Symmetry and Dynamical Analysis of a Discrete Time Model: The Higher Order Berverton-Holt Equation

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Abstract

In this paper, we study the higher-order Beverton-Holt equation. We derive non trivial symmetries, and thereafter, solutions are obtained. For constant rate and carrying capacity, we study the periodic nature of the solution and analyze the stability of the equilibrium points have been analyzed.

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1 Introduction

Population studies is a scientific study of animal and human populations. More often, this study involves parameters that are highly connected to the age, gender, geographic distribution as well as the evolution of the population understudy. Many models have been developed: logistic model, the Ricker model, the Beverton-Holt model, etc. The latter was first introduced in the topic of fisheries by Berverton and Holt in the twentieth century and is known to be [1]

$$z_{n+1} = \frac{\mu K z_n}{K + (1-\mu)z_n},$$

where $K > 0, \mu > 1$ are respectively, the carrying capacity and the inherent growth rate. Later, many authors were attracted by the periodically forced nonautonomous delay higher-order Beverton-Holt model [2]

$$z_{n+k} = \frac{\mu_n K_n z_n}{K_n + (\mu_n - 1) z_n}, \qquad n \in \mathbb{N}_0,$$
(1.1)

where $\mu_n > 1$, $K_n > 0$ are non-negative *p*-periodic sequences and the initial conditions $z_i, i = 0, \ldots, k - 1$, are all positive and interesting results were obtained.

In this work, we study the invariance properties of (1.1) and we construct its solutions via the invariant of their group of transformations. As one would expect, the study of existence of periodic solutions and the stability character of the solutions becomes easier once the form of the solutions are known. Higher-order difference equations have been studied from different angles by many researchers [3–6, 9, 10, 12].

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1.1 Preliminaries

For a deeper knowledge of symmetry analysis of recurrence equations, one can refer to [8] from which most of our notation and theorems are picked. To start, let $\mathbf{x} = (x_1, \ldots, x_q)$ be some continuous variables in a given differential equation and

$$\mathcal{T}: \mathbf{x} \to \bar{\mathbf{x}}(\mathbf{x})$$

be a (local) point transformations.

Definition 1.1. A parameterized set of transformations

$$\mathcal{T}_{\varepsilon}(\boldsymbol{x}) \equiv \hat{\boldsymbol{x}}(\boldsymbol{x};\varepsilon) \tag{1.2}$$

is a one-parameter local Lie group of transformations if the set of conditions below is satisfied:

- 1. \mathcal{T}_0 is the identity map, so that $\hat{\boldsymbol{x}} = \boldsymbol{x}$ when $\varepsilon = 0$.
- 2. $\mathcal{T}_{\gamma}\mathcal{T}_{\varepsilon} = \mathcal{T}_{\gamma+\varepsilon}$ for every γ, ε sufficiently close to 0.
- 3. Every \hat{x}_{α} can be represented as a Taylor series in ε , that is,

$$\hat{x}_{\alpha}(\boldsymbol{x};\varepsilon) = x_{\alpha} + \varepsilon \eta_{\alpha}(\boldsymbol{x}) + O(\varepsilon^2), \qquad \alpha = 0, 1, \dots, q$$

Definition 1.2. The infinitesimal generator of the one-parameter Lie group of point transformations (1.2) is the operator

$$X = X(\boldsymbol{x}) = \eta(\boldsymbol{x}) \cdot \Delta = \sum_{\alpha=1}^{q} \eta_{\alpha}(\boldsymbol{x}) \frac{\partial}{\partial x_{\alpha}}, \qquad (1.3)$$

and Δ is the gradient operator.

Theorem 1.1. F(x) is invariant under the Lie group of transformations (1.2) if and only if

$$XF(x) = 0. \tag{1.4}$$

Consider a forward difference equation

$$z_{n+k} = \mathcal{A}(n, z_n, z_{n+1}, \dots, z_{n+k-1}), \qquad n \in D$$

$$(1.5)$$

of order k where D is a regular domain. We strive to find a Lie group of point transformations

$$\Phi(n, z_n) = (n, z_n + \varepsilon \zeta(n, z_n)).$$
(1.6)

Observe that ε is the group parameter and ζ is the characteristic of the group of transformations. Let

$$G = Q(n, z_n)\frac{\partial}{\partial z_n} + Q(n+1, z_{n+1})\frac{\partial}{\partial z_{n+1}} + \dots + Q(n+k-1, z_{n+k-1})\frac{\partial}{\partial z_{n+k-1}}$$
(1.7)

be the prolonged generator admitted by the group of point transformations (1.6). Then the invariance condition reads

$$\zeta(n+k, z_{n+k}) - \frac{\partial \mathcal{A}}{\partial z_{n+k-1}} \zeta(n+k-1, z_{n+k-1}) - \dots - \frac{\partial \mathcal{A}}{\partial z_n} \zeta(n, z_n) = 0 \qquad (1.8)$$

subject to (1.5). Symmetries are powerful tools for reduction of order of differential and difference equations. In this paper, the reduction of order will be achieved using symmetries and the well-known canonical coordinate [11].

$$S_n = \int \frac{dz_n}{\zeta(n, z_n)}.$$
(1.9)

Definition 1.3. The equilibrium point z of (1.5) is stable (locally) if

$$\forall \epsilon > 0, \exists \delta > 0 : \sum_{i=0}^{k-1} |z_i - z| < \delta \implies |z_n - z| < \epsilon$$
(1.10)

for all solutions $\{z_n\}_{n=0}^{\infty}$ of (1.5).

Definition 1.4. The equilibrium point z of (1.5) is a global attractor if $z_n \to z$, as $n \to \infty$, for any solution $\{z_n\}_{n=0}^{\infty}$ of (1.5).

Definition 1.5. The equilibrium point z of (1.5) is globally asymptotically stable if z is locally stable and it is a global attractor of (1.5).

The polynomial

$$\lambda^{k} - p_{k-1}\lambda^{k-1} - p_{k-2}\lambda^{k-2} - \dots - p_{1}\lambda^{1} - p_{0} = 0$$
(1.11)

where

$$p_i = \frac{\partial \mathcal{A}}{\partial z_{n+i}}(z, z, \dots, z, z)$$
(1.12)

is referred to as the characteristic equation of (1.5) near z.

Theorem 1.2. Suppose \mathcal{A} is a smooth function defined on some neighborhood of z. Then,

- (i) If all the roots, λ_i , of (1.11) are such that $|\lambda_i| < 1$, then the equilibrium point \bar{x} is locally asymptotically stable.
- (ii) If at least one root of (1.11) has absolute value greater than one, then the equilibrium point \bar{x} of (1.11) is unstable.

Definition 1.6. The equilibrium point z of (1.5) is called non-hyperbolic if there exists a root of (1.11) with absolute value equal to one.

Theorem 1.3. Suppose the p_i 's are real numbers satisfying

$$|p_0| + |p_1| + \dots + |p_{k-1}| < 1.$$

Then, the roots of (1.11) lie inside the open unit disk $|\lambda| < 1$.

The above definitions and theorems can be found in [7, 8].

2 Main results

2.1 Symmetries

For the sake of aesthetics, we rewrite the Beverton-holt equation as

$$z_{n+k} = \frac{z_n}{A_n + B_n z_n},\tag{2.1}$$

with

$$A_n = \frac{1}{\mu_n} \qquad \text{and} \qquad B_n = \frac{\mu_n - 1}{K_n \mu_n}.$$
(2.2)

In the Beverton-Holt model, μ_n is set to be greater than one. This implies that A_n should be less that one. In this paper, we investigate solutions that are mathematically correct, so without loss of generality, we will will assume that A_n is simply a real number. Seeking for Lie symmetries, we force the criterion of invariance (1.8) on (1.1). This yields

$$\zeta(n+k,\mathcal{A}) - \frac{A_n}{\left(A_n + B_n z_n\right)^2} \zeta(n,u_n) = 0.$$
(2.3)

Solving the functional equation above, we obtain (after a set of lengthy computations) the infinitesimals:

(i)

$$\zeta_1(n, z_n) = \alpha_n + \frac{B_n}{A_n} \alpha_n z_n \tag{2.4}$$

where $A_n \alpha_{n+k} - \alpha_n = 0$, that is to say,

$$\alpha_{kn+j} = \alpha_j \left(\prod_{k_1=0}^{n-1} \frac{1}{A_{kk_1+j}} \right), \qquad j = 0, \dots, k-1;$$
(2.5)

(ii)

$$\zeta_2(n, z_n) = \beta_n z_n^2 \tag{2.6}$$

where $\beta_{n+k} - A_n \beta_n = 0$, that is to say,

$$\beta_{kn+j} = \beta_j \left(\prod_{k_1=0}^{n-1} A_{kk_1+j} \right), \qquad j = 0, \dots, k-1;$$
(2.7)

(iii)

$$\zeta_3(n, z_n) = \lambda_n z_n + \gamma_n z_n^2, \qquad (2.8)$$

where $\lambda_{n+k} - \lambda_n = 0$ and $\gamma_{n+k} - A_n \gamma_n + B_n \lambda_n = 0$, that is to say,

$$\lambda_n = e^{i\frac{2pn\pi}{k}} \tag{2.9}$$

and

$$\gamma_{kn+j} = \gamma_j \left(\prod_{k_1=0}^{n-1} A_{kk_1+j}\right) + \sum_{l=0}^{n-1} B_{kn+j} \lambda_{kn+j} \prod_{k_2=l+1}^{n-1} A_{kk_1+j}$$
(2.10)

with p = 0, ..., k - 1 and j = 0, ..., k - 1.

Using these infinitesimals, we obtain the following symmetries:

$$X_1 = \left(\alpha_n + \frac{B_n}{A_n} \alpha_n z_n\right) \frac{\partial}{\partial z_n}$$
(2.11)

$$X_2 = \beta_n z_n^2 \frac{\partial}{\partial z_n} \tag{2.12}$$

$$X_3 = \left(\lambda_n z_n + \gamma_n z_n^2\right) \frac{\partial}{\partial z_n} \tag{2.13}$$

where α_n , β_n , γ and λ_n are given in equations (2.5), (2.7), (2.9) and (2.10), respectively.

2.2 Canonical coordinate closed form solution

We select the infinitesimal ζ_2 to lower the order (1.1). Thus, the canonical coordinate takes the form

$$S_n = \int \frac{dz_n}{\zeta_2(n, z_n)} \tag{2.14}$$

$$= -\frac{1}{\beta_n z_n} \tag{2.15}$$

and therefore

$$S_{n+k}\beta_{n+k} - A_n S_n \beta_n = -\frac{1}{z_{n+k}} + \frac{A_n}{z_n}$$
(2.16)

$$= -B_n. (2.17)$$

Setting $\tilde{S}_n = -\beta_n S_n$, we find that

$$\tilde{S}_{n+k} = A_n \tilde{S}_n + B_n \tag{2.18}$$

and we take notice of

$$\tilde{S}_n = \frac{1}{z_n}.\tag{2.19}$$

By iterating (2.18), we get

$$\tilde{S}_{kn+j} = \tilde{S}_j \left(\prod_{k_1=0}^{n-1} A_{kk_1+j} \right) + \sum_{l=0}^{n-1} \left(B_{kl+j} \prod_{k_2=l+1}^{n-1} A_{kk_2+j} \right), \qquad (2.20)$$

for j = 0, ..., k - 1. We reverse the order created by the change of variables to derive the solution of (1.1). We observe from (2.19) that

$$z_n = \frac{1}{\tilde{S}_n}$$

from which we obtain

$$z_{kn+j} = \frac{1}{\tilde{S}_{kn+j}} = \frac{z_j}{\left(\prod_{k_1=0}^{n-1} A_{kk_1+j}\right) + z_j \sum_{l=0}^{n-1} \left(B_{kl+j} \prod_{k_2=l+1}^{n-1} A_{kk_2+j}\right)}, \quad j = 0, \dots, k-1.$$
(2.21)

Recall that $A_n = \frac{1}{\mu_n}$ and $B_n = \frac{\mu_n - 1}{K_n \mu_n}$. So, the closed form solution to the Beverton-Bolt (1.1) is given by

$$z_{kn+j} = \frac{z_j}{\left(\prod_{k_1=0}^{n-1} \frac{1}{\mu_{kk_1+j}}\right) + z_j \sum_{l=0}^{n-1} \left(\frac{\mu_{kl+j}-1}{K_{kl+j}\mu_{kl+j}} \prod_{k_2=l+1}^{n-1} \frac{1}{\mu_{kk_2+j}}\right)}, \quad j = 0, \dots, k-1. \quad (2.22)$$

provided the denominators are non-zero.

2.3 Periodicity in the growth rate and the carrying capacity

The goal of the next section is to investigate the form of the solution when the growth rate and the carrying capacity are 1 or k- periodic sequences.

2.3.1 The case when μ_n and K_n are k-periodic

In this case, invoking (2.2), we have that $A_{k+j} = A_j$ and $B_{k+j} = B_j$ for all j. This reduces (2.21) into (after using the formula for a geometric progression)

$$z_{kn+j} = \frac{z_j}{A_j^n + z_j B_j \left(\frac{1-A_j^n}{1-A_j}\right)}$$
$$= \frac{(1-A_j)z_j}{(1-A_j)A_j^n + B_j (1-A_j^n) z_j}, \quad j = 0, \dots, k-1$$
(2.23)

when $A_j \neq 1$ and $B_j \neq 1$.

2.3.2 The case when μ_n and K_n are 1-periodic (constant)

Here, thanks to (2.2), we may assume that $(A_n) = (A, A, ...)$ and $B_n = (B, B, ...)$.

(i) When A = 1, the solution given in equation (2.21) becomes

$$z_{kn+j} = \frac{z_j}{1+nBz_j}, \quad j = 0, \dots, k-1.$$
 (2.24)

(ii) When $A \neq 1$, the formula solution (2.22) reduces to

$$z_{kn+j} = \frac{z_j}{A^n + z_j B\left(\frac{1-A^n}{1-A}\right)}, \quad j = 0, \dots, k-1.$$
 (2.25)

2.4 Periodicity in the solution and analysis of the stability of the equilibrium points

In this section, we investigate the existence of periodic solutions and the nature of the equilibrium points of the model under study. We demonstrate that periodic solutions do exist under certain restrictions.

Theorem 2.1. The solution z_n of

$$z_{n+k} = \frac{z_n}{A_n + B_n z_n},\tag{2.26}$$

where $A_n \neq 1$, is k-periodic if and only the following conditions are met:

- (i) The sequence A_n and B_n periodic with period k.
- (ii) The initial conditions, x_j , $j = 0, \ldots, k-1$, satisfy $z_j = \frac{1-A_j}{B_j}$.

Proof. Suppose the initial conditions satisfy the conditions $z_j = \frac{1-A_j}{B_j}$. Using the latter in (2.23), we have that:

$$z_{kn+j} = \frac{z_j}{A_j^n + \frac{1 - A_j}{B_j} B_j \left(\frac{1 - A_j^n}{1 - A_j}\right)}$$

= $z_j, \quad j = 0, \dots, k - 1.$ (2.27)

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 z_{n+8}

It follows z_n is periodic with period divisible by k. With the choice of the restriction on the initial conditions, the period can not be less that k. Therefore, z_n is periodic with period k.



In Figure 1, we used the initial conditions

$$z_j = \frac{-2 - \sin(\frac{j\pi}{8})}{2 + \cos(\frac{j\pi}{8})},$$

 $j = 0, \ldots, 15$. These initial conditions satisfy the conditions in Theorem 2.1. As predicted, we have 16-periodic solutions.

In Figure 2, we used the initial conditions

$$z_j = \frac{-2 - \sin\left(\frac{j\pi}{4}\right)}{2 + \cos\left(\frac{j\pi}{4}\right)},$$

 $j = 0, \ldots, 7$. These initial conditions satisfy the conditions in Theorem 2.1. As predicted, we have 8-periodic solutions.

Theorem 2.2. The solution z_n of

$$z_{n+k} = \frac{z_n}{A + Bz_n},\tag{2.28}$$

where $A \neq 1$, is 1-periodic if and only the initial conditions, x_j , $j = 0, \ldots, k-1$, satisfy $z_j = \frac{1-A}{B}$.

Proof. The proof is identical to the proof of Theorem 2.1 and is omitted. \Box

Theorem 2.3. The solution z_n of

$$z_{n+k} = \frac{z_n}{-1 + Bz_n},$$
(2.29)

is 2k periodic and contains two cycles of length k.

Proof. We recall that the solution of the model for constant coefficients A_n and B_n is given in (2.25). Setting A = -1 in (2.25), we get:

$$z_{kn+j} = \frac{z_j}{(-1)^n + z_j B\left(\frac{1-(-1)^n}{1+1}\right)}, \quad j = 0, \dots, k-1.$$
(2.30)

It follows that

$$z_{2kn+j} = \frac{z_j}{(-1)^{2n} + z_j B\left(\frac{1-(-1)^{2n}}{1+1}\right)}, \quad j = 0, \dots, k-1$$
(2.31)

$$= z_j. \tag{2.32}$$

In other words,

$$\left\{\dots, z_0, z_1, \dots, z_{k-1}, \frac{z_0}{-1 + z_0 B}, \frac{z_1}{-1 + z_1 B}, \dots, \frac{z_{k-1}}{-1 + z_{k-1} B}, z_0, z_1, \dots\right\}$$

are the 2k periodic solutions with the two cycles of length k.



In Figure 3, we used the initial conditions $z_0 = 1$; $z_1 = 2$; $z_2 = 1$; $z_3 = -1/2$; $z_4 = 1$; $z_5 = 1/2$; $z_6 = -1/4$; $z_7 = 1/2$. As predicted, we have 16-periodic solutions. In Figure 4, we used the initial conditions $z_0 = 1$; $z_1 = 2$; $z_2 = 1$; $z_3 = -1/2$; $z_4 = 1$; $z_5 = 1/2$; $z_6 = -4$; $z_7 = 1/2$; $z_8 = 11$; $z_9 = 5z_{10} = 8$; $z_{11} = -4$; $z_{12} = -1/2$; $z_{13} = 1$. As predicted, we have 28-periodic solutions.

Theorem 2.4. If $A \neq 1$, then the equilibrium point $\bar{z} = 0$ of (1.1) is asymptotically stable when |A| > 1 and unstable when |A| < 1. Moreover, the non zero equilibrium point z = (1 - A)/B of (1.1) is asymptotically stable when |A| < 1 and unstable when |A| > 1.

Proof. The equilibrium points of (1.1) are the solutions of the equation $\bar{z}(A+B\bar{z}-1)=0$. If we let

$$z_{n+k} = f(z_n) = \frac{z_n}{A + Bz_n},$$
(2.33)

then:

- For $\bar{z} = 0$, it is easy to see that $f_{,z_n}(0) = 1/A$ and so, the characteristic equation of (1.1) around this equilibrium point is given by $\lambda^k \frac{1}{A} = 0$. It follows that $|\lambda_i| < 1$ when |A| > 1, in order words, $\bar{z} = 0$ is locally asymptotically stable. On the other hand, $|\lambda_i| > 1$ when |A| < 1 or in order words, $\bar{z} = 0$ unstable.
- For $\bar{z} = (1 A)/B$, $f_{,z_n}(0) = A$ and the characteristic equation of (1.1) around this equilibrium point is $\lambda^k - A = 0$. Consequently, when |A| < 1, the roots λ_i 's of this characteristic equation are such that $\lambda_i | < 1$ and therefore the equilibrium is asymptotically stable in this case. Similarly, when |A| > 1, the moduli of the roots are greater than one and therefore the equilibrium is unstable in this case.



In Figures 3 and 4, we used the initial conditions $z_0 = 1$; $z_1 = 2$; $z_2 = 1$; $z_3 = -1/2$; $z_4 = 1$; $z_5 = 1/2$; $z_6 = -4$; $z_7 = 1/2$; $z_8 = 1$; $z_9 = 15$, $z_{10} = 8$; $z_{11} = 4$; $z_{12} = -1/2$; $z_{13} = 1$.

Theorem 2.5. If A = 1, then the equilibrium point $\bar{z} = 0$ of (1.1) is non-hyperbolic.

Proof. Here, the only equilibrium point of (1.1) is $\overline{z} = 0$. One can readily check that the characteristic equation of (1.1) around this equilibrium point is given by $\lambda^k - 1 = 0$. There exists a root of $\lambda^k - 1 = 0$ with modulus one. This completes the proof. \Box

3 Conclusion

We performed the invariance analysis of the higher-order Beverton-Holt difference equation. Symmetries and the formula solutions are presented. We utilized the canonical coordinate to derive invariants that have been used reduce linearize the equation and eventually obtained the solutions in closed form. We have also presented periodic solutions that satisfy certain ansatz. Lastly, we studied the stability of the equilibrium points of the model.

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