ON GAMMA FUNCTIONS WITH RESPECT TO THE ALTERNATING HURWITZ ZETA FUNCTIONS

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Abstract. In 1730, Euler defined the Gamma function $\Gamma(x)$ by the integral representation. It possesses many interesting properties and has wide applications in various branches of mathematics and sciences. According to Lerch, the Gamma function $\Gamma(x)$ can also be defined by the derivative of the Hurwitz zeta function

$$\zeta(z,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^z}$$

at $z=0$. Recently, Hu and Kim [11] defined the corresponding Stieltjes constants $\widetilde{\gamma}_k(x)$ and Euler constant $\widetilde{\gamma}_0$ from the Taylor series of the alternating Hurwitz zeta function $\zeta_E(z,x)$

$$\zeta_E(z,x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)^z}.$$

And they also introduced the corresponding Gamma function $\widetilde{\Gamma}(x)$ which has the following Weierstrass–Hadamard type product

$$\widetilde{\Gamma}(x)=\frac{1}{x}e^{\widetilde{\gamma}_0 x}\prod_{k=1}^{\infty}\left(e^{-\frac{x}{k}}\left(1+\frac{x}{k}\right)\right)^{(-1)^{k+1}}.$$

In this paper, we shall further investigate the function $\widetilde{\Gamma}(x)$, that is, we obtain several properties in analogy to the classical Gamma function $\Gamma(x)$, including the integral representation, the limit representation, the recursive formula, the special values, the log-convexity, the duplication formula and the reflection equation. Furthermore, we also prove a Lerch-type formula, which shows that the derivative of $\zeta_E(z,x)$ can be representative by $\widetilde{\Gamma}(x)$. As an application to Stark's conjecture in algebraic number theory, we will explicit calculate the derivatives of the partial zeta functions for the maximal real subfield of cyclotomic fields at $z=0$.

1. Introduction

The gamma function $\Gamma(x)$ is defined by Euler from the integral representation

$$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt \tag{1.1}$$

for $\mathrm{Re}(x)>0$. It is an extension of the factorial function $\Pi(n)=n!$ from the integral variable to the real and complex variables, and it has many

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applications in various branches of sciences, such as number theory, physics, statistics, and so on. According to Lerch, Euler's gamma function $\Gamma(x)$ can also be defined by the derivative of the Hurwitz zeta function

$$\zeta(z,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^z} \tag{1.2}$$

at $z=0$, that is, we have

$$\log\Gamma(x)=\zeta'(0,x)-\zeta'(0,1)=\zeta'(0,x)-\zeta'(0) \tag{1.3}$$

(see [3, Proposition 9.6.13(1)]). Here

$$\zeta(z)=\zeta(z,1)=\sum_{n=1}^{\infty}\frac{1}{n^z}$$

is the Riemann zeta function. In addition, $\Gamma(x)$ has the following well-known Weierstrass–Hadamard product

$$\Gamma(x)=\frac{1}{x}e^{-\gamma x}\prod_{k=1}^{\infty}\left(e^{\frac{x}{k}}\left(1+\frac{x}{k}\right)^{-1}\right), \tag{1.4}$$

where $\gamma=0.5772156649\cdots$ is the Euler constant. Furthermore, for $\mathrm{Re}(x)>0$, let

$$\psi(x):=\frac{d}{dx}\log\Gamma(x) \tag{1.5}$$

be the digamma function (see [3, Definition 9.6.13(2)] and [5, p. 32])). And if setting

$$\psi^{(n)}(x):=\left(\frac{d}{dx}\right)^n\psi(x),\quad n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}, \tag{1.6}$$

then we have the following formula which represents the special values of $\zeta(z,x)$:

$$\psi^{(n)}(x)=(-1)^{n+1}n!\zeta(n+1,x),\quad n\in\mathbb{N} \tag{1.7}$$

(see [3, Proposition 9.6.41]).

For $\mathrm{Re}(z)>0$ and $x\neq 0,-1,-2,\ldots$, let

$$\zeta_E(z,x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+x)^z} \tag{1.8}$$

be the alternating form of the Hurwitz zeta function (also known as the Hurwitz-type Euler zeta function). In particular, setting $x=1$ we obtain the alternating zeta function (also known as Dirichlet's eta function),

$$\zeta_E(z)=\zeta_E(z,1)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^z}=\eta(z). \tag{1.9}$$

And $\zeta(z)$ and $\zeta_E(z)$ satisfy

$$\zeta_E(z)=\left(1-\frac{1}{2^{z-1}}\right)\zeta(z). \tag{1.10}$$

During the recent years, the Fourier expansion, power series and asymptotic expansions, integral representations, special values, and convexity properties of $\zeta_E(z,x)$ have been systematically studied (see [4, 9, 10, 11, 12]). In algebraic number theory, it is found that $\zeta_E(z,x)$ can be used to represent a partial zeta function of cyclotomic fields in one version of Stark's conjectures (see [13, p. 4249, (6.13)]). In addition, the alternating zeta function $\zeta_E(z)$ is a particular case of Witten's zeta functions in mathematical physics (see [14, p. 248, (3.14)]). And for a form containing in a handbook of mathematical functions by Abramowitz and Stegun, the left hand side gives the special values of the Riemann zeta functions $\zeta(z)$ at positive integers, and the right hand side gives the special values of $\zeta_E(z)$ at the corresponding points (see [2, p. 811]).

In analogy to the classical situation (see [11, (1.3)]), in 2021 Hu and Kim [11] defined the corresponding generalized Stieltjes constant $\widetilde{\gamma}_k(x)$ and the Euler constant $\widetilde{\gamma}_0$ from the Taylor expansion of $\zeta_E(z,x)$ at $z=1$. That is, they designated a modified Stieltjes constants $\widetilde{\gamma}_k(x)$ by the Taylor expansion of $\zeta_E(z,x)$ at $z=1$,

$$\zeta_E(z,x)=\sum_{k=0}^{\infty}\frac{(-1)^k\widetilde{\gamma}_k(x)}{k!}(z-1)^k \tag{1.11}$$

(see [11, (1.18)]). From the above expansion, we see that

$$\widetilde{\gamma}_0(x)=\zeta_E(1,x). \tag{1.12}$$

Let

$$\widetilde{\gamma}_k:=\widetilde{\gamma}_k(1) \tag{1.13}$$

for $k\in\mathbb{N}_0$, we have

$$\widetilde{\gamma}_0=\widetilde{\gamma}_0(1)=\zeta_E(1)=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}=\log 2 \tag{1.14}$$

(also see [11, (1.17) and (1.20)]). And they also defined the corresponding gamma function $\widetilde{\Gamma}(x)$ and the digamma function $\widetilde{\psi}(x)$, which satisfy the differential equation

$$\widetilde{\psi}(x)=\frac{d}{dx}\log\widetilde{\Gamma}(x) \tag{1.15}$$

(see [11, p. 5]). They proved an analogue of the Weierstrass–Hadamard product

$$\widetilde{\Gamma}(x)=\frac{1}{x}e^{\widetilde{\gamma}_0 x}\prod_{k=1}^{\infty}\left(e^{-\frac{x}{k}}\left(1+\frac{x}{k}\right)\right)^{(-1)^{k+1}} \tag{1.16}$$

(see [11, Theorem 3.12]). Furthermore, if setting

$$\widetilde{\psi}^{(n)}(x):=\left(\frac{d}{dq}\right)^n\widetilde{\psi}(x),\quad n\in\mathbb{N}_0, \tag{1.17}$$

then as in the classical situation (1.7), $\widetilde{\psi}^{(n)}(x)$ also represents the special values of $\zeta_E(z,x)$:

$$\widetilde{\psi}^{(n)}(x) = (-1)^{n+1} n! \zeta_E(n+1,x), \quad n \in \mathbb{N}_0 \tag{1.18}$$

(see [7, p. 957, 8.374] and [11, (1.25)]).

It is known that the classical gamma function $\Gamma(x)$ has many interesting properties, such as the integral representation, the limit representation, the recursive formula, the log-convexity, the duplication formula and the reflection equation. These properties lead it wide applications in analysis, number theory, mathematical physics and probability. In section 2, we will show that these properties are all established for $\widetilde{\Gamma}(x)$. Then in section 3, we will further investigate the properties of $\widetilde{\psi}(x)$, including its recursive formula and the reflection equation. As pointed out in [11, p. 5], $\widetilde{\psi}(x)$ is equivalent to Nilsen's β-function $\beta(x)$, that is, we have

$$\beta(x) = \frac{1}{2}\left(\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right) = -\widetilde{\psi}(x) \tag{1.19}$$

(comparing [11, (2.5)] and [16, (1.4)]). Furthermore, as shown by Nantomah, Nilsen's β-function $\beta(x)$ can be represented by the derivative of the classical Beta function $\mathrm{B}(x,y)$:

$$\beta(x) = -\frac{d}{dx}\log \mathrm{B}\left(\frac{x}{2},\frac{1}{2}\right). \tag{1.20}$$

for $x > 0$ (see [16, Proposition 1.1]).

In section 4, we will study the relation between $\widetilde{\Gamma}(x)$ and $\zeta_E(z,x)$. In concrete, we show that Lerch's formula (1.3) is also established in our situation, that is, we have

$$\log \widetilde{\Gamma}(x) = \zeta_E'(0,x) + \zeta_E'(0). \tag{1.21}$$

Then in section 5, we shall consider an application of the above formula to algebraic number theory. Let m be an odd positive integer, $K = \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field and $K^+ = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ be its maximal real subfield. Let $S = \{\infty, p|m\}$ and $T = \{2\}$ be sets of places of $\mathbb{Q}$. Applying Lerch's formula (1.3) and its variant (1.21), we explicit compute the derivatives of the S-partial zeta function $\zeta_{K^+/\mathbb{Q},S}(z,\sigma)$ and the (S,T)-partial zeta function $\zeta_{K^+/\mathbb{Q},S,T}(z,\sigma)$ at $z=0$ respectively, where $\sigma \in G = \mathrm{Gal}(K^+/\mathbb{Q})$. That is, we obtain

$$\zeta'_{K^+/\mathbb{Q},S}(0,\sigma) = -\log\left(2\sin\frac{\pi l}{m}\right)$$

and

$$\zeta'_{K^+/\mathbb{Q},S,T}(0,\sigma) = (-1)^{l+1}\log\left(\cot\frac{\pi l}{2m}\right),$$

where $0 < l < m, (l,m) = 1$ and $\sigma = \sigma_{\pm l} \in G$. (See (5.12) and (5.24) below).

After posting the present paper on the arXiv, we received an email from Nakamura on a related work [15]. In that paper, he defined the functions $\Psi_\Phi(a,z)$ and $\Gamma_\Phi(a,z)$, which are generalizations of the classical digamma and gamma functions in terms of the Hurwitz-Lerch zeta function $\Phi(z,s,a)$.

And he also obtained several properties of $\Gamma_\Phi(a,z)$, including its limit representation, a Lerch-type formula, the integral representation and the asymptotic expansion of the corresponding log-gamma function $\log\Gamma_\Phi(a,z)$ (see [15, Theorems 11 and 12]). Since his definition of the digamma function is different from us in the case of $z=-1$ (by comparing [15, (34)] and [11, (1.22)]), these properties appear in a different way in his paper.

2. The properties of $\widetilde{\Gamma}(x)$

In this section, we will investigate the properties of $\widetilde{\Gamma}(x)$. First, we obtain its special value at 1.

Lemma 2.1. *We have*

$$\widetilde{\Gamma}(1)=\frac{\pi}{2}. \tag{2.1}$$

Remark 2.2. Recall the following special values for the classical gamma function $\Gamma(x)$:

$$\Gamma(1)=1 \text{ and } \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Proof of Lemma 2.1. We have the expansion

$$\log\widetilde{\Gamma}(1)=\widetilde{\gamma}_0+\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{k}-\log\left(1+\frac{1}{k}\right)\right)$$

(see [11, (4.31)]). Since

$$\widetilde{\gamma}_0=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} \tag{2.2}$$

(see (1.14)), by Wallis' formula we see that

$$\begin{aligned}\log\widetilde{\Gamma}(1)&=\sum_{k=1}^{\infty}(-1)^{k+1}\left(\log(k+1)-\log(k)\right)\\&=\lim_{n\to\infty}\log\left(\frac{(2n!!)^2}{(2n-1)!!(2n+1)!!}\right)\\&=\log\frac{\pi}{2},\end{aligned}$$

which is equivalent to (2.1). □

Now we have the following integral representation of $\widetilde{\Gamma}(x)$.

Theorem 2.3 (Integral representation)**.** *For* $\mathrm{Re}(x)>0$*, we have*

$$\widetilde{\Gamma}(x)=\frac{1}{2}\mathrm{B}\left(\frac{x}{2},\frac{1}{2}\right)=\frac{1}{2}\int_0^1\frac{t^{\frac{x}{2}-1}}{\sqrt{1-t}}dt,$$

where $B(x,y)$ is the Beta function (also known as the first Eulerian integral).

Remark 2.4. Recall the integral representation for the classical gamma function $\Gamma(x)$:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \int_0^1 \left(\log \frac{1}{t}\right)^{x-1} dt.$$

Proof of Theorem 2.3. Comparing (1.15), (1.19) and (1.20), we get

$$\frac{d}{dx} \log \widetilde{\Gamma}(x) = \widetilde{\psi}(x) = -\beta(x) = \frac{d}{dx} \log \mathrm{B}\left(\frac{x}{2}, \frac{1}{2}\right),$$

thus

$$\widetilde{\Gamma}(x) = C\mathrm{B}\left(\frac{x}{2}, \frac{1}{2}\right) = C \int_0^1 t^{\frac{x}{2}-1}(1-t)^{-\frac{1}{2}} dt = C \int_0^1 \frac{t^{\frac{x}{2}-1}}{\sqrt{1-t}} dt, \tag{2.3}$$

where C is a constant. Then letting $x = 1$ in the above equation and by noticing (2.1), we have

$$\frac{\pi}{2} = \widetilde{\Gamma}(1) = C\mathrm{B}\left(\frac{1}{2}, \frac{1}{2}\right) = C\pi.$$

From which, we determine $C = \frac{1}{2}$. Then substituting into (2.3), we get the desired result. □

From the above theorem and the relation between $\Gamma(x)$ and $\mathrm{B}(x, y)$ (see [3, Proposition 9.6.39]), we immediately get the following result.

Corollary 2.5. *For* $\mathrm{Re}(x) > 0$*, we have*

$$\widetilde{\Gamma}(x) = \frac{1}{2}\mathrm{B}\left(\frac{x}{2}, \frac{1}{2}\right) = \frac{\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{x+1}{2}\right)}.$$

Then we have the following limit representation of $\widetilde{\Gamma}(x)$.

Theorem 2.6 (Limit representation)**.** *For* $\mathrm{Re}(x) > 0$*, we have*

$$\widetilde{\Gamma}(x) = \begin{cases} \lim\limits_{n\to\infty} \dfrac{n!!}{(n-1)!!} \displaystyle\prod_{k=0}^{n} \left(\frac{1}{k+x}\right)^{(-1)^k} & \text{if } n \text{ is even,} \\ \lim\limits_{n\to\infty} \dfrac{(n-1)!!}{n!!} \displaystyle\prod_{k=0}^{n} \left(\frac{1}{k+x}\right)^{(-1)^k} & \text{if } n \text{ is odd.} \end{cases} \tag{2.4}$$

Remark 2.7. Recall the limit representation of the classical gamma function $\Gamma(x)$:

$$\Gamma(x) = \lim_{n\to\infty} \frac{n^x n!}{x(x+1)\cdots(x+n-1)(x+n)}$$

(see [3, Proposition 9.6.17]).

According to [22, p. 101], the above limit representation of $\Gamma(x)$ firstly appeared in a letter by Euler to Goldbach in 1729. Then it was independently found by Gauss, who showed it in a letter to Bessel in 1811 (see [17, p. 33–34]).

Proof of Theorem 2.6. First, we have the following limit representation of γ_0:

$$\widetilde{\gamma}_0(1) = \widetilde{\gamma}_0 = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \frac{1}{k+1}$$

(see (2.2)). Substitute into (1.16), we get

$$\begin{aligned} \widetilde{\Gamma}(x) &= \lim_{n\to\infty} \frac{1}{x} \exp\left(\sum_{k=1}^{n} (-1)^k \left(\log k - \log(k+x)\right) \right) \\ &= \lim_{n\to\infty} \exp\left(\sum_{k=1}^{n} (-1)^k \log k \right) \prod_{k=0}^{n} \left(\frac{1}{k+x} \right)^{(-1)^k}. \end{aligned} \tag{2.5}$$

In what follows, we shall calculate the sum $\sum_{k=1}^{n} (-1)^k \log k$ by cases.

If n is even, that is, for $n = 2m$, we have

$$\begin{aligned} \sum_{k=1}^{n} (-1)^k \log k &= \sum_{k=1}^{2m} (-1)^k \log k \\ &= -\log 1 + \log 2 - \cdots - \log(2m-1) + \log(2m) \\ &= \log \frac{(2m)!!}{(2m-1)!!} \\ &= \log \frac{n!!}{(n-1)!!}. \end{aligned}$$

Then substituting to (2.5), we get

$$\widetilde{\Gamma}(x) = \lim_{n\to\infty} \frac{n!!}{(n-1)!!} \prod_{k=0}^{n} \left(\frac{1}{k+x} \right)^{(-1)^k}. \tag{2.6}$$

If n is odd, that is, for $n = 2m+1$, we have

$$\begin{aligned} \sum_{k=1}^{n} (-1)^k \log k &= \sum_{k=1}^{2m+1} (-1)^k \log k \\ &= -\log 1 + \log 2 - \cdots + \log(2m) - \log(2m+1) \\ &= \log \frac{(2m)!!}{(2m+1)!!} \\ &= \log \frac{(n-1)!!}{n!!}. \end{aligned}$$

Then substituting to (2.5), we get

$$\widetilde{\Gamma}(x) = \lim_{n\to\infty} \frac{(n-1)!!}{n!!} \prod_{k=0}^{n} \left(\frac{1}{k+x} \right)^{(-1)^k}. \tag{2.7}$$

Combing (2.6) and (2.7), we have (2.4). □

Corollary 2.8. *For* $\mathrm{Re}(x) > 0$, *we have*

$$\widetilde{\Gamma}(x) = \frac{1}{x} \prod_{k=1}^{\infty} \left(\frac{2k}{x+2k} \cdot \frac{x+2k-1}{2k-1} \right).$$

Proof. By Theorem 2.6, if n is even, that is, for $n = 2m$, we have

$$\begin{aligned}
\widetilde{\Gamma}(x) &= \lim_{n\to\infty} \frac{n!!}{(n-1)!!} \prod_{k=0}^{n} \left(\frac{1}{k+x}\right)^{(-1)^k} \\
&= \lim_{m\to\infty} \frac{(2m)!!}{(2m-1)!!} \prod_{k=0}^{2m} \left(\frac{1}{k+x}\right)^{(-1)^k} \\
&= \frac{1}{x} \lim_{m\to\infty} \left(\frac{2}{x+2} \cdot \frac{4}{x+4} \cdot \cdots \cdot \frac{2m}{x+2m}\right) \\
&\quad \times \left(\frac{x+1}{1} \cdot \frac{x+3}{3} \cdot \cdots \cdot \frac{x+2m-1}{2m-1}\right) \\
&= \frac{1}{x} \lim_{m\to\infty} \prod_{k=1}^{m} \left(\frac{2k}{x+2k}\right) \left(\frac{x+2k-1}{2k-1}\right) \\
&= \frac{1}{x} \prod_{k=1}^{\infty} \left(\frac{2k}{x+2k} \cdot \frac{x+2k-1}{2k-1}\right).
\end{aligned}$$

If n is odd, that is for $n = 2m+1$, the same reasoning leads to

$$\widetilde{\Gamma}(x) = \frac{1}{x} \prod_{k=1}^{\infty} \left(\frac{2k}{x+2k} \cdot \frac{x+2k-1}{2k-1}\right).$$

This is the desired assertion. □

We have the following recursive formula for $\Gamma(x)$.

Theorem 2.9 (Recursive formula)**.** *For* $\mathrm{Re}(x) > 0$ *and* $n \in \mathbb{N}$*, we have*

$$\left(\widetilde{\Gamma}(x+n)\right)^{(-1)^n} = \prod_{k=0}^{n-1} \left(\frac{2(x+k)}{\pi}\right)^{(-1)^k} \widetilde{\Gamma}(x). \tag{2.8}$$

In particular, for $n = 1$ *and* 2*, we have*

$$\widetilde{\Gamma}(x+1)\widetilde{\Gamma}(x) = \frac{\pi}{2x} \tag{2.9}$$

and

$$\widetilde{\Gamma}(x+2) = \frac{x}{x+1}\widetilde{\Gamma}(x), \tag{2.10}$$

respectively.

Remark 2.10. Recall the recursive formula of the classical gamma function $\Gamma(x)$:

$$\Gamma(x+1) = x\Gamma(x)$$

(see [3, Proposition 9.6.14]).

Proof of Theorem 2.9. It is known that

$$(-1)^{n-1}\widetilde{\psi}(x+n) + \widetilde{\psi}(x) = \sum_{k=1}^{n} \frac{(-1)^k}{x+k-1}$$

(see [11, (2.8)]), which is equivalent to

$$(-1)^n\widetilde{\psi}(x+n)-\widetilde{\psi}(x)=\sum_{k=0}^{n-1}\frac{(-1)^k}{x+k}.$$

So by (1.15) we get

$$\frac{d}{dx}\log\frac{\widetilde{\Gamma}(x+n)^{(-1)^n}}{\widetilde{\Gamma}(x)}=\sum_{k=0}^{n-1}\frac{(-1)^k}{x+k}$$

and

$$\begin{aligned}\log\frac{\widetilde{\Gamma}(x+n)^{(-1)^n}}{\widetilde{\Gamma}(x)}&=\int\sum_{k=0}^{n-1}\frac{(-1)^k}{x+k}dx\\&=\sum_{k=0}^{n-1}(-1)^k\left(\log(x+k)-\log C_k\right)\\&=\sum_{k=0}^{n-1}\log\left(\frac{x+k}{C_k}\right)^{(-1)^k},\end{aligned}$$

here C_k are constants depending on k. Thus

$$\left(\widetilde{\Gamma}(x+n)\right)^{(-1)^n}=\prod_{k=0}^{n-1}\left(\frac{x+k}{C_k}\right)^{(-1)^k}\widetilde{\Gamma}(x). \tag{2.11}$$

In particular, for $n=1$, we get

$$\widetilde{\Gamma}(x+1)\widetilde{\Gamma}(x)=\frac{C_0}{x}. \tag{2.12}$$

Now we determine the constant C_0. By letting $x=1$ in (2.12), we get

$$\widetilde{\Gamma}(2)\widetilde{\Gamma}(1)=C_0. \tag{2.13}$$

From Theorem 2.3, we have

$$\widetilde{\Gamma}(2)=\frac{1}{2}\mathrm{B}\left(1,\frac{1}{2}\right)=1, \tag{2.14}$$

and by Lemma 2.1,

$$\widetilde{\Gamma}(1)=\frac{\pi}{2}. \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.13), we determine

$$C_0=\frac{\pi}{2}.$$

So (2.12) becomes to

$$\widetilde{\Gamma}(x+1)\widetilde{\Gamma}(x)=\frac{\pi}{2x}$$

Inductively, we have

$$\left(\widetilde{\Gamma}(x+n)\right)^{(-1)^n}=\prod_{k=0}^{n-1}\left(\frac{2(x+k)}{\pi}\right)^{(-1)^k}\widetilde{\Gamma}(x),$$

which is the desired result. □

From the above result, we get the special values of $\widetilde{\Gamma}(x)$ at the integers.

Theorem 2.11 (Special values)**.** *For $n \in \mathbb{N}$, we have*

(1)

$$\widetilde{\Gamma}(n) = \begin{cases} \dfrac{(n-2)!!}{(n-1)!!} & \text{if } n \text{ is even,} \\ \dfrac{(n-2)!!}{(n-1)!!} \cdot \dfrac{\pi}{2} & \text{if } n \text{ is odd.} \end{cases} \tag{2.16}$$

(2)

$$\widetilde{\Gamma}(-n) = \begin{cases} \infty & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{2.17}$$

Remark 2.12. Recall the special values for the classical gamma function $\Gamma(x)$:

$$\Gamma(n) = (n-1)!$$

(see [3, Proposition 9.6.14]).

Proof of Theorem 2.11. (1) We first consider the case for n being odd. If $n = 3$, then by (2.10) and Lemma 2.1, we have

$$\widetilde{\Gamma}(3) = \frac{1}{2}\widetilde{\Gamma}(1) = \frac{(3-2)!!}{(3-1)!!} \cdot \frac{\pi}{2}.$$

Suppose that the result has been established for $n-2$, that is,

$$\widetilde{\Gamma}(n-2) = \frac{(n-4)!!}{(n-3)!!} \cdot \frac{\pi}{2}.$$

Then by Theorem 2.6, we have

$$\begin{aligned} \widetilde{\Gamma}(n) &= \frac{n-2}{n-1} \cdot \widetilde{\Gamma}(n-2) \\ &= \frac{n-2}{n-1} \cdot \frac{(n-4)!!}{(n-3)!!} \cdot \frac{\pi}{2} \\ &= \frac{(n-2)!!}{(n-1)!!} \cdot \frac{\pi}{2}. \end{aligned} \tag{2.18}$$

Now we consider even n. If $n = 2$, then by Theorem 2.3 we have

$$\widetilde{\Gamma}(2) = \frac{1}{2}\mathrm{B}\left(1, \frac{1}{2}\right) = 1 = \frac{0!!}{1!!}.$$

Suppose that the result has been established for $n-2$, that is,

$$\widetilde{\Gamma}(n-2) = \frac{(n-4)!!}{(n-3)!!}.$$

Then by Theorem 2.6, we have

$$\begin{aligned}\widetilde{\Gamma}(n) &= \frac{n-2}{n-1}\cdot\widetilde{\Gamma}(n-2)\\ &= \frac{n-2}{n-1}\cdot\frac{(n-4)!!}{(n-3)!!}\\ &= \frac{(n-2)!!}{(n-1)!!}.\end{aligned} \tag{2.19}$$

Combing (2.18) and (2.19), we get (2.16).

(2) Recall that the classical gamma function $\Gamma(x)$ has simple poles just at the non-positive integers (see [3, Proposition 9.6.19]). For n being even, that is $n = 2m$ ($m \in \mathbb{N}$), we have

$$\widetilde{\Gamma}\left(-\frac{n}{2}\right) = \widetilde{\Gamma}(-m) = \infty,$$

and by Theorem 2.14,

$$\widetilde{\Gamma}(-n) = \widetilde{\Gamma}(-2m) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(-m)}{\Gamma\left(-m+\frac{1}{2}\right)} = \infty. \tag{2.20}$$

For n being odd, that is $n = 2m+1$ ($m \in \mathbb{N}$), by Theorem 2.14 we have

$$\widetilde{\Gamma}(-n) = \widetilde{\Gamma}(-2m-1) = \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(-m-\frac{1}{2}\right)}{\Gamma(-m)} = 0. \tag{2.21}$$

Combing (2.20) and (2.21), we get (2.17). □

It is known that the classical gamma function $\Gamma(x)$ is log-convex. And according to Artin [1, p. 7], a function $f(x)$ defined on an interval is log-convex (or weakly log-convex) if the function $\log f(x)$ is convex (or weakly convex). Now we show the weakly log-convexity of $\widetilde{\Gamma}(x)$.

Theorem 2.13 (Weakly log-convexity)**.** *For $x > 0$, $\widetilde{\Gamma}(x)$ is weakly log-convex.*

Proof. Since for $x > 0$, we have $\Gamma(x) > 0$, thus by Corollary 2.5,

$$\widetilde{\Gamma}(x) = \frac{\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{x+1}{2}\right)} > 0.$$

Recall that (see (1.15))

$$\widetilde{\psi}(x) = \frac{d}{dx}\log\widetilde{\Gamma}(x).$$

Since

$$\widetilde{\psi}(x) = -\frac{1}{x} + \widetilde{\gamma}_0 + \sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{k} - \frac{1}{k+x}\right) \tag{2.22}$$

(see [11, Theorem 3.12]), we have

$$\left(\frac{d}{dx}\right)^2 \log \widetilde{\Gamma}(x) = \frac{d}{dx}\widetilde{\psi}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+x)^2}. \tag{2.23}$$

Obviously, for any $x \in (0, +\infty)$,

$$\left|\frac{(-1)^k}{(k+x)^2}\right| = \frac{1}{(k+x)^2} < \frac{1}{k^2}, \quad k \in \mathbb{N}.$$

Thus by Weierstrass' test, the series

$$S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+x)^2}$$

uniformly convergent for $x \in (0, +\infty)$. Denote $S_n(x)$ by the partial sum of $S(x)$, we have

$$\lim_{n\to\infty} S_n(x) = S(x).$$

Now considering the even terms of $S_n(x)$, for $x > 0$, we have

$$\begin{aligned} S_{2m-1}(x) &= \left(\frac{1}{x^2} - \frac{1}{(x+1)^2}\right) + \cdots + \left(\frac{1}{(x+2m-2)^2} - \frac{1}{(x+2m-1)^2}\right) \\ &> 0 \end{aligned}$$

and

$$S(x) = \lim_{m\to\infty} S_{2m-1}(x) \geq 0.$$

Thus by (2.23), we see that

$$\left(\frac{d}{dx}\right)^2 \log \widetilde{\Gamma}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+x)^2} = S(x) \geq 0$$

and $\widetilde{\Gamma}(x)$ is weakly log-convex. □

We have the following duplication formula for $\widetilde{\Gamma}(x)$.

Theorem 2.14 (Duplication formula)**.**

$$\widetilde{\Gamma}(2x) = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(x)}{\Gamma\left(x + \frac{1}{2}\right)}.$$

Remark 2.15. Recall the duplication formula for the classical gamma function $\Gamma(x)$:

$$\Gamma(x)\Gamma\left(x + \frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x-1}}\Gamma(2x)$$

(see [3, Proposition 9.6.33]).

Proof of Theorem 2.14. From Corollary 2.5, we have

$$\widetilde{\Gamma}(2x) = \frac{\Gamma(x)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(x + \frac{1}{2}\right)} = \frac{\sqrt{\pi}\,\Gamma(x)}{2\,\Gamma\left(x + \frac{1}{2}\right)}$$

by noticing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. □

We have the following reflection equation for $\widetilde{\Gamma}(x)$.

Theorem 2.16 (Reflection equation)**.** *For* $0 < \mathrm{Re}(x) < 1$*, we have*

$$\frac{\widetilde{\Gamma}(x)}{\widetilde{\Gamma}(1-x)} = \cot\left(\frac{\pi x}{2}\right). \tag{2.24}$$

Remark 2.17. Recall the reflection equation for the classical gamma function $\widetilde{\Gamma}(x)$:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \tag{2.25}$$

(see [3, Proposition 9.6.34]).

Proof of Theorem 2.16. From Corollary 2.5 and the reflection formula of $\Gamma(x)$ (2.25), we have

$$\begin{aligned}
\frac{\widetilde{\Gamma}(x)}{\widetilde{\Gamma}(1-x)} &= \frac{1}{2}\frac{\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{x+1}{2}\right)} \cdot 2\frac{\Gamma\left(\frac{1-x}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{1-x}{2}\right)\Gamma\left(\frac{1}{2}\right)} \\
&= \frac{\Gamma\left(\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)}{\Gamma\left(\frac{x+1}{2}\right)\Gamma\left(1-\frac{x+1}{2}\right)} \\
&= \cot\left(\frac{\pi x}{2}\right).
\end{aligned}$$

This is the desired assertion. □

3. The properties of $\widetilde{\psi}(x)$

In this section, we will investigate the properties of $\widetilde{\psi}(x)$. First, we state its recursive formula and reflection equation.

Theorem 3.1 (Recursive formula and reflection equation)**.** *For* $\mathrm{Re}(x) > 0$ *and* $n \in \mathbb{N}$*, we have the recursive formula*

$$\widetilde{\psi}(x+n) = \begin{cases} \widetilde{\psi}(x) + \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k}{x+k} & \text{if } n \text{ is even,} \\ -\widetilde{\psi}(x) + \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{x+k} & \text{if } n \text{ is odd,} \end{cases} \tag{3.1}$$

and for $0 < \mathrm{Re}(x) < 1$*, we have the reflection equation*

$$\widetilde{\psi}(x) + \widetilde{\psi}(1-x) = -\frac{\pi}{\sin \pi x}. \tag{3.2}$$

Remark 3.2. Recall the recursive formula and the reflection equation for the classical digamma function $\psi(x)$:

$$\psi(x+n) = \psi(x) + \sum_{k=0}^{n-1} \frac{1}{x+k},$$

$$\psi(x) - \psi(1-x) = -\pi \cot(\pi x).$$

See [3, Proposition 9.6.41(3)]) and [3, Proposition 9.6.41(5)], respectively.

Proof of Theorem 3.1. If n is even, then by Theorem 2.9 we have

$$\widetilde{\Gamma}(x+n) = \frac{x+n-2}{x+n-1} \cdot \frac{x+n-4}{x+n-3} \cdot \cdots \cdot \frac{x}{x+1} \cdot \widetilde{\Gamma}(x) \tag{3.3}$$

and

$$\begin{aligned} \log \widetilde{\Gamma}(x+n) &= (\log(x+n-2) + \cdots + \log x) \\ &\quad - (\log(x+n-1) + \cdots + \log(x+1)) + \log \widetilde{\Gamma}(x). \end{aligned} \tag{3.4}$$

Recall that

$$\widetilde{\psi}(x) = \frac{d}{dx} \log \widetilde{\Gamma}(x)$$

(see (1.15)). Derivating both sides of (3.4), we get

$$\begin{aligned} \widetilde{\psi}(x+n) &= \frac{1}{x+n-2} + \cdots + \frac{1}{x} - \left(\frac{1}{x+n-1} + \cdots + \frac{1}{x+1} \right) + \widetilde{\psi}(x) \\ &= \widetilde{\psi}(x) + \sum_{k=0}^{n-1} \frac{(-1)^k}{x+k}. \end{aligned} \tag{3.5}$$

Now we consider the odd n. By Theorem 2.9, we have

$$\widetilde{\Gamma}(x+n) = \frac{x+n-2}{x+n-1} \cdot \frac{x+n-4}{x+n-3} \cdot \cdots \cdot \frac{x+1}{x+2} \cdot \widetilde{\Gamma}(x+1)$$

and

$$\begin{aligned} \log \widetilde{\Gamma}(x+n) &= (\log(x+n-2) + \cdots + \log(x+1)) \\ &\quad - (\log(x+n-1) + \cdots + \log(x+2)) + \log \widetilde{\Gamma}(x+1). \end{aligned} \tag{3.6}$$

By (2.9),

$$\log \widetilde{\Gamma}(x+1) + \log \widetilde{\Gamma}(x) = \log \left(\frac{\pi}{2x} \right).$$

Substitute to (3.6) we get

$$\begin{aligned} \log \widetilde{\Gamma}(x+n) &= (\log(x+n-2) + \cdots + \log(x+1)) \\ &\quad - (\log(x+n-1) + \cdots + \log(x+2)) + \log \left(\frac{\pi}{2x} \right) - \log \widetilde{\Gamma}(x). \end{aligned}$$

Derivating both sides of the above equation, we have

$$\begin{aligned} \widetilde{\psi}(x+n) &= \frac{1}{x+n-2} + \cdots + \frac{1}{x+1} - \left(\frac{1}{x+n-1} + \cdots + \frac{1}{x+2} \right) - \frac{1}{x} - \widetilde{\psi}(x) \\ &= \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{x+k} - \frac{1}{x} - \widetilde{\psi}(x) \\ &= -\widetilde{\psi}(x) + \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{x+k}. \end{aligned} \tag{3.7}$$

Combing (3.5) and (3.7), we obtain (3.1). For $0 < \mathrm{Re}(x) < 1$, by the reflection equation of $\widetilde{\Gamma}(x)$ (see (2.24)), we have

$$\log \widetilde{\Gamma}(x) - \log \widetilde{\Gamma}(1-x) = \log \cot\left(\frac{\pi x}{2}\right).$$

Derivating both sides of the above equation, we get

$$\widetilde{\psi}(x) + \widetilde{\psi}(1-x) = -\frac{\pi}{\sin(\pi x)},$$

which is (3.2). □

The following result is [11, Corollary 3.14], which gives the special values of $\widetilde{\psi}(x)$ at the positive integers. We state it here for completeness.

Theorem 3.3 (Special values at the positive integers)**.** *For $n \in \mathbb{N}$, we have*

$$\widetilde{\psi}(n) = \begin{cases} \widetilde{\gamma}_0 + \displaystyle\sum_{k=1}^{n-1} \frac{(-1)^k}{k} & \text{if } n \text{ is even,} \\ -\widetilde{\gamma}_0 + \displaystyle\sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k} & \text{if } n \text{ is odd,} \end{cases}$$

where γ_0 is the Euler constant respect to $\zeta_E(z,x)$ (see (1.14)).

Remark 3.4. Recall the special values for the classical digamma function $\psi(x)$ at the positive integers (see [3, Proposition 9.6.41(3)]):

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} \frac{1}{k} = -\gamma + H_{n-1},$$

where γ is the classical Euler constant and H_n is the harmonic series.

By the recursive formula (3.1), we immediately get the following result on the special values of $\widetilde{\psi}(x)$ at the rational numbers.

Theorem 3.5 (Special values at the rational numbers)**.** *Let $p, q \in \mathbb{Z}$ and $q \neq 0$, $j = \frac{p}{q}$, we have*

$$\widetilde{\psi}(n+j) = \begin{cases} \widetilde{\psi}\left(\dfrac{p}{q}\right) + \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k q}{qk+p} & \text{if } n \text{ is even,} \\ -\widetilde{\psi}\left(\dfrac{p}{q}\right) + \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k q}{qk+p} & \text{if } n \text{ is odd.} \end{cases}$$

Remark 3.6. Recall the corresponding result on the special values of the classical digamma function $\psi(x)$ at the rational numbers:

$$\psi(n+j) = \psi\left(\frac{p}{q}\right) + \sum_{k=0}^{n-1} \frac{q}{p+kq}.$$

The following is [11, (2.5)], which gives the integral representation of $\widetilde{\psi}(x)$. We state it here for completeness.

Theorem 3.7 (Integral representation)**.** *For $x > 0$, we have*

$$\begin{aligned}\widetilde{\psi}(x) &= \widetilde{\gamma}_0 + \int_0^\infty \frac{-e^{-t} - e^{-xt}}{1+e^{-t}} dt \\ &= \widetilde{\gamma}_0 - \int_0^1 \frac{1+t^{x-1}}{1+t} dt \\ &= -\int_0^\infty \frac{e^{-xt}}{1+e^{-t}} dt,\end{aligned}$$

where γ_0 is the Euler constant respect to $\zeta_E(z,x)$ (see (1.14)).

Remark 3.8. Recall the integral representation for the classical digamma function $\psi(x)$ (see [3, Proposition 9.6.43]):

$$\begin{aligned}\psi(x) &= -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1-e^{-t}} dt \\ &= -\gamma + \int_0^1 \frac{1-t^{x-1}}{1-t} dt \\ &= \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right) dt,\end{aligned}$$

where γ is the classical Euler constant.

4. Lerch-type formula

In [11], the authors first designed the modified Stieltjes constants $\widetilde{\gamma}_k(x)$ from the Taylor expansion of $\zeta_E(z,x)$ at $z=1$,

$$\zeta_E(z,x) = \sum_{k=0}^\infty \frac{(-1)^k \widetilde{\gamma}_k(x)}{k!} (z-1)^k \tag{4.1}$$

(see [11, (1.18)]), then defined the corresponding digamma function $\widetilde{\psi}(x)$ by

$$\widetilde{\psi}(x) := -\widetilde{\gamma}_0(x) = -\zeta_E(1,x). \tag{4.2}$$

(see [11, (1.22)]), which is essentially Nilsen's beta function $\beta(x)$ (see (1.19)). And the corresponding gamma function $\widetilde{\Gamma}(x)$ is defined from the differential equation (see [11, p. 5])

$$\widetilde{\psi}(x) = \frac{d}{dx} \log \widetilde{\Gamma}(x), \tag{4.3}$$

which is in analogy to the classical formula (1.5). In this section, we will show that Lerch's idea on defining of the classical gamma function $\Gamma(x)$ as a derivative of the Hurwitz zeta function $\zeta(z,x)$ at $z=0$ (see (1.3)) is also applicable in this case. That is, we shall prove that

$$\log \widetilde{\Gamma}(x) = \zeta_E'(0,x) + \zeta_E'(0). \tag{4.4}$$

First we need the following lemma.

Lemma 4.1. *For* $\mathrm{Re}(x) > 0$*, we have*

$$\zeta_E'(0,x) = \log\Gamma\left(\frac{x}{2}\right) - \log\Gamma\left(\frac{x+1}{2}\right) - \frac{1}{2}\log 2 \tag{4.5}$$

and

$$\zeta_E'(0) = \log\sqrt{\frac{\pi}{2}}. \tag{4.6}$$

Proof. For $\mathrm{Re}(x) > 0$, from the definition of the Hurwitz zeta function $\zeta(z,x)$ (1.2), we have

$$\zeta\left(z,\frac{x}{2}\right) = \sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{x}{2}\right)^z} = \sum_{n=0}^{\infty}\frac{2^z}{(2n+x)^z}$$

and

$$\zeta\left(z,\frac{x+1}{2}\right) = \sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{x+1}{2}\right)^z} = \sum_{n=0}^{\infty}\frac{2^z}{(2n+1+x)^z}.$$

Then by comparing definition of the alternating Hurwitz zeta function $\zeta_E(z,x)$ (1.8), we get

$$\zeta_E(z,x) = 2^{-z}\left(\zeta\left(z,\frac{x}{2}\right) - \zeta\left(z,\frac{x+1}{2}\right)\right).$$

Finally, by derivating both sides of the above equation and applying Lerch's formula (1.3), we see that

$$\zeta_E'(0,x) = \log\Gamma\left(\frac{x}{2}\right) - \log\Gamma\left(\frac{x+1}{2}\right) - \frac{1}{2}\log 2,$$

which is (4.5).

Since for $\mathrm{Re}(z) > 0$, we have

$$\zeta_E(z) = \zeta_E(z,1).$$

Then from the above equation and noticing that $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we get

$$\begin{aligned}\zeta_E'(0) &= \zeta_E'(0,1)\\ &= \log\Gamma\left(\frac{1}{2}\right) - \log\Gamma(1) - \frac{1}{2}\log 2\\ &= \log\sqrt{\frac{\pi}{2}},\end{aligned}$$

which is (4.6). □

Theorem 4.2. *For* $\mathrm{Re}(x) > 0$*, we have*

$$\log\widetilde{\Gamma}(x) = \zeta_E'(0,x) + \zeta_E'(0). \tag{4.7}$$

Proof. From Corollary 2.5, we have

$$\begin{aligned}\log\widetilde{\Gamma}(x) &= \log\left(\frac{1}{2}\right) + \log\Gamma\left(\frac{1}{2}\right) + \log\Gamma\left(\frac{x}{2}\right) - \log\Gamma\left(\frac{x+1}{2}\right)\\ &= -\log 2 + \frac{1}{2}\log\pi + \log\Gamma\left(\frac{x}{2}\right) - \log\Gamma\left(\frac{x+1}{2}\right)\end{aligned}$$

by noticing that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Then applying (4.5) and (4.6), we obtain

$$\begin{aligned}\log\widetilde{\Gamma}(x) &= \zeta_E'(0,x) + \frac{1}{2}\log\pi - \frac{1}{2}\log 2\\ &= \zeta_E'(0,x) + \log\sqrt{\frac{\pi}{2}}\\ &= \zeta_E'(0,x) + \zeta_E'(0),\end{aligned} \tag{4.8}$$

which is the desired result. □

5. Derivatives of partial zeta functions

In number theory, Hilbert's 12th problem is a problem on the explicit constructions of the abelian extensions for a given number fields. It is generally believed that the class fields may be explicit constructed by adding the special values of analytic functions. Let K/k be a finite abelian extension of number fields with Galois group $G = \mathrm{Gal}(K/k)$. Let S be a finite set of places of k containing the archimedean and the ramified places of k. In 1970s, Stark [18, 19] formulated conjectures on explicit constructions of class fields of K/k from the derivatives of L-functions $L_{K/k}(z,\chi)$ or the S-partial zeta function $\zeta_{K/k,S}(z,\sigma)$ at $z=0$, where $\sigma \in G$. Let T be another finite and non-empty sets of places of k satisfying $S \cap T = \emptyset$. Then Gross [6] gave a (S,T)-refinement of Stark's conjecture by considering the derivative of the (S,T)-partial zeta function $\zeta_{K/k,S,T}(z,\sigma)$ at $z=0$ (also see [21, Conjecture 2.8]).

In this section, let m be an odd positive integer, $K = \mathbb{Q}(\zeta_m)$ be the mth cyclotomic field and $K^+ = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ be its maximal real subfield. Let $G = \mathrm{Gal}(K^+/\mathbb{Q})$. Let $S = \{\infty, p|m\}$ and $T = \{2\}$. In what follows, by applying Lerch's formula (1.3) and its refinement (4.7), we give explicit calculations of the derivatives of the S-partial zeta function $\zeta_{K^+/\mathbb{Q},S}(z,\sigma)$ and the the (S,T)-partial zeta function $\zeta_{K^+/\mathbb{Q},S,T}(z,\sigma)$ at $z=0$, where $\sigma \in G$.

The S-partial zeta function associated to $\sigma \in G = \mathrm{Gal}(K^+/\mathbb{Q})$ is defined by

$$\zeta_{K^+/\mathbb{Q},S}(z,\sigma) = \sum_{\substack{(n.m)=1\\(n,K^+/\mathbb{Q})=\sigma}} \frac{1}{n^z}, \tag{5.1}$$

where $(n, K^+/\mathbb{Q})$ denotes the Artin symbol for n in $K^+/\mathbb{Q}$. For $l \in \mathbb{N}$ and $(l,m)=1$, we have

$$(l, K/\mathbb{Q}) = \sigma_l : \ \zeta_m \longmapsto \zeta_m^l. \tag{5.2}$$

And for any $\sigma \in G = \mathrm{Gal}(K^+/\mathbb{Q})$, we have $\sigma = \sigma_{\pm l}$, for some $l \in \mathbb{N}$ with $0 < l < m$ and $(l, m) = 1$, that is,

$$\sigma = \sigma_{\pm l} : \ \zeta_m + \zeta_m^{-1} \longmapsto \zeta_m^l + \zeta_m^{-l}. \tag{5.3}$$

Thus

$$\begin{aligned}
\zeta_{K^+/\mathbb{Q},S}(z, \sigma_{\pm l}) &= \sum_{\substack{n=1 \\ n \equiv \pm l \pmod{m}}} \frac{1}{n^z} \\
&= \sum_{\substack{n=1 \\ n \equiv l \pmod{m}}} \frac{1}{n^z} + \sum_{\substack{n=1 \\ n \equiv -l \pmod{m}}} \frac{1}{n^z} \\
&= \sum_{k=0}^{\infty} \frac{1}{(km+l)^z} + \sum_{k=0}^{\infty} \frac{1}{(km+m-l)^z} \\
&= \frac{1}{m^z} \sum_{k=0}^{\infty} \frac{1}{\left(k + \frac{l}{m}\right)^z} + \frac{1}{m^z} \sum_{k=0}^{\infty} \frac{1}{\left(k + 1 - \frac{l}{m}\right)^z} \\
&= \frac{1}{m^z} \zeta\left(z, \frac{l}{m}\right) + \frac{1}{m^z} \zeta\left(z, 1 - \frac{l}{m}\right),
\end{aligned} \tag{5.4}$$

where $\zeta(s, x)$ is the Hurwitz zeta function defined in (1.2). So

$$\begin{aligned}
\zeta'_{K^+/\mathbb{Q},S}(0, \sigma_{\pm l}) &= (-\log m)\zeta\left(0, \frac{l}{m}\right) + \zeta'\left(0, \frac{l}{m}\right) \\
&\quad + (-\log m)\zeta\left(0, 1 - \frac{l}{m}\right) + \zeta'\left(0, 1 - \frac{l}{m}\right).
\end{aligned} \tag{5.5}$$

It is known that for $k \in \mathbb{N}$,

$$\zeta(1-k, x) = -\frac{B_k(x)}{k}, \tag{5.6}$$

where $B_k(x)$ is the kth Bernoulli polynomial defined by the generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

and

$$B_1(x) = x - \frac{1}{2}$$

(see [3, Corollary 9.6.10]). Thus

$$\zeta\left(0, \frac{l}{m}\right) = \frac{l}{m} - \frac{1}{2} \tag{5.7}$$

and

$$\zeta\left(0, 1 - \frac{l}{m}\right) = 1 - \frac{l}{m} - \frac{1}{2}. \tag{5.8}$$

Recall Riemann's formula

$$\zeta'(0) = -\frac{1}{2} \log 2\pi$$

(see [3, p. 78]). Then from Lerch's formula (1.3), we have

$$\zeta'\left(0,\frac{l}{m}\right)=\log\Gamma\left(\frac{l}{m}\right)-\frac{1}{2}\log 2\pi \tag{5.9}$$

and

$$\zeta'\left(0,1-\frac{l}{m}\right)=\log\Gamma\left(1-\frac{l}{m}\right)-\frac{1}{2}\log 2\pi. \tag{5.10}$$

So by substituting (5.8)–(5.10) into (5.5), we have

$$\zeta'_{K^+/\mathbb{Q},S}(0,\sigma_{\pm l})=\log\left(\Gamma\left(\frac{l}{m}\right)\Gamma\left(1-\frac{l}{m}\right)\right)-\log 2\pi \tag{5.11}$$

and from the reflection equation of $\Gamma(x)$ (2.25), we further get that

$$\zeta'_{K^+/\mathbb{Q},S}(0,\sigma_{\pm l})=-\log\left(2\sin\frac{\pi l}{m}\right). \tag{5.12}$$

If T be another finite set of finite places of $\mathbb{Q}$, then for $\sigma\in G=\mathrm{Gal}(K^+/\mathbb{Q})$, the (S,T)-partial zeta function $\zeta_{K^+/\mathbb{Q},S,T}(z,\sigma)$ is defined by the following equality in $\mathbb{C}[\mathrm{Gal}(K^+/\mathbb{Q})]$

$$\begin{aligned}&\sum_{\sigma\in\mathrm{Gal}(K^+/\mathbb{Q})}\zeta_{K^+/\mathbb{Q},S,T}(z,\sigma)\sigma^{-1}\\&\qquad=\prod_{\mathfrak{p}\in T}(1-\sigma_{\mathfrak{p}}^{-1}\mathrm{N}(\mathfrak{p})^{1-z})\sum_{\sigma\in\mathrm{Gal}(K^+/\mathbb{Q})}\zeta_{K^+/\mathbb{Q},S}(z,\sigma_{\pm l})\sigma^{-1},\end{aligned} \tag{5.13}$$

where $\sigma_{\mathfrak{p}}$ is the Frobenius automorphism associated to $\mathfrak{p}$ in $\mathrm{Gal}(K^+/\mathbb{Q})$. For details, we refer to [21, p. 2537].

Now let $T=\{2\}$, by (5.13) and (5.3), we have

$$\begin{aligned}\sum_{\sigma\in G}\zeta'_{K^+/\mathbb{Q},S,T}(z,\sigma)\sigma^{-1}&=\sum_{(l,m)=1}\zeta'_{K^+/\mathbb{Q},S,T}(z,\sigma_{\pm l})\sigma_{\pm l}^{-1}\\&=(1-\sigma_2^{-1}2^{1-z})\sum_{(l,m)=1}\zeta'_{K^+/\mathbb{Q},S}(z,\sigma_{\pm l})\sigma_{\pm l}^{-1}.\end{aligned} \tag{5.14}$$

Since m is odd, we have

$$\sum_{(l,m)=1}\zeta_{K^+/\mathbb{Q},S}(z,\sigma_{\pm l})\sigma_{\pm l}^{-1}=\sum_{(l,m)=1}\zeta_{K^+/\mathbb{Q},S}(z,\sigma_{2^{-1}\pm l})\sigma_{\pm 2^{-1}l}^{-1}.$$

Thus

$$\begin{aligned}\sum_{(l,m)=1}\zeta_{K^+/\mathbb{Q},S,T}(z,\sigma_{\pm l})\sigma_{\pm l}^{-1}&=(1-\sigma_2^{-1}\cdot 2^{1-z})\sum_{(l,m)=1}\zeta_{K^+/\mathbb{Q},S}(z,\sigma_{\pm l})\sigma_{\pm l}^{-1}\\&=\sum_{(l,m)=1}\zeta_{K^+/\mathbb{Q},S}(z,\sigma_{\pm l})\sigma_{\pm l}^{-1}\\&\quad-2^{1-z}\sum_{(l,m)=1}\zeta_{\mathbb{Q}(K^+/\mathbb{Q},S}(z,\sigma_{\pm 2^{-1}l})\sigma_2^{-1}\sigma_{\pm 2^{-1}l}^{-1}\\&=\sum_{(l,m)=1}\left(\zeta_{K^+/\mathbb{Q},S}(z,\pm\sigma_l)-2^{1-z}\zeta_{K^+/\mathbb{Q},S}(z,\sigma_{\pm 2^{-1}l})\right)\sigma_{\pm l}^{-1}.\end{aligned} \tag{5.15}$$

So the (S,T)-partial zeta function is

(5.16)

$$\begin{aligned}
\zeta_{K^+/\mathbb{Q},S,T}(z,\sigma_{\pm l}) &= \zeta_{K^+/\mathbb{Q},S}(z,\sigma_{\pm l}) - 2^{1-s}\zeta_{K^+/\mathbb{Q},S}(z,\sigma_{\pm 2^{-1}l}) \\
&= \sum_{\substack{n=1\\ n\equiv \pm l \pmod m}}^{\infty} \frac{1}{n^z} - \sum_{\substack{n=1\\ 2n\equiv \pm l \pmod m}}^{\infty} \frac{2}{(2n)^z} \\
&= \sum_{\substack{n=1\\ n\equiv l \pmod m}}^{\infty} \frac{1}{n^z} - \sum_{\substack{n=1\\ 2n\equiv l \pmod m}}^{\infty} \frac{2}{(2n)^z} \\
&\quad + \sum_{\substack{n=1\\ n\equiv -l \pmod m}}^{\infty} \frac{1}{n^z} - \sum_{\substack{n=1\\ 2n\equiv -l \pmod m}}^{\infty} \frac{2}{(2n)^z} \\
&= \Bigg(\sum_{\substack{n=1\\ 2n\equiv l \pmod m}}^{\infty} \frac{1}{(2n)^z} + \sum_{\substack{n=1\\ 2n-1\equiv l \pmod m}}^{\infty} \frac{1}{(2n-1)^z} \\
&\quad - \sum_{\substack{n=1\\ 2n\equiv l \pmod m}}^{\infty} \frac{2}{(2n)^z} \Bigg) \\
&\quad + \Bigg(\sum_{\substack{n=1\\ 2n\equiv -l \pmod m}}^{\infty} \frac{1}{(2n)^z} + \sum_{\substack{n=1\\ 2n-1\equiv -l \pmod m}}^{\infty} \frac{1}{(2n-1)^z} \\
&\quad - \sum_{\substack{n=1\\ 2n\equiv -l \pmod m}}^{\infty} \frac{2}{(2n)^z} \Bigg) \\
&= \sum_{\substack{n=1\\ n\equiv l \pmod m}}^{\infty} \frac{(-1)^{n-1}}{n^z} + \sum_{\substack{n=1\\ n\equiv -l \pmod m}}^{\infty} \frac{(-1)^{n-1}}{n^z}.
\end{aligned}$$

Since m is an odd integer, we have

$$\begin{aligned}
\sum_{\substack{n=1\\ n\equiv l \pmod m}}^{\infty} \frac{(-1)^{n-1}}{n^z} &= -\sum_{k=0}^{\infty} \frac{(-1)^{km+l}}{(km+l)^z} \\
&= \frac{(-1)^{l+1}}{m^z} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k+\frac{l}{m}\right)^z} \\
&= \frac{(-1)^{l+1}}{m^z} \zeta_E\left(z, \frac{l}{m}\right)
\end{aligned} \tag{5.17}$$

and

$$\sum_{\substack{n=1\\ n\equiv -l \pmod{m}}}^{\infty} \frac{(-1)^{n-1}}{n^z} = -\sum_{k=0}^{\infty} \frac{(-1)^{km+m-l}}{(km+m-l)^z}$$

$$= \frac{(-1)^l}{m^z} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k+1-\frac{l}{m}\right)^z} \tag{5.18}$$

$$= \frac{(-1)^l}{m^z} \zeta_E\left(z, 1-\frac{l}{m}\right),$$

where $\zeta_E(s,x)$ is the alternating Hurwitz zeta function defined in (1.8). Then substituting (5.17) and (5.18) into (5.16), we get

$$\zeta_{K^+/\mathbb{Q},S,T}(z,\sigma_{\pm l}) = \frac{(-1)^{l+1}}{m^z}\zeta_E\left(z,\frac{l}{m}\right) + \frac{(-1)^l}{m^z}\zeta_E\left(z,1-\frac{l}{m}\right).$$

Thus

(5.19)

$$\zeta'_{K^+/\mathbb{Q},S,T}(0,\sigma_{\pm l}) = (-\log m)(-1)^{l+1}\zeta_E\left(0,\frac{l}{m}\right) + (-1)^{l+1}\zeta'_E\left(0,\frac{l}{m}\right)$$
$$+ (-\log m)(-1)^l\zeta_E\left(0,1-\frac{l}{m}\right) + (-1)^l\zeta'_E\left(0,1-\frac{l}{m}\right).$$

Recall that for $k \in \mathbb{N}_0$,

$$\zeta_E(-k,x) = \frac{1}{2}E_k(x), \tag{5.20}$$

where $E_k(x)$ is the kth Euler polynomial defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{k=0}^{\infty} E_k(x)\frac{t^k}{k!}$$

and

$$E_0(x) = 1$$

(see [20, p. 520, (3.20)]). Thus

$$\zeta_E\left(0,\frac{l}{m}\right) = \zeta_E\left(0,1-\frac{l}{m}\right) = \frac{1}{2}. \tag{5.21}$$

By Lerch's formula for $\zeta_E(z,x)$ (see (4.7) and (4.8)), we have

$$\zeta'_E\left(0,\frac{l}{m}\right) = \log\widetilde{\Gamma}\left(\frac{l}{m}\right) - \log\sqrt{\frac{\pi}{2}} \tag{5.22}$$

and

$$\zeta'_E\left(0,1-\frac{l}{m}\right) = \log\widetilde{\Gamma}\left(1-\frac{l}{m}\right) - \log\sqrt{\frac{\pi}{2}}. \tag{5.23}$$

Then substituting (5.21)–(5.23) into (5.19), we get

$$\zeta'_{K^+/\mathbb{Q},S,T}(0,\sigma_{\pm l}) = (-1)^{l+1}\log\left(\frac{\widetilde{\Gamma}\left(\frac{l}{m}\right)}{\widetilde{\Gamma}\left(1-\frac{l}{m}\right)}\right).$$

Finally, from the reflection equation for $\widetilde{\Gamma}(x)$ (2.24), we further get that

$$\zeta'_{K^+/\mathbb{Q},S,T}(0,\sigma_{\pm l}) = (-1)^{l+1}\log\left(\cot\frac{\pi l}{2m}\right). \tag{5.24}$$

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