A GEOMETRIC REALIZATION FOR MAXIMAL ALMOST PRE-RIGID REPRESENTATIONS OVER TYPE $\mathbb{D}$ QUIVERS

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ABSTRACT. We focus on a class of special representations over a type $\mathbb{D}$ quiver Q_D with n vertices and directional symmetry, namely, maximal almost pre-rigid representations. By using the equivariant theory of group actions, we give a geometric model for the category of finite dimensional representations over Q_D via centrally-symmetric polygon $P(Q_D)$ with a puncture, and show that the dimension of extension group between indecomposable representations can be interpreted as the crossing number on $P(Q_D)$. Furthermore, we provide a geometric realization for maximal almost pre-rigid representations over Q_D. As an application, we illustrate their general form and prove that each maximal almost pre-rigid representation will determine two or four tilting objects over the path algebra $\Bbbk Q_{\overline{D}}$, where $Q_{\overline{D}}$ is a quiver obtained by adding $n-2$ new vertices and $n-2$ arrows to the quiver Q_D.

1. Introduction

The study of maximal rigid representations holds significant importance in algebraic representation theory. Particularly, these representations have extensive applications in quiver representation theory and tilting theory [2, 4, 7, 14, 17, 22, 23]. Maximal almost rigid representations were firstly introduced in [3] to offer Cambrian lattices a representation-theoretic interpretation. As a generalization of maximal rigid representations, it is a class of important objects from algebraic, geometric and combinatorial perspective. Let Q be a type $\mathbb{A}$ quiver with n vertices. The authors in [3] gave a geometric model for the category of finite dimensional representations over Q in terms of a polygon $P(Q)$ with $n+1$ vertices. Based on this, realized maximal almost rigid representations over Q as triangulations of $P(Q)$ and proved that their endomorphism algebra are tilted algebras of type $\mathbb{A}$. Furthermore, by defining a partial order on the set of maximal almost rigid representations, they showed that this partial order is a Cambrian lattice coming from Q. Thus it can be seen that the maximal almost rigid representations over type $\mathbb{A}$ quiver possess great value in algebra, geometry, and combinatorics. As such, they are worthy for generalization and exploration in other quivers.

In this paper, we investigate the representations over type $\mathbb{D}$ quiver. Notice that, although Barnard-Gunawan-Meehan-Schiffler in [3] have shown that maximal almost rigid representations over Q do correspond to the triangulations of $P(Q)$, this correlation cannot extend to other types of quivers beyond the type $\mathbb{A}$ quivers. Inspired by this discovery, we will give an appropriate polygon as a geometric model for the category of finite dimensional representations over type $\mathbb{D}$ quivers and seek out a class of representations which may play a similar combinatorial role as maximal rigid representations over type $\mathbb{A}$ quiver, that is, they can be realized as triangulations of the geometric model.

2020 *Mathematics Subject Classification.* Primary 05E10,16G20; Secondary 16D90,16G70.

Key words and phrases. almost pre-rigid representation, geometric realization, type $\mathbb{D}$ quiver, group action, extension group.

Throughout this paper, we consider the type $\mathbb{D}$ quiver Q_D with n vertices and directional symmetry. By [18], there exist a type $\mathbb{A}$ quiver Q_A with $2n-3$ vertices and an action of a group G with order 2 on Q_A such that the corresponding skew algebra $(\Bbbk Q_A)G$ is Morita equivalent to $\Bbbk Q_D$ (see Sect. 3 for more details). According to the theory of group action and equivariant, the induced category $(\Bbbk Q_A\text{-mod})^G$ is equivalent to the left module category of Q_D (c.f. [9]). Inspired from this equivalence of categories and [5, 6, 16, 21], we derive a geometric model $P(Q_D)$ for the left module category of Q_D via adding a puncture O in the center of the centrally-symmetric polygon $P(Q_A)$, where $P(Q_A)$ is the polygon model for the module category of Q_A constructed in [3, 11]. Specifically, we define a translation quiver (Γ_D, R_D) of $P(Q_D)$, where the vertices are all the tagged line segments of $P(Q_D)$; the arrows are induced by pivots of tagged line segments; R_D acts on tagged line segments by clockwise rotation. Let $\mathcal{C}_D$ be the mesh category with respect to (Γ_D, R_D). We get the following result.

Theorem A (Theorem 4.15)**.** *The category of indecomposable representations over Q_D, as an abelian category, is equivalent to the tagged line segment category $\mathcal{C}_D$.*

Employing this equivalence of categories, we can easily determine the dimension vectors of indecomposable representations over Q_D with the assistance of tagged line segments. Such an approach greatly simplifies the computation of the Auslander-Reiten quiver of $\Bbbk Q_D$. It should be pointed out that although the form of Theorem A is similar to Theorem 4.6 in [3], their equivalent functors have distinct definitions. Precisely, the equivalent functor in [3] is defined via the hook and cohook of arrows in Q_A, whereas the equivalent functor in this paper is introduced with reference to the exact functor between $\Bbbk Q_A$-mod and $\Bbbk Q_D$-mod in [18]. The distinction arises from the fact that while the path algebra $\Bbbk Q_A$ is a string algebra, $\Bbbk Q_D$ is not.

Using the geometric model $P(Q_D)$, we define the crossing number of two indecomposable representations over Q_D, and prove the following theorem.

Theorem B (Theorem 4.27)**.** *The dimension of extension group of degree 1 and the crossing number of two indecomposable representations over Q_D are equal.*

We then introduce the maximal almost pre-rigid representations over Q_D. Comparing with the definition of maximal almost rigid representations, the prefixion "pre-" indicates that we allow the middle term of extension between two indecomposable summands of almost pre-rigid representations to have the form $\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n$, for some $k \in \{0, 1, \dots, n-2\}$, while for almost rigid representations, the middle term must be indecomposable. By discussing the properties of maximal almost pre-rigid representations over Q_D, we give a geometric description of them in terms of triangulations of $P(Q_D)$, where the tagged line segments intersect at the puncture O and the boundary of $P(Q_D)$, deviating from conventional triangulations (see Definition 6.5).

Theorem C (Theorem 6.9)**.** *There is a one-to-one correspondence between the maximal almost pre-rigid representations over Q_D and the triangulations of $P(Q_D)$.*

Due to the fact that the geometric model for the category $\Bbbk Q_D$-mod is derived from the equivariant theory of group action, we use equivariant correspondence to prove Theorem C. Divide it into two steps: in the first step, we introduce a class of special representations over Q_A, which are referred to as F_g-stable maximal approximate rigid representations. Using the relation between categories $(\Bbbk Q_A\text{-mod})^G$ and $\Bbbk Q_D$-mod, we establish a bijection between these special representations over Q_A and maximal almost pre-rigid representations over Q_D; see Sect. 5. In the second step, we verify that there also exists a one-to-one correspondence between these special representations over Q_A and the triangulations of $P(Q_D)$ through the characteristics of the triangulations in $P(Q_D)$ and the structures of extension spaces in type $\mathbb{A}$ quivers; see Sect. 6. Therefore, Theorem C holds.

Besides, if $\mathcal{T}$ is a triangulation of $P(Q_D)$ with d tagged line segments passing through O, then $\mathcal{T}$ exactly has $2n-2+d'$ tagged line segments, where $d=2d'$ and $d'\in\{2,\ldots,n-1\}$(see Proposition 6.7). Combine this property with Theorem C, we give the general form of maximal almost pre-rigid representations over Q_D.

Theorem D (Corollary 6.10)**.** *Let $\overline{T}$ be a maximal almost pre-rigid representation over Q_D. Then $\overline{T}$ is of the form*

$$\overline{T}=\bigoplus_{i=1}^{2n-2-d'}\tau^{-m_i}\overline{P}_{r_i}\bigoplus_{j=1}^{d'}(\tau^{-k_j}\overline{P}_{n-1}\oplus\tau^{-k_j}\overline{P}_n),$$

where $\overline{P}_i$ is the projective module corresponding to $i\in Q_{D,\,0}$, $m_i,k_j\in\{0,\ldots,n-2\}$, $r_i\in\{1,\ldots,n-2\}$ for all $i\in\{1,\ldots,2n-2-d'\}$ and $j\in\{1,\ldots,d'\}$.

Based on this general form, we deduce the following application of maximal almost pre-rigid representations in tilting theory: let $Q_{\overline{D}}$ be the quiver constructed by adding $n-2$ new vertices and $n-2$ arrows to the quiver Q_D (for detailed construction process, see Section 7).

Theorem E (Theorem 7.3)**.** *Each maximal almost pre-rigid representation over Q_D will determine two or four tilting objects over the path algebra $\Bbbk Q_{\overline{D}}$.*

The paper is organized as follows. After a brief introduction in Sect. 1, we recall some needed concepts in Sect. 2, and then we investigate four functors: F_g, η, Ind and ψ in Sect. 3. Sect. 4 is devoted to the construction of the category $\mathcal{C}_D$ of tagged line segments on $P(Q_D)$ and the proof of Theorem A. Furthermore, we introduce the notion of crossing number of indecomposable representations over Q_D and prove Theorem B. In Sect. 5, we define the maximal almost pre-rigid representations over Q_D and F_g-stable maximal approximate rigid representations over Q_A, and show the one-to-one relation between them; see Theorem 5.15. In Sect. 6, we introduce the triangulations of $P(Q_D)$ and prove Theorem C. As a direct consequence of Theorem C, we give the general form of maximal almost pre-rigid representations over Q_D. In Sect. 7, we construct an additive functor G_D and prove Theorem E.

2. Preliminaries

2.1. **Representations over quiver.** In this subsection, we recall some notations and a property about representations over a quiver from [1,3].

Let $\Bbbk$ be an algebraically closed field whose characteristic is not 2. Given a finite connected quiver $Q=(Q_0,Q_1)$, where Q_0 is the set of vertices and Q_1 is the set of arrows. Denote by $\operatorname{rep}_{\Bbbk}(Q)$ be the category of finite dimensional representations over Q and by ind Q a full subcategory of $\operatorname{rep}_{\Bbbk}(Q)$ whose objects are one representative of the isoclass of each indecomposable representation. As is known that, the category $\Bbbk Q$-mod of finitely generated left modules over the path algebra $\Bbbk Q$ is equivalent to the category $\operatorname{rep}_{\Bbbk}(Q)$. Therefore, we always identify these two categories. The Auslander-Reiten quiver $\Gamma_{\mathrm{rep}_{\Bbbk}Q}$ of $\Bbbk Q$ has the isoclasses of indecomposable representations as vertices and irreducible morphisms as arrows.

2.2. **Translation quivers and mesh categories.** We recall the definitions of translation quivers and mesh categories from [1, 3, 15, 19].

A *translation quiver* (Γ, τ) is a locally finite quiver $\Gamma = (\Gamma_0, \Gamma_1)$ without loops together with an injection $\tau\colon \Gamma_0^{'} \to \Gamma_0$ from a subset $\Gamma_0^{'}$ of Γ_0 to Γ_0 satisfying for all vertices $x \in \Gamma_0^{'}$ and $y \in \Gamma_0$, the number of arrows from $y \to x$ is equal to the number of arrows from $\tau x \to y$. The injection τ is called the *translation*.

Given a translation quiver (Γ, τ), a *polarization of* Γ is an injection $\sigma : \Gamma_1^{'} \to \Gamma_1$, where $\Gamma_1^{'}$ is the set of all arrows $\alpha\colon y \to x$ with $x \in \Gamma_0^{'}$, such that $\sigma(\alpha)\colon \tau x \to y$ for each arrow $\alpha\colon y \to x \in \Gamma_1$. Clearly, if Γ has no multiple arrows, there is a unique polarization of Γ.

Assume that Γ has no multiple arrows. The *path category* of translation quiver (Γ, τ) is the category whose objects are the vertices Γ_0 of Γ, and given $x, y \in \Gamma_0$, the $\Bbbk$-vector space of morphisms from x to y is given by the $\Bbbk$-vector space with basis consisting of all paths from x to y. The composition of morphisms is induced from the usual composition of paths. The *mesh ideal* in the path category of Γ is the ideal generated by the *mesh relations*

$$m_x = \sum_{\alpha : y \to x} \sigma(\alpha)\alpha.$$

for all $x \in {\Gamma_0}'$. The *mesh category* $\mathcal{M}(\Gamma, \tau)$ of (Γ, τ) is the quotient of the path category of (Γ, τ) by the mesh ideal.

Remark 2.1. Let Q be a quiver of Dynkin type and $\Gamma_{\mathrm{rep}_{\Bbbk}Q}$ be the Auslander-Reiten quiver of $\Bbbk Q$. Denote by $\Gamma_0^{'}$ the set of all non-projective indecomposable representations over Q, then the quiver $\Gamma_{\mathrm{rep}_{\Bbbk}Q}$ together with Auslander-Reiten translation $\tau : \Gamma_0^{'} \to \Gamma_0$ is a translation quiver. In this case, the mesh category $\mathcal{M}(\Gamma_{\mathrm{rep}_{\Bbbk}Q}, \tau)$ is equivalent to the category $\mathrm{ind}\, Q$, and the additive closure of $\mathcal{M}(\Gamma_{\mathrm{rep}_{\Bbbk}Q}, \tau)$ is equivalent to $\mathrm{rep}_{\Bbbk}Q$.

2.3. **Sectional path and maximal slanted rectangle.** We review the definitions of sectional path and maximal slanted rectangle as introduced in [20, Chapter 3].

Let Q be a type $\mathbb{A}$ quiver, that is, the underlying graph $\overline{Q}$ over Q is a Dynkin graph $\mathbb{A}_n$ with $n \geq 1$. A path $M_0 \to M_1 \to \cdots \to M_s$ in the Auslander-Reiten quiver of path algebra $\Bbbk Q$ is called a *sectional path* if $\tau M_{i+1} \neq M_{i-1}$ for all $i = 1, \ldots, s-1$. Denote by $\Sigma_{\to}(M)$ the set of all indecomposable representations that can be reached from M by a sectional path, see Figure 1. And $\Sigma_{\leftarrow}(M)$ be the set of all indecomposable representations from which one can reach M by a sectional path.

Denote by $\mathscr{R}_{\to}(M)$ the set of all indecomposable representations whose position in the Auslander-Reiten quiver is in the slanted rectangle region whose left boundary is $\Sigma_{\to}(M)$, see Figure 1. Then $\mathscr{R}_{\to}(M)$ is called the *maximal slanted rectangle* in the Auslander-Reiten quiver whose leftmost point is M. Dual, $\mathscr{R}_{\leftarrow}(M)$ is called the *maximal slanted rectangle* in the Auslander-Reiten quiver whose rightmost point is M.

FIGURE 1. $\Sigma_{\to}(M)$(left) and $\mathscr{R}_{\to}(M)$(right) of M

In [20], the following properties of maximal slanted rectangles are provided. For the convenience of referencing, we summarize these properties into a proposition.

Proposition 2.2. *Let M, N be two indecomposable representations over Q. Then*

(1) *For the projective representation P_i corresponding to $i \in Q_0$, $\mathscr{R}_{\to}(P_i) = \mathscr{R}_{\leftarrow}(I_i)$, where I_i is the injective representation corresponding to $i \in Q_0$;*
(2) $\dim_{\Bbbk}\mathrm{Hom}(M, N)$ *is either* 0 *or* 1, *and it is equal to* 1 *if and only if N lies in $\mathscr{R}_{\to}(M)$;*
(3) $\dim_{\Bbbk}\mathrm{Ext}^1(M, N)$ *is either* 0 *or* 1, *and it is equal to* 1 *if and only if τM lies in $\mathscr{R}_{\to}(N)$. If* $\mathrm{Ext}^1(M, N) \neq 0$, *then the nonzero element of* $\mathrm{Ext}^1(M, N)$ *can be represented by a non-split short exact sequence of the form $0 \to N \to E \to M \to 0$. In this case, $\Sigma_{\to}(N)$ and $\Sigma_{\leftarrow}(M)$ have either 1 or 2 points in common, and these points corresponds to the indecomposable summands of E.*

2.4. **The category of equivariant objects and skew group algebra.** In this subsection, we summarize some fundamental knowledge about the category of equivariant objects and skew group algebra from [9, 10, 12, 13, 18] .

Let $\mathscr{A}$ be an additive category and G be a finite group whose unit is denoted by e. A *G-action* on $\mathscr{A}$ consists of the data $\{F_g, \varepsilon_{g,h} | g, h \in G\}$, where each $F_g : \mathscr{A} \to \mathscr{A}$ is an auto-equivalence and each $\varepsilon_{g,h} : F_g F_h \to F_{gh}$ is a natural isomorphism such that

$$\varepsilon_{gh,l} \circ \varepsilon_{g,h} F_l = \varepsilon_{g,hl} \circ F_g \varepsilon_{h,l}$$

holds for all $g, h, l \in G$. The G-action $\{F_g, \varepsilon_{g,h} | g, h \in G\}$ is *strict* if each $F_g : \mathscr{A} \to \mathscr{A}$ is automorphism and each isomorphism $\varepsilon_{g,h}$ is the identity transformation.

Let $\{F_g, \varepsilon_{g,h} |\ g, h \in G\}$ be a G-action on $\mathscr{A}$. A *G-equivariant object* in $\mathscr{A}$ is a pair (X, α), where X is an object in $\mathscr{A}$ and $\alpha := \{\alpha_g\}_{g \in G}$ assigns to each $g \in G$ an isomorphism $\alpha_g : X \to F_g(X)$ that satisfies $\alpha_{gg'} = (\varepsilon_{g,g'})_X \circ F_g(\alpha_{g'}) \circ \alpha_g$. A morphism $f \colon (X, \alpha) \to (Y, \beta)$ is a morphism $f \colon X \to Y$ in $\mathscr{A}$ such that $\beta_g \circ f = F_g(f) \circ \alpha_g$ for all $g \in G$. This gives rise to the category $\mathscr{A}^G$ of G-equivariant objects, and the *forgetful functor* $U \colon \mathscr{A}^G \to \mathscr{A}$ defined by $U(X, \alpha) = X$ and $U(f) = f$. The forgetful functor U admits a left adjoint $\mathrm{Ind} \colon \mathscr{A} \to \mathscr{A}^G$, which is known as the *induction functor*. For an object X, set $\mathrm{Ind}(X) = (\oplus_{h \in G} F_h(X), \varepsilon(X))$, where for each $g \in G$, the isomorphism $\varepsilon(X)_g : \oplus_{h \in G} F_h(X) \to F_g(\oplus_{h \in G} F_h(X))$ is induced by the isomorphism $(\varepsilon_{g,g^{-1}h})_X^{-1} : F_h(X) \to F_g(F_{g^{-1}h}(X))$ and for a morphism $\theta : X \to Y$ to $\mathrm{Ind}(\theta) = \oplus_{h \in G} F_h(\theta) : \mathrm{Ind}(X) \to \mathrm{Ind}(Y)$.

When focus on the category A-mod of left A-modules, where A is a finite dimensional $\Bbbk$-algebra, let $\mathrm{Aut}_k(A)$ be the group of $\Bbbk$-algebra automorphisms on A. We say that G acts on A by $\Bbbk$-algebra automorphisms, if there is a group homomorphism $G \to \mathrm{Aut}_{\Bbbk}(A)$. In this case, we identify elements in G with their images under this homomorphism. The corresponding *skew group algebra* AG is defined as follows: $AG = \oplus_{g \in G} A u_g$ is a free left A-module with basis $\{u_g | g \in G\}$ and the multiplication is given by $(au_g)(bu_h) = ag(b)u_{gh}$.

For a $\Bbbk$-algebra automorphism g on A and an A-module M, the twisted module ${}^gM$ is defined such that ${}^gM = M$ as a vector space and that the new A-action " $\circ$ " is given by $a \circ m = g(a)m$. This gives rise to a $\Bbbk$-linear automorphism

$$ {}^g(-) : A\text{-mod} \to A\text{-mod}, $$

which acts on morphism by the identity. It is easy to see that for two $\Bbbk$-algebra automorphisms g and h on A, ${}^h({}^gM) = {}^{gh}M$ for any A-module M. Then there is a strict $\Bbbk$-linear G-action on A-mod by setting $F_g = {}^{g^{-1}}(-)$. Moreover, there is an equivalence of categories

$$ \eta : (A\text{-mod})^G \to AG\text{-mod}, $$

by sending a G-equivariant object (X, α) to the AG-module X, where the AG-module structure is given by $(au_g)x = a\alpha_{g^{-1}}(x)$. Using this equivalence, the induction functor $\mathrm{Ind} \colon \mathscr{A} \to \mathscr{A}^G$ is identified with $AG \otimes_A - : A\text{-mod} \to AG\text{-mod}$.

3. Some important functors

Let $Q_{A_{2n-3}}$ denote the quiver with the underlying graph depicted in Figure 2 and directional symmetry, that is, arrows α_i and α_{2n-3-i} have the same directions, for all $i \in Q_{A_{2n-3},\,0}$. Let G be a cyclic group of order 2 with generator g, where g is the automorphism of $Q_{A_{2n-3}}$

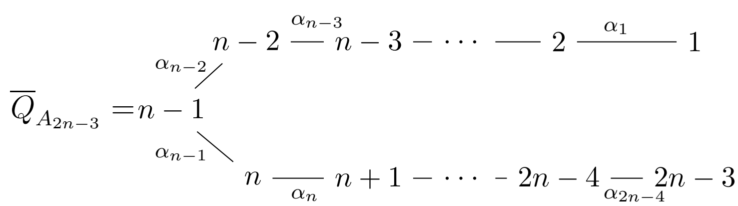


FIGURE 2. The underlying graph of $Q_{A_{2n-3}}$

given by $g(i) = 2n-2-i$ and $g(\alpha_i) = \alpha_{2n-3-i}$ for each $i \in Q_{A_{2n-3},\,0}$. By [18], the skew group algebra $(\Bbbk Q_{A_{2n-3}})G$ is Morita equivalent to the path algebra $\Bbbk Q_{D_n}$, where the underlying graph of the quiver Q_{D_n} is shown in Figure 3 and the direction of the arrow β_i is the same as α_i for $i = 1, \dots, n-1$. Therefore, Q_{D_n} is a type $\mathbb{D}$ quiver with directional symmetry.

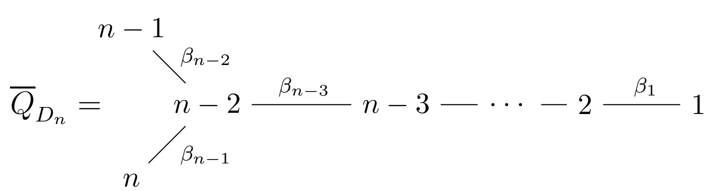


FIGURE 3. The underlying graph of Q_{D_n}

For simplification of the notations, we abbreviate $Q_{A_{2n-3}}$ as Q_A and Q_{D_n} as Q_D, denote the category $\Bbbk Q_A$-mod by $\mathcal{A}$ and the category $\Bbbk Q_D$-mod by $\mathcal{D}$ in the rest of the paper.

According to the theory of group actions, the action of G on Q_A induce a strict action $\{F_g, \varepsilon_{g,h} | g, h \in G\}$ of G on $\mathcal{A}$, where the automorphism $F_g : \mathcal{A} \to \mathcal{A}$ is dertermined thus: for any projective module P_i corresponding to $i \in Q_{A,\,0}$ and $k \in \{0, 1, \dots, n-2\}$

$$ F_g(\tau^{-k} P_i) = \tau^{-k} P_{2n-2-i}. \tag{3.1} $$

Next, we will discuss the functors discussed in Section 2.4 that associated with the strict action $\{F_g, \varepsilon_{g,h} | g, h \in G\}$ on the category $\mathcal{A}$. To characterize these functors better, we initially provide the following two observations.

Remark 3.1. (1) Denote by e_j the primitive orthogonal idempotent of $\Bbbk Q_A$ corresponding to $j \in Q_{A,0}$. By the definition of F_g, we get that equation (3.2) holds for all $j \in \{1, \dots, 2n-3\}$

$$\dim_{\Bbbk} \tau^{-k} P_i e_j = \dim_{\Bbbk} F_g(\tau^{-k} P_i) e_{2n-2-j}. \tag{3.2}$$

(2) Denote by $\overline{e}_j$ the primitive orthogonal idempotent of $\Bbbk Q_D$ corresponding to $j \in Q_{D,0}$ and by $\overline{P}_i$ the projective module corresponding to $i \in Q_{D,\ 0}$. Then for $i \in Q_{D,\ 0} \setminus \{n-1, n\}$, we have

$$\dim_{\Bbbk} \tau^{-k} \overline{P}_i \overline{e}_j = \begin{cases} \dim_{\Bbbk} \tau^{-k} P_i e_j + \dim_{\Bbbk} \tau^{-k} P_i e_{2n-2-j}, & \text{for } j = 1, \dots, n-2, \\ \dim_{\Bbbk} \tau^{-k} P_i e_j, & \text{for } j = n-1, \\ \dim_{\Bbbk} \tau^{-k} P_i e_{j-1}, & \text{for } j = n, \end{cases}$$

for $i = n-1$, we have

$$\dim_{\Bbbk} \tau^{-k} \overline{P}_{n-1} \overline{e}_j = \begin{cases} \dim_{\Bbbk} \tau^{-k} P_{n-1} e_j, & \text{for } j = 1, \dots, n-2, \\ 1 - r, & \text{for } j = n-1, \\ r, & \text{for } j = n, \end{cases}$$

and for $i = n$, we have

$$\dim_{\Bbbk} \tau^{-k} \overline{P}_n \overline{e}_j = \begin{cases} \dim_{\Bbbk} \tau^{-k} P_{n-1} e_j, & \text{for } j = 1, \dots, n-2, \\ r, & \text{for } j = n-1, \\ 1 - r, & \text{for } j = n, \end{cases}$$

where $k \in \{0, 1, \dots, n-2\}$ and $r = k \bmod 2$.

Based on the above remark, by performing calculations, we can obtain the definition of the following functors on objects.

(1) The equivalent functor $\eta : \mathcal{A}^G \to \mathcal{D}$ satisfies

$$\eta\,(P_{n-1}, \alpha) = \overline{P}_{n-1},\ \eta\,(P_{n-1}, \beta) = \overline{P}_n,\ \eta\,(P_i \oplus P_{2n-2-i}, \delta) = \overline{P}_i,\ \forall i \in \{1, \dots, n-2\}.$$

(2) The induction functor $\mathrm{Ind}\colon \mathcal{A} \to \mathcal{A}^G$ is determined by

$$\mathrm{Ind}(\tau^{-k} P_i) = \begin{cases} (\tau^{-k} P_{n-1}, \alpha) \oplus (\tau^{-k} P_{n-1}, \beta), & \text{if } i = n-1, \\ (\tau^{-k} P_i \oplus \tau^{-k} P_{2n-2-i}, \delta), & \text{otherwise}, \end{cases}$$

where

$$\begin{cases} \alpha_e = 1, \\ \alpha_g = 1, \end{cases} \quad \begin{cases} \beta_e = 1, \\ \beta_g = -1, \end{cases} \quad \text{and} \quad \begin{cases} \delta_e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \\ \delta_g = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right). \end{cases}$$

(3) The functor $\psi = (\Bbbk Q_A) G \otimes_{\Bbbk Q_A} - : \mathcal{A} \to \mathcal{D}$ is exact by [18] and we have

$$\psi(\tau^{-k} P_{n-1}) = \tau^{-k} \overline{P}_{n-1} \oplus \tau^{-k} \overline{P}_n,\ \psi(\tau^{-k} P_i) = \tau^{-k} \overline{P}_i = \psi(\tau^{-k} P_{2n-2-i}),\ \forall i \in \{1, \dots, n-2\}.$$

Towards the end of this section, we investigate the relationship between extension groups of degree 1 under the action of F_g, yielding the following properties.

Proposition 3.2. *Let M and N be two indecomposable $\Bbbk Q_A$-modules. Then we have*

(1) $\mathrm{Ext}^1_{\mathcal{A}}(M, F_g(M)) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(F_g(M), M)$;
(2) *The functor* F_g *induce a bijection between* $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ *and* $\mathrm{Ext}^1_{\mathcal{A}}(F_g(M), F_g(N))$. *In particular,* $\mathrm{Ext}^1_{\mathcal{A}}(M, N) = 0$ *if and only if* $\mathrm{Ext}^1_{\mathcal{A}}(F_g(M), F_g(N)) = 0$.

Proof. (1) For each indecomposable module $M \in \mathcal{A}$, $\tau M \notin \mathcal{R}_{\rightarrow}(F_g(M))$ and $\tau F_g(M) \notin \mathcal{R}_{\rightarrow}(M)$. Thus this statement is a direct consequence of Proposition 2.2(3).

(2) Noting that for any two indecomposable $\Bbbk Q_A$-modules M, N, $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ is either 0 or 1. We have $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(F_g(M), F_g(N))$ is either 0 or 1 since $F_g(M)$ and $F_g(N)$ are also indecomposable $\Bbbk Q_A$-modules. This proof will be completed by showing that $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M, N) = 1$ if and only if $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(F_g(M), F_g(N)) = 1$.

On the one hand, if $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M, N) = 1$, set ξ be a generator of $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$. Since F_g is exact, $F_g(\xi)$ is a non-split short exact sequence in $\mathrm{Ext}^1_{\mathcal{A}}(F_g(M), F_g(N))$. On the other hand, if $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(F_g(M), F_g(N)) = 1$, then the generator ξ' of $\mathrm{Ext}^1_{\mathcal{A}}(F_g(M), F_g(N))$ has the form $0 \to F_g(N) \to F \to F_g(M) \to 0$. Since F_g satisfies $F_g \circ F_g = \mathrm{Id}_{\mathscr{A}}$, we have that $F_g(\xi') : 0 \to N \to F_g(F) \to M \to 0$ is a non-split short exact sequence in $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$. $\square$

4. Geometric models for $\mathrm{rep}_{\Bbbk}Q_A$ and $\mathrm{rep}_{\Bbbk}Q_D$

In this section, we recall a geometric construction for the category $\mathrm{rep}_{\Bbbk}Q_A$ in terms of a convex polygon from [3, 11]. And then we combine this polygon with the relationship between Q_A and Q_D provided in Section 3, to give a geometric model for the category $\mathrm{rep}_{\Bbbk}Q_D$. Furthermore, we use this geometric model to interpret the dimension of extension group between two indecomposable representations over Q_D.

4.1. A geometric model for $\mathrm{rep}_{\Bbbk}Q_A$.

In [11], Chang-Qiu-Zhang construct a $(2n-2)$-gon $P(Q_A)$ via Q_A as follows: First, define the vertices, for any $s \in \{0, \ldots, 2n-3\}$, set

$$X_s = (x_s, y_s) \in \{(x, y) | x^2 + y^2 = 1\}$$

with

$$y_s = -1 + \frac{2s}{2n-3} \quad \text{and} \quad x_s = \mathrm{sign}(X_s)\sqrt{1 - y_s^2},$$

where $x_0 = 0 = x_{2n-3}$ and for $s \in \{1, \ldots, 2n-4\}$,

$$\mathrm{sign}(X_s) = \begin{cases} +, & \text{if } s \xrightarrow{\alpha_s} s+1 \in Q_{A,\,1}; \\ -, & \text{if } s \xleftarrow{\alpha_s} s+1 \in Q_{A,\,1}. \end{cases}$$

Next, starting from vertex X_0, connect the vertices in the set $\{X_s | s = 0, 1, \ldots, 2n-3\}$ sequentially in a clockwise direction to form a closed figure, denoted as $P(Q_A)$. It is evident that $P(Q_A)$ is a centrally symmetric convex $(2n-2)$-gon. Precisely, the vertices X_s and X_{2n-3-s} are always centrosymmetric in $P(Q_A)$ for all $s \in \{0, \ldots, 2n-3\}$. See Figure 4 for an example.

Now we recall from [3] the geometric realization for the category ind Q_A in terms of $(2n-2)$-gon $P(Q_A)$. First, let's establish some notation conventions: for any vertex X_s of $P(Q_A)$, denote by $X_{R_A(s)}$ and by $X_{R_A^{-1}(s)}$ the clockwise and counterclockwise neighbor of X_s on the boundary of $P(Q_A)$, respectively. Define the set ω as follows

$$\omega = \{\gamma(s, t) \mid -1 \le y_s < y_t \le 1\},$$

where $\gamma(s, t)$ denotes the line segment of $P(Q_A)$ connecting the vertices X_s and X_t.

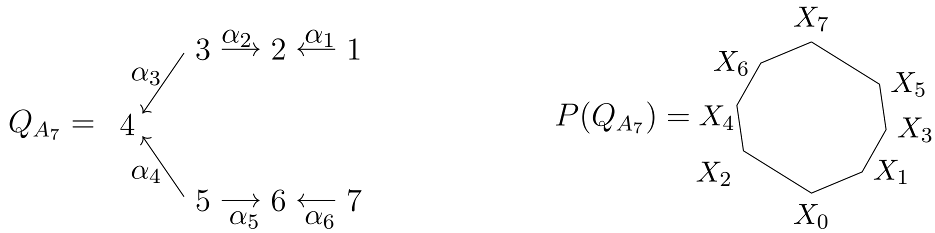


FIGURE 4. The quiver Q_{A_7} and the polygon $P(Q_{A_7})$

For each $\gamma(s,t) \in \omega$, the line segment $\gamma(s, R_A^{-1}(t))$ is called a *pivot* of $\gamma(s,t)$ if it lies in the set ω. Similarly, the line segment $\gamma(R_A^{-1}(s), t)$ is a *pivot* of $\gamma(s,t)$ if it lies in ω.

Remark 4.1. Keep the notations as above. For each $\gamma(s,t) \in \omega$, $\gamma(u,v)$ is a pivot of $\gamma(s,t)$ if and only if $\gamma(2n-3-v, 2n-3-u)$ is a pivot of $\gamma(2n-3-t, 2n-3-s)$.

Define a translation quiver (Γ_A, R_A) of $P(Q_A)$ with respect to pivots, where the vertices of Γ_A are all the line segments in ω; there is an arrow $\gamma(s,t) \to \gamma(u,v)$ in Γ_A if and only if $\gamma(u,v)$ is a pivot of $\gamma(s,t)$, and R_A is a translation on the set ω with

$$R_A(\gamma(s,t)) = \begin{cases} \gamma(R_A(s), R_A(t)) & \text{if } \gamma(R_A(s), R_A(t)) \in \omega; \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}_A$ be the mesh category of the translation quiver (Γ_A, R_A). Then there is a functor F_A from $\mathcal{C}_A$ to the abelian category ind Q_A defined as follows: On objects,

$$F_A(\gamma(s,t)) = M(s+1,t),$$

where $M(s+1,t)$ is the indecomposable representation supported on the vertices between $s+1$ and t. On morphisms, define $F_A\left(\gamma(s,t) \to \gamma(R_A^{-1}(s),t)\right)$ to be the irreducible morphism $M(s+1,t) \to M(R_A^{-1}(s)+1,t)$. Similarly, let $F_A\left(\gamma(s,t) \to \gamma(s, R_A^{-1}(t))\right)$ be the irreducible morphism $M(s+1,t) \to M(s+1, R_A^{-1}(t))$, see [3, Definition 4.5] for a detailed definition.

Theorem 4.2 ([3]). *The functor $F_A : \mathcal{C}_A \to \operatorname{ind} Q_A$ is an equivalence of categories.*

4.2. **A geometric model for** $\operatorname{rep}_{\Bbbk} Q_D$**.** In this subsection, we will derive a geometric model for the category $\operatorname{rep}_{\Bbbk} Q_D$ via $P(Q_A)$ and equivariant theory of group actions.

4.2.1. *The punctured polygon $P(Q_D)$.* Recall that there is a unique Q_A associated with the given Q_D and $P(Q_A)$ is the geometric model for the category $\operatorname{rep}_{\Bbbk} Q_A$. We relabel the vertices $\{X_s | s = 0, 1, \ldots, 2n-3\}$ of $P(Q_A)$ according to the following rule $(*)$:

- if $y_s < 0$, then relabel the vertex X_s by Y_{s-n+1};
- if $y_s > 0$, then relabel the vertex X_s by Y_{s-n+2}.

Denote the punctured polygon with vertices $\{Y_{\pm s} | s = 1, \ldots, n-1\}$ and a puncture $O = (0,0)$ by $P(Q_D)$. Then the vertices Y_s and Y_{-s} are symmetric about O for all $s \in \{1, \ldots, n-1\}$.

To illustrate the construction of $P(Q_D)$, we present an example here.

Example 4.3. Let Q_{D_5} be the quiver on the left side of Figure 5. Then the associate quiver Q_{A_7} and polygon $P(Q_{A_7})$ are shown in Figure 4. After relabeling the vertices of $P(Q_{A_7})$ and adding a puncture at its center, we obtain the punctured polygon $P(Q_{D_5})$, illustrated on the right side of Figure 5.

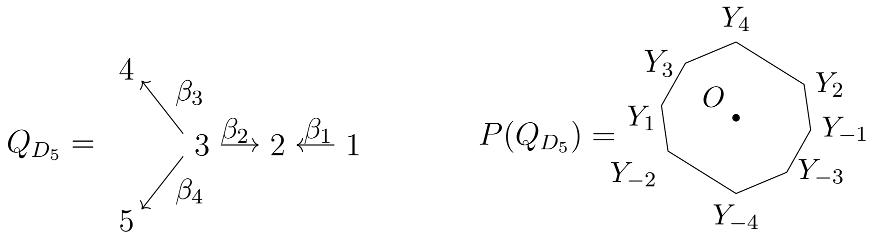


FIGURE 5. The quiver Q_{D_5} and the polygon $P(Q_{D_5})$

4.2.2. *The category of line segments $\mathcal{C}_D$.* Now we will define a category $\mathcal{C}_D$ with respect to the line segments of $P(Q_D)$ and prove the equivalence between $\mathcal{C}_D$ and ind Q_D. We first construct a set Ω as follows:

$$\Omega = \{\gamma_s^t \mid -(n-1) \leq s < t \leq n-1, s \neq -t\}/_\sim \cup \{\gamma_{-t}^{t,-1}, \gamma_{-t}^{t,1} \mid 1 \leq t \leq n-1\},$$

where γ_s^t is the unique line segment between Y_s and Y_t of $P(Q_D)$ with $\gamma_s^t \sim \gamma_{-t}^{-s}$, and $\gamma_{-t}^{t,\epsilon}$ ($\epsilon = 1, -1$) denotes these two overlapping line segments between Y_{-t} and Y_t. For distinguishing the two segments, we always draw the line segment $\gamma_{-t}^{t,-1}$ in picture with a tag "|" on it and the line segment $\gamma_{-t}^{t,1}$ in picture with no tag.

For example, consider the punctured polygon $P(Q_{D_5})$, there exists $\gamma_1^4 \sim \gamma_{-4}^{-1}$ in Ω. And in Figure 6, $\gamma_{-3}^{3,-1}$ and $\gamma_{-3}^{3,1}$ of $P(Q_{D_5})$ are shown in red. Specifically, $\gamma_{-3}^{3,-1}$ has a tag "|", while $\gamma_{-3}^{3,1}$ is not tagged.

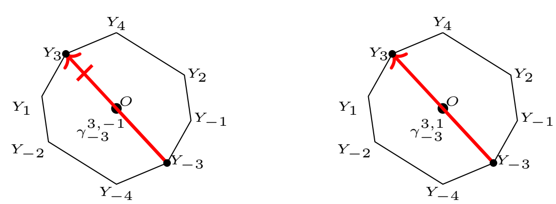


FIGURE 6. The line segment $\gamma_{-3}^{3,-1}$(left) and $\gamma_{-3}^{3,1}$(right) of $P(Q_{D_5})$

Remark 4.4. According to the definition of Ω, it is easy to calculate that there are $n(n-1)$ elements in Ω. For convenience to state the other definitions, we denote the elements in Ω by $\gamma_s^{t,l}$ and call them *tagged line segments*. Precisely, $\gamma_s^{t,l} = \gamma_s^t$ for the cases $s \neq -t$ and $\gamma_s^{t,l} = \gamma_s^{t,\epsilon}$ for the cases $s = -t$.

Let $[s,t]$ denote the number of the vertices on the minimal path from Y_s to Y_t along the boundary of $P(Q_D)$ in countercolockwise direction (including Y_s and Y_t). For example, see the right side of Figure 5, the minimal path from Y_{-2} to Y_4 along the boundary of $P(Q_{D_5})$ in countercolockwise direction is $Y_{-2} \to Y_{-4} \to Y_{-3} \to Y_{-1} \to Y_2 \to Y_4$, which consists of 6 vertices, so $[-2,4] = 6$ on $P(Q_{D_5})$.

Notice that the equation $[s,t] + [-t,-s] = 2n$ holds for all $-(n-1) \leq s < t \leq n-1$. Consequently, either $[s,t]$ or $[-t,-s]$ must be greater than or equal to n, while the other is less than or equal to n. Additionally, the tagged line segments γ_s^t and γ_{-t}^{-s} satisfy $\gamma_s^t = \gamma_{-t}^{-s}$ in Ω. Hence, it suffices to consider only one representative from each pair, when we discuss the line segments in Ω. In the following discussions, we always select the representative element, where $[-,-]$ is less than or equal to n.

Definition 4.5. Let $\gamma_s^{t,l}$ be a tagged line segment in Ω, and Y_u (resp. Y_v) be the counter-clockwise neighbor of Y_s (resp. Y_t). Define the *pivot* of $\gamma_s^{t,l}$ as follows:

Case I: $2 \leq [s,t] \leq n-2$. In this case, $\gamma_s^{t,l} = \gamma_s^t$.

- If γ_s^v lies in Ω, then γ_s^v is called a pivot of γ_s^t;
- If γ_u^t lies in Ω, then γ_u^t is called a pivot of γ_s^t.

Case II: $[s,t] = n-1$. In this case, $\gamma_s^{t,l} = \gamma_s^t$.

- If $s < 0$, then both $\gamma_s^{-s,1}$ and $\gamma_s^{-s,-1}$ are elements in Ω, therefore they are both considered as pivots of γ_s^t;
- If γ_u^t lies in Ω, then γ_u^t is called a pivot of γ_s^t.

Case III: $[s,t] = n$. In this case, $\gamma_s^{t,l} = \gamma_{-t}^{t,\epsilon}$ for $\epsilon = -1$ or 1.

- If γ_u^t lies in Ω, then γ_u^t is called a pivot of $\gamma_{-t}^{t,\epsilon}$.

Example 4.6. In Figure 5, γ_{-2}^{-1} has the three pivots: $\gamma_{-2}^{2,-1}$, $\gamma_{-2}^{2,1}$ and γ_{-4}^{-1}; γ_{-3}^{2} has the two pivots: γ_{-3}^{4} and γ_{-1}^{2}; $\gamma_{-4}^{4,-1}$ has only one pivot: γ_{-3}^{4}; γ_2^3 has no pivots.

Now we define a translation quiver (Γ_D, R_D) with respect to pivots, where Γ_D is a quiver whose vertices are all elements in Ω, and for two tagged line segments $\gamma_s^{t,l}, \gamma_{s'}^{t',l'} \in \Omega$, there is an arrow $\gamma_s^{t,l} \to \gamma_{s'}^{t',l'}$ if and only if $\gamma_{s'}^{t',l'}$ is a pivot of $\gamma_s^{t,l}$; the translation R_D over Γ_D is defined as follows: for a tagged line segment $\gamma_u^{v,l}$ in Ω, assume Y_s (resp. Y_t) is the clockwise neighbor of Y_u (resp. Y_v), then

$$R_D(\gamma_u^v) = \begin{cases} \gamma_s^t, & \text{if } \gamma_s^t \in \Omega; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and} \quad R_D(\gamma_{-v}^{v,l}) = \begin{cases} \gamma_{-t}^{t,-l}, & \text{if } \gamma_{-t}^{t,-l} \in \Omega; \\ 0, & \text{otherwise.} \end{cases}$$

Intuitively, R_D acts on tagged line segments by clockwise rotation.

Definition 4.7. Let $\mathcal{C}_D$ be the mesh category of the translation quiver (Γ_D, R_D). We call $\mathcal{C}_D$ the *category of tagged line segments* of $P(Q_D)$.

Next, we construct a functor from $\mathcal{C}_D$ to ind Q_D, outlined in three specific steps. The first step is to define two functors from the category $\mathcal{A}'$ to ind Q_D, where $\mathcal{A}'$ be the full subcategory of $\mathcal{A}$ with objects $\{\tau^{-k}P_i \in \text{ind } Q_A | i = 1, \ldots, n-1\}$.

Definition 4.8. Let $\psi_1 : \mathcal{A}' \to \text{ind } Q_D$ be the functor define on objects by

$$\psi_1\left(\tau^{-k}P_i\right) = \begin{cases} \tau^{-k}\overline{P}_i, & \text{if } i = 1, \ldots, n-2; \\ \tau^{-k}\overline{P}_{n-1}, & \text{if } i = n-1 \text{ and } k \bmod 2 = 0; \\ \tau^{-k}\overline{P}_n, & \text{if } i = n-1 \text{ and } k \bmod 2 = 1, \end{cases}$$

and on irreducible morphism $h : M \to N$ by irreducible morphism $\psi_1(h) : \psi_1(M) \to \psi_1(N)$.

Definition 4.9. Let $\psi_2 : \mathcal{A}' \to \text{ind } Q_D$ be the functor define on objects by

$$\psi_2\left(\tau^{-k}P_i\right) = \begin{cases} \tau^{-k}\overline{P}_i, & \text{if } i = 1, \ldots, n-2; \\ \tau^{-k}\overline{P}_{n-1}, & \text{if } i = n-1 \text{ and } k \bmod 2 = 1; \\ \tau^{-k}\overline{P}_n, & \text{if } i = n-1 \text{ and } k \bmod 2 = 0, \end{cases}$$

and on irreducible morphism $h : M \to N$ by irreducible morphism $\psi_2(h) : \psi_2(M) \to \psi_2(N)$.

The second step is to construct two functors from the full subcategories of $\mathcal{C}_D$ to the category $\mathcal{C}_A$. We start by defining two subsets of Ω. Let

$$\Omega_1 = \Omega \setminus \{\gamma_{-t}^{t,-1} \mid t = 1, \dots, n-1\} \text{ and } \Omega_2 = \Omega \setminus \{\gamma_{-t}^{t,1} \mid t = 1, \dots, n-1\}.$$

Set $\mathcal{C}_D^1$ and $\mathcal{C}_D^2$ the full subcategory of $\mathcal{C}_D$ consisting of objects from Ω_1 and Ω_2, respectively.

Observe that if the vertex $X_{h'}(h' \in \{0, \dots, 2n-3\})$ of $P(Q_A)$ is relabeled as Y_h of $P(Q_D)$, then an arrow $\gamma_u^v \to \gamma_{-t}^{t,l}$(resp. $\gamma_u^v \to \gamma_s^t$) exists in Γ_D if and only if there exists an arrow $\gamma(u', v') \to \gamma(2n-3-t', t')$ (resp. $\gamma(u', v') \to \gamma(s', t')$) in Γ_A, where $l = -1$ or 1. Based on this observation, we can define functors from $\mathcal{C}_D^1$ and $\mathcal{C}_D^2$ to $\mathcal{C}_A$ as follows:

Definition 4.10. Let $f_1 : \mathcal{C}_D^1 \to \mathcal{C}_A$ be the functor defined on objects by

$$f_1(\gamma_s^{t,l}) = \begin{cases} \gamma(s', t'), & \text{if } s \neq -t; \\ \gamma(2n-3-t', t'), & \text{if } s = -t \text{ and } l = 1, \end{cases}$$

and on pivot $\gamma_s^{t,l} \to \gamma_u^{v,l'}$ by pivot $f_1(\gamma_s^{t,l}) \to f_1(\gamma_u^{v,l'})$.

Definition 4.11. Let $f_2 : \mathcal{C}_D^2 \to \mathcal{C}_A$ be the functor defined on objects by

$$f_2(\gamma_s^{t,l}) = \begin{cases} \gamma(s', t'), & \text{if } s \neq -t; \\ \gamma(2n-3-t', t'), & \text{if } s = -t \text{ and } l = -1, \end{cases}$$

and on pivot $\gamma_s^{t,l} \to \gamma_u^{v,l'}$ by pivot $f_2(\gamma_s^{t,l}) \to f_2(\gamma_u^{v,l'})$.

In the third step, we define a functor from the category $\mathcal{C}_D$ to $\operatorname{ind} Q_D$ using the functors constructed in the previous two steps, together with the equivalent functor $F_A : \mathcal{C}_A \to \operatorname{ind} Q_A$ recalled in Section 4.1.

Proposition 4.12. *Assume that the vertex $X_{h'}$ ($h' \in \{0, 1, \dots, 2n-3\}$) of $P(Q_A)$ is relabeled by Y_h of $P(Q_D)$. For each $\gamma(s', t') \in \omega$,*

1. *there exists $k \in \{0, 1, \dots, n-2\}$ such that $F_A(\gamma(s', t')) = \tau^{-k} P_{n-1}$ if and only if $s' + t' = 2n-3$.*
2. *there exist $k \in \{0, 1, \dots, n-2\}$ and $i \in \{1, \dots, n-2\}$ such that $F_A(\gamma(s', t')) = \tau^{-k} P_i$ if and only if $\gamma(s', t')$ satisfies $2 \leq [s, t] \leq n-1$.*
3. *if there exist $k \in \{0, 1, \dots, n-2\}$ and $i \in Q_{A,0} \setminus \{n-1\}$ such that $F_A(\gamma(s', t')) = \tau^{-k} P_i$, then $F_A(\gamma(2n-3-t', 2n-3-s')) = \tau^{-k} P_{2n-2-i}$.*

Proof. It follows immediately from Remark 3.1, Theorem 4.2 and the definition of F_A. □

According to this proposition, we can obtain the following conclusion.

Proposition 4.13. *The images of functors $F_A f_1 : \mathcal{C}_D^1 \to \operatorname{ind} Q_A$ and $F_A f_2 : \mathcal{C}_D^2 \to \operatorname{ind} Q_A$ are both $\mathcal{A}'$. That is,*

$$F_A(f_1(\mathcal{C}_D^1)) = \mathcal{A}' = F_A(f_2(\mathcal{C}_D^2)).$$

Proof. By the definition of f_1 and f_2, we get $f_1(\mathcal{C}_D^1) = f_2(\mathcal{C}_D^2)$. Thus this statement is proved by combining the arguments of Proposition 4.12 (1) and (2). □

Therefore, we are ready to define the functor from $\mathcal{C}_D$ to $\operatorname{ind} Q_D$.

Definition 4.14. Let $F_D : \mathcal{C}_D \to \operatorname{ind} Q_D$ be the functor defined on objects by

$$F_D(\gamma_s^{t,l}) = \begin{cases} \psi_1(F_A(f_1(\gamma_s^{t,l}))) & \text{if } \gamma_s^{t,l} \in \Omega_1; \\ \psi_2(F_A(f_2(\gamma_{-t}^{t,-1}))) & \text{otherwise,} \end{cases}$$

and on pivot $\gamma_s^{t,l} \to \gamma_u^{v,l'}$ by irreducible morphism $F_D(\gamma_s^{t,l}) \to F_D(\gamma_u^{v,l'})$.

Now we are going to show that $\operatorname{ind} Q_D$, as an abelian category, is equivalent to $\mathcal{C}_D$.

Theorem 4.15. *The functor $F_D : \mathcal{C}_D \to \operatorname{ind} Q_D$ is an equivalence of categories. Particularly,*

(1) *F_D induces an isomorphism of translation quivers $(\Gamma_D, R_D) \to (\Gamma_{\mathcal{D}}, \tau)$;*
(2) *F_D induces bijections*

$$\{\text{line segments in } P(Q_D)\} \to \operatorname{ind} Q_D;$$

$$\{\text{pivots in } P(Q_D)\} \to \{\text{irreducible morphisms in } \operatorname{ind} Q_D\};$$

(3) *R_D corresponds to the Auslander-Reiten translation τ in the following sense*

$$F_D \circ R_D = \tau \circ F_D;$$

(4) *F_D is an exact functor with regard to the induced abelian structure on $\mathcal{C}_D$.*

Proof. According to the definition of F_D, it is easy to check that F_D is a bijection between the objects of the categories $\mathcal{C}_D$ and $\operatorname{ind} Q_D$ and thus a bijection between the vertices of the quivers Γ_D and $\Gamma_{\mathcal{D}}$.

Now we consider the relationship between the pivots in $P(Q_D)$ and irreducible morphisms in $\operatorname{ind} Q_D$. By the definition of f_1 and f_2, γ_2 is a pivot of γ_1 in $P(Q_D)$ if and only if $f_1(\gamma_2)$ (resp. $f_2(\gamma_2)$) is a pivot of $f_1(\gamma_1)$ (resp. $f_2(\gamma_1)$) in $P(Q_A)$. In addition, ψ_1 and ψ_2 give a bijection between irreducible morphisms in $\mathcal{A}'$ and $\operatorname{ind} Q_D$ since $(\Bbbk Q_{A_{2n-3}})G$ is Morita equivalent to $\Bbbk Q_D$. Therefore, we know from Theorem 4.2 and Proposition 4.12 that F_D is a bijection between the pivots in $P(Q_D)$ and irreducible morphisms in $\operatorname{ind} Q_D$.

Furthermore, we consider the correspondence between the Auslander-Reiten translation τ on $\operatorname{rep}_{\Bbbk} Q_D$ and the translation R_D. We only discuss the case γ_s^t with $2 \leq [s,t] \leq n-2$, the others are similar. As we know that in this case, $R_D^{-1}(\gamma_s^t)$ is either 0 or $\gamma_u^v \in \Omega$, where Y_u (resp. Y_v) is the counterclockwise neighbor of Y_s (resp. Y_t). If $R_D^{-1}(\gamma_s^t) = 0$, then γ_s^t has no pivots. As F_D is a bijection between pivots in $P(Q_D)$ and irreducible morphisms in $\operatorname{ind} Q_D$,

$$\tau^{-1} F_D(\gamma_s^t) = 0 = F_D R_D^{-1}(\gamma_s^t).$$

For $R_D^{-1}(\gamma_s^t) = \gamma_u^v$, we consider the irreducible morphisms $h : \overline{M} \to F_D(\gamma_u^v)$ ending at $F_D(\gamma_u^v)$. Then there exists $\gamma \in \Omega$ such that $\overline{M} = F_D(\gamma)$ since F_D is a bijection, which implies that $\gamma \to \gamma_u^v$ is an arrow in Γ_D. Since $2 \leq [u,v] \leq n-2$, by Definition 4.5, we get that γ is either γ_s^v or γ_u^t. If $\gamma = \gamma_s^v$, then $\gamma_s^t \to \gamma_s^v$ is also an arrow in Γ_D(c.f. Figure 7). Applying the functor

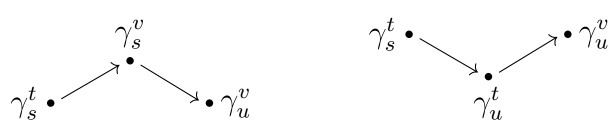


FIGURE 7. Local graph of quiver Γ_D

F_D gives an irreducible morphism from $F_D(\gamma_s^t)$ to $F_D(\gamma_s^v)$. If $\gamma = \gamma_u^t$, then it is analogous to prove that there is an irreducible morphism from $F_D(\gamma_s^t)$ to $F_D(\gamma_u^t)$. Thus, according to the definition of Auslander-Reiten translation τ, we have

$$F_D R_D^{-1}(\gamma_s^t) = F_D(\gamma_u^v) = \tau^{-1} F_D(\gamma_s^t).$$

Therefore, statement (1), (2) and (3) hold.

Since $\mathcal{C}_D$ (resp. ind Q_D) is the mesh category of translation quiver (Γ_D, R_D) (resp. $(\Gamma_{\mathcal{D}}, \tau)$), statement (1) implies that F_D is an equivalence of categories. In particular, this equivalence induces the structure of an abelian category on $\mathcal{C}_D$. With respect to this structure, F_D is exact since every equivalence between abelian categories is exact. $\square$

Consequently, we provide a geometric realization of the category ind Q_D via $\mathcal{C}_D$. Next, we present an application of this geometric realization: using the equivalent functor F_D, we can easily compute the dimension vectors of all indecomposable $\Bbbk Q_D$-modules. This will provide a great convenience for illustrating the Auslander-Reiten quivers of $\Bbbk Q_D$.

Theorem 4.16. *For any tagged line segment in Ω, we have*

$$\mathbf{dim}\ F_D(\gamma_{-t}^{t,l}) = \begin{cases} \sum\limits_{i=n-t}^{n-1} \mathbf{dim}\ \overline{S}_i, & \text{if } l=-1, \\ \sum\limits_{i=n-t}^{n-2} \mathbf{dim}\ \overline{S}_i + \mathbf{dim}\ \overline{S}_n, & \text{if } l=1; \end{cases} \tag{4.1}$$

$$\mathbf{dim}\ F_D(\gamma_s^t) = \begin{cases} \sum\limits_{i=n+s}^{n+t-1} \mathbf{dim}\ \overline{S}_i, & \text{if } s<t<0; \\ \sum\limits_{i=s+n}^{n-t-1} \mathbf{dim}\ \overline{S}_i + 2\sum\limits_{i=n-t}^{n-2} \mathbf{dim}\ \overline{S}_i + \sum\limits_{i=n-1}^{n} \mathbf{dim}\ \overline{S}_i, & \text{if } s<-t<0; \\ \sum\limits_{i=n-t}^{n-s-1} \mathbf{dim}\ \overline{S}_i, & \text{if } 0<s<t, \end{cases} \tag{4.2}$$

where **dim** *denotes the dimension vector and $\overline{S}_i$ is the simple module supported on $i \in Q_{D,0}$.*

Proof. We only discuss the case $s<t<0$ with $[s,t]<n$, the other cases are similar. In this case, the vertical coordinates of the vertices Y_s and Y_t, denoted as y_s and y_t, respectively, are both negative. Therefore, by rule $(*)$ and the definition of functor f_1, it follows that

$$f_1(\gamma_s^t) = \gamma(s+n-1, t+n-1).$$

Furthermore, according to the definition of F_A, we obtain

$$F_A f_1(\gamma_s^t) = M(s+n, t+n-1).$$

Follows from Remark 3.1, we immediately get that

$$\mathbf{dim}\ F_D(\gamma_s^t) = \mathbf{dim}\ \psi_1(M(s+n, t+n-1)) = \sum_{i=s+n}^{t+n-1} \mathbf{dim}\ \overline{S}_i.$$

Therefore, the statement holds. $\square$

Remark 4.17. The formula (4.1) and (4.2) of Theorem 4.16 may seem complex. Essentially, we identify the tagged segments corresponding to the simple modules, and then derive these two formulas through “vector addition”.

To illustrate the correspondence between tagged line segments and indecomposable $\Bbbk Q_D$-modules, we give an example below, which also shows the equivalence in Theorem 4.15.

Example 4.18. Consider the quiver Q_{D_5} in Figure 5. The translation quiver Γ_{D_5} with respect to pivots is shown in Figure 8 and the Auslander-Reiten quiver $\Gamma_{\mathrm{rep}_{\Bbbk} Q_{D_5}}$ for the quiver Q_{D_5} is shown in Figure 9. By comparing Γ_{D_5} with $\Gamma_{\mathrm{rep}_{\Bbbk} Q_{D_5}}$, we find that replacing the points $\gamma_s^{t,l}$ in the quiver Γ_{D_5} with $\mathbf{dim} F_D(\gamma_s^{t,l})$ is sufficient to obtain $\Gamma_{\mathrm{rep}_{\Bbbk} Q_{D_5}}$.

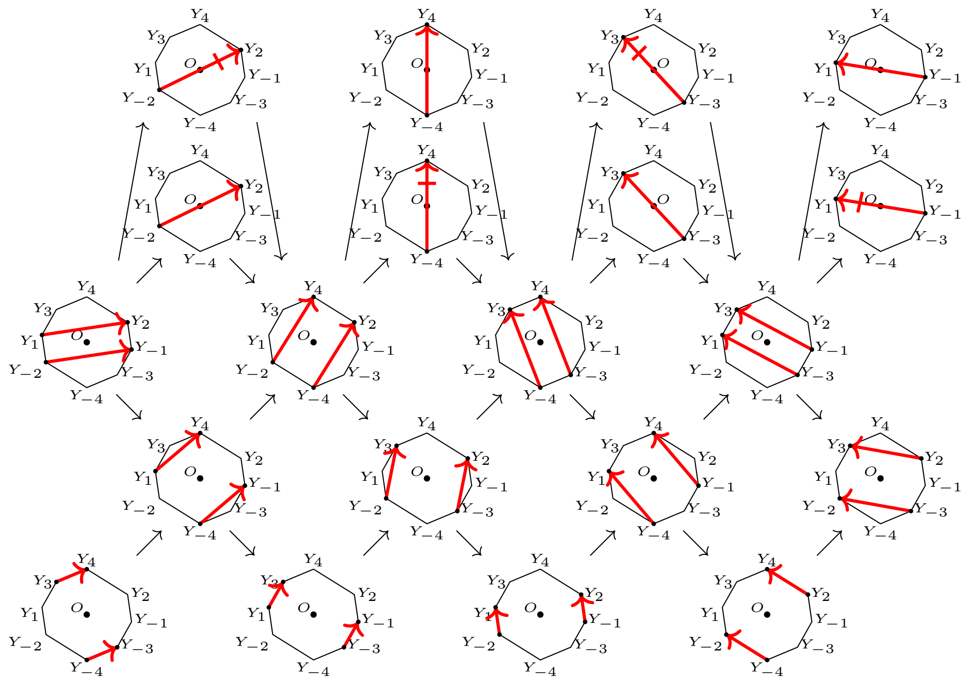


FIGURE 8. The translation quiver Γ_{D_5} for the quiver Q_D in Figure 5

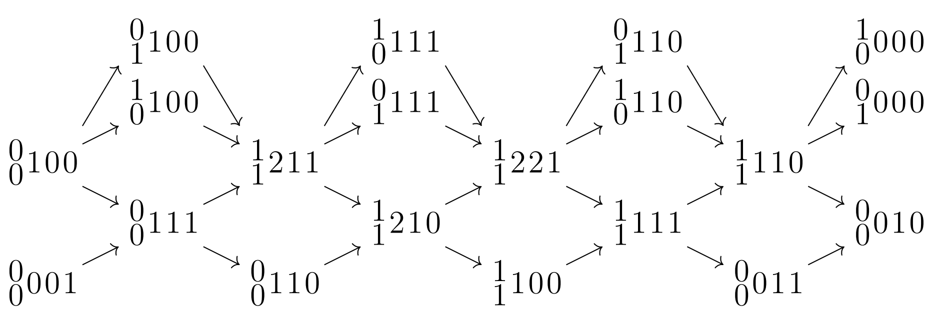


FIGURE 9. The Auslander-Reiten quiver $\Gamma_{\mathrm{rep}_{\Bbbk} Q_{D_5}}$ for the quiver Q_{D_5} in Figure 5

4.3. **The geometric interpretation of the dimension of extension group.** In this subsection, we want to determine the dimension of extension group between two indecomposable representations over Q_D via the tagged line segments of $P(Q_D)$. Since $\mathrm{rep}_{\Bbbk} Q_D$ is a hereditary category, the extension groups of degree i are zero for all $i \geq 2$. Through Serre duality, the dimension of extension groups of degree 0 can be determined by the corresponding extension groups of degree 1. Hence, we only consider the case of degree 1.

Definition 4.19. Let γ_1 and γ_2 be two tagged line segments of $P(Q_D)$. A point of intersection of γ_1 and γ_2 is called *positive intersection* of γ_1 and γ_2, if γ_1 intersects γ_2 from the left or the endpoint of γ_1 (resp. γ_2) coincides with the starting point of γ_2(resp. γ_1)(c.f. Figure 10). Denote by $\mathrm{Int}^+(\gamma_1,\gamma_2)$ the number of the positive intersections of γ_1 and γ_2.

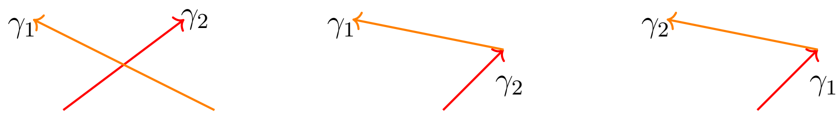


FIGURE 10. Positive intersection of γ_1 and γ_2

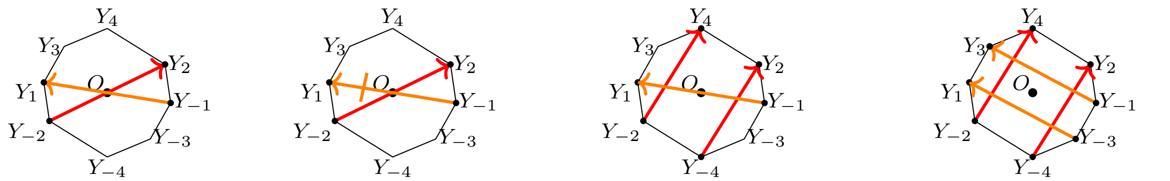
Remark 4.20. The number of positive intersections between two tagged line segments is determined by their position on $P(Q_D)$, regardless of whether they contain the tag "|". For any two tagged line segments γ_s^{t,l_1} and γ_u^{v,l_2} on $P(Q_D)$ with $[s,t]\le n$ and $[u,v]\le n$,

(1) it is easy to check that
$$\mathrm{Int}^+(\gamma_s^{t,l_1},\gamma_u^{v,l_2})=\mathrm{Int}^+(\gamma_{-t}^{-s,l_1},\gamma_{-v}^{-u,l_2}).$$
(2) assume that the vertex $X_{h'}$ ($h'\in\{0,1,\dots,2n-3\}$) of $P(Q_A)$ is relabeled by Y_h of $P(Q_D)$, $F_A(\gamma(s',t')=M$ and $F_A(\gamma(u',v'))=N$. By [3, Proposition 6.5], there exists a positive intersection of γ_s^{t,l_1} and γ_u^{v,l_2} if and only if $\mathrm{Ext}^1_{\mathcal{A}}(M,N)$ is generated by a non-split short exact sequence. Moreover, if $[u,v]<n$, then there exists a positive intersection of γ_s^{t,l_1} and γ_{-v}^{-u} if and only if $\mathrm{Ext}^1_{\mathcal{A}}(M,F_g(N))$ is also generated by a non-split short exact sequence.

Definition 4.21. Let $\overline{M}$ and $\overline{N}$ be two indecomposable representations over Q_D and assume that $F_D(\gamma_s^{t,l_1})=\overline{M}$ and $F_D(\gamma_u^{v,l_2})=\overline{N}$, where γ_s^{t,l_1} and γ_u^{v,l_2} are in Ω. The *crossing number* $\mathrm{Int}(\overline{M},\overline{N})$ of $\overline{M}$ and $\overline{N}$ is defined via:
$$\mathrm{Int}(\overline{M},\overline{N})=\begin{cases}\mathrm{Int}^+(\gamma_s^t,\gamma_u^v)|\frac{l_1-l_2}{2}| & \text{if } s=-t \text{ and } u=-v;\\ \mathrm{Int}^+(\gamma_s^t,\gamma_u^v)+\mathrm{Int}^+(\gamma_s^t,\gamma_{-v}^{-u}) & \text{if } s\neq -t \text{ and } u\neq -v;\\ \mathrm{Int}^+(\gamma_s^t,\gamma_u^v) & \text{otherwise,}\end{cases}$$
where $|\frac{l_1-l_2}{2}|$ denote the absolute value of $\frac{l_1-l_2}{2}$.

Example 4.22. Consider the quiver Q_{D_5} in Figure 5. According to Example 4.18, we have $F_D(\gamma_{-2}^{2,1})=\overline{P}_4,\ F_D(\gamma_{-1}^{1,1})=\tau^{-3}\overline{P}_5,\ F_D(\gamma_{-1}^{1,-1})=\tau^{-3}\overline{P}_4,\ F_D(\gamma_{-4}^{2})=\tau^{-1}\overline{P}_3,\ F_D(\gamma_{-1}^{3})=\tau^{-3}\overline{P}_3.$
By Definition 4.19, it is easy to obtain that
$$\mathrm{Int}^+(\gamma_{-1}^{1,1},\ \gamma_{-2}^{2,1})=\mathrm{Int}^+(\gamma_{-1}^{1,-1},\ \gamma_{-2}^{2,1})=\mathrm{Int}^+(\gamma_{-1}^{1,1},\ \gamma_{-4}^{2})=\mathrm{Int}^+(\gamma_{-1}^{3},\gamma_{-4}^{2})=\mathrm{Int}^+(\gamma_{-1}^{3},\gamma_{-2}^{4})=1$$
(c.f. Figure 11). Thus by Definition 4.21, the crossing numbers
$$\mathrm{Int}(\tau^{-3}\overline{P}_5,\overline{P}_4)=0,\ \mathrm{Int}(\tau^{-3}\overline{P}_4,\overline{P}_4)=1,\ \mathrm{Int}(\tau^{-3}\overline{P}_5,\tau^{-1}\overline{P}_3)=1,\ \mathrm{Int}(\tau^{-3}\overline{P}_3,\tau^{-1}\overline{P}_3)=2.$$

Next, we will explore the relationship between the extension group of degree 1 on $\mathcal{D}$ and the crossing number. Before that, we need to study the extension groups and properties of short exact sequences in the category $\mathcal{A}$.



$\operatorname{Int}(\tau^{-3}\overline{P}_5, \overline{P}_4) = 0$ $\operatorname{Int}(\tau^{-3}\overline{P}_4, \overline{P}_4) = 1$ $\operatorname{Int}(\tau^{-3}\overline{P}_5, \tau^{-1}\overline{P}_3) = 1$ $\operatorname{Int}(\tau^{-3}\overline{P}_3, \tau^{-1}\overline{P}_3) = 2$

FIGURE 11. Crossing number

Lemma 4.23. *Let $\tau^{-l}P_{n-1}$ and $\tau^{-k}P_{n-1}$ be two representations over Q_A with $0 \leq k, l \leq n-2$. If $l < k$, then there exist $m \in \{0, 1, \dots, n-2\}$ and $r \in \{1, \dots, n-2\}$ such that*

$$0 \longrightarrow \tau^{-l}P_{n-1} \longrightarrow \tau^{-m}P_r \oplus \tau^{-m}P_{2n-2-r} \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0$$

is a short exact sequence.

Proof. Since $\mathscr{R}_{\rightarrow}(P_{n-1}) = \mathscr{R}_{\leftarrow}(I_{n-1})$ and $I_{n-1} = \tau^{-(n-2)}P_{n-1}$, $\tau^{-k+1}P_{n-1} \in \mathscr{R}_{\rightarrow}(\tau^{-l}P_{n-1})$, the sets $\Sigma_{\rightarrow}(\tau^{-l}P_{n-1})$ and $\Sigma_{\leftarrow}(\tau^{-k}P_{n-1})$ have two points in common (cf. the blue points in Figure 12). Assume that E_1 and E_2 are indecomposable representations corresponding to

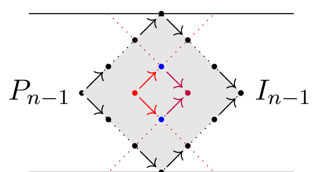


FIGURE 12. The gray-filled rectangle represents $\mathscr{R}_{\rightarrow}(P_{n-1})$

these points, respectively. By Proposition 2.2(3), $\operatorname{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, \tau^{-l}P_{n-1})$ is generated by the following non-split short exact sequence

$$\xi : 0 \longrightarrow \tau^{-l}P_{n-1} \longrightarrow E_1 \oplus E_2 \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0.$$

Applying F_g to ξ, we obtain a short exact sequence

$$F_g(\xi) : 0 \longrightarrow \tau^{-l}P_{n-1} \longrightarrow F_g(E_1) \oplus F_g(E_2) \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0.$$

Again by Proposition 2.2(3), $F_g(E_1) = E_2$. Thus the proof is completed by (3.1). $\square$

Corollary 4.24. For a non-split short exact sequence ξ in the category $\mathcal{A}$,

(1) If ξ is of the form

$$\xi : 0 \longrightarrow N \xrightarrow{\iota_1} E \xrightarrow{\nu_1} \tau^{-k}P_{n-1} \longrightarrow 0,$$

where $k \in \{0, 1, \dots, n-2\}$, N and E are indecomposable representations. Then there exists $h \in \{0, 1, \dots, k-1\}$ such that

$$0 \longrightarrow N \oplus F_g(N) \longrightarrow \tau^{-h}P_{n-1} \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0,$$

is a short exact sequence.

(2) If ξ is of the form

$$\xi : 0 \longrightarrow N \longrightarrow E \oplus F \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0,$$

where N, E and F are indecomposable representations with N not belonging to the τ-orbit of P_{n-1}. Then exists $h \in \{0, 1, \ldots, k-1\}$ such that

$$0 \longrightarrow \tau^{-h}P_{n-1} \longrightarrow F \oplus F_g(F) \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow N \oplus F_g(N) \longrightarrow \tau^{-h}P_{n-1} \oplus E \oplus F_g(E) \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0$$

are short exact sequences.

Proof. We only prove (1); the proof of (2) is analogous to part (1). Since ξ is a non-split short exact sequence, E is an element of $\Sigma_{\leftarrow}(\tau^{-k}P_{n-1})$ by Proposition 2.2. Thus, according to Lemma 4.23, there exists $h \in \{0, 1, \ldots, k-1\}$ such that

$$0 \longrightarrow \tau^{-h}P_{n-1} \xrightarrow{\binom{\mu_1}{\mu_2}} E \oplus F_g(E) \xrightarrow{(\nu_1\ \nu_2)} \tau^{-k}P_{n-1} \longrightarrow 0$$

is a short exact sequence. Since $F_g(\xi)$ is a short exact sequence, the following diagram

$$\begin{tikzcd}
0 \arrow[r] & N \oplus F_g(N) \arrow[r, "(\lambda_1\ \lambda_2)"] \arrow[d, equal] & \tau^{-h}P_{n-1} \arrow[r, "\nu_1\mu_1"] \arrow[d, "\binom{\mu_1}{\mu_2}"] & \tau^{-k}P_{n-1} \arrow[r] \arrow[d, "\binom{1}{-1}"] & 0 \\
0 \arrow[r] & N \oplus F_g(N) \arrow[r, "\begin{pmatrix} \iota_1 & 0 \\ 0 & \iota_2 \end{pmatrix}"] & E \oplus F_g(E) \arrow[r, "\begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}"] & \tau^{-k}P_{n-1}^2 \arrow[r] & 0
\end{tikzcd}$$

has an exact bottom row and a commutative right square. We claim that the left square is commutative. In fact, because $\tau^{-m}P_r$ is in $\mathscr{R}_{\rightarrow}(B)$ but $F_g(\tau^{-m}P_r)$ is not, we conclude that

$$\begin{aligned}
\dim_{\Bbbk}\mathrm{Hom}_{\mathcal{A}}(F_g(B), F_g(\tau^{-m}P_r)) &= \dim_{\Bbbk}\mathrm{Hom}_{\mathcal{A}}(B, \tau^{-m}P_r) = 1, \\
\dim_{\Bbbk}\mathrm{Hom}_{\mathcal{A}}(B, F_g(\tau^{-m}P_r)) &= \dim_{\Bbbk}\mathrm{Hom}_{\mathcal{A}}(F_g(B), \tau^{-m}P_r) = 0.
\end{aligned}$$

Therefore, without loss of generality, we can assume that $\mu_1\lambda_2 = 0 = \mu_2\lambda_1$, $\mu_1\lambda_1 = \iota_1$ and $\mu_2\lambda_2 = \iota_2$. This makes the left square commutative. Since ν_1 and ν_2 are epimorphisms, it follows from [1, Proposition 3.9.6(1)] that

$$0 \longrightarrow N \oplus F_g(N) \xrightarrow{(\lambda_1\ \lambda_2)} \tau^{-h}P_{n-1} \xrightarrow{\nu_1\mu_1} \tau^{-k}P_{n-1} \longrightarrow 0$$

is a short exact sequence. The details will be omitted here. $\square$

Let ξ be an element in $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$. Then it can be represented by a short exact sequence of the form

$$\xi\colon 0 \to Y \to E \to X \to 0.$$

The relationship between the extension groups of degree 1 on $\mathcal{A}$ and the category of equivariant objects $\mathcal{A}^G$ was established in [8] as follows: for a morphism $p\colon X' \to X$, the pullback of ξ along p is denoted by $\xi.p$. Similarly, for a morphism $p'\colon Y \to Y'$, the pushout of ξ along p' is denoted by $p'.\xi$. For two objects (X, α) and (Y, β) in $\mathcal{A}^G$, the space $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ carries a $\Bbbk$-linear G-action associated to these two objects

$$G \times \mathrm{Ext}^1_{\mathcal{A}}(X, Y) \to \mathrm{Ext}^1_{\mathcal{A}}(X, Y),\ (g, \xi) \mapsto \beta_g^{-1}.F_g(\xi).\alpha_g.$$

Denote by $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)^G$ the invariant subspace of $\mathrm{Ext}^1_{\mathcal{A}}(X,Y)$ with the $\Bbbk$-linear G-action.

Lemma 4.25 ([8]). *Let (X,α) and (Y,β) be in $\mathcal{A}^G$. Then the forgetful functor U induces a $\Bbbk$-linear isomorphism*

$$U : \mathrm{Ext}^1_{\mathcal{A}^G}((X,\alpha),(Y,\beta)) \to \mathrm{Ext}^1_{\mathcal{A}}(X,Y)^G.$$

Indeed, for an extension $\xi\colon 0 \to (Y,\beta) \to (E,\delta) \to (X,\alpha) \to 0$ in $\mathcal{A}^G$ the corresponding extension $U(\xi)$ in $\mathcal{A}$ satisfies $\beta_g.F_gU(\xi) = U(\xi).\alpha_g$ by the following commutative diagram

$$\begin{tikzcd}
U(\xi): & 0 \arrow[r] & Y \arrow[r] \arrow[d, "\beta_g"] & E \arrow[r] \arrow[d, "\delta_g"] & X \arrow[r] \arrow[d, "\alpha_g"] & 0 \\
F_gU(\xi): & 0 \arrow[r] & F_g(Y) \arrow[r] & F_g(E) \arrow[r] & F_g(X) \arrow[r] & 0.
\end{tikzcd}$$

Furthermore, we study the properties of the extension groups of degree 1 on $\mathcal{D}$, yielding the following lemma.

Lemma 4.26. *Suppose $k,l,m \in \{0,1,\dots,n-2\}$, $l<k$ and $m<k$. If $(k-l)\mathrm{mod}\ 2 = 1$ and $(k-m)\mathrm{mod}\ 2 = 0$, then*

$$\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \tau^{-l}\overline{P}_n) \cong \Bbbk \cong \mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \tau^{-m}\overline{P}_n),$$
$$\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \tau^{-l}\overline{P}_n) = 0 = \mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \tau^{-m}\overline{P}_n),$$
$$\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \tau^{-l}\overline{P}_{n-1}) = 0 = \mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \tau^{-m}\overline{P}_{n-1}),$$
$$\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \tau^{-l}\overline{P}_{n-1}) \cong \Bbbk \cong \mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \tau^{-m}\overline{P}_{n-1}).$$

Proof. It is easy to check by using the Auslander-Reiten theory on the category $\mathcal{D}$. □

Now we are ready to give a geometric interpretation for the dimension of extension group of degree 1 between two indecomposable representations over Q_D.

Theorem 4.27. *Let $\overline{M}$ and $\overline{N}$ be two indecomposable representations over Q_D. Then*

$$\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{D}}(\overline{M},\overline{N}) = \mathrm{Int}(\overline{M},\overline{N}).$$

Proof. Since $\overline{M}$ and $\overline{N}$ are indecomposable, there exist $\gamma_s^{t,l_1}, \gamma_u^{v,l_2} \in \Omega$ such that $F_D(\gamma_s^{t,l_1}) = \overline{M}$ and $F_D(\gamma_u^{v,l_2}) = \overline{N}$. Assume that the vertex $X_{h'}$ ($h' \in \{0,1,\dots,2n-3\}$) of $P(Q_A)$ is relabeled by Y_h of $P(Q_D)$, $F_A(\gamma(s',t')) = M$ and $F_A(\gamma(u',v')) = N$. If both $s=-t$ and $u=-v$, then it follows from Definition 4.21 and Lemma 4.26 that $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{D}}(\overline{M},\overline{N}) = \mathrm{Int}(\overline{M},\overline{N})$. If otherwise, then this proof can be divided into the following two cases.

(1) If $s \neq -t$ and $u \neq -v$, then $F_g(N) \neq N$ and $F_g(M) \neq M$. From Definition 4.21 and Remark 4.20(2), it follows that

$$\mathrm{Int}(\overline{M},\overline{N}) = \dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M,N) + \dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M,F_g(N)).$$

Thus by Lemma 4.25, if we can prove (4.3), then the statement follows immediately.

$$\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M\oplus F_g(M), N\oplus F_g(N))^G = \dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M,N) + \dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(M,F_g(N)) \quad (4.3)$$

If the right-hand side of (4.3) equals 0, then Proposition 3.2 implies that (4.3) holds.

If the right-hand side of (4.3) equals 1, then we may assume that $\mathrm{Ext}^1_{\mathcal{A}}(M,N)$ is generated by a non-split short exact sequence ξ_1. Note that an element of $\mathrm{Ext}^1_{\mathcal{A}}(M\oplus F_g(M), N\oplus F_g(N))$

has the form $a\xi_1 \oplus bF_g(\xi_1)$, where $a, b \in \Bbbk$. Moreover, it is simple to show that $a\xi_1 \oplus bF_g(\xi_1)$ is an element of $\mathrm{Ext}^1_{\mathcal{A}}(M \oplus F_g(M), N \oplus F_g(N))^G$ if and only if $a = b$. Hence (4.3) holds.

If the right-hand side of (4.3) equals 2, assume that $\mathrm{Ext}^1_{\mathcal{A}}(M, N)$ and $\mathrm{Ext}^1_{\mathcal{A}}(M, F_g(N))$ are generated by non-split short exact sequences ξ_1 and ξ_2, respectively. Then $\xi_1 \oplus F_g(\xi_1)$ and $\xi_2 \oplus F_g(\xi_2)$ are elements of $\mathrm{Ext}^1_{\mathcal{A}}(M \oplus F_g(M), N \oplus F_g(N))^G$ from above discussion. Due to the automorphic functor F_g, $\xi_1 \oplus F_g(\xi_1)$ is $\Bbbk$-linearly independent of $\xi_2 \oplus F_g(\xi_2)$. Since $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{D}}(\overline{M}, \overline{N})$ is at most equal to 2, it follows from Lemma 4.25 that (4.3) is valid.

(2) Either $s = -t$ or $u = -v$, but not both. Without loss of generality, we may assume that $\overline{M} = F_D(\gamma^t_{-t}) = \tau^{-l}\overline{P}_n$. An argument similar to the one used in (1) shows that our problem reduces to

$$\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-l}P_{n-1}, N \oplus F_g(N))^G = \dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-l}P_{n-1}, N). \tag{4.4}$$

Since both sides of (4.4) are less than or equal to 1, it suffices to prove $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-l}P_{n-1}, N) = 0$ if and only if $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-l}P_{n-1}, N \oplus F_g(N))^G = 0$. Thanks to Proposition 3.2, the necessity is obvious. The converse implications are also ture. If not, $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-l}P_{n-1}, N)$ is generated by a non-split short exact sequence. Then Corollary 4.24 implies that the right-hand side of (4.4) is non-zero. This leads to a contradiction.

The proof of the theorem is now complete. □

5. Maximal almost pre-rigid representations

In this section, we introduce the notation of maximal almost pre-rigid representations over Q_D which is the analogues of maximal almost rigid representations over Q_A. And we seek out a class of special representations over Q_A, which correspond to maximal almost pre-rigid representations over Q_D.

5.1. **Almost rigid representation and its properties over Q_A.** Recall that a representation is called *basic* if it has no repeated direct summands. Recall further that in [3], a representation T over quiver Q is *almost rigid*, if it is basic and satisfies that for each pair X, Y of indecomposable summands of T, either $\mathrm{Ext}(X, Y) = 0$ or $\mathrm{Ext}(X, Y)$ is generated by a short exact sequence of the form $0 \to Y \to E \to X \to 0$ whose middle term is indecomposable. An almost rigid representation T over Q is called *maximal almost rigid*, if for every representation M, the representation $T \oplus M$ is not almost rigid.

In this subsection, we focus on the almost rigid representations over Q_A. By observing the relationship between the categories $\mathcal{A}$ and $\mathcal{D}$, we consider the almost rigid representations T with $F_g(T) = T$. From now on, we call a representation $M \in \mathcal{A}$ is F_g-stable if $F_g(M) = M$.

According to the definition of F_g in Section 3, any F_g-stable representation over Q_A must contain either $A \oplus F_g(A)$ or $\tau^{-k}P_{n-1}$ as a direct summand, where A is an indecomposable representation in $\mathcal{A}'$ with $F_g(A) \neq A$ and $k \in \{0, 1, \ldots, n-2\}$. Given that $A \oplus F_g(A)$ is almost rigid, our examination of whether a F_g-stable representation is almost rigid only necessitates considering the following two classes of representations:

$$A \oplus F_g(A) \oplus B \oplus F_g(B), \quad \tau^{-k}P_{n-1} \oplus B \oplus F_g(B),$$

where B is also an indecomposable representation in $\mathcal{A}'$ with $F_g(B) \neq B$.

Remark 5.1. According to the definition of F_g, in the Auslander-Reiten quiver $\Gamma_{\mathcal{A}}$, both the M and $F_g(M)$ lie on the line $h(M)$ passing through M and perpendicular to the τ-orbit of M. Therefore, we can assume that neither A nor $\tau^{-k}P_{n-1}$ lies to the left of $h(B)$. In fact, if A was to the left of $h(B)$ or on $h(B)$, then B is not to the left of $h(A)$. Therefore, without loss of generality, we can assume that A is not to the left of $h(B)$. Similarly, we can assume that $\tau^{-k}P_{n-1}$ is not to the left of $h(B)$.

Now we give the necessary and sufficient condition for F_g-stable representations over Q_A to be almost rigid. Referring to Remark 5.1, it suffices to demonstrate the following proposition.

Proposition 5.2. *Assume that neither A nor $\tau^{-k}P_{n-1}$ lies to the left of $h(B)$, where $h(B)$ is the line passing through B and perpendicular to the τ-orbit of B in $\Gamma_{\mathcal{A}}$. Then*

(1) *$T = A \oplus F_g(A) \oplus B \oplus F_g(B)$ is almost rigid if and only if one of the following conditions holds:*
- $\mathrm{Ext}^1_{\mathcal{A}}(T,T) = 0$;
- $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B) = 0$ *and* $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$ *is generated by a short exact sequence whose middle term is an indecomposable representation E with $E \in \mathcal{A}'$;*
- $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0$ *and* $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B)$ *is generated by a short exact sequence whose middle term is $\tau^{-m}P_r$ for some $r \in Q_{A,0}\backslash\{n-1\}$ and $m \in \{0, 1, \dots, n-2\}$.*

(2) *$T' = \tau^{-k}P_{n-1} \oplus B \oplus F_g(B)$ is almost rigid if and only if $\mathrm{Ext}^1_{\mathcal{A}}(T', T') = 0$ or $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B)$ is generated by a short exact sequence whose middle term is $\tau^{-m}P_r$ with $r \in Q_{A,0} \setminus \{n-1\}$ and $m \in \{0, 1, \dots, n-2\}$.*

Proof. Owing to the definition of almost rigid representations, the sufficiency of (1) and (2) are obvious. Since neither A nor $\tau^{-k}P_{n-1}$ lies to the left of $h(B)$(c.f. Figure 13),

$$\mathrm{Ext}^1_{\mathcal{A}}(B, A) = \mathrm{Ext}^1_{\mathcal{A}}(B, F_g(A)) = \mathrm{Ext}^1_{\mathcal{A}}(B, \tau^{-k}P_{n-1}) = 0.$$

Thus we only need to consider $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$, $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B)$ and $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B)$.

(1) If $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0 = \mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B)$, then by Proposition 3.2(1), $\mathrm{Ext}^1_{\mathcal{A}}(T, T) = 0$. Otherwise, since A, $F_g(A)$ and B are indecomposable representations over Q_A, at least one of $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(A, B)$ and $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B)$ is equal to 1, by Proposition 2.2.

Suppose $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 1$. Since T is almost rigid, $\Sigma_{\rightarrow}(B)$ and $\Sigma_{\leftarrow}(A)$ have only one point in common, which is of the form $\tau^{-m}P_r$ with $r \in Q_{A,0}$ and $m \in \{0, 1, \dots, n-2\}$. We divide this proof into three cases according to the location of $F_g(A)$ (c.f. Figure 13).

If $F_g(A)$ is located on $l(B)$, then $r = n-1$ and $\tau F_g(A) \in \mathscr{R}_{\rightarrow}(B)$. Thus by Proposition 2.2, $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B) = 0$; By the same token, if $F_g(A)$ is on the left side of $l(B)$, then $r < n-1$ and $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B) = 0$; If $F_g(A)$ is on the right side of $l(B)$, then $r > n-1$, $\Sigma_{\rightarrow}(B)$ and $\Sigma_{\leftarrow}(F_g(A))$ have two points in common. Also by Proposition 2.2, there exists a short exact sequence starting with B and ending with $F_g(A)$ whose middle term has two indecomposable summands. Contradicts to the condition that T is almost rigid.

Suppose $\dim_{\Bbbk}\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B) = 1$. Then $\Sigma_{\rightarrow}(B)$ and $\Sigma_{\leftarrow}(F_g(A))$ have only one point $\tau^{-m}P_r$ in common since T is almost rigid. Thus $\tau F_g(A)$ is on the right boundary of $\mathscr{R}_{\rightarrow}(B)$ (c.f. the blue segments in Figure 14). It is easy to see that τA is not in $\mathscr{R}_{\rightarrow}(B)$ and $r \neq n-1$. Hence, $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0$.

From the above discussion, we can see that statement (1) is held.

(2) Since $\tau^{-k}P_{n-1}$ and B are indecomposable representations over Q_A, the dimension of $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B)$ is 0 or 1. Thus the statement(2) holds by Proposition 2.2. $\square$

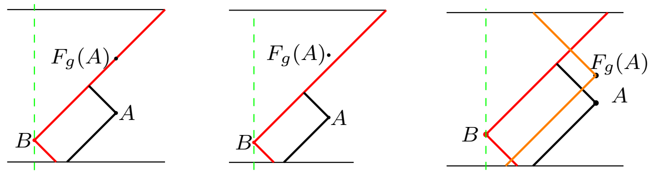


FIGURE 13. The line $h(B)$(green dashed line) and $\Sigma_{\rightarrow}(B)$(red lines) in $\Gamma_{\mathcal{A}}$

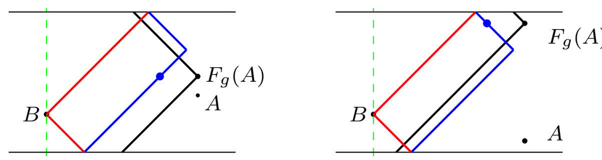


FIGURE 14. $\Sigma_{\rightarrow}(B)$(red lines) and $\tau F_g(A)$(blue dot) in $\Gamma_{\mathcal{A}}$

5.2. **The introduction of maximal almost pre-rigid representations.** In this subsection, we introduce the notaion of maximal almost pre-rigid representations over Q_D, and define a class of special representations over Q_A, thereby establishing their connection.

We start with defining almost pre-rigid representations over Q_D as follows:

Definition 5.3. A representation $\overline{T}$ over Q_D is called *almost pre-rigid* if it is basic and for each pair $\overline{X}, \overline{Y}$ of indecomposable summands of $\overline{T}$, $\mathrm{Ext}^1_{\mathcal{D}}(\overline{X}, \overline{Y})$ is either 0 or generated by a short exact sequence whose middle term is indecomposable or $\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n$, for some $k \in \{0, 1, \ldots, n-2\}$.

Definition 5.4. An almost pre-rigid representation $\overline{T}$ over Q_D is *maximal almost pre-rigid*, if for every nonzero representation $\overline{M}$ over Q_D, $\overline{T} \oplus \overline{M}$ is not almost pre-rigid. And let $\overline{mar}(Q_D)$ denote the set of all maximal almost pre-rigid representations over Q_D.

Combining the structure of Auslander-Reiten quiver $\Gamma_{\mathcal{D}}$ with Definition 5.4, we readily obtain the following property of maximal almost pre-rigid representations.

Proposition 5.5. *Let $\overline{T}$ be a maximal almost pre-rigid representation over Q_D. For any $k \in \{0, 1, \ldots, n-2\}$, $\tau^{-k}\overline{P}_{n-1}$ is a summand of $\overline{T}$ if and only if $\tau^{-k}\overline{P}_n$ is a summand of $\overline{T}$.*

To establish the link of the (maximal) almost pre-rigid representations over Q_D with the representations over Q_A, we introduce the following definitions.

Definition 5.6. A representation T over Q_A is *approximate rigid* if it is basic and for each pair X, Y of indecomposable summands of T, $\mathrm{Ext}^1_{\mathcal{A}}(X, Y)$ is either 0 or generated by a short exact sequence which is one of the following form:

- $0 \to Y \to E \to X \to 0$ whose middle term E is indecomposable;
- $0 \to \tau^{-l}P_{n-1} \to \tau^{-m}P_i \oplus \tau^{-m}P_{2n-2-i} \to \tau^{-k}P_{n-1} \to 0$ with $k, l, m \in \{0, 1, \ldots, n-2\}$ and $i \in Q_{A,0}$.

Definition 5.7. An approximate rigid representation T over Q_A is called *maximal approximate rigid*, if for every nonzero representation M over Q_A, $T \oplus M$ is not approximate rigid. And let $\overline{mar}(Q_A)$ denote the set of all maximal approximate rigid representations over Q_A.

In the remaining part of this section, we will establish the one-to-one correspondence between F_g-stable maximal approximate rigid representations over Q_A and maximal almost pre-rigid representations over Q_D. Denote by $\overline{smar}(Q_A)$ the subset of $\overline{mar}(Q_A)$ consisting of F_g-stable maximal approximate rigid representations over Q_A, and let $\psi' : \overline{smar}(Q_A) \to \mathcal{D}$ be the mapping determined additively by

$$\psi'\left(\tau^{-m}P_r \oplus \tau^{-m}P_{2n-2-r}\right) = \tau^{-m}\overline{P}_r \text{ and } \psi'\left(\tau^{-m}P_{n-1}\right) = \tau^{-m}\overline{P}_{n-1} \oplus \tau^{-m}\overline{P}_n,$$

where $r \in \{1, \ldots, n-2\}$ and $m \in \{0, 1, \ldots, n-2\}$. We will show that $\psi'(\overline{smar}(Q_A))$ is a subset of $\overline{mar}(Q_D)$. This proof is divided into two steps. In the first step, we will prove that for any $T \in \overline{smar}(Q_A)$, the image $\psi'(T)$ is an almost pre-rigid representation over Q_D. According to the definition of F_g, we only need to prove the following two lemmas.

Lemma 5.8. *Assume that neither A nor $\tau^{-k}P_{n-1}$ lies to the left of $h(B)$, where $h(B)$ is the line passing through B and perpendicular to the τ-orbit of B in $\Gamma_{\mathcal{A}}$.*

1. *If $T = A \oplus F_g(A) \oplus B \oplus F_g(B)$ is an almost rigid representation over Q_A, then $\psi'(T)$ is an almost pre-rigid representation over Q_D.*
2. *If $T' = \tau^{-k}P_{n-1} \oplus B \oplus F_g(B)$ is an almost rigid representation over Q_A, then $\psi'(T')$ is an almost pre-rigid representation over Q_D.*

Proof. By the definition of ψ', we may assume that

$$\psi'(A \oplus F_g(A)) = \overline{A} \text{ and } \psi'(B \oplus F_g(B)) = \overline{B},$$

where $\overline{A}$ and $\overline{B}$ are indecomposable representations over Q_D. Then $\psi'(T) = \overline{A} \oplus \overline{B}$ and $\psi'(T') = \tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n \oplus \overline{B}$. Since neither A nor $\tau^{-k}P_{n-1}$ lies to the left of $h(B)$,

$$\mathrm{Ext}^1_{\mathcal{D}}(\overline{B}, \overline{A}) = 0, \ \mathrm{Ext}^1_{\mathcal{D}}(\overline{B}, \tau^{-k}\overline{P}_{n-1}) = 0, \ \mathrm{Ext}^1_{\mathcal{D}}(\overline{B}, \tau^{-k}\overline{P}_n) = 0.$$

(1) According to Proposition 5.2(1), we divide this proof into three case. If $\mathrm{Ext}^1_{\mathcal{A}}(T, T) = 0$, then by Lemma 4.25,

$$\mathrm{Ext}^1_{\mathcal{D}}(\psi'(T), \psi'(T)) \cong \mathrm{Ext}^1_{\mathcal{D}}(\overline{A}, \overline{B}) \cong \mathrm{Ext}^1_{\mathcal{A}}(A \oplus F_g(A), B \oplus F_g(B))^G = 0.$$

If $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B) = 0$ and $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$ is generated by a short exact sequence ξ, whose middle term is of the form $\tau^{-m}P_r$ with $r \in \{1, \ldots, n-1\}$ and $m \in \{0, 1, \ldots, n-2\}$. Then it can be verified that $\mathrm{Ext}^1_{\mathcal{A}}(A \oplus F_g(A), B \oplus F_g(B))^G$ is generated by $\xi \oplus F_g(\xi)$, since

$$\mathrm{Ext}^1_{\mathcal{A}}(A \oplus F_g(A), B \oplus F_g(B)) \cong \mathrm{Ext}^1_{\mathcal{A}}(A, B) \oplus \mathrm{Ext}^1_{\mathcal{A}}(F_g(A), F_g(B)).$$

Thus Lemma 4.25 implies that $\mathrm{Ext}^1_{\mathcal{D}}(\overline{A}, \overline{B})$ is generated by

$$0 \longrightarrow \overline{B} \longrightarrow \psi(\tau^{-m}P_r) \longrightarrow \overline{A} \longrightarrow 0,$$

where $\psi(\tau^{-m}P_r) = \tau^{-m}\overline{P}_r$ for $r \leq n-2$ and $\psi(\tau^{-m}P_{n-1}) = \tau^{-m}\overline{P}_{n-1} \oplus \tau^{-m}\overline{P}_n$. If $\mathrm{Ext}^1_{\mathcal{A}}(A, B) = 0$ and $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B)$ is generated by a short exact sequence, whose middle term is $\tau^{-m}P_r$ with $r \neq n-1$. Then similarly, we obtain that $\mathrm{Ext}^1_{\mathcal{D}}(\overline{A}, \overline{B})$ is generated by

$$0 \longrightarrow \overline{B} \longrightarrow \psi(\tau^{-m}P_r) \longrightarrow \overline{A} \longrightarrow 0.$$

Consequently, we get that $\psi'(T) = \overline{A} \oplus \overline{B}$ is almost pre-rigid.

(2) Since T' is almost rigid, there are two possibilities according to Proposition 5.2(2). If $\mathrm{Ext}^1_{\mathcal{A}}(T', T') = 0$, then also by Lemma 4.25

$$\mathrm{Ext}^1_{\mathcal{D}}(\psi'(T'), \psi'(T')) \cong \mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n, \overline{B}) \cong \mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B \oplus F_g(B))^G = 0.$$

If $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B)$ is generated by a short exact sequence whose middle term is $\tau^{-m}P_r$ with $r \in Q_{A,0} \setminus \{n-1\}$. Then by Corollary 4.24, there exists h such that

$$\xi_1 : 0 \longrightarrow B \oplus F_g(B) \xrightarrow{(\lambda_1\ \lambda_2)} \tau^{-h}P_{n-1} \xrightarrow{\nu\mu} \tau^{-k}P_{n-1} \longrightarrow 0$$

is a short exact sequence. Note that

$$\xi_2 : 0 \longrightarrow B \oplus F_g(B) \xrightarrow{\left(\begin{smallmatrix} \iota & 0 \\ 0 & 1 \end{smallmatrix}\right)} E \oplus F_g(B) \xrightarrow{(\nu\ 0)} \tau^{-k}P_{n-1} \longrightarrow 0$$

is a short exact sequence, and ξ_1 is $\mathbb{k}$-linearly independent of ξ_2. Thus $\{\xi_1, \xi_2\}$ is a $\mathbb{k}$-basis of $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B \oplus F_g(B))$. It also can be verified that $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B \oplus F_g(B))^G$ is generated by ξ_1. Hence it follows from Lemma 4.25 that $\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \overline{B})$ (resp. $\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \overline{B})$) is generated by a non-split short exact sequence whose middle term is $\tau^{-h}\overline{P}_{n-1}$ (resp. $\tau^{-h}\overline{P}_n$). Therefore, $\psi'(T')$ is almost pre-rigid. □

Lemma 5.9. *Let $T = \tau^{-k}P_{n-1} \oplus \tau^{-l}P_{n-1}$ be a representation over Q_A with $k, l \in \{0, 1, \dots, n-2\}$. Then T is approximate rigid and $\psi'(T)$ is an almost pre-rigid representation over Q_D.*

Proof. Without loss of generality, we may assume that $l < k$. By Lemma 4.23, there exist $m \in \{0, 1, \dots, n-2\}$ and $r \in \{1, \dots, n-2\}$ such that

$$0 \longrightarrow \tau^{-l}P_{n-1} \longrightarrow \tau^{-m}P_r \oplus \tau^{-m}P_{2n-2-r} \longrightarrow \tau^{-k}P_{n-1} \longrightarrow 0 \tag{5.1}$$

is a short exact sequence. Thus T is approximate rigid. Applying the exact functor ψ to (5.1), we obtain a short exact sequence

$$0 \to \tau^{-l}\overline{P}_{n-1} \oplus \tau^{-l}\overline{P}_n \to \tau^{-m}\overline{P}_r \oplus \tau^{-m}\overline{P}_r \to \tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n \to 0.$$

According to Lemma 4.26, if $(k-l) \bmod 2 = 1$, then

$$\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \tau^{-l}\overline{P}_{n-1}) = 0 = \mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \tau^{-l}\overline{P}_n),$$

$\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \tau^{-l}\overline{P}_n)$ (resp. $\mathrm{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \tau^{-l}\overline{P}_{n-1})$) is generated by

$$0 \longrightarrow \tau^{-l}\overline{P}_n \longrightarrow \tau^{-m}\overline{P}_r \longrightarrow \tau^{-k}\overline{P}_n \longrightarrow 0$$

$$(\text{resp. } 0 \longrightarrow \tau^{-l}\overline{P}_{n-1} \longrightarrow \tau^{-m}\overline{P}_r \longrightarrow \tau^{-k}\overline{P}_{n-1} \longrightarrow 0\ ).$$

Therefore, $\psi'(T) = \tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n \oplus \tau^{-l}\overline{P}_{n-1} \oplus \tau^{-l}\overline{P}_n$ is almost pre-rigid in this case. The proof of the case where $(k-l) \bmod 2 = 0$ follows in a similar manner. □

In the second step, we will show that if T is a F_g-stable representation over Q_A and $\psi'(T)$ is an almost pre-rigid representation over Q_D, then T is approximate rigid. To prove this assertion, we need the following lemma.

Lemma 5.10. *If η_1 and η_2 are two short exact sequences of the form*

$$\eta_1 : 0 \longrightarrow M_1 \xrightarrow{\left(\begin{smallmatrix} \iota_1 \\ \iota_2 \end{smallmatrix}\right)} E \oplus M_2 \xrightarrow{(\nu_1\ \nu_2)} M_3 \longrightarrow 0$$

and

$$\eta_2 :\ 0 \longrightarrow M_2 \xrightarrow{\left(\begin{smallmatrix} \nu_2 \\ \nu_3 \end{smallmatrix}\right)} M_3 \oplus M_4 \xrightarrow{(\mu_1\ \mu_2)} M_5 \longrightarrow 0,$$

where E, M_i are representations over Q and M_i are indecomposable for $i = 1, 2, 3, 4, 5$, then

$$0 \longrightarrow M_1 \xrightarrow{\begin{pmatrix} \iota_1 \\ \nu_3\iota_2 \end{pmatrix}} E \oplus M_4 \xrightarrow{(\mu_1\nu_1 \ -\mu_2)} M_5 \longrightarrow 0$$

is also a short exact sequence.

Proof. Since η_2 is a short exact sequences, then

$$0 \longrightarrow E \oplus M_2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & \nu_3 \\ 0 & \nu_2 \end{pmatrix}} E \oplus M_4 \oplus M_3 \xrightarrow{(0 \ \mu_2 \ \mu_1)} M_5 \longrightarrow 0$$

is also a short exact sequence. Hence

$$0 \longrightarrow E \oplus M_2 \xrightarrow{\begin{pmatrix} \nu_1 & \nu_2 \\ 1 & 0 \\ 0 & \nu_3 \end{pmatrix}} M_3 \oplus E \oplus M_4 \xrightarrow{(\mu_1 \ -\mu_1\nu_1 \ \mu_2)} M_5 \longrightarrow 0$$

is a short exact sequence. Therefore, according to [1, Proposition 3.9.6(1)], the following diagram has exact rows and is commutative.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, "\begin{pmatrix} \iota_1 \\ \iota_2 \end{pmatrix}"] \arrow[d, equal] & E \oplus M_2 \arrow[r, "(\nu_1 \ \nu_2)"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & \nu_3 \end{pmatrix}"] & M_3 \arrow[r] \arrow[d, "\mu_1"] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "\begin{pmatrix} \iota_1 \\ \nu_3\iota_2 \end{pmatrix}"] & E \oplus M_4 \arrow[r, "(\mu_1\nu_1 \ -\mu_2)"] & M_5 \arrow[r] & 0
\end{tikzcd}$$

We have thus proved the statement.

□

Remark 5.11. In Lemma 5.10, the representation E could be decomposable, indecomposable or zero.

Lemma 5.12. *Assume that neither A nor $\tau^{-k}P_{n-1}$ lies to the left of $h(B)$, where $h(B)$ is the line passing through B and perpendicular to the τ-orbit of B in $\Gamma_{\mathcal{A}}$.*

(1) *Let $T = A \oplus F_g(A) \oplus B \oplus F_g(B)$ be a representation over Q_A. If $\psi'(T)$ is an almost pre-rigid representation over Q_D, then T is almost rigid.*

(2) *Let $T' = \tau^{-k}P_{n-1} \oplus B \oplus F_g(B)$ be a representation over Q_A. If $\psi'(T')$ is an almost pre-rigid representation over Q_D, then T' is almost rigid.*

Proof. Similar to the proof of Lemma 5.8, we may assume that

$$\psi'(T) = \overline{A} \oplus \overline{B}, \ \psi'(T') = \tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n \oplus \overline{B}.$$

Because neither A nor $\tau^{-k}P_{n-1}$ lies to the left of $h(B)$, to obtain the statements (1) and (2), we only need to compute $\mathrm{Ext}^1_{\mathcal{A}}(A, B)$, $\mathrm{Ext}^1_{\mathcal{A}}(F_g(A), B)$ and $\mathrm{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B)$.

(1) Since $\psi'(T)$ is almost pre-rigid, according to Definition 5.3 there are two cases.

(i) If $\mathrm{Ext}^1_{\mathcal{D}}(\overline{A}, \overline{B}) = 0$, then by the proof of Proposition 5.2 and Lemma 5.8, $\mathrm{Ext}^1_{\mathcal{A}}(T, T) = 0$.

(ii) If $\mathrm{Ext}^1_{\mathcal{D}}(\overline{A}, \overline{B})$ is generated by a short exact sequence whose middle term is $\tau^{-m}\overline{P}_r$ or $\tau^{-m}\overline{P}_{n-1} \oplus \tau^{-m}\overline{P}_n$ with $r \in \{1, \ldots, n-2\}$ and $m \in \{0, 1, \ldots, n-2\}$. We consider the case where $\overline{E} = \tau^{-m}\overline{P}_r$, with the remainder being similar. Then it follows from Lemma 4.25 that $\mathrm{Ext}^1_{\mathcal{A}}(A \oplus F_g(A), B \oplus F_g(B))^G$ is generated by

$$0 \longrightarrow B \oplus F_g(B) \longrightarrow \tau^{-m}P_r \oplus \tau^{-m}P_{2n-2-r} \longrightarrow A \oplus F_g(A) \longrightarrow 0.$$

By the discussion of Lemma 5.8(1) and Proposition 5.2(1), we have $\operatorname{Ext}^1_{\mathcal{A}}(F_g(A), B) = 0$ and $\operatorname{Ext}^1_{\mathcal{A}}(A, B)$ is generated by a short exact sequence whose middle term is of the form $\tau^{-m}P_r$, or $\operatorname{Ext}^1_{\mathcal{A}}(A, B) = 0$ and $\operatorname{Ext}^1_{\mathcal{A}}(F_g(A), B)$ is generated by a short exact sequence whose middle term is $\tau^{-m}P_r$ or $\tau^{-m}P_{2n-2-r}$. Thus T' is almost rigid.

(2) Since $\psi'(T')$ is almost pre-rigid, according to Definition 5.3 there are two cases.

(i) If $\operatorname{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \overline{B}) = 0$, then $\operatorname{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \overline{B}) = 0$. Thus, $\operatorname{Ext}^1_{\mathcal{A}}(T, T) = 0$ also by the proof of Lemma 5.8.

(ii) If $\operatorname{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_{n-1}, \overline{B})$ is generated by a non-split short exact sequence with middle term $\overline{E}$, then we get from Lemma 5.10 and Remark 5.11 that $\overline{E} = \tau^{-m}\overline{P}_{n-1}$, and $\operatorname{Ext}^1_{\mathcal{D}}(\tau^{-k}\overline{P}_n, \overline{B})$ is generated by a short exact sequence whose middle term is $\tau^{-m}\overline{P}_n$. Thus $\operatorname{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B\oplus F_g(B))^G$ is generated by a short exact sequence with middle term $\tau^{-m}P_{n-1}$. Similar to the proof of Corollary 4.24, we employ [1, Proposition 3.9.6(1)] to obtain that there exists indecomposable representation F over Q_A such that

$$0 \longrightarrow B \oplus F_g(B) \xrightarrow{\begin{pmatrix} \iota_1 & 0 \\ 0 & \iota_2 \end{pmatrix}} F \oplus F_g(F) \xrightarrow{\begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}} \tau^{-k}P^2_{n-1} \longrightarrow 0 \tag{5.2}$$

is also a short exact sequence. By using similar discussion as (1), it follows from (5.2) that $\operatorname{Ext}^1_{\mathcal{A}}(\tau^{-k}P_{n-1}, B)$ is generated by $0 \to B \to F \to \tau^{-k}P_{n-1} \to 0$. Thus T' is almost rigid. □

Thanks to Lemma 5.8, Lemma 5.12 and Lemma 5.9, we obtain the following proposition.

Proposition 5.13. *Let T be a F_g-stable representation over Q_A. We have*

(1) *If T is approximate rigid, then $\psi'(T)$ is an almost pre-rigid representation over Q_D.*
(2) *If $\psi'(T)$ is an almost pre-rigid representation over Q_D, then T is approximate rigid.*

Proof. Now that T is a F_g-stable representation over Q_A, it follows from equation(3.1) that

$$T = \bigoplus_{r\in S_1,\ m\in S_2} (\tau^{-m}P_r \oplus F_g(\tau^{-m}P_r)) \bigoplus_{k\in S_3} \tau^{-k}\overline{P}_{n-1},$$

where S_1 is a subset of $\{1, \dots, n-2\}$, both S_2 and S_3 are the subset of $\{0, 1, \dots, n-2\}$. Thus according to the definition of ψ', we get

$$\psi'(T) = \bigoplus_{m\in S_1,\ r\in S_2} \tau^{-m}\overline{P}_r \bigoplus_{k\in S_3} (\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n).$$

(1) For a F_g-stable approximate rigid representation T, applying repeatedly Lemma 5.8 and Lemma 5.9, we have $\psi'(T)$ is almost pre-rigid.

(2) For a F_g-stable representation T, if $\psi'(T)$ is an almost pre-rigid representation over Q_D, applying repeatedly Lemma 5.12 and Lemma 5.9, we have that T is approximate rigid. □

Corollary 5.14. Let T be a F_g-stable representation over Q_A. We have

(1) If T is maximal approximate rigid, then $\psi'(T)$ is a maximal almost pre-rigid representation over Q_D.
(2) If $\psi'(T)$ is a maximal almost pre-rigid representation over Q_D, then T is maximal approximate rigid.

An immediate consequence of this corollary is that $\psi'\colon \overline{smar}(Q_A) \to \overline{mar}(Q_D)$ is well-defined. Next, we prove that $\psi'\colon \overline{smar}(Q_A) \to \overline{mar}(Q_D)$ is a bijection. In this way, we establish a one-to-one correspondence between the F_g-stable maximal approximate rigid representations over Q_A and the almost pre-rigid representations over Q_D.

Theorem 5.15. *$\psi' \colon \overline{smar}(Q_A) \to \overline{mar}(Q_D)$ is a bijection.*

Proof. For a F_g-stable maximal approximate rigid representation T, we have

$$\psi'(T) = \bigoplus_{r\in S_1,\ m\in S_2} \tau^{-m}\overline{P}_r \bigoplus_{k\in S_2} (\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n),$$

where S_1 is a subset of $\{1, \dots, n-2\}$, both S_2 and S_3 are the subset of $\{0, 1, \dots, n-2\}$. By the definition of ψ', the preimage of $\tau^{-m}\overline{P}_r$ is $\tau^{-m}P_r \oplus \tau^{-m}P_{2n-2-r}$ and the preimage of $\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n$ is $\tau^{-k}P_{n-1}$, which are uniquely determined by $\tau^{-m}\overline{P}_r$ and $\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n$, respectively. Therefore, ψ' is injective.

Combining Proposition 5.5 with the definition of ψ', we get that if $\overline{T}$ is a maximal almost pre-rigid representation over Q_D, then there exists a F_g-stable representation T' over Q_A such that $\psi'(T') = \overline{T}$. Since $\psi'(T)$ is maximal almost pre-rigid, then by Corollary 5.13(2), T' is maximal approximate rigid. This shows that ψ' is surjective and finishes the proof. ☐

6. The geometric realization of maximal almost pre-rigid representations

In this section, we will give a geometric realization of maximal almost pre-rigid representations over Q_D. Recall that in [3], the authors realize a maximal almost rigid representation over Q_A as a triangulation of $P(Q_A)$ via F_A (which is introduced in Section 4.1) as follows.

Theorem 6.1 ([3]). *F_A induces a bijection also denoted by F_A*

$$F_A \colon \{\text{triangulations of } P(Q_A)\} \to mar(Q_A) : \mathcal{T} \mapsto F_A(\mathcal{T}).$$

where $mar(Q_A)$ is the set of all maximal almost rigid representations over Q_A.

Combining Definition 5.6 with Proposition 4.12, we can further extend the bijection F_A in Theorem 6.1. Before that, we define an approximate triangulation on $P(Q_A)$, as follows.

Definition 6.2. An *approximate triangulation* of $P(Q_A)$ is a maximal set of line segments (including the boundary edges) of $P(Q_A)$ satisfying that they only intersect at the center point O and the boundary of $P(Q_A)$.

Proposition 6.3. *F_A induces a bijection also denoted by F_A*

$$F_A \colon \{\text{approximate triangulations of } P(Q_A)\} \to \overline{mar}(Q_A) : \mathcal{T}' \mapsto F_A(\mathcal{T}').$$

Proof. Assume that $F_A(\gamma) = M$ and $F_A(\gamma') = M'$ for $\gamma,\ \gamma' \in \omega$. By [3, Proposition 6.5], γ and γ' do not intersect within $P(Q_A)$ if and only if $M \oplus M'$ is almost rigid. Note that γ and γ' intersect at O if and only if $M = \tau^{-l}P_{n-1}$ and $M' = \tau^{-k}P_{n-1}$ for some $k, l \in \{0, 1, \dots, n-2\}$. Therefore, this conclusion can be deduced from Lemma 4.23 and Definition 5.6. ☐

Remark 6.4. Thanks to Proposition 6.3 and Proposition 4.12(2)(3), it follows that

$$F_A \colon \{\text{approximate triangulations of } P(Q_A) \text{ with central symmetry}\} \to \overline{smar}(Q_A)$$

is also a bijection.

Now we introduce the definition of triangulation of $P(Q_D)$ as follows.

Definition 6.5. A *triangulation* of the punctured $(2n-2)$-gon $P(Q_D)$ is a maximal set of tagged line segments (including the boundary edges) that only intersect at the boundary of $P(Q_D)$ and O.

Remark 6.6. (1) If a triangulation of $P(Q_D)$ contains $\gamma_{-t}^{t,l}$ for some $t = 1, \dots, n-1$, then $\gamma_{-t}^{t,-l}$ is also in this triangulation. Thus if $\mathcal{T}$ is a triangulation of $P(Q_D)$ with d tagged line segments passing through O, then $d = 2d'$ for some $d' \in \{2 \dots, n-1\}$.

(2) Define an additive mapping $f : \{\text{triangulations of } P(Q_D)\} \to \omega$ satisfying

$$\begin{cases} f(\gamma_s^t) = \gamma(s', t') + \gamma(2n-3-t', 2n-3-s'), & \gamma_s^t \in \Omega; \\ f(\gamma_{-t}^{t,1} + \gamma_{-t}^{t,-1}) = \gamma(2n-3-t', t'), & t \in \{1, \dots, n-1\}. \end{cases}$$

It is clear that f induce the following bijection

$f : \{\text{triangulations of } P(Q_D)\} \to \{\text{approximate triangulations of } P(Q_A) \text{ with central symmetry}\}.$

Proposition 6.7. *If $\mathcal{T}$ is a triangulation of $P(Q_D)$ with d tagged line segments passing through O, then $\mathcal{T}$ exactly has $2n-2+d'$ tagged line segments, where $d = 2d'$ and $d' \geq 2$.*

Proof. Without loss of generality, we may assume that $\mathcal{T}$ contains $\gamma_{1-n}^{n-1,\epsilon}$ $(\epsilon = 1, -1)$. Denote by P the "$(n+1)$-polygon" shown in the right of Figure 15, which contains $n+1$ vertices, including O and the vertices $Y_s = (x_s, y_s)$ of $P(Q_D)$ where $x_s \geq 0$. Evidently, there exists a triangulation $\overline{\mathcal{T}}$ of P such that there is a one-to-one correspondence between the diagonals of $\overline{\mathcal{T}}$ and the tagged line segments of $\mathcal{T}$ in the inner of $P(Q_D)$ expect $\gamma_{1-n}^{n-1,\epsilon}$ $(\epsilon = 1, -1)$. It is well-konwn that each triangulation of P has $n-2$ diagonals. Thus, by Remark 6.6, $d = 2d'$ with $d' \geq 2$ and the number of tagged line segments $\mathcal{T}$ containing is

$$2d' + (n-2) - (d'-1) + \frac{2n-2}{2} = 2n-2+d'.$$

$\square$

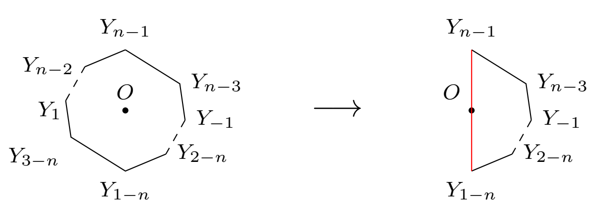


FIGURE 15. The punctured $(2n-2)$-gon $P(Q_D)$(left) and " polygon" P(right)

Corollary 6.8. The number of triangulations of the punctured convex $(2n-2)$-gon $P(Q_D)$ is $C_{n-1} - C_{n-2}$, where C_k denotes Catalan numbers $\frac{1}{k+1}\binom{2k}{k}$.

Proof. Because $P(Q_D)$ exhibits central symmetry, it follows from Definition 6.5 that if $\overline{\mathcal{T}}$ is a triangulation of P containing a diagonal passing through O, then $\overline{\mathcal{T}}$ also uniquely determines a triangulation of $P(Q_D)$. Therefore, by combining the proof process of Proposition 6.7, it can be inferred that the triangulations of $P(Q_D)$ correspond one-to-one with the triangulations of P containing diagonals passing through O. Furthermore, there are $C_{n-1} - C_{n-2}$ triangulations on $P(Q_D)$, where C_k is the Catalan numbers $\frac{1}{k+1}\binom{2k}{k}$.

$\square$

Now we realize the maximal almost pre-rigid representations over Q_D as triangulations of the punctured polygon $P(Q_D)$.

Theorem 6.9. *F_D induces a bijection also denoted by F_D*

$$F_D\colon \{\text{triangulations of } P(Q_D)\} \to \overline{mar}(Q_D).$$

Proof. It follows from Remark 6.6(2),Theorem 5.15 and Remark 6.4 that

$$\psi' \circ F_A \circ f\colon \{\text{triangulations of } P(Q_D)\} \to \overline{mar}(Q_D) : \mathcal{T} \mapsto \psi' \circ F_A \circ f(\mathcal{T}) \tag{6.1}$$

is a bijection. According to the definition of F_D, we can rewrite (6.1) as

$$F_D : \{\text{triangulations of } P(Q_D)\} \to \overline{mar}(Q_D) : \mathcal{T} \mapsto F_D(\mathcal{T}).$$

$\square$

Corollary 6.10. Let $\overline{T}$ be a maximal almost pre-rigid representation over Q_D. Then $\overline{T}$ is of the form

$$\overline{T} = \bigoplus_{i=1}^{2n-2-d} \tau^{-m_i}\overline{P}_{r_i} \bigoplus_{j=1}^{d}(\tau^{-k_j}\overline{P}_{n-1} \oplus \tau^{-k_j}\overline{P}_n),$$

where $d \in \{2, \ldots, n-1\}$, $m_i, k_j \in \{0, 1, \ldots, n-2\}$, $r_i \in \{1, \ldots, n-2\}$ for all $1 \le i \le 2n-2-d$ and $1 \le j \le d$. Furthermore, the number of maximal almost pre-rigid representation over Q_D is $C_{n-1} - C_{n-2}$, where C_k denotes the Catalan number $\frac{1}{k+1}\binom{2k}{k}$.

Proof. It is follows from Proposition 6.7, Theorem 6.9 and Corollary 6.8 immediately. $\square$

7. A property of maximal almost pre-rigid representations

In this chapter, we will provide an application of maximal almost pre-rigid representations in tilting theory based on their general form given in the previous chapter.

Recall from [3] that the authors established a link between Q_A and a type $\mathbb{A}$ quiver $Q_{\overline{A}}$ containing $4n-7$ vertices and possessing directional symmetry. Precisely, through replacing each arrow $i \to (i+1)$ in Q_A by a path of length two $i \to \frac{2i+1}{2} \to (i+1)$ and replacing each arrow $i \leftarrow (i+1)$ in Q_A by a path of length two $i \leftarrow \frac{2i+1}{2} \leftarrow (i+1)$, one can obtain $Q_{\overline{A}}$.

Similarly as [3], for the quiver Q_D, we associate Q_D with a type $\mathbb{D}$ quiver $Q_{\overline{D}}$ containing $2n-2$ vertices as follows:

- replacing arrow $i \xrightarrow{\beta_i} (i+1)$ in Q_D by a path of length two $i \xrightarrow{\beta(i)} \frac{2i+1}{2} \xrightarrow{\beta(\frac{2i+1}{2})} (i+1)$, for $i = 1, \ldots, n-3$;
- replacing arrow $i \xleftarrow{\beta_i} (i+1)$ in Q_D by a path of length two $i \xleftarrow{\beta(i)} \frac{2i+1}{2} \xleftarrow{\beta(\frac{2i+1}{2})} (i+1)$, for $i = 1, \ldots, n-3$;
- replacing two arrows $(n-1) \xrightarrow{\beta_{n-2}} (n-2)$ and $n \xrightarrow{\beta_{n-1}} (n-2)$ in Q_D by three arrows $(n-1) \xrightarrow{\beta(\frac{2n-3}{2})} \frac{2n-3}{2}$, $n \xrightarrow{\beta(\frac{2n-1}{2})} \frac{2n-3}{2}$ and $\frac{2n-3}{2} \xrightarrow{\beta(n-2)} (n-2)$;
- replacing two arrows $(n-1) \xleftarrow{\beta_{n-2}} (n-2)$ and $n \xleftarrow{\beta_{n-1}} (n-2)$ in Q_D by three arrows $(n-1) \xleftarrow{\beta(\frac{2n-3}{2})} \frac{2n-3}{2}$, $n \xleftarrow{\beta(\frac{2n-1}{2})} \frac{2n-3}{2}$ and $\frac{2n-3}{2} \xleftarrow{\beta(n-2)} (n-2)$,

we can obtain the quiver $Q_{\overline{D}}$. See Figure 16 as an example.

$$Q_{D_4} = \begin{tikzcd} 3 \arrow[dr, "\beta_2"] & & \\ & 2 & 1 \arrow[l, "\beta_1"'] \\ 4 \arrow[ur, "\beta_3"'] & & \end{tikzcd} \qquad Q_{\overline{D_4}} = \begin{tikzcd} 3 \arrow[dr, "\beta(\frac{5}{2})"] & & & & \\ & \frac{5}{2} \arrow[r, "\beta(2)"] & 2 & \frac{3}{2} \arrow[l, "\beta(\frac{3}{2})"'] & 1 \arrow[l, "\beta(1)"'] \\ 4 \arrow[ur, "\beta(\frac{7}{2})"'] & & & & \end{tikzcd}$$

FIGURE 16. From the quiver Q_{D_4} to the quiver $Q_{\overline{D_4}}$

Remark 7.1. If the group $\overline{G} = \{\overline{e}, \overline{g}\}$ acts on $\Bbbk Q_{\overline{A}}$ via $\overline{g}(i) = 2n-2-i$ and $\overline{g}(\alpha_i) = \alpha_{2n-3-i}$ for each $i \in Q_{\overline{A},\, 0}$, then the skew group algebra $(\Bbbk Q_{\overline{A}})\overline{G}$ is Morita equivalent to $\Bbbk Q_{\overline{D}}$.

By investigating the representations over the quiver Q_D and $Q_{\overline{D}}$, we find an additive functor $G_D\colon \mathrm{rep}_{\Bbbk} Q_D \to \mathrm{rep}_{\Bbbk} Q_{\overline{D}}$, which is determined by the following:

- On objects. For each indecomposable object $X = (X_i, \varphi_{\beta_i})_{i \in Q_{D,0}, \beta_i \in Q_{D,1}}$ in $\mathrm{rep}_{\Bbbk} Q_D$. Define $G_D(X) = (\widetilde{X_i}, \widetilde{\varphi}_{\beta(i)})$, where $\widetilde{X_i}$ is given as follows: for $i \in Q_{D,0}$, $\widetilde{X_i} = X_i$; for $i \in \{1, \dots, n-3\}$,
$$\widetilde{X}_{\frac{2i+1}{2}} = \begin{cases} \Bbbk^2 & \text{if } \ X_i = \Bbbk^2 = X_{i+1}, \\ \Bbbk & \text{if } \ X_i = \Bbbk = X_{i+1}, \\ \Bbbk & \text{if } \ X_i = \Bbbk \text{ and } X_{i+1} = \Bbbk^2, \\ 0 & \text{otherwise;} \end{cases}$$
for $i = \frac{2n-3}{2}$,
$$\widetilde{X}_{\frac{2n-3}{2}} = \begin{cases} \Bbbk^2 & \text{if } \ X_n = \Bbbk = X_{n-1} \text{ and } X_{n-2} = \Bbbk^2, \\ \Bbbk & \text{if } X_n = \Bbbk = X_{n-1} \text{ and } X_{n-2} = \Bbbk, \\ \Bbbk & \text{if } X_n \oplus X_{n-1} = \Bbbk \text{ and } X_{n-2} = \Bbbk, \\ 0 & \text{otherwise.} \end{cases}$$
And $\widetilde{\varphi}_{\beta(i)}$ is given as follows: for $i \in \{1, \dots, n-2\}$, $\widetilde{\varphi}_{\beta(\frac{2i+1}{2})} = \varphi_{\beta_i}$; for $i = n-2$,
$$\widetilde{\varphi}_{\beta(n-2)} = \begin{cases} 0 & \text{if } \ \varphi_{\beta_{n-2}} = 0 = \varphi_{\beta_{n-1}}, \\ 1 & \text{otherwise;} \end{cases}$$
for $i \in \{1, \dots, n-3\}$,
$$\widetilde{\varphi}_{\beta(i)} = \begin{cases} 0 & \text{if } \ \varphi_{\beta_i} = 0, \\ 1 & \text{otherwise.} \end{cases}$$
- On morphisms. Assume $\zeta\colon X \to Y$ is a morphism between indecomposable objects $X = (X_i, \varphi_{\beta_i})$ and $Y = (Y_i, \varphi'_{\beta_i})$ in $\mathcal{D}$. Define $G_D(f) = \widetilde{\zeta}\colon \widetilde{X} \to \widetilde{Y}$, where $\widetilde{\zeta_i}$ is given as follows: for $i \in Q_{D,0}$, $\widetilde{\zeta_i} = \zeta_i$; for $i \in \{1, \dots, n-3\}$,
$$\widetilde{\zeta}_{\frac{2i+1}{2}} = \begin{cases} \zeta_i & \text{if } \ \zeta_{i+1} \neq 0, \\ 0 & \text{if } \ \zeta_{i+1} = 0; \end{cases}$$
and for $i = \frac{2n-3}{2}$,
$$\widetilde{\zeta}_{\frac{2n-3}{2}} = \begin{cases} \zeta_{n-2} & \text{if } \ \zeta_n = 1 \text{ or } \zeta_{n-1} = 1, \\ 0 & \text{if } \ \zeta_n = 0 = \zeta_{n-1}. \end{cases}$$

Remark 7.2. (1) Let $\widetilde{P}_i$ be the projective module corresponding to $i \in Q_{\overline{D},\,0}$. Then

$$G_D(\tau^{-k}\overline{P}_i) = \tau^{-2k}\widetilde{P}_i \text{ and } G_D(\tau^{-k}\overline{P}_{n-1} \oplus \tau^{-k}\overline{P}_n) = \tau^{-2k}\widetilde{P}_{n-1} \oplus \tau^{-2k}\widetilde{P}_n,$$

where $i \in \{1, \dots, n-2\}$ and $k \in \{0, \dots, n-2\}$. An irreducible morphism $\zeta\colon X \to Y$ in $\mathrm{rep}_{\Bbbk}Q_D$ is mapped under G_D to the composition of two irreducible morphisms between the indecomposable objects $G_D(X)$ and $G_D(Y)$ in $\mathrm{rep}_{\Bbbk}Q_{\overline{D}}$.

(2) Denote by $\overline{\psi}$ the exact functor $(\Bbbk Q_{\overline{A}})\overline{G} \otimes_{\Bbbk Q_{\overline{A}}} - : \Bbbk Q_{\overline{A}}\text{-mod} \to \Bbbk Q_{\overline{D}}\text{-mod}$. It is easy to check that $\overline{\psi} \cdot G_A = G_D \cdot \psi$, where ψ is the exact functor $(\Bbbk Q_A)G \otimes_{\Bbbk Q_A} - : \mathcal{A} \to \mathcal{D}$ and G_A is the functor from $\mathrm{rep}_{\Bbbk}Q_A$ to $\mathrm{rep}_{\Bbbk}Q_{\overline{A}}$ defined in [3, Section 7].

For a representation M over some quiver, denote by $|M|$ the number of indecomposable direct summands of M. The following theorem shows a relationship between the image of a maximal almost pre-rigid representation over Q_D under G_D and a tilting module over $\Bbbk Q_{\overline{D}}$.

Theorem 7.3. *Let $\overline{T}$ be a maximal almost pre-rigid representation over Q_D. If $\widetilde{T}$ is a maximal rigid summand of $G_D(\overline{T})$, then $|\widetilde{T}|$ is equal to $2n-2$ or $2n-d$. More precisely, if $|\widetilde{T}| = 2n-2$, then $\widetilde{T}$ is a tilting module over $\Bbbk Q_{\overline{D}}$.*

Proof. By Corollary 6.10, $\overline{T}$ is of the form

$$\overline{T} = \overline{N} \oplus \overline{T'} = \overline{N} \oplus \tau^{-k_1}\overline{P}_{n-1} \oplus \cdots \oplus \tau^{-k_d}\overline{P}_{n-1} \oplus \tau^{-k_1}\overline{P}_n \oplus \cdots \oplus \tau^{-k_d}\overline{P}_n,$$

where $|\overline{N}| = 2n-2-d$ and $k_i \in \{0, 1, \dots, n-2\}$ for all $i \in \{1, \dots, d\}$. Assume that $N \in \mathcal{A}$ satisfies $\psi(N) = \overline{N}$. For ease of notations, we denote $\overline{\mathcal{A}}$ by the category $\Bbbk Q_{\overline{A}}$-mod and $\overline{\mathcal{D}}$ by the category $\Bbbk Q_{\overline{D}}$-mod in the following proof.

By [3, Proposition 6.5], we know that $N \oplus \tau^{-k_i}P_{n-1}$ is an almost rigid representation over Q_A for all $i \in \{1, \dots, d\}$. Then by [3, Theorem 7.3(1)],

$$\mathrm{Ext}^1_{\overline{\mathcal{A}}}(G_A(N \oplus \tau^{-k_i}P_{n-1}), G_A(N \oplus \tau^{-k_i}P_{n-1})) = 0.$$

Denote $\tau^{-k_i}\overline{P}_{n-1} \oplus \tau^{-k_i}\overline{P}_n$ by $\overline{M}_i$ for each $i \in \{1, \dots, d\}$. Therefore, thanks to the exactness of $\overline{\psi}$ and Remark 7.2, we obtain that

$$\begin{aligned} & \mathrm{Ext}^1_{\overline{\mathcal{D}}}(G_D(\overline{N} \oplus \overline{M}_i), G_D(\overline{N} \oplus \overline{M}_i)) \\ = \; & \mathrm{Ext}^1_{\overline{\mathcal{D}}}(G_D\psi(N \oplus \tau^{-k_i}P_{n-1}), G_D\psi(N \oplus \tau^{-k_i}P_{n-1})) \\ = \; & \mathrm{Ext}^1_{\overline{\mathcal{D}}}(\overline{\psi}G_A(N \oplus \tau^{-k_i}P_{n-1}), \overline{\psi}G_A(N \oplus \tau^{-k_i}P_{n-1})) \\ = \; & 0. \end{aligned} \tag{7.1}$$

for $i \in \{1, \dots, d\}$. This implies that each maximal rigid summand of $G_D(\overline{T})$ is of the form

$$G_D(\overline{N}) \oplus G_D(\overline{M}),$$

where $G_D(\overline{M})$ is a rigid summand of $G_D(\overline{T'})$. We claim that $G_D(\overline{M})$ is a maximal rigid summand of $G_D(\overline{T'})$. In fact, if not, then by (7.1), there exists $\widetilde{M'}$ which is a direct summand of $G_D(\overline{T'})$ but not a direct summand of $G_D(\overline{M})$ such that

$$\mathrm{Ext}^1_{\overline{\mathcal{D}}}(G_D(\overline{N} \oplus \overline{M}) \oplus \widetilde{M'}, G_D(\overline{N} \oplus \overline{M}) \oplus \widetilde{M'}) = 0.$$

Contradict to the maximality of $G_D(\overline{N}) \oplus G_D(\overline{M})$.

Note that if $\widetilde{T'}$ is a maximal rigid summand of $G_D(\overline{T'})$, then $\widetilde{T'}$ is of the form

$$\bigoplus_{i\in S_1} G_D(\tau^{-k_i}\overline{P}_{n-1}) \bigoplus_{j\in S_2} G_D(\tau^{-k_j}\overline{P}_n),$$

where S_1 and S_2 are subsets of $\{1,\ldots,d\}$, and $\widetilde{T'}$ satisfies $\mathrm{Ext}^1_{\overline{\mathcal{D}}}(\widetilde{T'},\widetilde{T'}) = 0$. Again by (7.1),

$$\mathrm{Ext}^1_{\overline{\mathcal{D}}}(G_D(\overline{M}_i), G_D(\overline{M}_i)) = 0,\ \forall i \in \{1,\ldots,d\}.$$

Hence we can describe the precise direct sum decomposition of $\widetilde{T'}$ according to whether there exists $i \in \{1,\ldots,d\}$ such that $G_D(\overline{M}_i)$ is a direct summand of $\widetilde{T'}$, as follows.

If $G_D(\overline{M}_i)$ is a direct summand of $\widetilde{T'}$, then it follows from Lemma 4.26 that

$$\mathrm{Ext}^1_{\overline{\mathcal{D}}}(\tau^{-2k_i}\widetilde{P}_{n-1}\oplus\tau^{-2k_j}\widetilde{P}_n, \tau^{-2k_i}\widetilde{P}_{n-1}\oplus\tau^{-2k_j}\widetilde{P}_n) \neq 0,\ \forall j \in \{1,\ldots,d\}\setminus\{i\}.$$

Thus for all $j \in \{1,\ldots,d\}$ with $j \neq i$ and $l \in \{n-1,n\}$,

$$\mathrm{Ext}^1_{\overline{\mathcal{D}}}(G_D(\overline{M}_i\oplus\tau^{-k_j}\overline{P}_l), G_D(\overline{M}_i\oplus\tau^{-k_j}\overline{P}_l)) \neq 0$$

Therefore, $\widetilde{T'} = G_D(\overline{M}_i)$ in this case.

If none of the $G_D(\overline{M}_i)$ for $i \in \{1,\ldots,d\}$ are direct summands of $\widetilde{T'}$, then $\widetilde{T'}$ has the form

$$\bigoplus_{i\in S_1}\tau^{-2k_i}\widetilde{P}_{n-1}\bigoplus_{j\in S_2}\tau^{-2k_j}\widetilde{P}_n,$$

where S_1 and S_2 are subsets of $\{1,\ldots,d\}$ with $S_1\cap S_2 = \emptyset$. We claim that $S_1 = \emptyset$ or $S_2 = \emptyset$. In fact, if otherwise, then there exist $i \in S_1$ and $i' \in S_2$. Similarly, by Lemma 4.26

$$\mathrm{Ext}^1_{\overline{\mathcal{D}}}(\tau^{-2k_i}\widetilde{P}_{n-1}\oplus\tau^{-2k_{i'}}\widetilde{P}_n, \tau^{-2k_i}\widetilde{P}_{n-1}\oplus\tau^{-2k_{i'}}\widetilde{P}_n) \neq 0,$$

which is a contradiction. Owing to the maximality of $\widetilde{T'}$, $\widetilde{T'}$ has the form

$$\bigoplus_{j=1}^{d}\tau^{-2k_j}\widetilde{P}_{n-1} \quad\text{or}\quad \bigoplus_{j=1}^{d}\tau^{-2k_j}\widetilde{P}_n.$$

In conclusion, if $\widetilde{T}$ is a maximal rigid summand of $G_D(\overline{T})$, then $|\widetilde{T}| = 2n-2$ or $|\widetilde{T}| = 2n-d$. Since $\Bbbk Q_{\overline{\mathcal{D}}}$ is a hereditary algebra and $Q_{\overline{\mathcal{D}}}$ contains $2n-2$ vertices, we get that if $|\widetilde{T}| = 2n-2$, then $\widetilde{T}$ is a tilting module over $\Bbbk Q_{\overline{\mathcal{D}}}$. □

Therefore, according to Theorem 7.3 and its proof, we conclude that a maximal almost pre-rigid representation $\overline{T}$ over Q_D can determine at least two tilting objects over $Q_{\overline{\mathcal{D}}}$. In fact, $\overline{T}$ can determine the following two tilting objects over $\Bbbk Q_{\overline{\mathcal{D}}}$:

$$G_D(\overline{N})\bigoplus_{j=1}^{d}\tau^{-2k_j}\widetilde{P}_{n-1} \quad\text{and}\quad G_D(\overline{N})\bigoplus_{j=1}^{d}\tau^{-2k_j}\widetilde{P}_n.$$

Especially, in the case of $d = 2$, $\overline{T}$ can additionally determine two tilting objects over $\Bbbk Q_{\overline{\mathcal{D}}}$:

$$G_D(\overline{N}\oplus\tau^{-k_1}\overline{P}_{n-1}\oplus\tau^{-k_1}\overline{P}_{n-1}) \quad\text{and}\quad G_D(\overline{N}\oplus\tau^{-k_1}\overline{P}_n\oplus\tau^{-k_1}\overline{P}_n).$$

Acknowledgements. The authors would like to thank Shiquan Ruan for helpful discussions. This work was partially supported by the National Natural Science Foundation of China (Grant Nos. 12371040, 12131018 and 12161141001).

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