{5}{8} & \frac{3}{8} & \frac{1}{4} & -\frac{1}{8} & -\frac{1}{8} & \frac{3}{8} & 0 & -\frac{3}{8} & \frac{1}{8} & 0 & 0 \\
-\frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} & \frac{3}{8} & \frac{1}{4} & 0 & -\frac{1}{8} & \frac{3}{8} & 0 & -\frac{3}{8} & \frac{1}{8} & 0 \\
\frac{1}{4} & -\frac{1}{8} & \frac{1}{8} & -\frac{1}{4} & \frac{5}{8} & \frac{3}{8} & \frac{1}{8} & 0 & -\frac{1}{8} & \frac{3}{8} & 0 & -\frac{3}{8} & 0 \\
0 & -\frac{3}{8} & \frac{1}{8} & 0 & -\frac{1}{8} & \frac{3}{8} & \frac{5}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & 0 \\
\frac{3}{8} & 0 & -\frac{3}{8} & \frac{1}{8} & 0 & -\frac{1}{8} & \frac{3}{8} & \frac{5}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & 0 \\
-\frac{1}{8} & \frac{3}{8} & 0 & -\frac{3}{8} & \frac{1}{8} & 0 & \frac{1}{4} & \frac{3}{8} & \frac{5}{8} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{8} & 0 \\
0 & -\frac{1}{8} & \frac{3}{8} & 0 & -\frac{3}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{5}{8} & -\frac{1}{4} & \frac{1}{8} & 0 \\
\frac{1}{8} & 0 & -\frac{1}{8} & \frac{3}{8} & 0 & -\frac{3}{8} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{5}{8} & -\frac{1}{4} & 0 \\
-\frac{3}{8} & \frac{1}{8} & 0 & -\frac{1}{8} & \frac{3}{8} & 0 & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{5}{8} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right) .
$$

The reduced parameters that enter Fig. $2(\mathrm{~h})$ and Eq. (56) are of the form

$$
\begin{equation*}
a^{(4)}=\frac{1}{\Delta} \sum_{j=0}^{9} \alpha^{(j)} E^{j}, \quad v_{i}^{(4)}=\frac{1}{\Delta} \sum_{j=0}^{9} \nu_{i}^{(j)} E^{j}, \quad \Delta=\sum_{j=0}^{9} \delta^{(j)} E^{j} \tag{B8}
\end{equation*}
$$

The expressions for $\alpha^{(j)}, \nu_{i}^{(j)}$, and $\delta^{(j)}$ are lengthy, but easily obtainable; they are not given here.


[^0]:    ${ }^{1}$ For simplicity, from now on we will use $\mathfrak{h}(k)$ instead of $\mathcal{R}_{S}[h(k), E]$ to denote the ISR of $h(k)$. The energy dependence and reduction to $S$ are implied.

[^1]:    ${ }^{2}$ Recall that through the ISR, the reduced parameters depend on both energy $E$ and the original system parameters $\mathbf{g}$.

[^2]:    ${ }^{3} \Pi(\Pi)$ should be viewed as a 'placeholder'. It will take the values $\mathbf{X}, \mathbf{Y}, \mathbf{M}, \mathbf{K}$ and $\mathbf{K}^{\prime}\left(X, Y, M, K\right.$ and $\left.K^{\prime}\right)$. For example, $\left[K_{2}^{(3)}\right]$ represents the number of eigenvalues of $\hat{C}_{3}|u(\mathbf{K})\rangle$ that are equal to $\exp \{2 \pi i / 3\}$, minus the number of eigenvalues evaluated at the $\Gamma$ point.

[^3]:    ${ }^{4}$ We note that there is a one-to-one mapping between a graph and its adjacency matrix $H$, which for many graphs is Hermitian and could thus be interpreted as a Hamiltonian.
    5 This can be shown by performing the isospectral reduction on any $n$ sites that are mapped onto each other by the corresponding symmetry operator $C_{n}$ commuting with the underlying Hamiltonian.

[^4]:    ${ }^{6}$ For brevity, we have dropped the energy dependence in the notation of the reduced hopping parameter and onsite potential: $v_{i}^{(n)}=v_{i}^{(n)}(E)$ and $a^{(n)}=a^{(n)}(E)$.

