Self-adjointness of unbounded time operators

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May 7, 2024

Abstract

Time operators for an abstract semi-bounded self-adjoint operator H with purely discrete spectrum is considered. The existence of a bounded self-adjoint time operator T for H is known as Galapon time operator. In this paper, a self-adjoint but *unbounded* time operator T of H is constructed.

1 Introduction

We give the definition of time operators and conjugate operators.

Definition 1.1. Let H be a self-adjoint operator on a Hilbert space \mathcal{H} and T an operator on \mathcal{H} . If H and T satisfy the canonical commutation relation

$$[H,T] = HT - TH = -i\mathbb{1}$$

on $D_{H,T} \subset D(HT) \cap D(TH)$ but $D_{H,T} \neq \{0\}$, then T is called a conjugate operator of H and $D_{H,T}$ is called a CCR-domain. Here D(A) is the domain of the operator A. If T is a symmetric operator on \mathcal{H} , then T is called a time operator of H.

Time operators and/or conjugate operators for H are in general not unique. In the series of papers [5, 4] we construct time operators and/or conjugate operators for 1D-harmonic oscillator. It is known that the so-called Galapon time operator $T_{\rm G}$ for the 1D-harmonic oscillator is bounded self-adjoint operator and the CCR-domain is dense.

We introduce Galapon operator. A self-adjoint operator H considering the Galapon operator usually imposes the following conditions.

Assumption 1.2. An operator H on a separable Hilbert space \mathcal{H} is positive, unbounded and self-adjoint. The spectrum $\sigma(H)$ of H consists of only simple eigenvalues and H^{-1} is Hilbert-Schmidt.

Let H be an operator which satisfies Assumtion 1.2, e_n an eigenvector of H for an eigenvalue E_n for $n \in \mathbb{N}$. Note that

$$\sum_{n=0}^{\infty} \frac{1}{E_n^2} < \infty$$

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Define the Galapon time operator $T_{\rm G}$ associated with the operator H by

$$D(T_G) = LH\{e_n \in \mathcal{H} \mid n \in \mathbb{N}\},\$$

$$T_G\varphi = i\sum_{n=0}^{\infty} \left(\sum_{m \neq n} \frac{(e_m, \varphi)}{E_n - E_m}\right)e_n, \quad \varphi \in D(T).$$

Here, for a subset \mathcal{A} of \mathcal{H} , LH \mathcal{A} means the linear hull of \mathcal{A} .

Proposition 1.3 ([3, 2]). Suppose that H satisfies Assumption 1.2. Then T_G is a densely defined time operator of H with a CCR-domain LH $\{e_n - e_m \mid n, m \in \mathbb{N}\}$.

It is established in [2, Theorem 4.5] that if

$$E_n - E_m \ge C(n^\lambda - m^\lambda) \tag{1.1}$$

for some constants C > 0 and $\lambda > 1$, then T_G is a bounded time operator of H, and hence it is self-adjoint. In particular, if

$$E_n = an^{\lambda} + b, \quad n \in \mathbb{N}, \quad \lambda > 1$$

with some constants a, b > 0, then $T_{\rm G}$ is a bounded self-adjoint time operator.

Let

$$E_n = an^{\lambda} + b, \quad n \in \mathbb{N}, \quad 1/2 < \lambda < 1$$

with constants a, b > 0. Then it can be shown that $T_{\rm G}$ is unbounded time operator. The self-adjointness of $T_{\rm G}$ is however unknown. It is also pointed out in [1, Remark 4.7] that no examples of *unbounded* self-adjoint time operator of the form $T_{\rm G}$ is constructed.

The purpose of this paper is to construct unbounded self-adjoint time operators for some abstract self-adjoint operator H. We organize this paper as follows. In Section 2 we unitarily transform $T_{\rm G}$ to an operator T_f on $\ell^2(\mathbb{N})$. Section 3 is devoted to constructing unbounded self-adjoint time operators. The main results are stated in Theorems 3.11 and 3.13.

2 Galapon time operator on $\ell^2(\mathbb{N})$

In this paper, the investigation of time operators is carried out on $\ell^2(\mathbb{N})$ instead of \mathcal{H} . Then we shall show first of all that T_G is unitarily equivalent to an operator T_f on $\ell^2(\mathbb{N})$. Denote by $\ell^2(\mathbb{N})$ the set of square summable functions on \mathbb{N} and let $\xi_n \in \ell^2(\mathbb{N})$ be the function on \mathbb{N} defined by

$$\xi_n(m) = \delta_{nm}, \quad m \in \mathbb{N},$$

where δ_{nm} is the Kronecker delta function. We write $\ell_{\text{fin}}^2(\mathbb{N})$ for the set of $\varphi \in \ell^2(\mathbb{N})$ which has a finite support, i.e., there exists $m \in \mathbb{N}$ and $(c_n)_{n=0}^m \in \mathbb{C}^{m+1}$ such that φ is represented by $\sum_{n=0}^m c_n \xi_n$. Note that $\ell_{\text{fin}}^2(\mathbb{N})$ is dense in $\ell^2(\mathbb{N})$. Let L be the left shift operator on $\ell^2(\mathbb{N})$ and L^* the adjoint operator of L;

$$L^*\xi_n = \xi_{n+1}, \quad n \in \mathbb{N}$$

Let N be the number operator on $\ell^2(\mathbb{N})$. Then $N\xi_n = n\xi_n$ for $n \in \mathbb{N}$. It is well known that N is a self-adjoint operator, $\ell_{\text{fin}}^2(\mathbb{N})$ is a core for N and N satisfies [N, L] = -L and $[N, L^*] = L^*$ on $\ell_{\text{fin}}^2(\mathbb{N})$.

We introduce notations \mathcal{K} and \mathcal{K}^- .

Definition 2.1 (\mathcal{K} and \mathcal{K}^-). Let us denote by \mathcal{K} the set of all real valued functions on \mathbb{N} which satisfy the following conditions:

- $(1) \ f(0) > 0,$
- (2) f(n) < f(n+1) for all $n \in \mathbb{N}$.

Write $\mathcal{K}^- = \{ f \in \mathcal{K} \mid 1/f \in \ell^2(\mathbb{N}) \}.$

To define T_f for $f \in \mathcal{K}$, we set

$$\Delta_k(f,n) = f(n+k) - f(n)$$

Lemma 2.2. Let $f \in \mathcal{K}$. Then $\ell_{\text{fin}}^2(\mathbb{N}) \subset D(\Delta_k(f, N)^{-1})$ for all natural number $k \geq 1$.

Proof. Since f is strictly increasing, $\Delta_k(f, N)$ is injective. Clearly $\ell_{\text{fin}}^2(\mathbb{N}) \subset D(\Delta_k(f, N))$ and ξ_n is an eigenvector of $\Delta_k(f, N)$:

$$\Delta_k(f,N)\xi_n = \Delta_k(f,n)\xi_n.$$

This implies that $\ell_{\text{fin}}^2(\mathbb{N}) \subset \mathcal{D}(\Delta_k(f, N)^{-1}).$

Remark 2.3. Note that $\inf_{n \in \mathbb{N}} \Delta_k(f, n) > 0$ if and only if $\Delta_k(f, N)^{-1}$ is a bounded operator.

Definition 2.4. Let $f \in \mathcal{K}$. We define operators $T_{f,m}$ and T_f on $\ell^2(\mathbb{N})$ by

$$T_{f,m} = i \sum_{k=1}^{m} \left(L^{*k} \Delta_k(f, N)^{-1} - \Delta_k(f, N)^{-1} L^k \right),$$

$$D(T_f) = \left\{ \varphi \in \bigcap_{m \ge 1} D(T_{f,m}) \middle| \lim_{m \to \infty} T_{f,m} \varphi \text{ exists in } \ell^2(\mathbb{N}) \right\},$$

$$T_f \varphi = \lim_{m \to \infty} T_{f,m} \varphi, \quad \varphi \in D(T_f).$$

Lemma 2.5. Suppose that $f \in \mathcal{K}^-$. Then $\ell_{\text{fin}}^2(\mathbb{N}) \subset D(T_f)$.

Proof. It is sufficient to show that $\lim_{m\to\infty} T_{f,m}\xi_n$ exists for all $n \in \mathbb{N}$. For any $n \leq m_1 \leq m_2$,

$$\|(T_{f,m_2} - T_{f,m_1})\xi_n\|^2 = \left\| \sum_{k=m_1+1}^{m_2} \left(L^{*k} \Delta_k (f,N)^{-1} - \Delta_k (f,N)^{-1} L^k \right) \xi_n \right\|^2$$
$$= \sum_{k=m_1+1}^{m_2} \frac{1}{(f(n+k) - f(n))^2}$$
$$\leq \left(1 - \frac{f(n)}{f(n+1)} \right)^{-2} \sum_{k=m_1+1}^{m_2} \frac{1}{f(n+k)^2} \to 0$$

as $m_1, m_2 \to \infty$. Hence $\{T_{f,m}\xi_n\}_{m\in\mathbb{N}}$ is a Cauchy sequence. Therefore $\lim_{m\to\infty} T_{f,m}\xi_n$ exists and $\xi_n \in \mathcal{D}(T_f)$.

From Lemma 2.5, we see that T_f is a densely defined symmetric operator. The relationship between T_f and T_G is given by the following theorem.

Theorem 2.6. Suppose that H satisfies Assumption 1.2. Then there exists a unitary operator $U: \mathcal{H} \to \ell^2(\mathbb{N})$ and a function $f \in \mathcal{K}^-$ such that $f(N) = UHU^*$ and T_f is unitary equivalent to T_G on $\ell^2_{\text{fin}}(\mathbb{N})$, i.e.,

$$UT_{\mathcal{G}}U^* = T_f \quad on \ \ell_{\mathrm{fin}}^2(\mathbb{N}).$$

Proof. Let $f: \mathbb{N} \to \mathbb{R}$ be a function such that $f(n) = E_n$. Then $f \in \mathcal{K}^-$. Let $U: \mathcal{H} \to \ell^2(\mathbb{N})$ be the unitary operator defined by $Ue_n = \xi_n$ for any $n \in \mathbb{N}$. For arbitrary $\varphi \in D(T)$, we see that

$$UT_{G}\varphi = i\sum_{n=0}^{\infty} \left(\sum_{m < n} \frac{(\xi_{m}, U\varphi)}{E_{n} - E_{m}} + \sum_{m > n} \frac{(\xi_{m}, U\varphi)}{E_{n} - E_{m}}\right) \xi_{n}$$
$$= i\sum_{n=0}^{\infty} \left(\sum_{m < n} \frac{(L^{n-m}\xi_{n}, \varphi)}{E_{n} - E_{m}} + \sum_{m > n} \frac{(L^{*m-n}\xi_{n}, U\varphi)}{E_{n} - E_{m}}\right) \xi_{n}$$
$$= i\sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \frac{(L^{k}\xi_{n}, U\varphi)}{E_{n} - E_{n-k}} - \sum_{k=1}^{\infty} \frac{(L^{*k}\xi_{n}, U\varphi)}{E_{n+k} - E_{n}}\right) \xi_{n}.$$

Since $f(N)\xi_n = E_n\xi_n$, it follows that

$$(E_n - E_{n-k})^{-1} L^k \xi_n = \Delta_k (f, N)^{-1} L^k \xi_n$$

and

$$(E_{n+k} - E_n)^{-1}\xi_n = \Delta_k(f, N)^{-1}\xi_n.$$

From Lemma 2.5, we see that $\xi_n, U\varphi \in D(T_f)$. Thus

$$UT_{\mathcal{G}}\varphi = i\sum_{n=0}^{\infty} \left(\sum_{k=1}^{\infty} \left(\Delta_k(f,N)^{-1}L^k - L^{*k}\Delta_k(f,N)^{-1}\right)\xi_n, U\varphi\right)\xi_n$$
$$= \sum_{n=0}^{\infty} \left(\xi_n, i\sum_{k=1}^{\infty} \left(L^{*k}\Delta_k(f,N)^{-1} - \Delta_k(f,N)^{-1}L^k\right)U\varphi\right)\xi_n.$$

This shows that $UT_G\varphi = T_f U\varphi$ for any $\varphi \in D(T_G) = U^* \ell_{fin}^2(\mathbb{N})$. Then the theorem is proven.

Corollary 2.7. For all $f \in \mathcal{K}^-$, the operator T_f is a time operator of f(N) with a CCR domain $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$.

Proof. By the definition of \mathcal{K}^- , the operator f(N) is a positive and unbounded self-adjoint operator, and $\sigma(f(N))$ consists of only simple eigenvalues and $f(N)^{-1}$ is Hilbert-Schmidt. Thus T_f is a time operator of f(N) by Theorem 2.6.

By Theorem 2.6 and Corollary 2.7, the set $\{T_f \mid f \in \mathcal{K}^-\}$ includes Galapon time operators T_G . So in what follows we consider time operator T_f .

3 Self-adjointness of time operators

3.1 Bounded cases

Let us recall the case where the operator T_f is bounded.

Lemma 3.1. Let $f \in \mathcal{K}$. Suppose that $0 \notin \sigma(\Delta_k(f, N))$ for all $k \geq 1$ and

$$\sum_{k\geq 1} \left\| \Delta_k(f, N)^{-1} \right\| < \infty.$$

Then the operator T_f is bounded. In particular, T_f is a self-adjoint operator.

Proof. For any $\varphi \in \ell^2(\mathbb{N})$ and $1 \leq m_1 < m_2$,

$$\|(T_{f,m_2} - T_{f,m_1})\varphi\| \le \sum_{k=m_1+1}^{m_2} \left(\left\| L^{*k} \Delta_k(f,N)^{-1} \right\| + \left\| \Delta_k(f,N)^{-1} L^k \right\| \right) \|\varphi\| \le 2\|\varphi\| \sum_{k=m_1+1}^{m_2} \left\| \Delta_k(f,N)^{-1} \right\|.$$

This shows that $\{T_{f,m}\varphi\}_{m\in\mathbb{N}}$ is a Cauchy sequence. Therefore $D(T_f) = \ell^2(\mathbb{N})$ and T_f is bounded.

A similar result to Lemma 3.1 is obtained in [2, Theorem 4.5].

Example 3.2. Let $\lambda > 1$ and $f(x) = x^{\lambda} + 1$. Then $f \in \mathcal{K}^-$. Since $\Delta_k(f, n) \ge \Delta_k(f, 0) = k^{\lambda}$, we have

$$\sum_{k\geq 1} \left\| \Delta_k(f,N)^{-1} \right\| \leq \sum_{k\geq 1} k^{-\lambda} < \infty.$$

Therefore T_f is bounded self-adjoint time operator of f(N).

3.2 Unbounded cases

Next proposition is a sufficient condition for T_f to be unbounded.

Proposition 3.3. Suppose that $f \in \mathcal{K}^-$ and $0 \in \sigma(\Delta_1(f, N))$. Then T_f is unbounded.

Proof. See [2, Theorem 5.1].

Let $f: \operatorname{dom}(f) \to \mathbb{C}$. In this paper, we denote by f^2 the function $f^2: \operatorname{dom}(f) \to \mathbb{C}$, $f^2(x) = f(x)^2$ for each $x \in \operatorname{dom}(f)$. In what follows we consider operators of the form $f(N)T_{f^2} + T_{f^2}f(N)$.

Lemma 3.4. Let $f \in \mathcal{K}^-$. Then $\ell_{\text{fin}}^2(\mathbb{N}) \subset D(f(N)T_{f^2})$ and

$$\lim_{m \to \infty} f(N) T_{f^2, m} \xi_n = f(N) T_{f^2} \xi_n$$

for all $n \in \mathbb{N}$.

Proof. Similarly to the proof of Lemma 2.5, for any $n \leq m_1 \leq m_2$, we have

$$\begin{split} \left\| f(N) \left(T_{f^2, m_2} - T_{f^2, m_1} \right) \xi_n \right\|^2 &= \sum_{k=m_1+1}^{m_2} \frac{f(n+k)^2}{(f(n+k)^2 - f(n)^2)^2} \\ &\leq \left(1 - \frac{f(n)}{f(n+1)} \right)^{-2} \sum_{k=m_1+1}^{m_2} \frac{1}{f(n+k)^2}. \end{split}$$

Therefore $\lim_{m\to\infty} f(N)T_{f^2,m}\xi_n$ exists. Since f(N) is a closed operator, we obtain desired conclusion.

The next lemma shows that T_f is identical to $f(N)T_{f^2} + T_{f^2}f(N)$ on $\ell_{\text{fin}}^2(\mathbb{N})$.

Lemma 3.5. Let $f \in \mathcal{K}^-$. Then

$$f(N)T_{f^2} + T_{f^2}f(N) = T_f (3.1)$$

on $\ell_{\text{fin}}^2(\mathbb{N})$ and

$$[f(N), f(N)T_{f^2} + T_{f^2}f(N)] = -i1$$

on $(1 - L^*)\ell_{\text{fin}}^2(\mathbb{N}).$

Proof. From Lemma 3.4, for any $\varphi \in \ell_{\text{fin}}^2(\mathbb{N})$,

$$\left(f(N)T_{f^2} + T_{f^2}f(N)\right)\varphi = \lim_{m \to \infty} \left(f(N)T_{f^2,m} + T_{f^2,m}f(N)\right)\varphi$$

For each m, we obtain

$$(f(N)T_{f^{2},m} + T_{f^{2},m}f(N)) \varphi = i \sum_{k=1}^{m} \left(L^{*k} \left(f(N+k) + f(N) \right) \Delta_{k} \left(f^{2}, N \right)^{-1} - \Delta_{k} \left(f^{2}, N \right)^{-1} \left(f(N+k) + f(N) \right) L^{k} \right) \varphi = i \sum_{k=1}^{m} \left(L^{*k} \Delta_{k}(f,N)^{-1} - \Delta_{k}(f,N)^{-1} L^{k} \right) \varphi = T_{f,m} \varphi.$$

Hence we see that $\varphi \in D(T_f)$ and $f(N)T_{f^2} + T_{f^2}f(N) = T_f$ on $\ell_{\text{fin}}^2(\mathbb{N})$. Since T_f is a time operator of f(N) with a CCR-domain $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$,

$$\left[f(N), f(N)T_{f^2} + T_{f^2}f(N)\right] = \left[f(N), T_f\right] = -i\mathbb{1}$$

on $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$.

Intuitively it may be hard to show the (essentially) self-adjointness of $f(N)T_{f^2} + T_{f^2}f(N)$ or T_f themselves, since operators $f(N)T_{f^2} + T_{f^2}f(N)$ and T_f are unbounded both from above and below, and a CCR-domain $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$ is not a core of f(N). So we add extra term $f(N)^{\beta}$ to $f(N)T_{f^2} + T_{f^2}f(N)$. Note that $[N, f(N)^{\beta}] \subset 0$. Hence we consider $f(N)T_{f^2} + T_{f^2}f(N) + \gamma f(N)^{\beta}$ instead of $f(N)T_{f^2} + T_{f^2}f(N)$ and show that $f(N)T_{f^2} + T_{f^2}f(N) + \gamma f(N)^{\beta}$ is self-adjoint by the fact that $f(N)T_{f^2} + T_{f^2}f(N)$ is relatively small with respect to $f(N)^{\beta}$.

We introduce classes $\mathcal{M}(\beta)$ and $\mathcal{M}_{s}(\beta)$ of functions on N.

Definition 3.6 $(\mathcal{M}(\beta) \text{ and } \mathcal{M}_{s}(\beta))$. Let $\beta \geq 0$. Denote by $\mathcal{M}(\beta)$ the set of all functions $f \in \mathcal{K}^{-}$ such that there exist functions $g: \mathbb{N} \to (0, \infty)$ and $h \in \ell^{1}(\mathbb{N}_{\geq 1}, \mathbb{R})$ satisfying the following conditions:

- (1) $f^2/g \in \ell^1(\mathbb{N}),$
- (2) for any $n \in \mathbb{N}$ and $k \ge 1$,

$$\frac{g(n)}{(f(n)^{\beta}\Delta_k(f^2, n))^2} \le h(k)^2.$$
(3.2)

Write $\mathcal{M}_{s}(\beta)$ the set of functions $f \in \mathcal{M}(\beta)$ that, for the above function g, there exists a constant C > 0 such that

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{n} \frac{g(n)}{\left(f(n-k)^{\beta} \left(f(n)^{2} - f(n-k)^{2}\right)\right)^{2}} < C.$$
(3.3)

Lemma 3.7. Let $f \in \mathcal{M}(1)$. Then T_{f^2} is bounded.

Proof. By (1) of Definition 3.6, $\sup_{n \in \mathbb{N}} f(n)^2/g(n)$ is finite. From (3.2), we have

$$\left\|\Delta_k(f^2,N)^{-1}\right\| \leq \sup_{n\in\mathbb{N}}\Delta_k(f^2,n)^{-1} \leq \sup_{n\in\mathbb{N}}\left(f(n)^2/g(n)\right)^{1/2}h(k).$$

Since $h \in \ell^1(\mathbb{N}_{\geq 1}, \mathbb{R})$, by Lemma 3.1, we see that T_{f^2} is bounded .

Lemma 3.8. Let $f \in \mathcal{M}_{s}(\beta)$ Then $f(N)T_{f^{2}}$ is relatively bounded with respect to $f(N)^{\beta}$, i.e., there exists some constant a > 0 such that for all $\varphi \in D(f(N)^{\beta})$

$$\left\|\overline{f(N)T_{f^2}}\varphi\right\| \le a \left\|f(N)^{\beta}\varphi\right\|.$$

Here \overline{A} denotes the closure of the linear operator A.

Proof. It is sufficient to show that $\overline{f(N)T_{f^2}}f(N)^{-\beta}$ is bounded. For any $\varphi \in \ell_{\text{fin}}^2(\mathbb{N})$,

$$\begin{split} \left\| f(N)T_{f^{2}}f(N)^{-\beta}\varphi \right\|^{2} &= \sum_{n=0}^{\infty} \left(\xi_{n}, f(N)T_{f^{2}}f(N)^{-\beta}\varphi \right)^{2} \\ &\leq \|\varphi\|^{2} \sum_{n=0}^{\infty} \left\| f(N)^{-\beta}T_{f^{2}}f(N)\xi_{n} \right\|^{2} \\ &= \|\varphi\|^{2} \sum_{n=0}^{\infty} f(n)^{2} \left\| f(N)^{-\beta}T_{f^{2}}\xi_{n} \right\|^{2}. \end{split}$$

For all $n \in \mathbb{N}$, we see that

$$\begin{split} \left\| f(N)^{-\beta} T_{f^2} \xi_n \right\|^2 &= \left\| \sum_{k \ge 1} f(N)^{-\beta} \left(L^{*k} \Delta_k (f^2, N)^{-1} - \Delta_k (f^2, N)^{-1} L^k \right) \xi_n \right\|^2 \\ &= \sum_{k \ge 1} \frac{1}{f(n+k)^{2\beta} \Delta_k (f^2, n)^2} + \sum_{k=1}^n \frac{1}{f(n-k)^{2\beta} (f(n)^2 - f(n-k)^2)^2} \\ &\leq \frac{1}{g(n)} \left(\sum_{k \ge 1} h(k)^2 + C \right). \end{split}$$

Thus we have

$$\left\| f(N)T_{f^2}f(N)^{-\beta}\varphi \right\|^2 \le \|\varphi\|^2 \left(\|h\|_{\ell^2}^2 + C \right) \sum_{n=0}^{\infty} f(n)^2 g(n)^{-1}.$$

Therefore $\overline{f(N)T_{f^2}}f(N)^{-\beta}$ is bounded.

Remark 3.9. Let $\beta \geq 1$, $f \in \mathcal{M}_{s}(\beta)$ and T_{f}^{2} be bounded. Since the operator T_{f} is equal to $f(N)T_{f^{2}}+T_{f^{2}}f(N)$ on $\ell_{\text{fin}}^{2}(\mathbb{N})$ by Lemma 3.5, it is easy to see that $\overline{T_{f}}$ is also relatively bounded with respect to $f(N)^{\beta}$, i.e., there exists a constant a > 0 such that for all $\varphi \in D(f(N)^{\beta})$

$$\left\|\overline{T_f}\varphi\right\| \le a \left\|f(N)^{\beta}\varphi\right\|.$$

Proposition 3.10. Let $f \in \mathcal{M}_{s}(1)$. Then (1) and (2) follow.

(1) $f(N)T_{f^2} + T_{f^2}f(N) + \gamma f(N)$ is self-adjoint if $|\gamma| \gg 1$.

(2) $f(N)T_{f^2} + T_{f^2}f(N) + \gamma f(N)^2$ is self-adjoint if $\gamma \in \mathbb{R} \setminus \{0\}$.

Proof. (1) From Lemma 3.8, there exists a relative bound a of $f(N)T_{f^2} + T_{f^2}f(N)$ with respect to f(N). Thus if $|\gamma| > a$, by the Kato-Rellich theorem, the operator is self-adjoint.

(2) From Lemma 3.8, we see that $f(N)T_{f^2}+T_{f^2}f(N)$ is relatively bounded with respect to f(N). Since f(N) is infinitesimally small with respect to $f(N)^2$, by the Kato-Relich theorem, $\gamma f(N)^2 + f(N)T_{f^2} + T_{f^2}f(N)$ is a self-adjoint operator.

We are in the position to state the main theorem in this paper.

Theorem 3.11. Let $f \in \mathcal{M}_{s}(1)$. Then $\overline{T_{f}} + \gamma f(N)^{2}$ is a self-adjoint time operator of f(N) with a dense CCR-domain for all $\gamma \in \mathbb{R} \setminus \{0\}$.

Proof. From (3.1), Lemma 3.7, Lemma 3.8 and Remark 3.9, we see that $\overline{T_f} + \gamma f(N)^2$ is self-adjoint for all $\gamma \in \mathbb{R} \setminus \{0\}$ and $[f(N), T_f + \gamma f(N)^2] = -i\mathbb{1}$ on $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$.

Example 3.12. Let $f(x) = x^{\lambda} + 1$ for $\lambda \in (3/4, 1)$. We show that $f \in \mathcal{M}_{s}(1)$. Firstly, it is immediate that $f \in \mathcal{K}^{-}$. Let $\alpha \in (1 + 2\lambda, 6\lambda - 2)$, $g(x) = x^{\alpha} + 1$ and $\delta = 6\lambda - 2 - \alpha$. Then the condition (1) of $\mathcal{M}(1)$ is satisfied.

Secondly, by the mean value theorem, we have

$$f(n+k) - f(n) \ge \frac{\lambda k}{(n+k)^{1-\lambda}}.$$

Thus we see that

$$\frac{g(n)}{f(n)^2 \Delta_k (f^2, n)^2} = \frac{n^{\alpha} + 1}{f(n)^2 (f(n+k)^2 - f(n)^2)^2} \le \frac{(n^{\alpha} + 1)(n+k)^{2(1-\lambda)}}{\lambda^2 (n^{\lambda} + 1)^2 (n+k)^{2\lambda} k^2} \le \frac{2}{\lambda^2 k^{2+\delta}}.$$

Thus the condition (2) of $\mathcal{M}(1)$ is satisfied and $f \in \mathcal{M}(1)$.

Finally we see that

$$\begin{split} \lim_{n \to \infty} \sum_{k=1}^{n} \frac{g(n)}{f(n-k)^2 (f(n)^2 - f(n-k)^2)^2} \\ &= \lim_{n \to \infty} \left(\sum_{k=1}^{[n/2]} \frac{g(n)}{f(n-k)^2 (f(n)^2 - f(n-k)^2)^2} + \sum_{k=[n/2]+1}^{n} \frac{g(n)}{f(n-k)^2 (f(n)^2 - f(n-k)^2)^2} \right) \\ &\leq \lim_{n \to \infty} \frac{4^{\lambda} (n^{\alpha} + 1) n^{2(1-\lambda)}}{\lambda^2 (n^{\lambda} + 1)^2 n^{2\lambda}} \left(\sum_{k=1}^{[n/2]} \frac{1}{k^2} + \frac{4^{1-\lambda}}{n^{2(1-\lambda)}} \sum_{k=[n/2]+1}^{n} \frac{1}{f(n-k)^2} \right) < \infty. \end{split}$$

Then the condition (3.3) is satisfied and $f \in \mathcal{M}_{s}(1)$.

We see that T_f is unbounded by Proposition 3.3 and that, from Theorem 3.11, f(N) has a self-adjoint time operator with a dense CCR-domain.

Similarly to Proposition 3.10 and Theorem 3.11, we can show the following statements. We omit proofs.

Proposition 3.13. Let $f \in \mathcal{M}_{s}(2)$ and $T_{f^{2}}$ is bounded. Then $f(N)T_{f^{2}} + T_{f^{2}}f(N) + \gamma f(N)^{2}$ is a self-adjoint time operator of f(N) if $|\gamma| \gg 1$.

Theorem 3.14. Let $f \in \mathcal{M}_{s}(2)$ and $T_{f^{2}}$ is bounded. Then $\overline{T_{f}} + \gamma f(N)^{3}$ is a self-adjoint time operator of f(N) with a dense CCR-domain for all $\gamma \in \mathbb{R} \setminus \{0\}$.

Example 3.15. Let $f(x) = x^{\lambda} + 1$ for $\lambda \in (1/2, 1)$. Then T_{f^2} is bounded by Lemma 3.1. Similar to Example 3.12, we can see that $f \in \mathcal{M}_s(2)$. Therefore f(N) has an unbounded self-adjoint time operator with a dense CCR-domain by Theorem 3.14.

3.3 Extensions

In this section we consider the case that $f^2 \in \mathcal{K}^-$ but not necessarily $f \in \mathcal{K}^-$.

Proposition 3.16. Let $f \in \mathcal{K}$. If $D(f(N)T_{f^2}) \cap D(f(N)^2)$ is dense and T_{f^2} is bounded, then $f(N)T_{f^2} + T_{f^2}f(N) + \gamma f(N)^2$ of f(N) has a self-adjoint extension for all $\gamma \ge 1$.

Proof. From

$$f(N)T_{f^2} + T_{f^2}f(N) + \gamma f(N)^2 \subset \left(f(N) + T_{f^2}\right)^2 + (\gamma - 1)f(N)^2 - T_{f^2}^2,$$

we see that $f(N)T_{f^2} + T_{f^2}f(N) + \gamma f(N)^2$ is bounded from below. Thus it has the Friedrichs extension.

Lemma 3.17. Let $f^2 \in \mathcal{K}^-$. Then $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N}) \subset D(f(N)^2 T_{f^2}) \cap D(T_{f^2}f(N))$ and the operator $f(N)T_{f^2} + T_{f^2}f(N)$ is symmetric.

Proof. From Lemma 3.5, T_{f^2} satisfies $[f(N)^2, T_{f^2}] = -i\mathbb{1}$ on $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$. This implies that $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N}) \subset D(f(N)^2T_{f^2}) \cap D(T_{f^2}f(N))$.

Lemma 3.18. Let $f \in \mathcal{K}$. Then $f(N)(\mathbb{1} - L^*)\Delta_1(f, N)^{-1}(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N}) \subset (\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$.

Proof. On $\ell_{\text{fin}}^2(\mathbb{N})$, we have

$$f(N)(\mathbb{1} - L^*)\Delta_1(f, N)^{-1}(\mathbb{1} - L^*)$$

= $(f(N)\Delta_1(f, N)^{-1} - L^*f(N + \mathbb{1})\Delta_1(f, N)^{-1})(\mathbb{1} - L^*)$
= $(f(N)\Delta_1(f, N)^{-1} - L^* - L^*f(N)\Delta_1(f, N)^{-1})(\mathbb{1} - L^*)$
= $(\mathbb{1} - L^*)(f(N)\Delta_1(f, N)^{-1}(\mathbb{1} - L^*) - L^*).$

Therefore we obtain the desired result.

Theorem 3.19. Let $f^2 \in \mathcal{K}^-$. Then $f(N)T_{f^2} + T_{f^2}f(N)$ and T_f are time operators of f(N) with an infinite dimensional CCR-domain.

Proof. From Lemmas 3.5, 3.17 and 3.18, we see that

$$(\mathbb{1} - L^*)\Delta_1(f, N)^{-1}(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N}) \subset \mathcal{D}(f(N)^2 T_f) \cap \mathcal{D}(f(N) T_f f(N)) \cap \mathcal{D}(T_f f(N)^2).$$

Therefore the symmetric operator $f(N)T_{f^2} + T_{f^2}f(N)$ satisfies

$$[f(N), f(N)T_{f^2} + T_{f^2}f(N)] = -i1$$

on $(\mathbb{1} - L^*)\Delta_1(f, N)^{-1}(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$. By the proof of Lemma 3.5, we can see that

$$f(N)T_{f^2} + T_{f^2}f(N) = T_f$$

on $(\mathbb{1} - L^*)\ell_{\text{fin}}^2(\mathbb{N})$. Hence T_f is also a time operator of f(N) with an infinite dimensional CCR-doamin.

Example 3.20. Let $f(x) = \sqrt{x+1}$. Clearly, $f^2 \in \mathcal{K}^-$. From Theorem 3.19, we see that $f(N)T_{f^2} + T_{f^2}f(N) + f(N)^2$ is a time operator of f(N). It is known that T_{f^2} is bounded by [2, Theorem 4.6]. Since it has a self-adjoint extension by Proposition 3.16, f(N) has a self-adjoint time operator with an infinite dimensional CCR-domain.

Acknowledgements: FH is financially supported by JSPS KAKENHI 20K20886 and JSPS KAKENHI 20H01808.

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