Null controllability for stochastic fourth order semi-discrete parabolic equations*

Yu Wang† and Qingmei Zhao‡

Abstract

This paper is devoted to studying null controllability for a class of stochastic fourth order semi-discrete parabolic equations, where the spatial variable is discretized with finite difference scheme and the time is kept as a continuous variable. For this purpose, we establish a new global Carleman estimate for a backward stochastic fourth order semi-discrete parabolic operators, in which the large parameter is connected to the mesh size. A relaxed observability estimate is established for backward stochastic fourth order semi-discrete parabolic equations by this new Carleman estimate, with an explicit observability constant that depends on the discretization parameter and coefficients of lower order terms. Then, the ϕ-null controllability of the stochastic fourth order semi-discrete parabolic equations is proved using the standard duality technique.

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Key Words. Stochastic fourth order semi-discrete parabolic equation, null controllability, observability, global Carleman estimate.

1 Introduction

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ with $\mathbf{F} = \{\mathcal{F}_t\}_{t\geq 0}$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$ is defined and $\mathbf{F}$ is the natural filtration generated by $W(\cdot)$, augmented by all the $\mathbb{P}$ null sets in $\mathcal{F}$. Write $\mathbb{F}$ for the progressive σ-field with respect to $\mathbf{F}$. Let $\mathcal{H}$ be a Banach space. Denote by $L^2_{\mathcal{F}_\tau}(\Omega;\mathcal{H})$ the space of all $\mathcal{F}_\tau$-measurable random variables ξ such that $\mathbb{E}|\xi|^2_{\mathcal{H}} < \infty$; by $L^2_{\mathbb{F}}(0,T;\mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;\mathcal{H})}) < \infty$, with the canonical norm; by $L^\infty_{\mathbb{F}}(0,T;\mathcal{H})$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted essentially bounded processes; and by $L^2_{\mathbb{F}}(\Omega; C([0,T];\mathcal{H}))$ the Banach space consisting of all $\mathcal{H}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{C([0,T];\mathcal{H})}) < \infty$.

Let $G = (0,1)$ and G_0 be a nonempty open subset of G. As usual, χ_{G_0} denotes the characteristic function of G_0. Firstly, we introduce the following controlled fourth order stochastic parabolic equation in a continuous framework:

$$
\begin{cases}
dy + \partial_x^4 y dt = (a_1 y + \chi_{G_0} u)dt + (a_2 y + v)dW(t) & \text{in } (0,T)\times G,\\
y(t,0) = y(t,1) = 0 & \text{on } (0,T),\\
\partial_x y(t,0) = \partial_x y(t,1) = 0 & \text{on } (0,T),\\
y(0) = y_0 & \text{in } G.
\end{cases}
\tag{1.1}
$$

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†School of Mathematics, Southwest Jiaotong University, Chengdu, P. R. China. Email: yuwangmath@163.com.

‡Corresponding author. School of Mathematics Sciences, Sichuan Normal University, Chengdu, P. R. China. E-mail: qmmath@163.com.

Here, the initial datum $y_0 \in L^2(G)$, the coefficients a_1, a_2 in $L^{\infty}_{\mathbb{F}}(0,T;L^{\infty}(G))$ and the control $(u,v) \in L^2_{\mathbb{F}}(0,T;L^2(G_0))\times\ L^2_{\mathbb{F}}(0,T;L^2(G))$. By the classical well-posedness result for stochastic evolution equations, we know that (1.1) admits a unique weak solution $y \in L^2_{\mathbb{F}}(\Omega;C([0,T];L^2(G)))\times L^2_{\mathbb{F}}(0,T;H^2(G)\cap H^1_0(G))$ (e.g., [16, Theorem 3.24]). The system (1.1) is said to be null controllable at time T if for any given y_0, there exist controls $(u,v) \in L^2_{\mathbb{F}}(0,T;L^2(G_0))\times\ L^2_{\mathbb{F}}(0,T;L^2(G))$ such that the corresponding solution satisfies $y(x,T)=0$, $\mathbb{P}$-a.s.

Stochastic fourth order parabolic equation in a continuous framework is widely used to describe some fast diffusion phenomenon under stochastic perturbations (e.g.,[8]). Further, it is the linearized version of the stochastic Cahn-Hilliard equation, which is used to model phase separation phenomena in a melted alloy that is quenched to a temperature (e.g.,[5]). Compared to second order parabolic equation, to derive controllability for higher order parabolic equation is not an easy task even in the continuous framework. To the best of our knowledge, the controllability of deterministic and stochastic fourth order parabolic equations has been extensively studied in many work (see [7, 10–12, 17, 22] and the references cited therein).

As the numerical approximation of this control system, it is natural to consider the null controllability of discrete systems.

For $N \in \mathbb{N}$, put the space discretization parameter $h \triangleq 1/(N+1)$. We consider the pairs (t, x_i) with $t \in (0,T)$ and $x_i = ih$ for $i = 0, \cdots, N+1$. Utilizing the centered finite difference method to the space variable for the system (1.1), and considering two additional points $x_{-1} = -h$ and $x_{N+2} = (N+2)h$ to discretize the boundary conditions, we consider the following stochastic semi-discrete system:

$$
\begin{cases}
dy^i + \dfrac{1}{4}(y^{i+2} - 4y^{i+1} + 6y^i - 4y^{i-1} + y^{i-2})dt, & \\
\qquad = (a_1^i y^i + \chi^i_{G_0} u^i)dt + (a_2^i y^i + v^i)dW(t) & t \in (0,T), \quad i = 1, \cdots, N,\\
y^0(t) = y^{N+1}(t) = 0 & t \in (0,T),\\
y^{-1}(t) = y^{N+2}(t) = 0 & t \in (0,T),\\
y^i(0) = y_0^i & i = 1, \cdots, N.
\end{cases}
\tag{1.2}
$$

It is well known that the uniform null controllability property is hard to address. For instance, as demonstrated by the counterexample provided in [23], the uniform null controllability property may not hold even for the deterministic second order semi-discrete equation in the two dimensional case. To the best of our knowledge, there are also only a few results on the uniform null controllability of deterministic second order semi-discrete parabolic equations for some special cases; see [1, 13, 15, 19].

Therefore, there are papers consider other controllability rather than uniform null controllability. In [2, 3], a uniform null controllability result for the lower part of the spectrum of deterministic semi-discrete second order parabolic problem is established. Later, [4] proved that the deterministic second order semi-discrete parabolic equations are ϕ-null controllable by establishing a discrete global Carleman estimate directly for semi-discrete parabolic operator. This ϕ-null controllability is defined as the norm of the semi-discrete solution at time T can be dominated by $\sqrt{\phi(h)}$, where ϕ is a real-valued function that tends to zero as the space discretization parameter h tends to zero.

Since then, there have been many results in the study of controllability problems and inverse problems of deterministic semi-discrete parabolic systems by using discrete Carleman estimates. We refer readers to [6, 14, 18]. However, as far as we know, [21] is only one paper which are concerned with controllability of the stochastic second order semi-discrete parabolic equations.

To state our main results, we first introduce some notations. We define the following regular partition of the interval $[0,1]$ as $\mathcal{K} \triangleq \{x_i \mid x_i \mid i = 0, 1, \ldots, N, N+1\}$. Considering any set of points

$\mathcal{W} \subset \mathcal{K}$, we define the dual meshes $\mathcal{W}'$ and $\mathcal{W}^*$, respectively, by

$$\mathcal{W}' \triangleq \tau_+(\mathcal{W}) \cap \tau_-(\mathcal{W}), \quad \mathcal{W}^* \triangleq \tau_+(\mathcal{W}) \cup \tau_-(\mathcal{W})$$

where

$$\tau_\pm(\mathcal{W}) \triangleq \left\{ x \pm \frac{h}{2} \middle| x \in \mathcal{W} \right\}$$

We denote $\overline{\mathcal{W}} \triangleq (\mathcal{W}^*)^*$ and $\mathring{\mathcal{W}} \triangleq (\mathcal{W}')'$. Note that for two consecutive points $x_i, x_{i+1} \in \mathcal{W}$, we have $x_{i+1} - x_i = h$ provided $\mathring{\overline{\mathcal{W}}} = \mathcal{W}$. We call such a subset $\mathcal{W} \subset \mathcal{K}$ a regular mesh. Define the boundary of a regular mesh $\mathcal{W}$ as $\partial\mathcal{W} \triangleq \overline{\mathcal{W}} \backslash \mathcal{W}$.

Then, we introduce the semi-discrete sets $\mathcal{Q} \triangleq (0,T) \times \mathcal{W}$. Moreover, we say that the semi-discrete set is regular if the space variable is a regular mesh. We also define the dual semi-discrete sets by $\mathcal{Q}' \triangleq (0,T) \times \mathcal{W}'$ and $\mathcal{Q}^* \triangleq (0,T) \times \mathcal{W}^*$. Similarly, the semi-discrete boundary is given by $\partial\mathcal{Q} = (0,T) \times \partial\mathcal{W}$.

Next, we define the average operator A_h and the difference operator D_h by

$$\begin{aligned} A_h(u)(t,x) &\triangleq \frac{\tau_+ u(t,x) + \tau_- u(t,x)}{2}, \\ D_h(u)(t,x) &\triangleq \frac{\tau_+ u(t,x) - \tau_- u(t,x)}{h}, \end{aligned}$$

where $\tau_\pm u(t,x) \triangleq u\big(t, x \pm \frac{h}{2}\big)$.

Here and in the sequel, $L(\mathcal{W})$ denotes the set of real-valued functions defined in $\mathcal{W}$. We also denote by $L_h^2(\mathcal{W})$ the Hilbert space with the inner product

$$\langle u, v \rangle_{L_h^2(\mathcal{W})} \triangleq \int_{\mathcal{W}} uv \triangleq h \sum_{x \in \mathcal{W}} u(x)v(x)$$

For $u \in L(\mathcal{W})$, we define its $L_h^\infty(\mathcal{W})$-norm as

$$|u|_{L_h^\infty(\mathcal{W})} \triangleq \max_{x \in \mathcal{W}} \{|u(x)|\}.$$

With above notations, defining $Q \triangleq (0,T) \times \mathcal{M}$ where $\mathcal{M} \triangleq \mathring{\mathcal{K}}$, the controlled semi-discrete system (1.2) can be written as

$$\begin{cases} dy + D_h^4 y dt = (a_1 y + \chi_{G_0} u)dt + (a_2 y + v)dW(t) & \text{in } Q, \\ y(t,0) = y(t,1) = 0 & \text{on } (0,T), \\ D_h y\big(t, -\frac{h}{2}\big) = D_h y\big(t, 1 + \frac{h}{2}\big) = 0 & \text{on } (0,T), \\ y(0,x) = y_0 & \text{in } \mathcal{M}, \end{cases} \tag{1.3}$$

where the initial datum $y_0 \in L_h^2(\mathcal{M})$, the coefficients a_1, a_2 in $L_{\mathbb{F}}^\infty(0,T; L_h^\infty(\mathcal{M}))$ and the control $(u,v) \in L_{\mathbb{F}}^2(0,T; L_h^2(G_0 \cap \mathcal{M})) \times L_{\mathbb{F}}^2(0,T; L_h^2(\mathcal{M}))$. According to the classical theory of stochastic differential equations (see [16, Theorem 3.2]), we know that there is a unique solution $y \in L_{\mathbb{F}}^2(\Omega; C([0,T]; L_h^2(\mathcal{M})))$ to (1.3).

The main null controllability result of this paper is the following.

Theorem 1.1. *There exist positive constants h_0 and C such that for all $T > 0$ and $h \le \min\{h_0, h_1\}$ with*

$$h_1 = C(1 + T^{-1} + \mathcal{H}^{2/7})^{-1},$$

where $\mathcal{H} = |a_1|_{L^\infty_{\mathbb{F}}(0,T;L^\infty_h(\mathcal{M}))} + |a_2|_{L^\infty_{\mathbb{F}}(0,T;L^\infty_h(\mathcal{M}))}$, there exist $(u, v) \in L^2_{\mathbb{F}}(0, T; L^2_h(G_0 \cap \mathcal{M})) \times L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$ such that the solution y to (1.3) satisfies

$$\mathbb{E}\int_Q |v|^2 dt + \mathbb{E}\int_0^T \int_{G_0\cap\mathcal{M}} |u|^2 dt \le C_{obs} \int_{\mathcal{M}} |y_0|^2 ,$$

and

$$\mathbb{E}\int_{\mathcal{M}} |y(T)|^2 \le C_{obs} e^{-\frac{C}{h}} \int_{\mathcal{M}} |y_0|^2 . \tag{1.4}$$

Here, $C_{obs} = e^{C(1+T^{-1}+\mathcal{H}^{2/7}+T+\mathcal{H}^2 T)}$.

Here and in what follows, unless otherwise stated, C stands for a generic positive constant depending only on G and G_0 whose value may vary from line to line.

Remark 1.1. Choosing $\phi(h)$ equals the right hand side of (1.4), we have $|y(T)|_{L^2_{\mathcal{F}_T}(\Omega;L^2_h(\mathcal{M}))} \le \sqrt{\phi(h)}$ with $\lim_{h\to 0} \phi(h) = 0$, which means that system (1.3) is ϕ-null controllable at time T. It would be quite interesting to study the uniform null controllability; i.e., for all $h > 0$, there exist (u, v) which have uniform bounds, such that the corresponding solution to (1.3) satisfies $y(T, x) = 0$, $\mathbb{P}$-a.s for any $x \in \mathcal{M}$. It has been done for the deterministic semi-discrete heat equation on a uniform mesh in [1]. However, we currently do not have a method for achieving this for the stochastic case.

Remark 1.2. Similar to proving Theorem 1.1, employing the techniques presented in this paper allows us to obtain ϕ-null controllability results for the backward stochastic fourth order semi-discrete parabolic equation.

Remark 1.3. Due to randomness, extra control is needed in the diffusion term, as demonstrated in [20]. Therefore, the control of (1.3) is a pair of function (u, v). It is an interesting question to study the ϕ-null controllability for system (1.3) with only one control in the drift.

To prove the controllability result of (1.3), we introduce the adjoint equation:

$$\begin{cases} dz - D_h^4 z dt = -(a_1 z + a_2 Z)dt + Z dW(t) & \text{in } Q, \\ z(t,0) = z(t,1) = 0 & \text{on } (0,T), \\ D_h z\big(t, -\frac{h}{2}\big) = D_h z\big(t, 1+\frac{h}{2}\big) = 0 & \text{on } (0,T), \\ z(T,x) = z_T & \text{in } \mathcal{M}. \end{cases} \tag{1.5}$$

Due to the classical theory of backward stochastic differential equations (see [16, Theorem 4.2]), we know that there is a unique solution $(z, Z) \in L^2_{\mathbb{F}}(\Omega; C([0, T]; L^2_h(\mathcal{M}))) \times L^2_{\mathbb{F}}(0, T; L^2_h(\mathcal{M}))$ to (1.5).

We have the following observability estimate

Theorem 1.2. *There exist positive constants h_0 and C such that for all $T > 0$ and $h \le \min\{h_0, h_1\}$, the solution (z, Z) to system (1.5) satisfies*

$$\mathbb{E}\int_{\mathcal{M}} |z(0)|^2 \le C_{obs}\Big(\mathbb{E}\int_Q |Z|^2 dt + \mathbb{E}\int_0^T \int_{G_0\cap\mathcal{M}} |z|^2 dt + e^{-\frac{C}{h}} \int_{\mathcal{M}} |z|^2\Big|_{t=T}\Big),$$

where h_1 and C_{obs} is given in Theorem 1.1.

Remark 1.4. Note that the last term in the right hand side of observability estimate in Theorem 1.2 is novel compared to the stochastic continuous case. And so far, we don't know how to remove it.

We will use discrete Carleman estimates to prove Theorem 1.2. To state our Carleman estimate, we first introduce the weight functions. Let $\widetilde{G}$ be an open interval contains $[0,1]$, $G_1 \subset \overline{G_1} \subset G_0$ and $G_2 \subset \overline{G_2} \subset G_1$ be two open sets. By [9, Lemma 1.1], we know that there exists a function $\psi \in C^4(\widetilde{G})$ satisfies

$$\psi > 0 \text{ in } \widetilde{G}, \quad |\partial_x \psi| \geq C_0 > 0 \text{ in } \overline{G \backslash G_2}, \quad \partial_x \psi(0) > 0, \text{ and } \partial_x \psi(1) < 0. \tag{1.6}$$

Furthermore, we assume $\overline{\mathcal{M} \cap G_2} \subset \mathcal{M} \cap G_1$. For any positive parameters λ and μ, choosing $0 < \delta < \frac{1}{2}$, put

$$\begin{cases} \phi(x) = e^{\mu(|2\psi|_{C(\overline{G})} + \psi(x))} - e^{4\mu|\psi|_{C(\overline{G})}}, \quad \varphi(x) = e^{\mu(|2\psi|_{C(\overline{G})} + \psi(x))}, \\ s(t) = \lambda\theta(t), \quad \theta(t) = \frac{1}{(t+\delta T)(T+\delta T - t)}, \\ r(t,x) = e^{s(t)\phi(x)}, \quad \rho(t,x) = (r(t,x))^{-1}. \end{cases} \tag{1.7}$$

We have the following Carleman estimate.

Theorem 1.3. *Let the function s and ϕ be defined according to (1.7). There exists a positive constant μ_0 such that for any $\mu \geq \mu_0$, one can find positive constants $C = C(\mu)$, h_0 and ε_0 with $0 < \varepsilon_0 \leq 1$ and a sufficiently large $\lambda_0 \geq 1$ such that for all $\lambda \geq \lambda_0 \left(T + T^2\right), 0 < h \leq h_0$ and $\lambda h \left(\delta T^2\right)^{-1} \leq \varepsilon_0$, it holds that*

$$\begin{aligned} &\mathbb{E}\int_Q s^7 e^{2s\phi}|w|^2 dt + \mathbb{E}\int_{Q^*} s^5 e^{2s\phi} \left|D_h w\right|^2 dt + \mathbb{E}\int_{\overline{Q}} s^3 e^{2s\phi} \left|D_h^2 w\right|^2 dt + \mathbb{E}\int_{Q^*} s e^{2s\phi} \left|D_h^3 w\right|^2 dt \\ &\leq C\Big(\mathbb{E}\int_0^T \int_{G_0 \cap \mathcal{M}} s^7 e^{2s\phi}|w|^2 dt + \mathbb{E}\int_Q e^{2s\phi}|f|^2 dt + \mathbb{E}\int_Q s^4 e^{2s\phi}|g|^2 dt \\ &\qquad + h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|w|^2\Big|_{t=0} + h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|w|^2\Big|_{t=T}\Big), \end{aligned} \tag{1.8}$$

for all $w \in L^2_{\mathbb{F}}(\Omega; C([0,T]; L^2_h(\mathcal{M})))$ and $f, g \in L^2_{\mathbb{F}}(0,T; L^2_h(\mathcal{M}))$ satisfying

$$-dw + D_h^4 w dt = f dt + g dW(t) \tag{1.9}$$

with $w = 0$ on $\partial\mathcal{M}$ and $D_h w = 0$ on $\partial\mathcal{M}^*$.

The rest of this paper is organized as follows. In section 2, we give some preliminaries. The Carleman estimate (1.8) is proved in section 3. In section 4, we prove Theorem 1.2. Finally, we prove the main controllability result Theorem 1.1 based on the Hilbert uniqueness method in section 5.

2 Some preliminary results

This section is devoted to presenting some preliminary results we needed.

Lemma 2.1. *[6, Lemma 2.1] For any $u, v \in C(\overline{\mathcal{W}})$, we have for the difference operator*

$$D_h(uv) = D_h u A_h v + A_h u D_h v, \text{ on } \mathcal{W}^*. \tag{2.1}$$

Similarly, the average of the product gives

$$A_h(uv) = A_h u A_h v + \frac{h^2}{4} D_h u D_h v, \text{ on } \mathcal{W}^*. \tag{2.2}$$

Finally, on $\mathring{\mathcal{W}}$ *we have*

$$u = A_h^2 u - \frac{h^2}{4} D_h^2 u. \tag{2.3}$$

Corollary 2.2. *[6, Corollary 2.2] Let* $\overline{\mathcal{W}} \subseteq \mathcal{M}$ *be a regular mesh. For* $u \in L(\overline{\mathcal{W}})$*,*

$$A_h(u^2) = (A_h u)^2 + \frac{h^2}{4}(D_h u)^2, \text{ on } \mathcal{W}^*. \tag{2.4}$$

In particular, for all $u \in L(\overline{\mathcal{W}})$*,*

$$A_h(u^2) \geq (A_h u)^2, \text{ on } \mathcal{W}^*. \tag{2.5}$$

For $u \in L(\overline{\mathcal{W}})$

$$D_h(u^2) = 2 D_h u A_h u. \tag{2.6}$$

To deal with the boundary conditions, we define the outward normal for $x \in \partial\mathcal{W}$ as

$$\nu(x) \triangleq \begin{cases} 1, & \text{if } \tau_-(x) \in \mathcal{W}^* \text{ and } \tau_+(x) \notin \mathcal{W}^*, \\ -1, & \text{if } \tau_-(x) \notin \mathcal{W}^* \text{ and } \tau_+(x) \in \mathcal{W}^*, \\ 0, & \text{otherwise}. \end{cases} \tag{2.7}$$

We also introduce the trace operator for $u \in L(\mathcal{W}^*)$ as

$$\forall\, x \in \partial\mathcal{W}, \quad t_r(u) \triangleq \begin{cases} \tau_- u(x), & \text{if } \nu(x) = 1, \\ \tau_+ u(x), & \text{if } \nu(x) = -1, \\ 0, & \text{if } \nu(x) = 0. \end{cases}$$

Then by the definition of trace operator, for $x \in \partial\mathcal{W}$ and $u \in L(\overline{\mathcal{W}})$, it is easy to check that

$$t_r(|A_h u|^2)\nu - \frac{h^2}{4} t_r(|D_h u|^2)\nu = |u|^2\nu - h u t_r(D_h u). \tag{2.8}$$

Finally, let us introduce the discrete integration on the boundary for $u \in L(\partial\mathcal{W})$ as

$$\int_{\partial\mathcal{W}} u \triangleq \sum_{x \in \partial\mathcal{W}} u(x).$$

Proposition 2.3. *[6, Proposition 2.3] Let* $\mathcal{W}$ *be a semi-discrete regular mesh. For* $u \in L(\overline{\mathcal{W}})$ *and* $v \in L(\mathcal{W}^*)$ *we have*

$$\int_{\mathcal{W}} u D_h v = -\int_{\mathcal{W}^*} D_h u v + \int_{\partial\mathcal{W}} u t_r(v)\nu \tag{2.9}$$

and

$$\int_{\mathcal{W}} uA_h v = \int_{\mathcal{W}^*} A_h uv - \frac{h}{2}\int_{\partial\mathcal{W}} ut_r(v). \tag{2.10}$$

Corollary 2.4. *[6, Corollary 2.4] Let $\mathcal{W}$ be a semi-discrete regular mesh. For $u, v \in L(\overline{\mathcal{W}})$ we have*

$$\int_{\mathcal{W}} uD_h^2 v = \int_{\overline{\mathcal{W}}} vD_h^2 u - \int_{\partial\mathcal{W}^*} D_h ut_r(v)\nu + \int_{\partial\mathcal{W}} ut_r\left(D_h v\right)\nu, \tag{2.11}$$

$$\int_{\mathcal{W}} uA_h^2 v = \int_{\overline{\mathcal{W}}} vA_h^2 u - \frac{h}{2}\int_{\partial\mathcal{W}^*} A_h ut_r(v) - \frac{h}{2}\int_{\partial\mathcal{W}} ut_r\left(A_h v\right), \tag{2.12}$$

and

$$\begin{aligned}\int_{\mathcal{W}} uA_h D_h v &= -\int_{\overline{\mathcal{W}}} vA_h D_h u + \frac{h}{2}\int_{\partial\mathcal{W}^*} D_h ut_r(v) + \int_{\partial\mathcal{W}} ut_r\left(A_h v\right)\nu \\ &= -\int_{\overline{\mathcal{W}}} vA_h D_h u - \frac{h}{2}\int_{\partial\mathcal{W}} ut_r\left(D_h v\right) + \int_{\partial\mathcal{W}^*} A_h ut_r(v)\nu. \end{aligned}\tag{2.13}$$

Proposition 2.5. *[6, Corollary 2.7] Let f be a $(n+4)$-times differentiable function defined on $\mathbb{R}$ and $m, n \in \mathbb{N}$. Then*

$$A_h^m D_h^n f = f^{(n)} + R_{A_h^m}(f^{(n)}) + R_{D_h^n}(f) + R_{A_h^m D_h^n}(f),$$

where

$$R_{D_h^n}(f) := h^2 \sum_{k=0}^{n} \binom{n}{k} (-1)^k \left(\frac{(n-2k)}{2}\right)^{n+2} \int_0^1 \frac{(1-\sigma)^{n+1}}{(n+1)!} f^{(n+2)}\left(\cdot + \frac{(n-2k)h}{2}\sigma\right) d\sigma$$

$$R_{A_h^n}(f) := \frac{h^2}{2^{n+2}} \sum_{k=0}^{n} \binom{n}{k} (n-2k)^2 \int_0^1 (1-\sigma) f^{(2)}\left(\cdot + \frac{(n-2k)h}{2}\sigma\right) d\sigma,$$

and $R_{A_h^m D_h^n} := R_{A_h^m} \circ R_{D_h^n}$. Here, $f^{(n)}$ denotes the n-th derivative of f with respect to x.

In the sequel, for any $n \in \mathbb{N}$, we denote by $\mathcal{O}(s^n)$ a function of order s^n for sufficiently large s (which is independent of μ and h), by $\mathcal{O}_\mu(s^n)$ a function of order s^n for fixed μ and sufficiently large s (which is independent of h), and by $\mathcal{O}_\mu((sh)^n)$ a function of order $(sh)^n$ for fixed μ and sufficiently large s, sufficiently small h.

Lemma 2.6. *[14, Lemma 4.3] For $\alpha, \beta \in \mathbb{N}$, we have*

$$\begin{aligned}\partial_x^\beta(r\partial_x^\alpha\rho) &= \alpha^\beta(-s\varphi)^\alpha\mu^{\alpha+\beta}(\partial_x\psi)^{\alpha+\beta} + \alpha\beta(s\varphi)^\alpha\mu^{\alpha+\beta-1}\mathcal{O}(1) + s^{\alpha-1}\alpha(\alpha-1)\mathcal{O}_\mu(1) \\ &= s^\alpha\mathcal{O}_\mu(1).\end{aligned}\tag{2.14}$$

Corollary 2.7. *[14, Corollary 4.4] For $\alpha, \beta, \gamma \in \mathbb{N}$, we have*

$$\begin{aligned}\partial_x^\gamma(r^2(\partial_x^\alpha\rho)\partial_x^\beta\rho) &= (\alpha+\beta)^\gamma(-s\varphi)^{\alpha+\beta}\mu^{\alpha+\beta+\gamma}(\partial_x\psi)^{\alpha+\beta+\gamma} + \gamma(\alpha+\beta)(s\varphi)^{\alpha+\beta}\mu^{\alpha+\beta+\gamma-1}\mathcal{O}(1) \\ &\quad + s^{\alpha+\beta-1}[\alpha(\alpha-1)+\beta(\beta-1)]\mathcal{O}_\mu(1) \\ &= s^{\alpha+\beta}\mathcal{O}_\mu(1).\end{aligned}\tag{2.15}$$

By (1.7) and (2.14), it easy to see that the following result holds.

Lemma 2.8. *For $\alpha, \beta, \gamma, \sigma \in \mathbb{N}$, we have*

$$\begin{aligned}\partial_x^\sigma(r\partial_x^\alpha\rho\partial_x^\beta(r\partial_x^\gamma\rho)) &= \gamma^\beta(\alpha+\gamma)^\sigma(-s\varphi)^{\alpha+\gamma}\mu^{\alpha+\beta+\gamma+\sigma}(\partial_x\psi)^{\alpha+\beta+\gamma+\sigma} + (s\varphi)^{\alpha+\gamma}\mu^{\alpha+\beta+\gamma+\sigma-1}\mathcal{O}(1)\\ &\quad + s^{\alpha+\gamma-1}\mathcal{O}_\mu(1)\\ &= s^{\alpha+\gamma}\mathcal{O}_\mu(1). \end{aligned} \tag{2.16}$$

Lemma 2.9. *[14, Lemma 4.8] Let $\alpha, j, k, m, n \in \mathbb{N}$. Provided $\lambda h(\delta T^2)^{-1} \le 1$, we have*

$$A_h^j D_h^k \partial_x^\alpha(rA_h^m D_h^n\rho) = \partial_x^k\partial_x^\alpha(r\partial_x^n\rho) + s^n\mathcal{O}_\mu((sh)^2) = s^n\mathcal{O}_\mu(1). \tag{2.17}$$

Theorem 2.10. *[14, Theorem 4.9] Let $\alpha, \beta, j, k, l, m, n, p \in \mathbb{N}$. Provided $\lambda h(\delta T^2)^{-1} \le 1$, we have*

$$\begin{aligned} A_h^p D_h^l \partial_x^\beta(r^2 A_h^j D_h^k(\partial_x^\alpha\rho)A_h^m D_h^n(\rho)) &= \partial_x^l\partial_x^\beta(r^2\partial_x^k\partial_x^\alpha\rho\partial_x^n\rho) + s^{n+k+\alpha}\mathcal{O}_\mu((sh)^2)\\ &= s^{n+k+\alpha}\mathcal{O}_\mu(1). \end{aligned} \tag{2.18}$$

Theorem 2.11. *[6, Theorem 2.9] Let $\alpha, j, k, m, n \in \mathbb{N}$. Provided $\lambda h(\delta T^2)^{-1} \le 1$, we have*

$$\partial_t A_h^j D_h^k(rA_h^m D_h^n(\partial_x^\alpha\rho)) = T\theta s^{\alpha+n}\mathcal{O}_\mu(1). \tag{2.19}$$

By Proposition 2.5 and (2.17), it is easy to check that the following estimate holds.

Lemma 2.12. *Let $i, j, k, l, m, n, p, q \in \mathbb{N}$. Provided $\lambda h(\delta T^2)^{-1} \le 1$, we have*

$$\begin{aligned} A_h^i D_h^j(rA_h^k D_h^l\rho A_h^m D_h^n(rA_h^p D_h^q\rho)) &= \partial_x^j(r\partial_x^l\rho\partial_x^n(r\partial_x^q\rho)) + s^{l+q}\mathcal{O}_\mu((sh)^2)\\ &= s^{l+q}\mathcal{O}_\mu(1). \end{aligned} \tag{2.20}$$

3 Proof of the Carleman estimate

Proof of Theorem 1.3. Step 1. At first, we define

$$\mathcal{P}_h w = -dw + D_h^4 w. \tag{3.1}$$

Letting $v = rw$ and recalling that $\rho = r^{-1}$, utilizing Itô's formula, we have

$$-rdw = -rd(\rho v) = -dv + \lambda\phi\partial_t\theta v dt. \tag{3.2}$$

Thanks to Lemma 2.1 and $w = \rho v$, it holds that

$$D_h^4 w = D_h^4\rho A_h^4 v + 4A_h D_h^3\rho A_h^3 D_h v + 6A_h^2 D_h^2\rho A_h^2 D_h^2 v + 4A_h^3 D_h\rho A_h D_h^3 v + A_h^4\rho D_h^4 v. \tag{3.3}$$

Combining (3.1)–(3.3), we deduce that

$$r\mathcal{P}_h w = I_1 dt + I_2 + I_3 dt, \tag{3.4}$$

where

$$I_1 = 6rA_h^2 D_h^2\rho A_h^2 D_h^2 v + rD_h^4\rho A_h^4 v + rA_h^4\rho D_h^4 v + 6A_h D_h(rA_h^2 D_h^2\rho)A_h D_h v + \Phi(v),$$

$$I_2 = -dv + 4rA_h D_h^3\rho A_h^3 D_h v dt + 4rA_h^3 D_h\rho A_h D_h^3 v dt + 2A_h D_h(rA_h D_h^3\rho)v dt,$$

$$I_3 = \lambda\phi\partial_t\theta v - 6A_h D_h(rA_h^2 D_h^2\rho)A_h D_h v - 2A_h D_h(rA_h D_h^3\rho)v - \Phi(v)$$

$$\Phi(v) = 3h^2 D_h^2[A_h D_h(rA_h^2 D_h^2\rho)A_h D_h v] + \frac{h^2}{2}A_h^2[A_h D_h(rD_h^4\rho)A_h D_h v] + \frac{h^2}{4}D_h^2(rD_h^4\rho)A_h^2 v \\ - D_h^2(rA_h^4\rho)D_h^2 v + 2A_h D_h[A_h D_h(rA_h^4\rho)D_h^2 v]. \tag{3.5}$$

Using (3.4), it follows that

$$2I_1 r\mathcal{P}_h w = 2I_1^2 dt + 2I_1 I_2 + 2I_1 I_3 dt,$$

which, combined with Cauchy-Schwarz inequality, (1.9) and (3.1), implies

$$\mathbb{E}\int_Q |rf|^2 dt \geq 2\mathbb{E}\int_Q I_1 I_2 - \mathbb{E}\int_Q |I_3|^2 dt. \tag{3.6}$$

The next steps provide an estimate for the right hand side of (3.6). For convenience, we denote

$$\mathbb{E}\int_Q I_1 I_2 = \sum_{i=1}^{5}\sum_{j=1}^{4} I_{ij}, \tag{3.7}$$

where I_{ij} stands for the inner product in $L^2_{\mathbb{F}}(0,T;L^2_h(\mathcal{M}))$ between the i-th term of I_1 and j-th term of I_2.

In order to shorten the formulas used in the sequel, we define the following lower order terms

$$\begin{aligned}
\mathcal{A}_1 = {} & \mathbb{E}\int_Q [s^6\mathcal{O}_\mu(1) + s^7\mathcal{O}_\mu((sh)^2) + s^7\mu^7\varphi^7\mathcal{O}(1)]|v|^2 dt \\
& + \mathbb{E}\int_{Q^*} [s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2) + s^5\mu^5\varphi^5\mathcal{O}(1)]|D_h v|^2 dt \\
& + \mathbb{E}\int_{\overline{Q}} [s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2) + s^3\mu^3\varphi^3\mathcal{O}(1)]|D_h^2 v|^2 dt \\
& + \mathbb{E}\int_{Q^*} [\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2) + s\mu\varphi\mathcal{O}(1)]|D_h^3 v|^2 dt, \\
\mathcal{A}_2 = {} & \mathbb{E}\int_Q s^4\mathcal{O}_\mu(1)|dv|^2 + \mathbb{E}\int_{\overline{Q}} h^2\mathcal{O}_\mu(1)|D_h^2(dv)|^2, \\
\mathcal{A}_3 = {} & \mathbb{E}\int_{\mathcal{M}} s^4\mathcal{O}_\mu(1)|v|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}^*} s^2\mathcal{O}_\mu(1)|D_h v|^2\Big|_{t=0} + \mathbb{E}\int_{\overline{\mathcal{M}}} \mathcal{O}_\mu(1)|D_h^2 v|^2\Big|_{t=0} \\
& + \mathbb{E}\int_{\mathcal{M}} s^4\mathcal{O}_\mu(1)|v|^2\Big|_{t=T} + \mathbb{E}\int_{\mathcal{M}^*} s^2\mathcal{O}_\mu(1)|D_h v|^2\Big|_{t=T} + \mathbb{E}\int_{\overline{\mathcal{M}}} \mathcal{O}_\mu(1)|D_h^2 v|^2\Big|_{t=T}, \\
\mathcal{B} = {} & \mathbb{E}\int_{\partial Q} [s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)]t_r(|D_h v|^2) dt + \mathbb{E}\int_{\partial Q} [s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu(sh)]|D_h^2 v|^2 dt \\
& + \mathbb{E}\int_{\partial Q} [\mathcal{O}_\mu(1) + s\mathcal{O}_\mu(sh)]t_r(|D_h^3 v|^2) dt.
\end{aligned}$$

Step 2. Let us compute I_{i1} for $i = 1, \cdots, 5$.

Since $w = 0$ on $\partial\mathcal{M}$ and $D_h w = 0$ on $\partial\mathcal{M}^*$, noting that $v = rw$, we obtain

$$v = 0 \text{ on } \partial Q,\ D_h v = A_h v = 0 \text{ on } \partial Q^*, \tag{3.8}$$

and

$$t_r(v)=0 \text{ on } \partial Q^* \text{ and } \partial\overline{Q}^*,\ t_r(D_hv)=t_r(A_hv)=0 \text{ on } \partial\overline{Q}. \tag{3.9}$$

From (2.1)–(2.3), (2.13), (3.5) and (3.8), letting $q_{11}=rA_h^2D_h^2\rho$, we have

$$\begin{aligned}
I_{11}&=-6\mathbb{E}\int_Q q_{11}A_h^2D_h^2vdv=6\mathbb{E}\int_{\overline{Q}}A_hD_hvA_hD_h(q_{11}dv)\\
&=6\mathbb{E}\int_{\overline{Q}}A_hD_hq_{11}A_h^2(dv)A_hD_hv+6\mathbb{E}\int_{\overline{Q}}A_h^2q_{11}A_hD_h(dv)A_hD_hv\\
&\quad+\frac{3}{2}h^2\mathbb{E}\int_{\overline{Q}}D_h^2q_{11}A_hD_h(dv)A_hD_hv+\frac{3}{2}h^2\mathbb{E}\int_{\overline{Q}}A_hD_hq_{11}D_h^2(dv)A_hD_hv\\
&=6\mathbb{E}\int_Q A_hD_hq_{11}A_hD_hvdv+6\mathbb{E}\int_{\overline{Q}}\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)A_hD_hvA_hD_h(dv)\\
&\quad+3h^2\mathbb{E}\int_{\overline{Q}}A_hD_hq_{11}A_hD_hvD_h^2(dv).
\end{aligned}\tag{3.10}$$

By Itô's formula, it follows that

$$\begin{aligned}
&6\mathbb{E}\int_{\overline{Q}}\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)A_hD_hvA_hD_h(dv)\\
&=3\mathbb{E}\int_{\overline{M}}\left[\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)|A_hD_hv|^2\right]\Big|_0^T-3\mathbb{E}\int_{\overline{Q}}\partial_t\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)|A_hD_hv|^2dt\\
&\quad-3\mathbb{E}\int_{\overline{Q}}\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)|A_hD_h(dv)|^2.
\end{aligned}\tag{3.11}$$

By (2.5), (2.10), (2.17) and (3.8), for $\lambda h(\delta T^2)^{-1}\le 1$, we conclude that

$$\begin{aligned}
&3\mathbb{E}\int_{\overline{M}}\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)|A_hD_hv|^2\ge-\mathbb{E}\int_{\overline{M}}[s^2\mathcal{O}_\mu(1)+\mathcal{O}_\mu((sh)^2)]|A_hD_hv|^2\\
&\ge-\mathbb{E}\int_{\overline{M}}[s^2\mathcal{O}_\mu(1)+\mathcal{O}_\mu((sh)^2)]A_h(|D_hv|^2)\\
&\ge-\mathbb{E}\int_{M^*}[s^2\mathcal{O}_\mu(1)+\mathcal{O}_\mu((sh)^2)]|D_hv|^2.
\end{aligned}\tag{3.12}$$

Therefore, similar to (3.12), by (2.19), it holds that

$$\begin{aligned}
&-3\mathbb{E}\int_{\overline{Q}}\partial_t\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)|A_hD_hv|^2dt-3\mathbb{E}\int_{\overline{Q}}\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)|A_hD_h(dv)|^2\\
&\ge-\mathbb{E}\int_{Q^*}T\theta[s^2\mathcal{O}_\mu(1)+\mathcal{O}_\mu((sh)^2)]|D_hv|^2dt-\mathbb{E}\int_{Q^*}[s^2\mathcal{O}_\mu(1)+\mathcal{O}_\mu((sh)^2)]|D_h(dv)|^2.
\end{aligned}\tag{3.13}$$

By Cauchy-Schwarz inequality and combining (3.11)–(3.13), for $\lambda h(\delta T^2)^{-1}\le 1$ and $\lambda\ge T+T^2$, we obtain

$$6\mathbb{E}\int_{\overline{Q}}\left(q_{11}+\frac{h^2}{2}D_h^2q_{11}\right)A_hD_hvd(A_hD_hv)$$

$$
\begin{aligned}
&\geq -\mathbb{E}\int_{Q^*}[s^2\mathcal{O}_\mu(1)+\mathcal{O}_\mu((sh)^2)]|D_h(dv)|^2-\mathcal{A}_1-\mathcal{A}_3\\
&=\mathbb{E}\int_{Q}[s^2\mathcal{O}_\mu(1)+\mathcal{O}_\mu((sh)^2)]D_h^2(dv)dv-\mathcal{A}_1-\mathcal{A}_3\\
&\geq -\frac{1}{4}\mathbb{E}\int_Q|D_h^2(dv)|^2-\mathcal{A}_1-\mathcal{A}_2-\mathcal{A}_3.
\end{aligned}
\tag{3.14}
$$

Combining (2.11), (3.8)–(3.10) and (3.14), for $\lambda h(\delta T^2)^{-1}\leq 1$ and $\lambda\geq T+T^2$, we have

$$
\begin{aligned}
I_{11}\geq\ & 6\mathbb{E}\int_Q A_hD_h(rA_h^2D_h^2\rho)A_hD_hvdv+3h^2\mathbb{E}\int_{\overline{Q}}D_h^2[A_hD_h(rA_h^2D_h^2\rho)A_hD_hv]dv\\
&-\frac{1}{4}\mathbb{E}\int_Q|D_h^2(dv)|^2-\mathcal{A}_1-\mathcal{A}_2-\mathcal{A}_3.
\end{aligned}
\tag{3.15}
$$

From (2.2), (2.3), (2.12) and (3.8), letting $q_{21}=rD_h^4\rho$, we obtain

$$
\begin{aligned}
I_{21}&=-\mathbb{E}\int_Q q_{21}A_h^4vdv=-\mathbb{E}\int_{\overline{Q}}A_h^2(q_{21}dv)A_h^2v\\
&=-\mathbb{E}\int_{\overline{Q}}A_h^2q_{21}A_h^2(dv)A_h^2v-\frac{h^2}{2}\mathbb{E}\int_{\overline{Q}}A_hD_hq_{21}A_hD_h(dv)A_h^2v-\frac{h^2}{4}\mathbb{E}\int_{\overline{Q}}D_h^2q_{21}A_h^2(dv)A_h^2v\\
&\quad+\frac{h^2}{4}\mathbb{E}\int_{\overline{Q}}D_h^2q_{21}A_h^2vdv.
\end{aligned}
\tag{3.16}
$$

Thanks to Itô's formula and similar to (3.12), from (2.17) and (2.19), for $\lambda h(\delta T^2)^{-1}\leq 1$ and $\lambda\geq T+T^2$, we have

$$
\begin{aligned}
&-\mathbb{E}\int_{\overline{Q}}A_h^2q_{21}A_h^2(dv)A_h^2v\\
&=-\frac{1}{2}\mathbb{E}\int_{\overline{\mathcal{M}}}(A_h^2q_{21}|A_h^2v|^2)\Big|_0^T+\frac{1}{2}\mathbb{E}\int_{\overline{Q}}\partial_t(A_h^2q_{21})|A_h^2v|^2dt+\frac{1}{2}\mathbb{E}\int_{\overline{Q}}A_h^2q_{21}|A_h^2(dv)|^2\\
&\geq-\mathcal{A}_1-\mathcal{A}_2-\mathcal{A}_3.
\end{aligned}
\tag{3.17}
$$

Using the identity $A_hD_h(dv)A_h^2v=\frac{1}{2}d[D_h(|A_hv|^2)]-A_hD_hvA_h^2(dv)-A_hD_h(dv)A_h^2(dv)$ and Cauchy-Schwarz inequality, from (2.5), (2.9), (2.10), (2.17), (2.19) and (3.9), for $\lambda h(\delta T^2)^{-1}\leq 1$, we deduce that

$$
\begin{aligned}
&-\frac{h^2}{2}\mathbb{E}\int_{\overline{Q}}A_hD_hq_{21}A_hD_h(dv)A_h^2v\\
&=-\frac{h^2}{4}\mathbb{E}\int_{\overline{Q}}d[A_hD_hq_{21}D_h(|A_hv|^2)]+\frac{h^2}{4}\mathbb{E}\int_{\overline{Q}}\partial_t(A_hD_hq_{21})A_hD_hvA_h^2vdt\\
&\quad+\frac{h^2}{2}\mathbb{E}\int_{\overline{Q}}A_hD_hq_{21}A_hD_hvA_h^2(dv)+\frac{h^2}{2}\mathbb{E}\int_{\overline{Q}}A_hD_hq_{21}A_hD_h(dv)A_h^2(dv)\\
&=-\frac{h^2}{4}\mathbb{E}\int_{\overline{\mathcal{M}}}A_hD_hq_{21}D_h(|A_hv|^2)\Big|_0^T+\mathbb{E}\int_{\overline{Q}}T\theta s^2\mathcal{O}_\mu((sh)^2)A_hD_hvA_h^2vdt\\
&\quad+\frac{h^2}{2}\mathbb{E}\int_{\overline{Q}}A_hD_hq_{21}A_hD_hvA_h^2(dv)-\frac{h^2}{4}\mathbb{E}\int_{\overline{Q}}A_hD_h^2q_{21}|A_h(dv)|^2
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{h^2}{2}\mathbb{E}\int_{\overline{Q}} A_hD_hq_{21}A_hD_hvA_h^2(dv) - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3\\
&= \frac{h^2}{2}\mathbb{E}\int_{\overline{Q}} A_h^2(A_hD_hq_{21}A_hD_hv)dv - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. && (3.18)
\end{aligned}
$$

Thanks to Itô's formula, from (2.5), (2.10), (2.17) and (2.19), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, it holds that

$$
\begin{aligned}
&-\frac{h^2}{4}\mathbb{E}\int_{\overline{Q}} D_h^2q_{21}A_h^2(dv)A_h^2v\\
&= -\frac{h^2}{8}\mathbb{E}\int_{\overline{\mathcal{M}}} D_h^2q_{21}|A_h^2v|^2\Big|_0^T + \frac{h^2}{8}\mathbb{E}\int_{\overline{Q}} \partial_t(D_h^2q_{21})|A_h^2v|^2dt + \frac{h^2}{8}\mathbb{E}\int_{\overline{Q}} D_h^2q_{21}|A_h^2(dv)|^2\\
&\geq -\mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. && (3.19)
\end{aligned}
$$

Combining (3.16)–(3.19), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T + T^2$, it follows that

$$
I_{21} \geq \frac{h^2}{2}\mathbb{E}\int_{\overline{Q}} A_h^2[A_hD_h(rD_h^4\rho)A_hD_hv]dv + \frac{h^2}{4}\mathbb{E}\int_{Q} D_h^2(rD_h^4\rho)A_h^2vdv - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. \quad (3.20)
$$

From (2.1), (2.3), (2.11), (2.13), (3.8) and (3.9), letting $q_{31} = rA_h^4\rho$, we have

$$
\begin{aligned}
I_{31} &= -\mathbb{E}\int_{Q} q_{31}D_h^4vdv = -\mathbb{E}\int_{\overline{Q}} D_h^2(q_{31}dv)D_h^2v\\
&= -\mathbb{E}\int_{\overline{Q}} D_h^2q_{31}A_h^2(dv)D_h^2v - 2\mathbb{E}\int_{\overline{Q}} A_hD_hq_{31}A_hD_h(dv)D_h^2v - \mathbb{E}\int_{\overline{Q}} A_h^2q_{31}D_h^2(dv)D_h^2v\\
&= -\mathbb{E}\int_{\overline{Q}} D_h^2q_{31}D_h^2vdv + 2\mathbb{E}\int_{\overline{Q}} A_hD_h(A_hD_hq_{31}D_h^2v)dv\\
&\quad - \mathbb{E}\int_{\overline{Q}} \left(A_h^2q_{31} + \frac{h^2}{4}D_h^2q_{31}\right)D_h^2(dv)D_h^2v. && (3.21)
\end{aligned}
$$

Thanks to Itô's formula, from (2.17) and (2.19), for $\lambda h(\delta T^2)^{-1} \leq 1$, $h \leq 1$ and $\lambda \geq T + T^2$, it holds that

$$
\begin{aligned}
&-\mathbb{E}\int_{\overline{Q}} \left(A_h^2q_{31} + \frac{h^2}{4}D_h^2q_{31}\right)D_h^2(dv)D_h^2v\\
&= -\frac{1}{2}\mathbb{E}\int_{\overline{Q}} \left(A_h^2q_{31} + \frac{h^2}{4}D_h^2q_{31}\right)|D_h^2v|^2\Big|_0^T + \frac{1}{2}\mathbb{E}\int_{\overline{Q}} \partial_t\left(A_h^2q_{31} + \frac{h^2}{4}D_h^2q_{31}\right)|D_h^2v|^2dt\\
&\quad + \frac{1}{2}\mathbb{E}\int_{\overline{Q}} \left(A_h^2q_{31} + \frac{h^2}{4}D_h^2q_{31}\right)|D_h^2(dv)|^2\\
&\geq \frac{1}{2}\mathbb{E}\int_{\overline{Q}} |D_h^2(dv)|^2 - \mathcal{A}_1 - \mathcal{A}_2 - \mathcal{A}_3. && (3.22)
\end{aligned}
$$

Combining (3.21) and (3.22), for $\lambda h(\delta T^2)^{-1} \leq 1$, $h \leq 1$ and $\lambda \geq T + T^2$, we have

$$
I_{31} \geq -\mathbb{E}\int_{\overline{Q}} D_h^2(rA_h^4\rho)D_h^2vdv + 2\mathbb{E}\int_{\overline{Q}} A_hD_h[A_hD_h(rA_h^4\rho)D_h^2v]dv
$$

$$+\frac{1}{2}\mathbb{E}\int_{\overline{Q}}|D_h^2(dv)|^2-\mathcal{A}_1-\mathcal{A}_2-\mathcal{A}_3. \tag{3.23}$$

Hence, combining (3.5), (3.15), (3.20) and (3.23), for $\lambda h(\delta T^2)^{-1}\le 1$, $h\le 1$ and $\lambda\ge T+T^2$, we obtain

$$I_{11}+I_{21}+I_{31}+I_{41}+I_{51}\ge\frac{1}{4}\mathbb{E}\int_{\overline{Q}}|D_h^2(dv)|^2-\mathcal{A}_1-\mathcal{A}_2-\mathcal{A}_3. \tag{3.24}$$

Noting that (1.9) and $v=rw$, by Proposition 2.5 and (1.7) and (2.1), for $\lambda h(\delta T^2)^{-1}\le 1$, we have

$$\begin{aligned}\mathbb{E}\int_{\overline{Q}}|D_h^2(dv)|^2&=\mathbb{E}\int_{\overline{Q}}|D_h^2(rg)|^2dt\\&\ge\frac{1}{3}\mathbb{E}\int_{\overline{Q}}r^2|D_h^2g|^2dt-\mathbb{E}\int_{\overline{Q}}\mathcal{O}_\mu((sh)^4)r^2|D_h^2g|^2dt-\mathbb{E}\int_{Q^*}s^2\mathcal{O}_\mu(1)r^2|D_hg|^2dt\\&\quad-\mathbb{E}\int_Q s^4\mathcal{O}_\mu(1)r^2|g|^2dt.\end{aligned} \tag{3.25}$$

From (1.7), (2.1) and (2.9) and Proposition 2.5, thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1}\le 1$, we obtain

$$-\mathbb{E}\int_{Q^*}s^2\mathcal{O}_\mu(1)r^2|D_hg|^2dt\ge-\frac{1}{4}\mathbb{E}\int_{\overline{Q}}r^2|D_h^2g|^2dt-\mathbb{E}\int_Q s^4\mathcal{O}_\mu(1)r^2|g|^2dt. \tag{3.26}$$

Hence, combining (3.24)–(3.26), for all $\mu\ge 1$, one can find constant $h_1\le 1$, such that for all $\lambda\ge T+T^2$, $h\le h_1$ and $\lambda h(\delta T^2)^{-1}\le 1$, there exists a constant $C>0$ such that

$$I_{11}+I_{21}+I_{31}+I_{41}+I_{51}\ge\frac{1}{48}\mathbb{E}\int_{\overline{Q}}r^2|D_h^2g|^2dt-C(\mu)\mathbb{E}\int_Q s^4r^2|g|^2dt-\mathcal{A}_1-\mathcal{A}_3. \tag{3.27}$$

Step 3. In this step, we compute I_{ij} for $i=1,\cdots,5$ and $j=2,3,4$.

Noting that (1.6), (2.3), (2.6), (2.7), (2.9), (2.15) and (2.18), letting $q_{12}=r^2A_h^2D_h^2\rho A_hD_h^3\rho$, for $\lambda h(\delta T^2)^{-1}\le 1$, we have

$$\begin{aligned}I_{12}&=24\mathbb{E}\int_Q q_{12}A_h^2D_h^2vA_h^3D_hvdt=12\mathbb{E}\int_Q q_{12}D_h[(A_h^2D_hv)^2]dt\\&=-12\mathbb{E}\int_{Q^*}D_hq_{12}|A_h^2D_hv|^2dt+12\mathbb{E}\int_{\partial Q}q_{12}t_r(|A_h^2D_hv|^2)\nu dt\\&=-12\mathbb{E}\int_{Q^*}D_hq_{12}\left|D_hv+\frac{h^2}{4}D_h^3v\right|^2dt+12\mathbb{E}\int_{\partial Q}q_{12}t_r\left(\left|D_hv+\frac{h^2}{4}D_h^3v\right|^2\right)\nu dt\\&\ge 60\mathbb{E}\int_{Q^*}s^5\mu^6\varphi^5|\partial_x\psi|^6|D_hv|^2dt-\mathbb{E}\int_{Q^*}[s\mathcal{O}_\mu((sh)^2)+s\mathcal{O}_\mu((sh)^4)]|D_h^3v|^2dt\\&\quad-\mathbb{E}\int_{Q^*}[s^4\mathcal{O}_\mu(1)+s^5\mathcal{O}_\mu((sh)^2)+s^5\mu^5\varphi^5\mathcal{O}(1)]|D_hv|^2dt\\&\quad-12\mathbb{E}\int_{\partial Q}s^5\mu^5\varphi^5(\partial_x\psi)^5t_r(|D_hv|^2)\nu dt-\mathbb{E}\int_{\partial Q}[s\mathcal{O}_\mu((sh)^2)+s\mathcal{O}_\mu((sh)^4)]t_r(|D_h^3v|^2)dt\end{aligned}$$

$$
\begin{aligned}
&- \mathbb{E}\int_{\partial Q}[s^4\mathcal{O}_\mu(1)+s^5\mathcal{O}_\mu((sh)^2)]t_r(|D_hv|^2)dt\\
&\geq 60\mathbb{E}\int_{Q^*}s^5\mu^6\varphi^5|\partial_x\psi|^6|D_hv|^2dt+12\mathbb{E}\int_{\partial Q}s^5\mu^5\varphi^5|\partial_x\psi|^5t_r(|D_hv|^2)dt-\mathcal{A}_1-\mathcal{B}. \qquad (3.28)
\end{aligned}
$$

From (2.4), (2.6), (2.9) and (2.10), letting $q_{13}=r^2A_h^2D_h^2\rho A_h^3D_h\rho$, for, we obtain

$$
\begin{aligned}
I_{13}&=24\mathbb{E}\int_Q q_{13}A_h^2D_h^2vA_hD_h^3vdt=12\mathbb{E}\int_Q q_{13}D_h[|A_hD_h^2v|^2]dt\\
&=-12\mathbb{E}\int_{Q^*}D_hq_{13}|A_hD_h^2v|^2dt+12\mathbb{E}\int_{\partial Q}q_{13}t_r(|A_hD_h^2v|^2)\nu dt\\
&=-12\mathbb{E}\int_{Q^*}D_hq_{13}A_h(|D_h^2v|^2)dt+3h^2\mathbb{E}\int_{Q^*}D_hq_{13}|D_h^3v|^2dt+12\mathbb{E}\int_{\partial Q}q_{13}t_r(|A_hD_h^2v|^2)\nu dt\\
&=-12\mathbb{E}\int_{\overline{Q}}A_hD_hq_{13}|D_h^2v|^2dt+6h\mathbb{E}\int_{\partial Q^*}D_hq_{13}t_r(|D_h^2v|^2)dt+3h^2\mathbb{E}\int_{Q^*}D_hq_{13}|D_h^3v|^2dt\\
&\quad+12\mathbb{E}\int_{\partial Q}q_{13}t_r(|A_hD_h^2v|^2)\nu dt. \qquad (3.29)
\end{aligned}
$$

From (2.15) and (2.18), for $\lambda h(\delta T^2)^{-1}\leq 1$, we have

$$
\begin{aligned}
-12\mathbb{E}\int_{\overline{Q}}A_hD_hq_{13}|D_h^2v|^2dt\geq\ & 36\mathbb{E}\int_{\overline{Q}}s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2v|^2dt\\
&-\mathbb{E}\int_{\overline{Q}}[s^2\mathcal{O}_\mu(1)+s^3\mathcal{O}_\mu((sh)^2)+s^3\mu^3\varphi^3\mathcal{O}(1)]|D_h^2v|^2dt, \qquad (3.30)
\end{aligned}
$$

and

$$
\begin{aligned}
&6h\mathbb{E}\int_{\partial Q^*}D_hq_{13}t_r(|D_h^2v|^2)dt+3h^2\mathbb{E}\int_{Q^*}D_hq_{13}|D_h^3v|^2dt\\
&\geq-\mathbb{E}\int_{\partial Q}s^2\mathcal{O}_\mu(1)|D_h^2v|^2dt-\mathbb{E}\int_{Q^*}s\mathcal{O}_\mu((sh)^2)|D_h^3v|^2dt. \qquad (3.31)
\end{aligned}
$$

From (2.8), (2.15) and (2.18) and Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1}\leq 1$, it holds that

$$
\begin{aligned}
&12\mathbb{E}\int_{\partial Q}q_{13}t_r(|A_hD_h^2v|^2)\nu dt\\
&=12\mathbb{E}\int_{\partial Q}q_{13}|D_h^2v|^2\nu dt+3h^2\mathbb{E}\int_{\partial Q}q_{13}t_r(|D_h^3v|^2)\nu dt-12h\mathbb{E}\int_{\partial Q}q_{13}D_h^2vt_r(D_h^3v)dt\\
&\geq-12\mathbb{E}\int_{\partial Q}s^3\mu^3\varphi^3(\partial_x\psi)^3|D_h^2v|^2\nu dt-\mathbb{E}\int_{\partial Q}[s^2\mathcal{O}_\mu(1)+s^3\mathcal{O}_\mu(sh)]|D_h^2v|^2dt\\
&\quad-\mathbb{E}\int_{\partial Q}s\mathcal{O}_\mu(sh)t_r(|D_h^3v|^2)dt. \qquad (3.32)
\end{aligned}
$$

Combining (3.29)–(3.32), for $\lambda h(\delta T^2)^{-1}\leq 1$ and $\lambda\geq T^2$, we have

$$
I_{13}\geq 36\mathbb{E}\int_{\overline{Q}}s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2v|^2dt+12\mathbb{E}\int_{\partial Q}s^3\mu^3\varphi^3|\partial_x\psi|^3|D_h^2v|^2dt-\mathcal{A}_1-\mathcal{B}. \qquad (3.33)
$$

From (2.1), (2.3), (2.6), (2.9), (2.16), (2.20), (3.8) and (3.9), noting that Cauchy-Schwarz inequality, letting $q_{14} = rA_h^2 D_h^2 \rho A_h D_h (rA_h D_h^3 \rho)$, for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$
\begin{aligned}
I_{14} &= 12\mathbb{E}\int_Q q_{14} A_h^2 D_h^2 v v dt = 12\mathbb{E}\int_Q q_{14} D_h^2 v v dt + 3h^2\mathbb{E}\int_Q q_{14} D_h^4 v v dt \\
&= -12\mathbb{E}\int_{Q^*} D_h q_{14} A_h v D_h v dt - 12\mathbb{E}\int_{Q^*} A_h q_{14} |D_h v|^2 dt - 3h^2\mathbb{E}\int_{Q^*} D_h q_{14} A_h v D_h^3 v dt \\
&\quad - 3h^2\mathbb{E}\int_{Q^*} A_h q_{14} D_h v D_h^3 v dt \\
&= 6\mathbb{E}\int_{\overline{Q}} D_h^2 q_{14} |v|^2 dt - 12\mathbb{E}\int_{Q^*} A_h q_{14} |D_h v|^2 dt - 3h^2\mathbb{E}\int_{Q^*} D_h q_{14} A_h v D_h^3 v dt \\
&\quad - 3h^2\mathbb{E}\int_{Q^*} A_h q_{14} D_h v D_h^3 v dt \\
&\geq 36\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5 |\partial_x\psi|^6 |D_h v|^2 dt - \mathbb{E}\int_Q s^5\mathcal{O}_\mu(1)|v|^2 dt - \mathbb{E}\int_{Q^*} s\mathcal{O}_\mu((sh)^2)|D_h^3 v|^2 dt \\
&\quad - \mathbb{E}\int_{Q^*} [s^5\mu^5\varphi^5\mathcal{O}(1) + s^5\mathcal{O}_\mu((sh)^2) + s^4\mathcal{O}_\mu(1)]|D_h v|^2 dt \\
&\geq 36\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5 |\partial_x\psi|^6 |D_h v|^2 dt - \mathcal{A}_1.
\end{aligned} \tag{3.34}
$$

From (2.2), (2.3), (2.6), (2.9) and (2.13), letting $q_{22} = r^2 D_h^4 \rho A_h D_h^3 \rho$, we have

$$
\begin{aligned}
I_{22} &= 4\mathbb{E}\int_Q q_{22} A_h^4 v A_h^3 D_h v dt = 4\mathbb{E}\int_Q q_{22} A_h (A_h^3 v A_h^2 D_h v) dt - h^2\mathbb{E}\int_Q q_{22} A_h^3 D_h v A_h^2 D_h^2 v dt \\
&= 2\mathbb{E}\int_Q q_{22} A_h D_h (|A_h^2 v|^2) dt - \frac{h^2}{2}\mathbb{E}\int_Q q_{22} D_h (|A_h^2 D_h v|^2) dt \\
&= 2\mathbb{E}\int_Q q_{22} A_h D_h \left(\left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) dt - \frac{h^2}{2}\mathbb{E}\int_Q q_{22} D_h \left(\left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 \right) dt \\
&= -2\mathbb{E}\int_{\overline{Q}} A_h D_h q_{22} \left| v + \frac{h^2}{4} D_h^2 v \right|^2 dt + h\mathbb{E}\int_{\partial Q^*} D_h q_{22} t_r \left(\left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) dt \\
&\quad + 2\mathbb{E}\int_{\partial Q} q_{22} t_r \left(A_h \left| v + \frac{h^2}{4} D_h^2 v \right|^2 \right) \nu dt + \frac{h^2}{2}\mathbb{E}\int_{Q^*} D_h q_{22} \left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 dt \\
&\quad - \frac{h^2}{2}\mathbb{E}\int_{\partial Q} q_{22} t_r \left(\left| D_h v + \frac{h^2}{4} D_h^3 v \right|^2 \right) \nu dt.
\end{aligned} \tag{3.35}
$$

From (2.15) and (2.18), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$
\begin{aligned}
&- 2\mathbb{E}\int_{\overline{Q}} A_h D_h q_{22} \left| v + \frac{h^2}{4} D_h^2 v \right|^2 dt \\
&\geq 14\mathbb{E}\int_Q s^7\mu^8\varphi^7 |\partial_x\psi|^8 |v|^2 dt - \mathbb{E}\int_Q [s^6\mathcal{O}_\mu(1) + s^7\mathcal{O}_\mu((sh)^2) + s^7\mu^7\varphi^7\mathcal{O}(1)]|v|^2 dt \\
&\quad - \mathbb{E}\int_{\overline{Q}} s^3\mathcal{O}_\mu((sh)^2)|D_h^2 v|^2 dt.
\end{aligned} \tag{3.36}
$$

Thanks to (2.18) and (3.9), for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\begin{aligned}
&h\mathbb{E}\int_{\partial Q^*} D_h q_{22} t_r\left(\left|v + \frac{h^2}{4}D_h^2 v\right|^2\right)dt - \frac{h^2}{2}\mathbb{E}\int_{\partial Q} q_{22} t_r\left(\left|D_h v + \frac{h^2}{4}D_h^3 v\right|^2\right)\nu dt \\
&\geq -\mathbb{E}\int_{\partial Q} s^2\mathcal{O}_\mu((sh)^5)|D_h^2 v|dt - \mathbb{E}\int_{\partial Q} s^5\mathcal{O}_\mu((sh)^2)t_r(|D_h v|^2)dt \\
&\quad - \mathbb{E}\int_{\partial Q} s\mathcal{O}_\mu((sh)^6)t_r(|D_h^3 v|^2)dt. \qquad (3.37)
\end{aligned}$$

and

$$\begin{aligned}
&2\mathbb{E}\int_{\partial Q} q_{22} t_r\left(A_h\left|v + \frac{h^2}{4}D_h^2 v\right|^2\right)\nu dt \\
&\geq -\mathbb{E}\int_{\partial Q} s^3\mathcal{O}_\mu((sh)^2)|D_h^2 v|dt - \mathbb{E}\int_{\partial Q} s^5\mathcal{O}_\mu((sh)^2)t_r(|D_h v|^2)dt - \mathbb{E}\int_{\partial Q} s\mathcal{O}_\mu((sh)^2)t_r(|D_h^3 v|^2)dt. \qquad (3.38)
\end{aligned}$$

From (2.18), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\frac{h^2}{2}\mathbb{E}\int_{Q^*} D_h q_{22}\left|D_h v + \frac{h^2}{4}D_h^3 v\right|^2 dt \geq -\mathbb{E}\int_{Q^*} s^5\mathcal{O}_\mu((sh)^2)|D_h v|^2 dt - \mathbb{E}\int_{Q^*} s\mathcal{O}_\mu((sh)^6)|D_h^3 v|^2 dt, \qquad (3.39)$$

Hence, combining (3.35)–(3.39), for all $\mu \geq 1$, one can find constant $h_1' \leq h_1$, such that for all $\lambda \geq T + T^2$, $h \leq h_2$ and $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$I_{22} \geq 14\mathbb{E}\int_Q s^7\mu^8\varphi^7|\partial_x\psi|^8|v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \qquad (3.40)$$

From (2.3), letting $q_{23} = r^2 D_h^4\rho A_h^3 D_h\rho$, for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$I_{23} = 4\mathbb{E}\int_Q q_{23}A_h^4 v A_h D_h^3 v dt = 4\mathbb{E}\int_Q q_{23}\left(v + \frac{h^2}{4}D_h^2 v + \frac{h^2}{4}A_h^2 D_h^2 v\right)A_h D_h^3 v dt. \qquad (3.41)$$

From (2.1), (2.4), (2.6), (2.9)–(2.11), (2.15), (2.18), (3.8) and (3.9), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$\begin{aligned}
&4\mathbb{E}\int_Q q_{23} v A_h D_h^3 v dt = 4\mathbb{E}\int_{\overline{Q}} D_h^2(q_{23}v)A_h D_h v dt \\
&= 4\mathbb{E}\int_{\overline{Q}} D_h^2 q_{23} A_h^2 v A_h D_h v dt + 8\mathbb{E}\int_{\overline{Q}} A_h D_h q_{23}|A_h D_h v|^2 dt + 4\mathbb{E}\int_{\overline{Q}} A_h^2 q_{23} A_h D_h v D_h^2 v dt \\
&= -2\mathbb{E}\int_{\overline{Q^*}} D_h^3 q_{23}|A_h v|^2 dt + 6\mathbb{E}\int_{\overline{Q^*}} A_h^2 D_h q_{23}|D_h v|^2 dt - 2h^2\mathbb{E}\int_{\overline{Q}} A_h D_h q_{23}|D_h^2 v|^2 dt \\
&\geq -30\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|\partial_x\psi|^6|D_h v|^2 dt - \mathbb{E}\int_{\overline{Q}} s^3\mathcal{O}_\mu((sh)^2)|D_h^2 v|^2 dt - \mathbb{E}\int_Q s^5\mathcal{O}_\mu(1)|v|^2 dt \\
&\quad - \mathbb{E}\int_{Q^*}[s^5\mu^5\varphi^5\mathcal{O}(1) + s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)]|D_h v|^2 dt. \qquad (3.42)
\end{aligned}$$

From (2.2), (2.6), (2.9), (2.10) and (2.18), for $\lambda h(\delta T^2)^{-1} \le 1$, we get

$$
\begin{aligned}
& h^2 \mathbb{E} \int_Q q_{23} D_h^2 v A_h D_h^3 v dt \\
& = h^2 \mathbb{E} \int_{Q^*} A_h q_{23} A_h D_h^2 v D_h^3 v dt + \frac{h^2}{4} \mathbb{E} \int_{Q^*} D_h q_{23} |D_h^3 v|^2 dt - \frac{h^3}{2} \mathbb{E} \int_{\partial Q} q_{23} D_h^2 v t_r(D_h^3 v) dt \\
& = -\frac{h^2}{2} \mathbb{E} \int_{\overline{Q}} A_h D_h q_{23} |D_h^2 v|^2 dt + \frac{h^2}{2} \mathbb{E} \int_{\partial Q^*} A_h q_{23} t_r(|D_h^2 v|^2) \nu dt + \frac{h^4}{4} \mathbb{E} \int_{Q^*} D_h q_{23} |D_h^3 v|^2 dt \\
& \quad - \frac{h^3}{2} \mathbb{E} \int_{\partial Q} q_{23} D_h^2 v t_r(D_h^3 v) dt \\
& \ge -\mathbb{E} \int_{\overline{Q}} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{Q^*} s \mathcal{O}_\mu((sh)^4) |D_h^3 v|^2 dt - \mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu((sh)^2) t_r(|D_h^2 v|^2) dt \\
& \quad - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^4) t_r(|D_h^3 v|^2) dt.
\end{aligned}
\tag{3.43}
$$

From (2.5), (2.6), (2.8)–(2.10) and (2.18), for $\lambda h(\delta T^2)^{-1} \le 1$, it holds that

$$
\begin{aligned}
& h^2 \mathbb{E} \int_Q q_{23} A_h^2 D_h^2 v A_h D_h^3 v dt \\
& = -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{23} |A_h D_h^2 v|^2 dt + \frac{h^2}{2} \mathbb{E} \int_{\partial Q} q_{23} t_r(|A_h D_h^2 v|^2) \nu dt \\
& = -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{23} |A_h D_h^2 v|^2 dt + \frac{h^2}{2} \mathbb{E} \int_{\partial Q} q_{23} \Big(|D_h^2 v|^2 \nu - h D_h^2 v t_r(D_h^3 v) + \frac{h^2}{4} t_r(|D_h^3 v|^2) \nu \Big) dt \\
& \ge -\mathbb{E} \int_{\overline{Q}} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s^3 \mathcal{O}_\mu((sh)^2) |D_h^2 v|^2 dt - \mathbb{E} \int_{\partial Q} s \mathcal{O}_\mu((sh)^4) t_r(|D_h^3 v|^2) dt.
\end{aligned}
\tag{3.44}
$$

Combining (3.41)–(3.44), for $\lambda h(\delta T^2)^{-1} \le 1$ and $\lambda \ge T^2$, we have

$$
I_{23} \ge -30 \mathbb{E} \int_{Q^*} s^5 \mu^6 \varphi^5 |\partial_x \psi|^6 |D_h v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \tag{3.45}
$$

From (2.3), letting $q_{24} = r D_h^4 \rho A_h D_h (r A_h D_h^3 \rho)$, we have

$$
I_{24} = 2\mathbb{E} \int_Q q_{24} A_h^4 v v dt = 2\mathbb{E} \int_Q q_{24} \Big(v + \frac{h^2}{4} D_h^2 v + \frac{h^2}{4} A_h^2 D_h^2 v \Big) v dt. \tag{3.46}
$$

From (2.16) and (2.20), for $\lambda h(\delta T^2)^{-1} \le 1$, we have

$$
2\mathbb{E} \int_Q q_{24} v^2 dt \ge -6\mathbb{E} \int_Q s^7 \mu^8 \varphi^7 |\partial_x \psi|^8 v^2 dt - \mathbb{E} \int_Q [s^6 \mathcal{O}_\mu(1) + s^7 \mathcal{O}_\mu((sh)^2) + s^7 \mu^7 \varphi^7 \mathcal{O}(1)] |v|^2 dt. \tag{3.47}
$$

From (2.1), (2.6), (2.9), (2.17), (3.8) and (3.9), for $\lambda h(\delta T^2)^{-1} \le 1$, we obtain

$$
\frac{h^2}{2} \mathbb{E} \int_Q q_{24} D_h^2 v v dt = -\frac{h^2}{2} \mathbb{E} \int_{Q^*} D_h q_{24} A_h v D_h v dt - \frac{h^2}{2} \mathbb{E} \int_{Q^*} A_h q_{24} |D_h v|^2 dt
$$

$$
\begin{aligned}
&= \frac{h^2}{4}\mathbb{E}\int_{\overline{Q}} D_h^2 q_{24} v^2 dt - \frac{h^2}{2}\mathbb{E}\int_{Q^*} A_h q_{24} |D_h v|^2 dt \\
&\geq -\mathbb{E}\int_Q s^5 \mathcal{O}_\mu(1) v^2 dt - \mathbb{E}\int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt. \qquad (3.48)
\end{aligned}
$$

From (2.1), (2.3), (2.5), (2.6), (2.9), (2.10), (2.17) and (3.8), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$
\begin{aligned}
&\frac{h^2}{2}\mathbb{E}\int_Q q_{24} A_h^2 D_h^2 v v dt \\
&= -\frac{h^2}{2}\mathbb{E}\int_{Q^*} D_h q_{24} A_h v A_h^2 D_h v dt - \frac{h^2}{2}\mathbb{E}\int_{Q^*} A_h q_{24} D_h v A_h^2 D_h v dt \\
&= -\frac{h^2}{2}\mathbb{E}\int_{Q^*} D_h q_{24} A_h v D_h v dt - \frac{h^4}{8}\mathbb{E}\int_{Q^*} D_h q_{24} A_h v D_h^3 v dt - \frac{h^2}{2}\mathbb{E}\int_{Q^*} A_h q_{24} |D_h v|^2 dt \\
&\quad - \frac{h^4}{8}\mathbb{E}\int_{Q^*} A_h q_{24} D_h v D_h^3 v dt \\
&\geq -\mathbb{E}\int_Q s^5 \mathcal{O}_\mu((sh)^2) |v|^2 dt - \mathbb{E}\int_{Q^*} s^5 \mathcal{O}_\mu((sh)^2) |D_h v|^2 dt - \mathbb{E}\int_{Q^*} s \mathcal{O}_\mu((sh)^4) |D_h^3 v|^2 dt. \qquad (3.49)
\end{aligned}
$$

Combining (3.46)–(3.49), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$
I_{24} \geq -6\mathbb{E}\int_Q s^7 \mu^8 \varphi^7 |\partial_x \psi|^8 v^2 dt - \mathcal{A}_1. \qquad (3.50)
$$

From (2.1), (2.3) and (2.9), letting $q_{32} = r^2 A_h^4 \rho A_h D_h^3 \rho$, we have

$$
\begin{aligned}
I_{32} &= 4\mathbb{E}\int_Q q_{32} D_h^4 v A_h^3 D_h v dt = 4\mathbb{E}\int_Q q_{32} D_h^4 v A_h D_h v dt + h^2\mathbb{E}\int_Q q_{32} D_h^4 v A_h D_h^3 v dt \\
&= -4\mathbb{E}\int_{Q^*} D_h q_{32} A_h^2 D_h v D_h^3 v dt - 4\mathbb{E}\int_{Q^*} A_h q_{32} A_h D_h^2 v D_h^3 v dt + 4\mathbb{E}\int_{\partial Q} q_{32} A_h D_h v t_r(D_h^3 v) \nu dt \\
&\quad + h^2\mathbb{E}\int_Q q_{32} D_h^4 v A_h D_h^3 v dt. \qquad (3.51)
\end{aligned}
$$

From (2.1), (2.3), (2.6), (2.9), (2.15), (2.18), (3.8) and (3.9), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$
\begin{aligned}
&-4\mathbb{E}\int_{Q^*} D_h q_{32} A_h^2 D_h v D_h^3 v dt \\
&= -4\mathbb{E}\int_{Q^*} D_h q_{32} D_h v D_h^3 v dt - h^2\mathbb{E}\int_{Q^*} D_h q_{32} |D_h^3 v|^2 dt \\
&= 4\mathbb{E}\int_{\overline{Q}} D_h^2 q_{32} A_h D_h v D_h^2 v dt + 4\mathbb{E}\int_{\overline{Q}} A_h D_h q_{32} |D_h^2 v|^2 dt - h^2\mathbb{E}\int_{Q^*} D_h q_{32} |D_h^3 v|^2 dt \\
&= -2\mathbb{E}\int_{\overline{Q^*}} D_h^3 q_{32} |D_h v|^2 dt + 4\mathbb{E}\int_{\overline{Q}} A_h D_h q_{32} |D_h^2 v|^2 dt - h^2\mathbb{E}\int_{Q^*} D_h q_{32} |D_h^3 v|^2 dt \\
&\geq -12\mathbb{E}\int_{\overline{Q}} s^3 \mu^4 \varphi^3 |\partial_x \psi|^4 |D_h^2 v|^2 dt - \mathbb{E}\int_{Q^*} s^3 \mathcal{O}_\mu(1) |D_h v|^2 dt - \mathbb{E}\int_{Q^*} s \mathcal{O}_\mu((sh)^2) |D_h^3 v|^2 dt
\end{aligned}
$$

$$
-\mathbb{E}\int_{\overline{Q}}[s^2\mathcal{O}_\mu(1)+s^3\mathcal{O}_\mu((sh)^2)+s^3\mu^3\varphi^3\mathcal{O}(1)]|D_h^2v|^2dt. \tag{3.52}
$$

From (2.6), (2.9), (2.15) and (2.18), for $\lambda h(\delta T^2)^{-1}\le 1$, it holds that

$$
\begin{aligned}
&-4\mathbb{E}\int_{Q^*}A_hq_{32}A_hD_h^2vD_h^3vdt\\
&=2\mathbb{E}\int_{\overline{Q}}A_hD_hq_{32}|D_h^2v|^2dt-2\mathbb{E}\int_{\partial Q^*}A_hq_{32}t_r(|D_h^2v|^2)\nu dt\\
&\ge -6\mathbb{E}\int_{\overline{Q}}s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2v|^2dt+2\mathbb{E}\int_{\partial Q^*}s^3\mu^3\varphi^3(\partial_x\psi)^3t_r(|D_h^2v|^2)\nu dt\\
&\quad-\mathbb{E}\int_{\overline{Q}}[s^2\mathcal{O}_\mu(1)+s^3\mathcal{O}_\mu((sh)^2)+s^3\mu^3\varphi^3\mathcal{O}(1)]|D_h^2v|^2dt\\
&\quad-\mathbb{E}\int_{\partial Q^*}[s^2\mathcal{O}_\mu(1)+s^3\mathcal{O}_\mu((sh)^2)+s^3\mu^2\varphi^3\mathcal{O}(1)]t_r(|D_h^2v|^2)dt.
\end{aligned} \tag{3.53}
$$

From (2.15), (2.18) and (3.8), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1}\le 1$, it holds that

$$
\begin{aligned}
&4\mathbb{E}\int_{\partial Q}q_{32}A_hD_hvt_r(D_h^3v)\nu dt\ge -\mathbb{E}\int_{\partial Q}s^3\mathcal{O}_\mu(1)|A_hD_hv|t_r(|D_h^3v|)dt\\
&=-\frac{h}{2}\mathbb{E}\int_{\partial Q}s^3\mathcal{O}_\mu(1)|D_h^2v|t_r(|D_h^3v|)dt\\
&\ge -\mathbb{E}\int_{\partial Q}s^3\mathcal{O}_\mu(sh)|D_h^2v|^2dt-\mathbb{E}\int_{\partial Q}s\mathcal{O}_\mu(sh)t_r(|D_h^3v|^2)dt.
\end{aligned} \tag{3.54}
$$

From (2.6), (2.9) and (2.18), for $\lambda h(\delta T^2)^{-1}\le 1$, we obtain

$$
\begin{aligned}
h^2\mathbb{E}\int_Q q_{32}D_h^4vA_hD_h^3vdt&=-\frac{h^2}{2}\mathbb{E}\int_{Q^*}D_hq_{32}|D_h^3v|^2dt+\frac{h^2}{2}\mathbb{E}\int_{\partial Q}q_{32}t_r(|D_h^3v|^2)\nu dt\\
&\ge -\mathbb{E}\int_{Q^*}s\mathcal{O}_\mu((sh)^2)|D_h^3v|^2dt-\mathbb{E}\int_{\partial Q}s\mathcal{O}_\mu((sh)^2)t_r(|D_h^3v|^2)dt.
\end{aligned} \tag{3.55}
$$

Combining (3.51)–(3.55), for $\lambda h(\delta T^2)^{-1}\le 1$ and $\lambda\ge T^2$, we have

$$
I_{32}\ge -18\mathbb{E}\int_{\overline{Q}}s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2v|^2dt-2\mathbb{E}\int_{\partial Q}s^3\mu^3\varphi^3|\partial_x\psi|^3|D_h^2v|^2dt-\mathcal{A}_1-\mathcal{B}. \tag{3.56}
$$

From (2.6), (2.9), (2.14) and (2.18), letting $q_{33}=r^2A_h^4\rho A_h^3D_h\rho$, for $\lambda h(\delta T^2)^{-1}\le 1$, we have

$$
\begin{aligned}
I_{33}&=4\mathbb{E}\int_Q q_{33}D_h^4vA_hD_h^3vdt=-2\mathbb{E}\int_{Q^*}D_hq_{33}|D_h^3v|^2dt+2\mathbb{E}\int_{\partial Q}q_{33}t_r(|D_h^3v|)\nu dt\\
&\ge 2\mathbb{E}\int_{Q^*}s\mu^2\varphi|\partial_x\psi|^2|D_h^3v|^2dt+2\mathbb{E}\int_{\partial Q}s\mu\varphi|\partial_x\psi|t_r(|D_h^3v|)dt\\
&\quad-\mathbb{E}\int_{Q^*}[\mathcal{O}_\mu(1)+s\mathcal{O}_\mu((sh)^2)+s\mu\varphi\mathcal{O}(1)]|D_h^3v|^2dt\\
&\quad-\mathbb{E}\int_{\partial Q}[\mathcal{O}_\mu(1)+s\mathcal{O}_\mu(sh)+s\varphi\mathcal{O}(1)]t_r(|D_h^3v|^2)dt.
\end{aligned} \tag{3.57}
$$

Letting $q_{34} = rA_h^4\rho A_h D_h(rA_h D_h^3\rho)$, from (2.1), (2.3), (2.6), (2.9), (2.11), (2.16), (2.20), (3.8) and (3.9), for $\lambda h(\delta T^2)^{-1} \le 1$ and $\lambda \ge T^2$, we have

$$
\begin{aligned}
I_{34} &= 2\mathbb{E}\int_Q q_{34} D_h^4 v v dt = 2\mathbb{E}\int_{\overline{Q}} D_h^2(q_{34}v) D_h^2 v dt \\
&= 2\mathbb{E}\int_{\overline{Q}} D_h^2 q_{34} A_h^2 v D_h^2 v dt + 4\mathbb{E}\int_{\overline{Q}} A_h D_h q_{34} A_h D_h v D_h^2 v dt + 2\mathbb{E}\int_{\overline{Q}} A_h^2 q_{34}|D_h^2 v|^2 dt \\
&= 2\mathbb{E}\int_{\overline{Q}} D_h^2 q_{34} v D_h^2 v dt + \frac{h^2}{2}\mathbb{E}\int_{\overline{Q}} D_h^2 q_{34}|D_h^2 v|^2 dt - 2\mathbb{E}\int_{\overline{Q^*}} A_h D_h^2 q_{34}|D_h v|^2 dt \\
&\quad + 2\mathbb{E}\int_{\overline{Q}} A_h^2 q_{34}|D_h^2 v|^2 dt \\
&\ge -6\mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2 v|^2 dt - \mathbb{E}\int_{Q^*} s^3\mathcal{O}_\mu(1)|D_h v|^2 dt - \mathbb{E}\int_Q s^4\mathcal{O}_\mu(1)|v|^2 dt \\
&\quad - \mathbb{E}\int_{\overline{Q}} [s^2\mathcal{O}_\mu(1) + (s+s^3)\mathcal{O}_\mu((sh)^2) + s^3\mu^3\varphi^3\mathcal{O}(1)]|D_h^2 v|^2 dt \\
&\ge -6\mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2 v|^2 dt - \mathcal{A}_1.
\end{aligned}
\tag{3.58}
$$

Letting $q_{42} = A_h D_h(rA_h^2 D_h^2\rho) rA_h D_h^3\rho$ from (2.3)–(2.5), (2.10), (2.14) and (2.17), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \le 1$, we obtain

$$
\begin{aligned}
I_{42} &= 24\mathbb{E}\int_Q q_{42} A_h D_h v A_h^3 D_h v dt = 24\mathbb{E}\int_Q q_{42}|A_h D_h v|^2 dt + 6h^2\mathbb{E}\int_Q q_{42} A_h D_h v A_h D_h^3 v dt \\
&= 24\mathbb{E}\int_Q q_{42} A_h(|D_h v|^2) dt - 6h^2\mathbb{E}\int_Q q_{42}|D_h^2 v|^2 dt + 6h^2\mathbb{E}\int_Q q_{42} A_h D_h v A_h D_h^3 v dt \\
&= 24\mathbb{E}\int_{Q^*} A_h q_{42}|D_h v|^2 dt - 12h\mathbb{E}\int_{\partial Q} q_{42} t_r(|D_h v|^2) dt - 6h^2\mathbb{E}\int_Q q_{42}|D_h^2 v|^2 dt \\
&\quad + 6h^2\mathbb{E}\int_Q q_{42} A_h D_h v A_h D_h^3 v dt \\
&\ge -48\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|\partial_x\psi|^6|D_h v|^2 dt - \mathbb{E}\int_{Q^*}[s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2) + s^5\mu^5\varphi^5\mathcal{O}(1)]|D_h v|^2 dt \\
&\quad - \mathbb{E}\int_Q s^3\mathcal{O}_\mu((sh)^2)|D_h^2 v|^2 dt - \mathbb{E}\int_{Q^*} s\mathcal{O}_\mu((sh)^2)|D_h^3 v|^2 dt - \mathbb{E}\int_{\partial Q} s^4\mathcal{O}_\mu((sh)) t_r(|D_h v|^2) dt \\
&\ge -48\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|\partial_x\psi|^6|D_h v|^2 dt - \mathcal{A}_1 - \mathcal{B}.
\end{aligned}
\tag{3.59}
$$

Letting $q_{43} = A_h D_h(rA_h^2 D_h^2\rho) rA_h^3 D_h\rho$, from (2.1) and (2.9), we have

$$
\begin{aligned}
I_{43} &= 24\mathbb{E}\int_Q q_{43} A_h D_h v A_h D_h^3 v dt \\
&= -24\mathbb{E}\int_{Q^*} D_h q_{43} A_h^2 D_h v A_h D_h^2 v dt - 24\mathbb{E}\int_{Q^*} A_h q_{43}|A_h D_h^2 v|^2 dt \\
&\quad + 24\mathbb{E}\int_{\partial Q} q_{43} A_h D_h v t_r(A_h D_h^2 v)\nu dt.
\end{aligned}
\tag{3.60}
$$

From (2.4), (2.6), (2.8)–(2.10), (2.17), (3.8) and (3.9), for $\lambda h(\delta T^2)^{-1} \leq 1$, we obtain

$$
\begin{aligned}
&-24\mathbb{E}\int_{Q^*} D_h q_{43} A_h^2 D_h v A_h D_h^2 v dt \\
&= 12\mathbb{E}\int_{\overline{Q}} D_h^2 q_{43} |A_h D_h v|^2 dt - 12\mathbb{E}\int_{\partial Q^*} D_h q_{43} t_r(|A_h D_h v|^2)\nu dt \\
&= 12\mathbb{E}\int_{\overline{Q^*}} A_h D_h^2 q_{43} |D_h v|^2 dt - 3h^2\mathbb{E}\int_{\overline{Q}} D_h^2 q_{43} |D_h^2 v|^2 dt - 3h^2\mathbb{E}\int_{\partial Q^*} D_h q_{43} t_r(|D_h^2 v|^2)\nu dt \\
&\geq -\mathbb{E}\int_{Q^*} s^3\mathcal{O}_\mu(1)|D_h v|^2 dt - \mathbb{E}\int_{\overline{Q}} s\mathcal{O}_\mu((sh)^2)|D_h^2 v|^2 dt - \mathbb{E}\int_{\partial Q} s\mathcal{O}_\mu((sh)^2)|D_h^2 v|^2 dt. \qquad (3.61)
\end{aligned}
$$

From (2.4), (2.10), (2.14) and (2.17), for $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$
\begin{aligned}
&-24\mathbb{E}\int_{Q^*} A_h q_{43} |A_h D_h^2 v|^2 dt \\
&= -24\mathbb{E}\int_{\overline{Q}} A_h^2 q_{43} |D_h^2 v|^2 dt + 12h\mathbb{E}\int_{\partial Q^*} A_h q_{43} t_r(|D_h^2 v|^2) dt + 6h^2\mathbb{E}\int_{Q^*} A_h q_{43} |D_h^3 v|^2 dt \\
&\geq 48\mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2 v|^2 dt - \mathbb{E}\int_{\overline{Q}} [s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2) + s^3\mu^3\varphi^3\mathcal{O}(1)]|D_h^2 v|^2 dt \\
&\quad - \mathbb{E}\int_{Q^*} s\mathcal{O}_\mu((sh)^2)|D_h^3 v|^2 dt - \mathbb{E}\int_{\partial Q} s^2\mathcal{O}_\mu(sh)|D_h^2 v|^2 dt. \qquad (3.62)
\end{aligned}
$$

From (3.8) and Cauchy-Schwarz inequality, by $A_h D_h v = \frac{h}{2} D_h^2 v$, $t_r(A_h D_h^2 v) = D_h^2 v - \frac{h}{2} t_r(D_h^3)\nu$ on ∂Q, then for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$
\begin{aligned}
24\mathbb{E}\int_{\partial Q} q_{43} A_h D_h v t_r(A_h D_h^2 v)\nu dt \geq& -\mathbb{E}\int_{\partial Q} s^2\mathcal{O}_\mu(sh)|D_h^2 v|^2 dt \\
&- \mathbb{E}\int_{\partial Q} \mathcal{O}_\mu((sh)^2) t_r(|D_h^3 v|^2) dt. \qquad (3.63)
\end{aligned}
$$

Combining (3.60)–(3.63), for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we obtain

$$
I_{43} \geq 48\mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2 v|^2 dt - \mathcal{A}_1 - \mathcal{B}. \qquad (3.64)
$$

From (2.5), (2.10), (2.14) and (2.17), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \leq 1$ and $\lambda \geq T^2$, we have

$$
\begin{aligned}
I_{44} &= 12\mathbb{E}\int_{Q} A_h D_h(rA_h^2 D_h^2\rho) A_h D_h(rA_h D_h^3\rho) A_h D_h v v dt \\
&\geq -\mathbb{E}\int_{Q^*} s^4\mathcal{O}_\mu(1)|D_h v|^2 dt - \mathbb{E}\int_{Q} s^6\mathcal{O}_\mu(1)|v|^2 dt \\
&\geq -\mathcal{A}_1. \qquad (3.65)
\end{aligned}
$$

From (2.1)–(2.3), (2.14), (2.17) and (3.5), letting

$$
\begin{cases}
q_1 = A_h D_h(rA_h^2 D_h^2\rho), \quad q_2 = A_h D_h(rD_h^4\rho), \quad q_3 = D_h^2(rD_h^4\rho), \\
q_4 = D_h^2(rA_h^4\rho), \quad q_5 = A_h D_h(rA_h^4\rho),
\end{cases} \qquad (3.66)
$$

for $\lambda h(\delta T^2)^{-1} \le 1$, $h \le 1$, we obtain

$$
\begin{aligned}
\Phi(v) &= 3h^2 D_h^2(q_1 A_h D_h v) + \frac{h^2}{2} A_h^2(q_2 A_h D_h v) + \frac{h^2}{4} q_3 A_h^2 v - q_4 D_h^2 v + 2A_h D_h(q_5 D_h^2 v) \\
&= 3h^2(D_h^2 q_1 A_h^3 D_h v + 2A_h D_h q_1 A_h^2 D_h^2 v + A_h^2 q_1 A_h D_h^3 v) + \frac{h^2}{2} q_2 A_h D_h v \\
&\quad + \frac{h^4}{8}(D_h^2 q_2 A_h^3 D_h v + 2A_h D_h q_2 A_h^2 D_h^2 v + A_h^2 q_2 A_h D_h^3 v) + \frac{h^2}{4} q_3 A_h^2 v - q_4 D_h^2 v \\
&\quad + 2A_h D_h q_5 A_h^2 D_h^2 v + \frac{h^2}{2} D_h^2 q_5 A_h D_h^3 v + 2A_h(A_h q_5 D_h^3 v) \\
&\ge -\mathcal{O}_\mu((sh)^2)(|A_h^3 D_h v| + |A_h^2 D_h^2 v| + |A_h D_h^3 v| + |D_h^2 v|) - s^2 \mathcal{O}_\mu((sh)^2)(|A_h D_h v| + |A_h^2 v|) \\
&\quad + 2A_h(A_h q_5 D_h^3 v).
\end{aligned} \tag{3.67}
$$

From (2.5), (2.10), (2.14), (2.17), (3.66) and (3.67), thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \le 1$, $h \le 1$ and $\lambda \ge T^2$, we have

$$
\begin{aligned}
I_{52} &= 4\mathbb{E}\int_Q r A_h D_h^3 \rho A_h^3 D_h v \Phi(v) dt \\
&\ge -\mathbb{E}\int_Q s^5 \mathcal{O}_\mu((sh)^2)|v|^2 dt - \mathbb{E}\int_{Q^*}[(s^3 + s^5)\mathcal{O}_\mu((sh)^2) + s^4 \mathcal{O}_\mu(1) + s^5 \mu^5 \varphi^5 \mathcal{O}(1)]|D_h v|^2 dt \\
&\quad - \mathbb{E}\int_{\overline{Q}}[s^3 \mathcal{O}_\mu((sh)^2) + s^2 \mathcal{O}_\mu(1)]|D_h^2 v|^2 dt \\
&\quad - \mathbb{E}\int_{Q^*}[\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2) + s\mu\varphi\mathcal{O}(1)]|D_h^3 v|^2 dt \\
&\ge -\mathcal{A}_1,
\end{aligned} \tag{3.68}
$$

and

$$
\begin{aligned}
I_{53} &= 4\mathbb{E}\int_Q r A_h^3 D_h \rho A_h D_h^3 v \Phi(v) dt \\
&\ge -\mathbb{E}\int_Q s^3 \mathcal{O}_\mu((sh)^2)|v|^2 dt - \mathbb{E}\int_{Q^*}[s\mathcal{O}_\mu((sh)^2) + s^3 \mathcal{O}_\mu((sh)^2)]|D_h v|^2 dt \\
&\quad - \mathbb{E}\int_{\overline{Q}} s\mathcal{O}_\mu((sh)^2)|D_h^2 v|^2 dt - \mathbb{E}\int_{Q^*}[\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2) + s\mu\varphi\mathcal{O}(1)]|D_h^3 v|^2 dt \\
&\ge -\mathcal{A}_1.
\end{aligned} \tag{3.69}
$$

Further,

$$
\begin{aligned}
I_{54} &= 2\mathbb{E}\int_Q A_h D_h(r A_h D_h^3 \rho) v \Phi(v) dt \\
&\ge -\mathbb{E}\int_Q (s^5 + s^6)\mathcal{O}_\mu(1)|v|^2 dt - \mathbb{E}\int_{Q^*}(\mathcal{O}_\mu(1) + s^4 \mathcal{O}_\mu(1))|D_h v|^2 dt \\
&\quad - \mathbb{E}\int_{\overline{Q}} \mathcal{O}_\mu(1)]|D_h^2 v|^2 dt - \mathbb{E}\int_{Q^*} \mathcal{O}_\mu(1)|D_h^3 v|^2 dt \\
&\ge -\mathcal{A}_1.
\end{aligned} \tag{3.70}
$$

Combining (3.28), (3.33), (3.34), (3.40), (3.45), (3.50), (3.56)–(3.59), (3.64), (3.65) and (3.68)–

(3.70), for $\lambda h(\delta T^2)^{-1}\le 1$, $h\le 1$ and $\lambda\ge T^2$, we have

$$
\begin{aligned}
\sum_{i=1}^{5}\sum_{j=2}^{4} I_{ij} \ge\ & 8\mathbb{E}\int_Q s^7\mu^8\varphi^7|\partial_x\psi|^8|v|^2dt+18\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|\partial_x\psi|^6|D_hv|^2dt\\
&+60\mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|\partial_x\psi|^4|D_h^2v|^2dt+2\mathbb{E}\int_{Q^*} s\mu^2\varphi|\partial_x\psi|^2|D_h^3v|^2dt\\
&+12\mathbb{E}\int_{\partial Q} s^5\mu^5\varphi^5|\partial_x\psi|^5 t_r(|D_hv|^2)dt+10\mathbb{E}\int_{\partial Q} s^3\mu^3\varphi^3|\partial_x\psi|^3|D_h^2v|^2dt\\
&+2\mathbb{E}\int_{\partial Q} s\mu\varphi|\partial_x\psi| t_r(|D_h^3v|^2)dt-\mathcal{A}_1-\mathcal{B}. \qquad (3.71)
\end{aligned}
$$

Step 4. In this step, we return to the variable w.
From (2.5), (2.10), (2.18), (2.19), (3.5) and (3.67), for $\lambda h(\delta T^2)^{-1}\le 1$ and $\lambda\ge T+T^2$, we have

$$
\begin{aligned}
\mathbb{E}\int_Q |I_3|^2dt \le\ & \mathbb{E}\int_Q (s^4+s^6)\mathcal{O}_\mu(1)|v|^2dt+\mathbb{E}\int_Q (s^4\mathcal{O}_\mu(1)+\mathcal{O}_\mu(1))|D_hv|^2dt\\
&+\mathbb{E}\int_Q \mathcal{O}_\mu(1)|D_h^2v|^2dt+\mathbb{E}\int_Q \mathcal{O}_\mu(1)|D_h^3v|^2dt\\
\le\ & \mathcal{A}_1. \qquad (3.72)
\end{aligned}
$$

Combining (1.6), (3.6), (3.7), (3.27), (3.71) and (3.72), for all $\mu\ge 1$, there exists a positive constant $\varepsilon_1\le 1$ and a sufficiently large $\lambda_1\ge 1$, such that for $\lambda\ge\lambda_1(T+T^2)$, $h\le h_1'$, and $\lambda h(\delta T^2)^{-1}\le\varepsilon_1$, we have

$$
\begin{aligned}
& C\mathbb{E}\int_Q r^2|f|^2dt+C(\mu)\mathbb{E}\int_Q s^4r^2|g|^2dt+C\mathbb{E}\int_0^T\int_{\mathcal{M}\cap G_2} s^7\mu^8\varphi^7|v|^2dt\\
&+C\Big[\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_2} s^5\mu^6\varphi^5|D_hv|^2dt+\mathbb{E}\int_0^T\int_{\overline{\mathcal{M}}\cap G_2} s^3\mu^4\varphi^3|D_h^2v|^2dt+\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_2} s\mu^2\varphi|D_h^3v|^2dt\Big]\\
&\ge \mathbb{E}\int_Q s^7\mu^8\varphi^7|v|^2dt+\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_hv|^2dt+\mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|D_h^2v|^2dt+\mathbb{E}\int_{Q^*} s\mu^2\varphi|D_h^3v|^2dt\\
&\quad+\mathbb{E}\int_{\overline{Q}} r^2|D_h^2g|^2dt-\mathcal{A}_1-\mathcal{A}_3. \qquad (3.73)
\end{aligned}
$$

Noting $w=\rho v$, combining (2.1), (2.15) and (2.18), for $\lambda h(\delta T^2)^{-1}\le 1$, we obtain

$$
\begin{aligned}
&\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5r^2|D_hw|^2dt=\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5r^2|D_h(\rho v)|^2dt\\
&\le 2\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5r^2(|D_h\rho A_hv|^2+|A_h\rho D_hv|^2)dt\\
&\le C\mathbb{E}\int_{Q^*} s^7\mu^8\varphi^7|A_hv|^2dt+2\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_hv|^2dt+\mathbb{E}\int_Q [s^6\mathcal{O}_\mu(1)+s^7\mathcal{O}_\mu((sh)^2)]|v|^2dt\\
&\quad+\mathbb{E}\int_{Q^*} s^5\mathcal{O}_\mu((sh)^2)|D_hv|^2dt. \qquad (3.74)
\end{aligned}
$$

From (1.7), (2.5), (2.10) and (3.8) and Proposition 2.5, we have

$$
\begin{aligned}
\mathbb{E}\int_{Q^*} s^7\mu^8\varphi^7|A_hv|^2dt &\le \mathbb{E}\int_{Q^*} s^7\mu^8\varphi^7A_h(v^2)dt\\
&= \mathbb{E}\int_{Q} s^7\mu^8A_h(\varphi^7)v^2dt\\
&\le \mathbb{E}\int_{Q} s^7\mu^8\varphi^7|v|^2dt + \mathbb{E}\int_{Q} s^5\mathcal{O}_\mu((sh)^2)|v|^2dt.
\end{aligned}
\tag{3.75}
$$

Combining (3.74) and (3.75), for $\lambda h(\delta T^2)^{-1}\le 1$ and $\lambda\ge T^2$, it holds that

$$
\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5r^2|D_hw|^2dt \le 2\mathbb{E}\int_{Q} s^7\mu^8\varphi^7|v|^2dt + 2\mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_hv|^2dt + \mathcal{A}_1. \tag{3.76}
$$

Since

$$
r^2|D_h^2w|^2 = r^2|D_h^2(\rho v)|^2 \le Cr^2(|D_h^2\rho A_h^2v|^2 + |A_hD_h\rho A_hD_hv|^2 + |A_h^2\rho D_h^2v|^2),
$$

similar to (3.76), for $\lambda h(\delta T^2)^{-1}\le 1$ and $\lambda\ge T^2$, we have

$$
\begin{aligned}
&\mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3r^2|D_h^2w|^2dt\\
&\le C\Big[\mathbb{E}\int_{Q} s^7\mu^8\varphi^7|v|^2dt + \mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_hv|^2dt + \mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|D_h^2v|^2dt\Big]\\
&\quad + \mathbb{E}\int_{\overline{Q}} s^3\mathcal{O}_\mu((sh)^2)|D_h^2v|^2dt + \mathbb{E}\int_{Q^*}[s^5\mathcal{O}_\mu((sh)^2) + s^4\mathcal{O}_\mu(1))|D_hv|^2dt\\
&\quad + \mathbb{E}\int_{Q}[s^7\mathcal{O}_\mu((sh)^2) + s^6\mathcal{O}_\mu(1))|v|^2dt\\
&\le C\Big[\mathbb{E}\int_{Q} s^7\mu^8\varphi^7|v|^2dt + \mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_hv|^2dt + \mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|D_h^2v|^2dt\Big] + \mathcal{A}_1.
\end{aligned}
\tag{3.77}
$$

Noting that

$$
r^2|D_h^3w|^2 = r^2|D_h^3(\rho v)|^2 \le Cr^2(|D_h^3\rho A_h^3v|^2 + |A_hD_h^2\rho A_h^2D_hv|^2 + |A_h^2D_h\rho A_hD_h^2v|^2 + |A_h^3\rho D_h^3v|^2),
$$

similar to (3.76), for $\lambda h(\delta T^2)^{-1}\le 1$ and $\lambda\ge T^2$, we have

$$
\begin{aligned}
&\mathbb{E}\int_{Q^*} s\mu^2\varphi r^2|D_h^3w|^2dt\\
&\le C\Big[\mathbb{E}\int_{Q} s^7\mu^8\varphi^7|v|^2dt + \mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_hv|^2dt + \mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|D_h^2v|^2dt + \mathbb{E}\int_{Q^*} s\mu^2\varphi|D_h^3v|^2dt\Big]\\
&\quad + \mathbb{E}\int_{Q^*} s\mathcal{O}_\mu((sh)^2)|D_h^3v|^2dt + \mathbb{E}\int_{\overline{Q}}[s^3\mathcal{O}_\mu((sh)^2) + s^2\mathcal{O}_\mu(1)]|D_h^2v|^2dt\\
&\quad + \mathbb{E}\int_{Q^*}[s^5\mathcal{O}_\mu((sh)^2) + s^4\mathcal{O}_\mu(1))|D_hv|^2dt + \mathbb{E}\int_{Q}[s^7\mathcal{O}_\mu((sh)^2) + s^6\mathcal{O}_\mu(1))|v|^2dt\\
&\le C\Big[\mathbb{E}\int_{Q} s^7\mu^8\varphi^7|v|^2dt + \mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_hv|^2dt + \mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|D_h^2v|^2dt + \mathbb{E}\int_{Q^*} s\mu^2\varphi|D_h^3v|^2dt\Big]\\
&\quad + \mathcal{A}_1.
\end{aligned}
\tag{3.78}
$$

Hence, combining (3.73) and (3.76)–(3.78), for $\mu \geq 1$, $\lambda \geq \lambda_1(T+T^2)$, $h \leq h_1$, and $\lambda h(\delta T^2)^{-1} \leq \varepsilon_1$, we have

$$
\begin{aligned}
& C\mathbb{E}\int_Q r^2|f|^2dt + C(\mu)\mathbb{E}\int_Q s^4r^2|g|^2dt + C\mathbb{E}\int_0^T\int_{\mathcal{M}\cap G_2} s^7\mu^8\varphi^7|v|^2dt \\
& + C\Big[\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_2} s^5\mu^6\varphi^5|D_hv|^2dt + \mathbb{E}\int_0^T\int_{\overline{\mathcal{M}}\cap G_2} s^3\mu^4\varphi^3|D_h^2v|^2dt + \mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_2} s\mu^2\varphi|D_h^3v|^2dt\Big] \\
& \geq \mathbb{E}\int_Q s^7\mu^8\varphi^7r^2|w|^2dt + \mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5r^2|D_hw|^2dt + \mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3r^2|D_h^2w|^2dt \\
& \quad + \mathbb{E}\int_{Q^*} s\mu^2\varphi r^2|D_h^3w|^2dt + \mathbb{E}\int_{\overline{Q}} r^2|D_h^2g|^2dt - \mathcal{A}_1 - \mathcal{A}_3.
\end{aligned} \tag{3.79}
$$

From (2.1) and (2.17), for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$
rD_hw = rD_h\rho A_hv + rA_h\rho D_hv = rD_h\rho A_hv + D_hv + \mathcal{O}_\mu((sh)^2)D_hv,
$$

which implies

$$
\begin{aligned}
& \mathbb{E}\int_0^T\int_{M^*\cap G_2} s^5\mu^6\varphi^5|D_hv|^2dt \\
& \leq C\mathbb{E}\int_0^T\int_{M^*\cap G_2} s^5\mu^6\varphi^5(rD_h\rho)^2|A_hv|^2dt + C\mathbb{E}\int_0^T\int_{M^*\cap G_2} s^5\mu^6\varphi^5r^2|D_hw|^2dt \\
& \quad + \mathbb{E}\int_{Q^*} s^5\mathcal{O}_\mu((sh)^2)|D_hv|^2dt \\
& \leq C\mathbb{E}\int_0^T\int_{M\cap G_1} s^7\mu^8\varphi^7r^2|w|^2dt + C\mathbb{E}\int_0^T\int_{M^*\cap G_1} s^5\mu^6\varphi^5r^2|D_hw|^2dt \\
& \quad + \mathbb{E}\int_{Q^*} s^5\mathcal{O}_\mu((sh)^2)|D_hv|^2dt + \mathbb{E}\int_Q [s^7\mathcal{O}_\mu((sh)^2) + s^6\mathcal{O}_\mu(1)]|v|^2dt \\
& \leq C\mathbb{E}\int_0^T\int_{M\cap G_1} s^7\mu^8\varphi^7r^2|w|^2dt + C\mathbb{E}\int_0^T\int_{M^*\cap G_1} s^5\mu^6\varphi^5r^2|D_hw|^2dt + \mathcal{A}_1,
\end{aligned} \tag{3.80}
$$

where we use $|A_hv|^2 \leq C(|\tau_+v|^2 + |\tau_-v|^2)$.

Similarly, for $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$
D_h^2v = rD_h^2w - rD_h^2\rho A_h^2v - 2rA_hD_h\rho A_hD_hv - \mathcal{O}_\mu((sh)^2)D_h^2v,
$$

which implies

$$
\begin{aligned}
& \mathbb{E}\int_0^T\int_{\overline{\mathcal{M}}\cap G_2} s^3\mu^4\varphi^3|D_h^2v|^2dt \\
& \leq C\mathbb{E}\int_0^T\int_{M\cap G_1} s^7\mu^8\varphi^7r^2|w|^2dt + C\mathbb{E}\int_0^T\int_{M^*\cap G_1} s^5\mu^6\varphi^5r^2|D_hw|^2dt \\
& \quad + C\mathbb{E}\int_0^T\int_{\overline{\mathcal{M}}\cap G_1} s^3\mu^4\varphi^2r^2|D_h^2w|^2dt + \mathcal{A}_1.
\end{aligned} \tag{3.81}
$$

For $\lambda h(\delta T^2)^{-1} \le 1$, we also obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_2} s\mu^2\varphi|D_h^3 v|^2 dt \\
&\le C\mathbb{E}\int_0^T \int_{M\cap G_1} s^7\mu^8\varphi^7 r^2|w|^2 dt + C\mathbb{E}\int_0^T \int_{M^*\cap G_1} s^5\mu^6\varphi^5 r^2|D_h w|^2 dt \\
&\quad + C\mathbb{E}\int_0^T \int_{\overline{\mathcal{M}}\cap G_1} s^3\mu^4\varphi^2 r^2|D_h^2 w|^2 dt + C\mathbb{E}\int_0^T \int_{M^*\cap G_1} s\mu^2\varphi r^2|D_h^3 w|^2 dt + \mathcal{A}_1.
\end{aligned} \tag{3.82}$$

Noting that

$$|D_h v|^2 \le Ch^{-2}(|\tau_- v|^2 + |\tau_+ v|^2), \tag{3.83}$$

and

$$|D_h^2 v|^2 \le Ch^{-4}(|\tau_-^2 v|^2 + |\tau_+^2 v|^2 + |v|^2),$$

for all $\mu \ge 1$ and $\lambda h(\delta T^2)^{-1} \le 1$, we have

$$\mathcal{A}_3 \le C(\mu)h^{-4}\Big(\mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=T}\Big) \tag{3.84}$$

Combining (3.79)–(3.82) and (3.84), there exists a positive constant $\mu_1 \ge 1$ such that for all $\mu \ge \mu_1$, one can find positive constants $\varepsilon_2 \le \varepsilon_1$, $h_2 \le h_1'$ and $\lambda_2 \ge \lambda_1$, such that for $\lambda \ge \lambda_2(T+T^2)$, $h \le h_2$, and $\lambda h(\delta T^2)^{-1} \le \varepsilon_2$, we have

$$\begin{aligned}
&C\mathbb{E}\int_Q r^2|f|^2 dt + C(\mu)\mathbb{E}\int_Q s^4 r^2|g|^2 dt + C\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_1} s^7\mu^8\varphi^7 r^2|w|^2 dt \\
&+ C\mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_1} s^5\mu^6\varphi^5 r^2|D_h w|^2 dt + C\mathbb{E}\int_0^T \int_{\overline{\mathcal{M}}\cap G_1} s^3\mu^4\varphi^3 r^2|D_h^2 w|^2 dt \\
&+ C\mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_1} s\mu^2\varphi r^2|D_h^3 w|^2 dt + C(\mu)h^{-4}\Big(\mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=T}\Big) \\
&\ge \mathbb{E}\int_Q s^7\mu^8\varphi^7|w|^2 dt + \mathbb{E}\int_{Q^*} s^5\mu^6\varphi^5|D_h w|^2 dt + \mathbb{E}\int_{\overline{Q}} s^3\mu^4\varphi^3|D_h^2 w|^2 dt + \mathbb{E}\int_{Q^*} s\mu^2\varphi|D_h^3 w|^2 dt \\
&\quad + \mathbb{E}\int_{\overline{Q}} r^2|D_h^2 g|^2 dt.
\end{aligned} \tag{3.85}$$

Step 5. In this step, we estimate the local terms on $D_h w$, $D_h^2 w$ and $D_h^3 w$.

Choose a function $\xi \in C_0^\infty(G_0;[0,1])$ such that $\xi = 1$ in G_1. Thanks to Itô's formula and (1.9), we have

$$d(s^3\varphi^3\xi^4 r^2 w^2) = \varphi^3\xi^4\partial_t(s^3 r^2)w^2 dt + 2s^3\varphi^3\xi^4 r^2 w dw + s^3\varphi^3\xi^4 r^2 (dw)^2.$$

Hence, we get

$$\mathbb{E}\int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 w^2\Big|_{t=0} - \mathbb{E}\int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 w^2\Big|_{t=T}$$

$$= \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} \varphi^3\xi^4\partial_t(s^3r^2)w^2 dt + 2\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 w(D_h^4 w - f)dt$$

$$+ \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 |g|^2. \tag{3.86}$$

For all $\mu \geq 1$ and $\lambda h(\delta T^2)^{-1} \leq 1$, we have

$$\mathbb{E}\int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 w^2\Big|_{t=0} - \mathbb{E}\int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 w^2\Big|_{t=T} \leq Ch^{-3}\Big(\mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=T}\Big). \tag{3.87}$$

For $\lambda \geq T + T^2$, we obtain

$$\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} \varphi^3\xi^4\partial_t(s^3r^2)w^2 dt \leq \mathbb{E}\int_Q s^6\mathcal{O}_\mu(1) r^2 w^2 dt. \tag{3.88}$$

Thanks to Cauchy-Schwarz inequality, it holds that

$$\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 w f dt \leq \mathbb{E}\int_Q s^6\mathcal{O}_\mu(1) r^2 w^2 dt + \mathbb{E}\int_Q r^2 f^2 dt. \tag{3.89}$$

From (2.1) and (2.11), we have

$$\begin{aligned}
&2\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2 w D_h^4 w dt\\
&= 2\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3 D_h^2(\varphi^3\xi^4 r^2 w) D_h^2 w dt\\
&= 2\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3[D_h^2(\varphi^3\xi^4 r^2)A_h^2 w + 2A_h D_h(\varphi^3\xi^4 r^2)A_h D_h w + A_h^2(\varphi^3\xi^4 r^2)D_h^2 w]D_h^2 w dt.
\end{aligned} \tag{3.90}$$

Thanks to Proposition 2.5 and (2.5) and (2.10) and Cauchy-Schwarz inequality, for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \leq 1$, it holds that

$$\begin{aligned}
&2\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3 D_h^2(\varphi^3\xi^4 r^2)A_h^2 w D_h^2 w dt\\
&\leq \varepsilon\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2|D_h^2 w|^2 dt + C(\varepsilon)\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^7\mu^4\varphi^7\xi^4 r^2|w|^2 dt\\
&\quad + \mathbb{E}\int_Q [s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2)]r^2|D_h^2 w|^2 dt + \mathbb{E}\int_Q [s^6\mathcal{O}_\mu(1) + s^7\mathcal{O}_\mu((sh)^2)]r^2|w|^2 dt,
\end{aligned} \tag{3.91}$$

and

$$\begin{aligned}
&4\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3 A_h D_h(\varphi^3\xi^4 r^2)A_h D_h w D_h^2 w dt\\
&\leq \varepsilon\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2|D_h^2 w|^2 dt + C(\varepsilon)\mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_0} s^5\mu^2\varphi^5\xi^4 r^2|D_h w|^2 dt\\
&\quad + \mathbb{E}\int_Q [s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2)]r^2|D_h^2 w|^2 dt + \mathbb{E}\int_{Q^*} [s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)]r^2|D_h w|^2 dt.
\end{aligned} \tag{3.92}$$

Similarly, for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \le 1$, we get

$$\begin{aligned}
&2\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3 A_h^2(\varphi^3\xi^4 r^2)|D_h^2 w|^2 dt \\
&\ge 2\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\varphi^3\xi^4 r^2|D_h^2 w|^2 dt - \mathbb{E}\int_0^T \int_Q s^3\mathcal{O}_\mu((sh)^2)r^2|D_h^2 w|^2 dt. \qquad (3.93)
\end{aligned}$$

Combining (3.86)–(3.93), letting ε be sufficiently small, for $\mu \ge 1$, $\lambda \ge T + T^2$, $h \le 1$ and $\lambda h(\delta T^2)^{-1} \le 1$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T \int_{\overline{\mathcal{M}}\cap G_1} s^3\mu^4\varphi^3 r^2|D_h^2 w|^2 dt \\
&\le \mathbb{E}\int_0^T \int_{\overline{\mathcal{M}}\cap G_0} s^3\mu^4\varphi^3\xi^4 r^2|D_h^2 w|^2 dt \\
&\le C\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^7\mu^8\varphi^7 r^2|w|^2 dt + C\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^5\mu^6\varphi^5\xi^4 r^2|D_h w|^2 dt \\
&\quad + \mathbb{E}\int_Q [s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2)]r^2|D_h^2 w|^2 dt + \mathbb{E}\int_Q [s^6\mathcal{O}_\mu(1) + s^7\mathcal{O}_\mu((sh)^2)]r^2|w|^2 dt \\
&\quad + \mathbb{E}\int_{Q^*} [s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)]r^2|D_h w|^2 dt + C(\mu)\mathbb{E}\int_Q s^4 r^2|g|^2 \\
&\quad + C(\mu)h^{-4}\Big(\mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=T}\Big). \qquad (3.94)
\end{aligned}$$

From (2.1) and (2.9) and Proposition 2.5, thanks to Cauchy-Schwarz inequality, for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \le 1$, we have

$$\begin{aligned}
&\mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_0} s^5\mu^6\varphi^5\xi^4 r^2|D_h w|^2 dt \\
&= -\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^5 D_h(\mu^6\varphi^5\xi^4 r^2 D_h w) w dt \\
&= -\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^5 D_h(\mu^6\varphi^5\xi^4 r^2) A_h D_h w w dt - \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^5 A_h(\mu^6\varphi^5\xi^4 r^2) D_h^2 w w dt \\
&\le \varepsilon\mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_0} s^5\mu^6\varphi^5\xi^4 r^2|D_h w|^2 dt + \varepsilon\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^3\mu^4\varphi^3\xi^4 r^2|D_h^2 w|^2 dt \\
&\quad + C(\varepsilon)\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^7\mu^8\varphi^7 r^2|w|^2 dt + \mathbb{E}\int_{Q^*} [s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)]r^2|D_h w|^2 dt \\
&\quad + \mathbb{E}\int_Q [s^6\mathcal{O}_\mu(1) + s^7\mathcal{O}_\mu((sh)^2)]r^2|w|^2 dt. \qquad (3.95)
\end{aligned}$$

From (3.94) and (3.95), letting ε be sufficiently small, for $\mu \ge 1$, $\lambda \ge T+T^2$, $h \le 1$ and $\lambda h(\delta T^2)^{-1} \le 1$, we obtain

$$\mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_1} s^5\mu^6\varphi^5 r^2|D_h w|^2 dt$$

$$
\begin{aligned}
&\le \mathbb{E}\int_0^T \int_{\mathcal{M}^*\cap G_0} s^5\mu^6\varphi^5\xi^4 r^2 |D_h w|^2 dt \\
&\le C\mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} s^7\mu^8\varphi^7 r^2|w|^2 dt + \mathbb{E}\int_{Q^*} [s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)] r^2 |D_h w|^2 dt \\
&\quad + \mathbb{E}\int_Q [s^6\mathcal{O}_\mu(1) + s^7\mathcal{O}_\mu((sh)^2)] r^2|w|^2 dt + \mathbb{E}\int_Q [s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2)] r^2 |D_h^2 w|^2 dt \\
&\quad + C(\mu)\mathbb{E}\int_Q s^4 r^2 |g|^2 + C(\mu) h^{-4}\Big(\mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=T}\Big).
\end{aligned}
\tag{3.96}
$$

Thanks to Itô's formula and (1.9), we have

$$
d(s\varphi\xi^4 r^2|D_h w|^2) = \varphi\xi^4 \partial_t(sr^2)|D_h w|^2 dt + 2s\varphi\xi^4 r^2 D_h w d(D_h w) + s\varphi\xi^4 r^2 |d(D_h w)|^2.
$$

Hence, we get

$$
\begin{aligned}
&\mathbb{E}\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2|D_h w|^2\Big|_{t=0} - \mathbb{E}\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2|D_h w|^2\Big|_{t=T} \\
&= \mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} \varphi\xi^4\partial_t(sr^2)|D_h w|^2 dt + 2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2 D_h w(D_h^5 w - D_h f) dt \\
&\quad + \mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2 |D_h g|^2.
\end{aligned}
\tag{3.97}
$$

From (3.83), for all $\mu \ge 1$ and $\lambda h(\delta T^2)^{-1} \le 1$, we have

$$
\begin{aligned}
&\mathbb{E}\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2|D_h w|^2\Big|_{t=0} - \mathbb{E}\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2|D_h w|^2\Big|_{t=T} \\
&\le Ch^{-3}\Big(\mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=T}\Big).
\end{aligned}
\tag{3.98}
$$

For $\lambda \ge T + T^2$, we obtain

$$
\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} \varphi\xi^4\partial_t(sr^2)|D_h w|^2 dt \le \mathbb{E}\int_Q s^4\mathcal{O}_\mu(1) r^2|D_h w|^2 dt. \tag{3.99}
$$

Similar to (3.26), for $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \le 1$, we have

$$
\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2|D_h g|^2 \le \varepsilon\mathbb{E}\int_{\overline{Q}} r^2|D_h^2 g|^2 dt + C(\varepsilon,\mu)\mathbb{E}\int_Q s^3 r^2 |g|^2 dt. \tag{3.100}
$$

From (2.1), (2.5), (2.9) and (2.10) and Proposition 2.5, thanks to Cauchy-Schwarz inequality, for $\lambda h(\delta T^2)^{-1} \le 1$, it holds that

$$
\begin{aligned}
&-2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2 D_h w D_h f dt \\
&= \mathbb{E}\int_0^T\int_{\mathcal{M}\cap G_0} D_h(s\varphi\xi^4 r^2 D_h w) f dt \\
&\le \mathbb{E}\int_Q s^2\mathcal{O}_\mu(1) r^2|D_h^2 w|^2 dt + \mathbb{E}\int_{Q^*} s^4\mathcal{O}_\mu(1) r^2 |D_h w|^2 dt + \mathbb{E}\int_Q r^2 f^2 dt.
\end{aligned}
\tag{3.101}
$$

From (2.1) and (2.11), we have

$$
\begin{aligned}
&2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2 D_h w D_h^5 w dt\\
&= 2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} D_h^2(s\varphi\xi^4 r^2)A_h^2 D_h w D_h^3 w dt + 4\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} A_h D_h(s\varphi\xi^4 r^2)A_h D_h^2 w D_h^3 w dt\\
&\quad + 2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} A_h^2(s\varphi\xi^4 r^2)|D_h^3 w|^2 dt. \qquad (3.102)
\end{aligned}
$$

Similar to (3.91), for all $\varepsilon > 0$ and $\lambda h(\delta T^2)^{-1} \le 1$, it holds that

$$
\begin{aligned}
&2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} D_h^2(s\varphi\xi^4 r^2)A_h^2 D_h w D_h^3 w dt + 4\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} A_h D_h(s\varphi\xi^4 r^2)A_h D_h^2 w D_h^3 w dt\\
&\le \varepsilon\mathbb{E}\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2|D_h^3 w|^2 dt + C(\varepsilon)\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s^5\mu^4\varphi^5\xi^4 r^2|D_h w|^2 dt\\
&\quad + C(\varepsilon)\mathbb{E}\int_0^T\int_{\mathcal{M}\cap G_0} s^3\mu^2\varphi^3\xi^4 r^2|D_h^2 w|^2 dt + \mathbb{E}\int_{Q^*}[\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2)]r^2|D_h^3 w|^2 dt\\
&\quad + \mathbb{E}\int_Q[s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2)]r^2|D_h^2 w|^2 dt + \mathbb{E}\int_{Q^*}[s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)]r^2|D_h w|^2 dt,
\end{aligned}
\qquad (3.103)
$$

and

$$
\begin{aligned}
&2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} A_h^2(s\varphi\xi^4 r^2)|D_h^3 w|^2 dt\\
&\ge 2\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s\varphi\xi^4 r^2|D_h^3 w|^2 dt - \mathbb{E}\int_{Q^*} s\mathcal{O}_\mu((sh)^2)r^2|D_h^3 w|^2 dt. \qquad (3.104)
\end{aligned}
$$

Combining (3.94)–(3.104), letting ε be sufficiently small, for $\mu \ge 1$, $\lambda \ge T + T^2$, $h \le 1$ and $\lambda h(\delta T^2)^{-1} \le 1$, we obtain

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_1} s\mu^2\varphi r^2|D_h^3 w|^2 dt\\
&\le \mathbb{E}\int_0^T\int_{\mathcal{M}^*\cap G_0} s\mu^2\varphi\xi^4 r^2|D_h^3 w|^2 dt\\
&\le C\mathbb{E}\int_0^T\int_{\mathcal{M}\cap G_0} s^7\mu^8\varphi^7 r^2|w|^2 dt + \mathbb{E}\int_{Q^*}[s^4\mathcal{O}_\mu(1) + s^5\mathcal{O}_\mu((sh)^2)]r^2|D_h w|^2 dt\\
&\quad + \mathbb{E}\int_Q[s^6\mathcal{O}_\mu(1) + s^7\mathcal{O}_\mu((sh)^2)]r^2|w|^2 dt + \mathbb{E}\int_Q[s^2\mathcal{O}_\mu(1) + s^3\mathcal{O}_\mu((sh)^2)]r^2|D_h^2 w|^2 dt\\
&\quad + \mathbb{E}\int_{Q^*}[\mathcal{O}_\mu(1) + s\mathcal{O}_\mu((sh)^2)]r^2|D_h^3 w|^2 dt + C(\mu)\mathbb{E}\int_Q s^4 r^2|g|^2\\
&\quad + C(\mu)h^{-4}\Big(\mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}} r^2|w|^2\Big|_{t=T}\Big). \qquad (3.105)
\end{aligned}
$$

Combining (3.85), (3.94)–(3.96) and (3.105), there exists a positive constant $\mu_0 \ge \mu_1$ such that for all $\mu \ge \mu_2$, one can find positive constants $\varepsilon_0 \le \varepsilon_2$, $h_0 \le h_2$ and $\lambda_0 \ge \lambda_2$, such that for

$\lambda \geq \lambda_0(T+T^2)$, $0<h\leq h_0$, and $\lambda h(\delta T^2)^{-1}\leq \varepsilon_0$, we obtain (1.8). $\square$

4 Proof of the observability inequality

This section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. From Theorem 1.3, choosing $\mu=\mu_0$, for $\lambda \geq \lambda_0(T+T^2)$, $0<h\leq h_0$ and $\lambda h\left(\delta T^2\right)^{-1}\leq \varepsilon_0$, we have

$$\mathbb{E}\int_Q s^7 e^{2s\phi}|z|^2dt \leq C\Big(\mathbb{E}\int_0^T\int_{G_0\cap\mathcal{M}} s^7e^{2s\phi}|z|^2dt+\mathbb{E}\int_Q e^{2s\phi}|a_1z+a_2Z|^2dt+\mathbb{E}\int_Q s^4e^{2s\phi}|Z|^2dt$$
$$+h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|z|^2\Big|_{t=0}+h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|z|^2\Big|_{t=T}\Big),$$

From this, there exists $\lambda_1\geq\lambda_0$ such that for a fixed

$$\lambda=\lambda_1(T+T^2+\mathcal{H}^{2/7}T^2) \tag{4.1}$$

we obtain

$$\mathbb{E}\int_Q \theta^7 e^{2s\phi}|z|^2dt \leq C\Big(\mathbb{E}\int_0^T\int_{G_0\cap\mathcal{M}} \theta^7e^{2s\phi}|z|^2dt+\mathbb{E}\int_Q \theta^7e^{2s\phi}|Z|^2dt+h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|z|^2\Big|_{t=0}$$
$$+h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|z|^2\Big|_{t=T}\Big)\,. \tag{4.2}$$

Noting that (1.7), we have $\theta(t)\geq\theta(T/2)\geq T^{-2}$ for $t\in[0,T]$ and $\theta(t)\leq\theta(T/4)\leq\frac{16}{3T^2}$ for $t\in[T/4,3T/4]$. Then

$$\mathbb{E}\int_Q \theta^7e^{2s\phi}|z|^2dt\geq \mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_{\mathcal{M}}\theta^7e^{2s\phi}|z|^2dt\geq Ce^{-C\lambda T^{-2}}\mathbb{E}\int_{\frac{T}{4}}^{\frac{3T}{4}}\int_{\mathcal{M}}|z|^2dt. \tag{4.3}$$

From (1.5) and (2.11), thanks to Cauchy-Schwarz inequality, for any $0\leq t_1\leq t_2\leq T$, we have

$$\begin{aligned}\mathbb{E}\int_{\mathcal{M}}|z(t_2)|^2-\mathbb{E}\int_{\mathcal{M}}|z(t_1)|^2&=\mathbb{E}\int_{t_1}^{t_2}\int_{\mathcal{M}}d(z^2)=\mathbb{E}\int_{t_1}^{t_2}\int_{\mathcal{M}}[2zdz+(dz)^2]\\&=\mathbb{E}\int_{t_1}^{t_2}\int_{\mathcal{M}}[2z(D_h^4z-a_1z-a_2Z)+Z^2]dt\\&=\mathbb{E}\int_{t_1}^{t_2}\int_{\mathcal{M}}[2|D_h^2z|^2-2a_1z^2-a_2zZ+Z^2]dt\\&\geq -C(1+\mathcal{H}^2)\mathbb{E}\int_{t_1}^{t_2}\int_{\mathcal{M}}|z|^2dt.\end{aligned}$$

Thus, in terms of Gronwall inequality, for $0\leq t_1\leq t_2\leq T$, we have

$$\mathbb{E}\int_{\mathcal{M}}|z(t_1)|^2dt\leq Ce^{CT(1+\mathcal{H}^2)}\mathbb{E}\int_{\mathcal{M}}|z(t_2)|^2dt. \tag{4.4}$$

From it, since $\theta(T) = \theta(0) \geq \frac{1}{2}(\delta T^2)^{-1}$, we obtain

$$
\begin{aligned}
&h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|z|^2\Big|_{t=0} + h^{-4}\mathbb{E}\int_{\mathcal{M}} e^{2s\phi}|z|^2\Big|_{t=T} \\
&\leq Ch^{-4}e^{-C\lambda(\delta T^2)^{-1}}\left[\mathbb{E}\int_{\mathcal{M}}|z|^2\Big|_{t=0} + \mathbb{E}\int_{\mathcal{M}}|z|^2\Big|_{t=T}\right] \\
&\leq Ch^{-4}e^{-C\lambda(\delta T^2)^{-1}+CT(1+\mathcal{H}^2)}\mathbb{E}\int_{\mathcal{M}}|z|^2\Big|_{t=T}.
\end{aligned}
\tag{4.5}
$$

We have

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_{G_0\cap\mathcal{M}}\theta^7 e^{2s\phi}|z|^2dt + \mathbb{E}\int_Q \theta^7 e^{2s\phi}|Z|^2dt \\
&\leq Ce^{-C\lambda T^{-2}}\Big(\mathbb{E}\int_0^T\int_{G_0\cap\mathcal{M}}|z|^2dt + \mathbb{E}\int_Q|Z|^2dt\Big).
\end{aligned}
\tag{4.6}
$$

Combining (4.2)–(4.6), for $0 < h \leq h_0$ and $\lambda h\left(\delta T^2\right)^{-1} \leq \varepsilon_0$, it holds that

$$
\mathbb{E}\int_{\mathcal{M}}|z(0)|^2 \leq Ce^{C\lambda T^{-2}+CT(1+\mathcal{H}^2)}\Big(\mathbb{E}\int_0^T\int_{G_0\cap\mathcal{M}}|z|^2dt + \mathbb{E}\int_Q|Z|^2dt + h^{-4}e^{-C\lambda(\delta T^2)^{-1}}\int_{\mathcal{M}}|z|^2\Big|_{t=T}\Big)
\tag{4.7}
$$

Since we required $\lambda h\left(\delta T^2\right)^{-1} \leq \varepsilon_0$ and (4.1), define

$$
h_1 = \frac{\varepsilon_0 T^2}{\lambda_1(T + T^2 + \mathcal{H}^{2/7}T^2)}.
$$

Then, for every $h \leq \min\{h_0, h_1\}$, from (4.1) and (4.7), we obtain

$$
\mathbb{E}\int_{\mathcal{M}}|z(0)|^2 \leq e^{C(1+T^{-1}+\mathcal{H}^{2/7}+T+\mathcal{H}^2T)}\Big(\mathbb{E}\int_0^T\int_{G_0\cap\mathcal{M}}|z|^2dt + \mathbb{E}\int_Q|Z|^2dt + e^{-\frac{C}{h}}\int_{\mathcal{M}}|z|^2\Big|_{t=T}\Big).
$$

□

5 Proof of the controllability result

Proof of Theorem 1.1. For any $y_0 \in L^2_h(\mathcal{M})$, define

$$
J_\varepsilon(z_T) = \frac{1}{2}\mathbb{E}\int_Q|Z|^2dt + \frac{1}{2}\mathbb{E}\int_0^T\int_{\mathcal{M}\cap G_0}|z|^2dt + \frac{\varepsilon}{2}\mathbb{E}|z_T|^2_{L^2_h(\mathcal{M})} - \langle y_0, z(0)\rangle_{L^2_h(\mathcal{M})},
$$

where $\varepsilon = e^{-\frac{C}{h}}$ and (z, Z) solves (1.5) with final datum z_T. It is clearly that $J_\varepsilon(\cdot)$ is continuous and convex. From Theorem 1.2 and Young's inequality, for $h \leq \min\{h_0, h_1\}$, we obtain

$$
\begin{aligned}
&J_\varepsilon(z_T) \\
&\geq \frac{1}{2}\mathbb{E}\int_Q|Z|^2dt + \frac{1}{2}\mathbb{E}\int_0^T\int_{\mathcal{M}\cap G_0}|z|^2dt + \frac{\varepsilon}{2}\mathbb{E}|z_T|^2_{L^2_h(\mathcal{M})} - \frac{1}{4C_{obs}}\mathbb{E}|z(0)|^2_{L^2_h(\mathcal{M})} - C_{obs}|y_0|^2_{L^2_h(\mathcal{M})}
\end{aligned}
$$

$$\geq \frac{\varepsilon}{4}\mathbb{E}|z_T|^2_{L^2_h(\mathcal{M})} - C_{obs}|y_0|^2_{L^2_h(\mathcal{M})},$$

which implies that $J_\varepsilon(\cdot)$ is coercive. Hence, $J_\varepsilon(\cdot)$ admits a unique minimizer z^*_T. Denote (z^*, Z^*) the solution to (1.5) with final datum z^*_T. Then we obtain

$$\mathbb{E}\int_Q ZZ^* dt + \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} zz^* dt + \varepsilon\mathbb{E}\langle z_T, z^*_T\rangle_{L^2_h(\mathcal{M})} - \langle y_0, z(0)\rangle_{L^2_h(\mathcal{M})} = 0, \tag{5.1}$$

for any z_T with the associated solution (z, T) of (1.5). Choose $u = -z^*$ and $v = -Z^*$, by the duality of (1.3) with (1.5), we obtain

$$\mathbb{E}\langle y(T), z_T\rangle_{L^2_h(\mathcal{M})} - \langle y_0, z(0)\rangle_{L^2_h(\mathcal{M})} = \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} uz dt + \mathbb{E}\int_Q vZ dt.$$

Then $y(T) = \varepsilon z^*_T$. Noting that (5.1), we have

$$\mathbb{E}\int_Q |Z^*|^2 dt + \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} |z^*|^2 dt + \varepsilon\mathbb{E}|z^*_T|^2_{L^2_h(\mathcal{M})} \leq C_{obs}|y_0|^2_{L^2_h(\mathcal{M})} + \frac{1}{4C_{obs}}|z^*(0)|^2_{L^2_h(\mathcal{M})}. \tag{5.2}$$

From Theorem 1.2, for $h \leq \min\{h_0, h_1\}$, we obtain

$$|z^*(0)|^2_{L^2_h(\mathcal{M})} \leq C_{obs}\Big(\mathbb{E}\int_Q |Z^*|^2 dt + \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} |z^*|^2 dt + \varepsilon\mathbb{E}|z^*_T|^2_{L^2_h(\mathcal{M})}\Big) \tag{5.3}$$

Combining (5.2) and (5.3), for $h \leq \min\{h_0, h_1\}$, we have

$$\mathbb{E}\int_{\mathcal{M}} |y(T)|^2 = \varepsilon^2\mathbb{E}|z^*_T|^2_{L^2_h(\mathcal{M})} \leq C_{obs}e^{-\frac{C}{h}}\int_{\mathcal{M}} |y_0|^2,$$

and

$$\mathbb{E}\int_Q |v|^2 dt + \mathbb{E}\int_0^T \int_{\omega\cap\mathcal{M}} |u|^2 dt = \mathbb{E}\int_Q |Z^*|^2 dt + \mathbb{E}\int_0^T \int_{\mathcal{M}\cap G_0} |z^*|^2 dt \leq C_{obs}\int_{\mathcal{M}} |y_0|^2,$$

which implies that (u, v) satisfy our need. □

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