

Approximate Realizations for Outerplanar Degree Sequences*

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May 7, 2024

Abstract

We study the question of whether a sequence $d = (d_1, d_2, \dots, d_n)$ of positive integers is the degree sequence of some outerplanar (a.k.a. 1-page book embeddable) graph G . If so, G is an outerplanar realization of d and d is an outerplanar sequence. The case where $\sum d \leq 2n - 2$ is easy, as d has a realization by a forest (which is trivially an outerplanar graph). In this paper, we consider the family \mathcal{D} of all sequences d of even sum $2n \leq \sum d \leq 4n - 6 - 2\omega_1$, where ω_x is the number of x 's in d . (The second inequality is a necessary condition for a sequence d with $\sum d \geq 2n$ to be outerplanar.) We partition \mathcal{D} into two disjoint subfamilies, $\mathcal{D} = \mathcal{D}_{NOP} \cup \mathcal{D}_{2PBE}$, such that every sequence in \mathcal{D}_{NOP} is provably non-outerplanar, and every sequence in \mathcal{D}_{2PBE} is given a realizing graph G enjoying a 2-page book embedding (and moreover, one of the pages is also bipartite).

*This work was supported by US-Israel BSF grant no. 2022205.

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1 Introduction

Background. Our study is concerned with degree sequences of graphs. The degree of a vertex in a graph G is the number of its incident edges, and the sequence of vertex degrees of G is denoted by $\deg(G)$. Studied for more than 60 years, the DEGREE REALIZATION problem requires to decide, given a sequence d of positive integers, whether d has a *realizing graph*, namely, a graph G such that $\deg(G) = d$, and to find such a graph if exists. A sequence d that has a realizing graph is called *graphic*. One of the first characterizations of all graphic degree sequences was given by Erdős and Gallai [6]. About the same time, Havel and Hakimi [10, 11] described an algorithm that, given a sequence d , generates a realization if d is graphic or verifies that d is not graphic. Other studies focused on the uniqueness of the realization. It is known that a sequence can have several non-isomorphic realizations. As an example, consider the sequence¹ (2^6) , which can be realized by a 6-vertex cycle or by two 3-vertex cycles. This issue is not of our concern in the current paper.

The realization problem has also been investigated for various well known classes of graphs. In particular, realization by planar graphs was considered in several papers, but currently a complete solution seems out of reach. The survey by Rao [18] listed a number of partial results and studies on related questions. In this paper, we address the question whether a given sequence has an *outerplanar* realization. Outerplanar graphs are planar graphs that can be embedded in the plane so that edges do not intersect each other and additionally, each vertex lies on the outer face (i.e., no vertex is surrounded by edges) [21].

A sequence $d \in \mathcal{D}$ is *outerplanar* if it has a realizing outerplanar graph. Similarly, a sequence $d \in \mathcal{D}$ is *maximal outerplanar (MOP)* if it has a realizing *maximal* outerplanar graph (namely, an outerplanar graph with the property that adding any edge to it violates outerplanarity).

Throughout, we denote by $d = (d_1, d_2, \dots, d_n)$ a nonincreasing sequence of n positive integers and denote its *volume* by $\sum d = \sum_{i=1}^n d_i$. We use the shorthand a^k to denote a subsequence of k consecutive integers a , and denote the multiplicity of degree x in a sequence d by ω_x . By the well-known Handshaking Lemma, $\sum d$ is even for every graphic sequence d . If $\sum d \leq 2n - 2$ and $\sum d$ is even, then d has a forest realization with $(2n - \sum_i d_i)/2$ components [9]. Therefore, we focus on d with $\sum d \geq 2n$. We show later (Lemma 4) that if d is outerplanar but not forestic (i.e., $\sum d > 2n - 2$), then also $\sum d \leq 4n - 6 - 2\omega_1$, and moreover, $\sum d = 4n - 6$ if and only if d is maximal outerplanar. Consequently, we focus on the family of nonincreasing sequences of n positive integers

$$\mathcal{D} = \{d = (d_1, \dots, d_n) \mid \sum d \text{ is even and } 2n \leq \sum d \leq 4n - 6 - 2\omega_1\}.$$

Let $\mathcal{D}_{\leq 4} \subseteq \mathcal{D}$ be the subclass of sequences whose largest degree satisfies $d_1 \leq 4$.

While we are unable to fully characterize the sequences of the family \mathcal{D} with respect to outerplanarity, we obtain a classification result that can be thought of as an “approximate” solution to the outerplanarity problem. To do that, we employ the concept of *book embeddings* which is used in graph theory to measure how far a graph is *away* from being outerplanar. Given a graph $G = (V, E)$, let $E = E_1 \cup \dots \cup E_p$ be a partition of its edges such that each subgraph $G_i = (V, E_i)$ is outerplanar. For a book embedding of G , think of a book in which the pages (half-planes) are filled by outerplanar embeddings of the G_i ’s such that the vertices are embedded on the spine of the book and in the same location on each page. This constraint is equivalent to requiring that the

¹We use the shorthand a^k to denote a subsequence of k consecutive a ’s.

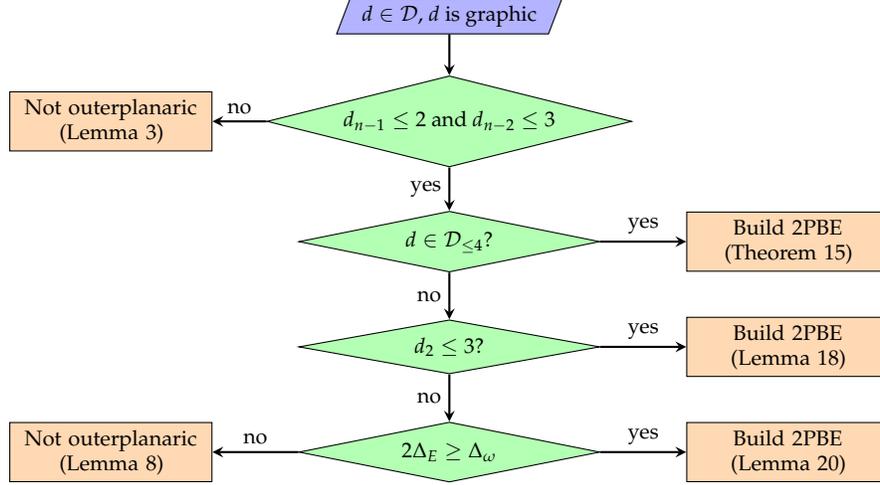


Figure 1: A flowchart of the resulting algorithm. $\Delta_E = (4n - 6 - \sum d)/2$ and $\Delta_\omega = 3\omega_1 + 2\omega_2 + \omega_3 - n$.

vertices appear in the same order along the cyclic order (defined later as the vertex ordering on the outer face) of each of the outerplanar embeddings of the G_i 's. The *book thickness* or *pagenumber* [2] is the minimal number of pages for which a graph has a valid book embedding. Note that a graph is outerplanar if and only if it has pagenumber 1, and it is known that the pagenumber of planar graphs is at most 4 [23].

Our contribution. Our main result is a classification of \mathcal{D} into two sub-families, $\mathcal{D} = \mathcal{D}_{NOP} \cup \mathcal{D}_{2PBE}$, such that

- every sequence in \mathcal{D}_{NOP} is provably non-outerplanar, and
- every sequence in \mathcal{D}_{2PBE} is given (in polynomial time) a realizing graph $G = (V, E)$, $E = E_1 \cup E_2$, enjoying a 2-page book embedding with pages G_1 and G_2 (hereafter referred to as a *2PBE realization*). Moreover, G_2 is bipartite (hereafter, an *OP+bi realization*), and in some cases, G_2 consists of at most one or two edges (hereafter, an *OP+1* or *OP+2 realization*).

Note that every *outerplanar* sequence in \mathcal{D} falls in \mathcal{D}_{2PBE} . Thus, the above result can be interpreted as yielding a polynomial time algorithm providing an “approximation” for outerplanar sequences, in the sense that if the given sequence d is outerplanar (i.e., has a 1-page realization), then the algorithm will produce a 2-page realization for it. We remark that for many of the outerplanar sequences in \mathcal{D} , we are in fact able to find an outerplanar realization, but as this refined classification is incomplete, it is omitted here.

Figure 1 gives an overview of the algorithm.

Additional related work. For the OUTERPLANAR DEGREE REALIZATION problem, a full characterization of *forcibly* outerplanar graphic sequences (namely, sequences each of whose realizations is outerplanar) was given in [5]. A characterization of the degree sequence of maximal outerplanar graphs having exactly two 2-degree nodes was provided in [17]. This was also mentioned in [3]. A characterization of the degree sequences of maximal outerplanar graphs with at most four vertices of degree 2 was given in [14].

Sufficient conditions for the realization of 2-trees were given in [15]. A full characterization of the degree sequences of 2-trees was presented in [3]. A graph G is a 2-tree if $G = K_3$ or G has a vertex v with degree 2, whose neighbors are adjacent and $G[V \setminus \{v\}]$ is a 2-tree. This definition implies, in particular, that 2-trees have $\sum d = 4n - 6$. By Theorem 1 of [3], if a sequence d is maximal outerplanar, i.e., it is outerplanar and satisfies $\sum d = 4n - 6$, then d can be realized by a 2-tree. In [19] it was shown that every 2-tree has a 2-page book embedding. Note that the class of 2-trees is not hereditary, i.e., subgraphs of 2-trees are not necessarily 2-trees. Therefore, the result of [3] does not apply to non-maximal degree sequences.

For the PLANAR DEGREE REALIZATION problem, regular planar graphic sequences were classified in [12], and planar bipartite biregular degree sequences were studied in [1]. In [20], Schmeichel and Hakimi determined which graphic sequences with $d_1 - d_n = 1$ are planar, and presented similar results for $d_1 - d_n = 2$, with a small number of unsolved cases. Some of the sequences left unsolved in [20] were later resolved in [7, 8]. Some additional studies on special cases of the planar degree realization problem were listed in [18].

2 Preliminaries

Definitions and notation. Given two n -integer sequences d and d' , denote their component-wise addition and difference, respectively, by $d \oplus d' = (d_1 + d'_1, \dots, d_n + d'_n)$ and $d \ominus d' = (d_1 - d'_1, \dots, d_n - d'_n)$. For a nonincreasing sequence d of n nonnegative integers, let $\text{pos}(d)$ denote the prefix consisting of the positive integers of d (discarding the zeros). Given two sequences d' and d'' of length n' and n'' respectively, let $d' \circ d'' = (d'_1, \dots, d'_{n'}, d''_1, \dots, d''_{n''})$.

Consider a simple (undirected) graph $G = (V, E)$. A (directed) *circuit* in G is an *ordered set* of vertices $C = \{v_0, v_1, \dots, v_{k-1}\}$ (with possible repetitions) such that $(v_i, v_{(i+1) \bmod k}) \in E$ for every $i = 0, \dots, k - 1$. The directed edge set of the circuit C is $E(C) = \{\langle v_i, v_{(i+1) \bmod k} \rangle \mid i = 0, \dots, k - 1\}$ and it is required that no directed edge may appear on it more than once. If the vertices of C are distinct then it is called a *cycle*. A *triangle* is a cycle consisting of three edges.

For a graph $G = (V, E)$, one may define an accompanying embedding in the plane, namely, a mapping of every vertex to a point in two-dimensional space, with a curve connecting the two endpoints u, w of every edge $(u, w) \in E$. An *outerplanar* graph G is a graph that has a planar embedding in which all the vertices occur on the outer face. (This embedding can be transformed into a 1-page book embedding of G , with all vertices placed on the *spine* of the book.) We refer to the order in which the vertices occur on the outer face as the *cyclic-order* of G , and denote it by $C_{\text{order}}(G)$. A *maximal outerplanar* (MOP) graph G is an outerplanar graph such that adding any new edge to it results in a non-outerplanar graph. It is folklore that for a 2-connected outerplanar graph, the cyclic order corresponds to a Hamiltonian cycle in G .

Hereafter, whenever we discuss a given outerplanar (resp., 2PBE) graph, we implicitly assume that some outerplanar (resp., 2PBE) embedding for it is given as well, and when constructing an outerplanar (resp., 2PBE) graph, we also give an outerplanar (resp., 2PBE) embedding for it.

Given an outer-planar embedding, an *external* edge is an edge residing on the outer face. That is, $e = (u, w)$ is external if u and w are neighbors on a circuit which is part of the outer face. All other edges of G are *internal*. An *outer* (respectively, *inner*) *cycle* in G is a cycle all of whose edges are external (resp., internal). Note that the outer face may be comprised of several outer-cycles (as can be seen in Figure 2).

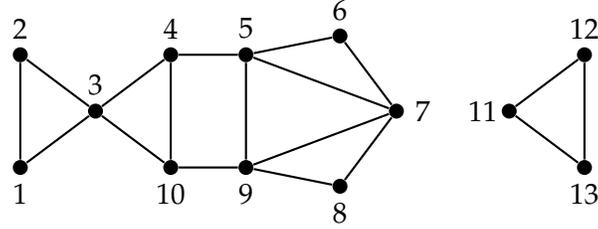


Figure 2: An outerplanar graph without degree-1 vertices. The triangle $(5, 7, 9)$ is the only internal cycle, while $(1, 2, 3)$, $(3, 4, 5, 6, 7, 8, 9, 10)$, and $(11, 12, 13)$ are outer-cycles. A possible cyclic-order is $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$.

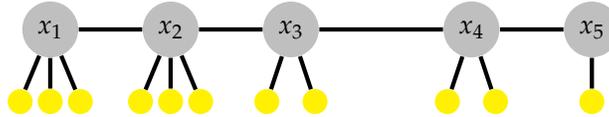


Figure 3: A caterpillar graph with degree sequence $(5, 4^3, 2, 1^{11})$. Spine vertices are gray, leaves are yellow.

Degree sequences of trees and forests. Consider a nonincreasing sequence $d = (d_1, \dots, d_n)$ of positive integers such that $\sum d$ is an even number. It is known that if $\sum d \leq 2n - 2$ then d is graphic, and moreover, d can be realized by an acyclic graph (forest). In this case, d is called a *forestic* sequence. If $\sum d = 2n - 2$, then d can be realized only by trees, and d is called a *treeic* sequence. A vertex of degree one is called a *leaf*.

We make use of a special type of realizations of treeic sequences by *caterpillar graphs*. In a caterpillar graph $G = (V, E)$, all the non-leaves vertices are arranged on a path, called the *spine* of the caterpillar. Formally, the spine of a caterpillar graph is an ordered sequence $S = (x_1, \dots, x_s) \subseteq V$, $s \geq 1$, in which $(x_i, x_{i+1}) \in E$ for $i = 1, \dots, s - 1$ (see Figure 3 for an example).

The other ingredient in the construction of a realization to a forestic sequence is a collection of disjoint edges, hereafter referred to somewhat informally as a *matching*.

Observation 1. A forestic sequence $d = (d_1, d_2, \dots, d_n)$ of positive integers can be realized by a union of a caterpillar graph and a matching.

Proof. First consider a treeic sequence d (such that $\sum d = 2n - 2$). Assume there are $n - s$ vertices of degree 1 in d . Denote by d^* the prefix of d that contains all the degrees $d_i > 1$. It follows that $\sum d^* + (n - s) = \sum d = 2n - 2$, or equivalently $\sum d^* - 2s + 2 = n - s$. To get the caterpillar realization of d , first arrange s vertices corresponding to the degrees of d^* in a path. The path edges contribute $2s - 2$ to the volume $\sum d^*$. The missing $\sum d^* - 2s + 2 = n - s$ degrees are satisfied by attaching the $n - s$ leaves to the path, which now forms the spine of the caterpillar realization. Note that the order of the vertices on the spine can be arbitrary.

Now assume that d is forestic but not treeic, i.e., $\sum d < 2n - 2$. In this case, remove pairs of 1 degrees from d until the volume of the reaming sequence d' with n' vertices is $2(n' - 2)$. This must happen because each pair removal reduces the volume by 2 while $2n - 2$ decreases by 4. Let G' be the caterpillar realization of d' as implied by the first part of the proof. To get the realization of d , add $(n - n')/2$ edges to G' to satisfy the $n - n'$ removed 1 degrees by a matching. \square

Some known results. We next summarize known conditions for sequence outerplanarity.

Lemma 2 ([22]). *If $d = (d_1, \dots, d_n)$ is an outerplanar degree sequence where $n \geq 2$, then $\sum d \leq 4n - 6$, with equality if and only if d is maximal outerplanar.*

Lemma 3 ([21]). *If $d = (d_1, \dots, d_n)$ is outerplanar, then (i) $d_{n-1} \leq 2$, and (ii) $d_{n-2} \leq 3$. Moreover, $d_{n-1} = d_n = 2$ in a maximal outerplanar sequence d .*

Our next observation is that Lemma 2 can be improved if ω_1 , the multiplicity of degree 1 in d is taken into consideration. Specifically, for the case $\sum d > 2n - 2$, we have the following.

Lemma 4. *If $d = (d_1, \dots, d_n)$ is outerplanar and $\sum d > 2n - 2$, then $\sum d \leq 4n - 6 - 2\omega_1$.*

Proof. Let d be as in the lemma and let $G = (V, E)$ be an outerplanar graph that realizes d . Construct a graph G' by deleting from G all the ω_1 vertices of degree 1 together with their incident edges, and subsequently deleting the vertices whose degree was reduced to zero. The graph G' may contain vertices whose degree became 1. Observe that G' is outerplanar, and let $d' = \deg(G')$. Note that $n' \leq n - \omega_1$, where n' the number of vertices in G' . Lemma 2 yields the claim, since

$$\sum d = \sum d' + 2\omega_1 \leq 4(n - \omega_1) - 6 + 2\omega_1 = 4n - 6 - 2\omega_1,$$

where the first equality comes from the fact that ω_1 vertices of degree 1 together with their incident edges are deleted, and the inequality is due to Lemma 2. \square

We remark that Lemma 4 is false without the assumption that $\sum d > 2n - 2$. To see this, consider the treeic sequence $d = (a, 1^a)$ which is (uniquely) realized by a star graph and therefore is outerplanar. The process in the proof of Lemma 4 yields G' consisting of one isolated vertex. Note that Lemma 2 cannot be applied in this case. Indeed, we have

$$\sum d = 2a > 2a - 2 = 4(a + 1) - 6 - 2a = 4n - 6 - 2\omega_1.$$

We now recall some facts concerning maximal outerplanar sequences (or MOP's).

Fact 1 ([16]). *A graph is a MOP if and only if it can be constructed from a triangle by a finite number of applications of repeatedly applying the following operation: to the graph already constructed, add a new vertex in the exterior face and join it to two adjacent vertices on that face.*

We rely on the construction method of Lemma 1 to prove the following simple but useful facts. Parts (1) and (2) are known (cf. [22, 4]). We could not find a reference for Part (3), so we give a proof for completeness.

Lemma 5. *Consider a maximal outerplanar graph G with $n \geq 4$ vertices.*

- (1) *Every two vertices of degree 2 in G are nonadjacent.*
- (2) *there is a outerplanar embedding of G such that the boundary of the outer face is a Hamiltonian cycle C and each inner face is a triangle.*
- (3) *Removing the edges of C from G and discarding the resulting degree-0 vertices yields a connected outerplanar graph.*

Proof. The lemma is proved by induction on the number of vertices n . The claim clearly holds for the base case of $n = 4$, where the 4-cycle augmented by a single chord is the only MOP. Now suppose the claim holds for all MOP graphs with at most n vertices, and consider a graph G with $n + 1$ vertices.

By Fact 1 G was obtained by adding a vertex v and edges (v, u) and (v, w) to some MOP graph G' in which $e = (u, w)$ is an external edge. By the inductive hypothesis, the claim holds for G' , so there exists an outer-cycle C' of G' which is Hamiltonian, and removing C' yields a connected graph $c(G')$. Note that u and w both appear in G' , so by Lemma 3 their degrees in G' are 2 or higher, and moreover, at least one of $\deg_{G'}(u)$ and $\deg_{G'}(w)$ is 3 or higher, since G' satisfies Property (1). This implies that at least one of u and w belongs to $c(G')$. Adding v to G and connecting it to u and w increases their degrees in G to 3 or higher, so v has no neighbor of degree 2 in G . Hence, property (1) holds in G by the inductive hypothesis. Also, inserting the vertex v between u and w in C' yields a Hamiltonian cycle C for G , implying Property (2). Finally, removing C from G yields the graph $c(G) = c(G') \cup \{e\}$, which is connected since $c(G')$ is connected and it contains u or w . \square

Jao and West [13] show that the number of internal triangles of a MOP is related to ω_2 .

Lemma 6. [13] *Let G be a maximal outerplanar graph on n vertices, let $d = \deg(G)$, and let t be the number of internal triangles. If $n \geq 4$, then $t = \omega_2 - 2$.*

Bipartite bracket balancing (BBB) algorithm. we make use of a simple variant of the BRACKET BALANCING problem, in which the matched brackets must fit in given positions. Formally, the problem, referred to as BIPARTITE BRACKET BALANCING (BBB), is defined as follows. We are given two ordered sets $P = \langle p_1, \dots, p_k \rangle$ and $Q = \langle q_1, \dots, q_k \rangle$, merged into a $2k$ -element sequence $R = \langle r_1, \dots, r_{2k} \rangle$ that contains all the elements from P and Q while maintaining their original order. The goal is to associate with every r_i , $1 \leq i \leq 2k$, a bracket $f(r_i) \in \{[,]\}$ with the following two properties:

- The sequence $f(R) = \langle f(r_1), \dots, f(r_{2k}) \rangle$ is balanced such that the number of opening brackets (“[”) in every prefix of R is no less than the number of closing brackets (“]”) in this prefix. Observe that in balanced sequences it is possible to match the opening brackets with the closing brackets so that the intervals associated with the matched brackets are either nested or disjoint.
- if $f(r_i) = [$ and $f(r_j) =]$ were matched, for $i < j$, then $r_i \in P$ and $r_j \in Q$ or vice versa.

This problem can be solved by a straightforward variant of the standard stack-based linear-time bracket balancing algorithm, named Algorithm BBB, described here for completeness. The algorithm uses a stack D and scans the sequence R from start to end. At any given time, the stack contains either only elements from P or only elements from Q (i.e., $D \subseteq P$ or $D \subseteq Q$). Upon reaching the element r_i , the algorithm acts based on the stack contents, as follows:

- $D = \emptyset$, or $D \subseteq P$ and $r_i \in P$, or $D \subseteq Q$ and $r_i \in Q$:
 $\text{PUSH}(r_i, D)$
- $D \subseteq P$ and $r_i \in Q$, or $D \subseteq Q$ and $r_i \in P$:
 $x \leftarrow \text{POP}(D)$; $f(x) \leftarrow [$; $f(r_i) \leftarrow]$

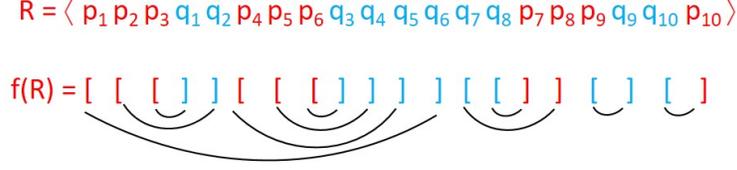


Figure 4: A merged sequence R and its corresponding bracket assignment.

It is easy to verify that the algorithm solves the BIPARTITE BRACKET BALANCING problem.

Figure 4 illustrates an example sequence R and the corresponding balanced bracket assignment generated by the algorithm. Notice that the stack switches its contents from time to time. In particular, it initially gets only elements of P , then becomes empty after inspecting $r_{12} = q_6$, then contains only elements of Q for a while, then becomes empty after inspecting $r_{16} = q_8$, and so on.

3 Necessary and Sufficient Conditions for Outerplanarity

This section presents some necessary and sufficient conditions for outerplanarity, which are used to prove the results of the main section concerning 2-page book embedding. The next lemma provides a necessary condition for maximal outerplanarity.

Lemma 7. *If $d = (d_1, \dots, d_n)$ is a maximal outerplanar sequence where $d_1 > 3$ then if $d_2 > 3$ or $\omega_2 > 2$ then $2\omega_2 + \omega_3 \leq n$ otherwise $2\omega_2 + \omega_3 \leq n + 1$.*

Proof. Let d be as in the lemma and let G be a maximal outerplanar realization of d . Observe that by Lemma 5(2), removing the outer-cycle from G and discarding the resulting 0-degree vertices results in an outerplanar graph G' with degree sequence $d' = \text{pos}(d \ominus (2^n))$ such that $n' = n - \omega_2$. It follows that d' is outerplanar. We consider the following three cases.

Case 1: $\omega_2 > 2$. G has an internal triangle due to Lemma 6. This triangle still exists in G' . Moreover, G' is connected by Lemma 5(3). It follows that $\sum d' > 2n' - 2$. By Lemma 4,

$$2n - 6 = \sum d - 2n = \sum d' \leq 4n' - 6 - 2\omega_1' = 4(n - \omega_2) - 6 - 2\omega_3.$$

Simplifying yields that $2\omega_2 + \omega_3 \leq n$.

Case 2: $\omega_2 = 2$ and $d_2 > 3$. We have that $\omega_3 \leq n - 4$ and $2\omega_2 + \omega_3 = 4 + \omega_3 \leq n$.

Case 3: $\omega_2 = 2$ and $d_2 \leq 3$. Since $d_1 > 3$, we have that $\omega_3 \leq n - 3$. It follows that $2\omega_2 + \omega_3 = 4 + \omega_3 \leq n + 1$.

The lemma follows. □

Note that this lemma does not hold for non-maximal outerplanar sequences. For example, the outerplanar sequence $(4, 2^4)$ is a counterexample for which $2\omega_2 + \omega_3 = 8 = n + 3$.

$\deg_{G_i}(u), \deg_{G_i}(v)$	δ
$(1, 1) \rightarrow (2, 2)$	-2
$(1, 2) \rightarrow (2, 3)$	-2
$(1, 3) \rightarrow (2, 4+)$	-2
$(1, 4+) \rightarrow (2, 5+)$	-1
$(2, 2) \rightarrow (3, 3)$	-2

$\deg_{G_i}(u), \deg_{G_i}(v)$	δ
$(2, 3) \rightarrow (3, 4+)$	-2
$(2, 4+) \rightarrow (3, 4+)$	-1
$(3, 3) \rightarrow (4+, 4+)$	-2
$(3, 4+) \rightarrow (4+, 4+)$	-1
$(4+, 4+) \rightarrow (4+, 4+)$	0

We next define two functions that play a key role in our analysis. For a sequence d , define the *edge deficit* and *low degree surplus* of d to be, respectively,

$$\Delta_E(d) = (4n - 6 - \sum d)/2, \quad \Delta_\omega(d) = 3\omega_1 + 2\omega_2 + \omega_3 - n.$$

Note that $\Delta_E(d) \geq 0$ for any outerplanar sequence. If $\Delta_E(d) > 0$ and G is a nonmaximal outerplanar realization of d , then $\Delta_E(d)$ is the number of edges that need to be added to G to generate a maximal outerplanar graph. If d is clear from the context we write simply Δ_E, Δ_ω instead of $\Delta_E(d), \Delta_\omega(d)$. The next lemma shows a connection between Δ_E and Δ_ω .

Lemma 8. *Let $d = (d_1, \dots, d_n)$ be an outerplanar sequence with $d_2 > 3$. Then $2\Delta_E \geq \Delta_\omega$.*

Proof. Let d be as in the lemma, and let G be an outerplanar realization of d . We can add Δ_E edges to G so as to transform it into a maximal outerplanar graph G' (without adding vertices). For $0 \leq i \leq \Delta_E$, let G_i be the graph G after adding i edges, and let d^i be the degree sequence of G_i . By definition, $G_0 = G$ and $G_{\Delta_E} = G'$. Define $f(G_i) = 2\Delta_E(d^i) - \Delta_\omega(d^i)$ for $0 \leq i \leq \Delta_E$. We show:

$$f(G_i) \geq f(G_{i+1}) \text{ for any } 0 \leq i \leq \Delta_E - 1. \quad (1)$$

Let $d' = \deg(G')$. Then $\sum d' = 4n - 6$ and $\omega'_1 = 0$. Lemma 7 implies that $2\omega'_2 + \omega'_3 \leq n$. Thus, $\Delta_E(d') = 0$ and $\Delta_\omega(d') = 3\omega'_1 + 2\omega'_2 + \omega'_3 - n \leq 0$. Hence,

$$f(G_{\Delta_E}) = 2\Delta_E(d') - \Delta_\omega(d') \geq 0. \quad (2)$$

We now compute how adding a single edge (u, v) to G_i decreases the value of $3\omega_1(G_i) + 2\omega_2(G_i) + \omega_3(G_i)$. There are ten possibilities for the changes in the degrees of u and v ($\deg_{G_i}(u)$ and $\deg_{G_i}(v)$) that cause $3\omega_1(G_i) + 2\omega_2(G_i) + \omega_3(G_i)$ to change by δ . We list the ten possibilities in the above table where we denote degrees of 4 or more by $4+$. It follows that adding a single edge decreases $3\omega_1(G_i) + 2\omega_2(G_i) + \omega_3(G_i)$ by at most 2 which implies that $\Delta_\omega(d^i)$ is also decreased by at most 2. Also, $2\Delta_E(d^i)$ is decreased by exactly 2 when adding a single edge to G_{i-1} . Therefore, Inequality (1) holds. Combining this with inequality (2) implies that $f(G_0) \geq 0$. The lemma follows because $f(G_0) = 2\Delta_E - \Delta_\omega$. \square

We start with a sequence d satisfying $d_n = 2$ and $\omega_2 = 2$. To prove our next result, we generate a reduced sequence d' of d where each degree is decreased by 2. We verify that d' is forestic and realize it by a graph G' constructed as the union of a caterpillar graph and a matching, following Observation 1. Then, a realization G of d is found by adding a Hamiltonian cycle to G' . The Hamiltonian cycle is the outer-cycle of G . It follows that in this case, there is a realization of d where each vertex appears exactly once on the outer-cycle.

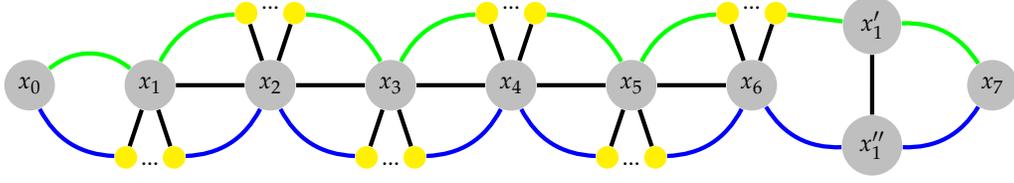


Figure 5: Illustration of G as constructed in Lemma 9. The vertices x_1, \dots, x_6 and the leaves in yellow (labels omitted) together with the black edges form the caterpillar graph G' . Note that x_5 has no leaves. Paths P_1 and P_2 are depicted in green and blue, respectively. Together, they form the cycle C .

Lemma 9. *Let $d \in \mathcal{D}$. If*

- (i) $d_n = 2$ and
- (ii) $\omega_2 = 2$,

then d is outerplanar, and has a realization with an outerplanar embedding that has a Hamiltonian outer-cycle.

Proof. Let d be as in the lemma. First, we construct the sequence $d' = (d'_1, \dots, d'_{n'})$ by subtracting 2 from each degree of d , i.e., let $d'_i = d_i - 2$ for $i \in \{1, \dots, n-2\}$. Note that $n' = n - 2$ and that $\sum d'$ is even. Moreover, note that d' is forestic, since $\sum d \leq 4n - 6$. Hence,

$$\sum d' = \sum d - 2n \leq 2n - 6 = 2(n' + 2) - 6 = 2n' - 2.$$

By Observation 1, d' can be realized by a graph $G' = (V', E')$ composed of the union of a caterpillar graph T' and a matching M' . Let $S = (x_1, \dots, x_s)$ be the vertices on the spine of T' , and let $X_i = \{\ell_{i,1}, \dots, \ell_{i,k_i}\} \subseteq V'$ be the leaves adjacent to the spine vertex x_i , for $i \in \{1, \dots, s\}$. Note that vertex x_i has degree $k_i + 1$ if $i \in \{1, s\}$, otherwise it has degree $k_i + 2$. Assume $M' = \{(x'_1, x''_1), \dots, (x'_t, x''_t)\}$. To find an outerplanar realization of d , we add a set of edges to G' that form a Hamiltonian cycle (including two additional vertices of degree 2). We explain our construction in four steps for which we assume that s is odd. If s is even, an analogous construction applies.

- (1) Add the edges along the following two paths $P_1 = (x_1, x_3, \dots, x_{s-2}, x_s)$ and $P_2 = (x_{s-1}, x_{s-3}, \dots, x_4, x_2)$ connecting spine vertices in odd and even positions, respectively.
- (2) Add two additional vertices x_0, x_{s+1} and use them to form a cycle C together with P_1 and P_2 . Let $C = (x_0, x_1, x_3, \dots, x_s, x_{s+1}, x_{s-1}, x_{s-3}, \dots, x_4, x_2, x_0)$.
- (3) The leaves of a vertex in P_1 are inserted into C between two vertices of P_2 . Specifically, the leaves of x_i , for $i = 1, \dots, s$, are inserted between x_{i-1} and x_{i+1} .
- (4) Insert vertices x'_1, \dots, x'_t on the segment between ℓ_{s,k_s} and x_{s+1} in clockwise order. Similarly, insert vertices x''_1, \dots, x''_t on the segment between x_{s+1} and x_s in a counter-clockwise order. Connect x'_i and x''_i for $1 \leq i \leq t$.

Figure 5 illustrates the construction of G . Following the four steps, we have

$$C = (x_0, x_1, \ell_{2,1}, \ell_{2,2}, \dots, \ell_{2,k_2}, x_3, \ell_{4,1}, \ell_{4,2}, \dots, \ell_{4,k_2}, x_5, \dots, x_{s-1}, \ell_{s,1}, \ell_{s,2}, \dots, \ell_{s,k_s}, x'_1, \dots, x'_t, x_{s+1}, x''_1, \dots, x''_t, x_s, \ell_{s-1,k_{s-1}}, \ell_{s-1,k_{s-1}-1}, \dots, x_2, \ell_{1,k_1}, \ell_{1,k_1-1}, \dots, \ell_{1,1}, x_0).$$

Formally, let $V = V' \cup \{x_0, x_{s+1}\}$. The set E' contains the path connecting the spine vertices, $\{(x_i, x_{i+1}) \mid i = 1, 2, \dots, s-1\}$, and the edges connecting each spine vertex to its leaves. The set L contains edges among the leaves while the sets A_1 and A_2 contain edges connecting spine and leaf vertices

$$\begin{aligned} L &= \{(\ell_{i,j}, \ell_{i,j+1}) \mid i = 1, \dots, s, \quad j = 1, \dots, k_i - 1\}, \\ A_2 &= \{(x_i, \ell_{i+1,1}) \cup (\ell_{i+1,k_{i+1}}, x_{i+2}) \mid i = 1, \dots, s \text{ and } k_{i+1} > 0\}, \\ A_1 &= \{(x_i, x_{i+2}) \mid i = 1, \dots, s \text{ and } k_{i+1} = 0\} \end{aligned}$$

Recall that $M' = \{(x'_1, x''_1), \dots, (x'_t, x''_t)\}$. Note that A_2, A_1, L, E', M' are pairwise disjoint. Let $E = E' \cup L \cup A_2 \cup A_1 \cup M'$, and let $G = (V, E)$. Notice that Figure 5 depicts an outerplanar embedding of G . By the construction of G , every vertex in V appears on the Hamiltonian outer-cycle. No two edges in $E(G)$ cross each other. Therefore, G is an outerplanar graph. By the definition of d' , two vertices of degree 2 are deleted, which correspond to x_0, x_{s+1} . Every vertex in $V \setminus \{x_0, x_{s+1}\}$ has its degree decreased by 2. Recall that $T' \cup M'$ realizes d' . After adding C to T' and inserting M' in step (4), every vertex $i \in V \setminus \{x_0, x_{s+1}\}$ has its degree increased by 2, making it equal to d_i . Combining this with the fact that x_0 and x_{s+1} have degree 2 in G , we have $\deg(G) = d$. The lemma follows. \square

Corollary 10. $d \in \mathcal{D}$ is outerplanaric if $\sum d \leq 4n - 2\omega_2 - 2$ and $d_{n-1} = d_n = 2$.

Proof. Let d'' be the prefix of d satisfying $d = d'' \circ (2^{\omega_2})$. Consider the sequence $d' = d'' \circ (2^2)$ of length $n' = n - \omega_2 + 2$. Observe that $\sum d'$ is even. As $\sum d \leq 4n - 2\omega_2 - 2$, it follows that

$$\sum d' = \sum d - 2(\omega_2 - 2) \leq 4n - 4\omega_2 + 2 = 4(n - \omega_2 + 2) - 6 = 4n' - 6.$$

Sequence d' satisfies the preconditions of Lemma 9. Hence, let G' be an outerplanar realization and embedding of d' with a Hamiltonian outer-cycle. An outerplanar realization of d is obtained by replacing an edge on this outer-cycle by a path of length $\omega_2 - 2$. \square

4 Approximate Realizations by Book Embeddings

This section concerns finding approximate realizations. Our main result is a classification of \mathcal{D} into two sub-families, $\mathcal{D} = \mathcal{D}_{NOP} \cup \mathcal{D}_{2PBE}$, such that

- every sequence in \mathcal{D}_{NOP} is provably non-outerplanaric, and
- every sequence in \mathcal{D}_{2PBE} is given a realizing graph enjoying a 2-page book embedding (hereafter referred to as a *2PBE realization*).

Note that if a graph has page number 2, then it is planar (the converse is false).

4.1 A 2PBE realization for sequences with $d_1 \leq 4$

In this section, we analyze the subclass $\mathcal{D}_{\leq 4} \subseteq \mathcal{D}$ of sequences whose largest degree satisfies $d_1 \leq 4$. We show that every $d \in \mathcal{D}_{\leq 4}$ enjoys a 2PBE realization. Lemmas 12, 13 and 14 imply Theorem 15.

First, consider the case where $\omega_3 > 0$ and $d_{n-1} = d_n = 2$.

Lemma 11. Every $d \in \mathcal{D}_{\leq 4}$ s.t. $d_n = d_{n-1} = 2$ and $\omega_3 > 0$ is outerplanar.

Proof. The degree sum $\sum d$ is even, so necessarily $\omega_3 \geq 2$. Since $n = \omega_4 + \omega_3 + \omega_2$, it follows that $\sum d = 4\omega_4 + 3\omega_3 + 2\omega_2 \leq 4n - 2\omega_2 - 2$, which satisfies the conditions of Corollary 10. Therefore, d is outerplanar. \square

Lemma 12. Every $d \in \mathcal{D}_{\leq 4}$ s.t. $d_n = d_{n-1} = 2$ and $d_1 \leq n - 1$ has an OP+2 realization.

Proof. When $\omega_3 > 0$, the claim follows by Lemma 11. So hereafter suppose $\omega_3 = 0$. We split the analysis into three cases.

Case 1: $\omega_4 = 0$. Then $d = (2^n)$, which for $n \geq 3$ can be realized by a cycle, hence it is outerplanar.

Case 2: $\omega_4 = 1$. Then $\omega_2 \geq 4$ since $\omega_3 = 0$ and $d_1 = 4 \leq n - 1$. In this case, construct an outerplanar graph G as follows. First, connect vertex v_1 of degree 4 to vertices v_2, v_3, v_4, v_5 of degree 2. Next, add edges $(v_2, v_3), (v_4, v_5)$ and insert $\omega_2 - 4$ vertices into the edge (v_2, v_3) , i.e., transform this edge into a path $(v_2, u_1, \dots, u_{\omega_2-4}, v_3)$. Notice that G is outerplanar.

Case 3: $\omega_4 \geq 2$. Here, d consists of only 2's and 4's. Since $\sum d \leq 4n - 6$, necessarily

$$\omega_2 \geq 3. \quad (3)$$

Generate a sequence d' by removing one degree 2 from d and reducing d_1, d_2 by 1. Then $d' = (4^{\omega_4-2}, 3^2, 2^{\omega_2-1})$. It is easy to verify, using Eq. (3), that d' satisfies the preconditions of Lemma 11. Hence, let G' be an outerplanar realization of d' by Lemma 11. We create a 2-page book embedding graph G realizing d based on $G' = (V', E')$. Let $v_1, v_2 \in V'$ be the vertices with $\deg_{G'}(v_1) = \deg_{G'}(v_2) = 3$. Add an isolated vertex u to G' and obtain an outerplanar graph $G_1 = (V, E')$ with $V = V' \cup \{u\}$. Let $G_2 = (V, E_2)$ with $E_2 = \{(v_1, u), (v_2, u)\}$. The position of u in the embeddings of G' and G_2 can be fixed arbitrarily since u is isolated in G_1 , and G_2 has only two edges that do not exist in \bar{G}' . Hence, G_1 and G_2 form a 2PBE realization for d . Notice that G_1 is outerplanar and G_2 consists of two edges.

The lemma follows. \square

Lemma 13. Every $d \in \mathcal{D}$ s.t. $d_n \geq 2$ and $d_{n-1} \geq 3$ has an OP+1 realization.

Proof. Since $\sum d \leq 4n - 6$ and $d_n \geq 2$, we have $d_{n-1} \leq 3$. Consider the two possible cases, $(d_{n-1}, d_n) \in \{(2, 3), (3, 3)\}$.

Case 1: $d_n = d_{n-1} = 3$. Then construct a sequence d' by replacing (d_{n-1}, d_n) with $(2, 2)$. Note that d' satisfies the conditions in Lemma 9 and it can be realized by an outerplanar graph G' with two nonadjacent vertices v', v'' of degree 2. Then d can be realized by a graph G obtained by adding the edge (v', v'') to G' . One can verify that G has a 2-page book embedding (with G' on one page and the added edge on the other).

Case 2: $d_n = 2$ and $d_{n-1} = 3$. In this case, $d_{n-2} = 3$ since $\sum d \leq 4n - 6$. Construct d' by replacing (d_{n-2}, d_{n-1}) with $(2, 2)$. Note that $\sum d' \leq 4n - 8$ and $\omega'_2 = 3$, so d' satisfies the conditions in Corollary 10 and there is an outerplanar graph G' realizing d' . By the construction of G' , it has two nonadjacent vertices v', v'' of degree 2. Therefore, we can construct a graph G by adding the edge (v', v'') to G' . Again, G has a 2-page book embedding.

Notice that in both cases G' is outerplanar and the other page consists of one edge. The lemma follows. \square

It remains to deal with sequences $d \in \mathcal{D}_{\leq 4}$ that contain also 1's. We show the following.

Lemma 14. *Every $d \in \mathcal{D}_{\leq 4}$ s.t. $d_n = 1$ and $d_1 \leq n - 1$ has an OP+2 realization.*

Proof. First, in ω_1 steps, delete one degree 1 from the sequence and decrease the current largest degree by 1, yielding the residual sequence d' , whose length is $n' = n - \omega_1$. Since $\sum d \leq 4n - 2\omega_1 - 6$ and $\sum d \geq 2n$,

$$\sum d' \leq 4(n - \omega_1) - 6, \quad d'_{n-\omega_1} \geq 2 \quad (4)$$

Consider the following two cases.

Case 1: $\omega_1 \leq \omega_4$. Then $d' = (4^{\omega_4 - \omega_1}, 3^{\omega_3 + \omega_1}, 2^{\omega_2})$. and that when $n' = 4$, d' is OP + 1. In this case, if $\omega_2 \geq 2$ (respectively, $\omega_2 \leq 1$), then d' has a 2PBE realization G' by Eq. (4) and Lemma 12, (resp., 13). Notice that for $\omega_2 \geq 2$, $d'_1 \leq n' - 1$ since $\omega_3 + \omega_1 \geq 2$.

Case 2: $\omega_1 > \omega_4$. Then

$$d' = (3^{\omega_3 + \omega_4 - (\omega_1 - \omega_4)}, 2^{\omega_2 + (\omega_1 - \omega_4)}) = (3^{\omega_3 + 2\omega_4 - \omega_1}, 2^{\omega_2 + \omega_1 - \omega_4}),$$

which satisfies the condition in Lemma 13 or 12, by Eq. (4), so again d' has a 2PBE realization G' . Notice that for $\omega'_2 = \omega_2 + \omega_1 - \omega_4 \geq 2$, $d'_1 \leq n' - 1$.

Finally, we obtain a 2PBE realization G for d by taking G' and connecting ω_1 new degree 1 vertices with degree 3 or 2 in G' . By above construction of G , one page is outerplanar and the other page consists of at most two edges. The lemma follows. \square

Combining Lemmas 12, 13 and 14, we have the following theorem.

Theorem 15. *Let $d \in \mathcal{D}_{\leq 4}$ and $d_1 \leq n - 1$. Then d has an OP+2 realization.*

4.2 A 2PBE realization for sequences with $d_1 \geq 5$

In this section, we analyze the subclass $\mathcal{D}_{\geq 5} = \mathcal{D} \setminus \mathcal{D}_{\leq 4}$ of sequences whose largest degree satisfies $d_1 \geq 5$. We first consider the special case of *maximal-volume* sequences in $\mathcal{D}_{\geq 5}$, namely, ones whose volume equals exactly $4n - 6$. By Lemma 4, $\omega_1 = 0$. Combining this with Lemma 3, $\omega_2 \geq 2$. Since Lemma 9 has already considered the case for $\omega_2 = 2$, it remains to consider sequences d where $\omega_2 > 2$. This class of sequences turns out to be the most challenging. Their characterization is covered by the next key lemma, whose proof is by far the most complex part of our analysis.

Lemma 16. *Let $d \in \mathcal{D}_{\geq 5}$. If (1) $\sum d = 4n - 6$, (2) $2\omega_2 + \omega_3 \leq n + 1$, (3) $\omega_2 > 2$, and (4) $d_n = 2$, Then d has an OP+bi realization.*

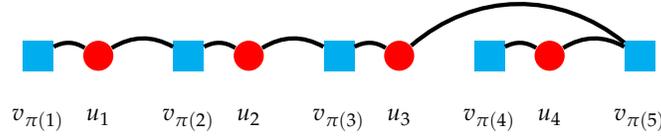


Figure 6: Illustration for the way in which the graph G' constructed in the proof of Case A of Lemma 16 can be embedded into the page of a book. The blue rectangles are B 's vertices. In the example, $j = 4$.

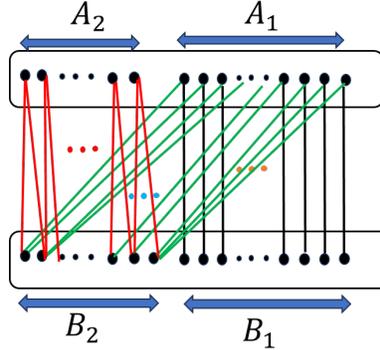


Figure 7: Illustration of the construction of $\text{Bar}(A, B)$. The black edges represent the matching between A_1 and B_1 . The green edges represent the union of disjoint stars between B_2 and A_1 . The red edges represent the zigzag path between A_2 and B_2 .

Proof. Let d be as in the lemma. To prove the claim, we define sequences d', d'' , of lengths n', n'' respectively, such that $d = d' \oplus d''$.

Intuitively, it would appear that if property (1) of the lemma holds, then d' and d'' can be defined such that $\sum d' = 2n' - 4$ and $\sum d'' = 2n'' - 2$, so both d' and d'' can be realized by a forest and embedded on its own page, implying a 2-page book embedding. Proving this intuition is complicated by the need to ensure two properties: (a) that no edge (u, w) occurs in *both* pages, and (b) that the vertices occur in the same order on the outer face of both pages.

Instead, we select the sequence d'' (ignoring the 0-degrees) so that it meets the preconditions of Lemma 9, i.e., d'' satisfies (i) $\omega_2'' = 2$, (ii) $d_{n''}'' = 2$, and (iii) $d'' \in \mathcal{D}$, namely, $\sum d'' \leq 4n'' - 6$ and $\sum d''$ is even. Then, Lemma 9 provides an outerplanar realization G'' of d'' and its embedding fills the first page of the book. The embedding of G'' imposes an ordering on the vertices, as they appear along the spine of the book (which is the same ordering as on the outer-cycle of the embedding). For the second page, we construct an outerplanar realization G' of d' such that the vertices follow the same ordering along the book spine. To obtain d'' , we apply the following two transformations to d .

(TR1) remove $q := \omega_2 - 2$ degree-2 entries from d to meet requirement (i).

(TR2) decrease the total volume of the remaining sequence by $2q$ to satisfy requirement (iii), while carefully avoiding the creation of new degrees 1 or 2.

Denote the number of degrees that are 5 or more by $\ell := \sum_{i \geq 5} \omega_i = n - \omega_2 - \omega_3 - \omega_4$. The remaining

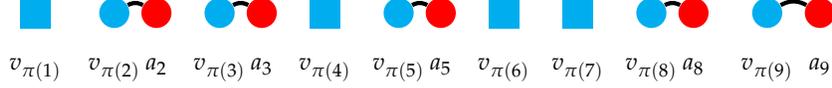


Figure 8: The graph G' after the first step of the construction. Throughout our illustration figures, vertices of A_1 (respectively, B) are colored red (resp., blue). Vertices of A_1, B_1 (resp., A_2, B_2) are circular (resp., rectangular).

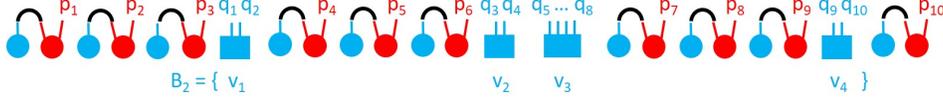


Figure 9: The situation before Step 2. Vertices of A (respectively, B) are colored red (resp., blue). Vertices of A_1 and B_1 (respectively, B_2) are depicted as circles (resp., rectangles). The free ports of a vertex are depicted as tags attached to it. In this example, the d' degrees of the vertices of B_2 are $d'(v_1) = 3$, $d'(v_2) = 4$, $d'(v_2) = 6$ and $d'(v_4) = 3$. The ports of B_2 and A_1 vertices that have to be connected in a non-intersecting way correspond to the input sequences of the example in Figure 4. This is done using Algorithm BBB of Section 2.

proof, describing the construction of G' for d' , is split into two cases, depending on the relation between q and ℓ .

Case A: $q < \ell$. In this easier case, a path sequence $d' = (2^{n'-2}, 1^2)$ is a suitable choice, and the construction of G' is simplified considerably.

We define the sequence d' to have length $n' = n$. It contains vertices of degree 0 and we do *not* consider it in non-increasing order, to satisfy the constraint $d = d' \oplus d''$. Once d' is defined, d'' is implicitly defined as well, as $d'' = d \ominus d'$. It can be regarded as having length $n'' = n$ where $d''_i = 0$ if $i > n - q$.

$$d'_i = \begin{cases} 1, & \text{if } i = 1, \\ 2, & \text{if } i = 2, \dots, q, \\ 1, & \text{if } i = q + 1, \\ 0, & \text{if } i = q + 2, \dots, n - q, \\ 2, & \text{if } i = n - q + 1, \dots, n. \end{cases} \quad d''_i = \begin{cases} d_i - 1, & \text{if } i = 1, \\ d_i - 2, & \text{if } i = 2, \dots, q, \\ d_i - 1, & \text{if } i = q + 1, \\ d_i, & \text{if } i = q + 2, \dots, n - q, \\ 0, & \text{if } i = n - q + 1, \dots, n. \end{cases}$$

We use the sequence $d = (6^2, 5^4, 4, 3^2, 2^6)$ as a running example for Case A. In this case, $n = 15$, $q = 4$, and $\ell = 6$. It is not hard to verify that d satisfies the conditions of this lemma and the condition of Case A. We have that $d' = (1, 2^3, 1, 0^2, 2^4)$, $d'' = (5, 4, 3^2, 4, 5, 4, 3^2, 2^2, 0^4)$.

For the next step, consider $\hat{d} = (d''_1, d''_2, \dots, d''_{n-q})$, which is d'' without the trailing 0 degrees. In the above example, we have that $\hat{d} = (5^2, 4^3, 3^4, 2^2)$. For this \hat{d} , whose length is $\hat{n} = n - q$, we show the following.

Claim 1. *Sequence \hat{d} satisfies the preconditions of Lemma 9.*

Proof. By construction of d'' , we have that $\sum \hat{d} = \sum d''$ is even, and that $\hat{d}_{\hat{n}} = d''_{n-q} = 2$. Observe that $\hat{\omega}_2 = \omega''_2 = 2$ as $q < \ell$ implies that $d_{q+1} \geq 5$. To complete the proof, observe that

$$\sum \hat{d} = \sum d - 4q \leq 4n - 6 - 4q = 4(n - q) - 6 = 4\hat{n} - 6. \quad \square$$

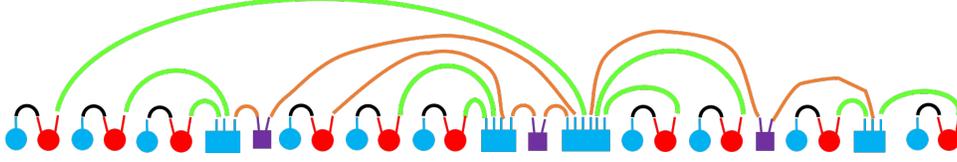


Figure 12: The graph after Algorithm 2 based on Figure 10. The purple squares represent the vertices in A_2 . The orange edges represent the modified edges based on Figure 10.

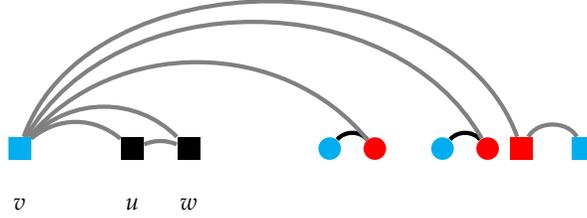


Figure 13: The black squares represent the added vertices u, w and they are connected to v .

$\deg_{G''}(v_i) = d_i$ for $i = 1, 2, \dots, n$. The vertices in $B \cup R$ form a path in G' where v_1 and $v_{\pi(j)}$ are the two ends (vertices of degree 1). Finally, observe that $E'' \cap E' = \emptyset$ since each edge of G' is incident to a vertex in R , which has vertices of degree 0 in G'' . It follows that G'' and G' form a valid book embedding of a graph that realizes d . Notice that G' is bipartite outerplanar and G'' is an outerplanar graph.

Notice that in our example $v_{\pi(1)} = v_1, v_{\pi(2)} = v_3, v_{\pi(3)} = v_4, v_{\pi(4)} = v_2, v_{\pi(5)} = v_5$. Then in G' , $d'(v_{\pi(1)}) = 1, d'(v_{\pi(2)}) = d'(v_{\pi(3)}) = d'(v_{\pi(4)}) = 2, d'(v_{\pi(5)}) = 1, R = \{u_1, \dots, u_q\}$. Figure 6 shows a valid embedding of G' into the page of a book. This completes the proof of Case A.

Case B: $q \geq \ell$. This case is significantly more complicated than case A. Note that as $2\omega_2 + \omega_3 \leq n + 1$, it follows that

$$\omega_4 = n - \omega_2 - \omega_3 - \ell \geq \omega_2 - 1 - \ell = q + 1 - \ell > 0.$$

Also note that $w_4 + \ell \geq q + 1$.

The construction generalizes that of case A. Similar to case A, we obtain sequence d'' from d by removing $q = \omega_2 - 2$ vertices of degree 2. The next steps, though, are more complicated than in case A. The reason is that, if we set (like above) $d''_i = d_i - 2$, for $i = 2, 3, \dots, q$, then some vertices of degree 4 in d are decreased to 2 in d'' and requirement (i), $\omega''_2 = 2$, might not be met.

Instead, the sequences d' and d'' have to be defined differently. In particular, to define d'' , we reduce only the degrees that are *larger* than 4 by 2 (or more), and we reduce $q - \ell + 1$ vertices of

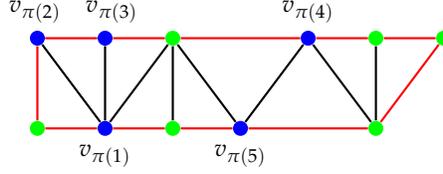


Figure 14: The graph realizing \hat{d} . Blue circles: vertices in B . Green circles: vertices in $V(\hat{G}) \setminus B$.

degree 4 to 3. At first, let sequence \bar{d} be such that

$$\bar{d}_i = \begin{cases} d_i - 2, & \text{if } i = 1, \dots, \ell, & (\text{where } d_i \geq 5) \\ d_i - 1, & \text{if } i = \ell + 1, \dots, q + 1 & (\text{where } d_i = 4) \\ d_i, & \text{if } i = q + 2, \dots, n - q, & (\text{where } 4 \geq d_i \geq 2) \\ 0, & \text{if } i = n - q + 1, \dots, n. & (\text{where } d_i = 2) \end{cases}$$

Recall that d satisfies $w_4 + \ell \geq q + 1$, which implies that $d_{q+1} \geq 4$ and therefore $\bar{d}_{q+1} \geq 3$. As $q = \omega_2 - 2$ out of the ω_2 vertices of degree 2 of d are transformed to 0, it follows that $\bar{\omega}_2 = 2$.

Note that $\sum \bar{d} \leq 4(n - q) - 6$ is not guaranteed, so requirement (iii) does not necessarily hold. However, observe that $\sum \bar{d} = \sum d - 2(\ell + q) - (q - \ell + 1)$, so

$$\sum \bar{d} = \sum d - 2(\ell + q) - (q - \ell + 1) = 4n - 6 - 3q - \ell - 1 = 4(n - q) - 6 + (q - \ell - 1) \quad (5)$$

does hold. To meet requirement (iii) of Lemma 9, we set $d'' = \bar{d}$ and gradually reduce $\sum_{i=1}^{\ell} d''_i$ further by decreasing the large degrees using Algorithm 1.

Algorithm 1: Generating sequence d'' from \bar{d} .

- 1 Set $d'' \leftarrow \bar{d}$
 - 2 **while** $\sum d'' > 4(n - q) - 6$ **do**
 - 3 $\lfloor d''_j \leftarrow d''_j - 1$, where $j = \operatorname{argmax}_{i \in [\ell]} d''_i$
 - 4 $\hat{d} \leftarrow \operatorname{pos}(d'')$.
-

Note that the output $\hat{d} = (d''_1, d''_2, \dots, d''_{n-q})$ of Algorithm 1 is the prefix of length $\hat{n} = n - q$ of the sequence d'' at the end of Algorithm 1, omitting the trailing zeros.

Claim 2. *The sequence \hat{d} satisfies the preconditions of Lemma 9.*

Proof. Requirement (iii) holds by construction. Note that when constructing \bar{d} we reduced 2ℓ from the volume of the “high” (first ℓ) elements, $\sum_{i=1}^{\ell} d_i$. Algorithm 1 reduces this volume further by $(q - \ell - 1)$. To show that \hat{d} satisfies requirements (i) and (ii), it suffices to show that the “high” degrees d_1, d_2, \dots, d_ℓ have sufficient volume so that $d''_i \geq 3$ for $i \leq \ell$, i.e., it remains to prove that $\sum_{i=1}^{\ell} (\bar{d}_i - 3) \geq q - \ell - 1$.

Recall that $\sum d = 4n - 6$, which can be rewritten as

$$\sum_{i=1}^{\ell} (\bar{d}_i - 3) = \sum_{i=1}^{\ell} (d_i - 5) = 4n - 6 - (4\omega_4 + 3\omega_3 + 2\omega_2) - 5\ell = n - 6 - \omega_4 + \omega_2 - 2\ell \quad (6)$$

By assumption (3) of the lemma, $\omega_2 > 2$, hence also $\omega_3 + \omega_2 \geq 3$, which can be written equivalently as $n - \omega_4 - 3 \geq \ell$. Equation (6), hence $\sum_{i=1}^{\ell} (\bar{d}_i - 3) \geq \omega_2 - \ell - 3 = q - \ell - 1$, and the claim follows. \square

Concluding that Lemma 9 applies, and noting that by construction $\sum \hat{d} = 4\hat{n} - 6$, it follows that \hat{d} is maximal outerplanar. Let \hat{G} be a maximal outerplanar realization of \hat{d} . We add q isolated vertices to $V(\hat{G})$ to generate an outerplanar realization $G'' = (V, E'')$ of d'' . Let $V = \{v_1, v_2, \dots, v_n\}$ such that $\deg_{G''}(v_i) = d''_i$, for $i = 1, 2, \dots, n$. Hence, the set $A := \{v_{n-q+1}, \dots, v_n\}$ contains the isolated vertices of degree 0 in G'' . For ease of notation, we relabel the vertices $a_i := v_{n-q+i}$, for $i = 1, \dots, q$. Consequently,

$$A = \{a_1, \dots, a_q\} = \{v_{n-q+1}, \dots, v_n\}.$$

Consider an outerplanar embedding of \hat{G} . Let π be the permutation by which the vertices of $B := \{v_1, v_2, \dots, v_{q+1}\}$ appear on the cyclic-order defined by the planar embedding of G'' . Without loss of generality, let $\pi(1) = 1$. It follows that G'' has an embedding into a book page where the vertices of B are arranged along the spine according to π . The vertices in A can be placed freely along the spine without interfering with the embedding of G'' .

Note that unlike in case A , here we did not define the sequence d'' explicitly, so the sequence d' too is not defined explicitly, but rather is determined as $d' := d \ominus d''$.

To complete the proof, we construct an outerplanar realization $G' = (V', E')$ of d' such that $V' = V$ and $\deg_{G'}(v_i) = d'_i$, for $i = 1, 2, \dots, n$. Only the vertices in A and B need to be connected in G' . Partition the vertices in B into the sets

$$B_1 = \{v_{\ell+1}, \dots, v_{q+1}\} \quad \text{and} \quad B_2 = \{v_1, \dots, v_{\ell}\}.$$

Note that $d'_i = 1$ for $v_i \in B_1$, and $d'_i \geq 2$ for $v_i \in B_2$, and that $d'_i = 2$ for every $v_i \in A$. G' and its book embedding are constructed simultaneously. The vertices in B are arranged according to π along the book spine. Hence, vertices of B_1 and B_2 may appear in any order and vertices in A can be placed freely between them. Since vertices in A have degree 0 in G'' , it follows that $E'' \cap E' = \emptyset$, which is necessary for G' and G'' to form a valid book embedding of a simple graph (rather than a multigraph).

It is essential that each edge in E' is incident to one vertex in A and one vertex in B , i.e., G' is a bipartite graph with parts A and B . Partition A into $A_1 \cup A_2$, where $|A_1| = |B_1| = q - \ell + 1$ and $|A_2| = |B_2| - 1 = \ell - 1$. Specifically, let

$$A_1 = \{a_1, \dots, a_{q-\ell+1}\} \quad \text{and} \quad A_2 = \{a_{q-\ell+2}, \dots, a_q\}.$$

The relationship between the volume of B_2 and the cardinalities of A_2 and A_1 is established in the following claim, which helps to show that G' is bipartite.

Claim 3. $\sum_{i=1}^{\ell} d'_i = 2|A_2| + |A_1|$.

Proof. Notice that $\sum_{i=1}^{\ell} d'_i$ consists of two parts. One comes from the decrease from d to \bar{d} on the elements of B_2 , and is equal to 2ℓ ; the other part comes from the decreases of d'' on the elements of B_2 in Algorithm 1, and is equal to $q - \ell - 1$ by (5). Also recall that $|A_1| = q - \ell + 1$ and $|A_2| = \ell - 1$. It follows that $\sum_{i=1}^{\ell} d'_i = 2\ell + (q - \ell - 1) = 2(\ell - 1) + q - \ell + 1 = 2|A_2| + |A_1|$. \square

By Claim 3, there is a bipartite graph $Bar(A, B)$ between A and B , defined as follows.

1. Connect a matching between B_1 and A_1 ;
2. Connect the vertices of B_2 to the vertices of A_1 by constructing a union of disjoint stars, with v_1 connected to $d'(v_1) - 1$ vertices in A_1 , v_i connected to $d'(v_i) - 2$ vertices in A_1 for $2 \leq i \leq \ell - 1$, and v_ℓ connected to $d'(v_\ell) - 1$ vertices in A_1 ;
3. Construct a path $v_1 - a_{q-\ell+2} - v_2 - \dots - a_q - v_\ell$ zigzagging between A_2 and B_2 .

The Figure 7 illustrates the structure of $Bar(A, B)$.

One can easily verify that assuming no restrictions on the ordering of the vertices, it is possible to find an outerplanar embedding of $Bar(A, B)$ in the plane. The only remaining obstacle is that the embedding must obey the ordering π dictated for the vertices of B by the graph G'' . The remainder of the proof concerns finding an outerplanar embedding obeying this ordering.

The construction of G' consists of three steps corresponding to those of the construction of $Bar(A, B)$. First, construct a matching between A_1 and B_1 . Next, construct disjoint stars between B_2 and A_1 . Finally, construct the path between A_2 and B_2 .

Our starting point is the spine defined for G'' , consisting of the vertices of B . This spine dictates the order of the B vertices, which in turn dictate the bipartite A - B connections in the constructed graph G' . It is convenient to separate the indices of B_1 and B_2 . Let $(h_1, \dots, h_{q-\ell+1})$ denote the increasing sequence of indices $\{i \mid v_{\pi(i)} \in B_1\}$, let (k_1, \dots, k_ℓ) denote the increasing sequence of indices $\{i \mid v_{\pi(i)} \in B_2\}$. The placement of the A_1 (respectively, A_2) vertices on the spine is fixed in Step 1 (resp., Step 3) based on these positions of the B_1 (resp., B_2) vertices.

In the first step, place one vertex of A_1 next to each vertex of B_1 in the embedding of G' (and G) and connect them. We place a_i to the right of $v_{\pi(h_i)}$, for $i = 1, \dots, q - \ell + 1$, and add the edge $(a_i, v_{\pi(h_i)})$ to E' . See Figure 8 for an illustration.

Note that after step 1, each of the vertices in A_1 is still one edge short, while each of the vertices in $A_2 := A \setminus A_1$ is still missing two edges. The goal of step 2 is to handle the remaining shortages of the vertices of A_1 by connecting them to the vertices of B_2 . Towards that, we partition the vertices in A_1 into subsets of vertices that fall in the gaps between consecutive vertices of B_2 . Formally, define $A_1^i = \{a_j \in A_1 : k_i < j < k_{i+1}\}$, for $i = 1, \dots, \ell$, as the vertices of A_1 that appear between $v_{\pi(k_i)}$ and $v_{\pi(k_{i+1})}$ along the spine. Consider Figure 8 where $k_1 = 1, k_2 = 4$, and $A_1^1 = \{2, 3\}$.

For the second step, we make use of the following tool.

In the second step towards building the graph G' , we connect vertices of B_2 to the vertices of A_1 by using the bipartite bracket balancing algorithm BBB. This is done as follows. Let $P = A_1$. For the other side, identify with every vertex $v_{\pi(k_i)} \in B_2$ a sequence of s_i "ports", or connection points, $\langle v_{\pi(k_i)}^1, \dots, v_{\pi(k_i)}^{s_i} \rangle$, where

$$s_i = d'(v_{\pi(k_i)}) - \begin{cases} 1, & i = 1, \\ 2, & 2 \leq i \leq \ell - 1, \\ 1, & i = \ell, \end{cases}$$

and let Q be the union of those sets, i.e., the set of all resulting ports,

$$Q = \langle v_{\pi(k_1)}^1, \dots, v_{\pi(k_1)}^{s_1}, v_{\pi(k_2)}^1, \dots, v_{\pi(k_2)}^{s_2}, \dots, v_{\pi(k_\ell)}^1, \dots, v_{\pi(k_\ell)}^{s_\ell} \rangle.$$

The ordering of the merged set R is dictated by the order of the vertices of A_1 and B_2 on the spine. Figure 9 illustrates the transformation to the instance of BIPARTITE BRACKET BALANCING problem.

After applying Algorithm BBB to this input, transform its output into a collection of edges on our original graph as follows. For each $1 \leq i \leq \ell$, merge the s_i “ports” corresponding to $v_{\pi(k_i)}$ into the point $v_{\pi(k_i)}$ and change all the edges that are incident to those s_i “ports” to be incident to $v_{\pi(k_i)}$. This yields an outerplanar graph \bar{G}' . Figure 10 illustrates the resulting graph after applying the algorithm.

After the construction of \bar{G}' , the end vertices $v_{\pi(k_1)}$ and $v_{\pi(k_\ell)}$ of B_2 are still missing one edge each, and the vertices $v_{\pi(k_{i'})}$ for $2 \leq i' \leq k_{\ell-1}$ are still missing two edges each. Our third step involves spreading the A_2 vertices between the B_2 vertices. Each A_2 vertex is placed between two consecutive B_2 vertices (according to π). Specifically, we place $a_{q-\ell+1+i}$ between $v_{\pi(k_i)}$ and $v_{\pi(k_{i+1})}$ for $i \in [\ell - 1]$, and construct the path

$$v_{\pi(k_1)} - a_{q-\ell+2} - v_{\pi(k_2)} - a_{q-\ell+3} - \dots - a_q - v_{\pi(k_\ell)}.$$

There are two difficulties involved in performing this step. The first concerns determining the exact position in which the vertex $a_{q-\ell+1+i}$ should be placed between $v_{\pi(k_i)}$ and $v_{\pi(k_{i+1})}$. The second is the need to eliminate possible crossings of edges after adding the path above.

Let $N_{\bar{G}'}(v) = \{v' \mid (v, v') \in E(\bar{G}')\}$ for any $v \in V$. To handle the first difficulty, $a_{q-\ell+1+i}$ should be placed between two vertices v and v' in $A_1^i \cup \{v_{\pi(k_i)}, v_{\pi(k_{i+1})}\}$ such that the point v to the left of $a_{q-\ell+1+i}$ satisfies that any vertex in $N_{\bar{G}'}(v)$ is to the left of v , and the point v' to the right of $a_{q-\ell+1+i}$ satisfies that any vertex in $N_{\bar{G}'}(v')$ is to the left of v' . As shown in Figure 11, $a_{q-\ell+1+i}$ is placed between $a_{i'+1}$ and $v_{\pi(k_{i+1})}$ since $a_{i'+1}$ has all its neighbors on its left and $v_{\pi(k_{i+1})}$ has all its neighbors on its right.

To handle the second difficulty, we first present some notations. The set E_{le}^i (respectively, E_{ri}^i) contains edges connecting a vertex $v_{\pi(j)}$, for $j < i$ (resp., $j > i + 1$) to a vertex in A_1^i . Formally,

$$\begin{aligned} E_{le}^i &= \{(v_{\pi(k_{j'})}, a_j) \in E(\bar{G}') \mid j' < i, a_j \in A_1^i\}, \\ E_{ri}^i &= \{(v_{\pi(k_{j'})}, a_j) \in E(\bar{G}') \mid j' > i + 1, a_j \in A_1^i\}. \end{aligned}$$

As shown in Figure 11, $E_{le}^i = \{(N'(a_{i'}), a_{i'}), (N'(a_{i'+1}), a_{i'+1})\}$ and $E_{ri}^i = \emptyset$. Note that $N'(a_{i'})$ is equal to $v_{\pi(k_{j'})}$ with some $j' < i$. Similar result can apply for $N'(a_{i'+1})$. When adding edges $(v_{\pi(k_i)}, a_{q-\ell+i+1})$ and $(a_{q-\ell+i+1}, v_{\pi(k_{i+1})})$, they may cross the existing edges in \bar{G}' . The following steps of Algorithm 2 eliminate the problem by replacing the set of crossing edges with noncrossing set of edges while preserving the vertex degrees.

Edge crossing happens when $E_{le}^i \neq \emptyset$ in line 3 or $E_{ri}^i \neq \emptyset$ in line 6. Notice that the right (respectively left) endpoints of edges in E_{le}^i (respectively E_{ri}^i) are in consecutive order by Algorithm BBB. Let the set of the right endpoints of edges in E_{le}^i be $\{a_{i'}, \dots, a_{i'+\ell'}\}$ with $\ell' = |E_{le}^i| - 1$. Similarly, let the set of the left endpoints of edges in E_{ri}^i be $\{a_{i''}, \dots, a_{i''+\ell''}\}$ with $\ell'' = |E_{ri}^i| - 1$. Since $N_{\bar{G}'}(v)$

consists only one element for any $v \in \{a_{i'}, \dots, a_{i'+\ell'}, a_{i''}, \dots, a_{i''+\ell''}\}$, let $N'(v)$ refer to the only element in $N_{\bar{G}'}(v)$. In the case where $E_{le}^i \neq \emptyset$, the edge $(v_{\pi(k_i)}, a_{q-\ell+i+1})$ crosses all the edges in E_{le}^i . The correction process exchanges the right endpoints of edges in $E_{le}^i \cup \{(v_{\pi(k_i)}, a_{q-\ell+i+1})\}$ such that there are no cross edges. In the other case, where $E_{ri}^i \neq \emptyset$, $(a_{q-\ell+i+1}, v_{\pi(k_{i+1})})$ crosses all the edges in E_{ri}^i . The correction process is similar to that of $E_{le}^i \neq \emptyset$.

Figure 11 illustrates the correction process between $v_{\pi(k_i)}$ and $v_{\pi(k_{i+1})}$ for the case where $E_{le}^i \neq \emptyset$. Figure 12 illustrates the corrected graph following the entire correction process performed by applying Algorithm 2, based on Figure 10.

Algorithm 2: Construction algorithm for G' third step

input : outerplanar graph \bar{G}' obtained from the second step
output: an outerplanar graph G'

- 1 **forall** $i \in [\ell - 1]$ **do**
- 2 Put $a_{q-\ell+i+1}$ between two vertices v and v' in $A_1^i \cup \{v_{\pi(k_i)}, v_{\pi(k_{i+1})}\}$ such that the point v to the left of $a_{q-\ell+i+1}$ satisfies that any vertex in $N_{\bar{G}'}(v)$ is to the left of v , and the point v' to the right of $a_{q-\ell+i+1}$ satisfies that any vertex in $N_{\bar{G}'}(v')$ is to the left of v' .
- 3 Connect $(v_{\pi(k_i)}, a_{q-\ell+i+1}), (a_{q-\ell+i+1}, v_{\pi(k_{i+1})})$
- 4 **if** $E_{le}^i \neq \emptyset$ **then**
- 5 Delete all the edges in E_{le}^i and $(v_{\pi(k_i)}, a_{q-\ell+i+1})$
- 6 Connect $(N'(a_{i'+\ell'}), a_{q-\ell+i+1}), (N'(a_{i'+\ell'-1}), a_{i'+\ell'}), \dots, (N'(a_{i'}), a_{i'+1}), (v_{\pi(k_i)}, a_{i'})$
- 7 **if** $E_{ri}^i \neq \emptyset$ **then**
- 8 Delete all the edges in E_{ri}^i and $(a_{q-\ell+i+1}, v_{\pi(k_{i+1})})$
- 9 Connect $(N'(a_{i''}), a_{q-\ell+i+1}), (N'(a_{i''+1}), a_{i''}), \dots, (N'(a_{i''+\ell''}), a_{i''+\ell''-1}), (v_{\pi(k_{i+1})}, a_{i''+\ell''})$

Claim 4. The output G' of Algorithm 2 is outerplanar and satisfies $\deg(G') = d'$.

Proof. Let us first prove the outerplanarity of G' . Consider a pair of vertices $v_{\pi(k_i)}, v_{\pi(k_{i+1})}$. If $E_{le}^i = \emptyset$ and $E_{ri}^i = \emptyset$, then for any edge $e \in E(\bar{G}')$ with one endpoint in A_1^i , the other endpoint of e is either $v_{\pi(k_i)}$ or $v_{\pi(k_{i+1})}$ by the definition of E_{le}^i and E_{ri}^i . Combining this with the choice of the position of $a_{q-\ell+i+1}$ in line 2 of the algorithm, $(v_{\pi(k_i)}, a_{q-\ell+i+1})$ and $(v_{\pi(k_{i+1})}, a_{q-\ell+i+1})$ do not cross any edges with one endpoint in A_1^i . Therefore, there are no crossing edges in this case.

If $E_{le}^i \neq \emptyset$, then $(v_{\pi(k_i)}, a_{q-\ell+i+1})$ crosses all the edges in E_{le}^i , which is the edge set with one endpoint before $v_{\pi(k_i)}$ and one in A_1^i . In this case, the correction process deletes the edges in E_{le}^i and $(v_{\pi(k_i)}, a_{q-\ell+i+1})$ and connects $(N'(a_{i'+\ell'}), a_{q-\ell+i+1}), (N'(a_{i'+\ell'-1}), a_{i'+\ell'}), \dots, (N'(a_{i'}), a_{i'+1}), (v_{\pi(k_i)}, a_{i'})$, which means that all the edges in E_{le}^i change their right endpoint to the closest vertex on the right of it in A , and $v_{\pi(k_i)}$ connects to the leftmost endpoint of edges in E_{le}^i . After this step, the newly added edges do not cross each other or the edges in $E(\bar{G}') \setminus E_{le}^i$ since \bar{G}' is outer planar. The argument for $E_{ri}^i \neq \emptyset$ is similar to that of $E_{le}^i \neq \emptyset$.

Next, we show that above correction processes are independent. Notice that the correction process in Algorithm 2 is based on exchanging the endpoint of e in A_1^i to another vertex in A_1^i

for some $e \in \bar{G}'$. Therefore, the correction processes applied to different pairs of $v_{\pi(k_i)}, v_{\pi(k_{i+1})}$ are independent. Also, in the case that $E_{l_e}^i \neq \emptyset$, Algorithm 2 exchanges the right endpoints of edges in $E_{l_e}^i$ and $a_{q-\ell+i+1}$. Similarly, in the case that $E_{r_i}^i \neq \emptyset$, Algorithm 2 exchanges the left endpoints of edges in $E_{r_i}^i$ and $a_{q-\ell+i+1}$. Since $E_{l_e}^i$ and $E_{r_i}^i$ are disjoint, the correction processes for $E_{l_e}^i \neq \emptyset$ and $E_{r_i}^i \neq \emptyset$ are independent. In conclusion, after performing all of the above corrections for each pair $v_{\pi(k_i)}, v_{\pi(k_{i+1})}$, there are no crossing edges left in G' .

Finally, we show that there are no parallel edges in the resulting G' . In the process of exchanging the endpoints, in the case that $E_{l_e}^i \neq \emptyset$, the added edges are $(N'(a_{i'+\ell}), a_{q-\ell+i+1}), (N'(a_{i'+\ell-1}), a_{i'+\ell}), \dots, (N'(a_{i'}), a_{i'+1}), (v_{\pi(k_i)}, a_{i'})$, which did not exist in \bar{G}' . A similar argument applies for $E_{r_i}^i \neq \emptyset$. Therefore, there are no parallel edges. It follows that G' is a simple outerplanar graph.

To prove that $\deg(G') = d'$, it suffices to show that G' has the same degree sequence as the bipartite graph $\text{Bar}(A, B)$, since $\deg(\text{Bar}(A, B)) = d'$. In the first and second step, the construction of G' is same as that of \bar{G}' . Comparing \bar{G}' with $\text{Bar}(A, B)$, the vertices $v_{\pi(k_1)}$ and $v_{\pi(k_\ell)}$ are each one edge short, while the vertices $v_{\pi(k_{i'})}$ for $2 \leq i' \leq \ell - 1$ are each two edges short. Every vertex in A_2 has degree 0. In the third step, the correction process in Algorithm 2 exchanges the endpoints of the cross edges, which keeps the degree 2 of the vertices in A_1 . Also, the degrees of $v_{\pi(k_1)}$ and $v_{\pi(k_\ell)}$ each increase by 1, and the degrees of $v_{\pi(k_i)}$ for $2 \leq i \leq \ell - 1$ each increase by 2. For $1 \leq i \leq \ell$, the vertex $a_{q-\ell+i+1}$ inserted between $v_{\pi(k_i)}$ and $v_{\pi(k_{i+1})}$ has degree 2 in G' . Therefore, $\deg(G') = \deg(\text{Bar}(A, B))$. \square

In summary, G' has an embedding into the page of a book where the vertices are ordered along book spine according to π . It follows that G and G' form a 2PBE realization of d . Note that G is outerplanar and G' is bipartite outerplanar. This completes the proof of Case B, and of Lemma 16. \square

Lemma 16 considers maximal-volume sequences. The following lemma mainly focuses on non-maximal-volume sequences. Recall that $\Delta_E(d) = (4n - 6 - \sum d)/2$ and $\Delta_\omega(d) = 3\omega_1 + 2\omega_2 + \omega_3 - n$.

Lemma 17. *Let $d \in \mathcal{D}_{\geq 5}$ and $d_1 \leq n - 1$. If (1) $2\Delta_E \geq \Delta_\omega$, (2) $\omega_2 > 2$ (3) $d_2 > 3$, and (4) $d_n = 2$, Then d has an OP+bi realization.*

Proof. Given d as in the lemma, apply to it the following Algorithm 3; let \bar{d} be its output.

Algorithm 3: generating \bar{d} from d

- 1 Set $\bar{d} \leftarrow d$
 - 2 **while** $\sum \bar{d} < 4\bar{n} - 6$ AND $\bar{\omega}_2 > 3$ AND $\bar{d}_1 > 4$ **do**
 - 3 $\bar{d}_j \leftarrow \bar{d}_j - 2$, where $j = \text{argmax}_{i \in [\bar{n}]} \bar{d}_i$
 - 4 Remove two degrees 2 from \bar{d}
-

To prove the lemma, we first show that \bar{d} has a realization \bar{G} that is outerplanar or a book embedding with two pages. Then, we modify \bar{G} to generate a realization G of d with the analogous properties.

Assume Algorithm 3 executes k iterations of its while-loop. Let \bar{d}^i be the intermediate sequence after iteration $i \in [0, k]$, i.e., $\bar{d}^0 = d$ and $\bar{d}^k = \bar{d}$. Observe that

- (i) $\bar{n}^i = \bar{n}^{i+1} + 2$,
- (ii) $\bar{\omega}_2^i = \bar{\omega}_2^{i+1} + 2$, and
- (iii) $\sum \bar{d}^i = \sum \bar{d}^{i+1} + 6$.

Next, we argue that $\bar{\Delta}_E^i = \bar{\Delta}_E^{i+1} + 1$. Verify that,

$$2\bar{\Delta}_E^i = 4\bar{n}^i - 6 - \sum \bar{d}^i = 4\bar{n}^{i+1} - 6 - \sum \bar{d}^{i+1} + 2 = 2(\bar{\Delta}_E^{i+1} + 1).$$

We proceed showing that $\bar{\Delta}_\omega^i = \bar{\Delta}_\omega^{i+1} + 2$. Verify that,

$$\bar{\Delta}_\omega^i = 2\bar{\omega}_2^i + \bar{\omega}_3^i - \bar{n}^i = 2\bar{\omega}_2^{i+1} + 4 + \bar{\omega}_3^i - \bar{n}^{i+1} - 2 = \bar{\Delta}_\omega^{i+1} + 2.$$

It follows that $2\bar{\Delta}_E \geq \bar{\Delta}_\omega$. Moreover, note that $\bar{d}_{\bar{n}-1} = \bar{d}_{\bar{n}} = 2$ holds.

There are three conditions under which Algorithm 3 stops. Each of them allows us to use a previous result and generate a realization of \bar{d} . We consider the three cases separately.

Case (a): $\bar{d}_1 \leq 4$. $\bar{d}_1 \leq \bar{n} - 1$ by Algorithm 3. Then Lemma 12, provides a 2PBE realization (\bar{G}, \bar{G}') of \bar{d} .

Case (b): $\bar{d}_1 > 4$ and $\sum \bar{d} = 4\bar{n} - 6$. Since $\bar{\Delta}_E = 0$ and $2\bar{\Delta}_E \geq \bar{\Delta}_\omega$ hold, we have that $2\bar{\omega}_2 + \bar{\omega}_3 \leq \bar{n}$. It follows that \bar{d} satisfies the preconditions of Lemma 16 yielding a 2PBE realization (\bar{G}, \bar{G}') .

Case (c): $\bar{d}_1 > 4$, $\sum \bar{d} < 4\bar{n} - 6$, and $\bar{\omega}_2 \leq 3$. If $\bar{\omega}_2 = 2$, then \bar{d} has an outerplanar realization \bar{G} due to Lemma 9. Otherwise, $\bar{\omega}_2 = 3$ holds, and Corollary 10 provides an outerplanar realization \bar{G} of \bar{d} .

In either of the cases, \bar{G} is a 2PBE graph that (partially) realizes \bar{d} . During the execution of Algorithm 3, the order of \bar{d} is kept the same and it is not necessarily in nonincreasing order. For $i \in [k]$, let ℓ_i be the index of the degree that was reduced by 2 in iteration i by Algorithm 3 (line(3)). To finish the proof, we modify \bar{G} by running Algorithm 4. Intuitively, it reverses the reduction, which was applied to sequence d by Algorithm 3, on the graph \bar{G} .

Algorithm 4: reversing \bar{d} to d

- 1 Set $G \leftarrow \bar{G}$
 - 2 **for** $i \in [k]$ **do**
 - 3 Let $v \in V(G)$ such that $\deg_G(v) = \bar{d}_{\ell_{k-i+1}}$
 - 4 $V(G) \leftarrow V(G) \cup \{u, w\}$
 - 5 Add edges (v, u) , (v, w) , and (u, w) to $E(G)$
 - 6 $\bar{d}_{\ell_{k-i+1}} = \bar{d}_{\ell_{k-i+1}} + 2$
-

Algorithm 4 starts constructing graph G by copying \bar{G} . Note that \bar{G} is guaranteed to have a vertex v with degree \bar{d}_{ℓ_k} . In the last iteration of Algorithm 3, \bar{d}_{ℓ_k} was reduced by 2. The degree of v ,

which is \bar{d}_{ℓ_k} , is increased by 2 in the first iteration of Algorithm 4. Additionally, two more vertices are added to G which are connected to each other and to v . Figure 13 shows how the new vertices can be added to an existing outerplanar embedding of G .

It follows that the graph G returned by Algorithm 4 is a (partial) realization of d . If \bar{G} is an outerplanar realization of \bar{G} , then G is an outerplanar realization of d . Otherwise, (\bar{G}, \bar{G}') is a 2PBE realization of \bar{d} . Observe that the vertices added by Algorithm 4 can be added to \bar{G}' as isolated vertices without invalidating the book embedding. Hence, (G, \bar{G}') is a 2PBE realization of d , where G is outerplanar and \bar{G}' is bipartite outerplanar. The lemma follows. \square

Lemma 18. *Every graphic $d \in \mathcal{D}_{\geq 5}$ with $d_2 \leq 3$ has an OP+bi realization.*

Proof. First, in ω_1 steps, connect a new vertex of degree 1 with the vertex of currently largest degree obtaining the residual degree sequence d' . Since $\sum d \geq 2n$ and $\sum d \leq 4n - 2\omega_1 - 6$,

$$\sum d' \leq 4(n - \omega_1) - 6, \quad d'_{n-\omega_1} \geq 2. \quad (7)$$

If $d'_1 \leq 4$, then d' has a 2PBE realization graph G' by Theorem 15 and Inequality (7). Add $d_i - d'_i$ leaves to the vertex in G' with degree d'_i for $1 \leq i \leq n - \omega_1$ and we get a graph G . Notice that G realizes d and has page number 2.

Assume $u_1, \dots, u_{n-\omega_1}$ are the vertices with decreasing degree in d' and $n' = n - \omega_1$. If $d'_1 \geq 5$, then $d'_1 > d_2$. All the ω_1 leaves are connected to the vertex with d'_1 since the leaf is connected with the vertex of currently largest degree. Therefore, $d_1 - d'_1 = \omega_1$. Then $d'_1 \leq n - 1 - \omega_1$ since $d_1 \leq n - 1$. Next connect u_1 with $u_2, \dots, u_{d'_1+1}$ and obtain the residual sequence d'' . Notice that d'' has the form of $(2^{n'-1})$ or $(3^{\omega'_3}, 2^{\omega'_2})$ or $(2^{\omega'_2}, 1^{\omega'_1})$ with $\omega'_3, \omega'_2, \omega'_1 > 0$.

Case 1: $d'' = (2^{n'-1})$. Since $d'_1 \geq 5$ and $d_2 \leq 3$, there are d'_1 vertices changing their degree from 3 in d' to 2 in d'' . Then $n' - 1 \geq 5$ and d'' can be realized by a cycle.

Case 2: $d'' = (3^{\omega'_3}, 2^{\omega'_2})$. Since $\omega'_3 > 0$ and $d_2 \leq 3$, there are d'_1 vertices changing their degree from 3 in d' to 2 in d'' . Then $\omega'_2 \geq 5$. Combining it with that ω'_3 is even, d'' can be realized by an outerplanar graph.

Case 3: For $d'' = (2^{\omega'_2}, 1^{\omega'_1})$, it can be realized by the union of a cycle and a matching or a forest. Therefore, d'' can be realized by an outerplanar graph G'' .

From above three cases, G'' is an outer planar graph. Add an isolated vertex u_1 to G'' and obtain an outerplanar graph $G'_1 = G'' \cup \{u_1\} = (V, E')$ with $V = V'' \cup \{u_1\}$. Let $G'_2 = (V, E'')$ with $E'' = \{(u_1, u_2), \dots, (u_1, u_{d'_1+1})\}$. The position of u_1 in the embedding of G'_1 and G'_2 can be fixed arbitrarily since u_1 is isolated in G'_1 , and G'_2 has edges that do not exist in G' . $G'_1 \cup G'_2$ realizes d' with 2 page embedding. Add $d_i - d'_i$ leaves to the vertex in G'_2 with degree d'_i for $1 \leq i \leq n - \omega_1$ and an outer planar graph G_2 is obtained. Notice that $G'_1 \cup G_2$ realizes d and has 2 page book embedding, where G'_1 is outerplanar and G_2 is bipartite outerplanar. \square

Corollary 19. *Let $d \in \mathcal{D}_{\geq 5}$ and $d_1 \leq n - 1$. If $2\Delta_E \geq \Delta_\omega$ and $d_n = d_{n-1} = 2$, then d has an OP+bi realization.*

Proof. If $\omega_2 = 2$, then by Lemma 9, d is outerplanar. If $d_2 \leq 3$, then d has a 2PBE realization by Lemma 18. Combining above observations with Lemma 17, this corollary is proved. \square

Lemma 20. *Let $d \in \mathcal{D}_{\geq 5}$ and $d_1 \leq n - 1$. If $2\Delta_E \geq \Delta_\omega$ and d satisfies $d_{n-1} \leq 2$ and $d_{n-2} \leq 3$, then d has an OP+bi realization.*

Proof. By the assumption on d_{n-1} , d_n is either 1 or 2. If $d_n = 2$, then $d_{n-1} = 2$ by the assumption on d_{n-1} , implying that d has a 2PBE realization by Corollary 19. So hereafter we focus on the case where $d_n = 1$.

We apply to d a sequence of transformations described in Algorithm 5, which gradually get rid of the 1 degrees.

Algorithm 5: Removing 1 degrees

```

1 Set  $\bar{d} \leftarrow d$ 
2 while  $\bar{\omega}_1 > 0$  AND  $\bar{d}_1 > 4$  do
3    $\bar{d}_1 \leftarrow \bar{d}_1 - 1$ 
4   Remove one degree 1 from  $\bar{d}$ 
5   Re-sort  $\bar{d}$ 

```

Assume Algorithm 5 executes k iterations of its while-loop before halting. Let \bar{d}^i be the intermediate sequence after iteration $i \in [0, k]$, i.e., $\bar{d}^0 = d$ and \bar{d}^k is the sequence \bar{d} reached upon termination of the algorithm. Observe that

- (i) $\bar{n}^i = \bar{n}^{i+1} + 1$,
- (ii) $\bar{\omega}_1^i = \bar{\omega}_1^{i+1} + 1$, $\bar{\omega}_2^i = \bar{\omega}_2^{i+1}$, $\bar{\omega}_3^i = \bar{\omega}_3^{i+1}$, and
- (iii) $\sum \bar{d}^i = \sum \bar{d}^{i+1} + 2$.

These observations imply also that,

$$\begin{aligned}
2\bar{\Delta}_E^i &= 4\bar{n}^i - 6 - \sum \bar{d}^i = 4(\bar{n}^{i+1} + 1) - 6 - (\sum \bar{d}^{i+1} + 2) \\
&= 4\bar{n}^{i+1} - 6 - \sum \bar{d}^{i+1} + 2 = 2\bar{\Delta}_E^{i+1} + 2.
\end{aligned} \tag{8}$$

By the above observations (i) and (ii),

$$\bar{\Delta}_\omega^i = 3\bar{\omega}_1^i + 2\bar{\omega}_2^i + \bar{\omega}_3^i - \bar{n}^i = 3(\bar{\omega}_1^{i+1} + 1) + 2\bar{\omega}_2^{i+1} + \bar{\omega}_3^{i+1} - \bar{n}^{i+1} - 1 = \bar{\Delta}_\omega^{i+1} + 2 \tag{9}$$

We next claim that throughout the execution of Algorithm 5,

$$2\bar{\Delta}_E^i \geq \bar{\Delta}_\omega^i, \quad \text{for } i = 0, \dots, k, \tag{10}$$

$$\sum \bar{d}^i \leq 4\bar{n}^i - 6 - 2\bar{\omega}_1^i, \quad \text{for } i = 0, \dots, k. \tag{11}$$

Both claims can be shown by induction on i , where the basis follows from the assumptions of the lemma. For the first inequality, the inductive progress step follows by Eq. (8) and (9). For the second inequality, the progress step follows from the inductive hypothesis since

$$\sum \bar{d}^{i+1} = \sum \bar{d}^i + 2 \leq 4\bar{n}^i - 6 - 2\bar{\omega}_1^i + 2 = 4\bar{n}^{i+1} - 6 - 2\bar{\omega}_1^{i+1}.$$

Algorithm 5 stops when either $\bar{\omega}_1 = 0$ or $\bar{d}_1 \leq 3$. Let the final sequence be d' and n is updated to n' . By Eq. (10) and (11), we have

$$2\Delta'_E \geq \Delta'_\omega, \quad (12)$$

$$\sum d' \leq 4n' - 6 - 2\omega'_1. \quad (13)$$

Next, we claim that the resulting sequence d' has a 2PBE realization. To prove this, consider the above two cases separately.

Case a: $\omega'_1 = 0$. By Eq. (13), we have $d'_{n'-1} \leq 3$. Consider two possible subcases, $(d'_{n'-1}, d'_{n'}) = (2, 2)$ and $(d'_{n'-1}, d'_{n'}) \in \{(2, 3), (3, 3)\}$.

SubCase (a1): $d'_{n'} = d'_{n'-1} = 2$. Then combining Eq. (12) and (13), the conditions in Corollary 19 are satisfied and d' has a 2PBE realization G' .

SubCase (a2): $d'_{n'} \in \{2, 3\}$ and $d'_{n'-1} = 3$. Then, d' has a 2PBE realization G' by Lemma 13.

Case (b): $d'_1 \leq 4$ and $\bar{\omega}_1 > 0$. Notice that $d'_1 \leq n' - 1$ by Algorithm 5. In this case, d' 2PBE realization G' by Lemma 14 and Eq. (13).

It follows that in all cases, d' has a 2PBE realization. We now transform this realization into a 2PBE realization G for the original sequence d by creating $d_i - d'_i$ new vertices of degree 1 and connecting them to vertex i in G' , for every i . (Clearly, these vertices can be embedded on the existing pages of G' while preserving their outerplanarity.) By the above construction, G consists of two graphs, where one is outerplanar and the other is bipartite outerplanar. \square

Combining Lemmas 3, 8 and 20, we conclude the following.

Corollary 21. *There is a polynomial time algorithm that, given an outerplanaric sequence $d \in \mathcal{D}_{\geq 5}$ with $d_2 > 3$, returns an OP+bi realization for d .*

4.3 Putting the pieces together

Combining Thm. 15, Lemma 18 and Cor. 21, we conclude with the following.

Theorem 22. *There is a polynomial time algorithm that given a sequence $d \in \mathcal{D}$ either returns “ d is not outerplanaric” or returns a 2PBE realization for d .*

Corollary 23. *There is a poly-time algorithm that constructs a 1PBE or 2PBE realization for any sequence in a superclass² \mathcal{S} of the class \mathcal{OP} of all outerplanaric sequences.*

²The class \mathcal{S} strictly contains the class \mathcal{OP} . For example, the sequence $(4^2, 2^4)$ is not outerplanaric yet it satisfies the conditions of Lemma 12, hence our algorithm succeeds in constructing a 2PBE realization for it.

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