Cosmological sector of Loop Quantum Gravity: a Yang-Mills approach

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Abstract

In this manuscript, we address the issue of the loss of $SU(2)$ internal symmetry in Loop Quantum Cosmology and its relationship with Loop Quantum Gravity. Drawing inspiration from Yang-Mills theory and employing the framework of fiber bundle theory, we successfully identify a cosmological sector for General Relativity in Ashtekar variables, which preserves the $SU(2)$ structure of the theory and the diffeomorphism gauge symmetry. Additionally, we proceed to quantize it using the machinery of Loop Quantum Gravity, uncovering that the resulting spin-network states exhibit distinctive symmetry properties and encompass the usual states of Loop Quantum Cosmology.

1 Introduction

Loop Quantum Gravity (LQG) is one of the most promising proposals for the quantization of the gravitational field [56, 52, 54]. The classical setup involves a reformulation of General Relativity as an $SU(2)$ gauge theory [7, 8, 14, 44], enabling us to quantize the theory using well-known techniques shared with particle physics. The main result of this approach is a discretization of geometry, where geometrical operators such as area and volume provide a discrete spectrum [53].

The natural set to stress a quantum gravity proposal is cosmology. The application of the techniques of LQG to the cosmological framework gives rise to the so-called Loop Quantum Cosmology (LQC) [9, 12]. This theory has enjoyed significant success in implementing dynamics and predicting new phenomena [20, 21, 48, 11], most notably in solving the singularity problem through the development of the Big Bounce [10, 9, 40, 39].
Despite the results obtained in the quantum theory of the classically symmetry-reduced minisuperspace, this approach has faced criticism [23]. Specifically, bridging the gap between LQC and LQG, i.e. identifying a proper cosmological sector within the full theory, remains a significant challenge and is crucial to eventually connect LQG with observable phenomena. Considerable effort has been devoted to this task over the past several years [4, 3, 5, 16]. A major technical point contributing to the ambiguity relation to the full theory is the loss of

$SU(2)$ internal symmetry due to gauge fixing and the emergence of second-class constraints [35, 34, 33, 49].
An interesting proposal to recover the $SU(2)$ gauge symmetry was put forward by M. Bojowald [17, 18, 19, 22]. However, after a deeper analysis of non-diagonal models in canonical LQC [30], a recent study on the Gauss constraint in these general models indicates that the 'Abelianization' of LQC is a characteristic inherent to the minisuperspace framework, with $U(1)$ symmetries emerging even within that proposal [29].

In this paper, our objective is to establish a notion of the cosmological sector within the classical theory without undermining the minisuperspace. To achieve this goal, we employ tools from Yang-Mills theories, borrowed from differential topology, which necessitates dealing with fiber bundles and connections.
After briefly reviewing the Arnowitt-Deser-Misner (ADM) formulation and the Ashtekar variables in General Relativity, we introduce the mathematical tools first presented in [28], which are pertinent to our current investigation. Additionally, we revisit some general aspects of cosmological models and discuss the properties of the minisuperspace.
Next, we delve into establishing a proper mathematical framework for cosmological models, wherein Lie groups are identified as the foundation, serving as the base manifold. This serves as the groundwork for discussing the homogeneous property of a suitable principal bundle and the connections on it. Drawing inspiration from works such as [27, 24], we approach homogeneity in line with Wang's theorem [57]. We demonstrate that this notion is sufficient to provide a homogeneous geometry on the base manifold, expressed in terms of ADM variables.
After establishing the appropriate configuration space and identifying the cosmological sector, we proceed with the kinematical quantization process. Notably, we observe that the set of constraints mirrors that of LQG. Therefore, by employing the same methodology, we derive the spin-network states.
Within this highly symmetric framework, spin networks exhibit new properties not evident in LQG, thereby revealing a link between them and the quantum states of the canonical approach in LQC.

2 Canonical variables and homogeneous space

The starting point of the Loop Quantum Gravity, such as the description of cosmological models, is the ADM splitting. The spacetime $\mathcal{M}$ is supposed to have a Cauchy surface, which is topological Σ. For Geroch's theorem [38], the spacetime is diffeomorphic to $\mathbb{R} \times \Sigma$. In this setting, there exists a set of coordinates (t, x^i) adapted to the splitting, in which x^i are coordinates on Σ, while t lies in $\mathbb{R}$.
Such a description allows us to give a Hamiltonian formulation of General Relativity [6]. The phase space is constituted by a couple of symmetric tensors (q_{ij}, K_{ij}) on Σ with a set of constraints, called supermomentum constraint

$\mathcal{D}(q, K) = 0$ and superhamiltonian constraint $H(q, K) = 0$. The couples that satisfy the constraints can be interpreted as the metric and the extrinsic curvature of an embedding of Σ in a Ricci flat spacetime [32].

From the Hamiltonian formulation, General Relativity can be recast as a Yang-Mills-like theory [7, 8]. The starting point is the tetrad formulation and the ADM splitting, in which the main variables are the dreibein e^i_a defined by $\delta_{ab} = q_{ij} e^i_a e^j_b$. Using connection A^a_i and electric field $E^i_a = \sqrt{q} e^i_a$ variables, a new constraint appears. This constraint is the Gauss constraint $G(A, E) = \partial_i E^i_a + \epsilon_{abc} A^b_i E^i_c = 0$, typical of the $SU(2)$ gauge theories. Thus, these variables, called Ashtekar variables, allow us to interpret the gravitational theory as a $SU(2)$ gauge theory [14, 44].

2.1 Mathematical structure of Ashtekar variables

The mathematical structure of Yang-Mills theory is given by the principal bundle theory. We are going to analyze General Relativity in Ashtekar variables in this framework. For a more extensive and deeper analysis, we refer the reader to the review [28].
We suppose that the Riemannian manifold (Σ, q) is spinnable, namely admits a spin structure. Fix once and for all an orientation for Σ. A spin structure is a couple $(P^{Spin}(\Sigma), \bar{\rho})$, where $P^{Spin}(\Sigma)$ is the principal $SU(2)$-bundle (called spin bundle) and $\bar{\rho} : P^{Spin}(\Sigma) \to P^{SO}(\Sigma)$ is a double covering map equivariant with respect to the group action of the bundles. $P^{SO}(\Sigma)$ is the orthonormal frame bundle of $\mathrm{T}\Sigma$ and has a principal $SO(3)$-bundle structure. Its fiber on a point $x \in \Sigma$ is the collection of linear isometries from $\mathbb{R}^3$ to $\mathrm{T}_x\Sigma$, equivalently, is the collection of the orthonormal basis in $\mathrm{T}_x\Sigma$.

The dreibein, or triad, is the data of a (global) section $e : \Sigma \to P^{SO}(\Sigma)$. While the Ashtekar connection A is the local field of a connection on $P^{Spin}(\Sigma)$. Here, a connection on $P^{Spin}(\Sigma)$ is, roughly, a $\mathfrak{su}(2)$-valued 1-form $\bar{\omega}$ on $P^{Spin}(\Sigma)$ and its local field is its pullback via a section $s : \Sigma \to P^{Spin}(\Sigma)$, namely a $\mathfrak{su}(2)$-valued 1-form on Σ

$$A = s^* \bar{\omega}. \tag{2.1}$$

Notice that the dreibein e induces a (not unique) section $\bar{e} : \Sigma \to P^{Spin}(\Sigma)$ that satisfies $\bar{\rho}(\bar{e}) = e$. Moreover, there exists a one-to-one correspondence between connection ω on $P^{SO}(\Sigma)$ and $\bar{\omega}$ on $P^{Spin}(\Sigma)$, i.e. $\bar{\omega} = \bar{\rho}^* \omega$. Because of this correspondence, the local field A does not depend on the particular lift $\bar{e}$ chosen.

2.2 Cosmological models and minisuperspace

A cosmological model is the description of a spatially homogeneous (eventually isotropic) Universe, namely a realization of the cosmological hypothesis in the context of General Relativity.

Under this hypothesis, the induced Riemannian metric $q_{ij}(t,x)$ on the hypersurfaces at constant time Σ_t factorizes as

$$q_{ij}(t,x) = \eta_{IJ}(t)\omega^I_i(x)\omega^J_j(x). \tag{2.2}$$

Here, $\omega^I = \omega^I_i(x)dx^i$ are a set of three 1-forms on Σ, called left-invariant 1-forms, which satisfy

$$d\omega^I + \frac{1}{2} f^I_{JK}\omega^J \wedge \omega^K = 0. \tag{2.3}$$

Their dual vector fields ξ_I are the generators of an algebra:

$$[\xi_I, \xi_J] = f^K_{IJ}\xi_K. \tag{2.4}$$

This provides a huge simplification of the phase space. We can consider as configuration variables the homogeneous part of the metric tensor η_{IJ}, leaving out the dependence on the point x of Σ. In such a way, the phase space becomes finite-dimensional and we usually refer to it as *minisuperspace.*

The minisuperspace plays a major role also in Loop Quantum Cosmology. The Ashtekar variables admit a similar factorization

$$A^a_i(t,x) = \phi^a_I(t)\omega^I_i(x),\ E^i_a(t,x) = |\det\omega|\, p^I_a(t)\xi^i_I(x). \tag{2.5}$$

So, in the canonical approach, we quantize the homogeneous part [9, 22].
However, this approach leads to an Abelianization of the quantum theory [34, 22]; indeed, the Gauss constraint can be recast into three Abelian constraints [29]. As we shall discuss in the remainder of the manuscript, this characteristic severs the connection with LQG.

3 Classical gauge field theory

We want to find a picture of the Ashtekar variables in the cosmological setting using the prescriptions of the gauge theories and the language presented in Sec.2.1 and introduced by [28].
We aim to analyze the classical description and identify the cosmological sector of General Relativity in Ashtekar variables. To achieve this, we must articulate the cosmological hypothesis and the Landau description in a pure group-theoretic form.

3.1 Homogeneous hypothesis on Cauchy hypersurface

The homogeneous hypothesis compelled us to regard Σ as a homogeneous space.

Definition 3.1. A homogeneous space is a pair $(\mathcal{X}, K)$ with $\mathcal{X}$ a topological space and K a group that acts transitively on $\mathcal{X}$. In this case $\mathcal{X} \cong K/H$, where H is the stabilizer of a fixed point.

In cosmology, we require that (Σ, q) is a Riemannian homogeneous manifold, namely, that the isometry group $\mathrm{Isom}(\Sigma, q)$ acts transitively.
However, the formalism presented in Sec.2.2 and proposed by L.Landau (c.f. [46]) describes the couple (Σ, q) as a 3-dimensional Lie group G equipped with a left-invariant metric η. In this case, we have a copy of G into the isometry group $G_L \lhd \mathrm{Isom}(G, \eta)$.
Moreover, considering a Lie group with left-invariant metric instead of a generic Riemannian homogeneous manifold has a well-posed mathematical aspect. Every class A [1] simply connected Riemannian homogeneous manifold admits a Lie group structure, hence it is isometric to a suitable group G equipped with a left-invariant metric η [1]. Furthermore, from simply connected Lie groups we can generate a 3-dimensional Lie group with non-trivial topology considering the quotient by a normal subgroup (two interesting examples are $\mathbb{R}^3/\mathbb{Z}^3 = \mathbb{T}^3$ and $SU(2)/\mathbb{Z}^2 = SO(3)$). In addition, there exist Lie groups associated with class B [2] Riemannian homogeneous manifolds.
Finally, the description through Lie groups is quite general and includes all the physically relevant models.

Once we deal with the couple (G, η), we need to discuss which group acts on this Riemannian manifold. In fact, on the same Riemannian manifold, different groups can act via isometries. In our case, on (G, η), there are two relevant groups: the group of orientation-preserving isometries $S = \mathrm{SIsom}(G, \eta)$ or the copy of G in it: G_L, that acts on G via left multiplication. We choose the smaller group of S that acts transitively on G, which is G_L. Then, our homogeneous space is the couple (G, G_L). This choice will be clear later.

Recalling the Definition 3.1 for homogeneous space, we notice that this choice does not break the diffeomorphism gauge symmetry. Consider a diffeomorphism $\varphi : \Sigma \to G$, and let η a left-invariant metric on G, the Riemannian manifold $(\Sigma, \varphi^*\eta)$ still admits a transitive action of G_L via isometries. Actually, let us consider $L_g \in \mathrm{Isom}(G, \eta)$, thus, $\varphi^{-1} \circ L_g \circ \varphi \in \mathrm{Isom}(\Sigma, \varphi^*\eta)$, which clearly defines an (equivalent) action of G_L on Σ. This discussion can be extended to all elements of the isometry group, hence $\mathrm{Isom}(\Sigma, \varphi^*\eta) = \varphi^{-1} \circ \mathrm{Isom}(G, \eta) \circ \varphi \cong \mathrm{Isom}(G, \eta)$. Moreover, in the whole paper, we deal only with geometrical objects and topological tools, so, the diffeomorphism gauge symmetry is always preserved.

3.2 Homogeneous spin bundle

We aim to analyze the properties required by the Ashtekar variables to understand a spatially homogeneous Universe. We aspire to utilize the Yang-Mills approach, where the connection is desired to be homogeneous in the sense of Wang's theorem. Thus, we need to begin studying the homogeneity of the corresponding principal bundle.

[1] A homogeneous Riemannian space is class A if it admits a compact quotient. If the space is a Lie group, it is class A if it is unimodular [51].

[2] A homogeneous Riemannian space is class B if it is not class A.

Considering the formulation in [28], we must work with the orthonormal frame bundle $P^{SO}(G)$. It is defined as the disjoint union $P^{SO}(G) := \bigsqcup_{g\in G} P^{SO}_g(G)$, where

$$P^{SO}_g(G) = \{h : \mathbb{R}^3 \to \mathrm{T}_g G \,|\, h \textit{ orientation-preserving linear isometry}\}.$$

The group $S = \mathrm{SIsom}(G, \eta)$ acts freely and transitive via automorphisms on $P^{SO}(G)$. In fact, let $f \in S$, the action

$$\begin{aligned} L_f : P^{SO}_g(G) &\to P^{SO}_{f(g)}(G); \\ h &\mapsto d_g f \circ h, \end{aligned}$$

preserves the principal bundle structure. Hence, the orthonormal frame bundle is homogeneous with respect to S (properly, it is S-invariant) and, so, the desirable connections would be too. Remark that $P^{SO}(G)$ is also G_L-invariant.

However, in the Ashtekar formulation, we are interested in the involvement of classical spinors. Hence, we must include in our analysis the spin bundle $P^{Spin}(G)$ (cf. [13, 47]).

Remark 3.2. On a homogeneous space (G, G_L) equipped with a left-invariant metric, always exists a unique G-invariant spin structure.

The previous property clearly does not hold for every homogeneous space, even with the same base manifold. The example of the 3-sphere $\mathbb{S}^3$ is enlightening [36, 2]. Considering the 3-sphere as a group $\mathbb{S}^3 = SU(2)$, in this case there exists a $SU(2)$-invariant spin structure, while, if $\mathbb{S}^3 \cong SO(4)/SO(3)$, a $SO(4)$-invariant spin structure does not exist. In our language, $\Sigma = \mathbb{S}^3 \cong SU(2) = G$ with the action of the groups $G_L = SU(2)$ and $S = SO(4)$, respectively.

Thus, we will examine the Riemannian Lie group G, the orthonormal frame bundle $P^{SO}(G)$, and the spin structure $P^{Spin}(G)$ equipped and invariant under the action of G_L.

3.3 Homogeneous Ashtekar variables

Such as the homogeneous space (G, G_L), we require the Ashtekar connection A to be G_L-invariant.
Motivated by the approach in Yang-Mills theory [27, 55] and in previous works in Loop Quantum Gravity [24], we demand that the associated connection 1-form ω^A on $P^{SO}(G)$ is G_L-invariant, namely $L_g^*\omega^A = \omega^A$. The same property holds for the associated connection 1-form $\bar{\omega}^A$ on $P^{Spin}(G)$.
These properties ensure that the principal bundle and the Ashtekar connection satisfy the hypotheses of Wang's theorem:

Theorem 3.3 (Wang's theorem [57])**.** *Let P be a G-invariant principal K-bundle over a manifold $\mathcal{X}$, where $\mathcal{X} \cong G/H$ is a homogeneous space and H is*

the stabilizer of a point $x_0 \in \mathcal{X}$. Let $p_0 \in P$ be a point in the fiber over x_0, and let $\lambda : H \to K$ denote the isotropy homomorphism associated with p_0.
Then, the K-invariant connections ω on P are in one to one correspondence with linear maps $\Lambda : \mathfrak{g} \to \mathfrak{k}$ such that

1. $\Lambda \circ Ad_h = Ad_{\lambda(h)} \circ \Lambda$, *for all* $h \in H$,

2. $\Lambda|_{\mathfrak{h}} = d\lambda$.

The correspondence is given by

$$\Lambda(v) =: \omega_{p_0}(X_v)$$

where X_v is the vector field on P induced by $v \in \mathfrak{g}$.

In our framework, the theorem is simplified a lot. Our homogeneous space is a Lie group (G, G_L) and, so, the stabilizer contains only the identity element $H = \{1\}$. The principal bundle we are considering is $P^{Spin}(G)$ as $SU(2)$-principal bundle and it is G_L-invariant, such as the desirable connections. Hence, the theorem can be recast in a simplified version:

Corollary 3.4. *Let $P^{Spin}(G)$ be the unique G_L-invariant spin structure on G. Then, the homogeneous connections ω are in one-to-one correspondence with linear maps $\phi : \mathfrak{g} \to \mathfrak{su}(2)$.*

Notice that considering the orthonormal frame bundle $P^{SO}(G)$ as a $SO(3)$-principal bundle, due to the isomorphism $\mathfrak{so}(3) \cong \mathfrak{su}(2)$, we obtain the same classification for the G_L-invariant connections on $P^{SO}(G)$.
For such connections there exists a preferred trivialization $s : G \to P^{Spin}(G)$ such that the correspondent local gauge field can be written as

$$A = s^*\omega = \phi \circ \theta_{MC} \tag{3.1}$$

where θ_{MC} is the Maurer-Cartan form on $\mathfrak{su}(2)$ [43].

Once we have clarified the properties of the Ashtekar connection, we need to discuss the dreibein. The restriction we want to impose is that the reconstructed physical quantities, the Riemannian metric q and the extrinsic curvature K (i.e. the ADM variables), be homogeneous.
We interpret the dreibein as a section $e : G \to P^{SO}(G)$, since $P^{SO}(G)$ is G_L-invariant, then q is homogeneous. Because q is the metric associated to $P^{SO}(G)$ and so it is G-invariant on the whole manifold G.
It is useful to notice that there exists a G_L-invariant dreibein, namely a section such that the following diagram commutes

$$\begin{tikzcd} P^{SO}(G) \arrow[r, "L_f"] \arrow[d, shift left, "\pi"] & P^{SO}(G) \arrow[d, shift right, "\pi"'] \\ G \arrow[r, "f"] \arrow[u, bend left, "e"] & G \arrow[u, bend right, "e"'] \end{tikzcd}$$

Thus, the equation for the G_L-invariant dreibein is

$$e_{f(g)} = d_g f \circ e_g,\ \forall g \in G,\ f \in G_L, \tag{3.2}$$

where e_g is the section e evaluated on the point $g \in G$, which is an element of $P^{SO}_g(G)$. Moreover, given a G_L-invariant dreibein e, $e(\mathfrak{v})$ (the image of a vector $\mathfrak{v}$ in $\mathbb{R}^3$ via e) is a left-invariant vector field and, so, an element of the Lie algebra $\mathfrak{g}$. It is easy to prove that every section is gauge equivalent to a G_L-invariant section and that a G_L-invariant section is the preferred section in Eq.(3.1).
Recall that in [28] the extrinsic curvature is defined starting from the Weingarten map $W : \mathrm{T}G \to \mathrm{T}G$ as the symmetric bilinear form K on $\mathrm{T}_g G$ such that $K(v, w) = q(W(v), w)$ for all $v, w \in \mathrm{T}_g G$. We can obtain the Weingarten map from the Ashtekar connection A and the local Levi-Civita connection Γ associated with e by

$$W = \varphi([\bar{e}, A - \Gamma]), \tag{3.3}$$

where $\varphi : \mathrm{ad}P^{Spin}(G) \xrightarrow{\sim} TG$ is the natural vector bundle isomorphism.
We now verify that the extrinsic curvature is homogeneous. Let ω^A be a homogeneous connections, then $\Omega = \omega^A - \omega^{LC}$ is G_L-invariant because the Levi-Civita connection ω^{LC} is so. Let e be a G_L-invariant dreibein and fix $g \in G$ and $v, w \in T_g G$ and $f \in G_L$

$$\begin{aligned}
&(e^*\Omega)_{f(g)}(d_g f(v)) = \Omega_{e_{f(g)}}(d_{f(g)} e \circ d_g f(v)) = \Omega_{e_{f(g)}}(d_g(e \circ f)(v)) \\
&= \Omega_{d_g f \circ e_g}(d_g(d_g f \circ e)(v)) = \Omega_{L_f(e_g)}(d_g(L_f \circ e)(v)) \\
&= \Omega_{L_f(e_g)}(d_{e_g} L_f \circ d_g e(v)) = \Omega_{e_g}(d_g e(v)) \\
&= (e^*\Omega)_x(v).
\end{aligned}$$

Since φ is an isomorphism between homogeneous fiber bundles, it preserves the homogeneous structure. Let V be a $\mathbb{R}^3$-valued 1-form on G such that $\varphi([e, e^*\Omega]) = [e, V]$, hence $f^*V = V$ when e is G_L-invariant. Thus, the Weingarten map satisfies

$$W_{f(g)}(d_g f(v)) = [e_{f(g)}, V_{f(g)}(d_g f(v))] = [d_g f \circ e, V_g(v)] = d_g f(W_g(v)). \tag{3.4}$$

From which descend that

$$\begin{aligned}
(f^*K)_g(v, w) &= q_{f(g)}(W_{f(g)}(d_g f(v)), d_g f(w)) = q_{f(g)}(d_g f(W_g(v)), d_g f(w)) \\
&= q_g(W_g(v), w) = K_g(v, w).
\end{aligned} \tag{3.5}$$

Since the last expression is gauge independent, it holds for every choice of the dreibein. Hence, from generality of $g \in G$, $v, w \in \mathrm{T}_g G$ and $f \in G_L$, the extrinsic curvature is homogeneous.

The data of the classical cosmological sector of General Relativity in Ashtekar variables consists of an invariant connection ω together with a section e in the homogeneous orthonormal frame bundle $P^{SO}(G)$. In particular, the reduced phase space is composed of the local fields $A = e^*\omega$, referred to as the homogeneous Ashtekar connections, and the densitized dreibein E of e as defined in [28]. The collection of homogeneous Ashtekar connections $\mathcal{A}^G$ will play the role of the configuration space.

4 The abelian artifact and the gauge fixing

In this framework, generically, the holonomy maintains its value in $SU(2)$ (cf. Theorem 4.1 of [45]). Therefore, it is capable of reintroducing the $SU(2)$ internal degree of freedom lost in the canonical approach in LQC [33, 35, 34, 23]. Moreover, this formulation does not require a minisuperspace, allowing gauge transformations to possess local degrees of freedom that do not adhere to the homogeneous property, as necessary to have a non-Abelian nor identically vanishing Gauss constraint [30, 29].
Furthermore, we can recover the classical formulation in minisuperspace. The description of the classical Ashtekar variables in cosmology by M.Bojowald [17, 22] coincide with the restriction of the theory to G_L-invariant dreibeins, as can be seen from Eq.(3.1). While the canonical description [9, 20, 21] is essentially a gauge fixing, namely the choice of a peculiar G_L-invariant dreibein, as shown explicitly in the Appendix A for the isotropic case.

In a previous work [29], we found that, in the minisuperspace, the Gauss constraint loses the divergence term $\partial_i E^i_a$, and we showed how to recast it into three Abelian constraints. The equation $\partial_i E^i_a = 0$ is proper for the minisuperspace model. Actually, a G_L-invariant dreibein means that e_a are left-invariant vector fields, which have a constant covariant divergence $\text{div}(e_a) = const$. For class A Bianchi models, the constant is zero. Thus, we can interpret $\partial_i E^i_a = 0$ as the minisuperspace constraint.

5 Quantum states

In the phase space presented in Sec.3.3, the algebra of the constraints remains the same as that of Loop Quantum Gravity, maintaining their original form. Since no minisuperspace is required, not only does the Gauss constraint persist, but even the Diffeomorphism constraint does not vanish, thereby recovering the full set of constraints of LQG.
Beginning with this observation, in this section, we propose that the quantum states of the cosmological theory can still be expressed in terms of spin-network states but restricted to the classical cosmological sector $\mathcal{A}^G$.
These states exhibit some intriguing properties that highlight connections with the standard approach in Loop Quantum Cosmology.

5.1 Spin networks in homogeneous spaces

The space of spin networks as colored graphs has quite good properties which can be linked to the homogeneity.
Citing the definition of graph by [56]:

Definition 5.1. A graph γ is a collection of piecewise analytic, continuous, oriented curve $c : [0,1] \to G$ (c is an embedding in G of an open subset of $\mathbb{R}$ that contains $[0,1]$) which intersect each other at their endpoints.

We recall that the classical notion of invariant subset cannot be properly implemented. Let $\sigma \subset G$ be a subset, it is invariant if $f(\sigma) \subset \sigma$, $\forall f \in G_L$. Since G_L acts transitively, the orbit of a point is the whole space G, then, no proper invariant subsets exist.
However, a finer property can be found for curves on G. On a Lie group is well-defined the exponential map $\exp : \mathfrak{g} \to G$. Locally, this map is a diffeomorphism, specifically, there exists an open neighborhood $U \subset G$ of the identity element and a ball of radius δ, $\mathbb{B}_\delta(0) \subset \mathfrak{g}$ such that for all $g \in U$ there exists a unique $v \in \mathbb{B}_\delta(0)$ such that $g = \exp(v)$.
This property can be extended to all couples of close enough points. Let g_1 and g_2 be two elements of G such that $g_1^{-1}g_2 \in U$, then $\exists! v \in \mathbb{B}_\delta(0)$ such that $g_1^{-1}g_2 = \exp(v)$, hence $g_2 = g_1 \exp(v)$.
The object $\exp(v)$ can be interpreted as the endpoint of the integral curve of the left-invariant vector field associated with v. Thus, between two close enough points g_1 and g_2 always exist an analytic curve $\zeta : [0,1] \to G$ such that $\zeta(0) = g_1$ and $\zeta(1) = g_2$, such curve is $\zeta(t) = g_1 \exp(tv)$. With such curves, we can approximate every smooth curve.
Consider a partition of the $[0,1]$ interval $\mathscr{P}_n = \{0, t_1, \dots, t_n, 1\}$, if the partition is fine enough, the points $g_i = c(t_i)$ and $g_{i-1} = c(t_{i-1})$ will be close enough and, so, there exists an integral curve that connects them $\zeta_i(t) = g_{i-1}\exp(tv_i)$. Thus every curve $c : [0,1] \to G$ is approximated by a continuous, piecewise analytic curve $\zeta_{\mathscr{P}_n} : [0,1] \to G$ composed by integral curves:

$$\zeta_{\mathscr{P}_n}(t) = \begin{cases} \zeta_1(\frac{t}{t_1}) & \textit{if } 0 \leq t < t_1, \\ \zeta_2(\frac{t-t_1}{t_2-t_1}) & \textit{if } t_1 < t < t_2, \\ \vdots & \\ \zeta_{n+1}(\frac{t-t_n}{1-t_n}) & \textit{if } t_n < t \leq 1. \end{cases} \tag{5.1}$$

The finer the partition, the better the approximation. Thus, every smooth curve c can be realized as the limit of some sequence composed of integral curves. In this sense, the set of curves composed of integral curves is dense in the set of piecewise analytic continuous curves.
We can consider collections of this kind of curve without losing generality because of the dense property.

Definition 5.2. A homogeneous graph γ^G is a collection of integral curves of left-invariant vector fields and their compositions.

Such as the curves, the set of homogeneous graphs is dense in the set of graphs. Poorly speaking, a homogeneous graph is made starting from a graph and substituting at each curve its approximation in integral curves. (Notice that this definition differs from the usual mathematical notion of a homogeneous graph as used in [15].)
Now, we can consider spin-networks states as supported on homogeneous graphs only, and as cylindrical functions on $\mathcal{A}^G$.

Definition 5.3. Given a homogeneous graph γ^G, let a n-valent vertex v be a point of intersection of n curves in the graph. Label each curve c with a triple (j_c, m_c, n_c) where j is the weight of an irreducible representation of $SU(2)$ and m, n the roots of the representation ρ_j associated to j. The collection of this data is called a homogeneous spin network $\mathscr{S}$

Fixing a representation ρ_j for each weight j. The function

$$\psi_{\mathscr{S}} : \mathcal{A}^G \to \mathbb{C};\, A \mapsto \prod_{c \in \gamma^G} \left[\sqrt{2j_c + 1} \rho_{j_c}(h_c(A))_{m_c n_c} \right]$$

is called a homogeneous spin-network state. Here, $h_c(A)$ is the holonomy along the curve c with Ashtekar connection A.

5.2 Invariant spin-network states and holonomy

The implementation of the Gauss constraint at the quantum level leads to the invariant spin-network states. Thus, the spin networks are enriched by the data of an intertwiner in each vertex. The intertwiner I_v is a projection from the tensor product of the representations carried by each edge linked to the vertex v into the trivial representation $j = 0$.
We maintain this characterization of invariant spin-network states in the homogeneous construction obtaining two interesting properties.
The first one, there not exist 2-valent vertices, such as in the general theory. A 2-valent vertex has two edges carrying representations j_1 and j_2, respectively. For the existence of the intertwiner, must be $0 \in j_1 \otimes j_2 = (j_1 + j_2) \oplus (j_1 + j_2 - 1) \oplus \cdots \oplus |j_1 - j_2|$. Then, must be $j_1 = j_2$ and the two edges can be considered as part of the same edge carrying the representation $j_1 = j_2$ [56].
The second property holds particular significance in describing the cosmological states within the LQC framework and establishing a strong connection with existing literature.

Proposition 5.4. *The holonomy along an edge of an invariant homogeneous spin-network state is given by pointwise holonomy.*

Proof. Considering $c : [0, 1] \to G;\, t \mapsto c(t)$ an integral curve of a left-invariant vector field X associated with a vector $v = X(1) \in \mathfrak{g}$. Hence,

$$\dot{c}(t) := d_t c(\partial_t) = X(c(t)).$$

Let $u : [0,1] \to SU(2)$ be the parallel transport along the curve c, fixing e a G_L-invariant dreibein, the parallel transport equation (cf. Eq.(5.1) from [28]) reads

$$\dot{u}(t) = -A(\dot{c}(t))u(t) = -\phi \circ \theta_{MC}(X(c(t))u(t) = -\phi(v)u(t).$$

The solution of this equation is $u(t) = \exp(-\phi(v)t)$. In the way the holonomy is defined, it is the inverse of the parallel transport operator evaluated in $t = 1$

$$h_c(A) = u(t=1)^{-1} = \exp(\phi(v)).$$

We have recovered the pointwise holonomy in a fixed gauge. Let $a : G \to SU(2)$ implement a gauge transformation: $A \mapsto A' = \mathrm{Ad}_{a^{-1}} A + a^{-1} da$, the holonomy transforms

$$h_c(A') = a(c(0))h_c(A)a(c(1))^{-1}.$$

Suppose that the curve c is an edge that links two vertices v and v' of an invariant spin network, the intertwiners I_v and $I_{v'}$ project the terms $a(c(0))$ and $a(c(1))$ into the trivial representation. Thus, we can always consider the holonomy given only in the G_L-invariant gauge. $\square$

Consequently, the quantum states in LQC, commonly interpreted as cubic spin-network states, are naturally included within this approach.

6 Relation with the previous literature

As mentioned previously, our formulation encompasses the canonical approach to LQC at both the classical and quantum levels. The usual form of Ashtekar variables in cosmology arises as a result of gauge fixing in our description, while our Hilbert space incorporates the LQC states.
We posit that implementing the minisuperspace constraint $\partial_i E^i_a = 0$ at the quantum level yields the Hilbert space of canonical LQC, namely the LQC Hilbert space is the kernel of the $\partial_i E^i_a$ operator. Such an implementation should adhere to the treatment of the second-class constraint in Reduced Loop Gravity [35, 4, 3, 5].
Furthermore, interpreting the canonical approach as a gauge fixing aligns with previous works on minisuperspace reduction, where a set of constraints similar to our minisuperspace constraint is imposed [49, 50].

We anticipate that a more thorough examination of the Diffeomorphism constraint in this context may reveal some similarities to modern approaches to diffeomorphism-invariant cosmological sectors, which are currently successful but limited to the isotropic case [16].
Moreover, with the implementation of the Hamiltonian constraint in future work, we aim to enhance theoretical understanding of certain effective models for cosmological dynamics derived from LQG [25, 26, 31], where particular symmetric graphs are considered, aligning with the direction of this study.

7 Conclusion

The mathematical concept of homogeneity, along with a rigorous formulation of Ashtekar variables that align more closely with Yang-Mills theory, enables the identification of a classical cosmological sector of General Relativity using Ashtekar variables, without the need to invoke the minisuperspace. This approach maintains the $SU(2)$ internal gauge symmetry, akin to the diffeomorphism gauge symmetry. Consequently, the theory's constraints and their algebra mirror those of Loop Quantum Gravity. Thus, quantization can be performed, leading to spin-network states as cylindrical functions on the classical cosmological sector. Furthermore, these states have analogous characteristics to quantum states in Loop Quantum Cosmology, with the latter being included within the new Hilbert space.

This approach holds promise for studying the kinematical space and developing the dynamics. While the Hamiltonian constraint is correctly implemented in Loop Quantum Cosmology, it remains unsolved in canonical Loop Quantum Gravity. We believe that this highly symmetric framework can lead to a suitable Hamiltonian operator in a context closer to the full theory, providing a theoretical argument for interpreting the Big Bounce as a robust prediction of Loop Quantum Gravity.

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A Isotropic case

The isotropic case deserves a peculiar treatment. The isotropic hypothesis prevents finding a preferred direction even during the dynamics evolution of the metric and the extrinsic curvature [37].
In this case, the usual mathematical definition of isotropic manifold does not catch the whole feature we desire.

Definition A.1. A Riemannian manifold (Σ, q) is isotropic if, for any $x \in \Sigma$ and any unit vectors $v, w \in \mathrm{T}_p\Sigma$, there exists an isometry $f : \Sigma \to \Sigma$ with $f(x) = x$ such that $f_*(v) = w$.

This definition means that Σ is a homogeneous space and the stabilizer group is isomorphic to the group of rotations. Herewith, G admits an action of $H \cong SO(3) \subset S$ with a fixed point (the fixed point will be the identity element $1 \in G$). However, we can see in Table 1 from [42] that the full isometry group

of $\mathbb{R}^3$ equipped with any left-invariant metric has a $SO(3)$ subgroup. Moreover, for every two left-invariant metric q, q' on $\mathbb{R}^3$ always exists an automorphism $F \in \mathrm{Aut}(\mathbb{R}^3)$ such that $q = F^* q'$ and their stabilizers are conjugate under it, $H_q = \mathrm{Ad}_F H_{q'}$ [41]. Roughly speaking, let $Q = \{q_{IJ}\}$ and $Q' = \{q'_{IJ}\}$ be the matrices associated with the two metrics in a given basis of generators of the Lie algebra $\mathbb{R}^3 \cong \mathrm{T}_1\mathbb{R}^3$, then, there exists a matrix M such that $Q = MQ'M^t$. Hence, given a transformation Λ that leaves the metric Q invariant $\Lambda Q \Lambda^t = Q$, we get an invariant transformation for Q', namely $M\Lambda M^{-1}$. Thus, the stabilizer group H_q is conjugate to the stabilizer of the canonical scalar product of $\mathbb{R}^3$, i.e. $SO(3)$.
Thinking about the two different metrics as a time evolution, the stabilizer group evolves in time but is still isomorphic to $SO(3)$. Nevertheless, this can not be a description of an isotropic universe because of the generality of q and q'. Then, we have to restrict our definition to be the stabilizer group not isomorphic to $SO(3)$ but exactly $H = SO(3)$, corresponding to fix Q as a positive multiple of the identity matrix.

Definition A.2. An isotropic model is a Riemannian homogeneous space (G, S) together with a metric q such that the stabilizer is $H = SO(3) \subset S$.

In our description of class A simply connected Bianchi models, there are only two Lie groups available for the isotropic model, $\mathbb{R}^3$ and $\mathbb{S}^3$ [42]. So they are homogeneous spaces together with the action of $E_0^3 = \mathbb{R}^3 \rtimes SO(3)$ and $SO(4)$, respectively. However, $\mathbb{S}^3$ does not admit an $SO(4)$-invariant spin structure. We can avoid this problem considering connection on $P^{SO}(\mathbb{S}^3)$ because in the usual formulation they are in one-to-one correspondence with the connection in the $SU(2)$-principal bundle.
Hence, we require that the connection is invariant under the action of the chosen group.

For consistency, we will show that there exists a gauge in which we can write the Ashtekar connection in the canonical form [20].
Consider $\mathbb{R}^3$ equipped with a positive multiple of the canonical scalar product $\alpha^2\langle\cdot,\cdot\rangle$ and $P^{SO}(\mathbb{R}^3)$ together with the natural action of E_0^3. The fiber on a point $x \in \mathbb{R}^3$ is canonically $SO(3)$

$$P^{SO}(\mathbb{R}^3) = \{U : \mathbb{R}^3_{\langle\cdot,\cdot\rangle} \to \mathrm{T}_x\mathbb{R}^3 \cong \mathbb{R}^3_{\alpha^2\langle\cdot,\cdot\rangle} \mid \text{orientation preserving isometry}\} = \alpha \cdot SO(3)$$

The action of $f = tR \in E_0^3$ on $P^{SO}(\mathbb{R}^3) = \mathbb{R}^3 \times SO(3)$ (forgetting the α factor), with t translation and R rotation, can be written explicitly

$$\begin{aligned} L_f : P_x^{SO}(\mathbb{R}^3) &\to P^{SO}(\mathbb{R}^3); \\ U &\mapsto d_x f \circ U, \qquad \Longrightarrow \; L_f(x, U) = (Rx + t, RU). \end{aligned}$$

In this case, a section invariant under the action of E_0^3 does not exist. But, invariant connections there exist due to Wang's theorem. A connection ω over $P^{SO}(\mathbb{R}^3)$ can be written as $\omega = a^{-1}da + a^{-1}\hat{\omega}a$, where $\hat{\omega}$ is a $\mathfrak{so}(3)$-valued 1-form over $\mathbb{R}^3$ and $a \in SO(3)$. Given a section $e : \mathbb{R}^3 \to P^{SO}(\mathbb{R}^3)$, we obtain the local connection 1-form $e^*\omega = e^{-1}de + e^{-1}\hat{\omega}e$. Fixing $e_x = (x, 1)$, $e^*\omega = \hat{\omega}$. The action of L_f on the connection is

$$L_f^*\omega = (Ra)^{-1}d(Ra) + (Ra)^{-1}f^*\hat{\omega}(Ra) = a^{-1}da + a^{-1}R^{-1}f^*\hat{\omega}Ra.$$

From which the condition of invariance is

$$(f^*\hat{\omega})_x(v) = \hat{\omega}_{Rx+t}(Rv) = R\hat{\omega}_x(v)R^{-1}.$$

That is, in the gauge fixing, $\hat{\omega}$ invariant under the adjoint action of $SO(3)$, hence, a possible solution is $\hat{\omega} = c\sum_a T_a \otimes \mathfrak{e}^a$, where $\{\mathfrak{e}^a\}$ is the canonical dual basis of $\mathbb{R}^3$ and T_a are the generators of $\mathfrak{so}(3)$. It corresponds on each $\mathrm{T}_x\mathbb{R}^3 \cong \mathbb{R}^3$ to the equivariant isomorphism $\mathbb{R}^3 \xrightarrow{\sim} \mathfrak{so}(3)$. In a matrix form, it is

$$\hat{\omega}_i^{IJ} = c\,\delta_i^a (T_a)^{IJ}. \tag{A.1}$$

Such as connection reconstruct an isotropic extrinsic curvature. Let ω^A be a E_0^3-invariant connection and ω^{LC} be the Levi-Civita one. Their difference is

$$\Omega = \omega^a - \omega^{LC} = a^{-1}\hat{\omega}^A a - a^{-1}\hat{\omega}^{LC} a = a^{-1}\hat{\Omega}a,$$

and it satisfies $f^*e^*(a^{-1}\hat{\Omega}a) = \mathrm{Ad}_R\hat{\Omega}$. Let V be a $\mathbb{R}^3$-valued 1-form such that $\varphi([e, e^*\Omega]) = [e, V]$, if $e = (x, 1)$, then $(f^*V)_x(v) = V_{Rx+t}(Rv) = RV_x(v)$, namely $V = \varphi(\hat{\Omega})$. Hence, the Weingarten map rotates under this transformation

$$W_{Rx+t}(Rv) = [e_{Rx+t}, V_{Rx+t}(Rv)] = [e_{Rx+t}, RV_x(v)] = RW_x(v). \tag{A.2}$$

While the extrinsic curvature keeps the E_0^3-invariant (i.e. isotropic and homogeneous) property

$$\begin{aligned}(f^*K)_x(v,w) = K_{Rx+t}(Rv, Rw) = \alpha\langle W_{Rx+t}(Rv), Rw\rangle = \alpha\langle RW_x(v), Rw\rangle \\ = \alpha\langle W_x(v), w\rangle = K_x(v,w).\end{aligned} \tag{A.3}$$

Thus, in our formulation, we are able to recover the canonical form of the Ashtekar connection in an isotropic case (A.1) and be consistent with the request of isotropic metric and extrinsic curvature. However, the isotropic case will be subjected to a deeper study to fully understand the classical description and the quantization procedure.

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