Extension groups for the C^*-algebras associated with λ-graph systems

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Abstract

A λ-graph system is a labeled Bratteli diagram with certain additional structure, which presents a subshift. The class of the C^*-algebras $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph systems is a generalized class of the class of Cuntz–Krieger algebras. In this paper, we will compute the strong extension groups $\mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}})$ for the C^*-algebras associated with λ-graph systems $\mathfrak{L}$ and study their relation with the weak extension group $\mathrm{Ext_w}(\mathcal{O}_{\mathfrak{L}})$.

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1 Introduction

Throughout the paper, $B(H)$ denotes the C^*-algebra of bounded linear operators on a separable infinite-dimensional Hilbert space H. Let us denote by $K(H)$ the C^*-subalgebra of $B(H)$ of compact operators on H, which is a closed two-sided ideal of $B(H)$. The quotient C^*-algebra $B(H)/K(H)$ is called the Calkin algebra, denoted by $Q(H)$. The quotient map $B(H) \longrightarrow Q(H)$ is denoted by π. Let $\mathcal{A}$ be a separable unital nuclear C^*-algebra. The extension group $\mathrm{Ext}_*(\mathcal{A})$ is defined by equivalence classes of short exact sequences

$$0 \longrightarrow K(H) \longrightarrow \mathcal{E} \longrightarrow \mathcal{A} \longrightarrow 0 \qquad \text{(exact)} \tag{1.1}$$

of C^*-algebras for which $K(H)$ is an essential ideal of $\mathcal{E}$. There are two kinds of extension groups $\mathrm{Ext_s}(\mathcal{A})$ and $\mathrm{Ext_w}(\mathcal{A})$, called the strong extension group and the weak extension group, respectively. They are defined by two different equivalence relations of the short exact sequences (1.1), respectively. The groups have been playing important role as one of K-theoretic invatriant in studying structure theory of C^*-algebras, classification of essentially normal operators, non commutative geometry, and so on. In [7], Cuntz–Krieger have computed the weak extension group $\mathrm{Ext_w}(\mathcal{O}_A)$ of Cuntz–Krieger algebra $\mathcal{O}_A$ as $\mathrm{Ext_w}(\mathcal{O}_A) = \mathbb{Z}^N/(I-A)\mathbb{Z}^N$ for an $N \times N$ irreducible matrix with entries in $\{0,1\}$, so that they found a lot of examples of unital simple purely infinite C^*-algebras which are mutually non-isomorphic. On the other hand, Paschke–Salinas [24] and Pimsner–Popa

[25] independently computed the groups $\mathrm{Ext_w}(\mathcal{O}_N)$ and $\mathrm{Ext_s}(\mathcal{O}_N)$ for Cuntz algebras as $\mathrm{Ext_w}(\mathcal{O}_N) = \mathbb{Z}/(1-N)\mathbb{Z}$ and $\mathrm{Ext_s}(\mathcal{O}_N) = \mathbb{Z}$. It is a remarkable fact that the group $\mathbb{Z}^N/(I-A)\mathbb{Z}^N$ appears as $\mathrm{Ext_w}(\mathcal{O}_A)$ from the view point of classification theory of symbolic dynamical systems, because the group $\mathbb{Z}^N/(I-A)\mathbb{Z}^N$ is an 'almost' complete invariant of flow equivalence of the associated two-sided topological Markov shift (Λ_A, σ_A) ([9], [3]).

In [20] (cf. [19]), the author recently computed the strong extension group $\mathrm{Ext_s}(\mathcal{O}_A)$ of Cuntz–Krieger algebras as $\mathrm{Ext_s}(\mathcal{O}_A) = \mathbb{Z}/(I-\widehat{A})\mathbb{Z}^N$, where $\widehat{A} = A + R_1 - AR_1$ and R_1 is the $N \times N$ matrix whose first row is $[1,1,\dots,1]$ and the other rows are zero vectors. In [20], the author also clarified exact relationship between the two groups $\mathrm{Ext_w}(\mathcal{O}_A)$ and $\mathrm{Ext_s}(\mathcal{O}_A)$ by presented the following cyclic six term exact sequence of abelian groups:

$$
\begin{array}{ccccc}
\mathrm{Ker}(I-\widehat{A}:\mathbb{Z}^N \longrightarrow \mathbb{Z}^N)/\iota(\mathbb{Z}) & \longrightarrow & \mathrm{Ker}(I-A:\mathbb{Z}^N \longrightarrow \mathbb{Z}^N) & \longrightarrow & \mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & \mathbb{Z}^N/(I-A)\mathbb{Z}^N & \longleftarrow & \mathbb{Z}^N/(I-\widehat{A})\mathbb{Z}^N
\end{array}
\tag{1.2}
$$

by translating an associated K-homology long exact sequence (cf. [11]), where $\iota : \mathbb{Z} \to \mathrm{Ker}(I-\widehat{A} : \mathbb{Z}^N \longrightarrow \mathbb{Z}^N)$ is given by $\iota(m) = [m, 0.\dots, 0]$, $m \in \mathbb{Z}$. The importance of the two groups $\mathrm{Ext_s}(\mathcal{O}_A)$ and $\mathrm{Ext_w}(\mathcal{O}_A)$ in the classification theory of C^*-algebras was shown in a recent joint paper [21]. It says that the two groups $\mathrm{Ext_s}(\mathcal{O}_A)$ and $\mathrm{Ext_w}(\mathcal{O}_A)$ are a complete set of invariants of the isomorphism class of the Cuntz–Krieger algebra $\mathcal{O}_A$. That is, the position $[1]_0$ in $\mathrm{K}_0(\mathcal{O}_A)$ of the unit of the $\mathcal{O}_A$ is determined by only the group structure of the two extension groups $\mathrm{Ext_s}(\mathcal{O}_A)$ and $\mathrm{Ext_w}(\mathcal{O}_A)$. The result will be generalized to a wider class of Kirchberg algebras in [22]. Hence it seems to be interesting and important to compute the two extension groups $\mathrm{Ext_s}(\,\cdot\,)$ and $\mathrm{Ext_w}(\,\cdot\,)$ for more general Kirchberg algebras.

In this paper we will generalize the above computations for $\mathrm{Ext_s}(\mathcal{O}_A)$ to more general C^*-algebras related to symbolic dynamical systems. Cuntz–Krieger algebra $\mathcal{O}_A$ is considered as the C^*-algebra associated to topological Markov shift defined by the matrix A. The C^*-algebras which we will consider in the present paper are the ones associated to general subshifts defined in [15] (cf. [18]). They are defined by λ-graph systems $\mathfrak{L}$ which are labeled Bratteli diagrams with some additional structure, and are regarded as generalizations of finite directed graphs. Let Σ be a finite set whose cardinality $|\Sigma| \ge 2$. A λ-graph sytems $\mathfrak{L} = (V, E, \lambda, \iota)$ over alphabet Σ consists of the vertex set $V = \cup_{l=0}^{\infty} V_l$, edge set $E = \cup_{l=0}^{\infty} E_{l,l+1}$, labeling map $\lambda : E \longrightarrow \Sigma$ and a sequence $\iota = \{\iota_{l,l+1}\}$ of surjective maps $\iota_{l,l+1} : V_{l+1} \longrightarrow V_l$ for each $l \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$. Each edge $e \in E_{l,l+1}$ has its source vertex $s(e) \in V_l$ and its terminal vertex $t(e) \in V_{l+1}$, with its label $\lambda(e) \in \Sigma$, so λ is a map from E to Σ. The first three (V, E, λ) expresses a labeled Bratteli diagram. Moreover the extra structure $\iota_{l,l+1} : V_{l+1} \longrightarrow V_l$, $l \in \mathbb{Z}_+$ of surjective maps satisfies a property called the local property of λ-graph system such that for every two vertices $u \in V_l$ and $v \in V_{l+2}$, there exists a bijective correspondence preserving their labels between the two sets of edges

$$\{e \in E_{l,l+1} \mid s(e) = u, t(e) = \iota(v)\}, \quad \{e \in E_{l+1,l+2} \mid \iota(s(e)) = u, t(e) = v\}.$$

Put $m(l) = |V_l|$ the cardinality $|V_l|$ of the finite set V_l, so that $m(l) \le m(l+1)$. A λ-graph system $\mathfrak{L}$ is said to be *left-resolving* if $e, f \in E_{l,l+1}$ satisfy $\lambda(e) = \lambda(f), t(e) = t(f)$, then $e = f$. We henceforth assume that a λ-graph system is left-resolving.

Any λ-graph system defines a subshift by gathering label sequences appearing on the concatenated labeled edges of the λ-graph system. Conversely any subshift can be presented by a λ-graph system ([13]). Hence a λ-graph system is regarded as a graph presentation of a subshift.

We fix a left-resolving λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ. Let us denote by the vertex set $V_l = \{v_1^l, \dots, v_{m(l)}^l\}$. Let $(A_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+}$ be the structure matrices of a given λ-graph system $\mathfrak{L}$ defined by for $\alpha \in \Sigma$

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } \exists e \in E_{l,l+1}; s(e) = v_i^l, t(e) = v_j^{l+1}, \lambda(e) = \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise,} \end{cases}$$

so $A_{l,l+1}, I_{l,l+1}$ are $m(l) \times m(l+1)$ matrices.

Proposition 1.1 ([15],[17]). *The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is realized as a universal C^*-algebra generated by partial isometries S_α indexed by $\alpha \in \Sigma$ and mutually commuting projections E_i^l indexed by $v_i^l \in V_l$ subject to the following operator relations called $(\mathfrak{L})$:*

$$1 = \sum_{\beta \in \Sigma} S_\beta S_\beta^* = \sum_{i=1}^{m(l)} E_i^l, \qquad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E_j^{l+1},$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) E_j^{m(l+1)}, \quad \alpha \in \Sigma, \ i = 1, \dots, m(l), l \in \mathbb{Z}_+.$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ was primarily constructed as the C^*-algebra $C^*(G_{\mathfrak{L}})$ of an étale amenable groupoid $G_{\mathfrak{L}}$ associated to the λ-graph system $\mathfrak{L}$. The class of the C^*-algebras $\mathcal{O}_{\mathfrak{L}}$ generalize the class of Cuntz–Krieger algebras $\mathcal{O}_A$. If $\mathfrak{L}$ satisfies condition (I), it is a unique C^*-algebra subject to the above operator relations $(\mathfrak{L})$. It becomes a simple C^*-algebra if $\mathfrak{L}$ satisfies condition (I) and is irreducible ([15],[17]). As in [17], lots of the C^*-algebras $\mathcal{O}_{\mathfrak{L}}$ are unital Kirchberg algebras.

Let $(A_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+}$ be the structure matrices of a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$. Let

$$\mathbb{Z}_I := \{([n_i^l]_{i=1}^{m(l)})_{l \in \mathbb{Z}_+} \in \prod_{l \in \mathbb{Z}_+} \mathbb{Z}^{m(l)} \mid I_{l,l+1}[n_j^l]_{i=1}^{m(l+1)} = [n_i^l]_{i=1}^{m(l)}, l \in \mathbb{Z}_+\}$$

the projective limit $\varprojlim\{I_{l,l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}\}$ of the projective system $I_{l.l+1} : \mathbb{Z}^{m(l+1)} \to \mathbb{Z}^{m(l)}, l \in \mathbb{Z}_+$ of abelian groups. The subgroup $\mathbb{Z}_{I,0}$ of $\mathbb{Z}_I$ is defined by

$$\mathbb{Z}_{I,0} = \{([n_i^l]_{i=1}^{m(l)})_{l \in \mathbb{Z}_+} \in \mathbb{Z}_I \mid \sum_{i=1}^{m(l)} n_i^l = 0, l \in \mathbb{Z}_+\}.$$

The family $I_{l,l+1} - A_{l,l+1}, l \in \mathbb{Z}_+$ of $m(l) \times m(l+1)$ matrices $I_{l,l+1} - A_{l,l+1}$ naturally give rise to an endomorphism on $\mathbb{Z}_I$ denoted by $I - A_{\mathfrak{L}}$. It satisfies $(I - A_{\mathfrak{L}})(\mathbb{Z}_{I,0}) \subset \mathbb{Z}_I$. Following

Higson–Roe [11], the reduced K-homology groups $\widetilde{K}^i(\mathcal{A}), i = 0,1$ and the unreduced K-homology groups $K^i(\mathcal{A}), i = 0,1$ for a separable C^*-algebra $\mathcal{A}$ are defined by using the dual C^*-algebra $\mathfrak{D}(\mathcal{A})$ (cf. [23]) for $\mathcal{A}$ such that

$$\widetilde{\mathrm{K}}^p(\mathcal{A}) = \mathrm{K}_{1-p}(\mathfrak{D}(\mathcal{A})) \quad \text{and} \quad \mathrm{K}^p(\mathcal{A}) = \mathrm{K}_{1-p}(\mathfrak{D}(\widetilde{\mathcal{A}}))$$

for $p = 0, 1$, where $\widetilde{\mathcal{A}}$ is the unitization of $\mathcal{A}$. They satisfy

$$\widetilde{\mathrm{K}}^1(\mathcal{A}) = \mathrm{Ext_s}(\mathcal{A}) \quad \text{and} \quad \mathrm{K}^1(\mathcal{A}) = \mathrm{Ext_w}(\mathcal{A}).$$

We write $\widetilde{\mathrm{K}}^i(\mathcal{A}) =: \mathrm{Ext}^{\mathrm{i}}_{\mathrm{s}}(\mathcal{A})$ and $\mathrm{K}^i(\mathcal{A}) =: \mathrm{Ext}^{\mathrm{i}}_{\mathrm{w}}(\mathcal{A})$ for $i = 0, 1$. There exists a cyclic six-term exact sequence:

$$\begin{array}{ccccc}
\mathrm{Ext}^0_{\mathrm{s}}(\mathcal{A}) & \longrightarrow & \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{A}) & \longrightarrow & \mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & \mathrm{Ext}^1_{\mathrm{w}}\mathcal{A} & \longleftarrow & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{A})
\end{array} \tag{1.3}$$

for a separable unital nuclear C^*-algebra $\mathcal{A}$ ([11, 5.2.10 Proposition]).

In the first half of the paper, we will prove the following theorem.

Theorem 1.2 (Theorem 2.15)**.** *Let $\mathfrak{L}$ be a left-resolving λ-graph system over Σ. There exist isomorphisms*

$$\begin{aligned}
\mathrm{Ind}^1_{\mathrm{w}} &: \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I, \\
\mathrm{Ind}^1_{\mathrm{s}} &: \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}, \\
\mathrm{Ind}^0_{\mathrm{w}} &: \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I), \\
\mathrm{Ind}^0_{\mathrm{s}} &: \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_{I,0} \longrightarrow \mathbb{Z}_I)
\end{aligned}$$

of abelian groups such that the K*-homology long exact sequence* (1.3) *for $\mathcal{A} = \mathcal{O}_{\mathfrak{L}}$ is given by the cyclic six term exact sequence*

$$\begin{array}{ccccc}
\mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_{I,0} \longrightarrow \mathbb{Z}_I) & \longrightarrow & \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I) & \longrightarrow & \mathbb{Z} \\
\uparrow & & & & \downarrow \\
0 & \longleftarrow & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I & \longleftarrow & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}.
\end{array} \tag{1.4}$$

In the second half of the paper, we will compute the extension groups $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}})$ and the cyclic six term exact sequence (1.4) for two examples of the λ-graph systems associated to subshifts. There are lots of examples of subshifts which are not topological Markov shifts. The first example of subshifts is the Markov coded systems S_G studied in [16]. The Markov coded system S_G is defined by a finite directed graph $G = (V, E)$ admitting multiple directed edges from a vertex to another vertex. Let $\{v_1, \dots, v_N\}$ be the vertex set V. Let us denote by $A = [A(i,j)]_{i,j=1}^N (= A_G)$ the $N \times N$ transition matrix of the directed graph such that $A(i,j)$ denotes the number of the directed from the vertex v_i to the vertex v_j. In [16], it was proved that the C^*-algebra $\mathcal{O}_{S_G}$ for the canonical λ-graph system $\mathfrak{L}_{S_G}$ of the subshift S_G is a unital purely infinite simple nuclear C^*-algebra, and its K-groups and the weak extension groups were computed. Since the torsion free part of $\mathrm{K}_0(\mathcal{O}_{S_G})$ is not isomorphic to $\mathrm{K}_1(\mathcal{O}_{S_G})$, the C^*-algebras $\mathcal{O}_{S_G}$ are never stably isomorphic to any of Cuntz–Krieger algebras.

The second example of subshifts are the class of Dyck shifts $D_N, 2 \le N \in \mathbb{N}$. They are interesting family of subshifts coming from automata theory and formal language theory which are located in the subshifts far from topological Markov shifts. They have minimal presentations of λ-graph systems which yield unital simple purely infinite C^*-algebras having infinite generators of its K-theory groups, so that they do not belong to the class of Cuntz–Krieger algebras. We will compute the strong extension groups for the two examples $\mathcal{O}_{S_G}$ and $\mathcal{O}_{D_N^{\min}}$ of C^*-algebras in the following way.

Theorem 1.3 (Proposition 3.2, Corollary 3.13)**.**

(i) *Assume that the transition matrix A of a finite directed graph G is aperiodic. Let $\mathcal{O}_{S_G}$ be the simple purely infinite C^*-algebra associated with the canonical λ-graph system of the Markov coded system S_G. Then we have*

$$\begin{aligned} \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G}) &\cong (\mathrm{Ker}(A) \text{ in } \mathbb{Z}^N) \oplus \mathbb{Z}^N, & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{S_G}) &\cong \mathbb{Z}^N/A\mathbb{Z}^N, \\ \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{S_G}) &\cong (\mathrm{Ker}(A) \text{ in } \mathbb{Z}^N) \oplus \mathbb{Z}^{N-1}, & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{S_G}) &\cong \mathbb{Z}^N/A\mathbb{Z}^N. \end{aligned}$$

(ii) *Let $\mathcal{O}_{D_N^{\min}}$ be the simple purely infinite C^*-algebra associated with the minimal presentation $\mathfrak{L}_{D_N^{\min}}$ of the Dyck shift D_N. Then we have*

$$\begin{aligned} \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) &\cong \mathrm{Hom}(C(\mathcal{C},\mathbb{Z}),\mathbb{Z}), & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) &\cong \mathbb{Z}/N\mathbb{Z}, \\ \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) &\cong \mathrm{Hom}(C(\mathcal{C},\mathbb{Z}),\mathbb{Z}), & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) &\cong \mathbb{Z}. \end{aligned}$$

where $C(\mathcal{C},\mathbb{Z})$ denotes the abelian group of integer valued continuous functions on a Cantor discontinuum $\mathcal{C}$.

We note that the computations $\mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G}) \cong (\mathrm{Ker}(A) \text{ in } \mathbb{Z}^N) \oplus \mathbb{Z}^N, \mathrm{Ext}^1_{\mathrm{w}}\mathcal{O}_{S_G}) \cong \mathbb{Z}^N/A\mathbb{Z}^N$ and $\mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) \cong \mathrm{Hom}(C(\mathcal{C},\mathbb{Z}),\mathbb{Z}), \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) \cong \mathbb{Z}/N\mathbb{Z}$ are already obtained in [16] and [12], respectively. In this paper, we will compute the strong extension groups $\mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{S_G}), \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{S_G})$ and $\mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}})$ and $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}})$.

2 $\mathrm{Ext}_*(\mathcal{O}_{\mathfrak{L}})$ and Fredholm indices

In what follows, H denotes a separable infinite dimensional Hilbert space. Let $\mathcal{A}$ be a separable unital nuclear C^*-algebra. An extension means a unital $*$-monomorphism $\sigma : \mathcal{A} \longrightarrow Q(H)$ from $\mathcal{A}$ to the Calkin algebra $Q(H)$. Two extensions $\sigma_i : \mathcal{A} \longrightarrow Q(H), i = 1,2$ are said to be weakly equivalent if there exists a unitary $u \in Q(H)$ such that $\sigma_2(a) = u\sigma_1(a)u^*, a \in \mathcal{A}$. If in particular the above unitary u in $Q(H)$ is taken as $u = \pi(U)$ for some unitary $U \in B(H)$, then the extensions $\tau_i : \mathcal{A} \longrightarrow Q(H), i = 1,2$ are said to be strongly equivalent. Let us denote by $\mathrm{Ext}_{\mathrm{w}}(\mathcal{A})$ and $\mathrm{Ext}_{\mathrm{s}}(\mathcal{A})$ the weak equivalence classes and the strong equivalence classes of extensions of $\mathcal{A}$, respectively. The class of a unital $*$-monomorphism $\tau : \mathcal{A} \longrightarrow Q(H)$ in $\mathrm{Ext}_{\mathrm{s}}(\mathcal{A})$ is denoted by $[\tau]_s$, and similarly $[\tau]_w$ in $\mathrm{Ext}_{\mathrm{w}}(\mathcal{A})$. Fix an identification between $H \oplus H$ and H. Through an embedding

$Q(H) \oplus Q(H) \hookrightarrow Q(H)$ defined by the identification, the sum of extensions $\tau_1 \oplus \tau_2$ are defined by a direct sum $\tau_1 \oplus \tau_2$. It is well-known that both $\mathrm{Ext_s}(\mathcal{A})$ and $\mathrm{Ext_w}(\mathcal{A})$ become abelian semigroups, and also they are abelian groups for nuclear C^*-algebra $\mathcal{A}$ (cf. [1], [5], [6], [8], etc.). They are called the strong extension group for $\mathcal{A}$ and the weak extension group for $\mathcal{A}$, respectively. Let us denote by $q_A : \mathrm{Ext_s}(\mathcal{A}) \longrightarrow \mathrm{Ext_w}(\mathcal{A})$ the natural quotient map. As in [11] and [25], there exists a homomorphism $\iota_{\mathcal{A}} : \mathbb{Z} \longrightarrow \mathrm{Ext_s}(\mathcal{A})$ such that the sequence

$$\mathbb{Z} \xrightarrow{\iota_{\mathcal{A}}} \mathrm{Ext_s}(\mathcal{A}) \xrightarrow{q_{\mathcal{A}}} \mathrm{Ext_w}(\mathcal{A}) \tag{2.1}$$

is exact at the middle, so the quotient group $\mathrm{Ext_s}(\mathcal{A})/\iota_{\mathcal{A}}(\mathbb{Z})$ is isomorphic to $\mathrm{Ext_w}(\mathcal{A})$.

For projections $e \in Q(H) = B(H)/K(H)$ and $E \in B(H)$ with $\pi(E) = e$, let $t \in Q(H)$ be an element such that ete is invertible in $eQ(H)e$. Let $T \in B(H)$ be a lift of t, which satisfies $\pi(T) = t$. The Fredholm index of ETE in EH is denoted by $\mathrm{ind}_e t$. The integer $\mathrm{ind}_e t$ does not depend on the choice of E and T.

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a left-resolving λ-graph system over Σ. Let $S_\alpha, \alpha \in \Sigma$ and $E_i^l, v_i^l \in V_l$ be the generating partial isometries and mutually commuting projections satisfying the relations $(\mathfrak{L})$ in Proposition 1.1. Let us denote by $\mathcal{A}_l$ the commutative C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections $E_i^l, i = 1, \dots, m(l)$. The commutative C^*-subalgebra $\mathcal{A}_{\mathfrak{L}}$ is defined by the one generated by $\mathcal{A}_l, l \in \mathbb{Z}_+$. Since $\mathcal{A}_l \subset \mathcal{A}_{l+1}, l \in \mathbb{Z}_+$, The algebra $\mathcal{A}_{\mathfrak{L}}$ is a commutative AF-algebra. Hence the projections $E_1^l, \dots, E_{m(l)}^l$ are the minimal projections in $\mathcal{A}_l$ satisfying $E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}$ for $i = 1, \dots, m(l)$.

Lemma 2.1. *For an extension $\sigma : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$, there exists a trivial extension $\tau : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\sigma|_{\mathcal{A}_{\mathfrak{L}}} = \pi \circ \tau|_{\mathcal{A}_{\mathfrak{L}}}$.*

Proof. Let us denote by $\hat{\sigma}$ the restriction $\sigma|_{\mathcal{A}_{\mathfrak{L}}}$ of σ to the subalgebra $\mathcal{A}_{\mathfrak{L}}$. As $\mathcal{A}_{\mathfrak{L}}$ is a commutative AF-algebra, the extension $\hat{\sigma}$ is trivial by [5, 1.15 Theorem]. Take a unital $*$-monomorphism $\rho : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ and put $\tilde{\rho} = \pi \circ \rho : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$ and

$$\hat{\rho} = \pi \circ \rho|_{\mathcal{A}_{\mathfrak{L}}} : \mathcal{A}_{\mathfrak{L}} \longrightarrow Q(H).$$

As the extensions $\hat{\sigma}, \hat{\rho}$ are both trivial, by Voiculescu's theorem [26], there exists a unitary $U \in B(H)$ such that

$$\hat{\sigma}(x) = \pi(U)\hat{\rho}(x)\pi(U)^*, \qquad x \in \mathcal{A}_{\mathfrak{L}}$$

so that $\sigma(x) = \pi(U)\pi(\rho(x))\pi(U)^*, x \in \mathcal{A}_{\mathfrak{L}}$. By putting $\tau = \mathrm{Ad}(U) \circ \rho : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$, we have $\sigma = \pi \circ \tau$ on $\mathcal{A}_{\mathfrak{L}}$. $\square$

A Fredholm module over a C^*-algebra $\mathcal{A}$ means a pair (u, ρ) of a unitary $u \in Q(H)$ and a $*$-homomorphism $\rho : \mathcal{A} \longrightarrow B(H)$ such that $\pi(\rho(a))u = u\pi(\rho(a))$ for all $a \in \mathcal{A}$. It is well-known that the K-homology group $\mathrm{K}^0(\mathcal{A})$ for $\mathcal{A}$ is realized as the homotopy equivalence classes of Fredholm modules over $\mathcal{A}$ (cf. [5], [8], [10], [11], etc.). The addition $[(u_1, \rho_1)] + [(u_2, \rho_2)]$ is defined by $[(u_1 \oplus u_2, \rho_1 \oplus \rho_2)]$, in particular, we have $[(u_1, \rho)] + [(u_2, \rho)] = [(u_1 u_2, \rho)]$, and hence $[(u^*, \rho)] + [(u, \rho)] = [(u^* u, \rho)] = [(1, \rho)]$, so that $-[(u, \rho)] = [(u^*, \rho)]$. Take a faithful representation $\tau_0 : \mathcal{A} \longrightarrow B(H)$ and a unitary $V_0 \in B(H)$ such that $\tau_0(a)V_0 = V_0\tau_0(a)$ for $a \in \mathcal{A}$. Since $(\pi(V_0), \tau_0)$ is a neutral element of $\mathrm{K}^0(\mathcal{A})$, $(u \oplus \pi(V_0), \rho \oplus \tau_0)$ is equivalent to (u, ρ) for any Fredholm module (u, ρ) over $\mathcal{A}$. Hence we can take a representative of $[(u,\rho)]$ as ρ being faithful. As $\mathcal{A}_{\mathfrak{L}} = \lim_{l\to\infty} \mathcal{A}_l$ is an AF-algebra, the formula

$$\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}}) = \mathrm{Hom}(\mathrm{K}_0(\mathcal{A}_{\mathfrak{L}}), \mathbb{Z}) = \varprojlim \mathrm{K}^0(\mathcal{A}_l)$$

holds (cf. [2], [11]). For $[(u,\rho)] \in \mathrm{K}^0(\mathcal{A}_l)$, the identity $E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j) E_j^{l+1}$ implies

$$\mathrm{ind}_{\pi\circ\rho(E_i^l)} u = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i,j)\mathrm{ind}_{\pi\circ\rho(E_j^{l+1})} u,$$

so that the correspondence for each $l \in \mathbb{Z}_+$

$$\mathrm{Ind}_l : [(u,\rho)] \in \mathrm{K}^0(\mathcal{A}_l) \longrightarrow (\mathrm{ind}_{\pi\circ\rho(E_i^l)} u)_{i=1}^{m(l)} \in \mathbb{Z}^{m(l)} \tag{2.2}$$

yields the isomorphism

$$\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}}) = \varprojlim \{ I_{l,l+1} : \mathbb{Z}^{m(l+1)} \longrightarrow \mathbb{Z}^{m(l)} \} \tag{2.3}$$

of abelian groups. The latter group is denoted by $\mathbb{Z}_I$. We denote by $\mathrm{Ind} : \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I$ the isomorphism induced by (2.2). We note the following lemma.

Lemma 2.2. *Let $\rho_i : \mathcal{A}_{\mathfrak{L}} \longrightarrow B(H), i = 1,2$ be representations and $u \in Q(H)$ a unitary satisfying $u\pi(\rho_i(a)) = \pi(\rho_i(a))u, a \in \mathcal{A}_{\mathfrak{L}}, i = 1,2$. Suppose that $\pi\circ\rho_2 = \pi\circ\rho_1$. Then $[(u,\rho_1)] = [(u,\rho_2)]$ as elements of $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$.*

Proof. Since $\pi\circ\rho_2(E_i^l) = \pi\circ\rho_1(E_i^l), i = 1,\dots,m(l)$, we have $\mathrm{ind}_{\pi\circ\rho_2(E_i^l)} u = \mathrm{ind}_{\pi\circ\rho_1(E_i^l)} u$ for $i = 1,\dots,m(l)$, so that $[(u,\rho_1)] = [(u,\rho_2)]$. ☐

For an extension $\sigma : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$, take a trivial extension $\tau : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\sigma|_{\mathcal{A}_{\mathfrak{L}}} = \pi\circ\tau|_{\mathcal{A}_{\mathfrak{L}}}$. We write $\tilde{\tau} = \pi\circ\tau : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$ and define $U_{\sigma,\tilde{\tau}} \in Q(H)$ by setting

$$U_{\sigma,\tilde{\tau}} := \sum_{\alpha\in\Sigma} \sigma(S_\alpha)\tilde{\tau}(S_\alpha^*).$$

Lemma 2.3. *$U_{\sigma,\tilde{\tau}}$ is a unitary in $Q(H)$ such that $U_{\sigma,\tilde{\tau}}\tilde{\tau}(a) = \tilde{\tau}(a)U_{\sigma,\tilde{\tau}}$ for all $a \in \mathcal{A}_{\mathfrak{L}}$. Hence the pair $(U_{\sigma,\tilde{\tau}}, \tau|_{\mathcal{A}_{\mathfrak{L}}})$ gives rise to an element of $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$.*

Proof. We have

$$U_{\sigma,\tilde{\tau}}U_{\sigma,\tilde{\tau}}^* = \sum_{\alpha,\beta\in\Sigma} \sigma(S_\alpha)\tilde{\tau}(S_\alpha^*)\tilde{\tau}(S_\beta)\sigma(S_\beta^*) = \sum_{\alpha\in\Sigma} \sigma(S_\alpha)\sigma(S_\alpha^*) = 1$$

and similarly $U_{\sigma,\tilde{\tau}}^*U_{\sigma,\tilde{\tau}} = 1$ so that $U_{\sigma,\tilde{\tau}}$ is a unitary in $Q(H)$. We also have for $a \in \mathcal{A}_{\mathfrak{L}}$

$$\begin{aligned} U_{\sigma,\tilde{\tau}}\tilde{\tau}(a) &= \sum_{\alpha\in\Sigma} \sigma(S_\alpha)\tilde{\tau}(S_\alpha^*)\tilde{\tau}(a) = \sum_{\alpha\in\Sigma} \sigma(S_\alpha)\tilde{\tau}(S_\alpha^* a S_\alpha)\tilde{\tau}(S_\alpha^*) \\ &= \sum_{\alpha\in\Sigma} \sigma(S_\alpha S_\alpha^* a S_\alpha)\tilde{\tau}(S_\alpha^*) = \sum_{\alpha\in\Sigma} \sigma(a S_\alpha)\tilde{\tau}(S_\alpha^*) = \tilde{\tau}(a)U_{\sigma,\tilde{\tau}}. \end{aligned}$$

☐

Lemma 2.4. *For an extension $\sigma : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$, take trivial extensions $\tau_i : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\sigma|_{\mathcal{A}_{\mathfrak{L}}} = \tilde{\tau}_i|_{\mathcal{A}_{\mathfrak{L}}}, i = 1, 2$. Then there exists a unitary $V \in B(H)$ such that*

$$\pi(V)\sigma(a) = \sigma(a)\pi(V), \quad a \in \mathcal{A}_{\mathfrak{L}} \quad \text{and} \quad U_{\sigma,\tilde{\tau}_2} = U_{\sigma,\tilde{\tau}_1}\phi_{\tilde{\tau}_1}(\pi(V))\pi(V)^* \tag{2.4}$$

where $\phi_{\tilde{\tau}_1}(\pi(V))$ is a unitary in $\mathcal{Q}(H)$ defined by

$$\phi_{\tilde{\tau}_1}(\pi(V)) = \sum_{\alpha\in\Sigma} \tilde{\tau}_1(S_\alpha)\pi(V)\tilde{\tau}_1(S_\alpha)^*.$$

Proof. By Voiculescu's theorem [26], one may find a unitary $V \in B(H)$ such that $\tilde{\tau}_2(X) = \pi(V)\tilde{\tau}_1(X)\pi(V)^*$ for $X \in \mathcal{O}_{\mathfrak{L}}$. For $a \in \mathcal{A}_{\mathfrak{L}}$, we have

$$\pi(V)\sigma(a) = \pi(V)\tilde{\tau}_1(a) = \tilde{\tau}_2(a)\pi(V) = \sigma(a)\pi(V),$$

so that $\pi(V)$ commutes with $\sigma(a)$ for all $a \in \mathcal{A}_{\mathfrak{L}}$. It then follows that

$$\begin{aligned}
U_{\sigma,\tilde{\tau}_2} =& \sum_{\alpha\in\Sigma} \sigma(S_\alpha)\pi(V)\tilde{\tau}_1(S_\alpha^*)\pi(V)^* \\
=& \sum_{\alpha\in\Sigma} \sigma(S_\alpha)\tilde{\tau}_1(S_\alpha^* S_\alpha)\pi(V)\tilde{\tau}_1(S_\alpha^*)\pi(V)^* \\
=& \sum_{\alpha\in\Sigma} \sigma(S_\alpha)\tilde{\tau}_1(S_\alpha^*) \sum_{\beta\in\Sigma} \tilde{\tau}_1(S_\beta)\pi(V)\tilde{\tau}_1(S_\beta^*)\pi(V)^* \\
=& U_{\sigma,\tilde{\tau}_1}\phi_{\tilde{\tau}_1}(\pi(V))\pi(V)^*.
\end{aligned}$$

It is routine to check that $\phi_{\tilde{\tau}_1}(\pi(V))$ is a unitary in $Q(H)$ by using the commutativity between $\pi(V)$ and $\tilde{\tau}_1(S_\alpha^* S_\alpha)$. □

Lemma 2.5. *For a Fredholm module (u, ρ) over $\mathcal{A}_{\mathfrak{L}}$, take a trivial extension $\tau_\rho : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\rho = \tau_\rho|_{\mathcal{A}_{\mathfrak{L}}}$. Then $\phi_{\tilde{\tau}_\rho}(u) = \sum_{\alpha\in\Sigma} \tilde{\tau}_\rho(S_\alpha)u\tilde{\tau}_\rho(S_\alpha^*)$ is a unitary in $Q(H)$ commuting with $\pi(\rho(\mathcal{A}_{\mathfrak{L}}))$. Hence the pair $(\phi_{\tilde{\tau}_\rho}(u), \rho)$ gives rise to an element of $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$ and its class $[(\phi_{\tilde{\tau}_\rho}(u), \rho)]$ in $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$ is independent of the choice of the extension $\tau_\rho : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ as long as $\tau_\rho|_{\mathcal{A}_{\mathfrak{L}}} = \rho|_{\mathcal{A}_{\mathfrak{L}}}$.*

Proof. We will show that $\phi_{\tilde{\tau}_\rho}(u)$ commutes with $\pi(\rho(a))$ for $a \in \mathcal{A}_{\mathfrak{L}}$. We have

$$\begin{aligned}
\phi_{\tilde{\tau}_\rho}(u)\pi(\rho(a)) =& \sum_{\alpha\in\Sigma} \tilde{\tau}_\rho(S_\alpha)u\tilde{\tau}_\rho(S_\alpha^* a) = \sum_{\alpha\in\Sigma} \tilde{\tau}_\rho(S_\alpha)u\tilde{\tau}_\rho(S_\alpha^* a S_\alpha)\tilde{\tau}_\rho(S_\alpha^*) \\
=& \sum_{\alpha\in\Sigma} \tilde{\tau}_\rho(S_\alpha)u\pi(\rho(S_\alpha^* a S_\alpha))\tilde{\tau}_\rho(S_\alpha^*) = \sum_{\alpha\in\Sigma} \tilde{\tau}_\rho(S_\alpha)\pi(\rho(S_\alpha^* a S_\alpha))u\tilde{\tau}_\rho(S_\alpha^*) \\
=& \sum_{\alpha\in\Sigma} \tilde{\tau}_\rho(S_\alpha S_\alpha^* a S_\alpha))u\tilde{\tau}_\rho(S_\alpha^*) = \sum_{\alpha\in\Sigma} \tilde{\tau}_\rho(a)\tilde{\tau}_\rho(S_\alpha)u\tilde{\tau}_\rho(S_\alpha^*) \\
=& \pi(\rho(a))\phi_{\tilde{\tau}_\rho}(u).
\end{aligned}$$

Take two trivial extensions $\tau_\rho, \tau'_\rho : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\rho = \tau_\rho|_{\mathcal{A}_{\mathfrak{L}}} = \tau'_\rho|_{\mathcal{A}_{\mathfrak{L}}}$. By (2.2), it suffices to show that

$$\mathrm{ind}_{\pi\circ\rho(E_i^l)}\phi_{\tau_\rho}(u) = \mathrm{ind}_{\pi\circ\rho(E_i^l)}\phi_{\tau'_\rho}(u). \tag{2.5}$$

It is straightforward to see that $\tilde{\tau}_\rho(S_\alpha)u\tilde{\tau}_\rho(S_\alpha^*)$ commutes with $\pi(\rho(E_i^l))$, and the equality

$$\mathrm{ind}_{\pi\circ\rho(E_i^l)}\phi_{\tau_\rho}(u) = \sum_{\alpha\in\Sigma}\mathrm{ind}_{\pi\circ\rho(E_i^l)}\tilde{\tau}_\rho(S_\alpha)u\tilde{\tau}_\rho(S_\alpha^*)$$

holds. One then has

$$\begin{aligned}\mathrm{ind}_{\pi\circ\rho(E_i^l)}\tilde{\tau}_\rho(S_\alpha)u\tilde{\tau}_\rho(S_\alpha^*) =& \mathrm{ind}_{\pi\circ\rho(E_i^l S_\alpha S_\alpha^*)}\tilde{\tau}_\rho(E_i^l S_\alpha)u\tilde{\tau}_\rho(S_\alpha^* E_i^l)\\ =& \mathrm{ind}_{\pi\circ\rho(S_\alpha^* E_i^l S_\alpha)}\tilde{\tau}_\rho(S_\alpha^* E_i^l S_\alpha)u\tilde{\tau}_\rho(S_\alpha^* E_i^l S_\alpha)\\ =& \mathrm{ind}_{\pi\circ\rho(S_\alpha^* E_i^l S_\alpha)}\tilde{\tau}'_\rho(S_\alpha^* E_i^l S_\alpha)u\tilde{\tau}'_\rho(S_\alpha^* E_i^l S_\alpha)\\ =& \mathrm{ind}_{\pi\circ\rho(E_i^l)}\tilde{\tau}'_\rho(S_\alpha)u\tilde{\tau}'_\rho(S_\alpha^*),\end{aligned}$$

proving the equality (2.5). ◻

Corollary 2.6. *For an extension $\sigma : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$, take trivial extensions $\tau, \tau' : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ satisfying $\sigma|_{\mathcal{A}_{\mathfrak{L}}} = \tilde{\tau}|_{\mathcal{A}_{\mathfrak{L}}} = \tilde{\tau}'|_{\mathcal{A}_{\mathfrak{L}}}$. Then one may find a unitary $V \in B(H)$ such that $(\pi(V), \tau|_{\mathcal{A}_{\mathfrak{L}}}) \in \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$ and the equality*

$$\mathrm{ind}[(U_{\sigma,\tilde{\tau}}, \tau|_{\mathcal{A}_{\mathfrak{L}}})] - \mathrm{ind}[(U_{\sigma,\tilde{\tau}'}, \tau'|_{\mathcal{A}_{\mathfrak{L}}})] = \mathrm{ind}[(\pi(V), \tau|_{\mathcal{A}_{\mathfrak{L}}})] - \mathrm{ind}[(\phi_{\tilde{\tau}}(\pi(V)), \tau|_{\mathcal{A}_{\mathfrak{L}}})] \tag{2.6}$$

holds.

Proof. Take a unitary $V \in B(H)$ satisfying (2.4). We then have $U_{\sigma,\tilde{\tau}'} = U_{\sigma,\tilde{\tau}}\phi_{\tilde{\tau}}(\pi(V))\pi(V)^*$ so that

$$\mathrm{ind}[(U_{\sigma,\tilde{\tau}'}, \tau|_{\mathcal{A}_{\mathfrak{L}}})] = \mathrm{ind}[(U_{\sigma,\tilde{\tau}}, \tau|_{\mathcal{A}_{\mathfrak{L}}})] + \mathrm{ind}[(\phi_{\tilde{\tau}}(\pi(V)), \tau|_{\mathcal{A}_{\mathfrak{L}}})] + \mathrm{ind}[(\pi(V)^*, \tau|_{\mathcal{A}_{\mathfrak{L}}})].$$

As $\mathrm{ind}[(U_{\sigma,\tilde{\tau}'}, \tau|_{\mathcal{A}_{\mathfrak{L}}})] = \mathrm{ind}[(U_{\sigma,\tilde{\tau}'}, \tau'|_{\mathcal{A}_{\mathfrak{L}}})]$, we get the equality (2.6). ◻

Define the subgroup $\mathbb{Z}_0^{m(l)}$ of $\mathbb{Z}^{m(l)}$ by

$$\mathbb{Z}_0^{m(l)} = \{(n_i^l)_{i=1}^{m(l)} \in \mathbb{Z}^{m(l)} \mid \sum_{i=1}^{m(l)} n_i^l = 0\}.$$

As $I_{l,l+1}\mathbb{Z}_0^{m(l+1)} \subset \mathbb{Z}_0^{m(l)}$, $l \in \mathbb{Z}_+$, we have a projective system $\{I_{l,l+1} : \mathbb{Z}_0^{m(l+1)} \longrightarrow \mathbb{Z}_0^{m(l)}, l \in \mathbb{Z}_+\}$ of abelian groups. Define the subgroup $\mathbb{Z}_{I,0}$ of $\mathbb{Z}_I$ by the abelian group $\varprojlim\{I_{l,l+1} : \mathbb{Z}_0^{m(l+1)} \longrightarrow \mathbb{Z}_0^{m(l)}\}$ of the projective limit, so that

$$\mathbb{Z}_{I,0} = \{(n^l)_{l\in\mathbb{Z}_+} \in \prod_{l=0}^{\infty}\mathbb{Z}_0^{m(l)} \mid I_{l,l+1}n^{l+1} = n^l\}.$$

The matrices $I_{l,l+1}, l \in \mathbb{Z}_+$ act on $\mathbb{Z}_{I,0}$ by $(n^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_{I,0} \longrightarrow (I_{l,l+1}n^{l+1})_{l\in\mathbb{Z}_+}$ as the identity denoted by I. Define the subgroup $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0$ of $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$ by

$$\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0 = \{[(\pi(V), \rho)] \in \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}}) \mid V \in U(B(H))\},$$

where $U(B(H))$ denotes the group of unitaries in $B(H)$.

Lemma 2.7. *The correspondence*

$$\mathrm{Ind}_l : [(u,\rho)] \in \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}}) \longrightarrow (\mathrm{ind}_{\pi\circ\rho(E_i^l)} u)_{i=1}^{m(l)} \in \mathbb{Z}^{m(l)} \quad \text{for } l \in \mathbb{Z}_+$$

yields an isomorphism $\mathrm{Ind} : \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0 \longrightarrow \mathbb{Z}_{I,0}$.

Proof. For $V \in U(B(H))$, we have

$$\sum_{i=1}^{m(l)} \mathrm{ind}_{\pi\circ\rho(E_i^l)}\pi(V) = \mathrm{ind}_{\sum_{i=1}^{m(l)} \pi\circ\rho(E_i^l)}\pi(V) = \mathrm{ind}_{\pi\circ\rho(1)}\pi(V) = 0$$

so that $(\mathrm{ind}_{\pi\circ\rho(E_i^l)}\pi(V))_{i=1}^{m(l)} \in \mathbb{Z}_0^{m(l)}$. Hence $\mathrm{Ind}([(u,\rho)]) \in \mathbb{Z}_{I,0}$ for $[(u,\rho)] \in \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0$. Conversely, for any $(n^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_{I,0}$ with $n^l = (n_i^l)_{i=1}^{m(l)} \in \mathbb{Z}_0^{m(l)}, l \in \mathbb{Z}_+$, one may find a Fredholm module (u,ρ) over $\mathcal{A}_{\mathfrak{L}}$ such that $\mathrm{Ind}[(u,\rho)] = (n^l)_{l\in\mathbb{Z}_+}$. As

$$0 = \sum_{i=1}^{m(l)} \mathrm{ind}_{\pi\circ\rho(E_i^l)} u = \mathrm{ind}_H(u),$$

there exists a unitary $V \in B(H)$ such that $\pi(V) = u$. Hence we have $\mathrm{Ind}[(\pi(V),\rho)] = (n^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_{I,0}$, proving $\mathrm{Ind}(\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0) = \mathbb{Z}_{I,0}$. $\square$

Let $(A_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ be the structure matrices for the λ-graph system $\mathfrak{L}$. Put $A^{\mathfrak{L}}_{l,l+1}(i,j) = \sum_{\alpha\in\Sigma} A_{l,l+1}(i,\alpha,j)$ for $i = 1,\dots,m(l)$, $j = 1,\dots,m(l+1)$, which satisfies the relation

$$I_{l,l+1}A^{\mathfrak{L}}_{l+1,l+2} = A^{\mathfrak{L}}_{l,l+1}I_{l+1,l+2}, \qquad l \in \mathbb{Z}_+. \tag{2.7}$$

Let $A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I$ be the endomorphism on the group $\mathbb{Z}_I$ defined by

$$A_{\mathfrak{L}}((x_l)_{l\in\mathbb{Z}_+}) = (A^{\mathfrak{L}}_{l,l+1}x_{l+1})_{l\in\mathbb{Z}_+}.$$

By the identity (2.7), we know that $(A^{\mathfrak{L}}_{l,l+1}x_{l+1})_{l\in\mathbb{Z}_+}$ belongs to $\mathbb{Z}_I$ for $(x_l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_I$, so that $A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I$ yields an endomorphism on $\mathbb{Z}_I$.

Lemma 2.8. *The map* $\phi : [(\pi(V),\rho)] \in \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0 \longrightarrow [(\phi_{\tilde{\tau}_\rho}(\pi(V)),\rho)] \in \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$ *gives rise to the commutative diagram*

$$\begin{array}{ccc} \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0 & \xrightarrow{\phi} & \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}}) \\ {\scriptstyle\mathrm{Ind}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{Ind}} \\ \mathbb{Z}_{I,0} & \xrightarrow{A_{\mathfrak{L}}} & \mathbb{Z}_I. \end{array} \tag{2.8}$$

Proof. We have

$$\begin{aligned}
\mathrm{ind}_{\pi\circ\rho(E_i^l)}\phi_{\tilde{\tau}_\rho}(\pi(V)) &= \sum_{\alpha\in\Sigma}\mathrm{ind}_{\pi\circ\rho(E_i^l)}\tilde{\tau}(S_\alpha)\pi(V)\tilde{\tau}(S_\alpha^*) \\
&= \sum_{\alpha\in\Sigma}\mathrm{ind}_{\pi\circ\rho(E_i^l)}\pi(\tau(S_\alpha)V\tau(S_\alpha^*)) \\
&= \sum_{\alpha\in\Sigma}\mathrm{ind}_{\pi\circ\rho(S_\alpha^* E_i^l S_\alpha)}\pi(\tau(S_\alpha^* E_i^l S_\alpha)V\tau(S_\alpha^* E_i^l S_\alpha)) \\
&= \sum_{\alpha\in\Sigma}\sum_{j=1}^{m(l+1)} A_{l,l+1}(i,\alpha,j)\mathrm{ind}_{\pi\circ\rho(E_j^{l+1})}\pi(\tau(E_j^{l+1})V\tau(E_j^{l+1})) \\
&= \sum_{j=1}^{m(l+1)} A_{l,l+1}^{\mathfrak{L}}(i,j)\mathrm{ind}_{\pi\circ\rho(E_j^{l+1})}\pi(V)
\end{aligned}$$

so that we have $A_{\mathfrak{L}}\circ\mathrm{Ind} = \mathrm{Ind}\circ\phi : \mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})_0 \longrightarrow \mathbb{Z}_I$. ∎

For an extension $\sigma : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$, take a trivial extension $\tau : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ satisfying $\sigma|_{\mathcal{A}_{\mathfrak{L}}} = \pi\circ\tau|_{\mathcal{A}_{\mathfrak{L}}}$ and consider the unitary $U_{\sigma,\tilde{\tau}} = \sum_{\alpha\in\Sigma}\sigma(S_\alpha)\tilde{\tau}(S_\alpha^*) \in Q(H)$. By Lemma 2.3, the pair $(U_{\sigma,\tilde{\tau}}, \tau|_{\mathcal{A}_{\mathfrak{L}}})$ defines an element of $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$. Put the Fredholm module

$$d(\sigma,\tau) = (U_{\sigma,\tilde{\tau}}, \tau|_{\mathcal{A}_{\mathfrak{L}}})$$

over $\mathcal{A}_{\mathfrak{L}}$ so that the class $[d(\sigma,\tau)]$ defines an element of $\mathrm{K}^0(\mathcal{A}_{\mathfrak{L}})$. Corollary 2.6 together with Lemma 2.8 says that for trivial extensions $\tau,\tau' : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\sigma|_{\mathcal{A}_{\mathfrak{L}}} = \tilde{\tau}|_{\mathcal{A}_{\mathfrak{L}}} = \tilde{\tau}'|_{\mathcal{A}_{\mathfrak{L}}}$, we have

$$\mathrm{Ind}[d(\sigma,\tau)] - \mathrm{Ind}[d(\sigma,\tau')] = (I - A_{\mathfrak{L}})\mathrm{ind}[(\pi(V), \tau|_{\mathcal{A}_{\mathfrak{L}}})].$$

As $\mathrm{ind}[(\pi(V),\tau|_{\mathcal{A}_{\mathfrak{L}}})] \in \mathbb{Z}_{I,0}$, the class

$$[\mathrm{Ind}[d(\sigma,\tau)]] \in \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}$$

is independent of the choice of τ as long as $\pi\circ\tau|_{\mathcal{A}_{\mathfrak{L}}} = \sigma|_{\mathcal{A}_{\mathfrak{L}}}$.

Lemma 2.9. *For an extension $\sigma_1 : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$, take a trivial extension $\tau_1 : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\sigma_1|_{\mathcal{A}_{\mathfrak{L}}} = \pi\circ\tau_1|_{\mathcal{A}_{\mathfrak{L}}}$. Let $\sigma_2 : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$ be an extension strongly equivalent to σ_1. Then there exists a trivial extension $\tau_2 : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ such that $\sigma_2|_{\mathcal{A}_{\mathfrak{L}}} = \pi\circ\tau_2|_{\mathcal{A}_{\mathfrak{L}}}$ and*

$$[\mathrm{Ind}[d(\sigma_1,\tau_1)]] = [\mathrm{Ind}[d(\sigma_2,\tau_2)]] \text{ in } \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}.$$

Proof. Since σ_2 is strongly equivalent to σ_1, one may take a unitary $V \in B(H)$ such that $\sigma_2 = \mathrm{Ad}(\pi(V))\circ\sigma_1$. Define a trivial extension $\tau_2 : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ by $\tau_2 = \mathrm{Ad}(V)\circ\tau_1$ satisfying $\sigma_2|_{\mathcal{A}_{\mathfrak{L}}} = \pi\circ\tau_2|_{\mathcal{A}_{\mathfrak{L}}}$. We then have

$$\begin{aligned}
\mathrm{Ind}[d(\sigma_2,\tau_2)] &= [(\mathrm{ind}_{\sigma_2(E_i^l)}U_{\sigma_2,\tilde{\tau}_2})_{i=1}^{m(l)})_{l\in\mathbb{Z}_+}] \\
&= [(\mathrm{ind}_{\sigma_2(E_i^l)}\sum_{\alpha\in\Sigma}\sigma_2(S_\alpha)\tilde{\tau}_2(S_\alpha^*))_{i=1}^{m(l)})_{l\in\mathbb{Z}_+}].
\end{aligned}$$

Now we have

$$\begin{aligned}\mathrm{ind}_{\sigma_2(E_i^l)}\sum_{\alpha\in\Sigma}\sigma_2(S_\alpha)\tilde{\tau}_2(S_\alpha^*) =&\mathrm{ind}_{\pi(V)\sigma_1(E_i^l)\pi(V^*)}\sum_{\alpha\in\Sigma}\pi(V)\sigma_1(S_\alpha)\tilde{\tau}_1(S_\alpha^*)\pi(V)^*\\ =&\mathrm{ind}_{\sigma_1(E_i^l)}\sum_{\alpha\in\Sigma}\sigma_1(S_\alpha)\tilde{\tau}_1(S_\alpha^*) = \mathrm{ind}_{\sigma_1(E_i^l)}U_{\sigma_1,\tilde{\tau}_1}\end{aligned}$$

so that $\mathrm{Ind}[d(\sigma_1,\tau_1)] = \mathrm{Ind}[d(\sigma_2,\tau_2)]$. □

Therefore we have

Proposition 2.10. *For the strong euivalence class $[\sigma]_s \in \mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}})$ of an extension $\sigma : \mathcal{O}_{\mathfrak{L}} \longrightarrow Q(H)$, the class $[\mathrm{Ind}[d(\sigma,\tau)]]$ in the group $\mathbb{Z}_I/(I-A_{\mathfrak{L}})\mathbb{Z}_{I,0}$ is independent of the choice of a trivial extension $\tau : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ as long as $\sigma|_{\mathcal{A}_{\mathfrak{L}}} = \tau|_{\mathcal{A}_{\mathfrak{L}}}$.*

We thus have a homomorphism

$$\mathrm{Ind_s} : \mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I-A_{\mathfrak{L}})\mathbb{Z}_{I,0}$$

defined by $\mathrm{Ind_s}([\sigma]_s) = [\mathrm{Ind}[d(\sigma,\tau)]]$.

Take a trivial extension $\tau : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ and a unitary $u_m \in Q(H)$ of Fredholm index $m \in \mathbb{Z}$ such that $\pi(\tau(a))u_m = u_m\pi(\tau(a))$ for $a \in \mathcal{A}_{\mathfrak{L}}$. Define an extension $\sigma_m : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ by $\sigma_m = \mathrm{Ad}(u_m)\circ(\pi\circ\tau)$ so that the class $[\sigma_m]_s$ in $\mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}})$ is defined for each $m \in \mathbb{Z}$. We then have a homomorphism $\iota_{\mathfrak{L}} : m \in \mathbb{Z} \longrightarrow [\sigma_m]_s \in \mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}})$ such that the sequence

$$\mathbb{Z} \xrightarrow{\iota_{\mathfrak{L}}} \mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}}) \xrightarrow{q} \mathrm{Ext_w}(\mathcal{O}_{\mathfrak{L}}) \tag{2.9}$$

is exact at the middle, where $q : \mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{Ext_w}(\mathcal{O}_{\mathfrak{L}})$ is the natural quotient map (cf. [25]).

We will introduce a homomorphism $\hat{\iota}_{\mathfrak{L}} : \mathbb{Z} \longrightarrow \mathbb{Z}_I/(I-A_{\mathfrak{L}})\mathbb{Z}_{I,0}$ in the following way. For $m \in \mathbb{Z}$, take an element $(n^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_I$ such that $n^l = (n_i^l)_{i=1}^{m(l)} \in \mathbb{Z}^{m(l)}$ and $m = \sum_{i=1}^{m(l)} n_i^l$ for each $l \in \mathbb{Z}_+$. One may take such a sequence as $n^l = (m,0,\dots,0) \in \mathbb{Z}^{m(l)}$. We then define

$$\hat{\iota}_{\mathfrak{L}}(m) = [(I-A_{\mathfrak{L}})[(n^l)_{\in\mathbb{Z}_+}]] \in \mathbb{Z}_I/(I-A_{\mathfrak{L}})\mathbb{Z}_{I,0}.$$

Let $(n'^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_I$ be another sequence such that $m = \sum_{i=1}^{m(l)} n'^l_i$. As $\sum_{i=1}^{m(l)}(n_i^l - n'^l_i) = m - m = 0$, we have $(n^l)_{l\in\mathbb{Z}_+} - (n'^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_{I,0}$ so that $[(I-A_{\mathfrak{L}})[(n^l)_{\in\mathbb{Z}_+}]] = [(I-A_{\mathfrak{L}})[(n'^l)_{\in\mathbb{Z}_+}]] \in \mathbb{Z}_I/(I-A_{\mathfrak{L}})\mathbb{Z}_{I,0}$. This shows that $\hat{\iota}_{\mathfrak{L}}(m)$ is independent of the choice of $(n^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_I$ as long as $m = \sum_{i=1}^{m(l)} n_i^l$.

Lemma 2.11. *The diagram*

$$\begin{array}{ccc}\mathbb{Z} & \xrightarrow{\iota_{\mathfrak{L}}} & \mathrm{Ext_s}(\mathcal{O}_{\mathfrak{L}})\\ \| & & \downarrow \mathrm{Ind_s}\\ \mathbb{Z} & \xrightarrow{\hat{\iota}_{\mathfrak{L}}} & \mathbb{Z}_I/(I-A_{\mathfrak{L}})\mathbb{Z}_{I,0}\end{array} \tag{2.10}$$

commutes.

Proof. Take a trivial extension $\tau : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ and a unitary $u_m \in Q(H)$ of Fredholm index $m \in \mathbb{Z}$ such that $\pi(\tau(a))u_m = u_m\pi(\tau(a))$ for $a \in \mathcal{A}_{\mathfrak{L}}$. The extension $\sigma_m : \mathcal{O}_{\mathfrak{L}} \longrightarrow B(H)$ is defined by $\sigma_m = \mathrm{Ad}(u_m) \circ (\pi \circ \tau)$. Put $k_i^l = \mathrm{ind}_{\sigma_m(E_i^l)} u_m = \mathrm{ind}_{\pi(\tau(E_i^l))} u_m$. As $\mathrm{ind}(u_m) = m$, we have $\sum_{i=1}^{m(l)} k_i^l = m$ for each $l \in \mathbb{Z}_+$. Now we have

$$\mathrm{Ind}_s([\sigma_m]_s) = [((\mathrm{ind}_{\sigma_m(E_i^l)} U_{\sigma_m,\tilde{\tau}})_{i=1}^{m(l)})_{l\in\mathbb{Z}_+}] = [((\mathrm{ind}_{\pi(\tau(E_i^l))} \sum_{\alpha\in\Sigma} \sigma_m(S_\alpha)\pi \circ \tau(S_\alpha^*)))_{l\in\mathbb{Z}_+}].$$

Since we have

$$\begin{aligned}
&\mathrm{ind}_{\pi(\tau(E_i^l))} \sum_{\alpha\in\Sigma} \sigma_m(S_\alpha)\pi(\tau(S_\alpha^*)) \\
=&\mathrm{ind}_{\pi(\tau(E_i^l))} u_m (\sum_{\alpha\in\Sigma} \pi(\tau(S_\alpha))u_m^* \pi(\tau(S_\alpha^*))) \\
=&\mathrm{ind}_{\pi(\tau(E_i^l))} u_m + \sum_{\alpha\in\Sigma} \mathrm{ind}_{\tilde{\tau}(E_i^l)} \tilde{\tau}(S_\alpha) u_m^* \tilde{\tau}(S_\alpha^*) \\
=&k_i^l + \sum_{\alpha\in\Sigma} \mathrm{ind}_{\tilde{\tau}(S_\alpha^* E_i^l S_\alpha)} u_m^* \\
=&k_i^l + \sum_{\alpha\in\Sigma} \sum_{j=1}^{m(l)} A_{l,l+1}(i,\alpha,j) \mathrm{ind}_{\tilde{\tau}(E_j^{l+1})} u_m^* \\
=&k_i^l - \sum_{j=1}^{m(l)} A_{l,l+1}^{\mathfrak{L}}(i,j) k_j^{l+1} = (I - A^{\mathfrak{L}})[(k_j^{l+1})_{j=1}^{m(l+1)}],
\end{aligned}$$

we get

$$\mathrm{Ind}_s([\sigma_m]_s) = (I - A^{\mathfrak{L}})[(k_j^{l+1})_{j=1}^{m(l+1)}].$$

As $\hat{\iota}_{\mathfrak{L}}(m) = [(I - A_{\mathfrak{L}})[(k^l)_{l\in\mathbb{Z}_+}]$, we conclude $\mathrm{Ind}_s([\sigma_m]_s) = \hat{\iota}_{\mathfrak{L}}(m)$, proving $\mathrm{Ind}_s(\iota_{\mathfrak{L}}(m)) = \hat{\iota}_{\mathfrak{L}}(m)$. □

Define a homomorphism $s_{\mathfrak{L}} : \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I) \longrightarrow \mathbb{Z}$ by setting $s_{\mathfrak{L}}((n^l)_{l\in\mathbb{Z}_+}) = \sum_{i=1}^{m(l)} n_i^l$ which is independent of $l \in \mathbb{Z}_+$.

Lemma 2.12. *We have a exact sequence*

$$\begin{array}{ccccc}
\mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_{I,0} \longrightarrow \mathbb{Z}_I) & \longrightarrow & \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I) & \xrightarrow{s_{\mathfrak{L}}} & \mathbb{Z} \\
\uparrow & & & & \downarrow \hat{\iota}_{\mathfrak{L}} \\
0 & \longleftarrow & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I & \longleftarrow & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}
\end{array}$$

Proof. It suffices to show that $\mathrm{Ker}(\hat{\iota}_{\mathfrak{L}}) = s_{\mathfrak{L}}(\mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I))$. For $m \in \mathrm{Ker}(\hat{\iota}_{\mathfrak{L}})$ in $\mathbb{Z}$, we have $\hat{\iota}_{\mathfrak{L}}(m) = (I - A_{\mathfrak{L}})[(k^l)_{l\in\mathbb{Z}_+}]$ where $k^l = (k_i^l)_{i=1}^{m(l)}, m = \sum_{i=1}^{m(l)} k_i^l$. Since $\hat{\iota}_{\mathfrak{L}}(m) \in (I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}$, one may find $(n^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_{I,0}$ such that $(I - A_{\mathfrak{L}})[(k^l)_{l\in\mathbb{Z}_+}] = (I - A_{\mathfrak{L}})[(n^l)_{l\in\mathbb{Z}_+}]$. We then have $(I - A_{\mathfrak{L}})[(k^l)_{l\in\mathbb{Z}_+}] - (I - A_{\mathfrak{L}})[(n^l)_{l\in\mathbb{Z}_+}] = 0$ so that $(k^l - n^l)_{l\in\mathbb{Z}_+} \in \mathrm{Ker}(I - A_{\mathfrak{L}})$ and $m = \sum_{i=1}^{m(l)} k_i^l = \sum_{i=1}^{m(l)} (k_i^l - n_i^l)$. This shows that

$$m = s_{\mathfrak{L}}((k^l - n^l)_{l\in\mathbb{Z}_+}) \in s_{\mathfrak{L}}(\mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I)).$$

Conversely, for $(n^l)_{l\in\mathbb{Z}_+} \in \operatorname{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I))$, we have

$$\hat{\iota}_{\mathfrak{L}}(s_{\mathfrak{L}}((n^l)_{l\in\mathbb{Z}_+})) = \hat{\iota}_{\mathfrak{L}}(\sum_{i=1}^{m(l)} n_i^l) = (I - A_{\mathfrak{L}})((n^l)_{l\in\mathbb{Z}_+}) = 0. \qquad \square$$

Following Higson[10] and Higson–Roe [11], for a separable unital nuclear C^*-algebra $\mathcal{A}$ the reduced K-homology groups $\widetilde{\mathrm{K}}^0(\mathcal{A}), \widetilde{\mathrm{K}}^1(\mathcal{A})$ and the unreduced K-homology groups $\mathrm{K}^0(\mathcal{A}), \mathrm{K}^1(\mathcal{A})$ are identified with their extension groups such as

$$\widetilde{\mathrm{K}}^0(\mathcal{A}) = \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{A}), \quad \widetilde{\mathrm{K}}^1(\mathcal{A}) = \mathrm{Ext}_s(\mathcal{A}), \quad \mathrm{K}^0(\mathcal{A}) = \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{A}), \quad \mathrm{K}^1(\mathcal{A}) = \mathrm{Ext}_w(\mathcal{A}),$$

respectively. The groups $\mathrm{Ext}_{\mathrm{s}}(\mathcal{A})$ and $\mathrm{Ext}_{\mathrm{w}}(\mathcal{A})$ are written as $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{A})$ and $\mathrm{Ext}^1_{\mathrm{w}}(\mathcal{A})$, respectively. The isomorphisms $\mathrm{Ind}_{\mathrm{s}} : \mathrm{Ext}_{\mathrm{s}}(\mathcal{A}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}$ and $\mathrm{Ind}_{\mathrm{w}} : \mathrm{Ext}_{\mathrm{w}}(\mathcal{A}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I$ are written as $\mathrm{Ind}^1_{\mathrm{s}}$ and $\mathrm{Ind}^1_{\mathrm{w}}$, respectively. A general theory of K-homology groups for a separable unital nuclear C^*-algebr $\mathcal{A}$ says that the following K-homology long exact sequence holds:

$$0 \longrightarrow \widetilde{\mathrm{K}}^0(\mathcal{A}) \longrightarrow \mathrm{K}^0(\mathcal{A}) \xrightarrow{\iota^*_{\mathbb{C}}} \mathrm{K}^0(\mathbb{C}) = \mathbb{Z} \xrightarrow{\iota} \widetilde{\mathrm{K}}^1(\mathcal{A}) \longrightarrow \mathrm{K}^1(\mathcal{A}) \longrightarrow 0. \tag{2.11}$$

By [14], we have already known that

$$\mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) = \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I, \qquad \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) = \operatorname{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I). \tag{2.12}$$

The homomorphism $\iota^*_{\mathbb{C}} : \mathrm{K}^0(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{K}^0(\mathbb{C})$ in the middle of (2.11) for $\mathcal{A} = \mathcal{O}_{\mathfrak{L}}$ is defined by the natural unital inclusion map $\iota_{\mathbb{C}} : \mathbb{C} \hookrightarrow \mathcal{O}_{\mathfrak{L}}$. As the number $\sum_{i=1}^{m(l+1)} n_i^l$ for $n^l = (n_i^l)_{i=1}^{m(l)} \in \operatorname{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I)$ does not depend on the choice of $l \in \mathbb{Z}_+$, the homomorphism $\iota^*_{\mathbb{C}} : \mathrm{K}^0(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{K}^0(\mathbb{C}) = \mathbb{Z}$ satisfies $\iota^*_{\mathbb{C}}((n^l)_{l\in\mathbb{Z}_+}) = \sum_{i=1}^{m(l+1)} n_i^l$ which does not depend on $l \in \mathbb{Z}_+$. Since $\widetilde{\mathrm{K}}^0(\mathcal{O}_{\mathfrak{L}}) = \operatorname{Ker}(\iota^*_{\mathbb{C}} : \mathrm{K}^0(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{K}^0(\mathbb{C}))$, we know that

$$\widetilde{\mathrm{K}}^0(\mathcal{O}_{\mathfrak{L}}) = \operatorname{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_{I,0} \longrightarrow \mathbb{Z}_I). \tag{2.13}$$

The cyclic six term exact sequence (2.11) says the following lemma.

Lemma 2.13. *The following diagram is commutative:*

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
\widetilde{\mathrm{K}}^0(\mathcal{O}_{\mathfrak{L}}) \arrow[r, equal] \arrow[d] & \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \arrow[r] \arrow[d] & \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_{I,0} \longrightarrow \mathbb{Z}_I) \arrow[d] \\
\mathrm{K}^0(\mathcal{O}_{\mathfrak{L}}) \arrow[r, equal] \arrow[d, "\iota^*_{\mathbb{C}}"'] & \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \arrow[r] \arrow[d, "s_{\mathfrak{L}}"] & \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I) \arrow[d, "s_{\mathfrak{L}}"] \\
\mathrm{K}^0(\mathbb{C}) \arrow[r, equal] \arrow[d, "\iota^*"'] & \mathbb{Z} \arrow[r, equal] \arrow[d, "\iota^*"'] & \mathbb{Z} \arrow[d, "\hat{\iota}_{\mathfrak{L}}"] \\
\widetilde{\mathrm{K}}^1(\mathcal{O}_{\mathfrak{L}}) \arrow[r, equal] \arrow[d] & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \arrow[r, "\mathrm{Ind}^1_{\mathrm{s}}"] \arrow[d] & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0} \arrow[d] \\
\mathrm{K}^1(\mathcal{O}_{\mathfrak{L}}) \arrow[r, equal] \arrow[d] & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \arrow[r, "\mathrm{Ind}^1_{\mathrm{w}}"] \arrow[d] & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I \arrow[d] \\
0 & 0 & 0
\end{tikzcd}
\tag{2.14}
$$

where $\hat{\iota}_{\mathfrak{L}} : \mathbb{Z} \longrightarrow \mathbb{Z}_I/)I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}$ *is defined by* $\hat{\iota}_{\mathfrak{L}}(m) = [(I - A_{\mathfrak{L}})[(n^l)_{l \in \mathbb{Z}_+}]]$ *for* $m = \sum_{i=1}^{m(l)} n_i^l$, $l \in \mathbb{Z}_+$, *and* $s_{\mathfrak{L}} : \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I) \longrightarrow \mathbb{Z}$ *is defined by* $s_{\mathfrak{L}}((n^l)_{l \in \mathbb{Z}_+}) = \sum_{i=1}^{m(l)} n_i^l$ *for* $(n^l)_{l \in \mathbb{Z}_+} \in \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I)$*.*

Corollary 2.14. $\mathrm{Ind}^1_{\mathrm{s}} : \mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}$ *is an isomorphism of abelian groups.*

Proof. By the commutative diagram (2.14), we have a commutative diagram of short exact sequences:

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/s_{\mathfrak{L}}(\mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}})) \arrow[r, "\iota"] \arrow[d] & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \arrow[r] \arrow[d, "\mathrm{Ind}^1_{\mathrm{s}}"'] & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \arrow[r] \arrow[d, "\mathrm{Ind}^1_{\mathrm{w}}"'] & 0 \\
0 \arrow[r] & \mathbb{Z}/s_{\mathfrak{L}}(\mathrm{Ker}(I - A_{\mathfrak{L}})) \arrow[r] & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0} \arrow[r] & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I \arrow[r] & 0.
\end{tikzcd}
$$

Since the two vertical arrows $\mathrm{Ind}^1_{\mathrm{w}} : \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I$ and $\mathbb{Z}/s_{\mathfrak{L}}(\mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}})) \longrightarrow \mathbb{Z}/s_{\mathfrak{L}}(\mathrm{Ker}(I - A_{\mathfrak{L}}))$ are isomorphisms, the five lemma says that the middle homomorphism $\mathrm{Ind}^1_{\mathrm{s}} : \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}$ is isomorphic. □

We therefore obtain the following theorem.

Theorem 2.15 (Theorem 1.2)**.** *Let* $\mathfrak{L}$ *be a left-resolving* λ*-graph system over* Σ*. There* *exist isomorphisms*

$$\begin{aligned}
\mathrm{Ind}^1_{\mathrm{w}} &: \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I,\\
\mathrm{Ind}^1_{\mathrm{s}} &: \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0},\\
\mathrm{Ind}^0_{\mathrm{w}} &: \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I),\\
\mathrm{Ind}^0_{\mathrm{s}} &: \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{\mathfrak{L}}) \longrightarrow \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_{I,0} \longrightarrow \mathbb{Z}_I)
\end{aligned}$$

of abelian groups such that the K*-homology long exact sequence* (2.11) *is computed to be the cyclic six term exact sequence*

$$\begin{tikzcd}
\mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_{I,0} \longrightarrow \mathbb{Z}_I) \arrow[r] & \mathrm{Ker}(I - A_{\mathfrak{L}} : \mathbb{Z}_I \longrightarrow \mathbb{Z}_I) \arrow[r] & \mathbb{Z} \arrow[d] \\
0 \arrow[u] & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_I \arrow[l] & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}. \arrow[l]
\end{tikzcd}$$

3 Examples

3.1 Markov coded systems

Let $G = (V, E)$ be a finite directed graph with vertex set $V = \{v_1, \dots, v_N\}$ and edge set $E = \{e_1, \dots, e_{N_1}\}$. We assume that the graph G is essential, which means that every vertex has at least one incoming edges and at least one outgoing edges. Let b, c be two letters. Consider the set

$$\mathcal{C}_G := \{\overbrace{b \cdots b}^{n}\overbrace{c \cdots c}^{m} e_k \mid k = 1, \dots, N_1,\ n \le m,\ n, m \in \mathbb{N}\} \tag{3.1}$$

which is called a code for G. Define the map $r : \mathcal{C}_G \to E$ by $r(b \cdots bc \cdots ce_k) = e_k$. Put $\Sigma = \{b, c, e_n \mid n = 1, \dots, N_1\}$. Let $\Omega_{(\mathcal{C}_G, r)}$ be a shift invariant set defined by setting

$$\begin{aligned}
\Omega_{(C_G,r)} :=&\{(\omega_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}} \mid \text{there exists } \cdots < k_{-1} < k_0 < k_1 < \cdots \text{ in } \mathbb{Z};\\
&\qquad \omega_{[k_i,k_{i+1})} \in \mathcal{C}_G, t(r(\omega_{[k_i,k_{i+1})})) = s(r(\omega_{[k_i,k_{i+1})})), i \in \mathbb{Z}\}
\end{aligned}$$

where $\omega_{[k_i,k_{i+1})} = \omega_{k_i} \cdots \omega_{k_{i+1}-1}$ and $t(e), s(e)$ for an edge $e \in E$ denote the target vertex and the source vertex, respectively. The set $\Omega_{(C_G,r)}$ is shift invariant but not necessarily closed in $\Sigma^{\mathbb{Z}}$. The closure $\overline{\Omega_{(C_G,r)}}$ is a shift space of a subshift. The subshift is called the Markov coded system and written S_G ([16]). It is a normal subshift in the sense of [18] and not any of topological Markov shifts for every finite directed graph G. There is a λ-graph system written $\mathfrak{L}^{S_G}$ canonically constructed from the Markov coded system S_G. It presents the subshift S_G and is minimal in the sense of [18]. The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{S_G}}$ associated with the λ-graph system $\mathfrak{L}^{S_G}$ is written as $\mathcal{O}_{S_G}$ in [16]. It is shown in [16] that the algebra $\mathcal{O}_{S_G}$ is simple purely infinite if the transition matrix A of the directed graph G is aperiodic, and the K-theory groups and the weak extension groups are such as

$$\begin{aligned}
\mathrm{K}_0(\mathcal{O}_{S_G}) &\cong \mathbb{Z}^N/A\mathbb{Z}^N \oplus \mathbb{Z}^N, & \mathrm{K}_1(\mathcal{O}_{S_G}) &\cong \mathrm{Ker}(A) \text{ in } \mathbb{Z}^N,\\
\mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G}) &\cong (\mathrm{Ker}(A) \text{ in } \mathbb{Z}^N) \oplus \mathbb{Z}^N, & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{S_G}) &\cong \mathbb{Z}^N/A\mathbb{Z}^N.
\end{aligned}$$

We note that in [16] the weak extension groups $\mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{S_G}), \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G})$ are written as $\mathrm{Ext}^0(\mathcal{O}_{S_G}), \mathrm{Ext}^1(\mathcal{O}_{S_G})$, respectively. In what follows, we will compute the strong extension groups $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{S_G}), \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{S_G})$.

Let us denote by 1_N and 0_N the identity matrix of size N and 0 matrix of size N. The canonical λ-graph system $\mathfrak{L}^{S_G}$ and its transition matrices $(A^{\mathfrak{L}^{S_G}}_{l,l+1}, I^{\mathfrak{L}^{S_G}}_{l,l+1})$, written as $(M_{l,l+1}, I_{l,l+1})$ in [16] for $\mathfrak{L}^{S_G}$ was concretely computed such as

$$M_{l,l+1} = \begin{bmatrix} 1_N & 0_N & \cdots & \cdots & \cdots & \cdots & \cdots & 0_N & 0_N & 1_N \\ 0_N & 1_N & \ddots & & & & \iddots & 0_N & 1_N & 0_N \\ \vdots & \ddots & \ddots & \ddots & & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0_N & \cdots & 0_N & 1_N & 0_N & 0_N & 1_N & 0_N & \cdots & 0_N \\ A^t & \cdots & \cdots & A^t & A^t & 0_N & 0_N & \cdots & 0_N & 0_N \\ 0_N & \cdots & \cdots & 0_N & 0_N & 1_N & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & 0_N & 0_N & 0_N \\ 0_N & \cdots & \cdots & 0_N & 0_N & \cdots & 0_N & 1_N & 1_N & 1_N \end{bmatrix}$$

$$I_{l,l+1} = \begin{bmatrix} 1_N & 1_N & 0_N & \cdots & \cdots & \cdots & \cdots & \cdots & 0_N & 0_N \\ 0_N & 0_N & 1_N & \ddots & & & & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & & & \ddots & 0_N & 1_N & 0_N & 0_N \\ 0_N & \cdots & \cdots & \cdots & \cdots & \cdots & 0_N & 0_N & 1_N & 1_N \end{bmatrix}$$

for $3 \le l \in \mathbb{N}$, where both $M_{l,l+1}, I_{l,l+1}$ are $2(l+1) \times 2(l+2)$-block matrices whose entries are $N \times N$-matrices, so they are $m(l) \times m(l+1)$ matrix with $m(l) = 2(l+1)N$. Hence we have

$$M_{l,l+1} - I_{l,l+1} = \begin{bmatrix} 0_N & -1_N & \cdots & \cdots & \cdots & \cdots & \cdots & 0_N & 0_N & 1_N \\ 0_N & 1_N & \ddots & & & & \iddots & 0_N & 1_N & 0_N \\ \vdots & \ddots & \ddots & \ddots & & \iddots & \iddots & \iddots & \iddots & \vdots \\ 0_N & \cdots & 0_N & 1_N & -1_N & 0_N & 1_N & 0_N & \cdots & 0_N \\ A^t & \cdots & \cdots & A^t & A^t & -1_N & 0_N & \cdots & 0_N & 0_N \\ 0_N & \cdots & \cdots & 0_N & 0_N & 1_N & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & -1_N & 0_N & 0_N \\ 0_N & \cdots & \cdots & 0_N & 0_N & \cdots & 0_N & 1_N & 0_N & 0_N \end{bmatrix}$$

Lemma 3.1. *The homomorphism $s_{\mathfrak{L}^{S_G}} : \mathrm{Ker}(I - A_{\mathfrak{L}^{S_G}} : \mathbb{Z}_I \to \mathbb{Z}_I) \to \mathbb{Z}$ in the upper right horizontal arrow in Lemma 2.12 for $\mathfrak{L} = \mathfrak{L}^{S_G}$ is surjective.*

Proof. Since the map $s_{\mathfrak{L}^{S_G}} : \mathrm{Ker}(I - A_{\mathfrak{L}^{S_G}} : \mathbb{Z}_I \to \mathbb{Z}_I) \to \mathbb{Z}$ is defined by $s_{\mathfrak{L}^{S_G}}((n^l)_{l\in\mathbb{Z}_+}) = \sum_{i=1}^{m(l)}$ that is independent of $l \in \mathbb{Z}_+$, we choose $l = 3$ and consider the kernel $\mathrm{Ker}(M_{3,4} - I_{3,4})$. The matrix $M_{3,4} - I_{3,4}$ is of the form:

$$
M_{3,4} - I_{3,4} = \begin{bmatrix}
0_N & -1_N & 0_N & 0_N & 0_N & 0_N & 0_N & 0_N & 0_N & 1_N \\
0_N & 1_N & -1_N & 0_N & 0_N & 0_N & 0_N & 0_N & 1_N & 0_N \\
0_N & 0_N & 1_N & -1_N & 0_N & 0_N & 0_N & 1_N & 0_N & 0_N \\
0_N & 0_N & 0_N & 1_N & -1_N & 0_N & 1_N & 0_N & 0_N & 0_N \\
A^t & A^t & A^t & A^t & A^t & -1_N & 0_N & 0_N & 0_N & 0_N \\
0_N & 0_N & 0_N & 0_N & 0_N & 1_N & -1_N & 0_N & 0_N & 0_N \\
0_N & 0_N & 0_N & 0_N & 0_N & 0_N & 1_N & -1_N & 0_N & 0_N \\
0_N & 0_N & 0_N & 0_N & 0_N & 0_N & 0_N & 1_N & 0_N & 0_N
\end{bmatrix}.
$$

It is easy to see that $[x_i]_{i=1}^{10}$ with $x_i \in \mathbb{Z}^N, i = 1, \dots, 10$ belongs to $\mathrm{Ker}(M_{3,4} - I_{3,4})$ if and only if

$$
\begin{gathered}
A^t(x_1 + x_2 + 3x_3) = 0, \\
x_6 = x_7 = x_8 = 0, \quad x_4 = x_5 = x_3, \quad x_9 = x_3 - x_2, \quad x_{10} = x_2.
\end{gathered}
$$

Hence the map $s_{\mathfrak{L}^{S_G}}((n^l)_{l\in\mathbb{Z}_+}) = \sum_{i=1}^{m(l)}$ is surjective. □

Since the cyclic six term exact sequence (1.3) is rephrased by (1.4), the upper right horizontal arrow $\mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G}) \longrightarrow \mathbb{Z}$ in (1.3) is surjective, so that we have the exact sequences:

$$
\begin{gathered}
0 \longrightarrow \mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{S_G}) \longrightarrow \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G}) \longrightarrow \mathbb{Z} \longrightarrow 0, \\
0 \longrightarrow \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{S_G}) \longrightarrow \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{S_G}) \longrightarrow 0
\end{gathered}
$$

which show that

$$
\mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{S_G}) \oplus \mathbb{Z} \cong \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G}), \qquad \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{S_G}) \cong \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{S_G}).
$$

Since we know that $\mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{S_G}) \cong \mathbb{Z}^N \oplus (\mathrm{Ker}(A) \text{ in } \mathbb{Z}^N)$ by [16], we have the strong extension groups $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{s}}(\mathcal{O}_{S_G}), i = 0, 1$ in the following way.

Proposition 3.2. *Let G be an essential finite directed graph. Suppose that its transition matrix A of G is aperiodic. Let $\mathcal{O}_{S_G}$ be the simple purely infinite C^*-algebra of the Markov coded system for G. Then we have*

$$
\mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{S_G}) \cong \mathbb{Z}^{N-1} \oplus (\mathrm{Ker}(A) \textit{ in } \mathbb{Z}^N), \qquad \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{S_G}) \cong \mathbb{Z}^N / A\mathbb{Z}^N.
$$

3.2 Dyck shifts

We will compute the extension groups for the C^*-algebra $\mathcal{O}_{D_N^{\min}}$ associated to the minimal presentation $\mathfrak{L}_{D_N^{\min}}$ of the Dyck shift D_N for $N \geq 2$. Let $\Sigma^- = \{\alpha_1, \dots, \alpha_N\}$ and $\Sigma^+ = \{\beta_1, \dots, \beta_N\}$ be two kinds of finite sets. Put $\Sigma = \Sigma^+ \cup \Sigma^-$. The Dyck shift D_N is defined to be the subshift over Σ in the following way. Equip the set of finite words of Σ with a monoid structure by

$$\alpha_i\beta_j = \begin{cases} \mathbf{1} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{3.2}$$

Let $\mathfrak{F}_N$ be the set of finite words $(\gamma_1, \dots, \gamma_n)$ of Σ such that the poduct $\gamma_1 \cdots \gamma_n$ is zero in the monoid. The Dyck shift D_N is defined by the subshift over Σ whose fobiddern words are $\mathfrak{F}_N$. This means that D_N is the set of bi-infinite sequences $(\gamma_n)_{n\in\mathbb{Z}}$ of Σ such that $(\gamma_n, \dots, \gamma_{n+1}, \dots, \gamma_{n+k})$ does not belong to $\mathfrak{F}_N$ for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. The subshift has a unique minimal presentation of λ-graph system called the Cantor horizon λ-graph system written $\mathfrak{L}^{\mathrm{Ch}}_{D_N}$ ([12]). In this paper, we call it the minimal presentation and write it as $\mathfrak{L}_{D_N^{\min}} = (V^{\min}, E^{\min}, \lambda^{\min}, \iota^{\min})$. It is constructed as in the following way. The vertex set $V_l^{\min}$ is

$$V_l^{\min} = \{(\beta_{\nu_1}, \dots, \beta_{\nu_l}) \in (\Sigma^+)^l \mid (\nu_1, \dots, \nu_l) \in \{1, \dots, N\}^l\}.$$

A labeled edge labeled β_j is defined as a directed edge from the vertex $(\beta_j, \beta_{\nu_1}, \dots, \beta_{\nu_{l-1}}) \in V_l^{\min}$ to the vertex $(\beta_{\nu_1}, \dots, \beta_{\nu_l}, \beta_{\nu_{l+1}}) \in V_{l+1}^{\min}$. A labeled edge labeled α_j is defined as a directed edge from the vertex $(\beta_{\nu_1}, \dots, \beta_{\nu_l}) \in V_l^{\min}$ to the vertex $(\beta_{\nu_0}, \beta_{\nu_1}, \dots, \beta_{\nu_l}) \in V_{l+1}^{\min}$ if and only if $j = \nu_0$. The set of such edges are denotd by $E_{l,l+1}^{\min}$. The map $\iota : V_{l+1}^{\min} \longrightarrow V_l^{\min}$ is defined by $\iota(\beta_{\nu_1}, \dots, \beta_{\nu_l}, \beta_{\nu_{l+1}}) = (\beta_{\nu_1}, \dots, \beta_{\nu_l})$. We then have a λ-graph system $\mathfrak{L}^{\min}_{D_N}$. It is irreducible and locally contracting in the sense of [17] so that the C^*-algebra $\mathcal{O}_{\mathfrak{L}_{D_N^{\min}}}$ is a unital separable nuclear simple purely infinite C^*-algebra. It is written as $\mathcal{O}_{D_N^{\min}}$. The K-groups were computed as

$$\mathrm{K}_0(\mathcal{O}_{D_N{}^{\min}}) = \mathbb{Z}/N\mathbb{Z} \oplus C(\mathcal{C}, \mathbb{Z}), \qquad \mathrm{K}_1(\mathcal{O}_{D_N{}^{\min}}) = 0$$

in [12], where $C(\mathcal{C}, \mathbb{Z})$ denotes the abelian group of integer valued continuous functions on a Cantor set $\mathcal{C}$. By the universal coefficient theorem

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{A}), \mathbb{Z}) \longrightarrow \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{A}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_1(\mathcal{A}), \mathbb{Z}) \longrightarrow 0,$$
$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(\mathcal{A}), \mathbb{Z}) \longrightarrow \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{A}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{A}), \mathbb{Z}) \longrightarrow 0$$

for a separable unital nuclear C^*-algebra $\mathcal{A}$ proved by L. Brown [4], we know the following proposition.

Proposition 3.3. $\mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) = \mathbb{Z}/N\mathbb{Z}, \qquad \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) = \mathrm{Hom}_{\mathbb{Z}}(C(\mathcal{C}, \mathbb{Z}), \mathbb{Z}).$

In this section, we will compute the other extension groups $\mathrm{Ext}_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}})$ and $\mathrm{Ext}_{\mathrm{s}}{}^0(\mathcal{O}_{D_N^{\min}})$. We will consider the cases for $N = 2$, so that the alphabet of D_2 is $\Sigma = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$. Let $(A_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ be the structre matrix for the minimal presentation $\mathfrak{L}_{D_N^{\min}}$. The cardinality $m(l)$ of the vertex set $V_l^{\min}$ is 2^l. As in [12], define $m(l) \times m(l+1)$ matrices

$J_{l,l+1},\ K_{l,l+1},\ L_{l,l+1},\ l \in \mathbb{Z}_+$ such as

$$J_{0,1} = [1,1], \qquad J_{1,2} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad J_{2,3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \dots$$

$$J_{l,l+1} = [J_{l,l+1}(i,j)]_{i=1,2,\dots,m(l)}^{j=1,2,\dots,m(l+1)} \quad \text{where } J_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } j = i,\ m(l)+i, \\ 0 & \text{otherwise,} \end{cases}$$

$$K_{0,1} = [1,1], \qquad K_{1,2} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad K_{2,3} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \dots$$

$$K_{l,l+1} = [K_{l,l+1}(i,j)]_{i=1,2,\dots,m(l)}^{j=1,2,\dots,m(l+1)}$$

$$\text{where } K_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } j = 4i-3,\ 4i-2,\ 4i-1,\ 4i \text{ for } 1 \le i \le m(l-1), \\ 1 & \text{if } j = 4i-3-m(l-1),\ 4i-2-m(l-1), \\ & \quad 4i-1-m(l-1),\ 4i-m(l-1) \text{ for } m(l-1)+1 \le i \le m(l) \\ 0 & \text{otherwise,} \end{cases}$$

$$L_{0,1} = [1,1], \qquad L_{1,2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \qquad L_{2,3} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \dots$$

$$L_{l,l+1} = [L_{l,l+1}(i,j)]_{i=1,2,\dots,m(l)}^{j=1,2,\dots,m(l+1)} \quad \text{where } L_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } j = 2i-1,\ 2i, \\ 0 & \text{otherwise,} \end{cases}$$

We directly see the following lemma.

Lemma 3.4. *$I_{l,l+1} = L_{l,l+1}$ and $A_{l,l+1} = J_{l,l+1} + K_{l,l+1}$ for $l \in \mathbb{Z}_+$ so that we have*

$$I_{l,l+1} - A_{l,l+1} = L_{l,l+1} - J_{l,l+1} - K_{l,l+1}, \qquad l \in \mathbb{Z}_+.$$

For example

$$I_{0,1} - A_{0,1} = [-2,-2], \qquad I_{1,2} - A_{1,2} = \begin{bmatrix} -1 & 0 & -2 & -1 \\ -1 & -2 & 0 & -1 \end{bmatrix},$$

$$I_{2,3} - A_{2,3} = \begin{bmatrix} -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & -1 & -2 & -1 & -1 \\ -1 & -1 & -2 & -1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 \end{bmatrix}, \dots$$

The following lemmas are straightforward.

Lemma 3.5. *Let $\xi : \mathbb{Z}_I \longrightarrow \mathbb{Z}$ be the homomorphism defined by $\xi(([n_i^l]_{i=1}^{m(l)})_{l \in \mathbb{Z}_+}) = \sum_{i=1}^{m(l)} n_i^l \in \mathbb{Z}$.*

(i) *The value* $\sum_{i=1}^{m(l)} n_i^l$ *for each* $l \in \mathbb{Z}_+$ *does not depend on* $l \in \mathbb{Z}_+$.

(ii) $\operatorname{Ker}(\xi) = \mathbb{Z}_{I,0}$, *so that* $\mathbb{Z}_I/\mathbb{Z}_{I,0}$ *is isomorphic to* $\mathbb{Z}$.

Lemma 3.6. (i) $I_{l,l+1}\mathbb{Z}_0^{m(l+1)} \subset \mathbb{Z}_0^{m(l)}$, $\quad A_{l,l+1}\mathbb{Z}_0^{m(l+1)} \subset \mathbb{Z}_0^{m(l)}$ *so that*

$$(I_{l,l+1} - A_{l,l+1})\mathbb{Z}_0^{m(l+1)} \subset \mathbb{Z}_0^{m(l)}.$$

(ii) *The diagram*

$$\begin{tikzcd} \mathbb{Z}^{m(l+1)} \arrow[r, "\xi_{l+1}"] \arrow[d, "I_{l,l+1}"'] & \mathbb{Z} \arrow[d, equal] \\ \mathbb{Z}^{m(l)} \arrow[r, "\xi_l"] & \mathbb{Z} \end{tikzcd}$$

commutes.

Lemma 3.7. *For any* $[n_i^l]_{i=1}^{m(l)} \in \mathbb{Z}_0^{m(l)}$ *and* $[m_i^l]_{i=1}^{m(l)} \in \mathbb{Z}_0^{m(l)}$, *there exists* $[m_j^{l+1}]_{j=1}^{m(l+1)} \in \mathbb{Z}_0^{m(l+1)}$ *such that*

(1) $[m_i^l]_{i=1}^{m(l)} = I_{l,l+1}[m_j^{l+1}]_{j=1}^{m(l+1)}$,

(2) $[n_i^l]_{i=1}^{m(l)} = (I_{l,l+1} - A_{l,l+1})[m_j^{l+1}]_{j=1}^{m(l+1)}$.

Proof. The condition (1) is rephrazed as

$$m_i^l = m_{2i-1}^{l+1} + m_{2i}^{l+1}, \qquad i = 1, \dots, m(l),$$

where $m(l) = 2^l$, $m(l+1) = 2^{l+1}$. Since

$$\begin{aligned} & K_{l,l+1}[m_j^{l+1}]_{j=1}^{m(l+1)} \\ =& [m_1^{l+1} + m_2^{l+1} + m_3^{l+1} + m_4^{l+1}, \dots, m_{m(l+1)-3}^{l+1} + m_{m(l+1)-2}^{l+1} + m_{m(l+1)-1}^{l+1} + m_{m(l+1)}^{l+1}] \\ =& [m_1^l + m_2^l, \dots, m_{m(l)-1}^l + m_{m(l)}^l] \end{aligned}$$

and

$$\begin{aligned} & L_{l,l+1}[m_j^{l+1}]_{j=1}^{m(l+1)} \\ =& [m_1^{l+1} + m_2^{l+1}, \dots, m_{m(l+1)-1}^{l+1} + m_{m(l+1)}^{l+1}] = [m_1^l, \dots, m_{m(l)}^l] \end{aligned}$$

it suffices to show that for given $[n_i^l]_{i=1}^{m(l)}, [m_i^l]_{i=1}^{m(l)} \in \mathbb{Z}_0^{m(l)}$, one may find $[m_j^{l+1}]_{j=1}^{m(l+1)} \in \mathbb{Z}^{m(l+1)}$ such that

$$m_i^l = m_{2i-1}^{l+1} + m_{2i}^{l+1}, \qquad n_i^l = -m_i^{l+1} - m_{i+m(l)}^{l+1}, \qquad i = 1, \dots, m(l).$$

Since $I_{l,l+1} - A_{l,l+1} = L_{l,l+1} - J_{l,l+1} - K_{l,l+1}$, this is possible because of the form of $J_{l,l+1}$ $\square$

Therefore we have

Lemma 3.8. *The equality* $(I_{l,l+1} - A_{l,l+1})\mathbb{Z}_0^{m(l+1)} = \mathbb{Z}_0^{m(l)}$ *holds for each* $l \in \mathbb{Z}_+$, *so that we have* $(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0} = \mathbb{Z}_{I,0}$.

We reach the following theorem.

Theorem 3.9. $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_2^{\min}}) \cong \mathbb{Z}$.

Proof. We have $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_2^{\min}}) = \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0}$. By Lemma 3.8 together with Lemma 3.5 (ii), we obtain $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_2^{\min}}) \cong \mathbb{Z}$. $\square$

Although the weak extension group $\mathrm{Ext}_{\mathrm{w}}(\mathcal{O}_{D_2^{\min}})$ for $\mathcal{O}_{D_2^{\min}}$ had been computed to be $\mathbb{Z}/2\mathbb{Z}$ in [12] through K-group computation $\mathrm{K}_*(\mathcal{O}_{D_2^{\min}})$ and the universal coefficient theorem, we may give another proof without using the K-group formulas in the following way.

Proposition 3.10. *The diagram*

$$\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\iota} & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_2^{\min}}) \\
\| & & \big\downarrow \mathrm{Ind}^1_{\mathrm{s}} \\
\mathbb{Z} & \xrightarrow{\hat{\iota}_{\mathfrak{L}}} & \mathbb{Z}_I/(I - A_{\mathfrak{L}})\mathbb{Z}_{I,0} = \mathbb{Z}
\end{array}$$

commutes, where $\hat{\iota}_{\mathfrak{L}}(m) = -2m$ *for* $m \in \mathbb{Z}$.

Proof. Recall that for $m = \sum_{j=1}^{m(l+1)} n_j^{l+1}$ with $n^{l+1} = [n_j^{l+1}]_{j=1}^{m(l+1)}$ and $(n^l)_{l\in\mathbb{Z}_+} \in \mathbb{Z}_I$, we have

$$\begin{aligned}
\hat{\iota}_{\mathfrak{L}}(m) =& \xi_l((I - A_{\mathfrak{L}})[[n_j^{l+1}]_{j=1}^{m(l+1)}]) \\
=& \sum_{i=1}^{m(l)} \sum_{j=1}^{m(l+1)} (I_{l,l+1}(i,j) - A_{l,l+1}(i,j))([[n_j^{l+1}]_{j=1}^{m(l+1)}]).
\end{aligned}$$

Since $\sum_{i=1}^{m(l)} (I_{l,l+1}(i,j) - A_{l,l+1}(i,j)) = -2$ for each $j = 1, \dots, m(l)$, we have

$$\hat{\iota}_{\mathfrak{L}}(m) = \sum_{j=1}^{m(l+1)} (-2) n_j^{l+1} = -2 \sum_{j=1}^{m(l+1)} n_j^{l+1} = -2m. \qquad \square$$

Corollary 3.11. *The diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_2^{\min}}) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_2^{\min}}) & \longrightarrow & 0 \\
 & & \| & & \mathrm{Ind}^1_{\mathrm{s}} \big\downarrow & & \mathrm{Ind}^1_{\mathrm{w}} \big\downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times(-2)} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0
\end{array}$$

is commutative such that the vertical arrows $\mathrm{Ind}^1_{\mathrm{s}}$ *and* $\mathrm{Ind}^1_{\mathrm{w}}$ *are both isomorphic so that we have* $\mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_2^{\min}}) \cong \mathbb{Z}/2\mathbb{Z}$.

As in the proof of [12, Section 5], one may easily generalize the above discussion of D_2 to general Dyck shifts $D_N, 2 \le N \in \mathbb{N}$ to get the following theorem. Since the proof of the generalization is direct and tedious, so we omit the proof.

Theorem 3.12. *There is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) & \longrightarrow & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle \mathrm{Ind}^1_{\mathrm{s}}}\downarrow & & {\scriptstyle \mathrm{Ind}^1_{\mathrm{w}}}\downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times(-N)} & \mathbb{Z} & \longrightarrow & \mathbb{Z}/N\mathbb{Z} & \longrightarrow & 0
\end{array}$$

such that the vertical arrows $\mathrm{Ind}^1_{\mathrm{s}}$ *and* $\mathrm{Ind}^1_{\mathrm{w}}$ *are both isomorphic so that we have* $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) \cong \mathbb{Z}$ *and* $\mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) \cong \mathbb{Z}/N\mathbb{Z}$.

Corollary 3.13. $\mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) = \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) = \mathrm{Hom}_{\mathbb{Z}}(C(\mathcal{C},\mathbb{Z}),\mathbb{Z})$, *where* $C(\mathcal{C},\mathbb{Z})$ *is the abelian group of integer valued continuous functions on a Cantor set* $\mathcal{C}$.

Proof. There exists a cyclic six term exact sequence

$$\begin{array}{ccccc}
\mathrm{Ext}^0_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) & \longrightarrow & \mathrm{Ext}^0_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) & \xrightarrow{\partial} & \mathbb{Z} \\
\uparrow & & & & \downarrow{\scriptstyle \times N} \\
0 & \longleftarrow & \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) & \longleftarrow & \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}})
\end{array} \tag{3.3}$$

for the C^*-algebra $\mathcal{O}_{D_N^{\min}}$. Since the map $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) \xleftarrow{\times N} \mathbb{Z}$ is injective, the connecting map $\partial : \mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) \longrightarrow \mathbb{Z}$ is zero map, so that we have $\mathrm{Ext}^1_{\mathrm{w}}(\mathcal{O}_{D_N^{\min}}) \cong \mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}})$. As in [12, Section 5], $\mathrm{Ext}^1_{\mathrm{s}}(\mathcal{O}_{D_N^{\min}}) \cong \mathrm{Hom}_{\mathbb{Z}}(C(\mathcal{C},\mathbb{Z}),\mathbb{Z})$, we get the assertioon. □

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