# The Ising Model Coupled to 2D Gravity: Higher-order Painlevé Equations/The (3,4) String Equation 

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#### Abstract

In continuation of the work [DHL23b], we study a higher-order Painlevé-type equation, arising as a string equation of the $3^{\text {rd }}$ order reduction of the KP hierarchy. This equation appears at the multi-critical point of the 2-matrix model with quartic interactions, and describes the Ising phase transition coupled to 2D gravity, cf. [CGM90; Dou90], and the forthcoming [DHL23b; DHL23a]. We characterize this equation in terms of the isomonodromic deformations of a particular rational connection on $\mathbb{P}^{1}$. We also identify the (nonautonomous) Hamiltonian structure associated to this equation, and write a suitable $\tau$-differential for this system. This $\tau$-differential can be extended to the canonical coordinates of the associated Hamiltonian system, allowing us to verify Conjectures 1. and 2. of [IP18]. We also present a fairly general formula for the $\tau$-differential of a special class of resonant connections, which is somewhat simpler than that of [BM05].


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## 1. Introduction.

In this work, we mainly study the following pair of equations for two functions $U=U\left(t_{5}, t_{2}, x\right), V=$ $V\left(t_{5}, t_{2}, x\right)$ :

$$
\left\{\begin{array}{l}
0=\frac{1}{2} V^{\prime \prime}-\frac{3}{2} U V+\frac{5}{2} t_{5} V+t_{2}  \tag{1.1}\\
0=\frac{1}{12} U^{(4)}-\frac{3}{4} U^{\prime \prime} U-\frac{3}{8}\left(U^{\prime}\right)^{2}+\frac{3}{2} V^{2}+\frac{1}{2} U^{3}-\frac{5}{12} t_{5}\left(3 U^{2}-U^{\prime \prime}\right)+x
\end{array}\right.
$$

Here (and throughout the present work), ${ }^{\prime}=\frac{\partial}{\partial x}$. We will also sometimes instead write $x:=t_{1}$, as this notation is more convenient in certain instances. We have tried to keep our notations for this equation close to those of [FGZ95]. The above is known as the $(3,4)$ string equation, and appears in the study of the Ising model coupled to 2D gravity, as we shall now make apparent.

This is a continuation of the recent work [DHL23b], in which we set up a Riemann-Hilbert analysis of the 2-matrix model with quartic interactions, corresponding to the Ising model on random quadrangulations. In [DHL23b], we replicated the results of [Kaz86; BK86] for the genus-zero partition function, thus providing a fully rigorous proof of their formula. The next task in our program is to investigate the multi-critical point of this model, which corresponds to the Ising $(3,4)$ minimal model of conformal field theory coupled to $2 D$-gravity. At the level of the steepest descent analysis, this amounts to finding the "right" model RiemannHilbert problem at the turning points, for which the matching condition is satisfied. Such a parametrix is presently absent from the literature; the current work aims to fill this gap.

In finding such a parametrix, we are not completely in the dark; as usual, physicists have already provided us the foundations. An equation characterizing this critical point was first derived in [Bré +90 ; CGM90; Dou90], and recognized to be a string equation to a $3^{r d}$ order reduction of the KP hierarchy. This equation is precisely (1.1). More generally, it is conjectured that all critical points of the 2-matrix model are characterized by the so-called ( $q, p$ )-string equations (see the discussion in $\S 2$ ), which arise as symmetry constraints of the KP hierarchy. We will not make any general statements about these string equations here. We continue with a more detailed description of the connection of Equation (1.1) with the 2-matrix model.

### 1.1. Connection to the 2-matrix model.

As previously mentioned, the above equation arises when studying the triple scaling limit of the 2-matrix model with quartic interactions. The partition function for this model is

$$
\begin{equation*}
Z_{n}(\tau, t, H ; N):=\iint e^{N \operatorname{tr}\left[\tau X Y-\frac{1}{2} X^{2}-\frac{e^{H_{t}}}{4} X^{4}-\frac{1}{2} Y^{2}-\frac{e^{-H}}{4} Y^{4}\right]} d X d Y \tag{1.2}
\end{equation*}
$$

where the integration here is carried out over the Cartesian product of the space of $n \times n$ Hermitian matrices with itself. This matrix model can be identified with the Ising model on random quadrangulations [Kaz86; BK86]. The multicritical point of this model, which characterizes the Ising spin-ordering transition coupled to gravity, occurs at

$$
\begin{equation*}
t=t_{c}=-\frac{5}{72}, \quad \quad \tau=\tau_{c}=\frac{1}{4}, \quad H=H_{c}=0 \tag{1.3}
\end{equation*}
$$

Evidently, since $t_{c}<0$, the matrix integral (1.2) is non-convergent. We must therefore make an appropriate analytic continuation of this integral in order to make sense of the multicritical point. This construction is demonstrated in [DHL23b]: here, by a slight abuse of notation, we shall denote both the partition function
and its analytic continuation by $Z_{n}(\tau, t, H ; N)$. In [CGM90; Dou90] (cf. the earlier work [Bré +90 ] for the model without the external field or temperature parameters), the following triple scaling limit is introduced:

$$
\begin{equation*}
\frac{n}{N}=\xi, \quad t-t_{c} \sim N^{-6 / 7}, \quad t \xi=t_{c}\left(1-N^{-6 / 7} x\right), \quad H=N^{-5 / 7} t_{2}, \quad \tau=\tau_{c}\left(1-N^{-2 / 7} t_{5}\right) \tag{1.4}
\end{equation*}
$$

After scaling, one finds that the partition function converges to (see the works [Gin $+90 ;$ FGZ95]), as $n \rightarrow \infty$,

$$
\begin{equation*}
C^{2} \frac{d^{2}}{d x^{2}} \log Z_{n}(\tau, t, H ; N) \rightarrow-U\left(t_{5}, t_{2}, x\right) \tag{1.5}
\end{equation*}
$$

for some constant $C^{2}>0$. Here, $U\left(t_{5}, t_{2}, x\right)$ is a solution to the string equation (1.1). This suggests that the multicritical partition function for this model is in fact a $\tau$-function of equation (1.1). It is the purpose of Part III of this series of works [DHL23a] to make rigorous sense of this scaling limit. In this work, we study the limiting object, i.e. the equation that results after performing this scaling limit.

One of the shortcomings of the work in the physics literature is that one is only able to identify that the multicritical partition function solves a particular integrable equation; there is no indication from this analysis which solution one has convergence to, or what properties the resulting solution has. An important consequence of our analysis is that one can identify the particular solution of (1.1) arising from the triple scaling limit of the 2-matrix model, and, since we furnish a Riemann-Hilbert formulation of the equation, this solution is amenable to asymptotic analysis.

### 1.2. Outline and Statement of Results.

The remainder of this work is organized as follows. We review the KP approach to this equation in §2, and recast the problem in matrices, which will give us the right framework to develop a Riemann-Hilbert formulation. The a version of the follow result is proven in $\S 4$.

Proposition 1.1. The string equation (1.1), and its compatibility with the (reduced) KP-flows $Q_{+}^{2 / 3}, Q_{+}^{5 / 3}$, is equivalent to the isomonodromy deformations with respect to $t_{5}, t_{2}, x$ of the following linear differential equation for a function $\Psi=\Psi\left(\lambda ; t_{5}, t_{2}, x\right)$ :

$$
\begin{equation*}
\frac{\partial \Psi}{\partial \lambda}=L\left(\lambda ; t_{5}, t_{2}, x\right) \Psi \tag{1.6}
\end{equation*}
$$

where

$$
\begin{align*}
& L\left(\lambda ; t_{5}, t_{2}, x\right)=\left(\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \lambda^{2}+\left(\begin{array}{ccc}
0 & 2 t_{5}+\frac{1}{4} Q_{U} & -Q_{V} \\
1 & 0 & 2 t_{5}+\frac{1}{4} Q_{U} \\
0 & 1 & 0
\end{array}\right) \lambda  \tag{1.7}\\
& +\left(\begin{array}{ccc}
\frac{1}{8} Q_{U}^{2}-P_{W}+\frac{1}{2} P_{V}-\frac{1}{4} t_{5} Q_{U}-\frac{1}{6} t_{5}^{2} & L_{12} & L_{13} \\
\frac{1}{2} Q_{V}-\frac{1}{4} Q_{W} & 2 P_{W}-\frac{1}{4} Q_{U}^{2}+\frac{1}{2} t_{5} Q_{U}+\frac{1}{3} t_{5}^{2} & L_{23} \\
t_{5}-\frac{1}{2} Q_{U} & \frac{1}{2} Q_{V}+\frac{1}{4} Q_{W} & \frac{1}{8} Q_{U}^{2}-P_{W}-\frac{1}{2} P_{V}-\frac{1}{4} t_{5} Q_{U}-\frac{1}{6} t_{5}^{2}
\end{array}\right),
\end{align*}
$$

where

$$
\begin{aligned}
L_{12} & :=\frac{5}{16} Q_{U} Q_{W}-P_{U}+\frac{1}{4} t_{5} Q_{W}-\frac{3}{8} Q_{U} Q_{V}-\frac{1}{2} t_{5} Q_{V}+t_{2}, \\
L_{13} & :=\frac{1}{16} Q_{W}^{2}+\frac{7}{32} Q_{U}^{3}+\frac{3}{4} Q_{V}^{2}-\frac{3}{2} P_{W} Q_{U}+\frac{5}{16} t_{5} Q_{U}^{2}-2 t_{5} P_{W}+\frac{1}{4} t_{5}^{2} Q_{U}+x+\frac{8}{27} t_{5}^{3}, \\
L_{23} & :=-\frac{5}{16} Q_{U} Q_{W}+P_{U}-\frac{1}{4} t_{5} Q_{W}-\frac{3}{8} Q_{U} Q_{V}-\frac{1}{2} t_{5} Q_{V}+t_{2} .
\end{aligned}
$$

From our setup, and some inspiration of where to look based on the classical Painlevé transcendents, we can write the string equation as a $3+3$ dimensional Hamiltonian system, in which the coordinates $\left(Q_{U}, Q_{V}, Q_{W} ; P_{U}, P_{V}, P_{W}\right)$ are canonical. The induced flows along the $t_{5}, t_{2}$, and $x:=t_{1}$ directions are
generated by (nonautonomous) Hamiltonians $H_{5}, H_{2}, H_{1}$, which pairwise commute with respect to the following Poisson bracket: if $f, g$ are functions of the variables $\left(Q_{U}, Q_{V}, Q_{W} ; P_{U}, P_{V}, P_{W}\right)$, we define

$$
\begin{equation*}
\{f, g\}:=\sum_{a \in\{U, V, W\}}\left(\frac{\partial f}{\partial Q_{a}} \frac{\partial g}{\partial P_{a}}-\frac{\partial f}{\partial P_{a}} \frac{\partial g}{\partial Q_{a}}\right) \tag{1.8}
\end{equation*}
$$

This is the essence of our next Proposition, which we will prove a version of in $\S 3$ :
Theorem 1.1. Let $\left(Q_{U}, Q_{V}, Q_{W} ; P_{U}, P_{V}, P_{W}\right)$ parameterize the solutions of the isomonodromy deformations of the connection (1.7). Then, there exist functions $H_{5}, H_{2}, H_{1}$, polynomially dependent on $\left(Q_{U}, Q_{V}, Q_{W} ; P_{U}, P_{V}, P_{W}\right)$ and $t_{5}, t_{2}, t_{1}$, such that

$$
\begin{equation*}
\frac{\partial Q_{a}}{\partial t_{k}}=\frac{\partial H_{k}}{\partial P_{a}}, \quad \frac{\partial P_{a}}{\partial t_{k}}=-\frac{\partial H_{k}}{\partial Q_{a}} \tag{1.9}
\end{equation*}
$$

where $a \in\{U, V, W\}, k=1,2,5$, and $x:=t_{1}$. We call $H_{k}$ Hamiltonians. These Hamiltonians commute with respect to the Poisson bracket (1.8):

$$
\begin{equation*}
\left\{H_{k}, H_{j}\right\}+\frac{\partial H_{k}}{\partial t_{j}}-\frac{\partial H_{j}}{\partial t_{k}}=0 \tag{1.10}
\end{equation*}
$$

where $k, j=1,2,5$.
We then set about defining a $\tau$-function for this system; as it will turn out, the differential equation (1.6) shares the same problem as the equivalent problem for the linear system associated to Painlevé I (PI): either the leading coefficient of the pole of $L$ is not diagonalizable, or (as we shall see) a transformed version of it does have diagonalizable leading coefficient at infinity, but carries a resonant Fuchsian singularity at the origin. Thus, the standard definition of the $\tau$-differential as given in [JMU81] does not apply. If we try to ignore the contribution from the resonant singularity (as is done for PI, cf. [JM81; LR17; ILP18]), it turns out the $\tau$-differential is not closed. Thus, we must provide an alternate definition of the $\tau$-differential; this is established in Section 5.3. Although most of this work is dedicated to the study of the string equation (1.1), we were able to derive a fairly general formula for the $\tau$-differential of a linear differential equation with polynomial coefficients whose leading term is not diagonalizable. The motivation for the class of equations we study arises from the so-called $(p, q)$ string equations (see [Gin +90 ] for an overview). An alternative formula was derived by Bertola and Mo [BM05] in terms of spectral invariants; the formula we present is in terms of a residue in the local gauge, and thus may merit interest, as it gives an alternative way to compute the $\tau$-differential. We thus present our result as a theorem:

Theorem 1.2. Fix $q \geq 2$, and consider the differential equation

$$
\begin{equation*}
\frac{\partial \Psi}{\partial \lambda}=A(\lambda ; \mathbf{t}) \Psi \tag{1.11}
\end{equation*}
$$

where $A(\lambda ; \mathbf{t})$ is a $q \times q$ matrix, polynomial in $\lambda$, whose leading term is the nondiagonalizable matrix

$$
\begin{equation*}
A(\lambda ; \mathbf{t})=\Lambda^{r} \lambda^{k}+\cdots \tag{1.12}
\end{equation*}
$$

for some $0<r<q, k \geq 0$, where $\Lambda=\Lambda(\lambda)$ is

$$
\Lambda(\lambda):=\left(\begin{array}{cccccc}
0 & 0 & \cdots & 0 & 0 & \lambda \\
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0
\end{array}\right) .
$$

Then, there is a formal solution of the form

$$
\begin{equation*}
\Psi(\lambda ; \mathbf{t})=\underbrace{g(\lambda)\left[\mathbb{I}+\frac{\Psi_{1}(\mathbf{t})}{\lambda^{1 / q}}+\mathcal{O}\left(\lambda^{-2 / q}\right)\right]}_{G(\lambda ; \mathbf{t})} e^{\Theta(\lambda ; \mathbf{t})} \tag{1.13}
\end{equation*}
$$

where $\Theta(\lambda ; \mathbf{t})$ is a diagonal matrix, polynomial in $\lambda^{1 / q}$, and $g(\lambda)=\lambda^{\Delta_{q}} \mathcal{U}_{q}$, for some constant, diagonal, traceless matrix $\Delta_{q}$, and constant matrix $\mathcal{U}_{q}$. Let $t_{\ell}$ be the collection of isomonodromic times. If we define

$$
\begin{equation*}
\hat{\omega}_{J M U}:=\sum_{\ell}\left(\left\langle A(\lambda ; \mathbf{t}) \frac{d G}{d t_{\ell}} G^{-1}\right\rangle-\left\langle\frac{\Delta_{q}}{\lambda} \frac{d G}{d t_{\ell}} G^{-1}\right\rangle\right) d t_{\ell} \tag{1.14}
\end{equation*}
$$

then we have that

$$
\begin{equation*}
\mathbf{d} \hat{\omega}_{J M U}=0 \tag{1.15}
\end{equation*}
$$

We can then formally define a $\tau$-function by $\hat{\omega}_{J M U}=\mathbf{d} \log \tau(\mathbf{t})$. Using this definition in the case of (1.7), we obtain the following proposition:

Proposition 1.2. The (modified) JMU isomonodromic tau function for the isomonodromic system defined by (1.7) is given by

$$
\begin{equation*}
\mathbf{d} \log \tau\left(t_{5}, t_{2}, t_{1}\right)=\frac{3}{2}\left(H_{5} d t_{5}+H_{2} d t_{2}+H_{1} d t_{1}\right) \tag{1.16}
\end{equation*}
$$

where $H_{5}, H_{2}$, and $H_{1}$ are the Hamiltonians of the Theorem 1.1.
We shall see that this definition coincides (up to an overall multiplicative factor) with the $\tau$-function as defined by Okamoto [Oka81; Oka99], justifying our modification of the isomonodromic $\tau$-function.

A "dressed" version of the $\tau$-function as defined here will be what appears as the critical partition function for the quartic 2-matrix model; this will be the main result of the forthcoming work [DHL23a]. This work is the analogy of the analyses of Painlevé I [Oka81; Fok+06], which were subsequently used for the analysis of the critical points of the quartic and cubic 1-matrix models [DK06; BD16].

There has been much interest in recent years concerning the dependence of the isomonodromic $\tau$-function on the monodromy data (equivalently, on any set of initial conditions for the isomonodromy equations) [Ber10; ILP18; LR17; IP18], in particular due to its applications in determining the constant factors in the asymptotics of $\tau$-functions. Building on earlier works, in [IP18] the authors greatly simplify the procedure for calculating these constant factors for the 6 Painlevé equations. They proposed two conjectures to this end, which we give the full statement of in Section 5.3. In our situation, these conjectures are equivalent to the following proposition, which we prove in Section 5.3:
Proposition 1.3. The extended $\tau$-differential $\omega_{0}$ for the system defined by (1.7) is given by

$$
\begin{equation*}
\omega_{0}=\frac{3}{2} \omega_{c l a}+d G \tag{1.17}
\end{equation*}
$$

where $\omega_{c l a}$ is

$$
\begin{equation*}
\omega_{c l a}=\sum_{a \in\{U, V, W\}} P_{a} d Q_{a}-\sum_{k \in\{1,2,5\}} H_{k} d t_{k} \tag{1.18}
\end{equation*}
$$

and $G$ is the polynomial

$$
\begin{equation*}
G=\frac{3}{7}\left[3 t_{1} H_{1}+\frac{5}{2} t_{2} H_{2}+t_{5} H_{5}-P_{U} Q_{U}-\frac{3}{2} P_{V} Q_{V}-\frac{3}{2} P_{W} Q_{W}\right] \tag{1.19}
\end{equation*}
$$

This result is in agreement with the conjectures of [IP18], adding further validity to these statements. We give the full definition of $\omega_{0}$ in Section 5.3 as well. Furthermore, there are currently not many explicit examples of multivariate isomonodromic $\tau$-functions in the literature, and so the above provides another non-trivial example of an isomonodromic $\tau$-function arising from an integrable equation.

### 1.3. Notations.

Throughout this work, we will frequently make use of several notations without comment. We list some of these notations here, for the convenience of the reader.

- $\omega=e^{\frac{2 \pi i}{3}}$ denotes the principal third root of unity,
- $E_{i j}$ will denote the $3 \times 3$ matrix with a 1 in the $i j^{\text {th }}$ position, and zeros elsewhere,
- If $A$ is a square matrix, the notation $\lambda^{A}$ is defined to mean $\lambda^{A}:=\exp (A \log \lambda)$, where "exp" here is the usual matrix exponential.
- Throughout, we make the identification of coordinates $t_{1} \equiv x$.


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## 2. Rational reductions of the KP hierarchy and the string equation.

In this section, we overview the derivation of the equation (1.1) as the string equation of an appropriate rational reduction of the KP hierarchy; we then recast this equation in matrix form, which sets up the framework for us to later realize the equation as arising from isomonodromy deformation. We do not attempt to give a comprehensive introduction to reductions of the KP hierarchy; for a full introduction, one should consult [Dic03], for example. We provide the necessary definitions that should allow the reader to follow the computations on their own, without any explanation of their origin or function. Further, we remark that what appears in this section can be treated as purely formal computation; we will use what we develop here as a objects to be compared to what comes later. For the reader uninterested in the origins of the equations we shall study later, this section can be safely ignored.

### 2.1. The basics of KP, rational reductions, and string equations.

We begin with a list of definitions:

- A pseudodifferential operator is an expression of the form $X=\sum_{k \in \mathbb{Z}} X_{i} \partial^{i}$, where the coefficients $X_{i}$ are functions of $t_{1}:=x$, and possibly a collection of other variables $\left\{t_{k}\right\}, \partial:=\frac{\partial}{\partial x}$, and the symbol $\partial^{-1}$ is defined using the generalized Leibniz rule

$$
\partial^{-1} \circ f=\sum_{k=0}^{\infty}(-1)^{k} f^{(k)} \partial^{-k-1}=f \partial^{-1}-f^{\prime} \partial^{-2}+f^{\prime \prime} \partial^{-3}+\ldots
$$

Note the relation $\partial^{-1} \circ \partial=\partial \circ \partial^{-1}=1$, the identity operator. Such operators are interpreted as acting on functions of $x$.

- The purely differential part or principal part of a pseudodifferential operator $X$ is written $X_{+}$, and is defined to be

$$
X_{+}:=\sum_{k \geq 0} X_{k} \partial^{k} .
$$

- The order of a pseudodifferential operator $X$ is the largest $k$ such that $X_{k} \neq 0$; if no such $k$ exists, we say the operator is of infinite order. One can interpret the order of $X_{+}$, with the word order standing for the usual definition of order of a differential operator.

We now define the KP operator

$$
\begin{equation*}
\mathfrak{L}:=\partial+\alpha_{1} \partial^{-1}+\alpha_{2} \partial^{-2}+\alpha_{3} \partial^{-3}+\cdots \tag{2.1}
\end{equation*}
$$

where the $\alpha_{k}$ are assumed to be functions of $t_{1}:=x$, and a (possibly infinite) collection of other "times" $\left\{t_{k}\right\}$. We define operators $\mathcal{A}_{k}:=\mathfrak{L}_{+}^{k}$; note that since $\mathfrak{L}$ is of order 1 , the operators $\mathcal{A}_{k}$ are of order $k$. The KP hierarchy is defined by the set of equations

$$
\begin{equation*}
\left[\mathfrak{L}-\lambda, \frac{\partial}{\partial t_{k}}-\mathcal{A}_{k}\right]=0, \quad k=1,2, \ldots \tag{2.2}
\end{equation*}
$$

with the assumption that the eigenvalue $\lambda$ is independent of the $t_{k}$ 's. It then follows that the flows $\frac{\partial}{\partial t_{k}}-\mathcal{A}_{k}$ pairwise commute (cf. [Dic03]):

$$
\begin{equation*}
\left[\frac{\partial}{\partial t_{k}}-\mathcal{A}_{k}, \frac{\partial}{\partial t_{k}}-\mathcal{A}_{j}\right]=0, \quad k, j=1,2, \ldots \tag{2.3}
\end{equation*}
$$

which implies the integrability of this collection of equations. A rational reduction of the KP hierarchy is obtained by requiring that a given power of the KP operator $\mathfrak{L}$ is purely differential, i.e. that

$$
\begin{equation*}
\mathfrak{L}^{q} \equiv \mathfrak{L}_{+}^{q}=\mathcal{A}_{q} \tag{2.4}
\end{equation*}
$$

The resulting hierarchy of equations retains the property of integrability. If we require that $\mathfrak{L}^{q}$ is purely differential, we call the hierarchy the $\mathrm{KdV}_{q}$ hierarchy (sometimes, this hierarchy is also called the $q^{t h}$ GelfandDickey hierarchy). These hierarchies also carry a natural bihamiltonian structure [Adl79; DS85; Dic03]. If $q=2$, the resulting hierarchy agrees with the well-known KdV hierarchy. For the $\mathrm{KdV}_{q}$ hierarchy, we define the differential operator $Q$ by

$$
\begin{equation*}
Q:=\mathfrak{L}^{q} . \tag{2.5}
\end{equation*}
$$

We sometimes express the original KP operator as $\mathfrak{L}=Q^{1 / q}$, when there is no cause for ambiguity. A string equation of the $\mathrm{KdV}_{q}$ hierarchy is obtained by the requirement that

$$
\begin{equation*}
[Q, P]=1 \tag{2.6}
\end{equation*}
$$

where the operator $P$ is a polynomial in the operator $Q_{+}^{1 / q}$, with of the form (cf. [FGZ95]):

$$
\begin{equation*}
P:=\sum_{\substack{k \geq 1 \\ k \bmod q \neq 0}}\left(1+\frac{k}{q}\right) t_{k+q} Q_{+}^{k / q}=\sum_{\substack{k \geq 1 \\ k \bmod q \neq 0}}\left(1+\frac{k}{q}\right) t_{k+q} \mathcal{A}_{k} . \tag{2.7}
\end{equation*}
$$

the order of a string equation is the largest index $k$ such that $c_{k}:=\left(1+\frac{k}{q}\right) t_{k+q} \neq 0$. If the order of the string equation is $p$, we call the string equation the $(q, p)$ string equation. Such equations generate additional symmetries of the $\mathrm{KdV}_{q}$ hierarchy, although these additional symmetries do not necessarily commute amongst themselves.

### 2.2. The $(3,4)$ string equation.

We now specialize to the case $q=3, p=4$, which is what is relevant for us. Consider the operator

$$
\begin{equation*}
Q:=\partial^{3}-\frac{3}{2} U \partial-\frac{3}{4} U^{\prime}+\frac{3}{2} V, \tag{2.8}
\end{equation*}
$$

where $U, V$ are functions of the variables $t_{5}, t_{2}$, and $x$. We take this operator to be the generator of the $\mathrm{KdV}_{3}$ hierarchy, and will be interested in the $(3,4)$ string equation.

Let us briefly explain the choice of parametrization of $Q$ (it essentially comes from [Dou90], and more generally [FIZ91]). We momentarily denote $Q:=Q(x)$ to stress that the variable of differentiation is $x$. Under any diffeomorphism $x \rightarrow x(\sigma)$, the composition

$$
\begin{equation*}
\tilde{Q}(\sigma):=\left(\frac{d x}{d \sigma}\right)^{2} \circ Q(x(\sigma)) \circ\left(\frac{d x}{d \sigma}\right)=\partial_{\sigma}^{3}-\frac{3}{2} \tilde{U} \partial_{\sigma}-\frac{3}{4} \tilde{U}_{\sigma}+\frac{3}{2} \tilde{V} \tag{2.9}
\end{equation*}
$$

retains the form of $Q(x)$, while $U, V$ transform as

$$
\begin{align*}
& \tilde{U}(\sigma)=U(x(\sigma))\left(\frac{d x}{d \sigma}\right)^{2}+2\{x, \sigma\}  \tag{2.10}\\
& \tilde{V}(\sigma)=V(x(\sigma))\left(\frac{d x}{d \sigma}\right)^{3} \tag{2.11}
\end{align*}
$$

i.e. as an projective connection ${ }^{1}$ and as a rank 3 tensor, respectively. The operator $Q$ is then seen to act covariantly from the space of rank 1 tensors to rank 2 tensors. At the physical level, this makes consistent our choice of parametrization of the operator $Q: U$ will act as the classical analog of the stress-energy tensor for the underlying conformal field theory, and $V$ will ultimately be responsible for the non-perturbative $\mathbb{Z}_{2^{-}}$ symmetry breaking of the model [Dou90], i.e. the shift in the magnetic field away from zero (see Subsection 4.3 for an interpretation of this statement).

From here on, we will not make any changes of coordinate, and so $\partial=\frac{\partial}{\partial x}$. Now, expanding $Q^{1 / 3}$ in pseudodifferential operators, we find that

$$
\begin{aligned}
Q^{1 / 3} & =\partial-\frac{1}{2} U \partial^{-1}+\frac{1}{2}\left[V+\frac{1}{2} U^{\prime}\right] \partial^{-2}-\frac{1}{4}\left[\frac{1}{3} U^{\prime \prime}+U^{2}+2 V^{\prime}\right] \partial^{-3}+\mathcal{O}\left(\partial^{-4}\right) \\
Q^{2 / 3} & =\partial^{2}-U+\mathcal{O}\left(\partial^{-1}\right) \\
Q^{4 / 3} & =\partial^{4}-2 U \partial^{2}+2\left[V-U^{\prime}\right] \partial+\left[V^{\prime}+\frac{1}{2} U^{2}-\frac{5}{6} U^{\prime \prime}\right]+\mathcal{O}\left(\partial^{-1}\right) \\
Q^{5 / 3} & =\partial^{5}-\frac{5}{2} U \partial^{3}+\frac{5}{2}\left[V-\frac{3}{2} U^{\prime}\right] \partial^{2}+\frac{5}{4}\left[U^{2}-\frac{7}{3} U^{\prime \prime}+2 V^{\prime}\right] \partial \\
& +\frac{5}{4}\left[\frac{4}{3} V^{\prime \prime}+U U^{\prime}-\frac{2}{3} U^{\prime \prime \prime}-2 U V\right]+\mathcal{O}\left(\partial^{-1}\right)
\end{aligned}
$$

The $(3,4)$ string equation is then given by $[Q, P]=1$, where

$$
\begin{equation*}
P:=Q_{+}^{4 / 3}+\frac{5}{3} t_{5} Q_{+}^{2 / 3}=\partial^{4}-\left[2 U-\frac{5}{3} t_{5}\right] \partial^{2}+2\left[V-U^{\prime}\right] \partial+\left[V^{\prime}+\frac{1}{2} U^{2}-\frac{5}{6} U^{\prime \prime}-\frac{5}{3} t_{5} U\right] \tag{2.12}
\end{equation*}
$$

(we set the flow along $Q_{+}^{1 / 3}$ to 0 , as it can be removed by an overall translation $x \rightarrow x+c$; further, we have set the flow $t_{7}:=\frac{3}{7}$ ). One can then verify that

Proposition 2.1. The $(3,4)$ string equation is equivalent to the following system on $U, V$ :

$$
\left\{\begin{array}{l}
0=\frac{1}{2} V^{\prime \prime}-\frac{3}{2} U V+\frac{5}{2} t_{5} V+t_{2}  \tag{2.13}\\
0=\frac{1}{12} U^{(4)}-\frac{3}{4} U^{\prime \prime} U-\frac{3}{8}\left(U^{\prime}\right)^{2}+\frac{3}{2} V^{2}+\frac{1}{2} U^{3}-\frac{5}{12} t_{5}\left(3 U^{2}-U^{\prime \prime}\right)+x
\end{array}\right.
$$

Proof. There is nothing deep happening here; the proof is a direct calculation, and so we omit it.

[^1]Since we will not consider any other string equations in what follows, we will refer to the (3,4) string equation as simply the string equation. The string equation is linearized on the Baker-Akhiezer function $\psi=\psi\left(\lambda ; t_{5}, t_{2}, x\right):$

$$
\left\{\begin{array}{l}
P \psi=\partial_{\lambda} \psi  \tag{2.14}\\
Q \psi=\lambda \psi
\end{array}\right.
$$

The compatibility of this linear system is equivalent to the string equation (1.1). This linearization is useful to us, since we can now write the action of the operators $P, Q$ as a closed-form system of linear differential equations on the functions $\psi, \psi^{\prime}$, and $\psi^{\prime \prime}$.
Proposition 2.2. Define the column vector ${ }^{2} \Psi\left(\lambda ; t_{5}, t_{2}, x\right):=\left\langle\psi^{\prime \prime}-\frac{3}{4} U \psi, \psi^{\prime}, \psi\right\rangle^{T}$. Then, the pair of equations on $\psi$ written above are equivalent to the following vector equations:

$$
\left\{\begin{array}{l}
\partial_{x} \Psi=\mathcal{Q} \Psi  \tag{2.15}\\
\partial_{\lambda} \Psi=\mathcal{P} \Psi
\end{array}\right.
$$

where the matrices $\mathcal{Q}\left(\lambda ; t_{5}, t_{2}, x\right), \mathcal{P}\left(\lambda ; t_{5}, t_{2}, x\right)$ are given by the expressions

$$
\left.\begin{array}{l}
\mathcal{Q}\left(\lambda ; t_{5}, t_{2}, x\right):=E_{13} \lambda+\left(\begin{array}{ccc}
0 & \frac{3}{4} U & -\frac{3}{2} V \\
1 & 0 & \frac{3}{4} U \\
0 & 1 & 0
\end{array}\right), \\
\mathcal{P}\left(\lambda ; t_{5}, t_{2}, x\right):=E_{13} \lambda^{2}+\left(\begin{array}{ccc}
0 & \frac{5}{3} t_{5}+\frac{1}{4} U & -V \\
1 & 0 & \frac{5}{3} t_{5}+\frac{1}{4} U \\
0 & 1 & 0
\end{array}\right) \lambda  \tag{2.17}\\
+\left(\begin{array}{ccc}
\frac{1}{2} V^{\prime}-\frac{1}{12} U^{\prime \prime}+\frac{1}{8} U^{2}-\frac{5}{12} t_{5} U & \frac{1}{12} U^{\prime \prime \prime} & -\frac{7}{16} U U^{\prime}-\frac{3}{8} U V+\frac{5}{12} t_{5} U^{\prime}+t_{2} \\
\frac{1}{2} V-\frac{1}{4}\left(U^{\prime}\right)^{2}-\frac{1}{8} U U^{\prime \prime}+\frac{7}{3} U^{3}+\frac{3}{4} V^{2}-\frac{5}{12} t_{5} U^{2}+x \\
\frac{5}{3} t_{5}-\frac{1}{2} U & \frac{1}{6} U^{\prime \prime}-\frac{1}{4} U^{2}+\frac{5}{6} t_{5} U & -\frac{1}{12} U^{\prime \prime \prime}+\frac{7}{16} U U^{\prime}-\frac{3}{8} U V-\frac{5}{12} t_{5} U^{\prime}+t_{2} \\
& & \frac{1}{2} V+\frac{1}{4} U
\end{array}\right.
\end{array}\right) .
$$

The compatibility condition $[Q, P]=1$ is equivalent to the compatibility condition for the linear system (2.15):

$$
[Q, P]=1 \Longleftrightarrow \frac{\partial \mathcal{P}}{\partial x}-\frac{\partial \mathcal{Q}}{\partial \lambda}+[\mathcal{P}, \mathcal{Q}]=0
$$

Proof. The proof here is again a direct calculation. The only point to remark on is that we have used the string equation to substitute for higher-order derivatives of $U, V$.

One also requires that the string equation is compatible with the other flows of the hierarchy; in our situation, we require that the string equation is compatible with the $t_{2}$ and $t_{5}$ flows. The linearization of these flows on $\psi$ are given by

$$
\begin{align*}
\frac{\partial}{\partial t_{2}} \psi & =Q_{+}^{2 / 3} \psi=\psi^{\prime \prime}-U \psi  \tag{2.18}\\
\frac{\partial}{\partial t_{5}} \psi & =Q_{+}^{5 / 3} \psi=(\lambda+V) \psi^{\prime \prime}+\frac{1}{12}\left(U^{\prime \prime}-3 U^{2}-6 V^{\prime}\right) \psi^{\prime}  \tag{2.19}\\
& +\frac{1}{2}\left(U U^{\prime}-\frac{1}{6} U^{\prime \prime \prime}-U V-2 \lambda U-\frac{5}{3} t_{5} V-\frac{2}{3} t_{2}\right) \psi
\end{align*}
$$

where we have already utilized compatibility of the string equation with the $t_{5}$ flow to reduce the order of the right hand side from 5 to 3 . We can similarly write the above two equations in matrix form:
Proposition 2.3. The equations (2.18), (2.19) are equivalent to the following pair of matrix equations:

$$
\begin{gather*}
\frac{\partial}{\partial t_{2}} \psi=Q_{+}^{2 / 3} \psi \Longleftrightarrow \frac{\partial \Psi}{\partial t_{2}}=M\left(\lambda ; t_{5}, t_{2}, x\right) \Psi,  \tag{2.20}\\
\frac{\partial}{\partial t_{5}} \psi=Q_{+}^{5 / 3} \psi \Longleftrightarrow \frac{\partial \Psi}{\partial t_{5}}=E\left(\lambda ; t_{5}, t_{2}, x\right) \Psi, \tag{2.21}
\end{gather*}
$$

[^2]where the vector $\Psi=\left\langle\psi^{\prime \prime}-\frac{3}{4} U \psi, \psi^{\prime}, \psi\right\rangle^{T}$, and the matrices $M\left(\lambda ; t_{5}, t_{2}, x\right), E\left(\lambda ; t_{5}, t_{2}, x\right)$ are defined to be
\[

$$
\begin{gather*}
M\left(\lambda ; t_{5}, t_{2}, x\right)=\left(\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right) \lambda+\left(\begin{array}{cc}
-\frac{1}{4} U & \frac{1}{4} U-\frac{3}{2} V \\
0 & \frac{9}{2} U^{2}-\frac{1}{4} U^{\prime \prime} \\
1 & 0 \\
1 & -\frac{1}{4} U-\frac{3}{2} V \\
-\frac{1}{4} U
\end{array}\right) .  \tag{2.22}\\
E\left(\lambda ; t_{5}, t_{2}, x\right)=\left(\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right) \lambda^{2}+\left(\begin{array}{ccc}
-\frac{1}{4} U & \frac{1}{4} U^{\prime}-\frac{1}{2} V & \frac{5}{16} U^{2}-\frac{1}{6} U^{\prime \prime} \\
0 & \frac{1}{2} U & -\frac{1}{4} U^{\prime}-\frac{1}{2} V \\
1 & 0 & -\frac{1}{4} U
\end{array}\right) \lambda  \tag{2.23}\\
+\left(\begin{array}{ccc}
\frac{1}{4} U V-\frac{1}{2} U U^{\prime} \frac{1}{12} U^{\prime \prime \prime}-\frac{5}{6} t_{5} V-\frac{1}{3} t_{2} & e_{12} & e_{13} \\
\frac{1}{12} U^{\prime \prime}-\frac{1}{4} U^{2}+\frac{1}{2} V & \frac{5}{3} t_{5} V-\frac{1}{2} U V+\frac{2}{3} t_{2} & e_{23} \\
V & \frac{1}{12} U^{\prime \prime}-\frac{1}{4} U^{2}-\frac{1}{2} V & \frac{1}{4} U V+\frac{1}{2} U U^{\prime}-\frac{1}{12} U^{\prime \prime \prime}-\frac{5}{6} t_{5} V-\frac{1}{3} t_{2}
\end{array}\right),
\end{gather*}
$$
\]

where

$$
\begin{aligned}
& e_{12}:=\frac{1}{8}\left(U^{\prime}\right)^{2}-\frac{3}{16} U^{\prime \prime} U-\frac{1}{4} U^{\prime} V+\frac{1}{8} V^{\prime} U+\frac{5}{16} U^{3}+\frac{5}{12} t_{5}\left(U^{\prime \prime}-3 U^{2}+2 V^{\prime}\right)+x, \\
& e_{23}:=\frac{1}{8}\left(U^{\prime}\right)^{2}-\frac{3}{16} U^{\prime \prime} U+\frac{1}{4} U^{\prime} V-\frac{1}{8} V^{\prime} U+\frac{5}{16} U^{3}+\frac{5}{12} t_{5}\left(U^{\prime \prime}-3 U^{2}-2 V^{\prime}\right)+x, \\
& e_{13}:=\frac{1}{8} U^{\prime \prime} V+\frac{1}{8} U^{\prime} V^{\prime}-\frac{9}{16} U^{2} V+\frac{25}{6} t_{5}^{2} V+t_{2} U+\frac{5}{3} t_{5} t_{2}
\end{aligned}
$$

Proof. Again, there is not much to show here: the proof is a direct calculation. We remark that one can make calculations slightly less tedious by first substituting with the string equation for higher order derivatives in $Q_{+}^{5 / 3}$, and then calculating derivatives of $\psi$ with respect to $x$.

Remark 2.1. Note that all of the matrices $\mathcal{P}, \mathcal{Q}, M$, and $E$ are traceless; this is a consequence of the fact that the generating operator $Q$ has no term of order $\partial^{2}$. We also comment here that in what follows, the matrices $\mathcal{Q}, M$, and $E$ can in fact be seen to arise on their own by requiring isomonodromy for the connection $\partial_{\lambda}-\mathcal{P}$. We present these matrices here for comparison to our results later.

The requirement that all of the above equations are compatible with one another further determines the derivatives of $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$ with respect to $t_{2}$ and $t_{5}$; this can either be done at the level of the operators $Q_{+}^{k / 3}$, or can be performed using the matrices $\mathcal{P}, \mathcal{Q}, M$, and $E$. The result is the following:
Proposition 2.4. The compatibility of the the operators $\lambda-Q, \frac{\partial}{\partial \lambda}-P, \frac{\partial}{\partial t_{2}}-Q_{+}^{2 / 3}, \frac{\partial}{\partial t_{5}}-Q_{+}^{5 / 3}$ (equivalently, the compatibility of the corresponding matrix equations) is equivalent to the string equation (1.1), and the following PDEs:

$$
\begin{align*}
& \frac{\partial U}{\partial t_{2}}=-2 V^{\prime}  \tag{2.24}\\
& \frac{\partial V}{\partial t_{2}}=\frac{1}{6} U^{\prime \prime \prime}-U U^{\prime}  \tag{2.25}\\
& \frac{\partial U}{\partial t_{5}}=\frac{\partial}{\partial x}\left[-\frac{1}{6} U U^{\prime \prime}+\frac{1}{8}\left(U^{\prime}\right)^{2}+\frac{1}{4} U^{3}-\frac{1}{2} V^{2}-\frac{5}{9} t_{5}\left(3 U^{2}-U^{\prime \prime}\right)+\frac{4}{3} x\right]  \tag{2.26}\\
& \frac{\partial V}{\partial t_{5}}=\frac{\partial}{\partial x}\left[\frac{1}{12} U^{\prime \prime} V-\frac{1}{4} U^{\prime} V^{\prime}+\frac{5}{16} U^{2} V-\left(\frac{5}{3} t_{5}+\frac{1}{4} U\right)^{2} V-t_{2} U\right] \tag{2.27}
\end{align*}
$$

Proof. The proof is once again a direct calculation. We remark only that the compatibility conditions

$$
\left[\lambda-Q, \frac{\partial}{\partial t_{2}}-Q_{+}^{2 / 3}\right]=0, \quad\left[\lambda-Q, \frac{\partial}{\partial t_{5}}-Q_{+}^{5 / 3}\right]=0
$$

are enough to infer (2.24)-(2.27); the remaining compatibility conditions are consistent with this calculation, and thus redundant.

The above proposition justifies our notation for ${ }^{\prime}=\frac{\partial}{\partial x}$ : all other derivatives can be rewritten in terms of $\frac{\partial}{\partial x}$. We will sometimes refer to equations (2.24)-(2.27) above, along with the string equation, collectively as the string equation, by a slight abuse of language. Also notice that, in the equations (2.24), (2.25), one can eliminate $V$ to obtain that $U$ satisfies a scaled version of the Boussinesq equation, with $t_{2}$ playing the role of the 'time' variable:

$$
\begin{equation*}
\frac{\partial^{2} U}{\partial t_{2}^{2}}=\frac{\partial}{\partial x}\left[\frac{1}{6} U^{\prime \prime \prime}-U U^{\prime}\right] \tag{2.28}
\end{equation*}
$$

This is not a surprise, as the Boussinesq equation comes exactly from this compatibility condition [Zak73]; for this reason, the $\mathrm{KdV}_{3}$ hierarchy is sometimes referred to as the Boussinesq hierarchy.

Remark 2.2. The structure of the KP hierarchy allowed us to derive an system of ODEs for the functions $U, V$, and a system of compatible PDEs which describe the dependence of $U, V$ on the auxiliary variables $t_{5}, t_{2}$. However, this structure is lacking on the analytical side; it is difficult to do any sort of analysis on the resulting equations, as they are nonlinear. The good news is that these are not just any nonlinear equations; they are in fact integrable. Much success has been found in the past for analyzing the solutions of other string equations using the isomonodromy method [Kap88; Kap04; KK93; FIK91; FIK92]. For example, the $(2,3)$ string equation is equivalent to Painlevé I:

$$
\begin{equation*}
u^{\prime \prime}(x)=6 u^{2}(x)+x, \tag{2.29}
\end{equation*}
$$

and the full Painlevé I hierarchy appears as the series of $(2,2 k+1)$ string equations [Tak07; Oka99]. Such equations admit a representation as the isomonodromy deformations of certain linear differential equations with rational coefficients [CV07; CIK10], have Hamiltonian structure [Tak07], carry a $\tau$-function, etc. Similarly, the six Painlevé equations also admit these features [Oka81; Oka80; HW93; Oka99; Fok+06], as does the Painlevé II hierarchy [CJM06; MM07]. The aim of the rest of this work is to develop these tools for the $(3,4)$ string equation.

## 3. Hamiltonian Structure of the $(3,4)$ String Equation.

Here, we develop a Hamiltonian formulation of the string equation (1.1), (2.24)-(2.27). We develop this formalism before moving to the isomonodromy setting, as the notations of this section will serve as convenient coordinates for parametrizing the solution to the isomonodromy problem. We will first define a set of Darboux coordinates, and show that the corresponding Hamilton equations are equivalent to the string equation. We further show that one can define a $\tau$-function in the sense of Okamoto [Oka81; Oka99] via this construction.

Our methods for finding Darboux coordinates here are admittedly a little ad hoc; we have a good "guess" of where things come from, and can develop a system of coordinates which match up with what we need. We will explain our procedure for identifying a "good" set of Darboux coordinates in Subsection 3.2. In this section, we shall revert to the notation

$$
\begin{equation*}
x=t_{1}, \tag{3.1}
\end{equation*}
$$

as it will be more convenient here when indexing sums.

### 3.1. Hamiltonian Structure and Okamoto $\tau$-function.

Here, we prove a version of Theorem (1.1).
Theorem 3.1. Given a solution $U=U\left(t_{5}, t_{2}, x\right), V=V\left(t_{5}, t_{2}, x\right)$ of the equations (1.1), (2.24)-(2.27), define functions

$$
\begin{align*}
Q_{U}:=U-\frac{4}{3} t_{5}, & Q_{V}:=V, & Q_{W}:=U^{\prime},  \tag{3.2}\\
P_{U}:=\frac{1}{4}\left(3 U U^{\prime}-\frac{1}{3} U^{\prime \prime \prime}-\frac{7}{3} t_{5} U^{\prime}\right), & P_{V}:=V^{\prime}, & P_{W}:=\frac{1}{12} U^{\prime \prime}-\frac{1}{6} t_{5} U+\frac{7}{18} t_{5}^{2}
\end{align*}
$$

(Here, we recall that $x:=t_{1}$ ). Then, there exist functions $H_{5}, H_{2}, H_{1}$, polynomially dependent on $\left(Q_{U}, Q_{V}, Q_{W}, P_{U}, P_{V}, P_{W}\right)$, and on $t_{5}, t_{2}, t_{1}$, such that

$$
\begin{equation*}
\frac{\partial Q_{a}}{\partial t_{k}}=\frac{\partial H_{k}}{\partial P_{a}}, \quad \frac{\partial P_{a}}{\partial t_{k}}=-\frac{\partial H_{k}}{\partial Q_{a}} \tag{3.4}
\end{equation*}
$$

for $a \in\{U, V, W\}, k=1,2,5$. These functions are defined up to the addition of an explicit function of the variables $t_{5}, t_{2}, t_{1}$; these "integration constants" can be chosen so that ${ }^{3}$

$$
\begin{equation*}
\left\{H_{k}, H_{j}\right\}+\frac{\partial H_{k}}{\partial t_{j}}-\frac{\partial H_{j}}{\partial t_{k}}=0 \tag{3.5}
\end{equation*}
$$

for $k, j=1,2,5$. Here, the Poisson bracket $\{\cdot, \cdot\}$ is defined by Equation (1.8). Furthermore, the Hamiltonians satisfy the stronger condition

$$
\begin{equation*}
\left\{H_{k}, H_{j}\right\}=\frac{\partial H_{k}}{\partial t_{j}}-\frac{\partial H_{j}}{\partial t_{k}}=0 \tag{3.6}
\end{equation*}
$$

Explicitly, these functions are given by

$$
\begin{align*}
H_{1} & =P_{U} Q_{W}+6 P_{W}^{2}+2 t_{5} Q_{U} P_{W}-2 t_{5}^{2} P_{W}-\frac{3}{8} Q_{U} Q_{W}^{2}-\frac{1}{8} t_{5} Q_{W}^{2}+\frac{1}{2} P_{V}^{2}-\frac{1}{8} Q_{U}^{4}-\frac{1}{4} t_{5} Q_{U}^{3} \\
& +\frac{1}{2} t_{5}^{2} Q_{U}^{2}-\frac{3}{2} Q_{U} Q_{V}^{2}+\frac{19}{27} t_{5}^{3} Q_{U}-t_{1} Q_{U}+\frac{1}{2} t_{5} Q_{V}^{2}+2 t_{2} Q_{V}-\frac{4}{3} t_{5} t_{1}+\frac{41}{54} t_{5}^{4},  \tag{3.7}\\
H_{2} & =-2 P_{U} P_{V}-6 P_{W} Q_{U} Q_{V}+2 t_{5} P_{W} Q_{V}+4 t_{2} P_{W}+\frac{1}{4} Q_{V} Q_{W}^{2}+\frac{1}{2} P_{V} Q_{U} Q_{W}-\frac{1}{2} t_{5} P_{V} Q_{W} \\
& +Q_{V}^{3}+Q_{U}^{3} Q_{V}+\frac{1}{2} t_{5} Q_{U}^{2} Q_{V}-t_{2} Q_{U}^{2}-2 t_{5} t_{2} Q_{U}-\frac{65}{27} t_{5}^{3} Q_{V}+2 t_{1} Q_{V}-\frac{22}{9} t_{5}^{2} t_{2},  \tag{3.8}\\
H_{5} & =\frac{3}{8} Q_{V}^{4}-\frac{1}{128} Q_{W}^{4}+4 P_{W}^{3}-\frac{1}{16} Q_{U}^{6}-\frac{65}{9} t_{5}^{2} t_{2} Q_{V}-P_{U} P_{V} Q_{V}+P_{U} P_{W} Q_{W}-\frac{3}{4} P_{U} Q_{W} Q_{U}^{2} \\
& -\frac{29}{18} t_{5}^{2} P_{U} Q_{W}-\frac{1}{2} t_{2} Q_{W} P_{V}-\frac{1}{4} t_{5} Q_{U} P_{V}^{2}-\frac{1}{4} t_{5} P_{W} Q_{W}^{2}-2 t_{5} P_{W} Q_{U}^{3}-\frac{3}{2} t_{5}^{2} P_{W} Q_{U}^{2} \\
& -\frac{71}{27} t_{5}^{3} P_{W} Q_{U}+2 t_{1} P_{W} Q_{U}-5 t_{5} P_{W} Q_{V}^{2}+2 t_{2} P_{W} Q_{V}-4 t_{5} t_{1} P_{W}-\frac{1}{16} t_{5} Q_{W}^{2} Q_{U}^{2}+\frac{29}{48} t_{5}^{2} Q_{W}^{2} Q_{U} \\
& +5 t_{5} P_{W}^{2} Q_{U}+\frac{3}{4} t_{5} Q_{U}^{2} Q_{V}^{2}+\frac{1}{2} t_{2} Q_{U}^{2} Q_{V}+t_{5} t_{1} Q_{U}^{2}+t_{5} P_{U} Q_{W} Q_{U}+\frac{1}{2} t_{5} Q_{W} P_{V} Q_{V} \\
& +\frac{1}{2} Q_{W} P_{V} Q_{U} Q_{V}+\frac{47}{12} t_{5}^{2} Q_{U} Q_{V}^{2}+\frac{19}{9} t_{5}^{2} t_{1} Q_{U}-2 t_{5} P_{U}^{2}-\frac{9}{2} P_{W}^{2} Q_{U}^{2}-\frac{20}{3} t_{5}^{2} P_{W}^{2}+\frac{49}{108} t_{5}^{3} Q_{U}^{3} \\
& -\frac{1}{8} Q_{U}^{3} Q_{V}^{2}-\frac{1}{4} t_{1} Q_{U}^{3}-\frac{299}{216} t_{5}^{4} Q_{U}^{2}-\frac{2173}{972} t_{5}^{5} Q_{U}-t_{2}^{2} Q_{U}+\frac{3}{16} t_{5} Q_{U}^{5}-\frac{14}{9} t_{5}^{2} P_{V}^{2}+\frac{7}{18} t_{5}^{2} Q_{U}^{4} \\
& +\frac{1}{8} Q_{U}^{2} P_{V}^{2}-P_{W} P_{V}^{2}+P_{W} Q_{U}^{4}+\frac{163}{27} t_{5}^{4} P_{W}+\frac{3}{32} Q_{W}^{2} Q_{U}^{3}-\frac{1}{16} Q_{W}^{2} Q_{V}^{2}+\frac{11}{108} t_{5}^{3} Q_{W}^{2} \\
& -\frac{1}{8} t_{1} Q_{W}^{2}-\frac{38}{27} t_{5}^{3} Q_{V}^{2}+\frac{1}{2} t_{1} Q_{V}^{2}+P_{U}^{2} Q_{U}-\frac{2}{3} t_{1}^{2}+\frac{82}{27} t_{5}^{3} t_{1}-\frac{556}{243} t_{5}^{6}-\frac{22}{9} t_{5} t_{2}^{2} . \tag{3.9}
\end{align*}
$$

Conversely, if one starts with the functions $H_{5}, H_{2}, H_{1}$, Hamilton's equations (3.4) for these functions are equivalent to the string equation (1.1), (2.24)-(2.27).

Proof. Let us prove that the equations

$$
\frac{\partial Q_{a}}{\partial t_{1}}=\frac{\partial H_{1}}{\partial P_{a}}, \quad \frac{\partial P_{a}}{\partial t_{1}}=-\frac{\partial H_{1}}{\partial Q_{a}}
$$

${ }^{3}$ Some caution must be taken here; the symbol $\frac{\partial}{\partial t_{k}}$ is taken to mean $=\left.\frac{\partial}{\partial t_{k}}\right|_{P_{a}, Q_{a}=\text { const. }}$
$a \in\{U, V, W\}$, can be integrated to a function $H_{1}$. By direct calculation,

$$
\frac{\partial Q_{U}}{\partial t_{1}}=U^{\prime}=Q_{W}
$$

On the other hand, Hamilton's equations tell us that

$$
Q_{W}=\frac{\partial Q_{U}}{\partial t_{1}}=\frac{\partial H_{1}}{\partial P_{U}}
$$

Integrating, we find that

$$
H_{1}=P_{U} Q_{W}+f\left(Q_{U}, Q_{V}, Q_{W}, P_{V}, P_{W} ; t_{5}, t_{2}, t_{1}\right)
$$

(note that $f$ is independent of the variable $P_{U}$ ). Next, we have that

$$
\begin{aligned}
\frac{\partial P_{U}}{\partial t_{1}} & =\frac{1}{4}\left(3\left(U^{\prime}\right)^{2}+3 U U^{\prime \prime}-\frac{1}{3} U^{\prime \prime \prime \prime}-\frac{7}{3} t_{5} U^{\prime \prime}\right) \\
& =\frac{3}{8}\left(U^{\prime}\right)^{2}+\frac{1}{2} U^{3}+\frac{3}{2} V^{2}-\frac{5}{4} t_{5} U^{2}-\frac{1}{6} t_{5} U^{\prime \prime}+t_{1} \\
& =\frac{3}{8} Q_{W}^{2}-2 t_{5} P_{W}+\frac{1}{2} Q_{U}^{3}+\frac{3}{4} t_{5} Q_{U}^{2}-t_{5}^{2} Q_{U}+\frac{3}{2} Q_{V}^{2}+t_{1}-\frac{19}{27} t_{5}^{3}
\end{aligned}
$$

where we have used the string equation (1.1) to rewrite $\frac{\partial P_{U}}{\partial t_{1}}$ in terms of the Hamiltonian variables $\left\{Q_{a}, P_{a}\right\}$, and $t_{5}, t_{2}, t_{1}$. Hamilton's equations tell us that

$$
\frac{3}{8} Q_{W}^{2}-2 t_{5} P_{W}+\frac{1}{2} Q_{U}^{3}+\frac{3}{4} t_{5} Q_{U}^{2}-t_{5}^{2} Q_{U}+\frac{3}{2} Q_{V}^{2}+t_{1}-\frac{19}{27} t_{5}^{3}=\frac{\partial P_{U}}{\partial t_{1}}=-\frac{\partial H_{1}}{\partial Q_{U}}=-\frac{\partial f}{\partial Q_{U}} .
$$

The left hand side of the above is independent of $P_{U}$, and so both sides can be integrated to obtain that
$f=-\frac{3}{8} Q_{U} Q_{W}^{2}+2 t_{5} Q_{U} P_{W}-\frac{1}{8} Q_{U}^{4}-\frac{1}{4} t_{5} Q_{U}^{3}+\frac{1}{2} t_{5}^{2} Q_{U}^{2}-\frac{3}{2} Q_{U} Q_{V}^{2}-x Q_{U}+\frac{19}{27} t_{5}^{3} Q_{U}+\tilde{f}\left(Q_{V}, Q_{W}, P_{V}, P_{W}\right)$.
Our expression for $H_{1}$ now reads

$$
H_{1}=P_{U} Q_{W}-\frac{3}{8} Q_{U} Q_{W}^{2}+2 t_{5} Q_{U} P_{W}-\frac{1}{8} Q_{U}^{4}-\frac{1}{4} t_{5} Q_{U}^{3}+\frac{1}{2} t_{5}^{2} Q_{U}^{2}-\frac{3}{2} Q_{U} Q_{V}^{2}-t_{1} Q_{U}+\frac{19}{27} t_{5}^{3} Q_{U}+\tilde{f}
$$

i.e. we have completely determined the dependence of $H_{1}$ on $Q_{U}, P_{U}$. Continuing in this fashion, one is able to determine the function $H_{1}$ up to an explicit function of the variables $t_{5}, t_{2}, t_{1}$; similar calculations for the Hamilton equations in the variables $t_{2}, t_{5}$ result in functions $H_{2}, H_{5}$, also defined up to the addition of an explicit function of the variables $t_{5}, t_{2}, t_{1}$. Denote these functions, which we will call "integration constants", by $c_{k}\left(t_{5}, t_{2}, t_{1}\right), k=1,2,5$. Calculating the Poisson brackets of the Hamiltonians in pairs, we obtain the equations

$$
\begin{aligned}
& \left\{H_{5}, H_{2}\right\}+\frac{\partial H_{5}}{\partial t_{2}}-\frac{\partial H_{2}}{\partial t_{5}}=\frac{\partial c_{5}}{\partial t_{2}}-\frac{\partial c_{2}}{\partial t_{5}} \\
& \left\{H_{5}, H_{1}\right\}+\frac{\partial H_{5}}{\partial t_{1}}-\frac{\partial H_{1}}{\partial t_{5}}=\frac{\partial c_{5}}{\partial t_{1}}-\frac{\partial c_{1}}{\partial t_{5}}+\frac{4}{3} t_{1}-\frac{82}{27} t_{5}^{3} \\
& \left\{H_{2}, H_{1}\right\}+\frac{\partial H_{2}}{\partial t_{1}}-\frac{\partial H_{1}}{\partial t_{2}}=\frac{\partial c_{2}}{\partial t_{1}}-\frac{\partial c_{1}}{\partial t_{2}}
\end{aligned}
$$

So, for example, we can take

$$
c_{5}=-\frac{2}{3} t_{1}^{2}+\frac{82}{27} t_{5}^{3} t_{1}-\frac{556}{243} t_{5}^{6}-\frac{22}{9} t_{5} t_{2}^{2}, \quad c_{2}=-\frac{22}{9} t_{5}^{2} t_{2}, \quad c_{1}=-\frac{4}{3} t_{5} t_{1}+\frac{41}{54} t_{5}^{4}
$$

From this calculation one can see that the condition $\left\{H_{k}, H_{j}\right\}=\frac{\partial H_{k}}{\partial t_{j}}-\frac{\partial H_{j}}{\partial t_{k}}=0$ holds, for $k, j=1,2,5$.
Conversely, suppose we start with the functions $H_{5}, H_{2}$, and $H_{1}$. We check only that the first Hamiltonian flow is equivalent to the string equation (1.1); the remaining equations can be obtained in an identical manner. Given $H_{1}$, Hamilton's equations in the variable $t_{1}$ read

$$
\begin{aligned}
& Q_{U}^{\prime}=\frac{\partial H_{1}}{\partial P_{U}}=Q_{W}, \quad Q_{V}^{\prime}=\frac{\partial H_{1}}{\partial P_{V}}=P_{V}, \quad Q_{W}^{\prime}=\frac{\partial H_{1}}{\partial P_{W}}=12 P_{W}+2 t_{5} Q_{U}-2 t_{5}^{2}, \\
& P_{U}^{\prime}=-\frac{\partial H_{1}}{\partial Q_{U}}=\frac{1}{2} Q_{U}^{3}+\frac{3}{2} Q_{V}^{2}+\frac{3}{8} Q_{W}^{2}+\frac{3}{4} t_{5} Q_{U}^{2}-2 t_{5} P_{W}-t_{5}^{2} Q_{U}-\frac{19}{27} t_{5}^{3}+t_{1}, \\
& P_{V}^{\prime}=-\frac{\partial H_{1}}{\partial Q_{V}}=3 Q_{U} Q_{V}-t_{5} Q_{V}-2 t_{2}, \quad \quad P_{W}^{\prime}=-\frac{\partial H_{1}}{\partial Q_{W}}=12 P_{W}-2 t_{5} Q_{U}-2 t_{5}^{2} .
\end{aligned}
$$

If we define $U:=Q_{U}+\frac{4}{3} t_{5}, V:=Q_{V}$, then the first three equations tell us that $Q_{W}=U^{\prime}, P_{V}=V^{\prime}$, and $P_{W}=\frac{1}{12} U^{\prime \prime}-\frac{1}{6} t_{5} U+\frac{7}{18} t_{5}^{2}$. Making these substitutions into the equation $P_{V}^{\prime}=-\frac{\partial H_{1}}{\partial Q_{V}}$, we obtain

$$
V^{\prime \prime}=3 U V-5 t_{5} V-2 t_{2}
$$

which is the second part of the string equation. Differentiating the equation $P_{W}^{\prime}=-\frac{\partial H_{1}}{\partial Q_{W}}$ once more with respect to $t_{1}$, and inserting the expression for $P_{U}^{\prime}$, we obtain the first part of the string equation.

Remark 3.1. (Homogeneous changes of coordinate.) Although the explicit equations for the Hamiltonians are rather unwieldy, the Hamiltonians themselves enjoy some nice properties, as we shall see in the subsequent remarks. The first observation one can make is that $H_{1}, H_{2}$, and $H_{5}$ are weighted homogeneous polynomials, in the following sense.
Proposition 3.1. Fix $\kappa \in \mathbb{C} \backslash\{0\}$. Under the change of variables

$$
\begin{equation*}
\left(Q_{U}, Q_{V}, Q_{W}, P_{U}, P_{V}, P_{W}, t_{1}, t_{2}, t_{5}\right) \mapsto\left(\kappa^{2} Q_{U}, \kappa^{3} Q_{V}, \kappa^{3} Q_{W}, \kappa^{5} P_{U}, \kappa^{4} P_{V}, \kappa^{4} P_{W}, \kappa^{6} t_{1}, \kappa^{5} t_{2}, \kappa^{2} t_{5}\right) \tag{3.10}
\end{equation*}
$$

the Hamiltonians $H_{1}, H_{2}, H_{5}$ transform as

$$
\begin{equation*}
\left(H_{1}, H_{2}, H_{5}\right) \mapsto\left(\kappa^{8} H_{1}, \kappa^{9} H_{2}, \kappa^{12} H_{5}\right) \tag{3.11}
\end{equation*}
$$

The proof of this proposition follows from direct calculation using formulae (3.7)-(3.9).
One should note that the calculation of the integration constants in the above does not determine them uniquely; we have made a choice which is consistent with the formulae we shall meet later, and the requirement that the Hamiltonians are weighted homogeneous polynomials.

Remark 3.2. ( $t_{2} \rightarrow 0$ limit.) There is a well-defined Hamiltonian system which emerges in the $t_{2} \rightarrow 0$ limit, obtained by simultaneously sending $\left(Q_{V}, P_{V}, t_{2}\right)$ to zero. The result is a $2+2$-dimensional completely integrable non-autonomous Hamiltonian system, in the variables $\left(Q_{U}, Q_{W}, P_{U}, P_{W}, t_{1}, t_{5}\right)$. The corresponding Hamiltonians are obtained by directly setting $Q_{V}=P_{V}=t_{2}=0$ in Formulas (3.7)-(3.9). This Hamiltonian system corresponds to the $\mathbb{Z}_{2}$-symmetric reduction that we shall study in Section 4.3.
Remark 3.3. (Okamoto $\tau$-function and a stronger integrability condition.) The above proposition also provides us with another useful object: since

$$
\begin{equation*}
\frac{\partial H_{j}}{\partial t_{k}}-\frac{\partial H_{k}}{\partial t_{j}}=0, \quad k, j=1,2,5 \tag{3.12}
\end{equation*}
$$

we have the following corollary:
Corollary 3.1. Consider the differential

$$
\begin{equation*}
\omega_{\text {Okamoto }}:=H_{5} d t_{5}+H_{2} d t_{2}+H_{1} d t_{1} . \tag{3.13}
\end{equation*}
$$

and let $\mathbf{d}$ denote the exterior differential in the variables $t_{5}, t_{2}, t_{1}$. Then, $\omega_{\text {Okamoto }}$ is closed:

$$
\begin{equation*}
\mathbf{d} \omega_{\text {Okamoto }}=0 \tag{3.14}
\end{equation*}
$$

Thus, we can locally integrate this differential up to a function on the parameter space, $\tau=e^{\int \omega_{\text {Okamoto }}}$. This observation for similar Painlevé systems was made by Okamoto in [Oka81; Oka99], and was used as the definition of the $\tau$-differential for such equations. This definition of the $\tau$-function is perhaps less familiar to the readership than the usual isomonodromic $\tau$-function defined by Jimbo, Miwa, and Ueno (JMU) [JMU81]. As we shall see in Section 5, the Okamoto definition coincides (up to an overall multiplicative constant) with the JMU isomonodromic $\tau$-function.

As a final remark, one can calculate that the coordinates we use here actually satisfy a stronger condition than (3.12) still: we have that

$$
\begin{equation*}
\left.\frac{\partial}{\partial t_{k}}\right|_{P, Q=\text { const. }} H_{j}=\frac{\partial H_{j}}{\partial t_{k}} \tag{3.15}
\end{equation*}
$$

for any $k, j=1,2,5$.

### 3.2. A formal derivation of the Darboux coordinates.

We did not explain how we found the Darboux coordinates (3.2),(3.3), we only demonstrated that they indeed work for the purposes of Proposition (3.1). Construction of Darboux coordinates for closely related systems has been performed in [BHH23] (see also the earlier work [BM05]). However, these methods do not apply directly to our system: The leading term of the matrix $\mathcal{P}\left(\lambda ; t_{5}, t_{2}, t_{1}\right)$ is not diagonalizable, and moreover (as we shall soon see) upon an appropriate gauge transformation which makes the leading term diagonal, the connection matrix develops a resonant Fuchsian singularity at the origin. This is similar to what occurs in the case of Painlevé I, cf. [JM81], formulae C2 and C5. Here, we furnish a formal argument on where this choice of coordinates comes from, and how one might develop a good set of such coordinates in a similar setting.

An outline of this procedure is as follows:
(i.) Construct an appropriate Hamiltonian $\tilde{H}_{1}$, i.e. a function satisfying $\frac{\partial \tilde{H}_{1}}{\partial t_{1}}=-U$.
(ii.) Require that $U, V$ are two of the canonical coordinates. Assume all other canonical coordinates are independent of $U, V$.
(iii.) Use Hamilton's equations to identify the variables canonically conjugate to $U, V$. If this procedure fails, then the ansatz that the other canonical coordinates are independent of $U, V$ is incorrect; find a minimal modification of this ansatz, and proceed.
(iv.) Identify the remaining independent coordinates $\tilde{Q}_{W}, \tilde{P}_{W}$,
(v.) Use the procedure of the preceding proof to construct the other Hamiltonians $\tilde{H}_{2}, \tilde{H}_{5}$. If these Hamiltonians satisfy $\left\{\tilde{H}_{k}, \tilde{H}_{j}\right\}=\frac{\partial \tilde{H}_{j}}{\partial t_{k}}-\frac{\partial \tilde{H}_{k}}{\partial t_{j}}=0$, for $k=1,2,5$, then we are done. If not, then proceed to the next step.
(vi.) Find a canonical transformation $(\tilde{Q}, \tilde{P}, \tilde{H}) \rightarrow(Q, P, H)$ such that, in the new coordinate system, $\left\{H_{k}, H_{j}\right\}=\frac{\partial H_{j}}{\partial t_{k}}-\frac{\partial H_{k}}{\partial t_{j}}=0$.
We begin with step (i.). We search for a function $\tilde{H}_{1}$ such that $\frac{\partial \tilde{H}_{1}}{\partial t_{1}}=-U$. From this, we can infer that

$$
\tilde{H}_{1}=-t_{1} U+f\left(U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, V, V^{\prime}, t_{5}, t_{2}, t_{1}\right)
$$

for some undetermined function $f$. The relation $\frac{\partial \tilde{H}_{1}}{\partial t_{1}}=-U$, along with the above formula, imply that

$$
\begin{aligned}
\frac{\partial}{\partial t_{1}} & {\left[f\left(U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, V, V^{\prime}, t_{5}, t_{2}, t_{1}\right)\right]=-t_{1} U^{\prime} } \\
& =-\left(\frac{1}{12} U^{(4)}-\frac{3}{4} U^{\prime \prime} U-\frac{3}{8}\left(U^{\prime}\right)^{2}+\frac{3}{2} V^{2}+\frac{1}{2} U^{3}-\frac{5}{12} t_{5}\left(3 U^{2}-U^{\prime \prime}\right)\right) U^{\prime} \\
& =\frac{\partial}{\partial t_{1}}\left[\frac{1}{24}\left(U^{\prime \prime}\right)^{2}-\frac{1}{12} U^{\prime \prime \prime} U^{\prime}+\frac{3}{8} U\left(U^{\prime}\right)^{2}-\frac{1}{8} U^{4}-\frac{5}{24} t_{5}\left[\left(U^{\prime}\right)^{2}-2 U^{3}\right]\right]-\frac{3}{2} U^{\prime} V^{2}
\end{aligned}
$$

where we have used the string equation (1.1) to substitute for $t_{1}$. Using the identity $\frac{3}{2} U^{\prime} V^{2}=\frac{3}{2} \frac{\partial}{\partial t_{1}}\left[U V^{2}\right]-$ $3 U V V^{\prime}$, and using the other half of the string equation to substitute $3 U V=V^{\prime \prime}+5 t_{5} V+2 t_{2}$, we obtain that

$$
\begin{aligned}
\frac{\partial}{\partial t_{1}} & {\left[f\left(U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, V, V^{\prime}, t_{5}, t_{2}, x\right)\right] } \\
& =\frac{\partial}{\partial t_{1}}\left[\frac{1}{24}\left(U^{\prime \prime}\right)^{2}-\frac{1}{12} U^{\prime \prime \prime} U^{\prime}+\frac{3}{8} U\left(U^{\prime}\right)^{2}-\frac{1}{8} U^{4}-\frac{5}{24} t_{5}\left[\left(U^{\prime}\right)^{2}-2 U^{3}\right]-\frac{3}{2} U V^{2}\right]+3 U V V^{\prime} \\
& =\frac{\partial}{\partial t_{1}}\left[\frac{1}{24}\left(U^{\prime \prime}\right)^{2}-\frac{1}{12} U^{\prime \prime \prime} U^{\prime}+\frac{3}{8} U\left(U^{\prime}\right)^{2}-\frac{1}{8} U^{4}-\frac{5}{24} t_{5}\left[\left(U^{\prime}\right)^{2}-2 U^{3}\right]-\frac{3}{2} U V^{2}\right]+\left(V^{\prime \prime}+5 t_{5} V+2 t_{2}\right) V^{\prime} \\
& =\frac{\partial}{\partial t_{1}}\left[\frac{1}{24}\left(U^{\prime \prime}\right)^{2}-\frac{1}{12} U^{\prime \prime \prime} U^{\prime}+\frac{3}{8} U\left(U^{\prime}\right)^{2}-\frac{1}{8} U^{4}-\frac{5}{24} t_{5}\left[\left(U^{\prime}\right)^{2}-2 U^{3}\right]-\frac{3}{2} U V^{2}+\frac{1}{2}\left(V^{\prime}\right)^{2}+\frac{5}{2} t_{5} V^{2}+2 t_{2} V\right]
\end{aligned}
$$

Thus, up to an overall integration constant (functionally independent of $U, V$ ), we can write $\tilde{H}_{1}$ as
$\tilde{H}_{1}=\frac{1}{24}\left(U^{\prime \prime}\right)^{2}-\frac{1}{12} U^{\prime \prime \prime} U^{\prime}+\frac{3}{8} U\left(U^{\prime}\right)^{2}-\frac{1}{8} U^{4}-\frac{5}{24} t_{5}\left[\left(U^{\prime}\right)^{2}-2 U^{3}\right]-\frac{3}{2} U V^{2}+\frac{1}{2}\left(V^{\prime}\right)^{2}+\frac{5}{2} t_{5} V^{2}+2 t_{2} V-t_{1} U$.
We now require that $\tilde{Q}_{U}:=U, \tilde{Q}_{V}:=V$. We must search for a third coordinate, as well as canonically conjugate variables to the $\tilde{Q}_{a}$ 's. We identify the variable $\tilde{P}_{V}$ conjugate to $\tilde{Q}_{V}$ first, as this turns out to be the simplest situation. We make the assumption that $\tilde{P}_{U}, \tilde{P}_{V}$, and the other variables $\tilde{Q}_{W}, \tilde{P}_{W}$ do not depend explicitly on $U, V$; thus, we can calculate derivatives of $H_{1}$ with respect to $\tilde{Q}_{U}, \tilde{Q}_{V}$ without any further knowledge of the form of the unknown coordinates. If Hamilton's equations are to hold, we must have that

$$
\frac{\partial}{\partial t_{1}} \tilde{P}_{V}=-\frac{\partial \tilde{H}_{1}}{\partial \tilde{Q}_{V}}=3 \tilde{Q}_{U} \tilde{Q}_{V}-5 t_{5} \tilde{Q}_{V}-2 t_{2}=\tilde{Q}_{V}^{\prime \prime}
$$

where we have used the string equation in the last equality. Thus, we see that we may take $\tilde{P}_{V}=V^{\prime}$. A similar calculation for $\tilde{Q}_{U}$ yields that

$$
\frac{\partial}{\partial t_{1}} \tilde{P}_{U}=-\frac{\partial \tilde{H}_{1}}{\partial \tilde{Q}_{U}}=-\frac{1}{12} U^{(4)}+\frac{5}{12} t_{5} U^{\prime \prime}-\frac{3}{4} U U^{\prime \prime}
$$

which cannot be integrated fully, due to the presence of the term $\frac{3}{4} U U^{\prime \prime}$. This suggests that our original assumption that the $\tilde{P}_{U}$ coordinate was independent of $U$ is incorrect, which effects our calculation of the partial derivative $\frac{\partial \tilde{H}_{1}}{\partial \tilde{Q}_{U}}$. A good replacement ansatz for $\tilde{P}_{U}$ is then $\tilde{P}_{U}=-\frac{1}{12} U^{\prime \prime \prime}+\alpha U U^{\prime}+\beta t_{5} U^{\prime}$, for some constants $\alpha, \beta$, to be determined ${ }^{4}$. With this definition of $\tilde{P}_{U}$, and still assuming that the remaining coordinates $\tilde{Q}_{W}, \tilde{P}_{W}$ are independent of $U, V$, we recalculate $\frac{\partial \tilde{H}_{1}}{\partial \tilde{Q}_{U}}$ :

$$
\frac{\partial}{\partial t_{1}} \tilde{P}_{U}=-\frac{\partial \tilde{H}_{1}}{\partial \tilde{Q}_{U}}=-\frac{1}{12} U^{(4)}+\frac{5}{12} t_{5} U^{\prime \prime}-\frac{3}{4} U U^{\prime \prime}-\alpha\left(U^{\prime}\right)^{2}
$$

When $\alpha=\frac{3}{4}$, the right hand side can be integrated. The result is that

$$
\frac{\partial}{\partial t_{1}} \tilde{P}_{U}=\frac{\partial}{\partial t_{1}}\left[-\frac{1}{12} U^{\prime \prime \prime}+\frac{5}{12} t_{5} U^{\prime}-\frac{3}{4} U U^{\prime}\right]
$$

which implies that $\beta=\frac{5}{12}$ in our ansatz. Thus, the only remaining independent functions that have not been accounted for are $U^{\prime}, U^{\prime \prime}$. We take the simplest choice: $\tilde{Q}_{W}:=U^{\prime}$, and search for a canonically conjugate variable. We have that

$$
\frac{\partial}{\partial t_{1}} \tilde{P}_{W}=-\frac{\partial \tilde{H}_{1}}{\partial \tilde{Q}_{W}}=-\frac{1}{12} U^{\prime \prime \prime}
$$

[^3]and so we may take $\tilde{P}_{W}:=-\frac{1}{12} U^{\prime \prime}$. Our system of coordinates is then
\[

$$
\begin{aligned}
\tilde{Q}_{U}: & =U, & \tilde{Q}_{V} & :=V,
\end{aligned}
$$ r \tilde{Q}_{W}:=U^{\prime} .
\]

One can readily check that the choice of coordinates we have made is consistent with the remaining Hamilton equations. In other words, one can also consistently construct functions $\tilde{H}_{2}, \tilde{H}_{5}$, such that the equations

$$
\frac{\partial \tilde{Q}_{a}}{\partial t_{k}}=\frac{\partial \tilde{H}_{k}}{\partial \tilde{P}_{a}}, \quad \frac{\partial \tilde{P}_{a}}{\partial t_{k}}=-\frac{\partial \tilde{H}_{k}}{\partial \tilde{Q}_{a}}
$$

hold, for $a \in\{U, V, W\}, k=1,2,5$. Furthermore, one can find integration constants (functions independent of the Hamiltonian variables) such that the following equations hold:

$$
\left\{\tilde{H}_{k}, \tilde{H}_{j}\right\}+\frac{\partial \tilde{H}_{k}}{\partial t_{j}}-\frac{\partial \tilde{H}_{j}}{\partial t_{k}}=0
$$

for $k, j=1,2,5$.
One might be tempted to think that we have completed our task, and the system of coordinates we have found here works just as well as the one we used in Proposition 3.1. However, there is an additional constraint we have imposed in Proposition 3.1 that is not satisfied here: we required that the quantities $\left\{H_{k}, H_{j}\right\}, \frac{\partial H_{k}}{\partial t_{j}}-\frac{\partial H_{j}}{\partial t_{k}}$ to vanish separately:

$$
\begin{equation*}
\left\{H_{k}, H_{j}\right\}=0=\frac{\partial H_{k}}{\partial t_{j}}-\frac{\partial H_{j}}{\partial t_{k}} \tag{3.16}
\end{equation*}
$$

One can readily check that the system of Hamiltonians $\tilde{H}_{k}$ do not satisfy this condition. Indeed, this condition is quite important, as if we try to define a $\tau$-differential as in (3.13) using the tilde-coordinates, one finds that this differential is not closed.

Of course, canonical coordinates for Hamiltonian systems are not unique: one can always make a canonical transformation to obtain a new set of coordinates. In other words, if we define the non-degenerate 2 -form

$$
\begin{equation*}
\Omega:=\sum_{a \in\{U, V, W\}} d P_{a} \wedge d Q_{a}-\sum_{k \in\{1,2,5\}} d H_{k} \wedge d t_{k} \tag{3.17}
\end{equation*}
$$

then any change of coordinates $\left(\tilde{Q}_{a}, \tilde{P}_{a}, \tilde{H}_{k}, t_{5}, t_{2}, t_{1}\right) \rightarrow\left(Q_{a}, P_{a}, H_{k}, t_{5}, t_{2}, t_{1}\right)$ which preserves $\Omega$ will retain the form of Hamilton's equations. One might hope to find a canonical transformation to some system of coordinates for which the Hamiltonians in these coordinates satisfy the condition (3.16). This is achievable; the result is stated in the following Proposition.

Proposition 3.2. The change of coordinates

$$
\begin{align*}
Q_{U} & :=\tilde{Q}_{U}-\frac{4}{3} t_{5}, \quad Q_{V}:=\tilde{Q}_{V}, \quad Q_{W}:=\tilde{Q}_{W}  \tag{3.18}\\
P_{U} & :=\tilde{P}_{U}-\frac{1}{6} t_{5} \tilde{Q}_{W}, \quad P_{V}:=\tilde{P}_{V}, \quad P_{W}:=\tilde{P}_{W}-\frac{1}{6} t_{5} \tilde{Q}_{U}+\frac{7}{18} t_{5}^{2}  \tag{3.19}\\
H_{1} & :=\tilde{H}_{1}, \quad H_{2}:=\tilde{H}_{2}, \quad H_{5}:=\tilde{H}_{5}-\frac{4}{3} \tilde{P}_{U}+\frac{1}{6} \tilde{Q}_{U} \tilde{Q}_{W}-\frac{5}{9} t_{5} \tilde{Q}_{W} \tag{3.20}
\end{align*}
$$

is canonical.

Proof. The proof is a straightforward calculation; we must show the equality $\tilde{\Omega}=\Omega$. On one hand, we have that

$$
\begin{aligned}
d H_{5} \wedge d t_{5} & =d \tilde{H}_{5} \wedge d t_{5} \underbrace{-\frac{4}{3} d \tilde{P}_{U} \wedge d t_{5}+\frac{1}{6} \tilde{Q}_{U} d \tilde{Q}_{W} \wedge d t_{5}+\frac{1}{6} \tilde{Q}_{W} d \tilde{Q}_{U} \wedge d t_{5}-\frac{5}{9} t_{5} d \tilde{Q}_{W} \wedge d t_{5}}_{\gamma} \\
& :=d \tilde{H}_{5} \wedge d t_{5}+\gamma
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
d P_{U} \wedge d Q_{U} & =d \tilde{P}_{U} \wedge d \tilde{Q}_{U}-\frac{4}{3} d \tilde{P}_{U} \wedge d t_{5}-\frac{1}{6} t_{5} d \tilde{Q}_{W} \wedge d \tilde{Q}_{U}+\frac{1}{6} \tilde{Q}_{W} d \tilde{Q}_{U} \wedge d t_{5}+\frac{2}{9} t_{5} d \tilde{Q}_{W} \wedge d t_{5} \\
d P_{W} \wedge d Q_{W} & =d \tilde{P}_{W} \wedge d \tilde{Q}_{W}-\frac{1}{6} t_{5} d \tilde{Q}_{U} \wedge d \tilde{Q}_{W}+\frac{1}{6} \tilde{Q}_{U} d \tilde{Q}_{W} \wedge d t_{5}-\frac{7}{9} d \tilde{Q}_{W} \wedge d t_{5}
\end{aligned}
$$

Summing these contributions, we see that

$$
d P_{U} \wedge d Q_{U}+d P_{W} \wedge d Q_{W}=d \tilde{P}_{U} \wedge d \tilde{Q}_{U}+d \tilde{P}_{W} \wedge d \tilde{Q}_{W}+\gamma
$$

Since $d P_{V} \wedge d Q_{V}=d \tilde{P}_{W} \wedge d \tilde{Q}_{V}, d H_{1} \wedge d t_{1}=d \tilde{H}_{1} \wedge d t_{1}, d H_{2} \wedge d t_{2}=d \tilde{H}_{2} \wedge d t_{2}$, we see that $\tilde{\Omega}=\Omega$.
One should immediately recognize the coordinates obtained here as those of Proposition 3.1.
Remark 3.4. Let us comment on how this canonical transformation might be hypothesized. One first notices that, in the tilde-coordinate system, the bracket $\left\{\tilde{H}_{1}, \tilde{H}_{2}\right\}=0$, and also that $\frac{\partial \tilde{H}_{1}}{\partial t_{2}}=\frac{\partial \tilde{H}_{2}}{\partial t_{1}}$. This suggests that the transformation we make should only involve the Hamiltonian $\tilde{H}_{\tilde{\sim}}$. Furthermore, one should first attempt to make changes of variables involving only $\tilde{P}_{U}, \tilde{Q}_{U}, \tilde{P}_{W}, \tilde{Q}_{W}$, as $\tilde{P}_{V}, \tilde{Q}_{V}$ do not interact with these variables. The requirement that the condition (3.16) then essentially fixes the form of the transformation.

To summarize, the philosophy is the following: find some set of Darboux coordinates that realize the string equation (1.1), (2.24)-(2.27) as a Hamiltonian system. There is a natural way to make an educated guess. Then, all one must do is make a canonical transformation to a more convenient set of coordinates, i.e. one satisfying the condition (3.16).

## 4. The Isomonodromy Approach.

In this section, we study the isomonodromy approach to the $(3,4)$ string equation. A Riemann-Hilbert formulation of this equation is given, and the various symmetries of the Riemann-Hilbert problem are studied.

We now want to study the monodromy preserving deformations of the equation

$$
\begin{equation*}
\frac{\partial \Psi}{\partial \lambda}=L\left(\lambda ; t_{5}, t_{2}, x\right) \Psi \tag{4.1}
\end{equation*}
$$

where $L\left(\lambda ; t_{5}, t_{2}, x\right)$ is given by the expression

$$
\begin{align*}
& L\left(\lambda ; t_{5}, t_{2}, x\right)=\left(\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \lambda^{2}+\left(\begin{array}{ccc}
0 & 2 t_{5}+\frac{1}{4} Q_{U} & -Q_{V} \\
1 & 0 & 2 t_{5}+\frac{1}{4} Q_{U} \\
0 & 1 & 0
\end{array}\right) \lambda  \tag{4.2}\\
& +\left(\begin{array}{ccc}
\frac{1}{8} Q_{U}^{2}-P_{W}+\frac{1}{2} P_{V}-\frac{1}{4} t_{5} Q_{U}-\frac{1}{6} t_{5}^{2} & L_{12} & L_{13} \\
\frac{1}{2} Q_{V}-\frac{1}{4} Q_{W} & 2 P_{W}-\frac{1}{4} Q_{U}^{2}+\frac{1}{2} t_{5} Q_{U}+\frac{1}{3} t_{5}^{2} & L_{23} \\
t_{5}-\frac{1}{2} Q_{U} & \frac{1}{2} Q_{V}+\frac{1}{4} Q_{W} & \frac{1}{8} Q_{U}^{2}-P_{W}-\frac{1}{2} P_{V}-\frac{1}{4} t_{5} Q_{U}-\frac{1}{6} t_{5}^{2}
\end{array}\right),
\end{align*}
$$

and

$$
\begin{aligned}
L_{12} & :=\frac{5}{16} Q_{U} Q_{W}-P_{U}+\frac{1}{4} t_{5} Q_{W}-\frac{3}{8} Q_{U} Q_{V}-\frac{1}{2} t_{5} Q_{V}+t_{2} \\
L_{13} & :=\frac{1}{16} Q_{W}^{2}+\frac{7}{32} Q_{U}^{3}+\frac{3}{4} Q_{V}^{2}-\frac{3}{2} P_{W} Q_{U}+\frac{5}{16} t_{5} Q_{U}^{2}-2 t_{5} P_{W}+\frac{1}{4} t_{5}^{2} Q_{U}+x+\frac{8}{27} t_{5}^{3} \\
L_{23} & :=-\frac{5}{16} Q_{U} Q_{W}+P_{U}-\frac{1}{4} t_{5} Q_{W}-\frac{3}{8} Q_{U} Q_{V}-\frac{1}{2} t_{5} Q_{V}+t_{2}
\end{aligned}
$$

(Note that this expression for $L$ coincides with the definition of $\mathcal{P}$ before, with the definitions (3.2), (3.3) taken into account).
Remark 4.1. Similarly to the works [BHH23; BM05], one can recover the Hamiltonians of the previous section from the spectral curve. The spectral curve corresponding to $L\left(\lambda ; t_{5}, t_{2}, x\right)$ admits an explicit representation in terms of these Hamiltonians:

$$
\begin{equation*}
0=\operatorname{det}\left(\xi \mathbb{I}-L\left(\lambda ; t_{5}, t_{2}, x\right)\right)=\xi^{3}-\left[5 t_{5} \lambda^{2}+2 t_{2} \lambda+H_{1}+\frac{5}{3} t_{5} x\right] \xi+\ell_{0}\left(\lambda ; t_{5}, t_{2}, x\right) \tag{4.3}
\end{equation*}
$$

where $\ell_{0}$ is the degree 4 polynomial

$$
\begin{equation*}
\ell_{0}\left(\lambda ; t_{5}, t_{2}, x\right)=\lambda^{4}+\left(\frac{125}{27} t_{5}^{3}+x\right) \lambda^{2}+\left(\frac{1}{2} H_{2}+\frac{50}{9} t_{5}^{2} t_{2}\right) \lambda+\frac{1}{2} H_{5}+\frac{25}{18} t_{5}^{2} H_{1}+\frac{20}{9} t_{5} t_{2}+\frac{1}{3} x^{2} \tag{4.4}
\end{equation*}
$$

However, as an ODE, Equation (4.1) is "defective"; the leading coefficient at the only singularity of $L$ $(\lambda=\infty)$ is not diagonalizable. Thus, the usual technology used for linear differential equations with rational coefficients [Was02; JMU81; Fok+06] does not directly apply. This situation is reminiscent of the situation for the so called "Fuchs-Garnier" Lax pair for Painlevé I (see C2 of [JM81]). The resolution in the case of Painlevé I, discovered in [JM81], is to make an appropriate gauge transformation which (after a change of variables $\lambda=\zeta^{2}$ ) diagonalizes the leading term at infinity, at the price of introducing a resonant Fuchsian singularity at the origin (see C5 of [JM81]).

The first goal of this section is to try and find an analogous transformation for the equation (4.1). We have the following Proposition:

Proposition 4.1. Define the matrix

$$
g(\lambda)=\frac{i}{\sqrt{3}} \underbrace{\left(\begin{array}{ccc}
\lambda^{1 / 3} & 0 & 0  \tag{4.5}\\
0 & 1 & 0 \\
0 & 0 & \lambda^{-1 / 3}
\end{array}\right)}_{\lambda^{\Delta / 3}} \underbrace{\left(\begin{array}{ccc}
1 & \omega & \omega^{2} \\
1 & 1 & 1 \\
1 & \omega^{2} & \omega
\end{array}\right)}_{-i \sqrt{3} \mathcal{U}}
$$

and set $\Psi:=g \Phi$ (note that $\operatorname{det} g(\lambda)=1, \Delta=\operatorname{diag}(1,0,-1)$, and that $\left.\mathcal{U}^{\dagger} \mathcal{U}=\mathcal{U} \mathcal{U}^{\dagger}=\mathbb{I}\right)$. Then, if $\Psi$ satisfies the ODE (4.1), after the change of variables $\lambda=\zeta^{3}$, the function $\Phi:=\Phi\left(\zeta ; t_{5}, t_{2}, x\right)$ satisfies the ODE

$$
\begin{equation*}
\frac{\partial \Phi}{\partial \zeta}=\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right) \Phi \tag{4.6}
\end{equation*}
$$

where

$$
\mathcal{L}(\zeta)=3\left(\begin{array}{ccc}
1 & 0 & 0  \tag{4.7}\\
0 & \omega & 0 \\
0 & 0 & \omega^{2}
\end{array}\right) \zeta^{6}+\sum_{k=0}^{4} \mathcal{L}_{k} \zeta^{k}+\frac{\frac{i}{\sqrt{3}}\left(\begin{array}{ccc}
0 & -1 & 1 \\
1 & 0 & -1 \\
-1 & 1 & 0
\end{array}\right)}{\zeta}
$$

Proof. The proof is a direct calculation. Explicitly, one has that $\mathcal{L}(\zeta)=3 \zeta^{2} \tilde{L}\left(\zeta^{3}\right)$, where

$$
\tilde{L}(\lambda)=\left[g^{-1} L(\lambda) g-g^{-1} \frac{d g}{d \lambda}\right]
$$

Remark 4.2. Although the proof of the above proposition is straightforward, some remarks are in order.

1. Note that the matrix $\mathcal{U}$ conjugates $\mathcal{L}$ from the outside; if we had instead simply defined the gauge transformation simply by $g(\lambda):=\lambda^{\Delta / 3}$, and subsequently made the change of variables $\lambda=\zeta^{3}$, the effect we set out for (making the leading coefficient at infinity diagonalizable) would still be achieved. In other words, after an appropriate change of variables, $\lambda^{\Delta / 3}$ makes the leading coefficient of $\mathcal{L}$ at infinity diagonalizable, and $\mathcal{U}$ makes this coefficient diagonal.
2. The choice of such $g$ is not unique; one may also additionally multiply $g(\lambda)$ on the right by any diagonal invertible matrix, and still achieve the desired result. This gauge freedom corresponds to the freedom in the choice of diagonalization matrix.
3. We stress that this transformation is analogous to the one made in [JM81] for Painlevé I; indeed, here we also see the appearance of a resonant singularity at the origin. Note that the residue at 0 of $\mathcal{L}$ has eigenvalues $\pm 1,0$. Thus, there is no monodromy around this singularity: the solution will have a first order pole at zero. The form of the solution near $\zeta=0$ is

$$
\begin{equation*}
\Phi(\zeta)=\left[\mathcal{U}^{-1}+\mathcal{O}(\zeta)\right] \zeta^{-\Delta}=\frac{\mathcal{U}^{-1} E_{11}}{\zeta}+\mathcal{O}(1), \quad \zeta \rightarrow 0 \tag{4.8}
\end{equation*}
$$

We can also bring the other Lax matrices into the new gauge:
Proposition 4.2. Under the gauge transformation induced by $g(\lambda)$, and the coordinate transformation $\lambda=\zeta^{3}$, the matrices $\mathcal{Q}, M$, and $E$ transform to
$\mathcal{N}\left(\zeta ; t_{5}, t_{2}, x\right):=g^{-1} \mathcal{Q}\left(\zeta^{3} ; t_{5}, t_{2}, x\right) g, \quad \mathcal{M}\left(\zeta ; t_{5}, t_{2}, x\right):=g^{-1} M\left(\zeta^{3} ; t_{5}, t_{2}, x\right) g, \quad \mathcal{E}\left(\zeta ; t_{5}, t_{2}, x\right):=g^{-1} E\left(\zeta^{3} ; t_{5}, t_{2}, x\right) g$,
where the matrices $\mathcal{N}, \mathcal{M}, \mathcal{E}$ are given by

$$
\begin{aligned}
\mathcal{N}\left(\zeta ; t_{5}, t_{2}, x\right) & :=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \omega & 0 \\
0 & 0 & \omega^{2}
\end{array}\right) \zeta+\frac{\mathcal{N}_{-1}}{\zeta}-\frac{\frac{1}{2} V \cdot D_{0}}{\zeta^{2}} \\
\mathcal{M}\left(\zeta ; t_{5}, t_{2}, x\right) & :=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \omega^{2} & 0 \\
0 & 0 & \omega
\end{array}\right) \zeta^{2}+\sum_{k=-1}^{0} \mathcal{M}_{k} \zeta^{k}+\frac{\frac{1}{12}\left(\frac{9}{4} U^{2}-U^{\prime \prime}\right) \cdot D_{0}}{\zeta^{2}} \\
\mathcal{E}\left(\zeta ; t_{5}, t_{2}, x\right) & :=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \omega^{2} & 0 \\
0 & 0 & \omega
\end{array}\right) \zeta^{5}+\sum_{k=-1}^{3} \mathcal{E}_{k} \zeta^{k}+\frac{\frac{1}{24}\left(U^{\prime \prime} V-\frac{9}{2} U^{2} V+U^{\prime} V^{\prime}+8 t_{2} U+\frac{10}{3} t_{5}\left(10 t_{5}-4 t_{2}\right)\right) \cdot D_{0}}{\zeta^{2}}
\end{aligned}
$$

where the matrix $D_{0}$ is the nilpotent matrix

$$
D_{0}:=3 \cdot \mathcal{U}^{-1} E_{13} \mathcal{U}=\left(\begin{array}{ccc}
1 & \omega^{2} & \omega  \tag{4.9}\\
\omega^{2} & \omega & 1 \\
\omega & 1 & \omega^{2}
\end{array}\right) .
$$

Note that all of the above matrices have a second-order pole at $\zeta=0$; this is due to the resonant singularity in the "spectral" matrix $\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right)$. Further, observe that the most singular terms in each of these matrices is a multiple of the same nilpotent matrix. We will re-derive these matrices from the isomonodromic deformations of the system (4.6) later.

We now return to the analysis of the equation (4.6). Since $\mathcal{L}$ has diagonal leading coefficient at infinity, many useful theorems in the theory of linear ODEs with rational coefficients now apply. For example, we can claim that

Proposition 4.3. The ODE (4.6) admits the formal series solution at $\zeta=\infty$

$$
\begin{equation*}
\Phi(\zeta)=\left[\mathbb{I}+\frac{\Phi_{1}}{\zeta}+\frac{\Phi_{2}}{\zeta^{2}}+\mathcal{O}\left(\zeta^{-3}\right)\right] e^{\Theta\left(\zeta ; t_{5}, t_{2}, x\right)} \tag{4.10}
\end{equation*}
$$

where $\Theta\left(\zeta ; t_{5}, t_{2}, x\right)=\operatorname{diag}\left(\xi_{1}\left(\zeta ; t_{5}, t_{2}, x\right), \xi_{2}\left(\zeta ; t_{5}, t_{2}, x\right), \xi_{3}\left(\zeta ; t_{5}, t_{2}, x\right)\right.$ ), and

$$
\begin{equation*}
\xi_{j}\left(\zeta ; t_{5}, t_{2}, x\right)=\frac{3}{7} \omega^{j-1} \zeta^{7}+\omega^{1-j} t_{5} \zeta^{5}+\omega^{1-j} t_{2} \zeta^{2}+\omega^{j-1} x \zeta \tag{4.11}
\end{equation*}
$$

$j=1,2,3$.

Proof. We refer to [Was02; Fok+06; JMU81] for the details. The exact form the exponential part of the asymptotics can be inferred by considering the eigenvalues of the matrix $\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right)$; indeed, one can readily check that the expressions $\xi_{j}\left(\zeta ; t_{5}, t_{2}, x\right)$ are the principal part of the eigenvalues of $\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right)$ at $\zeta=\infty$.
Remark 4.3. The previous proposition implies in turn that (by "undoing" the gauge transformation) that $\Psi$ admits the formal expansion

$$
\begin{equation*}
\Psi(\lambda)=g(\lambda)\left[\mathbb{I}+\frac{\Phi_{1}}{\lambda^{1 / 3}}+\frac{\Phi_{2}}{\lambda^{2 / 3}}+\mathcal{O}\left(\lambda^{-3}\right)\right] e^{\Theta\left(\lambda^{1 / 3} ; t_{5}, t_{2}, x\right)} \tag{4.12}
\end{equation*}
$$

Note that the coefficients in the subexponential part of the expansion agree with the corresponding coefficients of $\Phi$. These asymptotics are precisely what appear in the local parametrices of the critical quartic 2-matrix model. Thus, we can use results about $\Phi$ to construct our model Riemann-Hilbert problem.

The explicit form of the coefficient matrices $\mathcal{L}_{k}$ is not so important at this stage; the only immediately relevant information is the form of the formal asymptotic expansion for $\Phi(\zeta)$, as in Equation (4.10).
Remark 4.4. For completeness, we record the form of the "regularized" spectral curve here:

$$
\begin{equation*}
0=\operatorname{det}\left[\xi \mathbb{I}-\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right)\right]=\xi^{3}-\left[45 t_{5} \zeta^{10}+18 t_{2} \zeta^{7}+\left(\frac{9}{2} H_{1}+15 t_{5} t_{1}\right) \zeta^{4}-3 \frac{\partial H_{1}}{\partial P_{V}} \zeta+\frac{1}{\zeta^{2}}\right] \xi-\tilde{\ell}_{0} \tag{4.13}
\end{equation*}
$$

where $\tilde{\ell}_{0}=\tilde{\ell}_{0}\left(\lambda ; t_{5}, t_{2}, t_{1}\right)$ is

$$
\begin{align*}
\tilde{\ell}_{0}\left(\lambda ; t_{5}, t_{2}, t_{1}\right) & =27 \zeta^{18}+\left(125 t_{5}^{3}+27 x\right) \zeta^{12}+\left(\frac{27}{2} H_{2}+150 t_{5}^{2} t_{2}\right) \zeta^{9}+\left(\frac{27}{2} H_{5}+\frac{75}{2} t_{5}^{2} H_{1}-9 \frac{\partial H_{2}}{\partial P_{V}}\right. \\
& \left.+60 t_{5} t_{2}^{2}+9 x^{2}\right) \zeta^{6}-\frac{\partial}{\partial P_{V}}\left(9 H_{5}+25 t_{5}^{2} H_{1}\right) \zeta^{3}-6 P_{W}+\frac{3}{4} Q_{U}^{2}-\frac{3}{2} t_{5} Q_{U}-t_{5}^{2} \tag{4.14}
\end{align*}
$$

Before proceeding to the construction of an appropriate Riemann-Hilbert problem for $\Phi$, we first study some of the symmetries of the equation.

### 4.1. Symmetry of $\Phi(\zeta)$.

Proposition 4.4. Define the matrix

$$
\mathcal{S}:=\left(\begin{array}{lll}
0 & 1 & 0  \tag{4.15}\\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right) .
$$

Then, $\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right)$ satisfies the symmetry condition

$$
\begin{equation*}
\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right)=\omega \mathcal{S}^{T} \mathcal{L}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S} \tag{4.16}
\end{equation*}
$$

Proof. The proof of this fact follows almost immediately from the fact that $g(\omega \zeta(\lambda))=g(\zeta(\lambda)) \mathcal{S}^{T}$. By the definition of $\mathcal{L}(\zeta)$, we have that:

$$
\begin{aligned}
\omega \mathcal{L}(\omega \zeta) & =3 \zeta^{2}\left[g^{-1}(\omega \zeta) L\left(\zeta^{3}\right) g(\omega \zeta)-g^{-1}(\omega \zeta) \frac{d g}{d \lambda}(\omega \zeta)\right] \\
& =3 \zeta^{2} \mathcal{S}\left[g^{-1}(\zeta) L\left(\zeta^{3}\right) g(\zeta)-g^{-1}(\zeta) \frac{d g}{d \lambda}(\zeta)\right] \mathcal{S}^{T} \\
& =\mathcal{S} \mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right) \mathcal{S}^{T}
\end{aligned}
$$

multiplication on the left by $\mathcal{S}^{T}$ and the right by $\mathcal{S}$ yields the result.

Similar argumentation yields that the matrices $\mathcal{E}, \mathcal{M}$, and $\mathcal{N}$ have the symmetries

$$
\begin{align*}
\mathcal{S}^{T} \mathcal{E}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S} & =\mathcal{E}\left(\zeta ; t_{5}, t_{2}, x\right), \quad \mathcal{S}^{T} \mathcal{M}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S}=\mathcal{M}\left(\zeta ; t_{5}, t_{2}, x\right) \\
\mathcal{S}^{T} \mathcal{N}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S} & =\mathcal{N}\left(\zeta ; t_{5}, t_{2}, x\right) \tag{4.17}
\end{align*}
$$

Corollary 4.1. The formal expansion $\Phi\left(\zeta ; t_{5}, t_{2}, x\right)$ satisfies the symmetry condition

$$
\begin{equation*}
\Phi\left(\zeta ; t_{5}, t_{2}, x\right)=\mathcal{S}^{T} \Phi\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S} \tag{4.18}
\end{equation*}
$$

Proof. Denote the right hand side of the above equation by $\tilde{\Phi}\left(\zeta ; t_{5}, t_{2}, x\right):=\mathcal{S}^{T} \Phi\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S}$. Then, by direct calculation,

$$
\frac{d \tilde{\Phi}}{d \zeta}=\omega \mathcal{S}^{T} \frac{d \Phi}{d \zeta}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S}=\omega \mathcal{S}^{T} \mathcal{L}(\omega \zeta) \Phi(\omega \zeta) \mathcal{S}=\omega \mathcal{S}^{T} \mathcal{L}(\omega \zeta) \mathcal{S} \tilde{\Phi}(\zeta)=\mathcal{L}(\zeta) \tilde{\Phi}(\zeta)
$$

where the last line follows from Lemma (4.4). Thus, both $\tilde{\Phi}(\zeta)$ and $\Phi(\zeta)$ solve the same equation; furthermore, since $e^{\Theta(\zeta)}=\mathcal{S}^{T} e^{\Theta(\omega \zeta)} \mathcal{S}$, we see that $\tilde{\Phi}(\zeta)=\Phi(\zeta)$.

The above corollary in turn implies the coefficients $\Phi_{k}$ of the formal expansion satisfy

$$
\begin{equation*}
\Phi_{k}=\omega^{-k} \mathcal{S}^{T} \Phi_{k} \mathcal{S} \tag{4.19}
\end{equation*}
$$

This fact will become particularly useful when we begin solving for the coefficients $\Phi_{k}$ in terms of the coefficients of $\mathcal{L}$.

### 4.2. Proof of Proposition (1.1).

In this subsection, we prove a version of Proposition (1.1). What is contained here is the direct analog of Proposition 5.6, 5.7 and Theorem 5.3 of $[F o k+06]$ for the Painlevé I system. Before formulating the Proposition, we state a technical lemma:

Lemma 4.1. Consider the functions $\xi_{j}(\zeta)=\xi_{j}\left(\zeta ; t_{5}, t_{2}, x\right)$ defined by Equation (4.11), and fix $\epsilon>0$. For any $t_{5}, t_{2}, x$ in some fixed compact set $K \subset \mathbb{C}^{3}$, there exists a constant $M=M_{K}$ such that, for all $|\zeta|>M_{K}$, and for any $\ell \in \mathbb{Z}$,

$$
\begin{array}{ll}
\operatorname{Re} \xi_{1}(\zeta)<\operatorname{Re} \xi_{2}(\zeta), & \frac{\pi}{21}(6 \ell+2)+\epsilon<\arg \zeta<\frac{\pi}{21}(6 \ell+5)-\epsilon \\
\operatorname{Re} \xi_{2}(\zeta)<\operatorname{Re} \xi_{3}(\zeta), & \frac{\pi}{21}(6 \ell)+\epsilon<\arg \zeta<\frac{\pi}{21}(6 \ell+3)-\epsilon \\
\operatorname{Re} \xi_{1}(\zeta)<\operatorname{Re} \xi_{3}(\zeta), & \frac{\pi}{21}(6 \ell+1)+\epsilon<\arg \zeta<\frac{\pi}{21}(6 \ell+4)-\epsilon \tag{4.22}
\end{array}
$$

Proof. The lemma follows from straightforward calculation; one has that

$$
\frac{1}{|\zeta|^{7}} \operatorname{Re} \xi_{j}(\zeta)=\frac{3}{7} \cos \left[7 \arg \zeta+\frac{2 \pi}{3}(j-1)\right]\left(1+\mathcal{O}\left(|\zeta|^{-2}\right)\right)
$$

and so it is clear that for $|\zeta|$ taken to be sufficiently large, $\operatorname{Re} \xi_{j}(\zeta)$ is dominated by the first term. Comparison of the values of $\cos \left(7 \theta+\frac{2 \pi}{3}(j-1)\right)$ functions for different values of the argument $\theta$ yields the result.

We can now formulate and prove the following Proposition, which is a more precise statement of (1.1).
Proposition 4.5. Let $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$ solve the string equation (1.1), (2.24)-(2.27). Let $\Phi^{(k)}\left(\zeta ; t_{5}, t_{2}, x\right)$ be solutions to the linear ODE (4.6), which are uniquely determined by the condition that

$$
\begin{equation*}
\Phi^{(k)}\left(\zeta ; t_{5}, t_{2}, x\right)=\left[\mathbb{I}+\mathcal{O}\left(\zeta^{-1}\right)\right] e^{\Theta\left(\zeta ; t_{5}, t_{2}, x\right)}, \quad \zeta \rightarrow \infty, \quad \zeta \in \Omega_{k} \tag{4.23}
\end{equation*}
$$

where the open sectors $\Omega_{k}$ are defined to be

$$
\begin{equation*}
\Omega_{k}:=\left\{\zeta \in \mathbb{C}: \frac{\pi}{21}(k-3)<\arg \zeta<\frac{\pi}{21}(k+1)\right\}, \quad k=1, \ldots, 42 \tag{4.24}
\end{equation*}
$$

The functions $\Phi^{(k)}$ are related by
$\Phi^{(k+1)}\left(\zeta ; t_{5}, t_{2}, x\right)=\Phi^{(k)}\left(\zeta ; t_{5}, t_{2}, x\right) S_{k}, \quad k=1, \ldots, 41, \quad \Phi^{(1)}\left(\zeta ; t_{5}, t_{2}, x\right)=\Phi^{(42)}\left(e^{2 \pi i} \zeta ; t_{5}, t_{2}, x\right) S_{42}$ where the matrices $S_{k}$ have the form

$$
S_{k}=\mathbb{I}+s_{k}\left\{\begin{array}{lll}
E_{32}, & k \equiv 0 & \bmod 6  \tag{4.25}\\
E_{31}, & k \equiv 1 & \bmod 6 \\
E_{21}, & k \equiv 2 & \bmod 6 \\
E_{23}, & k \equiv 3 & \bmod 6 \\
E_{13} . & k \equiv 4 & \bmod 6 \\
E_{12}, & k \equiv 5 & \bmod 6
\end{array}\right.
$$

Furthermore, the $S_{k}$ satisfy the identities

$$
\begin{equation*}
S_{k+14}=\mathcal{S}^{T} S_{k} \mathcal{S}, \quad S_{1} \cdots S_{14}=\mathcal{S}^{T} \tag{4.26}
\end{equation*}
$$

In particular it follows from the above that, $s_{k+14}=s_{k}$, and that generically there are only 6 independent Stokes parameters. Furthermore, denote $\Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right)$ to be the solution of (4.6) near $\zeta=0$, normalized as follows:

$$
\begin{equation*}
\Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right)=\left[\mathcal{U}^{-1}+\mathcal{O}(\zeta)\right] \zeta^{-\Delta} \tag{4.27}
\end{equation*}
$$

The functions $\Phi^{(1)}\left(\zeta ; t_{5}, t_{2}, x\right)$ and $\Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right)$ are related by the unimodular constant matrix $\mathcal{C}$ :

$$
\begin{equation*}
\Phi^{(1)}\left(\zeta ; t_{5}, t_{2}, x\right)=\Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right) \mathcal{C}, \quad \operatorname{det} \mathcal{C}=1 \tag{4.28}
\end{equation*}
$$

The equations (4.27),(4.28) imply that $\mathcal{C}$ has three free parameters. Thus, the string equation (1.1), (2.24)(2.27) are associated with $6+3=9$ constant monodromy data.

Proof. Standard ODE theory [Was02; Fok+06; JMU81] establishes that the functions $\Phi^{(k)}\left(\zeta ; t_{5}, t_{2}, x\right)$ are indeed uniquely specified by the asymptotic condition (4.23). The structure of the Stokes matrices $S_{k}$ can be inferred as follows. Note that $\Phi^{(k)}, \Phi^{(k+1)}$ are both defined on the sector

$$
\delta \Omega_{k}:=\Omega_{k} \cap \Omega_{k+1}=\left\{\zeta \in \mathbb{C}: \frac{\pi}{21}(k-2)<\arg \zeta<\frac{\pi}{21}(k+1)\right\}
$$

Since both $\Phi^{(k)}, \Phi^{(k+1)}$ satisfy equation (4.6), their ratio $\left(\Phi^{(k)}\right)^{-1} \Phi^{(k+1)}=: S_{k}$ is a constant matrix. We have that

$$
S_{k}=\lim _{\substack{\zeta \rightarrow \infty \\ \zeta \in \delta \Omega_{k}}}\left(\Phi^{(k)}\right)^{-1} \Phi^{(k+1)}=\lim _{\substack{\zeta \rightarrow \infty \\ \zeta \in \delta \Omega_{k}}} e^{-\Theta(\zeta)}\left[\mathbb{I}+\mathcal{O}\left(\zeta^{-1}\right)\right] e^{\Theta(\zeta)}
$$

Equivalently, component-wise,

$$
\left(S_{k}\right)_{i j}=\lim _{\substack{\zeta \rightarrow \infty \\ \zeta \in \delta \Omega_{k}}} e^{\xi_{j}(\zeta)-\xi_{i}(\zeta)}\left[\delta_{i j}+\mathcal{O}\left(\zeta^{-1}\right)\right]
$$

where the functions $\xi_{j}(\zeta)=\xi_{j}\left(\zeta ; t_{5}, t_{2}, x\right)$ are defined by Equation (4.11). The above formula implies immediately that the diagonal components of $S_{k}$ are all identically 1 . The all but one of the remaining entries can be determined by taking the above limit in various parts of the sector. We furnish the proof here for the case $k=6 \ell+1$; the structure of the Stokes matrices in the other cases may be obtained in an identical manner.

If $k=6 \ell+1$, consider first the sector $\left\{\frac{\pi}{21}(6 \ell-1)<\arg \zeta<\frac{\pi}{21}(6 \ell)\right\} \subset \delta \Omega_{k}$. Using Lemma 4.1, we see that there is a definite ordering of $\operatorname{Re} \xi_{j}(\zeta)$ in this sector for $|\zeta|$ sufficiently large, given by $\operatorname{Re} \xi_{3}(\zeta)<$ $\operatorname{Re} \xi_{1}(\zeta)<\operatorname{Re} \xi_{2}(\zeta)$. This implies that $\left(S_{k}\right)_{21}=\left(S_{k}\right)_{23}=\left(S_{k}\right)_{13}=0$. On the other hand, in the sector $\left\{\frac{\pi}{21}(6 \ell)<\arg \zeta<\frac{\pi}{21}(6 \ell+1)\right\} \subset \delta \Omega_{k}$, for $|\zeta|$ sufficiently large the ordering $\operatorname{Re} \xi_{3}(\zeta)<\operatorname{Re} \xi_{2}(\zeta)<\operatorname{Re} \xi_{1}(\zeta)$ holds, and so we find that $\left(S_{k}\right)_{12}=0$ in addition. Finally, in the sector $\left\{\frac{\pi}{21}(6 \ell+1)<\arg \zeta<\frac{\pi}{21}(6 \ell+2)\right\} \subset$ $\delta \Omega_{k}$, for $|\zeta|$ sufficiently large the ordering $\operatorname{Re} \xi_{2}(\zeta)<\operatorname{Re} \xi_{3}(\zeta)<\operatorname{Re} \xi_{1}(\zeta)$ holds, and so we see that $\left(S_{k}\right)_{32}=$ 0 . The only entry which cannot be determined by the above line of argumentation is $\left(S_{k}\right)_{31}$; thus, the two solutions are related by

$$
\Phi^{(6 \ell+2)}=\Phi^{(6 \ell+1)}\left[\mathbb{I}+s_{6 j+1} E_{31}\right]
$$

Now, Proposition 4.4 and Equations (4.17) imply that, if $\Phi\left(\zeta ; t_{5}, t_{2}, x\right)$ is a solution to the linearization equations, then so is $\mathcal{S}^{T} \Phi\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S}$. Since $\zeta \in \Omega_{k}$ implies that $\omega \zeta \in \Omega_{k+14}$, we obtain the relations

$$
\Phi^{(k)}\left(\zeta ; t_{5}, t_{2}, x\right)=\mathcal{S}^{T} \Phi^{(k+14)}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S} \quad \Longleftrightarrow \quad \mathcal{S} \Phi^{(k)}\left(\omega^{2} \zeta ; t_{5}, t_{2}, x\right) \mathcal{S}^{T}=\Phi^{(k+14)}\left(\zeta ; t_{5}, t_{2}, x\right)
$$

This implies the relation $\mathcal{S}^{T} S_{k} \mathcal{S}=S_{k+14}$; one can further check that this is consistent with the formula (4.25) for the Stokes matrices, i.e. the only relation that this implies is the following one among the parameters: $s_{k+14}=s_{k}$. Furthermore, we have that

$$
\mathcal{S} \Phi^{(1)}\left(\zeta ; t_{5}, t_{2}, x\right) \mathcal{S}^{T}=\Phi^{(15)}\left(\omega \zeta ; t_{5}, t_{2}, x\right)=\Phi^{(1)}\left(\omega \zeta ; t_{5}, t_{2}, x\right) S_{1} \cdots S_{14}
$$

which implies the following identity for the solution $\Psi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right)$ in a neighborhood of $\zeta=0$ :

$$
\mathcal{S} \Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right) \mathcal{C} \mathcal{S}^{T}=\Phi^{(0)}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{C} S_{1} \cdots S_{14}
$$

Using Equation (4.8), we further see that $\Phi^{(0)}\left(\omega \zeta ; t_{5}, t_{2}, x\right)=\mathcal{S} \Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right)$, and so

$$
\mathcal{S} \Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right) \mathcal{C} \mathcal{S}^{T}=\Phi^{(0)}\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{C} S_{1} \cdots S_{14}=\mathcal{S} \Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right) \mathcal{C} S_{1} \cdots S_{14}
$$

Since $\mathcal{S} \Phi^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right) \mathcal{C}$ is invertible, we obtain the identity $\mathcal{S}^{T}=S_{1} \cdots S_{14}$. Equations (4.26) imply that there are only 6 independent Stokes parameters.

Now, let us show that the matrix $\mathcal{C}$ depends only on 3 independent parameters. Suppose $\Phi^{(0)}$, $\tilde{\Phi}^{(0)}$ are two different solutions in a neighborhood of $\zeta=0$ which connect to $\Phi^{(1)}$ through the matrices $\mathcal{C}, \tilde{\mathcal{C}}$. In other words,

$$
\Phi^{(1)}=\Phi^{(0)} \mathcal{C}=\tilde{\Phi}^{(0)} \tilde{\mathcal{C}}
$$

Now, the functions $\Phi^{(0)}(\zeta) \zeta^{\Delta}, \tilde{\Phi}^{(0)}(\zeta) \zeta^{\Delta}$ are holomorphic and invertible in a neighborhood of zero, and so it follows that the matrix

$$
\mathcal{J}(\zeta):=\zeta^{-\Delta} \mathcal{C} \tilde{\mathcal{C}}^{-1} \zeta^{\Delta}
$$

must be holomorphic and invertible as well. This places constraints on the matrix $K:=\mathcal{C} \tilde{\mathcal{C}}^{-1}$; we find that

$$
K=\left(\begin{array}{ccc}
k_{11} & 0 & 0 \\
k_{21} & k_{22} & 0 \\
k_{31} & k_{32} & k_{33}
\end{array}\right),
$$

where the diagonal elements are subject to the constraint $k_{11} k_{22} k_{33}=1$. Furthermore, one can utilize the gauge freedom (cf. Remark 4.2, point 2.) to further eliminate two of the parameters below the diagonal. This leaves three free parameters.

Remark 4.5. One can show that the generic solution to the constraint equations (4.26) is given by

$$
\begin{align*}
s_{7} & =\frac{s_{1} s_{3} s_{6}-s_{2} s_{6}+s_{1}+1}{W_{1}}, \quad s_{8}=\frac{-s_{1} s_{3} W_{2}+s_{2} W_{2}+W_{1}-s_{2} s_{4}-s_{3}}{W_{1} W_{2}} \\
s_{9} & =\frac{W_{1}-s_{2} s_{4}-s_{3}}{W_{2}}, \quad s_{10}=-W_{1}, \quad s_{11}=-W_{2}, \quad s_{12}=\frac{-W_{2}-s_{3} s_{5}+s_{4}}{W_{1}}  \tag{4.29}\\
s_{13} & =\frac{-s_{4} s_{6} W_{1}-s_{5} W_{1}+W_{2}+s_{3} s_{5}-s_{4}}{W_{1} W_{2}} \quad s_{14}=\frac{-s_{1} s_{4} s_{6}-s_{1} s_{5}-s_{6}+1}{W_{2}}
\end{align*}
$$

with $s_{1}, \ldots, s_{6}$ free parameters, if

$$
\begin{equation*}
W_{1}:=s_{1} s_{3} s_{5}-s_{1} s_{4}-s_{2} s_{5}-1 \neq 0 \quad \text { and } \quad W_{2}:=s_{2} s_{4} s_{6}+s_{2} s_{5}+s_{3} s_{6}+1 \neq 0 \tag{4.30}
\end{equation*}
$$

There are many subcases if $W_{1}$ or $W_{2}$ vanish; we shall save the study of these for later. We now see that there are generically 6 free Stokes parameters, which is consistent with the fact that the string equation (1.1) is of order $4+2=6$.

We now state the "converse" to the above: we formulate a Riemann-Hilbert problem associated to the string equation.

Proposition 4.6. Let $\left\{S_{k}\right\}_{k=1}^{42}$ be the constant $3 \times 3$ matrices defined by (4.25), satisfying relations (4.26). Furthermore, let $\mathcal{C}$ be a constant $3 \times 3$ matrix satisfying

$$
\operatorname{det} \mathcal{C}=1, \quad K \mathcal{C} \sim \mathcal{C}
$$

where $K$ is any unimodular lower-triangular matrix as prescribed by the previous proposition, with $\sim$ denoting similarity equivalence. For $\zeta, t_{5}, t_{2}, x \in \mathbb{C}$, define the $3 \times 3$ sectionally analytic function $X\left(\zeta ; t_{5}, t_{2}, x\right)$ as follows:

$$
X\left(\zeta ; t_{5}, t_{2}, x\right):= \begin{cases}X^{(0)}\left(\zeta ; t_{5}, t_{2}, x\right), & |\lambda|<1  \tag{4.31}\\ X^{(k)}\left(\zeta ; t_{5}, t_{2}, x\right), & \lambda \in \Omega_{k} \cap\{|\lambda|>1\}, \quad k=1, \ldots, 42\end{cases}
$$

where the sectors $\Omega_{k}$ are defined as in (4.24). Finally, let $X\left(\zeta ; t_{5}, t_{2}, x\right)$ solve the following Riemann-Hilbert problem:

$$
\begin{align*}
X^{(k+1)}\left(\zeta ; t_{5}, t_{2}, x\right) & =X^{(k)}\left(\zeta ; t_{5}, t_{2}, x\right) e^{\Theta\left(\zeta ; t_{5}, t_{2}, x\right)} S_{k} e^{-\Theta\left(\zeta ; t_{5}, t_{2}, x\right)}, \quad k=1, \ldots, 42, \quad X_{43}=X_{1} . \\
X^{(1)}\left(\zeta ; t_{5}, t_{2}, x\right) & =X_{0}\left(\zeta ; t_{5}, t_{2}, x\right) e^{\Theta\left(\zeta ; t_{5}, t_{2}, x\right)} \zeta^{-\Delta} \mathcal{C} e^{-\Theta\left(\zeta ; t_{5}, t_{2}, x\right)},  \tag{4.32}\\
X^{(1)}\left(\zeta ; t_{5}, t_{2}, x\right) & =\mathbb{I}+\mathcal{O}\left(\zeta^{-1}\right), \quad \zeta \rightarrow \infty,
\end{align*}
$$

where $\Theta\left(\zeta ; t_{5}, t_{2}, x\right), \Delta$ are as previously defined. Then, the above Riemann-Hilbert problem defined a unique matrix $X\left(\lambda ; t_{5}, t_{2}, x\right)$ which is meromorphic in $t_{5}, t_{2}, x$. Furthermore, if we denote

$$
\begin{equation*}
X^{(1)}\left(\zeta ; t_{5}, t_{2}, x\right)=\mathbb{I}+\frac{X_{1}^{(1)}\left(t_{5}, t_{2}, x\right)}{\zeta}+\frac{X_{2}^{(1)}\left(t_{5}, t_{2}, x\right)}{\zeta^{2}}+\mathcal{O}\left(\zeta^{-3}\right) \tag{4.33}
\end{equation*}
$$

then

$$
\begin{aligned}
U\left(t_{5}, t_{2}, x\right) & :=2 \frac{d}{d x}\left[X_{1}^{(1)}\left(t_{5}, t_{2}, x\right)\right]_{11} \\
V\left(t_{5}, t_{2}, x\right) & :=-2 \frac{d}{d x}\left[X_{2}^{(1)}\left(t_{5}, t_{2}, x\right)-\frac{1}{2} X_{1}^{(1)}\left(t_{5}, t_{2}, x\right)^{2}\right]_{11}=-\frac{d}{d t_{2}}\left[X_{1}^{(1)}\left(t_{5}, t_{2}, x\right)\right]_{11}
\end{aligned}
$$

Then $U, V$ are meromorphic in $t_{5}, t_{2}, x$, and satisfy the string equation (1.1), (2.24)-(2.27).
Proof. Uniqueness of the solution to this problem follows from the usual Liouville argument. Observe that if we set

$$
\Phi^{(k)}=X^{(k)} e^{\Theta}, \quad k=1, \ldots, 42, \quad \Phi^{(0)}=X^{(0)} e^{\Theta} \zeta^{-\Delta}
$$

Then the functions $\Phi^{(k)}$ satisfy the relations (4.25), (4.28). By construction, we can find a system of contours $\Gamma$ such that the jump matrix of the above Riemann-Hilbert problem is smooth, and decays exponentially for $\zeta \rightarrow \infty$ (for example, one may take the unit circle unioned with the rays $\left\{\arg \zeta=\frac{\pi}{21} k\right\}_{k=1}^{42} \cap\{|\zeta|>1\}$ ). By standard Riemann-Hilbert arguments (cf. [Fok+06]), we obtain that the solution to this RHP exists, and
depends meromorphically on its parameters $t_{5}, t_{2}, x$. Our next task is to extract the string equation from the isomonodromy/zero-curvature conditions. Write the asymptotic expansion for $\Phi\left(\zeta ; t_{5}, t_{2}, x\right)$ as

$$
\Phi\left(\zeta ; t_{5}, t_{2}, x\right)=\left(\mathbb{I}+\sum_{k=1}^{\infty} \frac{\Phi_{k}\left(t_{5}, t_{2}, x\right)}{\zeta^{k}}\right) e^{\Theta\left(\zeta ; t_{5}, t_{2}, x\right)}
$$

In general, we have the following procedure for determining the entries of the matrices $\Phi_{k}$ :

1. First, observe that we only have to determine the first row of $\Phi_{k}$ :

$$
\left[\Phi_{k}\right]_{1, \cdot}:=\left[a_{k}\left(t_{5}, t_{2}, x\right), b_{k}\left(t_{5}, t_{2}, x\right), c_{k}\left(t_{5}, t_{2}, x\right)\right] .
$$

The rest of the entries are determined by the symmetry constraint (4.19).
2. Using the formal expansion of $\Phi$, form the series

$$
\frac{d \Phi}{d \zeta} \Phi^{-1}=\mathcal{L}\left(\zeta ; t_{5}, t_{2}, x\right)+\sum_{k=2}^{\infty} \frac{R_{k}\left(t_{5}, t_{2}, x\right)}{\zeta^{k}}
$$

where we use our previous expression for $\mathcal{L}$ (4.7) to parameterize the entries of the above. The condition that $\Phi(\zeta)$ satisfies the differential equation $\frac{d \Phi}{d \zeta}=\mathcal{L} \Phi$ determines the coefficients $\left\{b_{k}, c_{k}\right\}_{k=1}^{7}$ as differential polynomials in the variables $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$, and (for the $k^{t h}$ function, $k \geq 3$ ) the functions $\left\{a_{j}\right\}_{j=1}^{k-2}$; it also imposes the constraint

$$
R_{k}\left(t_{5}, t_{2}, x\right) \equiv 0, \quad k=2,3, \ldots
$$

3. The condition that the coefficients $R_{k}$ vanish identically allows us to solve for the rest of the variables. More precisely, for $k=2,3, \ldots$, we have that
(a) $\left[R_{k}\right]_{11}$ can be solved for $a_{k-1}$ as a differential polynomial in $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$,
(b) $\left[R_{k}\right]_{12}$ can be solved for $b_{k+6}$ as a differential polynomial in $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$, and the functions $\left\{a_{j}\right\}_{j=1}^{k-2}$,
(c) $\left[R_{k}\right]_{13}$ can be solved for $c_{k+6}$ as a differential polynomial in $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$, and the functions $\left\{a_{j}\right\}_{j=1}^{k-2}$.
the symmetry constraint (4.19) implies that solving the above three equations makes $R_{k} \equiv 0$.
In particular, we obtain that

$$
\Phi_{1}=\left(\begin{array}{ccc}
-\frac{1}{2} H_{1} & 0 & 0  \tag{4.34}\\
0 & -\frac{\omega^{2}}{2} H_{1} & 0 \\
0 & 0 & -\frac{\omega}{2} H_{1}
\end{array}\right)
$$

where $\frac{d}{d x} H_{1}=-U$ is the Hamiltonian for the $x$-variable. Also,

$$
\Phi_{2}=\left(\begin{array}{ccc}
\frac{1}{8}\left(H_{1}\right)^{2}-\frac{1}{4} H_{2} & -\frac{i \omega^{2} \sqrt{3}}{12} U & \frac{i \omega \sqrt{3}}{12} U  \tag{4.35}\\
\frac{i \omega^{2} \sqrt{3}}{12} U & \omega\left(\frac{1}{8}\left(H_{1}\right)^{2}-\frac{1}{4} H_{2}\right) & -\frac{i \sqrt{3}}{12} U \\
-\frac{i \omega \sqrt{3}}{12} U & \frac{i \sqrt{3}}{12} U & \omega^{2}\left(\frac{1}{8}\left(H_{1}\right)^{2}-\frac{1}{4} H_{2}\right)
\end{array}\right),
$$

where $\frac{d}{d x} H_{2}=2 V$ is the Hamiltonian for the $t_{2}$-variable.
Direct calculation then shows that:

1. The zero-curvature equation between the $\zeta, x$ variables is equivalent to the string equation (1.1);
2. The zero-curvature equation between the $\zeta, t_{2}$ variables is equivalent to the equations (2.24), (2.25), modulo the string equation ${ }^{5}$,

[^4]3. The zero-curvature equation between the $\zeta, t_{5}$ variables is equivalent to the equations (2.26), (2.27), modulo the string equation,
4. Modulo the string equation (1.1), (2.24)-(2.27), the other zero-curvature equations between $\left(t_{5}, t_{2}\right),\left(t_{2}, x\right)$, and $\left(t_{5}, x\right)$, vanish identically. In other words, these equations result in no new differential conditions on the functions $U, V$.

Remark 4.6. We list the $1-1$ entries of the first few matrices $\Phi_{k}\left(t_{5}, t_{2}, x\right)$ here, for the convenience of the reader.

$$
\begin{aligned}
{\left[\Phi_{1}\right]_{11} } & =-\frac{1}{2} H_{1}, \\
{\left[\Phi_{2}\right]_{11} } & =\frac{1}{8}\left(H_{1}\right)^{2}-\frac{1}{4} H_{2}, \\
{\left[\Phi_{3}\right]_{11} } & =-\frac{1}{48}\left(H_{1}\right)^{3}+\frac{1}{2} H_{1} H_{2}, \\
{\left[\Phi_{4}\right]_{11} } & =\frac{1}{384}\left(H_{1}\right)^{4}+\frac{1}{32}\left(H_{1}\right)^{2} H_{2}+\frac{1}{32}\left(H_{2}\right)^{2}-\frac{5}{24} t_{5} H_{2}+\frac{1}{12} V^{\prime}-\frac{1}{96} U^{2}+\frac{1}{6} t_{2} x, \\
{\left[\Phi_{5}\right]_{11} } & =-\frac{1}{38400}\left(H_{1}\right)^{5}+\frac{1}{192}\left(H_{1}\right)^{3} H_{2}-\frac{1}{64} H_{1}\left(H_{2}\right)^{2}-\frac{1}{24}\left(V^{\prime}-\frac{1}{8} U^{2}\right) H_{1} \\
& -\frac{5}{48} t_{5} H_{1} H_{2}-\frac{1}{12} t_{2} x H_{1}+\frac{1}{90} U^{\prime \prime \prime}-\frac{1}{16} U U^{\prime}-\frac{1}{24} U V-\frac{1}{10} H_{5}, \\
{\left[\Phi_{6}\right]_{11} } & =\frac{1}{46080} H_{1}^{6}-\frac{1}{1536} H_{1}^{4} H_{2}+\frac{1}{256} H_{1}^{2} H_{2}^{2}-\frac{1}{384} H_{2}^{3}+\frac{1}{20} H_{5} H_{1}+\frac{5}{192} t_{5} H_{2} H_{1}^{2} \\
& -\frac{1}{768}\left(U^{2}-8 V^{\prime}-16 t_{2} x\right) H_{1}^{2}-\frac{5}{96} t_{5} H_{2}^{2}+\frac{1}{180}\left(\frac{15}{4} U V+\frac{45}{8} U U^{\prime}-U^{\prime \prime \prime}\right) H_{1} \\
& +\frac{1}{384}\left(U^{2}-8 V^{\prime}-16 t_{2} x\right) H_{2}+\frac{5}{192} U^{3}-\frac{1}{72} U^{\prime \prime} U+\frac{1}{12} V^{2}+\frac{1}{144}\left(U^{\prime}\right)^{2},
\end{aligned}
$$

As a final result of this subsection, we state without proof the equivalent Riemann-Hilbert formulation in the $\lambda$-plane.

Proposition 4.7. Define Stokes rays $\left\{\gamma_{k}\right\}, k= \pm 1, \ldots, \pm 7$, as shown in Figure 4.1. Explicitly, these rays are defined as

$$
\gamma_{ \pm k}:=\left\{\lambda \left\lvert\, \arg \lambda= \pm \frac{\pi}{14} \pm \frac{\pi}{7}(k-1)\right.\right\}, \quad k=1, \ldots, 7
$$

Furthermore, set $\rho:=(-\infty, 0)$; orient all of these rays outwards from the origin. Let $\left\{S_{k}, S_{-k}\right\}_{k=1, \ldots, 7}$ be a collection of constant matrices of the form given in 4.1, subject to the constraint

$$
\begin{equation*}
S_{-7} \cdots S_{-1} S_{1} \cdots S_{7}=\mathcal{S}^{T} \tag{4.36}
\end{equation*}
$$

Consider the following Riemann-Hilbert problem for a $3 \times 3$ sectionally analytic function $\Psi\left(\lambda ; t_{5}, t_{2}, x\right)$ :

$$
\begin{cases}\Psi_{+}\left(\lambda ; t_{5}, t_{2}, x\right)=\Psi_{-}\left(\lambda ; t_{5}, t_{2}, x\right) S_{k}, & \lambda \in \gamma_{k}, \quad k= \pm 1, \ldots, \pm 7  \tag{4.37}\\ \Psi_{+}\left(\lambda ; t_{5}, t_{2}, x\right)=\Psi_{-}\left(\lambda ; t_{5}, t_{2}, x\right) \mathcal{S}, & \lambda \in \rho \\ \Psi\left(\lambda ; t_{5}, t_{2}, x\right)=g(\lambda)\left[\mathbb{I}+\frac{\Phi_{1}}{\lambda^{1 / 3}}+\frac{\Phi_{2}}{\lambda^{2 / 3}}+\mathcal{O}\left(\lambda^{-1}\right)\right] e^{\Theta\left(\lambda^{1 / 3} ; t_{5}, t_{2}, x\right)}, & \lambda \rightarrow \infty\end{cases}
$$

where $g(\lambda)$ is as defined in (4.5), and $\Phi_{1}, \Phi_{2}$ are as defined in Equations (4.34), (4.35). Then, the solution to this Riemann-Hilbert problem is unique, provided the asymptotics (this includes $\Phi_{1}, \Phi_{2}!$ ) above are specified.


Figure 4.1: The Stokes lines $\gamma_{j}$, for the Riemann-Hilbert problem for $\Psi(\lambda)$. Each of the Stokes sectors is bisected by an anti-Stokes line, depicted by a dashed line. All contours are oriented outwards from the origin. Note that we have also labelled the anti-Stokes line $(-\infty, 0]$ by $\rho$. The Stokes matrix $S_{k}$ is the matrix associated to the parameter $s_{k}$; these parameters are not all independent, and must satisfy the equation $S_{-7} \cdots S_{-1} S_{1} \cdots S_{7}=\mathcal{S}^{T}$. The equivalent diagram for the $\Phi(\zeta)$-Riemann-Hilbert problem in the $\zeta$ plane has 42 rays.

Furthermore, The functions

$$
\begin{aligned}
& U\left(t_{5}, t_{2}, x\right):=2 \frac{d}{d x}\left[\Phi_{1}\left(t_{5}, t_{2}, x\right)\right]_{11} \\
& V\left(t_{5}, t_{2}, x\right):=-2 \frac{d}{d x}\left[\Phi_{2}\left(t_{5}, t_{2}, x\right)-\frac{1}{2} \Phi_{1}\left(t_{5}, t_{2}, x\right)^{2}\right]_{11}=-\frac{d}{d t_{2}}\left[\Phi_{1}\left(t_{5}, t_{2}, x\right)\right]_{11}
\end{aligned}
$$

satisfy the string equation.
Remark 4.7. It is important to note that the solution to the above Riemann-Hilbert problem is not unique unless the coefficients $\Phi_{1}, \Phi_{2}$ in the asymptotic expansion are specified. This is again a phenomenon shared by the Painlevé I system [Kap04; KK93; Fok+06], and is ultimately due to the gauge freedom arising from the resonant singularity at the origin. Since all choices of gauge lead to the same integrability condition, we are free to fix a gauge, and work in it. Fixing a gauge is equivalent to making a choice of the form of $\Phi_{1}$, $\Phi_{2}$, up to multiplication by a unimodular lower triangular matrix. This gauge freedom was first pointed out in the work of Drinfeld and Sokolov [DS85].

### 4.3. A NON-LOCAL $\mathbb{Z}_{2}$ SYMMETRY REDUCTION.

As it turns out, there is an additional (and generically, non-local) symmetry of the above Riemann-Hilbert problem, which can be identified with the $\mathbb{Z}_{2}$-parity of the magnetic field. This is summarized in the following Proposition:
Proposition 4.8. Let $\Phi\left(\zeta ; t_{5}, t_{2}, x\right)$ be a solution to the Riemann-Hilbert problem defined by Proposition 4.6. Then,

$$
\begin{equation*}
\Phi\left(-\zeta ; t_{5},-t_{2}, x\right)^{-T}=\Phi\left(\zeta ; t_{5}, t_{2}, x\right) \tag{4.38}
\end{equation*}
$$

provided the Stokes parameters satisfy

$$
\begin{equation*}
s_{k}\left(t_{5}, t_{2}, x ; U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, V, V^{\prime}\right)=-s_{k+7}\left(t_{5},-t_{2}, x ; U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime},-V,-V^{\prime}\right), \quad k \in \mathbb{Z}_{14} \tag{4.39}
\end{equation*}
$$

Proof. Note that $-\Theta\left(-\xi ; t_{5},-t_{2}, x\right)^{T}=\Theta\left(\xi ; t_{5}, t_{2}, x\right)$. This implies that the functions $\Phi\left(\zeta ; t_{5}, t_{2}, x\right)$ and $\Phi\left(-\zeta ; t_{5},-t_{2}, x\right)^{-T}$ both have the same leading-order asymptotics at infinity. Thus, if these two functions have the same jumps, then their ratio is holomorphic, and equal to the identity at infinity; the usual Liouville argument then implies that $\Phi\left(\zeta ; t_{5}, t_{2}, x\right)=\Phi\left(-\zeta ; t_{5},-t_{2}, x\right)^{-T}$. Comparing the jumps of the these two functions, we see that the Stokes matrices must satisfy

$$
S_{k}\left(t_{5}, t_{2}, x, U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, V, V^{\prime}\right)=S_{k+21}\left(t_{5},-t_{2}, x, \check{U}, \check{U}^{\prime}, \check{U}^{\prime \prime}, \check{U}^{\prime \prime \prime}, \check{V}, \check{V}^{\prime}\right)^{-T}, \quad k \in \mathbb{Z}_{42}
$$

where $\check{f}\left(t_{5}, t_{2}, x\right)=f\left(t_{5},-t_{2}, x\right)$. Using the relations (4.26), along with the formulae (4.25), this implies the following relation on the Stokes parameters:

$$
\begin{equation*}
s_{k}\left(t_{5}, t_{2}, x, U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, V, V^{\prime}\right)=-s_{k+7}\left(t_{5},-t_{2}, x, \check{U}, \check{U}^{\prime}, \check{U}^{\prime \prime}, \check{U}^{\prime \prime \prime}, \check{V}, \check{V}^{\prime}\right)^{T}, \quad k \in \mathbb{Z}_{14} \tag{4.40}
\end{equation*}
$$

In fact, the above is equivalent to (4.39). To see this, suppose that the relation (4.40) holds, and expand the solutions $\Phi\left(\zeta ; t_{5}, t_{2}, x\right), \Phi\left(-\zeta ; t_{5},-t_{2}, x\right)^{-T}$ at infinity. One finds that

$$
\begin{aligned}
\Phi\left(\zeta ; t_{5}, t_{2}, x\right) & =\left[\mathbb{I}+\frac{\Phi_{1}\left(t_{5}, t_{2}, x\right)}{\zeta}+\frac{\Phi_{2}\left(t_{5}, t_{2}, x\right)}{\zeta^{2}}+\mathcal{O}\left(\zeta^{-3}\right)\right] e^{\Theta\left(\zeta ; t_{5}, t_{2}, x\right)} \\
\Phi\left(-\zeta ; t_{5},-t_{2}, x\right)^{-T} & =\left[\mathbb{I}+\frac{\Phi_{1}^{T}\left(t_{5},-t_{2}, x\right)}{\zeta}+\frac{\Phi_{1}^{2 T}\left(t_{5},-t_{2}, x\right)-\Phi_{2}^{T}\left(t_{5},-t_{2}, x\right)}{\zeta^{2}}+\mathcal{O}\left(\zeta^{-3}\right)\right] e^{\Theta\left(\zeta ; t_{5}, t_{2}, x\right)}
\end{aligned}
$$

Equating the coefficients ${ }^{6}$, one finds that

$$
H_{1}\left(t_{5},-t_{2}, x\right)=H_{1}\left(t_{5}, t_{2}, x\right), \quad H_{2}\left(t_{5},-t_{2}, x\right)=-H_{2}\left(t_{5}, t_{2}, x\right), \quad H_{5}\left(t_{5},-t_{2}, x\right)=H_{5}\left(t_{5}, t_{2}, x\right)
$$

[^5]$$
U\left(t_{5},-t_{2}, x\right)=U\left(t_{5}, t_{2}, x\right), \quad V\left(t_{5},-t_{2}, x\right)=-V\left(t_{5}, t_{2}, x\right)
$$

In other words, $U, H_{1}, H_{5}$ are even functions of $t_{2}$, and $H_{2}, V$ are odd functions of $t_{2}$. This justifies the equivalence of (4.40) and (4.39).

As a consequence of the above Proposition, we obtain a number of important corollaries:
Corollary 4.2. If the Stokes parameters satisfy Relation (4.39), then functions $U, V$, and the Hamiltonians $H_{1}, H_{2}, H_{5}$ satisfy the following relations:

$$
\begin{gather*}
H_{1}\left(t_{5},-t_{2}, x\right)=H_{1}\left(t_{5}, t_{2}, x\right), \quad H_{2}\left(t_{5},-t_{2}, x\right)=-H_{2}\left(t_{5}, t_{2}, x\right), \quad H_{5}\left(t_{5},-t_{2}, x\right)=H_{5}\left(t_{5}, t_{2}, x\right)  \tag{4.41}\\
U\left(t_{5},-t_{2}, x\right)=U\left(t_{5}, t_{2}, x\right), \quad V\left(t_{5},-t_{2}, x\right)=-V\left(t_{5}, t_{2}, x\right) \tag{4.42}
\end{gather*}
$$

Furthermore, the Okamoto $\tau$-function, defined by $d \log \tau_{\text {Okamoto }}=H_{5} d t_{5}+H_{2} d t_{2}+H_{1} d x$, satisfies $\tau_{\text {Okamoto }}\left(t_{5},-t_{2}, x\right)=\tau_{\text {Okamoto }}\left(t_{5}, t_{2}, x\right)$.

Corollary 4.3. If the Stokes parameters satisfy Relation (4.39), and $t_{2}=0$, then $V \equiv 0, H_{2} \equiv 0$, and the string equation (1.1) reduces to

$$
\begin{equation*}
\frac{1}{12} U^{(4)}-\frac{3}{4} U^{\prime \prime} U-\frac{3}{8}\left(U^{\prime}\right)^{2}+\frac{1}{2} U^{3}-\frac{5}{12} t_{5}\left(3 U^{2}-U^{\prime \prime}\right)+x \tag{4.43}
\end{equation*}
$$

The only other nonzero part of the string equation is then

$$
\begin{equation*}
\frac{\partial U}{\partial t_{5}}=\frac{\partial}{\partial x}\left[-\frac{1}{6} U U^{\prime \prime}+\frac{1}{8}\left(U^{\prime}\right)^{2}+\frac{1}{4} U^{3}-\frac{5}{9} t_{5}\left(3 U^{2}-U^{\prime \prime}\right)+\frac{4}{3} x\right] \tag{4.44}
\end{equation*}
$$

Furthermore, the generic dimension of the Stokes manifold is reduced from 6 to 4 .
Proof. On the hyperplane $t_{2}=0$, the nonlocal equations (4.39), (4.41), and (4.42) become local. In particular, we see that

$$
V\left(t_{5}, 0, x\right)=V\left(t_{5},-0, x\right)=-V\left(t_{5}, 0, x\right)
$$

i.e. $V\left(t_{5}, 0, x\right) \equiv 0$. Similarly, we obtain that $H_{2}\left(t_{5}, 0, x\right) \equiv 0$. Finally, when $t_{2}=0$, the $\mathbb{Z}_{14}$-periodicity of the Stokes parameters further reduces to a $\mathbb{Z}_{7} \times \mathbb{Z}_{2}$-periodicity:

$$
s_{k+7}=-s_{k}
$$

and consequently the relation $S_{1} \cdots S_{14}=\mathcal{S}^{T}$ implies that the Stokes manifold is generically of dimension 4.

Remark 4.8. The generic solution to the constraint equations (4.26) on the $t_{2}=0$ hyperplane (which further implies $s_{k+7}=-s_{k}$ ) is given by

$$
\begin{equation*}
s_{5}=\frac{s_{1} s_{4}+s_{3}+1}{s_{1} s_{3}-s_{2}}, \quad s_{6}=-\frac{s_{1}\left(s_{2} s_{4}+s_{3}\right)-s_{4}\left(s_{1} s_{3}-s_{2}\right)-s_{2} s_{3}}{\left(s_{1} s_{3}-s_{2}\right)\left(s_{2} s_{4}+s_{3}\right)}, \quad s_{7}=\frac{s_{1} s_{4}-s_{2}+1}{s_{2} s_{4}+s_{3}} \tag{4.45}
\end{equation*}
$$

where $s_{1}, s_{2}, s_{3}, s_{4}$ are free parameters, and provided $s_{1} s_{3}-s_{2} \neq 0, s_{1} s_{3}-s_{2} \neq 0$. This reduction of the dimension of the Stokes manifold is consistent with the reduction of the order of the string equation from 6 to 4 .

This symmetry gives an interpretation to our previous statement that $V$ is responsible for the nonperturbative $\mathbb{Z}_{2}$ symmetry-breaking of the model: this function is only non-zero when $t_{2}$, the parameter that is identified with a shift in the magnetic field, is nonzero. We point out that this symmetry also passes through to the $\lambda$-gauge, without issue.

Remark 4.9. $\mathbb{Z}_{14}$ symmetry. Finally, we remark that the above symmetry is a special case of a more general $\mathbb{Z}_{14}$ symmetry, which is the analog of the $\mathbb{Z}_{5}$ symmetry possessed by Painlevé I [Kap88]; it is in fact just a realization of the subgroup $\mathbb{Z}_{2}$ of $\mathbb{Z}_{14}$. Let $\beta:=e^{\frac{2 \pi i}{42}}=e^{\frac{\pi i}{21}}$ denote the principal $42^{\text {nd }}$ root of unity, and note that

$$
\begin{equation*}
\mathcal{S}^{T} \Theta\left(\beta^{-1} \zeta ; \beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x\right) \mathcal{S}=-\Theta\left(\zeta ; t_{5}, t_{2}, x\right) \tag{4.46}
\end{equation*}
$$

An identical line of argumentation to the preceding section shows that $\Phi\left(\zeta ; t_{5}, t_{2}, x\right)$ satisfies the symmetry condition

$$
\begin{equation*}
\Phi\left(\zeta ; t_{5}, t_{2}, x\right)=\mathcal{S}^{T} \Phi\left(\beta^{-1} \zeta ; \beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x\right) \mathcal{S} \tag{4.47}
\end{equation*}
$$

Provided that the Stokes parameters satisfy

$$
\begin{equation*}
s_{k}\left(t_{5}, t_{2}, x ; U, U^{\prime}, U^{\prime \prime}, U^{\prime \prime \prime}, V, V^{\prime}\right)=-s_{k+1}\left(\beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x ; \check{U}, \check{U}^{\prime}, \check{U}^{\prime \prime}, \check{U}^{\prime \prime \prime}, \check{V}, \check{V}^{\prime}\right) \tag{4.48}
\end{equation*}
$$

where here $\check{f}\left(t_{5}, t_{2}, x\right)=f\left(\beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x\right)$. We also have the following relations:

$$
\begin{aligned}
& \beta^{2} U\left(\beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x\right)=U\left(t_{5}, t_{2}, x\right) \\
& \beta^{3} V\left(\beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x\right)=V\left(t_{5}, t_{2}, x\right)
\end{aligned}
$$

and finally that the Okamoto $\tau$-function satisfies

$$
\begin{equation*}
\tau_{\text {Okamoto }}\left(\beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x\right)=\tau_{\text {Okamoto }}\left(t_{5}, t_{2}, x\right) \tag{4.49}
\end{equation*}
$$

Based on the appearance of a $42^{n d}$ root of unity, one might be tempted to think that this system possesses a full $\mathbb{Z}_{42}=\mathbb{Z}_{7} \oplus \mathbb{Z}_{3} \oplus \mathbb{Z}_{2}$ symmetry group. In fact, as we shall now show, the subgroup $\mathbb{Z}_{3}$ appears in a trivial manner, and thus does not play a role. Let us denote by $\chi$ the operation of acting by this symmetry on $\Phi$, i.e. the map

$$
\begin{equation*}
\chi\left[\Phi\left(\zeta ; t_{5}, t_{2}, x\right)\right]:=\mathcal{S}^{T} \Phi\left(\beta^{-1} \zeta ; \beta^{12} t_{5}, \beta^{9} t_{2}, \beta^{-6} x\right) \mathcal{S} . \tag{4.50}
\end{equation*}
$$

Clearly, $\chi^{42}=1$, the identity map on $\Phi$. Note that $\chi^{6}$ is the generator of the subgroup $\mathbb{Z}_{7}, \chi^{14}$ is the generator of the subgroup $\mathbb{Z}_{3}, \chi^{21}$ is the generator of the subgroup $\mathbb{Z}_{2}$. Let us first see that $\chi^{14}=1$, the identity map. If we apply $\chi^{14}$, we obtain that

$$
\chi^{14}\left[\Phi\left(\zeta ; t_{5}, t_{2}, x\right)\right]=\mathcal{S}^{T} \Phi\left(\omega \zeta ; t_{5}, t_{2}, x\right) \mathcal{S}=\Phi\left(\zeta ; t_{5}, t_{2}, x\right)
$$

as we already observed in Subsection 4.1. So, the subgroup $\mathbb{Z}_{3}$ does not participate, and there is generically a $\mathbb{Z}_{14}$ symmetry acting on the solutions.

Note further that, if we apply the generator of the subgroup $\mathbb{Z}_{2}$ to $\Phi$, we obtain that

$$
\begin{equation*}
\chi^{21}\left[\Phi\left(\zeta ; t_{5}, t_{2}, x\right)\right]=\Phi\left(-\zeta ; t_{5},-t_{2}, x\right)^{-T} \tag{4.51}
\end{equation*}
$$

which is precisely the $\mathbb{Z}_{2}$ symmetry described in this subsection. The $\mathbb{Z}_{7}$ symmetry is nontrivial, i.e. the generator of this subgroup $\chi^{6}$ acts nontrivially on $\Phi$. However, there is no clear simplification or physical interpretation of this symmetry, as was the case for the $\mathbb{Z}_{2}$ subgroup.

## 5. The Isomonodromic and Extended $\tau$-Function.

Associated to almost any linear differential equation with rational coefficients is an object called the isomonodromic $\tau$-function. The $\tau$-function has many important properties. For example, its zeros determine where the inverse monodromy problem for the associated linear equation are not solvable [Mal83; Pal99]. Furthermore, the $\tau$-function itself is often the object that appears in many physical applications; this is also the case for the multi-critical quartic 2-matrix model. However, the word "almost" is the antagonist in this story. As we have seen, in many cases of interest, the leading coefficient of the singularity of the connection matrix is not diagonalizable, or, if we make a change of gauge, a resonant Fuchsian singularity manifests at the origin. In either case, the theory introduced in [JMU81] is not applicable. This motivates us to give a modified
definition of the $\tau$-differential. Most of this section can be read completely independently of the rest of this work.

In this first part of this section, we will work in slightly more generality, in order to show that our definition is indeed a sensible extension of the $\tau$-function, as defined by Jimbo, Miwa, and Ueno. Our definition is meant to address the case of the general $(p, q)$ string equations, which all share the feature that 1. the leading term of the polynomial connection matrix $A(\lambda)$ is not diagonalizable, and 2 . in a suitably regularized gauge, the connection matrix develops a resonant Fuchsian singularity at the origin.

We divide this section into the following parts: in Subsection 5.1, we lay out a set of assumptions for a model problem with a single non-diagonalizable singularity at infinity (or, equivalently, a resonant Fuchsian singularity at 0 ), for which we will define a suitable $\tau$-differential. In Subsection 5.2 , we will see the shortcoming of the original JMU definition, and show that the modified definition of the $\tau$-function (up to an irrelevant constant factor) indeed makes sense, and coincides with the Okamoto $\tau$-function (3.13) in the settings of the rest of this work. In Section 5.3, we extend the $\tau$-function to the initial data of the associated Hamiltonian system (cf. Proposition 3.1), and verify Conjectures 1. and 2. of [IP18] for the system at hand.

We adopt the following set of notations. First, let $q \geq 2$. If $X: \mathbb{C} \rightarrow M_{q}(\mathbb{C})$ is a matrix-valued function which admits a Laurent expansion at $\lambda=\infty$, we define

$$
\begin{equation*}
\langle X(\lambda)\rangle:=\underset{\lambda=\infty}{\operatorname{Res}} \operatorname{tr} X(\lambda) d \lambda \tag{5.1}
\end{equation*}
$$

We list some of the key properties of $\langle\cdot\rangle$ below, which the reader may readily check:

1. (Cyclicity) $\langle X(\lambda) Y(\lambda)\rangle=\langle Y(\lambda) X(\lambda)\rangle$,
2. (Integration by Parts) $\left\langle\frac{\partial}{\partial \lambda} X(\lambda)\right\rangle=0$, and, consequentially, $\left\langle X^{\prime}(\lambda) Y(\lambda)\right\rangle=-\left\langle X(\lambda) Y^{\prime}(\lambda)\right\rangle$.
3. If $X=X(\lambda ; \mathbf{t})$ depends on additional parameters $\mathbf{t}$, and $\mathbf{d}$ denotes the exterior differential in these parameters, then $\mathbf{d}\langle X(\lambda ; \mathbf{t})\rangle=\langle\mathbf{d} X(\lambda ; \mathbf{t})\rangle$.
4. If $A$ is a constant (in $\lambda$ ) matrix, then $\left\langle A \lambda^{-k-1}\right\rangle=(\operatorname{tr} A) \delta_{k, 0}$. In particular, this implies that if $X(\lambda)$ is a polynomial, then $\langle X(\lambda)\rangle=0$.
5. (Ad-invariance) If $A, B, C$ are matrix-valued functions, then $\langle A[B, C]\rangle=\langle[A, B] C\rangle$.

We now list a set of assumptions on which the remainder of our calculation will be based. The motivation for this assumptions comes from what one should expect out of the $(p, q)$ string equation in general. When restricted to the case $q=3, p=4$, these assumptions coincide with what we have derived in the preceding sections of the present work.

### 5.1. MAIN ASSUMPTIONS FOR THE MODEL PROBLEM.

Given $q \geq 2$, we fix an integer $p$ coprime to $q$. Setting $\omega_{q}:=e^{\frac{2 \pi i}{q}}$, we then define $q \times q$ matrices $\Delta_{q}, \mathcal{U}_{q}$ as follows:

$$
\begin{align*}
\left(\Delta_{q}\right)_{i j} & :=\frac{1}{2}\left(1-\frac{2 j-1}{q}\right) \delta_{i j},  \tag{5.2}\\
\left(\mathcal{U}_{q}\right)_{i j} & := \begin{cases}\omega_{q}^{\frac{1}{2}(q-2 i+1)(j-1)}, & q \text { odd } \\
\omega_{q}^{\frac{1}{2}(q-2 i)(j-1)+1}, & \text { q even }\end{cases} \tag{5.3}
\end{align*}
$$

where $i, j=1, \ldots, q$. As well as the shift matrix

$$
\mathcal{S}_{q}:=\left(\begin{array}{cccccc}
0 & 1 & 0 & \ldots & 0 & 0  \tag{5.4}\\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & : & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & \cdots & 0
\end{array}\right) \omega_{q}^{\frac{1}{2} \delta_{q}},
$$

where $\delta_{q}=0$ if $q$ is odd, and $\delta_{q}=1$ if $q$ is even. (Note that $\mathcal{S}_{q}$ is unitary: $\mathcal{S}_{q}^{\dagger}=\mathcal{S}_{q}^{-1}$ ). Finally, we define functions $\vartheta_{j}^{(q, p)}(\lambda ; \mathbf{t})$ as

$$
\begin{equation*}
\vartheta_{j}^{(q, p)}(\lambda ; \mathbf{t}):=\frac{q}{p+q} \omega_{q}^{(j-1) p} \lambda^{\frac{p+q}{q}}+\sum_{\substack{\ell=1 \\ \ell \\ \bmod q \neq 0}}^{p+q-1} t_{\ell} \omega_{q}^{(j-1) \ell} \lambda^{\ell / q} \tag{5.5}
\end{equation*}
$$

for $j=1, \ldots, q$. Subsequently, we define the matrix-valued functions

$$
\begin{equation*}
g_{q}(\lambda):=\lambda^{\Delta_{q}} \mathcal{U}_{q}, \quad \Theta(\lambda ; \mathbf{t}):=\operatorname{diag}\left(\vartheta_{1}^{(q, p)}(\lambda ; \mathbf{t}), \cdots, \vartheta_{q}^{(q, p)}(\lambda ; \mathbf{t})\right) \tag{5.6}
\end{equation*}
$$

If we denote $\Theta_{a}:=\frac{\partial \Theta}{\partial t_{a}}$, we can see that the conditions

$$
\begin{equation*}
\frac{\partial \Theta_{a}}{\partial t_{b}}-\frac{\partial \Theta_{b}}{\partial t_{a}}=\left[\Theta_{a}, \Theta_{b}\right]=0 \tag{5.7}
\end{equation*}
$$

hold trivially. By construction, these matrices have jumps on the negative real axis (with orientation taken outwards), given by:

Lemma 5.1. For $\lambda<0$,

$$
\begin{equation*}
g_{q,+}(\lambda)=g_{q,-}(\lambda) \mathcal{S}_{q}, \quad \Theta_{+}(\lambda ; \mathbf{t})=\mathcal{S}_{q}^{\dagger} \Theta_{-}(\lambda ; \mathbf{t}) \mathcal{S}_{q} \tag{5.8}
\end{equation*}
$$

The proof is a direct computation with the formulae given above, and so we omit it. Consider the following lemma:
Lemma 5.2. Let $g_{q}(\lambda)$ be an $\mathrm{SL}_{q}(\mathbb{C})$-valued function on $\mathbb{C} \backslash(-\infty, 0]$ such that $g_{q,+}(\lambda)=g_{q,-}(\lambda) \mathcal{S}_{q}$, where $\mathcal{S} \in \mathrm{SL}_{d}(\mathbb{C})$. Consider the series

$$
\begin{equation*}
R(\lambda):=\mathbb{I}+\sum_{m=1}^{\infty} \frac{\Psi_{m}}{\lambda^{m / q}} . \tag{5.9}
\end{equation*}
$$

Then, the function

$$
\begin{equation*}
\hat{R}(\lambda):=g_{q}(\lambda) R(\lambda) g_{q}^{-1}(\lambda) \tag{5.10}
\end{equation*}
$$

is holomorphic in a neighborhood of infinity if and only if the coefficients $\Psi_{m}$ satisfy the symmetry relation

$$
\begin{equation*}
\Psi_{m}=\omega_{q}^{-m} \mathcal{S}_{q}^{-1} \Psi_{m} \mathcal{S}_{q} \tag{5.11}
\end{equation*}
$$

Proof. Set $\Psi_{0}:=\mathbb{I}$. For each $r=0, \ldots, q-1$, set

$$
R_{r}(\lambda):=\sum_{k=0}^{\infty} \Psi_{k q+r} \lambda^{-k}
$$

Then, $R(\lambda)$ can be rewritten as

$$
R(\lambda)=\sum_{r=0}^{q-1} R_{r}(\lambda) \lambda^{r / q}
$$

The functions $R_{r}(\lambda)$ are analytic at infinity, and satisfy the relation $R_{r}(\lambda)=\omega_{q}^{r} \mathcal{S}_{q}^{-1} R_{r}(\lambda) \mathcal{S}_{q}$, since $\omega_{q}^{k q+r}=$ $\omega_{q}^{r}$. Since $\lambda_{+}^{r / q}=\lambda_{-}^{r / q} \omega_{q}^{r}$, it follows that

$$
\left[R_{r}(\lambda) \lambda^{r / q}\right]_{+}=\mathcal{S}_{q}^{-1}\left[R_{r}(\lambda) \lambda^{r / q}\right]_{-} \mathcal{S}_{q}
$$

for each $r=0, \ldots, q-1$, and thus that

$$
R_{+}(\lambda)=\mathcal{S}_{q}^{-1} R_{-}(\lambda) \mathcal{S}_{q}
$$

Therefore,

$$
\hat{R}_{+}(\lambda)=g_{+}(\lambda) R_{+}(\lambda) g_{+}^{-1}(\lambda)=g_{-}(\lambda) \mathcal{S}_{q} \mathcal{S}_{q}^{-1} R_{-}(\lambda) \mathcal{S}_{q} \mathcal{S}_{q}^{-1} g_{-}^{-1}(\lambda)=g_{-}(\lambda) R_{-}(\lambda) g_{-}^{-1}(\lambda)
$$

and thus $\hat{R}(\lambda)$ has no jumps near $\lambda=\infty$. Thus, $\hat{R}(\lambda)$ extends to a holomorphic function in a neighborhood of infinity. Reading the above proof from bottom to top yields the other direction of the lemma.

With this lemma in mind, we are motivated to define the function

$$
\begin{equation*}
\Psi(\lambda ; \mathbf{t}):=g(\lambda)\left[\mathbb{I}+\frac{\Psi_{1}(\mathbf{t})}{\lambda^{1 / q}}+\frac{\Psi_{2}(\mathbf{t})}{\lambda^{2 / q}}+\mathcal{O}\left(\lambda^{-3 / q}\right)\right] e^{\Theta(\lambda ; \mathbf{t})} \tag{5.12}
\end{equation*}
$$

where the coefficients $\Psi_{k}(\mathbf{t})$ satisfy the symmetry constraint $\omega_{q}^{-k} \mathcal{S}_{q}^{\dagger} \Psi_{k}(\mathbf{t}) \mathcal{S}_{q}=\Psi_{k}(\mathbf{t})$. This formal series therefore satisfies the jump condition

$$
\Psi_{+}(\lambda ; \mathbf{t})=\Psi_{-}(\lambda ; \mathbf{t}) \mathcal{S}_{q}, \quad \lambda<0
$$

as a consequence of the above lemma. Similarly, if we define the function $G(\lambda ; \mathbf{t}):=\Psi(\lambda ; \mathbf{t}) e^{-\Theta(\lambda ; \mathbf{t})}$, we can see that $G_{+}(\lambda ; \mathbf{t})=G_{-}(\lambda ; \mathbf{t}) \mathcal{S}_{q}$, for $\lambda<0$. We assert that $\Psi(\lambda ; \mathbf{t})$ is the (formal) solution to the following collection of differential equations

$$
\begin{equation*}
\frac{\partial \Psi}{\partial \lambda}=A(\lambda ; \mathbf{t}) \Psi(\lambda ; \mathbf{t}), \quad \frac{\partial \Psi}{\partial t_{\ell}}=B_{\ell}(\lambda ; \mathbf{t}) \Psi(\lambda ; \mathbf{t}), \quad \ell=1, \ldots, p+q-1, \ell \quad \bmod q \not \equiv 0 \tag{5.13}
\end{equation*}
$$

Here, all matrices $A(\lambda ; \mathbf{t}), B_{\ell}(\lambda ; \mathbf{t})$ are assumed to be polynomials in $\lambda$. Furthermore, by formal differentiation of the series (5.12), one can deduce that the leading coefficient of $A(\lambda ; \mathbf{t})$ is (for $p=k q+r$ )

$$
\begin{equation*}
A(\lambda ; \mathbf{t})=\Lambda^{r} \lambda^{k}+\cdots \tag{5.14}
\end{equation*}
$$

where $\Lambda=\Lambda(\lambda)$ is the matrix

$$
\Lambda(\lambda):=\left(\begin{array}{cccccc}
0 & 0 & \cdots & 0 & 0 & \lambda  \tag{5.15}\\
1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0
\end{array}\right)
$$

In particular, it is apparent that the leading coefficient of $A(\lambda ; \mathbf{t})$ is not diagonalizable. As we performed in the case of the $(3,4)$ string equation, if we perform a gauge transformation $\lambda=\zeta^{q}, \Psi=g_{q} \Phi$, then the transformed connection matrices have the following properties:

Proposition 5.1. Under the change of gauge $\lambda=\zeta^{q}, \Psi=g_{q} \Phi$, the matrix $\Phi$ satisfies the differential equations

$$
\begin{equation*}
\frac{\partial \Phi}{\partial \zeta}=\hat{A}(\zeta ; \mathbf{t}) \Phi(\zeta ; \mathbf{t}), \quad \frac{\partial \Phi}{\partial t_{\ell}}=\hat{B}_{\ell}(\zeta ; \mathbf{t}) \Phi(\zeta ; \mathbf{t}), \tag{5.16}
\end{equation*}
$$

for $\ell=1, . ., p+q-1, \ell \bmod q \not \equiv 0$, where the matrices $\hat{A}(\zeta ; \mathbf{t}), \hat{B}_{\ell}(\zeta ; \mathbf{t})$ are given by

$$
\begin{array}{r}
\hat{A}(\zeta ; \mathbf{t})=q \zeta^{q-1} \tilde{A}\left(\zeta^{q} ; \mathbf{t}\right) \\
\hat{B}_{\ell}(\zeta ; \mathbf{t})=g_{q}^{-1} B_{\ell}\left(\zeta^{q} ; \mathbf{t}\right) g_{q} \tag{5.18}
\end{array}
$$

where $\tilde{A}(\lambda ; \mathbf{t})=g_{q}^{-1} A(\lambda ; \mathbf{t}) g_{q}-g_{q}^{-1} \frac{d g_{q}}{d \lambda}$.
This proposition is a direct analog of Proposition 4.2, and so we omit the proof. The only point we want to emphasize about these matrices is that:

1. For $\zeta \rightarrow \infty, A(\zeta ; \mathbf{t})$ has asymptotics

$$
A(\zeta ; \mathbf{t})=q\left(\begin{array}{ccc}
1 & 0 & \cdots  \tag{5.19}\\
0 & \omega_{q}^{p} & \cdots \\
\vdots & \vdots & \ddots
\end{array}\right) \zeta^{p+q-1}+\mathcal{O}\left(\zeta^{p+q-2}\right)
$$

so we have indeed 'regularized' the singular point at infinity: the leading term is diagonal.
2. The matrix $A(\zeta ; \mathbf{t})$ has a first order pole at $\zeta=0$, which arises from the term $g_{q}^{-1} \frac{d g_{q}}{d \lambda}$. Note that this term does not depend on the coefficients of the matrix $A(\lambda ; \mathbf{t})$, and thus can be computed explicitly:

$$
\begin{equation*}
\lim _{\zeta \rightarrow 0} \zeta \hat{A}(\zeta ; \mathbf{t})=-q \mathcal{U}^{-1} \Delta \mathcal{U} \tag{5.20}
\end{equation*}
$$

where $\Delta$ was the diagonal matrix from before (cf. Equation (5.2)). We have that $q \Delta_{j j}-q \Delta_{i i}=j-i$, and

$$
\max _{i, j} q\left|\Delta_{j j}-\Delta_{i i}\right|=q-1
$$

Crucially, we observe that this singularity is resonant.
3. Due to the form of $g_{q}(\lambda(\zeta))$, the matrices $B_{\ell}(\zeta ; \mathbf{t})$ develop poles of order at most $q-1$ at $\zeta=0$.
4. The zero-curvature equations hold in the $\zeta$-gauge as well (this is just the trivial observation that the zero-curvature equations hold, independent of the choice of coordinate system).

Thus, the above isomonodromic system has analogous complications as the Painlevé I system in [JMU81] and the $(3,4)$ string equation discussed in this work. To conclude this section, let us summarize our main assumptions about the system we will be studying.

- Assumption 1. We are given a matrix-valued formal series $\Psi(\lambda ; \mathbf{t})$ of the form (5.12).
- Assumption 2. $\Psi(\lambda ; \mathbf{t})$ satisfies the differentials equations (5.13), for polynomial matrices $A(\lambda ; \mathbf{t})$, $B(\lambda ; \mathbf{t})$.
- Assumption 3. The zero-curvature equations $\frac{\partial A}{\partial t_{\ell}}-\frac{\partial B_{\ell}}{\partial \lambda}+\left[A, B_{\ell}\right]=0, \frac{\partial B_{r}}{\partial t_{\ell}}-\frac{\partial B_{\ell}}{\partial t_{r}}+\left[B_{r}, B_{\ell}\right]=0$, hold. One can see that the system we are studying in the present work emerges when we specialize to $q=3, p=4$.


### 5.2. Modification of $\omega_{J M U}$.

Before proceeding to discuss our modification of the $\tau$-differential, let us clarify why there is a need for such a modification. First, if we start with a connection matrix $A(\lambda ; \mathbf{t})$ whose leading term is not diagonalizable, then the standard definition given by Jimbo, Miwa and Ueno fails to hold. One can then attempt to transform into a gauge which resolves this problem, as we have discussed. In this new gauge, the situation can be treated by [JMU81] at $\zeta=\infty$. However, one finds that a new problem arises at $\zeta=0$ : a resonant Fuchsian singularity emerges, which again brings us out of the context of the work of [JMU81]. In the literature for Painlevé I [LR17; IP18], this problem is typically surmounted by simply ignoring any contributions from the resonant singularity, and one is able to proceed without further complications. However, in the present situation (and also the situation we outlined in the previous subsection), we are not afforded this luxury. Indeed, if we transform into the $\zeta$-gauge and directly apply the definition of [JMU81] to the connection (4.7), simply ignoring the contribution from the resonant singularity, we find that

$$
\begin{align*}
\mathbf{d} \omega_{J M U} & =\left(U^{3}+3 V^{2}+\frac{1}{4}\left(U^{\prime}\right)^{2}-\frac{1}{2} U U^{\prime \prime}+\frac{5}{2} t_{5}\left(\frac{1}{3} U^{\prime \prime}-U^{2}\right)+2 x\right) d t_{5} d t_{1} \\
& +\left(U^{\prime \prime} V-3 U^{2} V+2 t_{2} U+\frac{25}{3} t_{5}^{2} V+\frac{10}{3} t_{5} t_{2}\right) d t_{5} d t_{2} \neq 0 . \tag{5.21}
\end{align*}
$$

Of course, one can simply add to $\omega_{J M U}$ the differential

$$
\begin{equation*}
\alpha=-\frac{1}{3}\left(\frac{1}{6} U^{\prime \prime \prime}-U U^{\prime}\right) d t_{5}=-\frac{1}{3} \frac{\partial V}{\partial t_{2}} d t_{5} \tag{5.22}
\end{equation*}
$$

whose differential is precisely $-\mathbf{d} \omega_{J M U}$, so that this new, modified differential is indeed closed. However, it is not obvious where this term arises from, or how to treat closely related systems apart from ad-hoc analysis.

The aim of this subsection is to provide a general definition of a modified $\tau$-differential $\hat{\omega}_{J M U}$ for systems of the form discussed in the previous subsection, which has the following properties:

1. The modified differential is closed: $\mathbf{d} \hat{\omega}_{J M U}=0$,
2. When there are no resonant Fuchsian singularities, the modified differential coincides with the definition given in [JMU81]: $\hat{\omega}_{J M U}=\omega_{J M U}$.
With this in mind, we define the modified $\tau$-differential to be

## Definition 5.1.

$$
\begin{equation*}
\hat{\omega}_{J M U}=\sum_{\ell}\left(\left\langle A(\lambda ; \mathbf{t}) \frac{d G}{d t_{\ell}} G^{-1}\right\rangle-\left\langle\frac{\Delta_{q}}{\lambda} \frac{d G}{d t_{\ell}} G^{-1}\right\rangle\right) d t_{\ell} \tag{5.23}
\end{equation*}
$$

Equivalently, expressed in terms of local quantities in the $\zeta$-gauge,

$$
\begin{equation*}
\hat{\omega}_{J M U}=\sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \frac{d S}{d t_{\ell}} S^{-1}\right\rangle d t_{\ell} \tag{5.24}
\end{equation*}
$$

where $S(\zeta ; \mathbf{t})=\mathbb{I}+\frac{\Psi_{1}(\mathbf{t})}{\zeta}+\frac{\Psi_{2}(\mathbf{t})}{\zeta^{2}}+\mathcal{O}\left(\zeta^{-3}\right)$, where the residue is now taken at $\zeta=\infty$.
Let us remark that both definitions indeed make sense, in that the terms inside the brackets $\langle\cdot\rangle$ are formal Laurent series in $\lambda$ (respectively, $\zeta$ ). This requires no commentary in the latter case. In the former case, this is slightly more subtle: note that $\frac{d G}{d t_{\ell}}$ and $G$ both have jumps only on the left. Since the jump matrix for $G$ is constant, we see that the ratio $\frac{d G}{d t_{\ell}} G^{-1}$ is single-valued near infinity, and thus admits a Laurent series expansion there.

Our first important observation is that this definition agrees with the definition given in [JMU81] in the case when the resonant Fuchsian singularity at the origin vanishes (equivalently, when the leading term of $A(\lambda ; \mathbf{t})$ is diagonalizable). Recall that, if $A(\lambda ; \mathbf{t})$ is a polynomial in $\lambda$ with diagonalizable leading term, then we can write a formal series solution to the differential equation $\frac{\partial \Psi}{\partial \lambda}=A(\lambda ; \mathbf{t}) \Psi$ as

$$
\Psi(\lambda ; \mathbf{t})=\underbrace{\left[\mathbb{I}+\frac{\Psi_{1}(\mathbf{t})}{\lambda}+\frac{\Psi_{2}(\mathbf{t})}{\lambda^{2}}+\mathcal{O}\left(\lambda^{-3}\right)\right]}_{G(\lambda ; \mathbf{t})} e^{\Theta(\lambda ; \mathbf{t})}
$$

where $\Theta$ is a diagonal matrix whose entries are polynomials in $\lambda$. The definition of the Jimbo-Miwa-Ueno $\tau$-differential is then

$$
\begin{equation*}
\omega_{J M U}:=-\sum_{\ell}\left\langle G^{-1} \frac{d G}{d \lambda} \frac{\partial \Theta}{\partial t_{\ell}}\right\rangle d t_{\ell} \tag{5.25}
\end{equation*}
$$

An alternative, equivalent expression given later (cf. [ILP18; IP18]) is

$$
\begin{equation*}
\omega_{J M U}=\sum_{\ell}\left\langle A(\lambda ; \mathbf{t}) \frac{\partial G}{\partial t_{\ell}} G^{-1}\right\rangle d t_{\ell} . \tag{5.26}
\end{equation*}
$$

Comparing this definition to our definition of $\hat{\omega}_{J M U}$, we see that $\hat{\omega}_{J M U}$ indeed reduces to $\omega_{J M U}$ when we are in a standard case.

It remains to see that $\mathbf{d} \hat{\omega}_{J M U}=0$. We establish this through a sequence of lemmas.

## Lemma 5.3.

$$
\begin{equation*}
\hat{\omega}_{J M U}=\omega_{J M U}+\sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \hat{B}_{\ell}(\zeta ; \mathbf{t})\right\rangle d t_{\ell} \tag{5.27}
\end{equation*}
$$

where the residue here is taken at $\zeta=\infty$.
Proof. The proof mimics the calculation that $\hat{\omega}_{J M U}=\omega_{J M U}$ in the case where the matrices $\hat{A}(\zeta ; \mathbf{t}), \hat{B}_{\ell}(\zeta ; \mathbf{t})$ are all polynomials; we must take care to make sure that every step pushes through. Let us expand the expression

$$
\sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \hat{B}_{\ell}(\zeta ; \mathbf{t})\right\rangle d t_{\ell}
$$

note that when the Lax matrices are polynomials, this expression vanishes identically. Now, since $\Phi(\zeta ; \mathbf{t})=$ $S(\zeta ; \mathbf{t}) e^{\Theta(\zeta ; \mathbf{t})}$, we can rearrange the identities $\frac{\partial \Phi}{\partial \zeta}=\hat{A} \Phi, \frac{\partial \Phi}{\partial t_{\ell}}=\hat{B}_{\ell} \Phi$ to read $\hat{A}=S \Theta_{\zeta} S^{-1}+\frac{\partial S}{\partial \zeta} S^{-1}$, $\hat{B}_{\ell}=$ $S \Theta_{\ell} S^{-1}+\frac{\partial S}{\partial t_{\ell}} S^{-1}$. Inserting these expressions into our previous identity, we find that

$$
\begin{aligned}
\sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \hat{B}_{\ell}(\zeta ; \mathbf{t})\right\rangle d t_{\ell} & =\sum_{a}\left\langle\hat{A} \hat{B}_{\ell}\right\rangle d t_{\ell}=\sum_{\ell}\left\langle\left(S \Theta_{\zeta} S^{-1}+\frac{\partial S}{\partial \zeta} S^{-1}\right)\left(S \Theta_{\ell} S^{-1}+\frac{\partial S}{\partial t_{\ell}} S^{-1}\right)\right\rangle d t_{\ell} \\
& =\sum_{\ell}[\underbrace{\left\langle\Theta_{\zeta} \Theta_{\ell}\right\rangle}_{\text {polynom. }}+\left\langle\frac{\partial S}{\partial \zeta} S^{-1} \frac{\partial S}{\partial t_{\ell}} S^{-1}\right\rangle+\underbrace{\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \Theta_{\ell}\right\rangle}_{-\left[\omega_{J M U}\right]_{\ell}}+\left\langle\Theta_{\zeta} S^{-1} \frac{\partial S}{\partial t_{\ell}}\right\rangle] d t_{\ell} \\
& =-\omega_{J M U}+\sum_{\ell}\left[\left\langle\frac{\partial S}{\partial \zeta} S^{-1} \frac{\partial S}{\partial t_{\ell}} S^{-1}\right\rangle+\left\langle\Theta_{\zeta} S^{-1} \frac{\partial S}{\partial t_{\ell}}\right\rangle\right] d t_{\ell} \\
& =-\omega_{J M U}+\sum_{\ell}\left\langle\left(S \Theta_{\zeta} S^{-1}+\frac{\partial S}{\partial \zeta} S^{-1}\right) \frac{\partial S}{\partial t_{\ell}} S^{-1}\right\rangle d t_{\ell} \\
& =-\omega_{J M U}+\sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \frac{\partial S}{\partial t_{\ell}} S^{-1}\right\rangle d t_{\ell}
\end{aligned}
$$

Lemma 5.4. For the system on $\Phi(\zeta ; \mathbf{t})$,

$$
\begin{equation*}
\mathbf{d} \omega_{J M U}=\sum\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \hat{B}_{b}\right\rangle d t_{a} \wedge d t_{b} \tag{5.28}
\end{equation*}
$$

(Note that, since $\hat{B}_{a}$ are no longer polynomials in the $\zeta$-gauge, the expression on the right hand side does not necessarily vanish).

Proof. Let $\omega_{a}$ denote the coefficient of $d t_{a}$ in $\omega_{J M U}$. We shall first calculate $\frac{\partial \omega_{a}}{\partial t_{b}}$. By direct calculation,

$$
\frac{\partial \omega_{a}}{\partial t_{b}}=\left\langle S^{-1} \frac{\partial S}{\partial t_{b}} S^{-1} \frac{\partial S}{\partial \zeta} \Theta_{a}\right\rangle-\left\langle S^{-1} \frac{\partial^{2} S}{\partial t_{b} \partial \zeta} \Theta_{a}\right\rangle-\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \frac{\partial \Theta_{a}}{\partial t_{b}}\right\rangle
$$

Now, the equation $\frac{\partial \Phi}{\partial t_{b}}=\hat{B}_{b}(\zeta ; \mathbf{t}) \Phi$ implies that $\frac{\partial S}{\partial t_{b}}=\hat{B}_{b} S-S \Theta_{b}$. So, we can rewrite the above as

$$
\begin{aligned}
\frac{\partial \omega_{a}}{\partial t_{b}} & =\left\langle S^{-1} \hat{B}_{b} \frac{\partial S}{\partial \zeta} \Theta_{a}\right\rangle-\left\langle\Theta_{b} S^{-1} \frac{\partial S}{\partial \zeta} \Theta_{a}\right\rangle-\left\langle S^{-1} \frac{\partial}{\partial \zeta}\left[\hat{B}_{b} S-S \Theta_{b}\right] \Theta_{a}\right\rangle-\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \frac{\partial \Theta_{a}}{\partial t_{b}}\right\rangle \\
& =-\left\langle\Theta_{b} S^{-1} \frac{\partial S}{\partial \zeta} \Theta_{a}\right\rangle-\left\langle S^{-1} \frac{\partial \hat{B}_{b}}{\partial \zeta} S \Theta_{a}\right\rangle+\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \Theta_{b} \Theta_{a}\right\rangle+\left\langle\frac{\partial \Theta_{b}}{\partial \zeta} \Theta_{a}\right\rangle-\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \frac{\partial \Theta_{a}}{\partial t_{b}}\right\rangle
\end{aligned}
$$

Using the identity

$$
\begin{equation*}
\left\langle\frac{\partial \Theta_{b}}{\partial \zeta} \Theta_{a}\right\rangle=\left\langle\frac{\partial}{\partial \zeta}\left(S \Theta_{b} S^{-1}\right) S \Theta_{a} S^{-1}\right\rangle+\left\langle S^{-1} \frac{\partial S}{\partial \zeta}\left[\Theta_{a}, \Theta_{b}\right]\right\rangle \tag{5.29}
\end{equation*}
$$

cyclicity, and representing $S \Theta_{a} S^{-1}=\hat{B}_{a}-\frac{\partial S}{\partial t_{a}} S^{-1}$, the above can be arranged to read

$$
\begin{aligned}
\frac{\partial \omega_{a}}{\partial t_{b}} & =-\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \frac{\partial \Theta_{a}}{\partial t_{b}}\right\rangle-\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} S \Theta_{a} S^{-1}\right\rangle+\left\langle\frac{\partial}{\partial \zeta}\left(S \Theta_{b} S^{-1}\right) S \Theta_{a} S^{-1}\right\rangle \\
& =-\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \frac{\partial \Theta_{a}}{\partial t_{b}}\right\rangle-\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle+\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle+\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} B_{a}\right\rangle-\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle \\
& -\left\langle\frac{\partial}{\partial \zeta}\left(\frac{\partial S}{\partial t_{b}} S^{-1}\right) B_{a}\right\rangle+\left\langle\frac{\partial}{\partial \zeta}\left(\frac{\partial S}{\partial t_{b}} S^{-1}\right) \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle
\end{aligned}
$$

Using integration by parts on the second to last term, we obtain that

$$
\frac{\partial \omega_{a}}{\partial t_{b}}=-\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \frac{\partial \Theta_{a}}{\partial t_{b}}\right\rangle+\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \frac{\partial S}{\partial t_{b}} S^{-1}\right\rangle+\left\langle\frac{\partial}{\partial \zeta}\left(\frac{\partial S}{\partial t_{b}} S^{-1}\right) \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle
$$

Now, the argument of the last term is of order $\mathcal{O}\left(\zeta^{-2}\right)$, and thus has no residue. So, our final expression for $\frac{\partial \omega_{a}}{\partial t_{b}}$ is

$$
\frac{\partial \omega_{a}}{\partial t_{b}}=\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \frac{\partial S}{\partial t_{b}} S^{-1}\right\rangle-\left\langle S^{-1} \frac{\partial S}{\partial \zeta} \frac{\partial \Theta_{a}}{\partial t_{b}}\right\rangle
$$

Interchanging the roles of $a$ and $b$ allows one to compute $\frac{\partial \omega_{b}}{\partial t_{a}}$; the difference of these two quantities is

$$
\frac{\partial \omega_{a}}{\partial t_{b}}-\frac{\partial \omega_{b}}{\partial t_{a}}=\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \frac{\partial S}{\partial t_{b}} S^{-1}\right\rangle-\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle-\left\langle S^{-1} \frac{\partial S}{\partial \zeta}\left(\frac{\partial \Theta_{a}}{\partial t_{b}}-\frac{\partial \Theta_{b}}{\partial t_{a}}\right)\right\rangle
$$

Closedness of $\omega_{J M U}$ is equivalent to the vanishing of the above expression, for all $a, b$. Indeed, it is easy to see that the last term vanishes, by the integrability condition (5.7); it remains to see that the expression

$$
\frac{\partial \omega_{a}}{\partial t_{b}}-\frac{\partial \omega_{b}}{\partial t_{a}}=\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \frac{\partial S}{\partial t_{b}} S^{-1}\right\rangle-\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle
$$

vanishes. Here is the first place where the fact that the matrices $\hat{B}_{a}$ are not polynomials in $\zeta$ comes into play. Again writing the identity (5.29), and representing $S \Theta_{a} S^{-1}=\hat{B}_{a}-\frac{\partial S}{\partial t_{a}} S^{-1}$, we see that

$$
\begin{aligned}
\left\langle\frac{\partial \Theta_{b}}{\partial \zeta} \Theta_{a}\right\rangle & =\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle-\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle-\left\langle\frac{\partial}{\partial \zeta}\left(\frac{\partial S}{\partial t_{b}} S^{-1}\right) \hat{B}_{a}\right\rangle \\
& -\left\langle\frac{\partial}{\partial \zeta}\left(\frac{\partial S}{\partial t_{b}} S^{-1}\right) \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle+\left\langle S^{-1} \frac{\partial S}{\partial \zeta}\left[\Theta_{a}, \Theta_{b}\right]\right\rangle
\end{aligned}
$$

Now, the integrability condition (5.7) implies that the last term vanishes; we also have already seen that the second to last term vanishes, as its argument is residueless. Similarly, since the matrices $\Theta_{a}$ are polynomial in $\zeta$, the argument left hand side is residueless, and thus vanishes. Integrating the third term by parts, we obtain the identity

$$
\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \frac{\partial S}{\partial t_{a}} S^{-1}\right\rangle-\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \frac{\partial S}{\partial t_{b}} S^{-1}\right\rangle=\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle=-\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \hat{B}_{b}\right\rangle
$$

and so we see that

$$
\frac{\partial \omega_{a}}{\partial t_{b}}-\frac{\partial \omega_{b}}{\partial t_{a}}=\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \hat{B}_{b}\right\rangle
$$

If the matrices $\hat{B}_{a}$ are polynomials, then the right hand side vanishes identically; otherwise, we obtain the expression above.

## Lemma 5.5.

$$
\begin{equation*}
\mathbf{d} \sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \hat{B}_{\ell}(\zeta ; \mathbf{t})\right\rangle d t_{\ell}=-\sum_{a<b}\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \hat{B}_{b}\right\rangle d t_{a} \wedge d t_{b} \tag{5.30}
\end{equation*}
$$

Proof. Put

$$
\sigma:=\sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \hat{B}_{\ell}(\zeta ; \mathbf{t})\right\rangle d t_{\ell}
$$

and let $\sigma_{a}:=\left\langle\hat{A} \hat{B}_{a}\right\rangle$ denote the coefficient of $d t_{a}$ of $\sigma$. We then have that, using the integrability conditions for $\hat{B}_{\ell}, \hat{A}$,

$$
\begin{aligned}
\frac{\partial \sigma_{a}}{\partial t_{b}} & =\left\langle\frac{\partial \hat{A}}{\partial t_{b}} \hat{B}_{a}\right\rangle+\left\langle\hat{A} \frac{\partial \hat{B}_{a}}{\partial t_{b}}\right\rangle \\
& =\left\langle\left(\frac{\partial \hat{B}_{b}}{\partial \zeta}+\left[\hat{B}_{b}, \hat{A}\right]\right) \hat{B}_{a}\right\rangle+\left\langle\hat{A} \frac{\partial \hat{B}_{a}}{\partial t_{b}}\right\rangle \\
& =\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle+\left\langle\hat{A}\left[\hat{B}_{a}, \hat{B}_{b}\right]\right\rangle+\left\langle\hat{A} \frac{\partial \hat{B}_{a}}{\partial t_{b}}\right\rangle
\end{aligned}
$$

where in the last equality we have used the Ad-invariance of the bracket. Then, we can compute $[\mathbf{d} \sigma]_{b a}$ to be

$$
\begin{aligned}
{[\mathbf{d} \sigma]_{b a}=\frac{\partial \sigma_{a}}{\partial t_{b}}-\frac{\partial \sigma_{b}}{\partial t_{a}} } & =2\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle+2\left\langle\hat{A}\left[\hat{B}_{a}, \hat{B}_{b}\right]\right\rangle+\left\langle\hat{A}\left(\frac{\partial \hat{B}_{a}}{\partial t_{b}}-\frac{\partial \hat{B}_{b}}{\partial t_{a}}\right)\right\rangle \\
& =2\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle+\left\langle\hat{A}\left[\hat{B}_{a}, \hat{B}_{b}\right]\right\rangle
\end{aligned}
$$

In the last line, we have again used the integrability conditions for $\hat{B}_{\ell}, \hat{A}$. We claim that $\left\langle\hat{A}\left[\hat{B}_{a}, \hat{B}_{b}\right]\right\rangle=$ $-\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle$. On one hand, expanding $\left\langle\hat{A}\left[\hat{B}_{a}, \hat{B}_{b}\right]\right\rangle$,

$$
\begin{aligned}
\left\langle\hat{A}\left[\hat{B}_{a}, \hat{B}_{b}\right]\right\rangle & =\left\langle q \zeta^{q-1}\left(g_{q}^{-1} A\left(\zeta^{q}\right) g_{q}-g_{q}^{-1} \frac{d g_{q}}{d \lambda}\right)\left[g_{q}^{-1} B_{a}\left(\zeta^{q}\right) g_{q}, g_{q}^{-1} B_{b}\left(\zeta^{q}\right) g_{q}\right]\right\rangle \\
& =\left\langle q \zeta^{q-1}\left(g_{q}^{-1} A\left(\zeta^{q}\right) g_{q}-g_{q}^{-1} \frac{d g_{q}}{d \lambda}\right) g_{q}^{-1}\left[B_{a}\left(\zeta^{q}\right), B_{b}\left(\zeta^{q}\right)\right] g_{q}\right\rangle \\
& =\left\langle q \zeta^{q-1} A\left(\zeta^{q}\right)\left[B_{a}\left(\zeta^{q}\right), B_{b}\left(\zeta^{q}\right)\right]\right\rangle-\left\langle\frac{q \mathcal{U} \Delta \mathcal{U}^{-1}}{\zeta}\left[B_{a}\left(\zeta^{q}\right), B_{b}\left(\zeta^{q}\right)\right]\right\rangle \\
& =-\left\langle\frac{q \mathcal{U} \Delta \mathcal{U}^{-1}}{\zeta}\left[B_{a}\left(\zeta^{q}\right), B_{b}\left(\zeta^{q}\right)\right]\right\rangle
\end{aligned}
$$

where the last equality follows from the fact that all of the expressions inside the first bracket are polynomials. On the other hand, we calculate that

$$
\frac{\partial \hat{B}_{b}}{\partial \zeta}=-g_{q}^{-1} \frac{d g_{q}}{d \zeta} g_{q}^{-1} B_{b}\left(\zeta^{q}\right) g_{q}+g_{q}^{-1} \frac{\partial}{\partial \zeta} B_{b}\left(\zeta^{q}\right) g_{q}+g_{q}^{-1} B_{b}\left(\zeta^{q}\right) g_{q}
$$

and so

$$
\begin{aligned}
\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle & =\left\langle\frac{d g_{q}}{d \zeta} g_{q}^{-1}\left[B_{a}\left(\zeta^{q}\right), B_{b}\left(\zeta^{q}\right)\right]\right\rangle+\left\langle\frac{\partial}{\partial \zeta}\left(B_{b}\left(\zeta^{q}\right)\right) B_{a}\left(\zeta^{q}\right)\right\rangle \\
& =\left\langle\frac{q \mathcal{U} \Delta \mathcal{U}^{-1}}{\zeta}\left[B_{a}\left(\zeta^{q}\right), B_{b}\left(\zeta^{q}\right)\right]\right\rangle
\end{aligned}
$$

It follows that

$$
\begin{equation*}
[\mathbf{d} \sigma]_{b a}=2\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle+\left\langle\hat{A}\left[\hat{B}_{a}, \hat{B}_{b}\right]\right\rangle=\left\langle\frac{\partial \hat{B}_{b}}{\partial \zeta} \hat{B}_{a}\right\rangle \tag{5.31}
\end{equation*}
$$

an integration by parts yields that this is equal to $-\left\langle\frac{\partial \hat{B}_{a}}{\partial \zeta} \hat{B}_{b}\right\rangle$. This completes the proof.
As a result of these lemmas, we finally obtain the Theorem
Theorem 5.1. Closedness of the modified $\tau$-differential/analog of Theorem 1.2. Under the assumptions of Subsection 5.1, and given Definition 5.1,

$$
\begin{equation*}
\mathbf{d} \hat{\omega}_{J M U}=0 \tag{5.32}
\end{equation*}
$$

Proof. One simply must add the results of Lemmas 5.4 and 5.5, in accordance with the fact that

$$
\hat{\omega}_{J M U}=\omega_{J M U}+\sum_{\ell}\left\langle\hat{A}(\zeta ; \mathbf{t}) \hat{B}_{\ell}(\zeta ; \mathbf{t})\right\rangle d t_{\ell},
$$

as per Lemma 5.3.
Remark 5.1. Note that the explicit form of the matrices $\Delta_{q}, \mathcal{U}_{q}$ was not so important in the proof of this proposition. The only details that mattered were the fact that $g_{q}(\lambda)$ was of the form $g_{q}(\lambda)=\lambda^{\Delta_{q}} \mathcal{U}_{q}$, and the fact that $\Psi(\lambda ; \mathbf{t}), G(\lambda ; \mathbf{t})$ had jumps only on the right.
Remark 5.2. In principle, the above theorem/definition of the $\tau$-differential should follow from the work of Bertola and Mo [BM05] on isomonodromic deformations of resonant rational connections. We nevertheless feel our theorem is worth writing down, for the following reasons:

- Although our theorem is less general, the corresponding expression for the modified $\tau$-differential is more manageable,
- The expression for the $\tau$-differential in [BM05] is in terms of spectral invariants, whereas our expression is in terms of formal residues in the local gauge. This is more in line with the original expression for the $\tau$-differential provided by Jimbo, Miwa, and Ueno [JMU81]. These expressions should of course be equivalent.

Remark 5.3. Irrelevance of modification in the case of Painlevé $I$. This construction is unnecessary in the case of the usual Painlevé I Lax pair, and so the $\tau$-differential as defined by Jimbo, Miwa, and Ueno [JM81] or Lisovyy and Roussillon [LR17] agrees with the one given here. Recall that this Lax pair in the $\zeta$-gauge is given by (cf. [JM81], Formula C5, or [LR17], Formulae 2.4a and 2.4b)

$$
\begin{aligned}
& \hat{A}(\zeta ; t)=\left(4 \zeta^{4}+2 q^{2}+t\right) \sigma_{3}-\left(2 p \zeta+(2 \zeta)^{-1}\right) \sigma_{1}-\left(4 q \zeta^{2}+2 q^{2}+t\right) i \sigma_{2} \\
& \hat{B}(\zeta ; t)=(\zeta+q / \zeta) \sigma_{3}-i q \zeta^{-1} \sigma_{2}
\end{aligned}
$$

where $\sigma_{k}$ are the standard Pauli matrices. Here, $q$ solves Painlevé I, and $p=q^{\prime}$ (although the calculations we perform now are independent of this fact). Using the fact that $\operatorname{tr}\left(\sigma_{j} \sigma_{k}\right)=2 \delta_{j k}$, we find that

$$
\operatorname{tr} \hat{A}(\zeta ; t) \hat{B}(\zeta ; t)=4 \zeta\left(2 \zeta^{4}+2 q \zeta^{2}-q^{2}+t / 2\right)
$$

Hence $\langle\hat{A}(\zeta ; t) \hat{B}(\zeta ; t)\rangle=0$, and $\hat{\omega}_{J M U}=\omega_{J M U}$ by Lemma 5.3. However, as we have seen in the rest of the present work, this construction is nontrivial in general. Indeed, one can readily check that for the next entry of the Painlevé I hierarchy (the $(2,5)$ string equation), this contribution is indeed nontrivial.

By comparing the coefficients of the expression we obtained in the previous proposition, we can show that $\omega_{\text {Okamoto }}$ is a constant multiple of the modified differential $\hat{\omega}_{J M U}$ :
Proposition 5.2. The differentials $\omega_{\text {Okamoto }}, \hat{\omega}_{J M U}$ are related by

$$
\begin{equation*}
\omega_{\text {Okamoto }}=\frac{2}{3} \hat{\omega}_{J M U} \tag{5.33}
\end{equation*}
$$

Proof. The proof is straightforward, and follows from definitions. Note that $\mathcal{L}$ has terms of degree 7, and so we must compute $S(\zeta)=\mathbb{I}+\frac{\Phi_{1}}{\zeta}+\cdots$ to terms of order $\zeta^{-8}$. This calculation can be performed by applying our previous calculations (cf. Remark 4.6); one finds that the coefficients $\hat{\omega}_{t_{5}}, \hat{\omega}_{t_{2}}$, and $\hat{\omega}_{x}$ are differential polynomials in the variables $\left\{\left[\Phi_{k}\right]_{11}\right\}_{k=1}^{7}$, and the functions $U, V$. One can match these coefficients explicitly to the Hamiltonians from before, up to a proportionality factor of $2 / 3$, and so we can identify $\hat{\omega}_{J M U}$ with $\omega_{\text {Okamoto }}$.

Consequentially, if we define the corresponding $\tau$-functions by $\tau=e^{\int \omega}$, we see that the $\tau$-functions arising from these definitions are related by $\tau_{\text {Okamoto }}^{3}=\tau_{J M U}^{2}$.

### 5.3. THE $\tau$-FUNCTION ON THE EXTENDED MONODROMY DATA.

The $\tau$-function also depends intrinsically on the extended monodromy data of the system; in our case, the Stokes parameters $s_{1}, \ldots, s_{6}$. It is thus natural to ask the question What is the dependence of the $\tau$-function on the extended monodromy data? Such a question is by no means new, and has been addressed in the literature before by various sources [Mal83; Pal99; Ber10; LR17; ILP18]. This problem of determining the dependence of the $\tau$-function on the extended monodromy data has many important applications, one of the main ones being the problem of determining constant factors for the asymptotics of $\tau$-functions [LR17; ILP18; IP18]. This subsection will be organized as follows: we first introduce the definition of the extended JMU differential, in the context of the previous section. We then overview some of the main points given in [IP18] about the role of the Hamiltonian structure of Painlevé equations in the problem of computing constant factors. We also state their conjectures. We conclude by showing that their conjectures hold in the case of the isomonodromic system associated to the string equation.

Let us first define the extended JMU $\tau$-differential. We work again with the system (5.16). Let $\mathcal{T}$ denote the space of isomonodromic deformation parameters of this system ${ }^{7}$, and denote $\mathbf{d}_{\mathcal{T}}$ the differential in these parameters (note that we had previously used the notation $\mathbf{d}$ for this object). Associated to the system (5.16) are a number of parameters which we refer to collectively as monodromy data. For the system (5.16), the monodromy data will consist of a number of Stokes parameters. We denote these parameters by $\left\{m_{\ell}\right\}$, denote the space of these parameters by $\mathcal{M}$, and denote the differential in these parameters as $\mathbf{d}_{\mathcal{M}}$. A specification of a solution to the isomonodromy equations (the zero curvature conditions) depend intrinsically on the monodromy data $\left\{m_{\ell}\right\}$, and thus the JMU $\tau$-function also depends on these parameters. One is then led to wonder how $\tau$ depends on these parameters. This question can essentially be answered if one can extend the JMU differential from a closed differential on $\mathcal{T}$ to a closed differential on all of $\mathcal{T} \times \mathcal{M}$. This can be accomplished through the following steps [ILP18]:

1. Define the following 1 -form on $\mathcal{T} \times \mathcal{M}$ :

$$
\begin{equation*}
\omega_{0}:=\left\langle\hat{A}(\zeta) \mathbf{d}_{\mathcal{T}} S(\zeta) S^{-1}(\zeta)\right\rangle+\left\langle\hat{A}(\zeta) \mathbf{d}_{\mathcal{M}} S(\zeta) S^{-1}(\zeta)\right\rangle . \tag{5.34}
\end{equation*}
$$

This 1-form obviously has the property that its restriction to $\mathcal{T}$ coincides with the usual JMU $\tau$ differential. Furthermore, one can show that

$$
\begin{equation*}
\Omega_{0}:=\left(\mathbf{d}_{\mathcal{T}}+\mathbf{d}_{\mathcal{M}}\right) \omega_{0} \tag{5.35}
\end{equation*}
$$

is a 2 -form on $\mathcal{M}$ only. In other words, the restriction of $\Omega_{0}$ to $\mathcal{T}$ vanishes identically; this is equivalent to the fact that (i.) $\mathbf{d}_{\mathcal{T}} \omega_{J M U}=0$, and (ii.) $\Omega_{0}$ contains no cross-terms of the form $d t_{k} \wedge d m_{\ell}$.

[^6]2. By means of asymptotic analysis, one can calculate (at least in principle) $\Omega_{0}$ explicitly. Once this expression is obtained, construct a 1 -form $\omega_{\text {correction }}$ on $\mathcal{M}$ such that $d \omega_{\text {correction }}=\Omega$; then, put
\[

$$
\begin{equation*}
\hat{\omega}:=\omega_{0}-\omega_{\text {correction }} \tag{5.36}
\end{equation*}
$$

\]

The 1-form $\hat{\omega}$ then by construction is closed on $\mathcal{T} \times \mathcal{M}$, and its restriction to $\mathcal{T}$ agrees with the JMU $\tau$-differential. We are thus justified in calling $\hat{\omega}$ the extended $\tau$-differential. Such a differential is of course not unique, as our construction of $\omega_{\text {correction }}$ is defined only up to the addition of an exact differential on $\mathcal{M}$.

With the definition of the extended $\tau$-differential in place, we now proceed to discuss the Hamiltonian aspects of the problem. In [IP18], the central role of the Hamiltonian structure of Painlevé equations with regards to the problem of evaluation of constant factors was demonstrated. Let us briefly overview some of their main philosophical arguments; we will essentially be summarizing Section 2 of [IP18].

Consider a completely integrable Hamiltonian system with Darboux coordinates $\left\{P_{a}, Q_{a}\right\}$, with Hamiltonians $\left\{H_{k}\right\}$ with respect to the times $\left\{t_{k}\right\}$. Denote the parameter space of times by $\mathcal{T}$; suppose the Darboux coordinates depend additionally on a collection of monodromy parameters $\left\{m_{\ell}\right\}$,

$$
\begin{equation*}
Q_{a}=Q_{a}\left(t_{k}, m_{\ell}\right), \quad P_{a}=P_{a}\left(t_{k}, m_{\ell}\right) \tag{5.37}
\end{equation*}
$$

and denote the parameter space of the monodromy parameters by $\mathcal{M}$. We define the classical action differential on the total space $\mathcal{T} \times \mathcal{M}$ :

$$
\begin{equation*}
\omega_{c l a}:=\sum_{a} P_{a} d Q_{a}-\sum_{k} H_{k} d t_{k}=\sum_{k}\left(P_{a} \frac{\partial Q_{a}}{\partial t_{k}}-H_{k}\right) d t_{k}+\sum_{\ell}\left(\sum_{a} P_{a} \frac{\partial Q_{a}}{\partial m_{\ell}}\right) d m_{\ell} \tag{5.38}
\end{equation*}
$$

The fact that the system is a completely integrable Hamiltonian system implies that the differential is closed in the time parameters. In other words, if we define $\mathbf{d}_{\mathcal{T}}:=\sum_{k} d t_{k} \frac{\partial}{\partial t_{k}}$, then

$$
\begin{equation*}
\mathbf{d}_{\mathcal{T}}\left(\left.\omega_{c l a}\right|_{\left\{m_{\ell}=\text { const. }\right\}}\right)=0 \tag{5.39}
\end{equation*}
$$

This is nothing but the classical statement that the symplectic form defined by (3.17) vanishes along the trajectories of the Hamiltonian flows. Note that in many cases, including our own, there is already a connection between the classical action and the isomonodromic $\tau$-function: namely, we have that $d \log \tau=$ $\sum_{k} H_{k} d t_{k}$, and so the $\tau$-function appears as a "truncation" of the classical action integral. If we take the total differential (on the whole of $\mathcal{T} \times \mathcal{M}$ ) of formula (5.38), we find that

$$
\begin{equation*}
\left(\mathbf{d}_{\mathcal{T}}+\mathbf{d}_{\mathcal{M}}\right) \omega_{c l a}=\sum_{a} \mathbf{d}_{\mathcal{M}} P_{a} \wedge \mathbf{d}_{\mathcal{M}} Q_{a}=: \Omega \tag{5.40}
\end{equation*}
$$

which is reminiscent of formula (5.35). This observation led Its and Prokhorov to make the following conjectures:

Conjecture 1. ([IP18].) Suppose the parameter space $\mathcal{T} \times \mathcal{M}$ is equipped with a symplectic structure $\Omega$. Then, there exists a constant $\gamma \in \mathbb{C}$ such that

$$
\begin{equation*}
\Omega_{0}=\gamma \Omega \tag{5.41}
\end{equation*}
$$

where $\Omega_{0}$ is the 2 -form defined by (5.35).
Conjecture 2. ([IP18].) There exists a function $G\left(P_{a}, Q_{a}, t_{k}\right)$, rational in the variables $\left\{P_{a}\right\},\left\{Q_{a}\right\},\left\{t_{k}\right\}$, such that

$$
\begin{equation*}
\omega_{0}=\gamma \omega_{c l a}+d G \tag{5.42}
\end{equation*}
$$

These conjectures allow one to write a formula for the variation of the $\tau$-function in terms of the monodromy parameters, which in practice is much more efficient in application to the evaluation of constant factors than many earlier procedures. If we define the $\tau$-function as $\log \tau:=\int_{C} \hat{\omega}$, where $C \subset \mathcal{T}$ is a 'nice' curve in the deformation parameter space, the formula is (see Remark 3 of [IP18])

$$
\begin{equation*}
\frac{\partial}{\partial m_{\ell}} \log \tau=\left.\sum_{a} P_{a} \frac{\partial Q_{a}}{\partial m_{\ell}}\right|_{\partial C}+\left.\frac{\partial G}{\partial m_{\ell}}\right|_{\partial C} \tag{5.43}
\end{equation*}
$$

In [IP18], the authors were able to verify this conjecture for the classical Painlevé transcendents. In fact, these conjectures hold in our situation as well, as the next Proposition states.
Proposition 5.3. Consider the isomonodromic system defined by (4.6), and define the extended 1-form for this system by

$$
\begin{equation*}
\omega_{0}:=\left\langle\mathcal{L}(\zeta) \mathbf{d}_{\mathcal{T}} S(\zeta) S^{-1}(\zeta)\right\rangle+\left\langle\mathcal{L}(\zeta) \mathbf{d}_{\mathcal{M}} S(\zeta) S^{-1}(\zeta)\right\rangle \tag{5.44}
\end{equation*}
$$

where $\mathcal{L}$ is as defined in (4.7), and $S(\zeta)=\mathbb{I}+\sum_{k=1}^{\infty} \frac{\Phi_{k}}{\zeta^{k}}$. Then,

$$
\begin{equation*}
\omega_{0}=\frac{3}{2} \omega_{c l a}+d G \tag{5.45}
\end{equation*}
$$

where $G$ is the polynomial

$$
\begin{equation*}
G=\frac{3}{7}\left[3 t_{1} H_{1}+\frac{5}{2} t_{2} H_{2}+t_{5} H_{5}-P_{U} Q_{U}-\frac{3}{2} P_{V} Q_{V}-\frac{3}{2} P_{W} Q_{W}\right] \tag{5.46}
\end{equation*}
$$

This verifies Conjectures 1. and 2. for the system (4.6).
Proof. The proof of this proposition is a straightforward, albeit tedious, calculation. since $\mathcal{L}(\zeta)$ is degree 7 in $\zeta$, one must in principle compute terms up to order $\zeta^{-8}$ in the expansion of $S(\zeta)$; however, the symmetry of $\mathcal{L}, S$ under conjugation by the matrix $\mathcal{S}$ actually implies one must only compute up to terms of order $\zeta^{-7}$. This calculation involves (cf. the proof of Proposition 4.6) determining the off-diagonal terms of the matrices $\Phi_{k}$ up to order 13. Once one has successfully calculated the coefficients $\Phi_{1}, \ldots, \Phi_{7}$, one can use formula (5.34) with the Hamiltonian variables as coordinates on the monodromy manifold $\mathcal{M}$ to compute the coefficients of $\omega_{0}$. Calculating the $d P_{U} \wedge d Q_{U}$-coefficient of $\left(\mathbf{d}_{\mathcal{T}}+\mathbf{d}_{\mathcal{M}}\right) \omega_{0}$,

$$
\frac{\partial\left(\omega_{0}\right)_{Q_{U}}}{\partial P_{U}}-\frac{\partial\left(\omega_{0}\right)_{P_{U}}}{\partial Q_{U}}=\frac{3}{2} .
$$

Similarly, the coefficients of the $d P_{V} \wedge d Q_{V}, d P_{W} \wedge d Q_{W}$ terms in $\left(\mathbf{d}_{\mathcal{T}}+\mathbf{d}_{\mathcal{M}}\right) \omega_{0}$ are constant, and equal to $\frac{3}{2}$. On the other hand, we have the equalities

$$
\begin{aligned}
& -\frac{3}{2} \frac{\partial H_{k}}{\partial Q_{a}}=\frac{\partial\left(\omega_{0}\right)_{t_{k}}}{\partial Q_{a}}-\frac{\partial\left(\omega_{0}\right)_{Q_{a}}}{\partial t_{k}}, \\
& -\frac{3}{2} \frac{\partial H_{k}}{\partial P_{a}}=\frac{\partial\left(\omega_{0}\right)_{t_{k}}}{\partial P_{a}}-\frac{\partial\left(\omega_{0}\right)_{P_{a}}}{\partial t_{k}}
\end{aligned}
$$

for every $k \in\{1,2,5\}, a \in\{U, V, W\}$; all other coefficients of $\left(\mathbf{d}_{\mathcal{T}}+\mathbf{d}_{\mathcal{M}}\right) \omega_{0}$ vanish identically (Note that we could have also inferred this constant from the relation of $\hat{\omega}_{J M U}$ and $\omega_{\text {Okamoto }}$ ). Subtracting $\frac{3}{2} \omega_{\text {cla }}$ from $\omega_{0}$, we obtain the differential

$$
d G:=\omega_{0}-\frac{3}{2} \omega_{c l a}
$$

By construction, this differential is closed. Consequentially, it can be integrated up to a function $G=$ $G\left(Q_{U}, Q_{V}, Q_{W}, P_{U}, P_{V}, P_{W} ; t_{1}, t_{2}, t_{5}\right)$. Direct calculation shows that this function is

$$
G=\frac{3}{7}\left[3 t_{1} H_{1}+\frac{5}{2} t_{2} H_{2}+t_{5} H_{5}-P_{U} Q_{U}-\frac{3}{2} P_{V} Q_{V}-\frac{3}{2} P_{W} Q_{W}\right]
$$

as claimed.

## 6. Discussion and Outlook.

In summary, we have constructed a Riemann-Hilbert formulation of the $(3,4)$ string equation, which will appear as the model Riemann-Hilbert problem in the local analysis of the multi-critical quartic 2-matrix model [DHL23a]. The string equation is equivalent to a $3+3$-dimensional, completely integrable nonautonomous Hamiltonian system. Furthermore, we were able to calculate an appropriate $\tau$-function for this system. Upon extending this $\tau$-function to the canonical coordinates, we were able to verify Conjectures 1 and 2 of [IP18], lending them further validity.

Aside from the completion of the work [DHL23a], we hope to further investigate the large-parameter asymptotics of the above Riemann-Hilbert problem. This is a standard question whenever a RiemannHilbert problem such as the one described in this work arises. The physically relevant solution to the string equation, according to [FGZ95], should have asymptotic expansion of the form

$$
\begin{equation*}
U\left(t_{5}, t_{2}, x\right) \sim x^{1 / 3}\left(\sum_{k=0}^{\infty} u_{k}\left(t_{5}, t_{2}\right) x^{-k / 3}\right), \quad \quad V\left(t_{5}, t_{2}, x\right) \sim x^{-1 / 3}\left(\sum_{k=0}^{\infty} v_{k}\left(t_{5}, t_{2}\right) x^{-k / 3}\right) \tag{6.1}
\end{equation*}
$$

The work [DHL23a] shows that the partition function of the critical two matrix model can be written in terms of $U\left(t_{5}, t_{2}, x\right)$; it is not yet clear what the particular solution looks like.

There is also an additional physical motivation for the study of these asymptotics. As observed by Crnković, Ginsparg, and Moore [CGM90], there should exist a "renormalization group flow" between the multicritical points of the 2-matrix model. Formally, this observation says that, given a solution $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$ of the string equation (1.1), if we make the scaling

$$
\begin{equation*}
u\left(t_{5}, t_{2}, x\right):=t_{5}^{2 / 5} U\left(t_{5}, t_{2}, t_{5}^{1 / 5} x\right), \quad v\left(t_{5}, t_{2}, x\right):=t_{5}^{3 / 5} V\left(t_{5}, t_{2}, t_{5}^{1 / 5} x\right) \tag{6.2}
\end{equation*}
$$

and take a formal limit as $t_{5} \rightarrow \infty$, then $v \rightarrow 0$, and $u \rightarrow \hat{u}(x)$, where $\hat{u}$ solves the Painlevé I equation, after a rescaling of the variables. A full investigation of this statement can be performed via steepest descent analysis for the Riemann-Hilbert problem developed in this work. Some preliminary calculations suggest that this Riemann-Hilbert problem "flows" to a $3 \times 3$ version of the Painlevé I Riemann-Hilbert problem. The associated Lax pair has appeared in the literature before [JKT09], and this $3 \times 3$ problem also seems to appear in the local parametrices of the critical energy, critical temperature (but non-critical external field) quartic 2-matrix problem [DHL23b; DHL23a]. The analysis of this problem and the large-parameter asymptotics of the Riemann-Hilbert problem described in this paper will be the subject of a future work. We also remark that it would be interesting to see if this degeneration can be identified using the Hamiltonian formalism, in a similar manner to the $t_{2} \rightarrow 0$ limit discussed in $\S 2$.

The partition function of the 2-matrix model is identified with the partition function of a particular theory of minimal matter coupled to topological gravity [Kon92; Wit91; Wit92], which counts a class of intersection numbers on the moduli space of Riemann surfaces. This implies that the Riemann-Hilbert problem discussed above could be of use in enumeration of these intersection numbers; we hope to investigate this in the future.

In this work, we essentially gave no analysis of the solutions to the string equation. There are several fundamental questions that should be addressed:

- Irreducibility of the string equation. Due to the similar nature of the Riemann-Hilbert problems of the $(4,3)$ string equation and the Painleve I Riemann-Hilbert problem, it is natural to conjecture that the string equation admits no solutions in terms of classical functions, in the sense of [Oka99]). Indeed, there is a procedure ([Ume88], see also [Ume90]) by which one can infer the irreducibility of solutions of a given Hamiltonian system. This procedure applies in principle to the string equation; it would be interesting to see if this method can be applied practically.
- The space of initial conditions $\mathcal{E}$ Stokes manifold. Aside from determining its generic dimension, we provided essentially no analysis of the Stokes manifold associated to the string equation. The Stokes manifolds of the classical Painlevé equations, in particular PI and PII, have a rich mathematical structure, and carry their own Poisson tensor, as well as an association to certain cluster algebras
[LR17; BT22]. A more complete analysis of this Stokes manifold, as well as an accompanying analysis of the space of initial conditions (cf. [Oka99] for the equivalent analysis for PII) is certainly needed.
- Evaluation of constant factor in the $\tau$-function. So far, we have only calculated the $\tau$-differential, and thus the free energy of the multi-critical matrix model up to a multiplicative constant. This problem was first noticed in [Dou90], who believed the problem could be resolved by appealing to the general theory of $\tau$-functions. It would be interesting if one could apply the calculations in Section 5.3 of this work to this end.


## A. Singularity Analysis of the String Equation.

Here, we perform a rudimentary singularity analysis of the string equation (1.1). The main point of this work is to provide familiarity with the $(3,4)$ string equation. Some, if not all, of what is written here can be found in earlier physics literature, such as [Bré+90; FGZ95], and references therein. We record these results here, for the convenience of the reader, and also to bring attention to these results to a potentially new audience. Perhaps the only noteworthy observation here is that certain behaviors of the solution to the ODE (1.1) can be eliminated by using the fact that solutions must also satisfy (2.24)-(2.27).

Proposition A.1. Let $U, V$ be a solution to the ODE (1.1), meromorphic (and possibly multivalued) in a neighborhood of $x=x_{0}$. Further, let $u_{k}, v_{k}$ denote the $k^{t h}$ Laurent coefficient of $U, V$, respectively. Then, one of the following holds:

1. $U, V$ are holomorphic at $x=x_{0}$, with expansion starting with

$$
\begin{cases}U\left(t_{5}, t_{2}, x\right) & =u_{0}+u_{1}\left(x-x_{0}\right)+u_{2}\left(x-x_{0}\right)^{2}+u_{3}\left(x-x_{0}\right)^{3}+\mathcal{O}\left(\left(x-x_{0}\right)^{4}\right)  \tag{A.1}\\ V\left(t_{5}, t_{2}, x\right) & =v_{0}+v_{1}\left(x-x_{0}\right)+\mathcal{O}\left(\left(x-x_{0}\right)^{2}\right)\end{cases}
$$

and all subsequent entries determined as polynomials in $u_{0}, \ldots, u_{3}, v_{0}, v_{1}, x_{0}, t_{5}, t_{2}$.
2. $U, V$ have second and third order poles, respectively, with expansion starting with

$$
\left\{\begin{array}{l}
U\left(t_{5}, t_{2}, x\right)=\frac{4}{\left(x-x_{0}\right)^{2}}+\frac{5}{3} t_{5} \mp v_{0}\left(x-x_{0}\right) \mp v_{1}\left(x-x_{0}\right)^{2}+\mathcal{O}\left(\left(x-x_{0}\right)^{3}\right)  \tag{A.2}\\
V\left(t_{5}, t_{2}, x\right)=\frac{ \pm 4}{\left(x-x_{0}\right)^{3}}+v_{0}+v_{1}\left(x-x_{0}\right)+\mathcal{O}\left(\left(x-x_{0}\right)^{2}\right)
\end{array}\right.
$$

and all subsequent entries are determined in as polynomials in $v_{0}, v_{1}, v_{5}, v_{6}, x_{0}, t_{5}, t_{2}$.
3. $U, V$ both have second-order poles, with expansion starting with

$$
\left\{\begin{array}{l}
U\left(t_{5}, t_{2}, x\right)=\frac{2}{\left(x-x_{0}\right)^{2}}+\frac{1}{2} v_{-2}^{2}+u_{1}\left(x-x_{0}\right)+u_{2}\left(x-x_{0}\right)^{2}+\mathcal{O}\left(\left(x-x_{0}\right)^{3}\right),  \tag{A.3}\\
V\left(t_{5}, t_{2}, x\right)=\frac{v_{-2}}{\left(x-x_{0}\right)^{2}}-\frac{1}{4} v_{-2}^{3}+\frac{5}{6} t_{5} v_{-2}-\frac{1}{2} v_{-2} u_{1}\left(x-x_{0}\right)+\mathcal{O}\left(\left(x-x_{0}\right)^{2}\right),
\end{array}\right.
$$

and all subsequent entries are determined as polynomials in $v_{-2}, v_{3}, u_{1}, u_{2}, u_{6}, x_{0}, t_{5}, t_{2}$.
4. $U$ has a pole of order 2 and $V$ is regular, with expansions starting with

$$
\left\{\begin{array}{l}
U\left(t_{5}, t_{2}, x\right)=\frac{10}{\left(x-x_{0}\right)^{2}}+\frac{20}{21} t_{5}+\frac{25}{294} t_{5}^{2}\left(x-x_{0}\right)^{2}+\mathcal{O}\left(\left(x-x_{0}\right)^{3}\right)  \tag{A.4}\\
V\left(t_{5}, t_{2}, x\right)=\frac{t_{2}}{14}\left(x-x_{0}\right)^{2}+\frac{5}{588} t_{5} t_{2}\left(x-x_{0}\right)^{4}+v_{6}\left(x-x_{0}\right)^{6}+\mathcal{O}\left(\left(x-x_{0}\right)^{7}\right)
\end{array}\right.
$$

and all subsequent terms are determined as polynomials in $v_{6}, u_{6}, u_{10}, x_{0}, t_{5}, t_{2}$.
Proof. We follow the usual procedure of Painlevé-type analysis, cf. [ARS80], for example. By hypothesis, $U, V$ have expansions of the form $U=\sum_{k=0}^{\infty} u_{k}\left(x-x_{0}\right)^{k-\alpha}, V=\sum_{k=0}^{\infty} v_{k}\left(x-x_{0}\right)^{k-\beta}$, for some $\alpha, \beta \in \mathbb{C}$. This is not consistent with our indexing convention in the statement of the theorem, but it is a more convenient choice of labelling for the proof; we will make note of where we must relabel indices later on. We assume,
without loss of generality, that $u_{0}, v_{0} \neq 0$; otherwise, we could redefine $\alpha, \beta$ accordingly. The first part of the string equation reads

$$
0=\frac{1}{2} V^{\prime \prime}-\frac{3}{2} U V+\frac{5}{2} t_{5} V+t_{2}
$$

Note that the most singular terms in the above equation are $\frac{1}{2} V^{\prime \prime}$ and $-\frac{3}{2} U V$. Matching the most singular terms, we see that

$$
0=\frac{\frac{1}{2} v_{0} \beta(\beta+1)}{\left(x-x_{0}\right)^{\beta+2}}-\frac{\frac{3}{2} u_{0} v_{0}}{\left(x-x_{0}\right)^{\alpha+\beta}}+[\text { less singular terms }]
$$

and so, since $u_{0}, v_{0} \neq 0$, we see that $\beta+2=\alpha+\beta \Rightarrow \alpha=2$, and subsequently that

$$
\frac{1}{3} \beta(\beta+1)=u_{0}
$$

Now, the second part of the string equation reads

$$
0=\frac{1}{12} U^{(4)}-\frac{3}{4} U^{\prime \prime} U-\frac{3}{8}\left(U^{\prime}\right)^{2}+\frac{3}{2} V^{2}+\frac{1}{2} U^{3}-\frac{5}{12} t_{5}\left(3 U^{2}-U^{\prime \prime}\right)+x
$$

The first 6 terms are the most singular; inserting the $U=\frac{\frac{1}{3} \beta(\beta+1)}{\left(x-x_{0}\right)^{2}}+\ldots, V=v_{0}\left(x-x_{0}\right)^{-\beta}+\ldots$ into this equation, we obtain that

$$
0=\frac{\frac{2}{9} \beta(\beta-5)(\beta-2)(\beta+1)(\beta+3)(\beta+6)}{\left(x-x_{0}\right)^{6}}+\frac{18 v_{0}^{2}}{\left(x-x_{0}\right)^{2 \beta}}+[\text { less singular terms }] .
$$

Here, we see that there are several possibilities, some of which we can eliminate or identify as subcases of one another immediately:

- (Case 0.) $\beta=0$, implying $V$ is locally holomorphic near $x=x_{0}$, and we must try and match the next-most singular terms in the second equation. Following through this calculation, one finds that $U$ must also be holomorphic near $x=x_{0}$.
- (Case 1.) $\beta=3$, and $18 v_{0}^{2}=288 \Longrightarrow v_{0}= \pm 4$.
- (Case 2.) $\beta=2$, and we must try and match the next-most singular terms in the second equation, in this case.
- (Case 3.) $\beta=-1$, in which case $V$ is holomorphic near $x=x_{0}$; following through with the analysis, one finds that $U$ must be holomorphic as well. Thus, we can identify this as a specialization of Case 0 .
- (Case 4.) $\beta=-3$, for which a solution exists only if $t_{2}=0$. We then find that $V \equiv 0$, and there is a 3 -parameter family of solutions $U$, parameterized by the Laurent coefficients $u_{1}, u_{2}$, and $u_{6}$; this is in fact a specialization of Case 2.
- (Case 5.), $\beta=-6$, in which case $V$ is holomorphic, and $U=\frac{10}{\left(x-x_{0}\right)^{2}}+\ldots$

From the above, we see that the only independent cases are cases $0,1,2$, and 5 . One can verify case by case that the cases we have described here are as are described in the statement of the Proposition, by calculating the resonances in each case to determine where arbitrary constants may appear. Let us perform this analysis case by case.

Case 1. This case contains two subcases, corresponding to $v_{0}= \pm 4$; we perform the analysis for $v_{0}=+4$ only, as the ' - ' case is similar. Following [ARS80], to determine resonances, we substitute

$$
U=\frac{4}{\left(x-x_{0}\right)^{2}}\left[1+\epsilon_{1}\left(x-x_{0}\right)^{r}\right], \quad V=\frac{4}{\left(x-x_{0}\right)^{3}}\left[1+\epsilon_{2}\left(x-x_{0}\right)^{r}\right]
$$

into the string equation, and retain only the most singular terms, to leading order in $\epsilon_{1}, \epsilon_{2}$. The resonance matrix is

$$
M(r):=\left(\begin{array}{cc}
-4 r(r-7) & 48 \\
576 & 4 r^{4}-56 r^{3}+140 r^{2}+392 r-67
\end{array}\right) .
$$

Resonances are determined by the positive real solutions of $\operatorname{det} M(r)=0$; since $\operatorname{det} M(r)=-16(r-3)(r-$ $4)(r-8)(r-9)(r+2)(r+1)$, we see that the possible resonances are $r=3,4,8$, and 9 . These resonances can be resolved in terms of the coefficients $v_{0}, v_{1}, v_{5}$, and $v_{6}$, respectively (here, the indices differ from the resonance indices by a factor of 3 due to our labelling convention).

Case 2. Here, we substitute

$$
U=\frac{2}{\left(x-x_{0}\right)^{2}}\left[1+\epsilon_{1}\left(x-x_{0}\right)^{r}\right], \quad V=\frac{v_{-2}}{\left(x-x_{0}\right)^{2}}\left[1+\epsilon_{2}\left(x-x_{0}\right)^{r}\right]
$$

into the string equation, and retain only the most singular terms, to leading order in $\epsilon_{1}, \epsilon_{2}$. The determinant of the resonance matrix is

$$
\operatorname{det} M(r)=\operatorname{det}\left(\begin{array}{cc}
v_{-2} r(r-5) & -6 v_{-2} \\
0 & 2(r-3)(r-4)(r-8)(r+1)
\end{array}\right)=2 v_{-2} r(r-3)(r-4)(r-5)(r-8)(r+1)
$$

The presence of the roots $r=0,-1$ indicate the arbitrariness of the location of the pole $x_{0}$ and the coefficient $v_{-2}$. The resonances $r=3,4,5,8$ correspond to the arbitrariness of the coefficients $u_{1}, u_{2}, v_{3}$, and $u_{6}$, respectively (note that the indices of the coefficients differ from their corresponding resonances by a factor of 2 due to our labelling convention).

Case 5. Here, we substitute

$$
U=\frac{10}{\left(x-x_{0}\right)^{2}}\left[1+\epsilon_{1}\left(x-x_{0}\right)^{r}\right], \quad V=\frac{t_{2}}{14}\left(x-x_{0}\right)^{2}\left[1+\epsilon_{2}\left(x-x_{0}\right)^{r}\right]
$$

into the string equation, and retain only the most singular terms, to leading order in $\epsilon_{1}, \epsilon_{2}$. The determinant of the resonance matrix is

$$
\operatorname{det} M(r)=\operatorname{det}\left(\begin{array}{cc}
-\frac{15}{7} t_{2} \\
10(r+5)(r+1)(r-8)(r-12) & \frac{t_{2}(r+7)(r-4)}{14} \\
0
\end{array}\right)=\frac{5 t_{2}}{7}(r+7)(r+5)(r+1)(r-4)(r-8)(r-12)
$$

The resonances $r=4,8,12$ can be resolved in terms of the coefficients $v_{6}, u_{6}, u_{10}$ respectively.
So far we have only considered part of the full string equation, namely, the ODE (1.1). However the space of solutions of the ODE (1.1) is in general much large that the space of solutions to the string equation, which includes the equations (2.24)-(2.27) in addition to equation (1.1). In fact, only two of the local expansions described in Proposition A. 1 can be made consistent with the other components of the string equation (2.24)(2.27), as we shall now demonstrate. Suppose $U\left(t_{5}, t_{2}, x\right), V\left(t_{5}, t_{2}, x\right)$ is a meromorphic solution to the full string equation. In other words, $U, V$ solve (1.1) in addition to equations (2.24)-(2.27). If we fix $t_{5}$, $t_{2}$, then in particular we have that $U, V$ are a solution to the ODE (1.1). It follows that $U, V$ must have one of the expansions from Proposition A.1, where the undetermined coefficients of this expansion depending on $t_{2}, t_{5}$ meromorphically (for example, in Case (A.1), we would have that $u_{0}=u_{0}\left(t_{5}, t_{2}\right), \cdots, u_{3}=u_{3}\left(t_{5}, t_{2}\right), v_{0}=$ $\left.v_{0}\left(t_{5}, t_{2}\right), v_{1}=v_{1}\left(t_{5}, t_{2}\right)\right)$. Suppose $U, V$ indeed have one of the expansions (A.1)-(A.4). $U$ and $V$ must also satisfy the equation

$$
\frac{\partial U}{\partial t_{2}}=-2 \frac{\partial V}{\partial x}
$$

differentiating the series (A.1)-(A.4) term by term with respect to $x, t_{2}$ yields that (in cases (A.2),(A.3),
(A.4) additionally allowing $\left.x_{0}=x_{0}\left(t_{2}\right)\right)$ :

$$
\begin{aligned}
& (\mathrm{A} .1),(2.24) \Longrightarrow 0=\frac{\partial u_{0}}{\partial t_{2}}+2 v_{1}+\mathcal{O}\left(x-x_{0}\right) \\
& (\mathrm{A} .2),(2.24) \Longrightarrow 0=\frac{4}{\left(x-x_{0}\right)^{3}} \frac{\partial x_{0}}{\partial t_{2}} \pm \frac{12}{\left(x-x_{0}\right)^{4}}+\mathcal{O}(1) \\
& \text { (A.3), (2.24) } \Longrightarrow 0=\frac{4 \frac{\partial x_{0}}{\partial t_{2}}}{\left(x-x_{0}\right)^{3}}+\frac{4 v_{2}}{\left(x-x_{0}\right)^{3}}+\mathcal{O}(1) \\
& \text { (A.4), }(2.24) \Longrightarrow 0=-\frac{20}{\left(x-x_{0}\right)^{3}} \frac{\partial x_{0}}{\partial t_{2}}-\frac{25}{147} \frac{\partial x_{0}}{\partial t_{2}} t_{5}^{2}\left(x-x_{0}\right)-\frac{t_{2}}{7}\left(x-x_{0}\right)+\mathcal{O}\left(\left(x-x_{0}\right)^{2}\right) .
\end{aligned}
$$

We see that the first and third lines determine a consistent system of PDEs for the coefficients $u_{k}, v_{k}$, whereas the second and last lines lead to a contradiction, whether or not $\frac{\partial x_{0}}{\partial t_{2}}$ is nonzero. In other words, the requirement that $U, V$ additionally satisfy (2.24)-(2.27) implies that the behaviors (A.2), (A.4) cannot appear in the solution to the string equation, as they lead to the above contradictions. One can readily check that these expansions are also consistent with equations (2.26), (2.25), and (2.27).

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[^1]:    ${ }^{1}$ Here, $\{x, \lambda\}$ denotes the Schwarzian derivative of $x$ with respect to $\sigma:\{x, \sigma\}:=\frac{\dddot{x}}{\dot{x}}-\frac{3}{2}\left(\frac{\ddot{x}}{\dot{x}}\right)^{2}$, where $\dot{x}=\frac{d x}{d \sigma}$. This is the only place we will see the appearance of the Schwarzian derivative in this work; we hope our notation will not cause later confusion when $\{\cdot, \cdot\}$ will represent the Poisson bracket.

[^2]:    ${ }^{2}$ The factor of $-\frac{3}{4} U \psi$ added to the first entry is only to make the resulting matrices look more symmetric; this is an aesthetic choice, and is not essential. One can undo this by making an appropriate gauge transformation.

[^3]:    ${ }^{4}$ This ansatz is essentially what one might infer by integrating $-\frac{1}{12} U^{(4)}+\frac{5}{12} t_{5} U^{\prime \prime}-\frac{3}{4} U U^{\prime \prime}$, since $\frac{\partial}{\partial t_{1}} U U^{\prime}=\left(U^{\prime}\right)^{2}+U U^{\prime \prime}$.

[^4]:    ${ }^{5}$ By "modulo the string equation" we mean that we must use the string equation to replace higher order derivatives of the functions $U, V$ in the variable $x$.

[^5]:    ${ }^{6}$ One should calculate the coefficients up to $\Phi_{3}$; from here, all of the relations stated can be inferred. This inference is direct for all of the relations except the one for $H_{5}$; this relation can be inferred from the rest of the relations, and the formula (3.9) for $H_{5}$.

[^6]:    ${ }^{7}$ Actually, one can extend all the definitions naturally to the universal covering $\tilde{\mathcal{T}}$ of $\mathcal{T}$, and this is really where we want to define $\mathbf{d}_{\mathcal{T}}$. However, we will not be going far enough into this subject for this distinction to make much of a difference.

