A NOTE ON ADJOINT REALITY IN SIMPLE COMPLEX LIE ALGEBRAS

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Abstract. Let G be a Lie group with Lie algebra $\mathfrak{g}$. In [GM], an infinitesimal version of the notion of classical reality, namely adjoint reality, has been introduced. An element $X \in \mathfrak{g}$ is adjoint real if $-X$ belongs to the adjoint orbit of X in $\mathfrak{g}$. In this paper, we investigate the adjoint real and the strongly adjoint real semisimple elements in complex simple classical Lie algebras. We also prove that every element in a complex symplectic Lie algebra is adjoint real.

1. Introduction

In the group theoretical set-up an element g in a group G is called *real* or *reversible* if it is conjugate to g^{-1} in G. An element g is *strongly real* or *strongly reversible* in G if it is conjugate to g^{-1} by an involution. Classification of real and strongly real elements in a group is a problem of wide interest; see [OS], [ST].

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider the natural $\mathrm{Ad}(G)$-representation of G on its Lie algebra $\mathfrak{g}$

$$\mathrm{Ad}\colon G \longrightarrow \mathrm{GL}(\mathfrak{g})\,.$$

For $X \in \mathfrak{g}$, *the adjoint orbit* of X in $\mathfrak{g}$ is defined as $\mathcal{O}_X := \{\mathrm{Ad}(g)X \mid g \in G\}$. If $X \in \mathfrak{g}$ is semisimple, then $\mathcal{O}_X$ is called *semisimple orbit*. Understanding the adjoint orbits in a semisimple Lie group has been an intense area of research, cf. [CM], [Mc]. For various results related to semisimple orbits, see [CM, Chapter 2].

In [GM], the authors introduced the notion of *adjoint reality* which we recall now. Consider the $\mathrm{Ad}(G)$-representation of G on $\mathfrak{g}$. For a linear Lie group G, $\mathrm{Ad}(g)X = gXg^{-1}$.

Definition 1.1 ([GM, Definition 1.1])**.** An element $X \in \mathfrak{g}$ is called Ad_G*-real* if $-X \in \mathcal{O}_X$. An Ad_G-real element is called *strongly* Ad_G*-real* if $-X = \mathrm{Ad}(\tau)X$ for some $\tau \in G$ so that $\tau^2 = \mathrm{Id}$.

This is an infinitesimal analogue of the reality in Lie groups. It was shown in [GM] that the reality of the unipotent elements in a Lie group and the Ad_G-reality of the nilpotent elements in the corresponding Lie algebras are equivalent via the exponential map. This correspondence was used to classify unipotent real elements in classical Lie groups in [GM]. However, this correspondence does not necessarily hold in general. Nevertheless, classifying the adjoint reality in Lie agebras is a problem of independent algebraic interest, and it also helps to understand the real and strongly real elements in the image of the exponential map, thus providing understanding of reality in the Lie group to a large

2020 *Mathematics Subject Classification.* Primary 20E45; Secondary: 22E60, 17B08.
Key words and phrases. Adjoint orbits, real element, Strongly adjoint real element, semisimple element.
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extent. The above notion of adjoint reality has turned useful in classifying the strongly real elements in $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{GL}_n(\mathbb{D}) \ltimes \mathbb{D}^n$ for $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, respectively, cf. [GLM1], [GLM2].

With this motivation, it is a natural problem to investigate the adjoint reality for semisimple orbits. The aim of this note is to classify the adjoint real and strongly adjoint real semisimple elements in the complex simple classical Lie algebras; see Theorem 3.5, Theorem 3.9, Theorem 3.12. The adjoint real nilpotent elements in these Lie algebras are classified in [GM].

By the Jordan decomposition, every element in a semisimple Lie algebra decomposes as a unique sum of a semisimple and a unipotent element. Thus, classifying an arbitrary adjoint real element in a Lie algebra is intimately related to such classification of semisimple and nilpotent elements. We demonstrate this for the symplectic Lie algebras, i.e. semisimple Lie algebras of type C_n. Recall that symplectic group plays an vital role in many branch of Mathematics. Thus the characterisation of adjoint real elements in the symplectic Lie algebra are fundamentally important which is done in Theorem 4.2 by using description of the centralizers. We note here that classifying strongly real elements using this idea might require further technicalities as we have seen for the type A_n Lie algebras in [GLM3]. Other than Lie algebras of type A_n and C_n, adjoint reality for arbitrary elements in other semisimple Lie algebras are yet to be fully understood.

Given $X \in \mathfrak{g}$ one defines the following subsets of G. The *centralizer* and the *reverser* of an element X in G are respectively defined as

$$Z_G(X) := \{s \in G \mid sXs^{-1} = X\}, \text{ and } R_G(X) := \{r \in G \mid rXr^{-1} = -X\}.$$

Note that $Z_G(X)$ is a subgroup but the set $R_G(X)$ is a right coset of the centralizer $Z_G(X)$. Thus the reversing symmetry group or the extended centralizer $E_G(X) := Z_G(X) \cup R_G(X)$ is a subgroup of G in which $Z_G(X)$ has index 1 or 2. The group $E_G(X)$ is an extension of $Z_G(X)$ of degree at most two. In the group theoretical set-up we refer to [OS, §2.1.4], [BR] on reversing symmetries for groups.

To find the reversing symmetric group $E_G(X)$, it is enough to construct one reversing element which is not in the centralizer. We have explicitly constructed an element in $R_G(X)$ for each adjoint real semisimple element. Recall that for a simply connected complex semisimple Lie group G, the centralizer $Z_G(X)$ of a semisimple element X is connected; see [CM, Theorem 2.3.3, p.28]. Thus the centralizer is determined by its Lie algebra which has a nice description in terms of Cartan subalgebra and certain root vectors; see [CM, Lemma 2.1.2, p. 20]. Therefore, our construction also classifies the reverser of adjoint real semisimple elements in simple Lie groups.

2. Notation and background

The Lie groups will be denoted by the capital letters, while the Lie algebra of a Lie group will be denoted by the corresponding lower case German letter. For a subgroup H of G and a subset S of $\mathfrak{g}$, the subgroup $Z_H(S)$ of H that fixes S pointwise under the adjoint action is called the *centralizer* of S in H. Similarly, for a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a subset $S \subset \mathfrak{g}$, by $\mathfrak{z}_{\mathfrak{h}}(S)$ we will denote the subalgebra of $\mathfrak{h}$ consisting of all the elements that commute with every element of S. For $A \in \mathrm{M}_n(\mathbb{C})$, A^t denotes the transpose of the matrix A. Let I_n denote the $n \times n$ identity matrix, and

$$\mathrm{J}_n := \begin{pmatrix} & -\mathrm{I}_n \\ \mathrm{I}_n & \end{pmatrix}. \tag{2.1}$$

Here we will work with the following classical simple Lie groups and Lie algebras over $\mathbb{C}$:

$$\begin{aligned} \mathrm{SL}_n(\mathbb{C}) &:= \{g \in \mathrm{GL}_n(\mathbb{C}) \mid \det(g) = 1\}, & \mathfrak{sl}_n(\mathbb{C}) &:= \{z \in \mathrm{M}_n(\mathbb{C}) \mid \mathrm{tr}(z) = 0\}; \\ \mathrm{SO}(n,\mathbb{C}) &:= \{g \in \mathrm{SL}_n(\mathbb{C}) \mid g^t g = \mathrm{I}_n\}, & \mathfrak{so}(n,\mathbb{C}) &:= \{z \in \mathfrak{sl}_n(\mathbb{C}) \mid z^t \mathrm{I}_n + \mathrm{I}_n z = 0\}; \\ \mathrm{Sp}(n,\mathbb{C}) &:= \{g \in \mathrm{SL}_{2n}(\mathbb{C}) \mid g^t \mathrm{J}_n g = \mathrm{J}_n\}, & \mathfrak{sp}(n,\mathbb{C}) &:= \{z \in \mathfrak{sl}_{2n}(\mathbb{C}) \mid z^t \mathrm{J}_n + \mathrm{J}_n z = 0\}. \end{aligned}$$

For any group H, let H^n_Δ denote the diagonally embedded copy of H in the n-fold direct product H^n. Similarly, for a matrix $A \in \mathrm{M}_n(\mathbb{C})$, let A^n_Δ denote the diagonally embedded copy of A in the n-fold direct sum $A \oplus \cdots \oplus A$. For a pair of disjoint ordered sets $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$, the ordered set $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ will be denoted by

$$(v_1, \ldots, v_n) \vee (w_1, \ldots, w_m).$$

For a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, a subset $\{X, H, Y\} \subset \mathfrak{g}$ is said to be a $\mathfrak{sl}_2$-*triple* if $X \neq 0$, $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$. Note that for a $\mathfrak{sl}_2$-triple $\{X, H, Y\}$ in $\mathfrak{g}$, $\mathrm{Span}_{\mathbb{C}}\{X, H, Y\}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. We now recall a well-known result due to Jacobson and Morozov.

Theorem 2.1 (Jacobson-Morozov, cf. [CM, Theorem 9.2.1])**.** *Let $X \in \mathfrak{g}$ be a non-zero nilpotent element in a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Then there exist $H, Y \in \mathfrak{g}$ such that $\{X, H, Y\}$ is a $\mathfrak{sl}_2$-triple.*

3. Adjoint reality for semisimple elements

Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$. Let $H \in \mathfrak{g}$ be a semisimple element, and $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$. Further we may assume $H \in \mathfrak{h}$ as $\mathrm{Ad}(g)(H) \in \mathfrak{h}$ for some $g \in G$.

3.1. Semi-simple elements in $\mathfrak{sl}_n(\mathbb{C})$.

Let $\mathfrak{g} := \mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{h}$ be the subalgebra consisting of all diagonal matrices in $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra in $\mathfrak{g}$.

Lemma 3.1. *Let H be a semisimple element in $\mathfrak{gl}_n(\mathbb{C})$. Then H is $\mathrm{Ad}_{\mathrm{GL}_n(\mathbb{C})}$-real in $\mathfrak{gl}_n(\mathbb{C})$ if and only if whenever λ is an eigenvalue of H, $-\lambda$ is also an eigenvalue of H with the same multiplicity.*

Remark 3.2. A semisimple element $H \in \mathfrak{gl}_n(\mathbb{C})$ is $\mathrm{Ad}_{\mathrm{GL}_n(\mathbb{C})}$-real if and only if H is $\mathrm{Ad}_{\mathrm{SL}_n(\mathbb{C})}$-real in $\mathfrak{sl}_n(\mathbb{C})$. The analogous statement is also true in the group theoretic sense; see [OS, p. 77].

Theorem 3.3. *Every real semisimple element in* $\mathrm{Lie}(\mathrm{PSL}_n(\mathbb{C}))$ *is strongly* $\mathrm{Ad}_{\mathrm{PSL}_n(\mathbb{C})}$*-real.*

Proof. Let $H \in \mathrm{Lie}(\mathrm{PSL}_n(\mathbb{C}))$ be a $\mathrm{Ad}_{\mathrm{PSL}_n(\mathbb{C})}$-real semisimple element. We may assume $H = \mathrm{diag}(h_1, \ldots, h_m, -h_1, \ldots, -h_m, 0, \ldots, 0)$. Then $\sigma := \mathrm{diag}(\mathrm{J}_m, \sqrt{-1}\,\mathrm{I}_s)$ will conjugate H and $-H$, where $2m + s = n$, and J_n is as in (2.1). $\square$

Example 3.4. Consider the semi-simple element $H = \mathrm{diag}(x_1, -x_1)$. Let $g \in \mathrm{GL}_2(\mathbb{C})$ so that $gH = -Hg$. Then g is of the form $\left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$. Hence H is a strongly $\mathrm{Ad}_{\mathrm{GL}_2(\mathbb{C})}$-real element in $\mathfrak{gl}_2(\mathbb{C})$, and is an $\mathrm{Ad}_{\mathrm{SL}_2(\mathbb{C})}$-real element but not strongly $\mathrm{Ad}_{\mathrm{SL}_2(\mathbb{C})}$-real in $\mathfrak{sl}_2(\mathbb{C})$. Note that $g \in \mathrm{Sp}(1, \mathbb{C})$ if $bc = -1$. Similarly, H is a $\mathrm{Ad}_{\mathrm{Sp}(1,\mathbb{C})}$-real element but not strongly $\mathrm{Ad}_{\mathrm{Sp}(1,\mathbb{C})}$-real element in $\mathfrak{sp}(1, \mathbb{C})$. □

The next result classifies strong $\mathrm{Ad}_{\mathrm{SL}_n(\mathbb{C})}$-reality in the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$; see [GLM3, Proposition 2.5]. Here we provide a detailed proof.

Theorem 3.5. *An $\mathrm{Ad}_{\mathrm{SL}_n(\mathbb{C})}$-real semisimple element in $\mathfrak{sl}_n(\mathbb{C})$ is strongly $\mathrm{Ad}_{\mathrm{SL}_n(\mathbb{C})}$-real if and only if either 0 is an eigenvalue or $n \not\equiv 2 \pmod 4$.*

Proof. Let $H \in \mathfrak{sl}_n(\mathbb{C})$ be a $\mathrm{Ad}_{\mathrm{SL}_n(\mathbb{C})}$-real semisimple element. If 0 is an eigenvalue of H, then using Example 3.4 it follows that H is strongly $\mathrm{Ad}_{\mathrm{SL}_n(\mathbb{C})}$-real. Suppose 0 is not an eigenvalue of H, and $n \not\equiv 2 \pmod 4$, then $n = 4m$ for $m \in \mathbb{N}$ and we can assume $H = \mathrm{diag}(h_1, \dots, h_{2m}, -h_1, \dots, -h_{2m})$. Then, H and $-H$ will be conjugated by $g = \left(\begin{smallmatrix} & \mathrm{I}_{2m} \\ \mathrm{I}_{2m} & \end{smallmatrix}\right)$.

Next assume that H is strongly $\mathrm{Ad}_{\mathrm{SL}_n(\mathbb{C})}$-real and 0 is not an eigenvalue of H. We will show $n \in 4\mathbb{N}$. Without loss of generality, we can assume $H = \mathrm{diag}(h_1, \dots, h_m, -h_1, \dots, -h_m)$ and $gH = -Hg$ for some involution. Let e_j be the standard column vector in $\mathbb{C}^n$ with 1 in j^{th} place and 0 elsewhere. For $1 \le j \le m$,

$$Hge_j \;=\; -gHe_j \;=\; -gh_je_j \;=\; -h_jge_j.$$

Let $V_j := \mathbb{C}e_j \oplus \mathbb{C}ge_j$ and $\mathcal{C}_j := \{e_j, ge_j\}$. Since $g^2 = \mathrm{I}_2$, $g(V_j) \subset V_j$. Then $\{e_j, ge_j \mid 1 \le j \le m\}$ forms a basis of $\mathbb{C}^{2m}$. Set $\mathcal{C} := \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_m$. Then the matrix $[g]_{\mathcal{C}}$ is a 2×2 block-diagonal matrix and $\det\,[g]_{\mathcal{C}} = (-1)^m$. As $\det g = 1$, it follows that $m \in 2\mathbb{N}$. This completes the proof. □

3.2. **Semi-simple elements in $\mathfrak{o}(n, \mathbb{C})$ and $\mathfrak{so}(n, \mathbb{C})$.** Up to conjugacy, any semisimple element in $\mathfrak{o}(n, \mathbb{C})$ or $\mathfrak{so}(n, \mathbb{C})$ belongs to the following Cartan subalgebra $\mathfrak{h}$, see [Kn, p. 127]:

$$\mathfrak{h} := \begin{cases} \mathrm{diag}(H_1, \dots, H_m, 0) & \text{if } n = 2m+1 \\ \mathrm{diag}(H_1, \dots, H_m) & \text{if } n = 2m \end{cases}, \text{ where } H_j = \begin{pmatrix} 0 & x_j \\ -x_j & 0 \end{pmatrix}, x_j \in \mathbb{C}. \tag{3.1}$$

Example 3.6. Consider $H = \left(\begin{smallmatrix} 0 & x \\ -x & 0 \end{smallmatrix}\right) \in \mathfrak{so}(2, \mathbb{C}) = \mathfrak{o}(2, \mathbb{C})$, where $x \in \mathbb{C}$. Let $g \in \mathrm{GL}_2(\mathbb{C})$ so that $gHg^{-1} = -H$. Then g is of the form $\left(\begin{smallmatrix} a & b \\ b & -a \end{smallmatrix}\right)$. Hence, $\det g = -a^2 - b^2$, and $gg^t = g^2 = (-\det g)\mathrm{I}_2$. Thus, one can choose $g \in \mathrm{O}(2, \mathbb{C})$ with $\det g = -1$. This shows that H is a $\mathrm{Ad}_{\mathrm{O}(n,\mathbb{C})}$-real, as well as strongly $\mathrm{Ad}_{\mathrm{O}(n,\mathbb{C})}$-real, element in $\mathfrak{o}(2, \mathbb{C})$ but not $\mathrm{Ad}_{\mathrm{SO}(n,\mathbb{C})}$-real in $\mathfrak{so}(2, \mathbb{C})$. □

The following result follows from the construction done in the above example.

Lemma 3.7. *Every semisimple element in $\mathfrak{o}(n, \mathbb{C})$ is $\mathrm{Ad}_{\mathrm{O}(n,\mathbb{C})}$-real.*

Next we investigate strongly $\mathrm{Ad}_{\mathrm{O}(n,\mathbb{C})}$-real elements in $\mathfrak{o}(n, \mathbb{C})$.

Proposition 3.8. *Every semisimple element in $\mathfrak{o}(n, \mathbb{C})$ is strongly $\mathrm{Ad}_{\mathrm{O}(n,\mathbb{C})}$-real.*

Proof. Enough to consider the elements of $\mathfrak{h}$. For the elements in $\mathfrak{h}$ one can easily construct the required involution using the Example 3.6. $\square$

Now we classify the strongly $\mathrm{Ad}_{\mathrm{SO}(n,\mathbb{C})}$-real semisimple element in $\mathfrak{so}(n,\mathbb{C})$.

Theorem 3.9. *Let $H \in \mathfrak{so}(n,\mathbb{C})$ be a semisimple element. Then H is a strongly $Ad_{\mathrm{SO}(n,\mathbb{C})}$-real if and only if either 0 is an eigenvalue of H or $n \not\equiv 2 \pmod 4$.*

Proof. Since H is semisimple, there exists $\sigma \in \mathrm{SO}(n,\mathbb{C})$ such that $\mathrm{Ad}(\sigma)H = \mathrm{diag}(H_1,\dots,H_m,\underbrace{0,\dots,0}_{r\text{-many}})$, $r \geq 1$ where H_j is as in (3.1).

First assume that 0 is an eigenvalue of H. Let $g := \mathrm{diag}(\underbrace{\mathrm{I}_{1,1},\dots,\mathrm{I}_{1,1}}_{m\text{-many}},\mathrm{I}_r)$, where $\mathrm{I}_{1,1} := \mathrm{diag}(1,-1)$. Then $\mathrm{Ad}(\sigma^{-1}g\sigma)H = -H$. Note that $g^2 = \mathrm{I}_n$. If $\det g = 1$, then we are done. Otherwise replace g by $\mathrm{diag}(\mathrm{I}_{1,1},\dots,\mathrm{I}_{1,1},\mathrm{I}_{r-1},-1)$ to get the required involution in $\mathrm{SO}(n,\mathbb{C})$. Next assume that 0 is not an eigenvalue, and $n \not\equiv 2 \pmod 4$. Thus $n \equiv 0 \pmod 4$, and hence up to conjugacy H is of the form $\mathrm{diag}(H_1,\dots,H_{2k})$, $4k = n$. By choosing an involution g as above with $m = 2k, r = 0$, we have that H is strongly $\mathrm{Ad}_{\mathrm{SO}(n,\mathbb{C})}$-real.

Next suppose that H is strongly $\mathrm{Ad}_{\mathrm{SO}(n,\mathbb{C})}$-real. Further assume that 0 is not an eigenvalue of H, and $H = \mathrm{diag}(H_1,\dots,H_m)$ where H_j is as in (3.1). Let e_j be the standard column vector in $\mathbb{C}^n$. Then $e_{2j-1}+\sqrt{-1}e_{2j}$ and $g(e_{2j-1}+\sqrt{-1}e_{2j})$ are eigenvector of H corresponding to the eigenvalues $\sqrt{-1}x_j$ and $-\sqrt{-1}x_j$, respectively. Let $V_j := \mathbb{C}(e_{2j-1}+\sqrt{-1}e_{2j}) \oplus \mathbb{C}g(e_{2j-1}+\sqrt{-1}e_{2j})$, and $\mathcal{C}_j := \{e_{2j-1}+\sqrt{-1}e_{2j},\, g(e_{2j-1}+\sqrt{-1}e_{2j})\}$. Since $g^2 = \mathrm{I}_2$, $g(V_j) \subset V_j$. Then $\{e_{2j-1}+\sqrt{-1}e_{2j},\, g(e_{2j-1}+\sqrt{-1}e_{2j}) \mid 1 \leq j \leq m\}$ forms a basis of $\mathbb{C}^n$. Set $\mathcal{C} := \mathcal{C}_1 \vee \cdots \vee \mathcal{C}_m$. Then the matrix $[g]_{\mathcal{C}}$ is a 2×2 block-diagonal matrix and $\det [g]_{\mathcal{C}} = (-1)^m$. As $\det g = 1$, it follows that $m \in 2\mathbb{N}$. This completes the proof. $\square$

3.3. **Semi-simple elements in $\mathfrak{sp}(n,\mathbb{C})$.** Recall that in Example 3.4, $\mathrm{diag}(x,-x)$ is not strongly $\mathrm{Ad}_{\mathrm{Sp}(1,\mathbb{C})}$-real in but we can choose a conjugating involution g from $\mathrm{PSp}(1,\mathbb{C})$. Thus, we will first consider the semisimple element in the Lie algebra of $\mathrm{PSp}(n,\mathbb{C})$. Let $\mathfrak{h} := \{\mathrm{diag}(h_1,\dots,h_m,-h_1,\dots,-h_m) \mid h_j \in \mathbb{C}\}$.

Theorem 3.10. *Every semisimple element in* $\mathrm{Lie}\,(\mathrm{PSp}(n,\mathbb{C}))$ *is strongly* $\mathrm{Ad}_{\mathrm{PSp}(n,\mathbb{C})}$*-real.*

Proof. Let $X \in \mathrm{Lie}\,(\mathrm{PSp}(n,\mathbb{C}))$. Then $\mathrm{Ad}(g)X \in \mathfrak{h}$ for some $g \in \mathrm{PSp}(n,\mathbb{C})$. Thus we define required involution using Example 3.4. $\square$

The following corollary is immediate.

Corollary 3.11. *Every semisimple element in $\mathfrak{sp}(n,\mathbb{C})$ is* $\mathrm{Ad}_{\mathrm{Sp}(n,\mathbb{C})}$*-real.*

Theorem 3.12. *A semisimple element in $\mathfrak{sp}(n,\mathbb{C})$ is strongly* $\mathrm{Ad}_{\mathrm{Sp}(n,\mathbb{C})}$*-real if and only if the multiplicity of each non-zero eigenvalue is even.*

Proof. Let H be a strongly real semisimple element in $\mathfrak{sp}(n,\mathbb{C})$. We can assume $H = \mathrm{diag}(h_1,\dots,h_n,-h_1,\dots,-h_n)$, and $gH = -Hg$ for some involution $g \in \mathrm{Sp}(n,\mathbb{C})$. Suppose $h_j \neq 0$ and multiplicity of h_j is m. Let $\mathcal{C}_j := \{e_{j_1},\dots,e_{j_m}\}$ be an ordered basis of the eigenspace of H corresponding to the eigenvalue h_j. Then $\mathcal{C}_{n+j} := \{e_{n+j_1},\dots,e_{n+j_m}\}$

is an ordered basis of the eigenspace corresponding to the eigenvalue $-h_j$. Let V_j be the $\mathbb{C}$-span of $\mathcal{C}_j \vee \mathcal{C}_{n+j}$. Then the involution g keeps V_j invariant and $(g|_{V_j})^t \mathrm{J}_m (g|_{V_j}) = \mathrm{J}_m$, where J_m is as in (2.1). As $H(ge_{jl}) = -h_j ge_{jl}$ for $1 \leq l \leq m$, we can write

$$[g]_{\mathcal{C}_j \vee \mathcal{C}_{n+j}} := \left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right), \quad \text{for some } B, C \in \mathrm{GL}_m(\mathbb{C}). \tag{3.2}$$

Since $g|_{V_j}$ is an involution and $(g|_{V_j})^t \mathrm{J}_m (g|_{V_j}) = \mathrm{J}_m$, it follows that $C^{-1} = B = -B^t$. Hence, m (the multiplicity of h_j) has to be even.

Next assume that multiplicity of each non-zero eigenvalue is even. Can assume any semisimple element $H = \mathrm{diag}(h_1, \dots, h_n, -h_1, \dots, -h_n)$, where $h_{2j-1} = h_{2j}$ for $j = 1, \dots, n/2$. In this case to define a required involution g, set $B := \begin{pmatrix} & & \mathrm{J}_1 \\ & \iddots & \\ \mathrm{J}_1 & & \end{pmatrix}$ in (3.2). This completes the proof. □

4. $\mathrm{Ad}_{\mathrm{Sp}(n,\mathbb{C})}$-REALITY IN $\mathfrak{sp}(n, \mathbb{C})$

The aim of this section is to prove that every element in $\mathfrak{sp}(n, \mathbb{C})$ is adjoint real. For this, we need to recall some known construction and results, cf. [BCM1]. The structure of the centralizer of nilpotent elements play an important role here.

Let $0 \neq X \in \mathfrak{sp}(n, \mathbb{C})$ be a nilpotent element. Let $\mathfrak{a}$ be a $\mathfrak{sl}_2$-triple in $\mathfrak{sp}(n, \mathbb{C})$ containing X; see Theorem 2.1. Note that $\mathbb{C}^{2n}$ is $\mathbb{C}$-module over $\mathfrak{a}$. By decomposing $\mathbb{C}^{2n}$ as direct sum of irreducible $\mathfrak{a}$-module, let $\mathbb{N}_{\mathbf{d}} := \{d_1, \dots, d_s\}$ be the dimensions of irreducible $\mathfrak{a}$-modules. Let $M(d-1)$ denote the sum of all $\mathbb{C}$-subspaces of $\mathbb{C}^{2n}$ which are irreducible $\mathfrak{a}$-submodule of dimension d. Then $M(d-1)$ is the *isotypical component* of $\mathbb{C}^{2n}$ containing all the irreducible submodules of $\mathbb{C}^{2n}$ with highest weight $d-1$. Let

$$L(d-1) := \mathrm{Span}_{\mathbb{C}}\{v \in M(d-1) \mid \text{weight of } v \text{ is } 1-d\}.$$

Then it follows that $M(d-1) = L(d-1) \oplus XL(d-1) \oplus \dots \oplus X^{d-1}L(d-1)$; see [BCM1, Lemma A.1]. Let $t_d := \dim_{\mathbb{C}} L(d-1)$. Then $\sum_{d \in \mathbb{N}_{\mathbf{d}}} dt_d = 2n$, and let $\{X^l v_j^d \mid 0 \leq l < d,\ 1 \leq j \leq t_d, d \in \mathbb{N}_{\mathbf{d}}\}$ be a basis of $\mathbb{C}^{2n}$ as constructed in [BCM1, Lemma A.6]. We need to fix an ordering of the above basis. Let $(v_1^d, \dots, v_{t_d}^d)$ be an ordered $\mathbb{C}$-basis of $L(d-1)$ for $d \in \mathbb{N}_{\mathbf{d}}$. Then it follows that

$$\mathcal{B}^l(d) := (X^l v_1^d, \dots, X^l v_{t_d}^d)$$

is an ordered $\mathbb{C}$-basis of $X^l L(d-1)$ for $0 \leq l \leq d-1$ with $d \in \mathbb{N}_{\mathbf{d}}$. Define

$$\mathcal{B}(d) := \mathcal{B}^0(d) \vee \dots \vee \mathcal{B}^{d-1}(d) \ \forall\ d \in \mathbb{N}_{\mathbf{d}}, \quad \text{and} \quad \mathcal{B} := \mathcal{B}(d_1) \vee \dots \vee \mathcal{B}(d_s).$$

Let $\langle \cdot, \cdot \rangle : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \longrightarrow \mathbb{C}$ be the symplectic form given by $\langle x, y \rangle := x^t \mathrm{J}_n y$. Define another form on $L(d-1)$ below, as in [CM, p. 139],

$$(\cdot, \cdot)_d : L(d-1) \times L(d-1) \longrightarrow \mathbb{C} \quad ; \quad (v, u)_d := \langle v, X^{d-1} u \rangle. \tag{4.1}$$

The following result is due to Springer-Steinberg, which describes the structure of the centralizer of the $\mathfrak{sl}_2$-triple in $\mathrm{Sp}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$.

Lemma 4.1 (cf. [CM, Theorem 6.1.3], [BCM1, Lemma 4.4]). *The following isomorphisms hold:*

(1) *If $\mathfrak{a}$ is a $\mathfrak{sl}_2$-triple in $\mathfrak{sp}(n,\mathbb{C})$, then*

$$\mathcal{Z}_{\mathrm{Sp}(n,\mathbb{C})}(\mathfrak{a}) = \left\{ g \in \mathrm{SL}(\mathbb{C}^{2n}) \;\middle|\; \begin{array}{c} g(X^l L(d-1)) \subset X^l L(d-1), \\ \big[g|_{X^l L(d-1)}\big]_{\mathcal{B}^l(d)} = \big[g|_{L(d-1)}\big]_{\mathcal{B}^0(d)}, (gx,gy)_d = (x,y)_d \\ \textit{for all } d \in \mathbb{N}_{\mathbf{d}},\ 0 \le l \le d-1,\ \textit{and } x,y \in L(d-1) \end{array} \right\};$$

here $(\cdot,\,\cdot)_d$ is as in (4.1).

(2) *In particular,*

$$\begin{aligned} \mathcal{Z}_{\mathrm{Sp}(n,\mathbb{C})}(\mathfrak{a}) &\simeq \Big\{ g \in \prod_{d \in \mathbb{N}_{\mathbf{d}}} \mathrm{GL}(L(d-1)) \;\Big|\; (gx,gy)_d = (x,y)_d,\ \chi_{\mathbf{d}}(g) = 1 \Big\} \\ &\simeq \Big\{ \prod_{d \in \mathbb{O}_{\mathbf{d}}} \mathrm{Sp}(t_{\mathbf{d}}/2,\,\mathbb{C})^d_\Delta \times \prod_{d \in \mathbb{E}_{\mathbf{d}}} \mathrm{O}(t_{\mathbf{d}},\,\mathbb{C})^d_\Delta \Big\}. \end{aligned} \tag{4.2}$$

(3)

$$\mathfrak{z}_{\mathfrak{sp}(n,\mathbb{C})}(\mathfrak{a}) = \left\{ A \in \mathfrak{sl}(\mathbb{C}^{2n}) \;\middle|\; \begin{array}{c} A(X^l L(d-1)) \subset X^l L(d-1), \\ \big[A|_{X^l L(d-1)}\big]_{\mathcal{B}^l(d)} = \big[A|_{L(d-1)}\big]_{\mathcal{B}^0(d)}, (Ax,y)_d + (x,Ay)_d = 0 \\ \textit{for all } d \in \mathbb{N}_{\mathbf{d}},\ 0 \le l \le d-1,\ \textit{and } x,y \in L(d-1) \end{array} \right\}.$$

(4) *In particular,*

$$\mathfrak{z}_{\mathfrak{sp}(n,\mathbb{C})}(\mathfrak{a}) \simeq \Big\{ \Big(\bigoplus_{d \in \mathbb{O}_{\mathbf{d}}} \mathfrak{sp}(t_{\mathbf{d}}/2,\,\mathbb{C})^d_\Delta \Big) \bigoplus \Big(\bigoplus_{d \in \mathbb{E}_{\mathbf{d}}} \mathfrak{o}(t_{\mathbf{d}},\,\mathbb{C})^d_\Delta \Big) \Big\}.$$

Now we will characterize the $\mathrm{Ad}_{\mathrm{Sp}(n,\mathbb{C})}$-real elements in $\mathfrak{sp}(n,\mathbb{C})$.

Theorem 4.2. *Every element of $\mathfrak{sp}(n,\mathbb{C})$ is $\mathrm{Ad}_{\mathrm{Sp}(n,\mathbb{C})}$-real.*

Proof. Let $X \in \mathfrak{sp}(n,\mathbb{C})$, and $X = X_s + X_n$ be the Jordan decomposition of X where X_s and X_n are the semisimple and nilpotent part of X, respectively. In view of Corollary 3.11 and [GM, Lemma 4.2], we may further assume that $X_s \neq 0$, $X_n \neq 0$. We will construct below two elements σ and τ in $\mathrm{Sp}(n,\mathbb{C})$ so that

- $\sigma X_n \sigma^{-1} = -X_n$ and $\sigma X_s \sigma^{-1} = X_s$,
- $\tau X_s \tau^{-1} = -X_s$ and $\tau X_n \tau^{-1} = X_n$.

Then $\sigma\tau$ will do the job, i.e., $\sigma\tau X(\sigma\tau)^{-1} = -X$.

Construction of σ. Define $\sigma \in \mathrm{GL}\left(\mathbb{C}^{2n}\right)$ below, as done in [GM, (4.3)] :

$$\sigma(X^l v_j^d) := \begin{cases} (-1)^l X^l v_j^d & \text{if } d \in \mathbb{O}_{\mathbf{d}}\,, \\ (-1)^l \sqrt{-1} X^l v_j^d & \text{if } d \in \mathbb{E}_{\mathbf{d}}\,. \end{cases} \tag{4.3}$$

Note that $\sigma X_n = -X_n \sigma$, and $\langle \sigma x, \sigma y \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{C}^{2n}$. This shows that $\sigma \in \mathrm{Sp}(n,\mathbb{C})$; cf. [GM, Section 4.3].

The element X_s is a semisimple element and commutes with X_n, and $\mathfrak{z}_{\mathfrak{sp}(n,\mathbb{C})}(\mathfrak{a})$ is reductive part (Levi part) of $\mathfrak{z}_{\mathfrak{sp}(n,\mathbb{C})}(X_n)$; see [CM, Lemma 3.7.3]. Thus, $X_s \in \mathfrak{z}_{\mathfrak{sp}(n,\mathbb{C})}(\mathfrak{a})$. Write $[X_s|_{L(d-1)}]_{\mathcal{B}^0_d} := X_{sd}$. Then using Lemma 4.1(3), $X_s = \oplus_{d \in \mathbb{N}_{\mathbf{d}}} (X_{sd})^d_\Delta$. Also $[\sigma]_{\mathcal{B}^0(d)}$ is a scalar matrix; see (4.3). Thus from Lemma 4.1(3), it follows that $\sigma X_s \sigma^{-1} = X_s$.

Construction of τ. Note that

$$X_{sd} \in \begin{cases} \mathfrak{sp}(t_d/2, \mathbb{C}) & \text{for } \theta \in \mathbb{O}_{\mathbf{d}} \\ \mathfrak{o}(t_d, \mathbb{C}) & \text{for } \theta \in \mathbb{E}_{\mathbf{d}} \, . \end{cases}$$

In view of Corollary 3.11 and Proposition 3.8, there exists $\tau_d \in \begin{cases} \mathrm{Sp}(t_d/2, \mathbb{C}) & \text{for } \theta \in \mathbb{O}_{\mathbf{d}} \\ \mathrm{O}(t_d, \mathbb{C}) & \text{for } \theta \in \mathbb{E}_{\mathbf{d}} \end{cases}$ such that $\tau_d X_{sd} \tau_d^{-1} = -X_{sd}$ for all $d \in \mathbb{N}_{\mathbf{d}}$. Finally set $\tau := \chi(\tau_d)$, where χ is an isomorphism in (4.2); see [BCM2, Section 3.4] for such isomorphism. Then $\tau X_s \tau^{-1} = -X_s$ and $\tau X_n \tau^{-1} = X_n$. This completes the proof. ☐

Acknowledgment. Gongopadhyay acknowledges SERB grant CRG/2022/003680. The authors thank Tejbir for his comments.

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