

Non-singular flows with twisted saddle orbit on orientable 3-manifolds

Olga Pochinka, Danila Shubin*

National Research University “Higher School of Economics”

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Abstract

In this paper we consider non-singular Morse-Smale flows on closed orientable 3-manifolds, under the assumption that among the periodic orbits of the flow there is only one saddle orbit and it is twisted. It is found that any manifold admitting such flows is either a lens space, or a connected sum of a lens space with a projective space, or Seifert manifolds with base sphere and three special layers. A complete topological classification of the described flows is obtained and the number of their equivalence classes on each admissible manifold is calculated.

1 Introduction and formulation of results

In this paper we consider, *NMS-flows* f^t , i.e., *not-singular* (without fixed points) Morse-Smale flows defined on closed connected orientable 3-manifolds M^3 . The non-wandering set of such a flow consists of a finite number of periodic hyperbolic orbits. In the neighborhood of a hyperbolic periodic orbit $\mathcal{O}$, the flow admits a simple description (up to topological equivalence), namely, there exists its tubular neighborhood $V_{\mathcal{O}}$ homeomorphic to the solid torus $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$, in which the flow is topologically equivalent to the suspension over some linear diffeomorphism of the plane given by a matrix with positive determinant and real eigenvalues modulo different from unity (see e.g. [6]). If both eigenvalues are modulo greater than (less than) one, then the corresponding periodic orbit is *repelling* (*attracting*), otherwise it is *saddle*. A saddle orbit is called *twisted* if both eigenvalues are negative and *is non-twisted* otherwise.

The dynamics of NMS-flows has been studied in a number of papers: M. Wada [16] showed that the link consisting of periodic orbits of the NMS-flow on the sphere is obtained from the Hopf link by applying a finite number of certain

*Corresponding author. E-mail: schub.danil@yandex.ru

operations (Wada operations); J. Franks [2] described the flows on the three-dimensional sphere using the Lyapunov graph; It is known from the work of Azimov [1] that the ambient manifold in this case is a union of circular handles. However, in the case of a small number of periodic orbits, the topology of the manifold can be substantially refined. For example, NMS-flows with exactly two periodic orbits, attracting and repelling (such a pair of periodic orbits must be contained by any NMS-flow), admit only lens spaces. Moreover, in [9] it is proved that every lens space admits exactly two equivalence classes of such flows, except for the 3-sphere $\mathbb{S}^3$ and the projective space $\mathbb{R}P^3$, on which the equivalence class is one. In the case of a larger number of orbits, the topology of the ambient manifold is considerably richer: in the paper [13] we construct NMS-flows with three periodic orbits on small Seifert manifolds. The topology of compact orientable 3-manifolds admitting NMS-flows was studied in more detail by Morgan in [8].

In the present paper we obtain an exhaustive classification for the set $G_1^-(M^3)$ of NMS flows $f^t: M^3 \rightarrow M^3$ with a single saddle orbit, under the assumption that it is twisted. Note that such information cannot be obtained from the general classification of Morse-Smale flows on 3-manifolds obtained in the works of Umansky [15] and Prishlyak [10].

Since the ambient manifold of the flow $f^t \in G_1^-(M^3)$ is the union of stable (unstable) manifolds of all its periodic orbits [14], the flow must have at least one attracting and at least one repelling orbit. In the present paper the following fact is established.

Lemma 1. *The non-wandering set of any flow $f^t \in G_1^-(M^3)$ consists of exactly three periodic orbits S, A, R , saddle, attracting and repelling, respectively.*

Due to the equivalence of the flow f^t in the neighborhood of a periodic orbit to a suspension over a linear diffeomorphism, the unstable and stable manifolds of these orbits have the following topology:

- $W_S^u \cong W_S^s \cong \mathbb{R} \tilde{\times} \mathbb{S}^1$ (open Mobius band);
- $W_A^s \cong W_R^u \cong \mathbb{R}^2 \times \mathbb{S}^1$;
- $W_A^u \cong W_R^s \cong \mathbb{S}^1$.

A consequence of the topology of invariant manifolds of periodic orbits and the Lemma 1 is the following representation of the M^3 ambient manifold.

Lemma 2. *The ambient manifold M^3 of any flow $f^t \in G_1^-(M^3)$ is represented as a union of three solid tori*

$$M^3 = V_A \cup V_S \cup V_R$$

with non-intersecting interior, which are tubular neighborhoods of the orbits A, S, R , respectively, with the following properties:

- *torus $T_S = \partial V_S$ is the union of compact tubular neighborhoods K_A, K_R of nodes $\gamma_A = W_S^u \cap T_S, \gamma_R = W_S^s \cap T_S$, respectively, such that $K_A \cap K_R = \partial K_A \cap \partial K_R$ (see Fig. 1);*

- torus $T_A = \partial V_A$ is the union of the annulus K_A and the compact surface $K = T_A \setminus \text{int } K_A$,
- torus $T_R = \partial V_R$ is the union of the annulus K_R and the compact surface $K = T_R \setminus \text{int } K_R$.

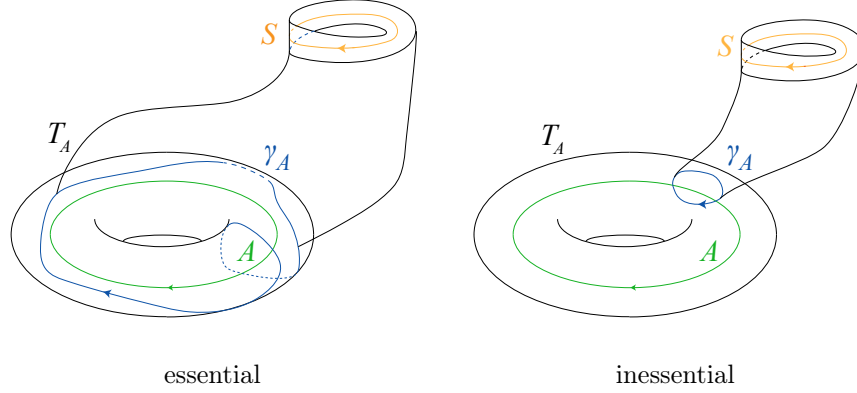


Figure 1: Knot γ_A

For $\mathcal{O} \in \{A, S, R\}$ we choose *parallel* $L_{\mathcal{O}}$ on the torus $T_{\mathcal{O}}$ (a curve homologous in $V_{\mathcal{O}}$ to the orbit of $\mathcal{O}$) and *meridian* $M_{\mathcal{O}}$ (a curve, homotopic to zero on $V_{\mathcal{O}}$ and essential on $T_{\mathcal{O}}$) such that the ordered pair of curves $L_{\mathcal{O}}, M_{\mathcal{O}}$ defines the outer side of the solid torus $V_{\mathcal{O}}$.

Let γ_S be the connected component of the set ∂K oriented coherently with the saddle orbit S . By virtue of the equivalence of the flow $f^t|_{V_S}$ to the suspension, the meridian M_S can be chosen such that γ_S intersects the meridian M_S at exactly two points. Then the generators L_S, M_S can be chosen so that with respect to them the node γ_S has homotopic type

$$\langle \gamma_S \rangle = \langle l_S, m_S \rangle = \langle 2, 1 \rangle.$$

We orient the nodes γ_R, γ_A consistent with the node γ_S . Let us write down the homotopy type of the node γ_R with respect to the L_R, M_R

$$\langle \gamma_R \rangle = \langle l_R, m_R \rangle$$

and the homotopy type of the node γ_A with respect to the L_A, M_A

$$\langle \gamma_A \rangle = \langle l_A, m_A \rangle.$$

Since $T_R \setminus \gamma_R = T_A \setminus \gamma_A$, then

$$(l_R, m_R) = (0, 0) \iff (l_A, m_A) = (0, 0).$$

If $(l_R, m_R) = (0, 0)$, then let us write the homotopy type of the meridian $M_R \subset K$ with respect to the formers L_A, M_A

$$\langle M_R \rangle = \langle p_A, q_A \rangle.$$

If $(l_R, m_R) \neq (0, 0)$, then choose a node $\sigma_S \subset T_S$ such that

$$\langle \sigma_S \rangle = \langle 1, 1 \rangle$$

. and σ_S intersects with each component of the ∂K connectivity at exactly one point (this can be done since the intersection index of the nodes γ_S and σ_S is 1).

Let us choose nodes $\sigma_R \subset T_R, \sigma_A \subset T_A$ coinciding with each other on the ring K and such that $\sigma_S = (\sigma_R \cup \sigma_A) \cap T_S$. Let us write their homotopy types with respect to generators

$$\langle \sigma_R \rangle = \langle b_R, c_R \rangle, \langle \sigma_A \rangle = \langle b_A, c_A \rangle.$$

Definition 1. By the flow $f^t \in G_1^-(M^3)$, we define set

$$C_{f^t} = (l_1, b_1, l_2, b_2)$$

as follows:

- $(l_1, b_1, l_2, b_2) = (l_R, b_R, l_A, b_A)$ if $(l_R, m_R) \neq (0, 0)$;
- $(l_1, l_2, b_1, b_2) = (0, 2, p_A, q_A)$ if $(l_R, m_R) = (0, 0)$ and the 2-disk bounded by knot γ_R remains on the left when moving along the knot;
- $(l_1, b_1, l_2, b_2) = (0, -2, -p_A, -q_A)$ if $(l_R, m_R) = (0, 0)$ and the 2-disk bounded by the knot γ_R remains on the right when moving along the knot.

Note that the set C_{f^t} of the flow $f^t \in G_1^-(M^3)$ is admissible in the sense of the following definition.

Definition 2. The set of integers $C = (l_1, b_1, l_2, b_2)$ is called admissible if

- $(l_1, b_1) = (0, \pm 2)$ or $\gcd(l_1, b_1) = 1$;
- $\gcd(l_2, b_2) = 1$.

Definition 3. We call the admissible sets $C = (l_1, b_1, l_2, b_2), C' = (l'_1, b'_1, l'_2, b'_2)$ consistent ($C \sim C'$) if:

- $l_i = l'_i, i = 1, 2,$

and exists $\delta \in \{-1, 1\}$ such that

- $b_i \equiv \delta b'_i \pmod{l_i}$;
- $l_1 l_2 (2l_2(b_1 - \delta b'_1) + 2l_1(b_2 - \delta b'_2) + l_1 l_2 (1 - \delta)) = 0$.

In the present work, the following classification result is established.

Theorem 1. *The flows $f^t, f^{t'} \in G_3^-(M^3)$ are topologically equivalent if and only if $C_{f^t} \sim C_{f^{t'}}$. Moreover, for any admissible set C there exists a flow $f^t \in G_3^-(M^3)$ such that $C \sim C_{f^t}$.*

We also managed to construct a correspondence between invariants and ambient manifolds of flows of the considered class.

Theorem 2. *Flows of class $G_1^-(M^3)$ admit all lens spaces $L_{p,q}$, all connected sums of the form $L_{p,q} \# \mathbb{R}P^3$ and all Seifert manifolds of the form $M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1))$. More precisely, let the flow $f^t \in G_1^-(M^3)$ have the invariant $C_{f^t} = (l_1, b_1, l_2, b_2)$. Then*

- 1) If $(|l_1| - 1)(|l_2| - 1) = 0$, then $M^3 \cong L_{p,q}$, thus:
 - i) if $l_1 l_2 = 0$, then $M^3 \cong \mathbb{R}P^3$;
 - ii) if $C_{f^t} = (\pm 1, b_1, l_2, b_2)$, $l_2 \neq 0$, then $M^3 \cong L_{l_2 - 2b_2, b_2}$;
 - iii) if $C_{f^t} = (l_1, b_1, \pm 1, b_2)$, $l_1 \neq 0$, then $M^3 \cong L_{l_1 - 2b_1, b_1}$;
- 2) If $l_1 l_2 = 0$ and $(|l_1| - 1)(|l_2| - 1) \neq 0$, then $M^3 \cong L_{p,q} \# \mathbb{R}P^3$, thus:
 - i) if $C_{f^t} = (0, b_1, l_2, b_2)$, then $M^3 \cong L_{l_2, b_2} \# \mathbb{R}P^3$;
 - ii) if $C_{f^t} = (l_1, b_1, 0, \pm 1)$, $l_1 \neq 0$, then $M^3 \cong L_{l_1, b_1} \# \mathbb{R}P^3$.
- 3) If $C_{f^t} = (l_1, b_1, l_2, b_2)$, $|l_1| > 1$, $|l_2| > 1$, then $M^3 \cong M(\mathbb{S}^2, (l_1, b_1), (l_2, b_2), (2, 1))$.

Due to the fact that the topological equivalence class and the topology of a manifold are defined using the same invariant, it becomes possible to compute the number of topological equivalence classes on each admissible manifold. For this purpose, for any pair p, q of coprime integers, let $\bar{p} = |p|$ and denote by $\bar{q}$ – the smallest non-negative of the numbers q' satisfying the condition $q \equiv \pm q' \pmod{p}$, and by $\tilde{q}$ – the smallest non-negative of the numbers q' satisfying the condition $qq' \equiv \pm 1 \pmod{p}$.

Theorem 3. *The set $G_1^-(L_{p,q})$, $|p| \neq 2$ decomposes into a countable number of equivalence classes, whereas the sets $G_1^-(\mathbb{R}P^3)$, $G_1^-(L_{p,q} \# \mathbb{R}P^3)$, $G_1^-(M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1)))$ consist of a finite number of classes. Namely,*

- 1) equivalence classes of the set $G_1^-(L_{p,q})$ depending on p, q are represented by flows with the following invariants:

a) $|p| > 2, q^2 \not\equiv \pm 1 \pmod{p}, n, k \in \mathbb{Z}$

$$\begin{aligned} & (\pm 1, n, \bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}), \\ & (\pm 1, n, \bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}), \\ & (\pm 1, n, \bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}), \\ & (\pm 1, n, \bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}), \\ & (\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n), \\ & (\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n), \\ & (\bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(\tilde{q} + k\bar{p}), \tilde{q} + k\bar{p}, \pm 1, n), \\ & (\bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(-\tilde{q} + k\bar{p}), -\tilde{q} + k\bar{p}, \pm 1, n); \end{aligned}$$

b) $|p| > 2, q^2 \equiv \pm 1 \pmod{p}, n, k \in \mathbb{Z}$

$$\begin{aligned} & (\pm 1, n, \bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}), \\ & (\pm 1, n, \bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}), (\pm 1, n, -\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}), \\ & (\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}, \pm 1, n), \\ & (\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n), (-\bar{p} + 2(-\bar{q} + k\bar{p}), -\bar{q} + k\bar{p}, \pm 1, n); \end{aligned}$$

c) $p = 0, n \in \mathbb{Z}$

$$(\pm 1, n, 2, 1), (\pm 1, n, -2, -1), (2, 1, \pm 1, n), (-2, -1, \pm 1, n);$$

d) $|p| = 1, n, k \in \mathbb{Z}$

$$(\pm 1, n, 1 + 2k, k), (1 + 2k, k, \pm 1, n);$$

e) $|p| = 2$

$$(\pm 1, 0, 0, 1), (0, 1, \pm 1, 0), (0, 2, \pm 1, 0);$$

2) equivalence classes of the set $G_1^-(L_{p,q} \# \mathbb{R}P^3)$ depending on p, q are represented by flows with the following invariants:

a) $|p| > 2, q^2 \not\equiv \pm 1 \pmod{p}$

$$\begin{aligned} & (0, 2, \bar{p}, \pm \bar{q}), (0, -2, \bar{p}, \pm \bar{q}), (0, -2, \bar{p}, \pm \bar{q}), (0, -2, -\bar{p}, \pm \bar{q}), \\ & (0, 2, \bar{p}, \pm \bar{q}), (0, 2, -\bar{p}, \pm \bar{q}), (0, -2, \bar{p}, \pm \bar{q}), (0, -2, -\bar{p}, \pm \bar{q}), \\ & (0, 1, \bar{p}, \pm \bar{q}), (0, 1, -\bar{p}, \pm \bar{q}), (0, -1, \bar{p}, \pm \bar{q}), (0, -1, -\bar{p}, \pm \bar{q}), \\ & (0, 1, \bar{p}, \pm \bar{q}), (0, 1, -\bar{p}, \pm \bar{q}), (0, -1, \bar{p}, \pm \bar{q}), (0, -1, -\bar{p}, \pm \bar{q}), \\ & (\bar{p}, \pm \bar{q}, 0, 1), (-\bar{p}, \pm \bar{q}, 0, 1), (\bar{p}, \pm \bar{q}, 0, -1), (-\bar{p}, \pm \bar{q}, 0, -1), \\ & (\bar{p}, \pm \bar{q}, 0, 1), (-\bar{p}, \pm \bar{q}, 0, 1), (\bar{p}, \pm \bar{q}, 0, -1), (-\bar{p}, \pm \bar{q}, 0, -1); \end{aligned}$$

b) $|p| > 2, q^2 \equiv \pm 1 \pmod{p}$

$$\begin{aligned} & (0, 2, \bar{p}, \pm \bar{q}), (0, 2, -\bar{p}, \pm \bar{q}), (0, -2, \bar{p}, \pm \bar{q}), (0, -2, -\bar{p}, \pm \bar{q}), \\ & (0, 1, \bar{p}, \pm \bar{q}), (0, 1, -\bar{p}, \pm \bar{q}), (0, -1, \bar{p}, \pm \bar{q}), (0, -1, -\bar{p}, \pm \bar{q}), \\ & (\bar{p}, \pm \bar{q}, 0, 1), (-\bar{p}, \pm \bar{q}, 0, 1), (\bar{p}, \pm \bar{q}, 0, -1), (-\bar{p}, \pm \bar{q}, 0, -1); \end{aligned}$$

c) $p = 0$

$$(0, 2, 0, \pm 1), (0, 1, 0, \pm 1);$$

d) $|p| = 2$

$$(0, 2, 2, \pm 1), (0, 2, -2, \pm 1), (0, 1, 2, \pm 1), (0, 1, -2, \pm 1), \\ (2, \pm 1, 0, 1), (-2, \pm 1, 0, 1);$$

3) *equivalence classes of the set $G_1^-(M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1)))$ depending on $\alpha_1, \beta_1, \alpha_2, \beta_2$ are represented by flows with the following invariants:*

a) $\alpha_1 = \alpha_2 = \alpha, \beta_1 = \beta_2 = \beta$

$$(\alpha, \beta, \alpha, \beta).$$

b) $|\alpha_1 - \alpha_2| + |\beta_1 - \beta_2| > 0$

$$(\alpha_1, \beta_1, \alpha_2, \beta_2), (\alpha_2, \beta_2, \alpha_1, \beta_1).$$

Note that on the three-dimensional sphere $\mathbb{S}^3$, the list of flows representing equivalence classes of the set $G_1^-(\mathbb{S}^3)$ listed in Theorem 3 is exactly the same as that obtained in Bin Yu's paper (see Proposition 7.4 in [17]).

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2 Topology of 3-manifolds

2.1 Lens spaces

Futhere, we will assume that the constituents of the homotopy types of knots on the boundary $\partial\mathbb{V}$ of the standard fullness $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$ are the meridian $\mathbb{M} = (\partial\mathbb{D}^2) \times \mathbb{S}^1$ with homotopy type $\langle 0, 1 \rangle$ and the parallel $\mathbb{L} = \{x\} \times \mathbb{S}^1, x \in \partial\mathbb{D}^2$ with homotopy type $\langle 1, 0 \rangle$.

A three-dimensional manifold $L_{p,q} = V_1 \cup_j V_2$ resulting from gluing two copies of a solid torus $V_1 = \mathbb{V}$ is called a *lens space*, $V_2 = \mathbb{V}$ by some homeomorphism $j: \partial V_1 \rightarrow \partial V_2$ such that $j_*(\langle 0, 1 \rangle) = \langle p, q \rangle$.

Statement 1 ([5]). *Two lens spaces $L_{p,q}, L_{p',q'}$ are homeomorphic if and only if $|p| = |p'|, q \equiv \pm q' \pmod{p}$ or $qq' \equiv \pm 1 \pmod{p}$. In this case,*

$$L_{0,1} \cong \mathbb{S}^2 \times \mathbb{S}^1, L_{1,0} \cong \mathbb{S}^3, L_{2,1} \cong \mathbb{RP}^3.$$

2.2 Dehn surgery along the knots and links

Let

- (a) a closed 3-manifold M ;
- (b) knot $\gamma \subset M$;
- (c) tubular neighborhood U_γ of node γ , which is a solid torus with standard generators on ∂U_γ – meridian M_γ and parallel L_γ ;
- (d) homeomorphism $h: \partial \mathbb{V} \rightarrow \partial U_\gamma$, which induces an isomorphism in the given generators such that $h_*(\langle 0, 1 \rangle) = \langle \beta, \alpha \rangle$.

Manifold

$$M_\gamma = (M \setminus \text{int } U_\gamma) \cup_h \mathbb{V}$$

is called the *manifold obtained from the manifold M by Dehn surgery along the knot γ with equipment β, α* .

Let us denote by $p_\gamma: (M \setminus \text{int } U_\gamma) \sqcup \mathbb{V} \rightarrow M_\gamma$ the natural projection. Let us put $\tilde{\gamma} = p_\gamma(\{0\} \times \mathbb{S}^1)$, $U_{\tilde{\gamma}} = p_\gamma(\mathbb{V})$, $\tilde{h} = p_\gamma h^{-1}: \partial U_\gamma \rightarrow \partial U_{\tilde{\gamma}}$. Then the manifold M is recovered from M_γ by the following inverse surgery.

Statement 2 ([11]). *Let $\gamma \subset M$ be a node with β, α and $\tilde{\gamma}$ be a node with $-\beta, \xi$, where $\alpha\xi \equiv 1 \pmod{\beta}$. Then*

$$M \cong (M_\gamma)_{\tilde{\gamma}}.$$

The Dehn surgery naturally generalizes to the case when $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r \subset M$ – disjunctive union (link) of equipped knots. The resulting manifold M_γ in this case is called *the manifold obtained from the manifold M^3 by Dehn surgery along the equipped link γ* . A link $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r \subset M$ is called *is trivial* if the knots $\gamma_1, \dots, \gamma_r$ bound the pairwise non-overlapping 2-discs $d_1, \dots, d_r \subset M$.

Statement 3 ([11]). *Let $\gamma = \gamma_1 \sqcup \dots \sqcup \gamma_r \subset M$ be a trivial link with equipment $\beta_1, \alpha_1; \dots; \beta_r, \alpha_r$. Then*

$$M_\gamma \cong M \# L_{\alpha_1, \beta_1} \# \dots \# L_{\alpha_r, \beta_r}.$$

2.3 Seifert fiber spaces

A solid torus $\mathbb{V}$ partitioned into fibers of the form $\{x\} \times \mathbb{S}^1$ is called a *trivially fibered solid torus*. Consider the solid torus $\mathbb{V} = \mathbb{D}^2 \times \mathbb{S}^1$ as a solid cylinder $\mathbb{D}^2 \times [0, 1]$ with bases glued together by virtue of rotation by an angle $2\pi\nu/\alpha$ for coprime integers α, ν , $\alpha > 1$. The partitioning of a solid cylinder into segments of the form $\{x\} \times [0, 1]$ determines the partitioning of this solid cylinder into circles called *fibers*. The segment $\{0\} \times [0, 1]$ generates a fiber called *special*, all other are (*non-special*) fibers of the solid torus are wrapped α times around the special layer and ν times around the meridian of the solid torus. The number α

is called the *multiplicity* of the singular fiber. A solid torus with such a partition into fibers is called a *nontrivially fibered solid torus* with *orbital invariants* (α, ν) .

An *Seifert fiber space* — is a compact, orientable 3-manifold M , partitioned into non-intersecting simple closed curves (fibers) such that each fiber has an tubular neighborhood entirely composed of fibers, layer-by-layer homeomorphic to the fibered solid torus. Such a partitioning is called *Seifert fibration*. Fibers that under some such homeomorphism pass to the center of a nontrivially fibered solid torus are called *special*.

The *Base* of a Seifert fiber space M is a compact surface $\Sigma = M/\sim$, where $\sim$ is an equivalence relation such that $x \sim y$ if and only if x and y belong to the same layer.

The base of any Seifert fiber space is a compact surface which is closed if and only if the manifold M is closed; in particular, the base of any fibered solid torus is a disk (see, e.g., [7]). Thus, any Seifert fibration M with a given base Σ and orbital invariants $(\alpha_1, \nu_1), \dots, (\alpha_r, \nu_r)$, $r \in \mathbb{N}$, is obtained from the manifold $\Sigma \times \mathbb{S}^1$ by Dehn surgery along the link $\gamma = \bigsqcup_{i=1}^r \gamma_i$, where $\gamma_i = \{s_i\} \times \mathbb{S}^1$, $s_i \in \Sigma$ — a knot with equipment β_i, α_i , $\nu_i \beta_i \equiv 1 \pmod{\alpha_i}$. Therefore, the generally accepted notation of such a Seifert fibration is as follows

$$M(\Sigma, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)).$$

Thus, *the orientation on the fibers* of the Seifert fibration is uniquely determined by the orientation of the circle $\mathbb{S}^1$ in the manifold $\Sigma \times \mathbb{S}^1$.

Two Seifert fibrations M, M' are called *isomorphic* if there exists a homeomorphism $h: M \rightarrow M'$ which maps the fibers of one fibration into the fibers of the other with preserving the orientation of the fibers. The homeomorphism h in this case is called *isomorphism of Seifert fibrations*. It is not difficult to see that the fibrations on solid tori with orbital invariants (α, ν) and (α', ν') are isomorphic if and only if, when $\alpha = \alpha'$ and $\nu \equiv \delta \nu' \pmod{\alpha}$ ($\delta = \pm 1$), and if $\delta = +1$, then the isomorphism preserves the orientation of the solid torus, otherwise it changes.

The following statement, which gives a criterion for isomorphism of two Seifert stratifications by their invariants, was proposed by Herbert Seifert in [12]. An exposition of this statement, in notations closer to those given in this section, but only for orientation-preserving isomorphisms, can be found in the notes by Allen Hatcher [5] and the textbook by Sergey Matveev and Anatoly Fomenko [7].

Statement 4. *The Seifert fibration $M(\Sigma, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$ and $M'(\Sigma', (\alpha'_1, \beta'_1), \dots, (\alpha'_r, \beta'_r))$ are isomorphic if and only if $r = r'$ and the following conditions are satisfied for $\delta = \pm 1$ and the permutation $\sigma: \{1, 2, \dots, r\} \rightarrow \{1, 2, \dots, r\}$:*

- Σ is homeomorphic to Σ' ;
- $\alpha_i = \alpha'_{\sigma(i)}$; $\beta_i \equiv \delta \beta'_{\sigma(i)} \pmod{\alpha_i}$ for $i \in \{1, \dots, r\}$;
- if the surface of Σ is closed, then $\sum_{i=1}^r \frac{\beta_i}{\alpha_i} = \delta \sum_{i=1}^r \frac{\beta'_i}{\alpha'_i}$.

Thus, if $\delta = +1$, the isomorphism is orientation-preserving, and if $\delta = -1$, the isomorphism is orientation-reversing.

Note that some manifolds admit non-isomorphic Seifert fibrations. All such manifolds are well known (see, for example [5]) and, as can be seen from the following statement, such manifolds include, for example, lens spaces.

Statement 5 ([3]). *3-manifolds admits a Seifert fibrations with a base homeomorphic to sphere and at most two special fibers if and only if it is homeomorphic to a lens space. Thus, the list of all Seifert fibrations on lens spaces is as follows:*

- only the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ admits fibrations without special fibers;
- $M(\mathbb{S}^2, (\alpha, \beta)) \cong L_{\beta, \alpha}$;
- $M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2)) \cong L_{p, q}$, where $p = \beta_1 \alpha_2 + \alpha_1 \beta_2$, $q = \beta_1 \nu_2 + \alpha_1 \xi_2$ and $\alpha_2 \xi_2 - \nu_2 \beta_2 = 1$.

The following statement, on the contrary, allows us to infer the non-homeomorphism of ambient manifolds from the non-isomorphism of Seifert stratifications.

Statement 6 ([5], Theorem 2.3). *If two Seifert fibrations with three special fibers and a base sphere are not isomorphic, then the manifolds on which they are defined are not homeomorphic.*

3 Dynamics of flows of class $G_1^-(M^3)$

In this section we prove the lemmas given in the introduction.

Let us start with the lemma 1: the non-wandering set of any flow $f^t \in G_1^-(M^3)$ consists of exactly three periodic orbits S, A, R , saddle, attracting and repelling, respectively.

Proof. The basis of the proof is the following representation of the ambient manifold M^3 of the NMS-flow f^t with a set of periodic orbits Per_{f^t} (see, e.g., [14])

$$M^3 = \bigcup_{\mathcal{O} \in Per_{f^t}} W_{\mathcal{O}}^u = \bigcup_{\mathcal{O} \in Per_{f^t}} W_{\mathcal{O}}^s, \quad (1)$$

as well as the asymptotic behavior of invariant manifolds

$$\begin{aligned} cl(W_{\mathcal{O}}^u) \setminus W_{\mathcal{O}}^u &= \bigcup_{\tilde{\mathcal{O}} \in Per_{f^t} : W_{\mathcal{O}}^u \cap W_{\tilde{\mathcal{O}}}^s \neq \emptyset} W_{\tilde{\mathcal{O}}}^u, \\ cl(W_{\mathcal{O}}^s) \setminus W_{\mathcal{O}}^s &= \bigcup_{\tilde{\mathcal{O}} \in Per_{f^t} : W_{\mathcal{O}}^s \cap W_{\tilde{\mathcal{O}}}^u \neq \emptyset} W_{\tilde{\mathcal{O}}}^s. \end{aligned}$$

In particular, it follows from the above relations that any NMS-flow has at least one attracting orbit and at least one repelling orbit. Moreover, if the NMS-flow

has a saddle periodic orbit, then the basin of any attracting orbit has a non-empty intersection with the unstable manifold of at least one saddle orbit (see Proposition 2.1.3 [4]) and the same situation with the basin of the repelling orbit.

Let now $f^t \in G_1^-(M^3)$ and S be its only saddle orbit. It follows from the relation (1) that $W_S^u \setminus S$ intersects only with basins of attracting orbits. Since the set $W_S^u \setminus S$ is connected and the basins of attracting orbits are open, W_S^u intersects exactly one such basin. Let us denote by A the corresponding attracting orbit. Since the saddle orbit is unique, the attracting orbit is unique. Similar reasoning for W_S^s leads to the existence of a single repelling orbit R . $\square$

3.1 Canonical neighborhoods of periodic orbits

The flows admit a simple description (up to topological equivalence) in the neighborhood of a hyperbolic periodic orbit, namely, they are suspensions over linear diffeomorphisms of the plane.

Let us recall the definition of suspension. Let $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a diffeomorphism. Let us define the diffeomorphism $\Phi: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by the formula

$$\Phi(x_1, x_2, x_3) = (\phi(x_1, x_2), x_3 - 1).$$

Then the group $\{\Phi^n\} \cong \mathbb{Z}$ acts freely and discontinuously on $\mathbb{R}^3$, by virtue of which the orbit space $\Pi_\phi = \mathbb{R}^3/\Phi$ is a smooth 3-manifold, and the natural projection $v_\phi: \mathbb{R}^3 \rightarrow \Pi_\phi$ is a covering. In this case, the flow $\xi^t: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by the formula

$$\xi^t(x_1, x_2, x_3) = (x_1, x_2, x_3 + t),$$

induces a flow $[\phi]^t = v_\phi \xi^t v_\phi^{-1}: \Pi_\phi \rightarrow \Pi_\phi$, called *suspension*.

We define the diffeomorphisms $a_{\pm 1}, a_2, a_0: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by the formulas

$$a_{\pm 1}(x_1, x_2) = (\pm 2x_1, \pm x_2/2), a_2(x_1, x_2) = (2x_1, 2x_2), a_0 = a_2^{-1}.$$

Suppose

$$V_0 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 4^{x_3} x_1^2 + 4^{x_3} x_2^2 \leq 1\},$$

$$V_{\pm 1} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 4^{-x_3} x_1^2 + 4^{x_3} x_2^2 \leq 1\},$$

$$V_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 4^{-x_3} x_1^2 + 4^{-x_3} x_2^2 \leq 1\},$$

For $i \in \{0, -1, +1, 2\}$, let

$$T_i = \partial V_i, V_i = v_{a_i}(V_i), \mathbb{T}_i = \partial \mathbb{V}_i, \mathbb{O}_i = v_{a_i}(Ox_3).$$

The following fact asserts canonical neighborhoods at hyperbolic periodic orbits.

Statement 7 ([6]). *For any hyperbolic periodic orbit $\mathcal{O}$ of a flow $f^t: M^3 \rightarrow M^3$ defined on a closed orientable manifold M^3 , there exists a tubular neighborhood $V_{\mathcal{O}}$ of the orbit $\mathcal{O}$ and a number $i_{\mathcal{O}} \in \{0, -1, +1, 2\}$ such that the flow $f^t|_{V_{\mathcal{O}}}$ is topologically equivalent, via some homeomorphism $H_{\mathcal{O}}$, to the flow $[a_{i_{\mathcal{O}}}]^t|_{\mathbb{V}_{i_{\mathcal{O}}}}$.*

Let us call the neighborhood $V_{\mathcal{O}} = H_{\mathcal{O}}(\mathbb{V}_{i_{\mathcal{O}}})$ *canonical neighborhood* of the periodic orbit of $\mathcal{O}$.

On the torus $\mathbb{T}_i$ we choose *longitude* $\mathbb{L}_i$ (a curve homologous in $\mathbb{V}_i$ to the orbit of $\mathbb{O}_i$) and *meridian* $\mathbb{M}_i$ (a curve, homotopic to zero on $\mathbb{V}_i$ and essential on $\mathbb{T}_i$) such that the ordered pair of curves $\mathbb{L}_i, \mathbb{M}_i$ defines the outer side of the solid torus $\mathbb{V}_i$.

In the proof of topological equivalence we will use the following fact, which follows from the proof of Theorem 4 and Lemma 4 in [9], and can also be found in [15] (Theorem 1.1).

Statement 8. *The homeomorphism $h: \mathbb{T}_i \rightarrow \mathbb{T}_i$ for $i \in \{0, 2\}$ continues up to the homeomorphism $H: \mathbb{V}_i \rightarrow \mathbb{V}_i$, realizing the equivalence of the flow $[a_i]^t$ with itself, if and only if the induced isomorphism is of the form ¹ $h_* = \begin{pmatrix} 1 & k \\ 0 & \delta \end{pmatrix}$, where $\delta \in \{-1, 1\}$, $k \in \mathbb{Z}$.*

The boundary of the canonical neighborhood of a saddle orbit, in contrast to an attracting or repulsing orbit, contains curves tangent to the suspension trajectories. Precisely, we denote by $\mathcal{O}_{x_1, x_2}$ the flow trajectory ξ^t intersecting the plane Ox_1x_2 at a point with coordinates $(x_1, x_2, 0)$. It is directly verified that the trajectory $\mathcal{O}_{x_1, x_2}$ intersects the surface $T_{\pm 1}$ if and only if $|x_1x_2| \leq \frac{1}{2}$ and $(x_1, x_2) \neq (0, 0)$. The trajectories touch the surface at one point if $|x_1x_2| = \frac{1}{2}$, transversally intersect the surface at one point if $x_1x_2 = 0$, and otherwise transversally intersect the surface at two points

$$\mathcal{O}_{x_1, x_2} \cap T_{\pm 1} = \{(x_1, x_2, x_3^s), (x_1, x_2, x_3^u)\}, \quad x_3^s < x_3^u.$$

Let $\Gamma = \{(x_1, x_2, x_3) \in T_{\pm 1} : |x_1x_2| = \frac{1}{2}\}$, $\Gamma^u = O_{x_1x_3} \cap T_{\pm 1}$ and $\Gamma^s = O_{x_2x_3} \cap T_{\pm 1}$. The sets Γ^u, Γ^s consist of two curves by construction, the set Γ consist of four curves deviding $T_{\pm 1}$ into four connected components. The closure T^u of two of these components contains Γ^u , the closure T^s of two other contains Γ^s (see. Pic. 2). We assume that Γ^u and Γ^s are oriented in ascending order of coordinate x_3 . For $i \in \{-1, 1\}$, let us put

$$\mathbb{T}_i^s = v_{a_i}(T^s), \quad \mathbb{T}_i^u = v_{a_i}(T^u), \quad \Gamma_i = v_{a_i}(\Gamma), \quad \Gamma_i^s = v_{a_i}(\Gamma^s), \quad \Gamma_i^u = v_{a_i}(\Gamma^u).$$

3.2 Trajectory mappings

In this section we prove the lemma 2: the ambient manifold M^3 of any flow $f^t \in G_1^-(M^3)$ is represented as the union of three solid tori

$$M^3 = V_A \cup V_S \cup V_R$$

with non-intersecting interior, which are tubular neighborhoods of the orbits A, S, R , respectively, with the following properties:

¹Throughout the paper, we assume that the string (l, m) is multiplied by the matrix on the left and the first element of the basis is the parallel of the torus.

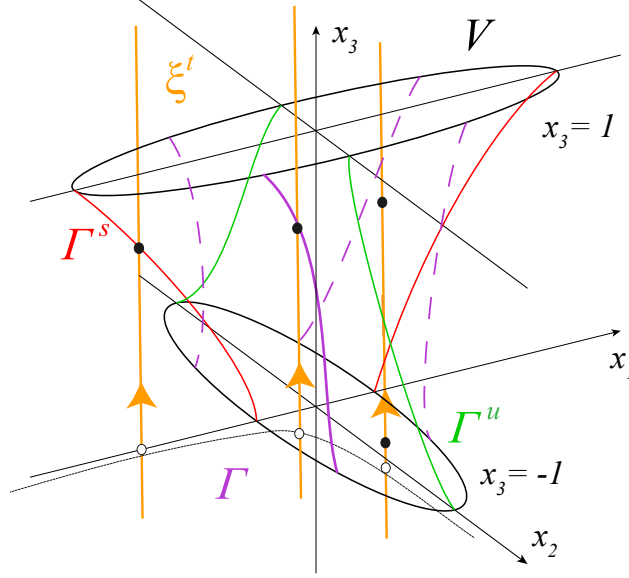


Figure 2: Cylinder $T_{\pm 1}$

- the torus $T_S = \partial V_S$ is the union of closed tubular neighborhoods K_A , K_R of knots $\gamma_A = W_S^u \cap T_S$, $\gamma_R = W_S^s \cap T_S$, respectively, such that $K_A \cap K_R = \partial K_A \cap \partial K_R$;
- the torus $T_A = \partial V_A$ is the union of the annulus K_A and the compact surface $K = T_A \setminus \text{int } K_A$,
- the torus $T_R = \partial V_R$ is the union of the ring K_R and the surface $K = T_R \setminus \text{int } K_R$.

Proof. Without reducing generality we will assume that the neighborhoods $\mathcal{V}_A = H_A(\mathbb{V}_0)$, $\mathcal{V}_S = H_S(\mathbb{V}_{-1})$, $\mathcal{V}_R = H_A(\mathbb{V}_2)$ of orbits A , S , R pairwise disjoint.

Note, that knots

$$\gamma_A = W_S^u \cap T_S = H_S(\Gamma_{-1}^u), \quad \gamma_R = W_S^s \cap T_S = H_S(\Gamma_{-1}^s)$$

have tubular neighbourhoods

$$K_A = H_S(\mathbb{T}_{-1}^u), \quad K_R = H_S(\mathbb{T}_{-1}^s)$$

respectively, which are homeomorphic to annuli with a common boundary

$$\Gamma_S = H_S(\Gamma_{-1}).$$

Next, we “blow up” the solid tori $\mathcal{V}_A$ and $\mathcal{V}_R$ along the trajectories so that they become “adjacent” to each other and to $\mathcal{V}_S$. For this purpose we introduce the following notations:

- let $\mathcal{T}_A = \partial\mathcal{V}_A$, $\mathcal{T}_R = \partial\mathcal{V}_R$,

$$K_R^s = \left(\bigcup_{t>0, w \in cl(K_R)} f^{-t}(w) \right) \cap \mathcal{T}_R, \quad K_R^u = \mathcal{T}_R \setminus K_R^s,$$

$$K_A^u = \left(\bigcup_{t>0, w \in cl(K_A)} f^t(w) \right) \cap \mathcal{T}_A, \quad K_A^s = \mathcal{T}_A \setminus K_A^u;$$

- define continuous function $\tau_R: \mathcal{T}_R \rightarrow \mathbb{R}^+$ such that $f^{\tau_R(r)}(r) \in K_R$ for $r \in K_R$ and the set $K = \bigcup_{r \in cl(K_R^u)} f^{\tau_R(r)}(r)$ disjoint with torus $\mathcal{T}_A$, let $T_R = K \cup K_R$ and define homeomorphism $\psi_R: \mathcal{T}_R \rightarrow T_R$ by the formula $\psi_R(r) = f^{\tau_R(r)}(r)$. Also, let V_R denote the connected component of $M^3 \setminus T_R$ which contains R ;
- define continuous function $\tau_A: \mathcal{T}_A \rightarrow \mathbb{R}^+$ such that $f^{-\tau_A(a)}(a) \in K_A$ for $a \in K_A^u$ and $f^{-\tau_A(a)}(a) \in K$ for $a \in K_A^s$, let $T_A = K \cup K_A$ $\psi_A: \mathcal{T}_A \rightarrow T_A$ and define homeomorphism $\psi_A(a) = f^{-\tau_A(a)}(a)$. Also, let V_A denote the connected component of $M^3 \setminus T_A$ which contains A ;
- define continuous function $\tau_{RA}: K_R \setminus \gamma_R \rightarrow \mathbb{R}^+$ such that $f^{\tau_{RA}(w)}(w) \in K_A \setminus \gamma_A$, define homeomorphism $\psi: T_R \setminus \gamma_R \rightarrow T_A \setminus \gamma_A$ by the formula

$$\psi(w) = \begin{cases} f^{\tau_{RA}(w)}(w), & w \in (K_R \setminus \gamma_R) \\ w, & w \in (T_R \setminus K_R). \end{cases}$$

Thus, the constructed solid tori V_A, V_S, V_R satisfy the conditions of the lemma. $\square$

4 Topological classification of flows $f^t \in G_1^-(M^3)$

Let us prove the first statement of Theorem 1: flows $f^t, f'^t \in G_1^-(M^3)$ are topologically equivalent if and only if their sets $C_{f^t} = (l_1, b_1, l_2, b_2)$, $C_{f'^t} = (l'_1, b'_1, l'_2, b'_2)$ are consistent, that is:

- $l_i = l'_i$, $i = 1, 2$,

and exists $\delta \in \{-1, 1\}$ such that

- $b_i \equiv \delta b'_i \pmod{l_i}$;
- $l_1 l_2 (2l_2(b_1 - \delta b'_1) + 2l_1(b_2 - \delta b'_2) + l_1 l_2(1 - \delta)) = 0$.

Proof. Recall that for a periodic orbit $\mathcal{O} \in \{A, S, R\}$ of the flow $f^t \in G_1^-(M^3)$ we denote by $V_{\mathcal{O}}$ its canonical neighborhood with boundary $T_{\mathcal{O}}$. In this case, the ambient manifold M^3 of the flow f^t is represented as a union of three solid

tori $M^3 = V_A \cup V_S \cup V_R$ with non-intersecting interior, torus T_S is the union of compact tubular neighborhoods of K_A , K_R knots $\gamma_A = W_S^u \cap T_S$, $\gamma_R = W_S^s \cap T_S$, $K = T_R \setminus \text{int } K_A = T_A \setminus \text{int } K_A$ and the knot γ_S is the connected component of the boundary of the ring K .

On the torus $T_{\mathcal{O}}$, $\mathcal{O} \in \{S, R, A\}$ we have chosen the longitude $L_{\mathcal{O}}$ (a curve homologous in $V_{\mathcal{O}}$ to the orbit of $\mathcal{O}$) and the meridian $M_{\mathcal{O}}$ (a curve, homotopic to zero on $V_{\mathcal{O}}$ and essential on $T_{\mathcal{O}}$) such that the ordered pair of curves $L_{\mathcal{O}}$, $M_{\mathcal{O}}$ defines the outer side of the solid torus of $V_{\mathcal{O}}$. The node $\gamma_{\mathcal{O}}$ is oriented coherently with the saddle orbit S and has homotopy type $\langle \gamma_{\mathcal{O}} \rangle = \langle l_{\mathcal{O}}, m_{\mathcal{O}} \rangle$ with respect to the generators $L_{\mathcal{O}}, M_{\mathcal{O}}$.

If $(l_R, m_R) = (0, 0)$, then we have written the homotopy type of the meridian $M_R \subset K$ with respect to the generators L_A, M_A

$$\langle M_R \rangle = \langle p_A, q_A \rangle.$$

If $(l_R, m_R) \neq (0, 0)$, then any knot $\sigma_{\mathcal{O}} \subset T_{\mathcal{O}}$ having homotopy type $\langle \sigma_{\mathcal{O}} \rangle = \langle b_{\mathcal{O}}, c_{\mathcal{O}} \rangle$ and the intersection index 1 with knot $\gamma_{\mathcal{O}}$ has the following property

$$l_{\mathcal{O}} c_{\mathcal{O}} - m_{\mathcal{O}} b_{\mathcal{O}} = 1. \quad (2)$$

Let $\Sigma_{\mathcal{O}}^{+1}$ ($\Sigma_{\mathcal{O}}^{-1}$) denote set of all knots on $T_{\mathcal{O}}$, having intersection index +1 (-1) with the knot $\gamma_{\mathcal{O}}$. Then

$$\tilde{\sigma}_{\mathcal{O}} \in \Sigma_{\mathcal{O}}^{\pm 1} \iff \langle \tilde{\sigma}_{\mathcal{O}} \rangle = \langle \pm b_{\mathcal{O}} + n_{\mathcal{O}} l_{\mathcal{O}}, \pm c_{\mathcal{O}} + n_{\mathcal{O}} m_{\mathcal{O}} \rangle, n_{\mathcal{O}} \in \mathbb{Z}. \quad (3)$$

It is easily verified that the intersection index of nodes $\sigma_{\mathcal{O}}, \tilde{\sigma}_{\mathcal{O}}$ is $-n_{\mathcal{O}}$. Then, if $\tilde{\sigma}_S = (\tilde{\sigma}_R \cup \tilde{\sigma}_A) \cap T_S$, then

$$n_A + n_R + n_S = 0. \quad (4)$$

Also recall that the generators L_S, M_S are chosen such that with respect to them the knot γ_S has homotopy type

$$\langle \gamma_S \rangle = \langle l_S, m_S \rangle = \langle 2, 1 \rangle. \quad (5)$$

If $(l_R, m_R) \neq (0, 0)$ let the knot $\sigma_S \subset T_S$ be chosen such that

$$\langle \sigma_S \rangle = \langle b_S, c_S \rangle = \langle 1, 1 \rangle \quad (6)$$

and σ_S intersects with each connected component of the ∂K at exactly one point (this can be done since the intersection index of the nodes γ_S and σ_S is 1). Let the knots $\sigma_R \subset T_R$, $\sigma_A \subset T_A$ be chosen such that

$$\sigma_S = (\sigma_R \cup \sigma_A) \cap T_S. \quad (7)$$

By definition $C_{f^t} = (l_1, b_1, l_2, b_2)$, where

- $(l_1, b_1, l_2, b_2) = (l_R, b_R, l_A, b_A)$, if $(l_R, m_R) \neq (0, 0)$;

- $(l_1, b_1, l_2, b_2) = (0, 2, p_A, q_A)$, if $(l_R, m_R) = (0, 0)$ and 2-ball, bounded by the knot γ_R remains to the left when traveling along the knot;
- $(l_1, b_1, l_2, b_2) = (0, -2, -p_A, -q_A)$, if $(l_R, m_R) = (0, 0)$ and 2-ball, bounded by the knot γ_R remains to the right when traveling along the knot.

Similar equalities with primes hold for the flow f^{tt} .

Let us prove separately the necessity and sufficiency of the conditions of the theorem 1.

Necessity. Let the flows f^t and f^{tt} with periodic orbits A, R, S and A', R', S' be topologically equivalent via the homeomorphism $h: M^3 \rightarrow M^3$. For $\mathcal{O} \in \{A, S, R\}$, without reducing generality, let $V_{\mathcal{O}'} = h(V_{\mathcal{O}})$. Let $h_{\mathcal{O}} = h|_{T_{\mathcal{O}}}: T_{\mathcal{O}} \rightarrow T_{\mathcal{O}'}$.

Since $h_{\mathcal{O}}$ is a restriction of a homeomorphism of a solid torus, the action of the homeomorphism $h_{\mathcal{O}}$ in the fundamental group $\pi_1(T_{\mathcal{O}})$ in the generators $L_{\mathcal{O}}, M_{\mathcal{O}}$ is given by a matrix:

$$h_{\mathcal{O}*} = \begin{pmatrix} 1 & k_{\mathcal{O}} \\ 0 & \delta_{\mathcal{O}} \end{pmatrix}, \quad k_{\mathcal{O}} \in \mathbb{Z}, \quad \delta_{\mathcal{O}} \in \{-1, +1\}. \quad (8)$$

Thus, since the tori T_A, T_S, T_R are pairwise intersecting two-dimensional manifolds, all numbers $\delta_A, \delta_S, \delta_R$ have the same sign, let

$$\delta_A = \delta_S = \delta_R = \delta_R = \delta.$$

From the properties of the conjugating homeomorphism it follows that $h_{\mathcal{O}}(\gamma_{\mathcal{O}}) = \gamma_{\mathcal{O}'}$, $\mathcal{O} \in \{S, A, R\}$, whence we obtain that

$$l_{\mathcal{O}} = l_{\mathcal{O}'} \quad (9)$$

and

$$(l_R, m_R) = (0, 0) \iff (l_{R'}, m_{R'}) = (0, 0). \quad (10)$$

Let us prove that the consistency conditions of the sets $C_{f^t}, C_{f^{tt}}$ holds separately for two cases: I) $(l_R, m_R) = (0, 0)$, II) $(l_R, m_R) \neq (0, 0)$.

In case I), it follows from the definition of the sets $C_{f^t}, C_{f^{tt}}$ that $l_1 = l_2 = 0, |b_2| = |b'_2| = 2$. Since the homeomorphism h_R maps the 2-disc bounded by the node γ_R into the 2-disc bounded by the knot $\gamma_{R'}$ with direction of knots preserved, then $b_2 = \delta b'_2$.

It follows from the equations (8) that $h_{R*}(\langle 0, 1 \rangle) = \langle 0, \delta \rangle$. Thus, $h_{A*}(\langle p_A, q_A \rangle) = \langle \delta p_{A'}, \delta q_{A'} \rangle$. Since $(l_2, b_2) = (\pm p_A, \pm q_A)$, we have $(l'_2, b'_2) = (\delta(\pm p_{A'}), \delta(\pm q_{A'}))$ which implies $h_{A*}(\langle l_2, b_2 \rangle) = \langle l'_2, b'_2 \rangle$. Also, it follows from the equations (8) that $h_{A*}(\langle l_2, b_2 \rangle) = \langle l_2, \delta b_2 + k_A l_2 \rangle$, whence $l_2 = l'_2$ and $b_2 \equiv \delta b'_2 \pmod{l_2}$.

In case II), the equality (9) is equivalent to the equality $l_i = l'_i, i = 1, 2$. Let $\tilde{\sigma}_{\mathcal{O}'} = h_{\mathcal{O}}(\sigma_{\mathcal{O}})$ and denote by

$$\langle \tilde{\sigma}_{\mathcal{O}'} \rangle = \langle \tilde{b}_{\mathcal{O}'}, \tilde{c}_{\mathcal{O}'} \rangle$$

. the homotopy type of the knot $\tilde{\sigma}_{\mathcal{O}'}$ with respect to the generators $L_{\mathcal{O}'}, M_{\mathcal{O}'}$. Then it follows from the formula (8) that

$$\tilde{b}_{\mathcal{O}'} = b_{\mathcal{O}} \quad (11)$$

Since the determinant of the matrix $h_{\mathcal{O}*}$ equals δ and $h_{\mathcal{O}}(\gamma_{\mathcal{O}}) = \gamma_{\mathcal{O}'}$, then $\tilde{\sigma}_{\mathcal{O}'} \in \Sigma_{\mathcal{O}'}^{\delta}$. Then from the formula (3) we obtain that

$$\tilde{b}_{\mathcal{O}'} = \delta b_{\mathcal{O}'} + n_{\mathcal{O}'} l_{\mathcal{O}'}, \tilde{c}_{\mathcal{O}} = \delta c_{\mathcal{O}'} + n_{\mathcal{O}'} m_{\mathcal{O}'}. \quad (12)$$

Whence, taking into account the equalities (9) and (11), we obtain that

$$b_{\mathcal{O}} = \delta b_{\mathcal{O}'} + n_{\mathcal{O}'} l_{\mathcal{O}}, \quad (13)$$

so

$$b_{\mathcal{O}} \equiv \delta b_{\mathcal{O}'} \pmod{l_{\mathcal{O}}}. \quad (14)$$

By construction $\tilde{\sigma}_S = (\tilde{\sigma}_R \cup \tilde{\sigma}_A) \cap T_S$, which, given the equality (4), entails equality

$$n_{A'} + n_{R'} + n_{S'} = 0. \quad (15)$$

If $l_A l_R \neq 0$, then by expressing $n_{\mathcal{O}'}$ from the equality (13) and substituting into the equality (15), given that $l_S = 2$, $b_S = b_{S'} = 1$, we arrive at

$$2l_R(b_A - \delta b_{A'}) + 2l_A(b_R - \delta b_{R'}) + l_A l_R(1 - \delta) = 0,$$

which is equivalent to

$$l_A l_R(2l_R(b_A - \delta b_{A'}) + 2l_A(b_R - \delta b_{R'}) + l_A l_R(1 - \delta)) = 0,$$

which holds when $l_A l_R = 0$.

Sufficiency. Let the sets $C_{f^t} = (l_1, b_1, l_2, b_2)$, $C_{f'^t} = (l'_1, b'_1, l'_2, b'_2)$ of flows f^t, f'^t are consistent via the parameter $\delta \in \{-1, 1\}$. We define the homeomorphism $Q_\delta: \mathbb{V}_{-1} \rightarrow \mathbb{V}_{-1}$ by the formula

$$Q_\delta = v_{a_{-1}} \bar{Q}_\delta v_{a_{-1}}^{-1}, \text{ where } \bar{Q}_\delta(x_1, x_2, x_3) = (\delta x_1, x_2, x_3): V_{-1} \rightarrow V_{-1}.$$

We check directly that the constructed homeomorphism Q_δ realizes the equivalence of the flow $[a_{-1}]^t$ with itself. Let

$$h_S = H_{S'} Q_\delta H_S^{-1}: V_S \rightarrow V_{S'}.$$

We show that the homeomorphism $h_S|_{K_A}$ can be extended to a homeomorphism $h_A: T_A \rightarrow T_{A'}$ inducing an isomorphism $h_{A*} = \begin{pmatrix} 1 & k_A \\ 0 & \delta \end{pmatrix}$ for some $k_A \in \mathbb{Z}$ and the homeomorphism $h_S|_{K_R}$ can be extended to the homeomorphism $h_R: T_R \rightarrow T_{R'}$ inducing isomorphism $h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$ for some $k_R \in \mathbb{Z}$ such that $h_A|_K = h_R|_K$. Then, by virtue of Proposition 8, the homeomorphisms h_A, h_R can be extended to homeomorphisms $h_A: V_A \rightarrow V_{A'}, h_R: V_R \rightarrow V_{R'}$

realizing the equivalence of the flows $f^t|_{V_A} \subset f^{t\prime}|_{V_{A'}}$ and $f^t|_{V_R}$ with $f^{t\prime}|_{V_{R'}}$, respectively, and the desired homeomorphism $h: M^3 \rightarrow M^3$ realizing the equivalence of $f^t, f^{t\prime}$ flows coincides with $h_{\mathcal{O}}$ on $V_{\mathcal{O}}$ for $\mathcal{O} \in \{S, A, R\}$.

Let's consider the cases separately: I) $(l_1, b_1) = (0, \pm 2)$, II) $(l_1, b_1) \neq (0, \pm 2)$.

In case I), it follows from the consistency condition of the sets $C_{f^t}, C_{f^{t\prime}}$ that $b_1 = \delta b'_1, l_2 = l'_2$ and $b'_2 = \delta b_2 + k_A l_2$ for some $k_A \in \mathbb{Z}$. Since the annuli $K_A, K_{A'}$ are contractible on tori $T_A, T_{A'}$, the homeomorphism $h_S|_{K_A}$ can be extended to the homeomorphism $h_A: T_A \rightarrow T_{A'}$ inducing an isomorphism

$$h_{A*} = \begin{pmatrix} 1 & k_A \\ 0 & \delta \end{pmatrix}.$$

Let us define the homeomorphism $h_R: T_R \rightarrow T_{R'}$ by the formula

$$h_R(x) = \begin{cases} h_S(x), & x \in K_R \\ h_A(x), & x \in K \end{cases}.$$

Since $h_{A*}(\langle l_2, b_2 \rangle) = \langle l'_2, b'_2 \rangle$, then $h_{R*}(\langle 0, 1 \rangle) = \langle 0, \delta \rangle$ so

$$h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$$

for some $k_R \in \mathbb{Z}$.

In case II), it follows from the consistency condition of the sets $C_{f^t}, C_{f^{t\prime}}$ that $l'_i = l_i, b'_i \equiv \delta b_i \pmod{l_i}, i = 1, 2$. So,

$$l_{R'} = l_R, b'_{R'} \equiv \delta b_R \pmod{l_R}; l_{A'} = l_A, b'_{A'} \equiv \delta b_A \pmod{l_A}. \quad (16)$$

Next, we consider separately cases IIa) $l_A l_R = 0$, IIb) $l_A l_R \neq 0$.

In case IIa) we assume without loss of generality that $l_R = 0$ (in case $l_A = 0$ the reasoning is similar). It follows from (16) and (2) that $m_{A'} = \delta m_A + k_A l_A$ for some $k_A \in \mathbb{Z}$. Then the homeomorphism $h_S|_{K_A}$ continues to a homeomorphism $h_A: T_A \rightarrow T_{A'}$ inducing an isomorphism

$$h_{A*} = \begin{pmatrix} 1 & k_A \\ 0 & \delta \end{pmatrix}.$$

Let us define the homeomorphism $h_R: T_R \rightarrow T_{R'}$ by the formula

$$h_R(x) = \begin{cases} h_S(x), & x \in K_R \\ h_A(x), & x \in K \end{cases}.$$

Since $l_R = 0$, then $m_R = \pm 1, m_{R'} = \pm \delta$ and hence $h_{R*}(\langle 0, \pm 1 \rangle) = \langle 0, \pm \delta \rangle$. Then

$$h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$$

for some $k_R \in \mathbb{Z}$.

In case IIb), the homeomorphisms $h_A: T_A \rightarrow T_{A'}$ and $h_R: T_R \rightarrow T_{R'}$ are constructed as in case IIa). Let us show that $h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}$, where $m_{R'} = \delta m_R + k_R l_R$.

From the equality (12) we obtain that the knot $h_S(\sigma_S)$ has the intersection index

$$n_{S'} = \frac{1 - \delta}{2}$$

with the knot $\sigma_{S'}$, and the knot $h_A(\sigma_A)$ has the intersection index

$$n_{A'} = \frac{b_A - \delta b_{A'}}{l_A}$$

with knot $\sigma_{A'}$. According to the equality (15), knot $h_R(\sigma_R)$ has intersection index $n_{R'} = -(n_{S'} + n_{A'})$. Then from the consistency condition of the sets, we obtain that

$$n_{R'} = \frac{b_R - \delta b_{R'}}{l_R}.$$

Whence $h_{R*}(\langle b_R, c_R \rangle) = \langle b_R, \tilde{c}_R \rangle$. Since $h_{R*}(\langle l_R, m_R \rangle) = \langle l_R, m_{R'} \rangle$, then

$$h_{R*} = \begin{pmatrix} 1 & k_R \\ 0 & \delta \end{pmatrix}.$$

□

5 Realization of flows $f^t \in G_1^-(M^3)$

In this section we prove the second part of Theorem 1: for any admissible invariant C there exists a flow $f^t \in G_1^-(M^3)$. Recall that an invariant C is called admissible if:

- $(l_1, b_1) = (0, \pm 2)$ or $\gcd(l_1, b_1) = 1$;
- $\gcd(l_2, b_2) = 1$.

Proof. Let $C = (l_1, b_1, l_1, l_1, b_1)$. We construct the three-dimensional manifold M^3 and the flow $f^t \in G_1^-(M^3)$ such that $C_{f^t} = C$ separately for the cases: I) $(l_1, b_1) = (0, \pm 2)$, II) $(l_1, b_1) \neq (0, \pm 2)$.

In case I), for $(l_1, b_1) = (0, \pm 2)$, let $(p, q) = (\pm l_2, \pm b_2)$. Let us define a homeomorphism $\psi: \mathbb{T}_2 \rightarrow \mathbb{T}_0$ inducing an isomorphism defined by the integer matrix $h_* = \begin{pmatrix} r & s \\ p & q \end{pmatrix}$ with determinant equals to -1. On the torus $\mathbb{T}_2$, we choose a γ_R essential knot with a tubular neighborhood $K_R \subset (\mathbb{T}_2 \setminus \mathbb{M}_2)$ and orient it so that the 2-disk bounded by it remains on the left in the case $b_1 = +2$ and - on the right in the case $b_1 = -2$. Let $\gamma_A = \psi(\gamma_R)$ and $K_A = \psi(K_R)$.

Let $\psi_R: K_R \rightarrow \mathbb{T}_{-1}^s$, $\psi_{iA}: K_A \rightarrow \mathbb{T}_{-1}^u$ be such homeomorphisms, that $\psi_A^{-1} \psi_R|_{\partial K_R} = \psi|_{\partial K_R}$,

$\psi_R(\gamma_R) = \Gamma^s$, $\psi_A(\gamma_A) = \Gamma^u$. Let $\sim$ be the minimal equivalence relation on $\mathbb{V}_0 \sqcup \mathbb{V}_{-1} \sqcup \mathbb{V}_2$ for which $x \sim \psi(x)$, $x \in (\mathbb{T}_2 \setminus \text{int } K_A)$, $x \sim \psi_A(x)$, $x \in K_A$, $x \sim \psi_R(x)$, $x \in K_R$. Then

$$M^3 = (\mathbb{V}_0 \sqcup \mathbb{V}_{-1} \sqcup \mathbb{V}_2) / \sim.$$

We denote by $\pi: \mathbb{V}_0 \sqcup \mathbb{V}_{-1} \sqcup \mathbb{V}_2 \rightarrow M^3$ the natural projection. Let the flow $f^t: M^3 \rightarrow M^3$ be given by the formula

$$f^t(x) = \begin{cases} \pi([a_0]^t(\pi^{-1}(x))), & x \in \pi(\mathbb{V}_0) \\ \pi([a_{-1}]^t(\pi^{-1}(x))), & x \in \pi(\mathbb{V}_{-1}) \\ \pi([a_2]^t(\pi^{-1}(x))), & x \in \pi(\mathbb{V}_2) \end{cases}$$

By construction, $C_{f^t} = C$.

In case II), we represent the sphere $\mathbb{S}^2$ as a union of three two-dimensional disks D_A , D_S , D_R with centers O_A , O_S , O_R , glued along the boundary as depicted in Picture 3 (glued segments are marked with the same color). Then

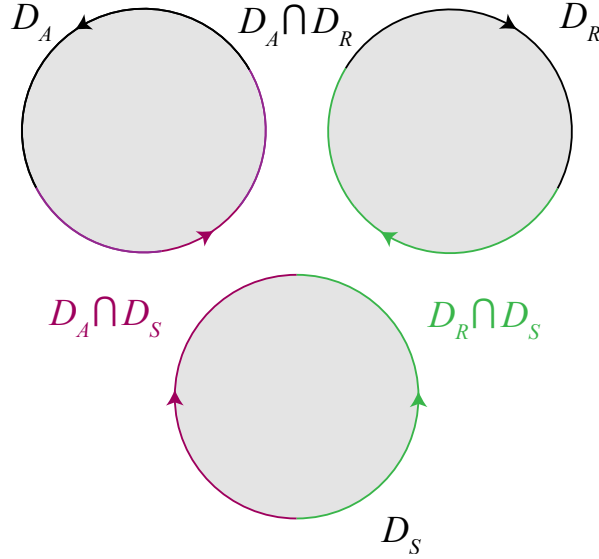


Figure 3: Disks D_A , D_S , D_R

the manifold $\mathbb{S}^2 \times \mathbb{S}^1$ is represented as a union of three solid tori $V_A = D_A \times \mathbb{S}^1$, $V_S = D_S \times \mathbb{S}^1$, $V_R = D_R \times \mathbb{S}^1$, which are tubular neighborhoods of the knots $\ell_A = O_A \times \mathbb{S}^1$, $\ell_S = O_S \times \mathbb{S}^1$, $\ell_R = O_R \times \mathbb{S}^1$, glued along the boundaries $T_A = \partial V_A$, $T_S = \partial V_S$, $T_R = \partial V_R$, along the annuli $K_A = T_A \cap T_S$, $K_R = T_R \cap T_S$, $K = T_A \cap T_R$, $K = T_A \cap T_R$, respectively.

Let $C = (l_1, b_1, l_2, b_2)$. Then the sought manifold M^3 is obtained by the Dehn surgery along the link $\ell_A \sqcup \ell_S \sqcup \ell_R$ with equipment $(-b_2, l_2), (-1, 2), (-b_1, l_1)$.

Moreover, the homeomorphisms of surgery $h_A: \mathbb{V}_0 \rightarrow V_A$, $h_S: \mathbb{V}_{-1} \rightarrow V_S$, $h_R: \mathbb{V}_{-1} \rightarrow V_S$, $h_R: \mathbb{V}_2 \rightarrow V_R$ are chosen such that $h_S(\mathbb{T}_{-1}^u) = K_A$, $h_S(\mathbb{T}_{-1}^s) = K_R$. Denote by $\pi: \mathbb{V}_0 \sqcup \mathbb{V}_{-1} \sqcup \mathbb{V}_2 \rightarrow M^3$ the natural projection. Let $f^t: M^3 \rightarrow M^3$ be defined by the formula

$$f^t(x) = \begin{cases} \pi([a_0]^t(\pi^{-1}(x))), & x \in \pi(\mathbb{V}_0) \\ \pi([a_{-1}]^t(\pi^{-1}(x))), & x \in \pi(\mathbb{V}_{-1}) \\ \pi([a_2]^t(\pi^{-1}(x))), & x \in \pi(\mathbb{V}_2) \end{cases}$$

By construction $C_{f^t} = C$. □

6 Topology of ambient manifolds of flows $f^t \in G_1^-(M^3)$

In this section we prove the theorem 2: flows of class $G_1^-(M^3)$ admit all lens spaces $L_{p,q}$, all connected sums of the form $L_{p,q} \# \mathbb{R}P^3$, and all Seifert fiber spaces of the form $M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1))$. Exactly, let the flow $f^t \in G_1^-(M^3)$ have the invariant $C_{f^t} = (l_1, b_1, l_2, b_2)$. Then

- 1) If $(|l_1| - 1)(|l_2| - 1) = 0$, then $M^3 \cong L_{p,q}$, herewith:
 - i) if $l_1 l_2 = 0$, then $M^3 \cong \mathbb{R}P^3$;
 - ii) if $C_{f^t} = (\pm 1, b_1, l_2, b_2)$, $l_2 \neq 0$, then $M^3 \cong L_{l_2 - 2b_2, b_2}$;
 - iii) if $C_{f^t} = (l_1, b_1, \pm 1, b_2)$, $l_1 \neq 0$, then $M^3 \cong L_{l_1 - 2b_1, b_1}$;
- 2) If $l_1 l_2 = 0$ and $(|l_1| - 1)(|l_2| - 1) \neq 0$, then $M^3 \cong L_{p,q} \# \mathbb{R}P^3$, herewith:
 - i) if $C_{f^t} = (0, b_1, l_2, b_2)$, then $M^3 \cong L_{l_2, b_2} \# \mathbb{R}P^3$;
 - ii) if $C_{f^t} = (l_1, b_1, 0, \pm 1)$, $l_1 \neq 0$, then $M^3 \cong L_{l_1, b_1} \# \mathbb{R}P^3$.
- 3) If $C_{f^t} = (l_1, b_1, l_2, b_2)$, $|l_1| > 1$, $|l_2| > 1$, then $M^3 \cong M(\mathbb{S}^2, (l_1, b_1), (l_2, b_2), (2, 1))$.

Proof. Let us prove the theorem separately for the cases: I) $l_1 l_2 = 0$, II) $l_1 l_2 \neq 0$.

In case I), we denote by M_S^3 the manifold obtained by Dehn surgery along knot S with equipment $(1, 1)$ in the generators L_S, M_S . Let $v_S: (M^3 \setminus \text{int } V_S) \sqcup \mathbb{V} \rightarrow M_S^3$ — the natural projection. For simplicity, we keep the labels of all objects on $v_S(M^3 \setminus \text{int } V_S)$ the same as they were on $M^3 \setminus \text{int } V_S$ and put $\tilde{S} = v_S(\{0\} \times \mathbb{S}^1)$, $V_{\tilde{S}} = v_S(\mathbb{V})$. Then $\tilde{V}_R = V_R \cup V_{\tilde{S}}$ is a solid torus with boundary $\tilde{T}_R$ and there exists an isotopy $\zeta_t: V_R \rightarrow \tilde{V}_R$, $t \in [0, 1]$ such that $\zeta_0 = \text{id}|_{V_A}$, $\zeta_t|_K = \text{id}|_K$, $t \in [0, 1]$, $\zeta_1(V_R) = \tilde{V}_R$, $\zeta_1(\sigma_S \cap K_R) = \sigma_S \cap K_A$. For any curve $c \subset T_R$, let

us put $\tilde{c} = \zeta_1(c) \subset \tilde{T}_R$. Then the isomorphism ζ_{1*} is identical in the generators $L_R, M_R; \tilde{L}_R, \tilde{M}_R$ and

$$M_S^3 = \tilde{V}_R \cup_\psi V_A, \quad (17)$$

where $\psi: \partial\tilde{V}_R \rightarrow \partial V_A$ is homeomorphism inducing isomorphism in generators $\tilde{L}_R, \tilde{M}_R; L_A, M_A$ and $\psi_* = \begin{pmatrix} r & s \\ p & q \end{pmatrix}$. Hence, $M_S^3 \cong L_{p,q}$. From the Statement 2, we obtain that $M^3 \cong (L_{p,q})_{\tilde{S}}$, where $\tilde{S}$ is a knot with equipment $(-1, 2)$. Since the knot $\tilde{S}$ bounds a 2-ball on at least one of the tori $\tilde{V}_A, V_R$, then, by virtue of the Statement 3 $(L_{p,q})_{\tilde{S}} \cong L_{p,q} \# L_{2,-1}$. Whence, by virtue of the Statement 1,

$$M^3 \cong L_{p,q} \# \mathbb{R}P^3. \quad (18)$$

Let us show how the proof of 2) and 1i) follows from the deductions made.

1i)+2i) $(l_1, b_1) = (0, \pm 2)$. It follows from the definition of the set C_{f^t} that $\psi_*(\langle 0, 1 \rangle) = \langle \pm l_2, \pm b_2 \rangle$. By virtue of Statement 3,

$$2i) M^3 \cong L_{l_2, b_2} \# \mathbb{R}P^3, |l_2| \neq 1;$$

$$1i) M^3 \cong \mathbb{R}P^3, |l_2| = 1.$$

1i)+2i) $(l_1, b_1) \neq (0, \pm 2)$. It follows from the definition of the set C_{f^t} that $\psi_*(\langle 0, 1 \rangle) = \langle \pm l_2, \pm m_A \rangle$, $\psi_*(\langle 1, \pm c_R \rangle) = \langle \pm b_2, \pm c_A \rangle$. By direct calculation we obtain that $\psi_*^{-1}(\langle 0, 1 \rangle) = \langle \pm l, b \rangle$, where $|l| = |l_2|$, $|b| \equiv |b_2| \pmod{l_2}$. By virtue of Statement 3,

$$2i) M^3 \cong L_{l_2, b_2} \# \mathbb{R}P^3, |l_2| \neq 1;$$

$$1i) M^3 \cong \mathbb{R}P^3, |l_2| = 1.$$

1i)+2ii). By reasoning analogous to the above, we obtain that

$$2ii) M^3 \cong L_{l_1, b_1} \# \mathbb{R}P^3, |l_1| \neq 1;$$

$$1i) M^3 \cong \mathbb{R}P^3, |l_1| = 1.$$

In case II), consider first the subcase $|l_1| = 1$. Then $\tilde{V}_S = V_S \cup V_R$ is a filled torus with boundary $\tilde{T}_S$ and there exists an isotopy $\zeta_t: V_S \rightarrow \tilde{V}_S$, $t \in [0, 1]$ such that $\zeta_0 = id|_{V_S}$, $\zeta_t|_{K_A} = id|_{K_A}$, $t \in [0, 1]$, $\zeta_1(V_S) = \tilde{V}_S$, $\zeta_1(\sigma_S \cap K_R) = \sigma_R \cap K$. For any curve $c \subset T_S$, let $\tilde{c} = \zeta_1(c) \subset \tilde{T}_S$. Then the isomorphism ζ_{1*} is identical in the generators $L_S, M_S; \tilde{L}_S, \tilde{M}_S$ and

$$M_S^3 = \tilde{V}_S \cup_\psi V_A, \quad (19)$$

where $\psi: \partial\tilde{V}_S \rightarrow \partial V_A$ is a homeomorphism, inducing in the generators $\tilde{L}_S, \tilde{M}_S; L_A, M_A$ isomorphism $\psi_* = \begin{pmatrix} r & s \\ p & q \end{pmatrix}$. Hence, $M_S^3 \cong L_{p,q}$. From the definition of the set C_{f^t} , it follows that $\psi_*(\langle 2, 1 \rangle) = \langle \pm l_2, m_A \rangle$, $\psi_*(\langle 1, 1 \rangle) = \langle \pm b_2, \pm c_A \rangle$. By direct calculation, we obtain that $\psi_*^{-1}(\langle 0, 1 \rangle) = \langle \pm l, b \rangle$, where $|l| = |l_2 - 2b_2|$, $|b| \equiv |b_2| \pmod{l_2}$. By virtue of the Statement 3,

$$1ii) M^3 \cong L_{l_2 - 2b_2, b_2}.$$

In the case $|l_2| = 1$ by similar reasoning, we obtain that

$$1\text{iii}) M^3 \cong L_{l_1-2b_1, b_1}.$$

In the case $|l_1| > 1$, $|l_2| > 1$, it follows from the procedure for realizing a flow over an admissible set (see the proof of the second part of Theorem 1 in Case II)) that M^3 is a Seifert fiber space with base sphere with three special fibers

$$M^3 \cong M(\mathbb{S}^2, (l_1, b_1), (l_2, b_2), (2, 1)).$$

□

7 Counting the number of topological equivalence classes

In this section we give a proof of Theorem 3. To do this, recall that for any pair p, q of integer prime numbers, we put $\bar{p} = |p|$ and denote by $\bar{q}$ the smallest non-negative of the numbers q' satisfying the condition $q \equiv \pm q' \pmod{p}$, and by $\tilde{q}$ the smallest non-negative of the numbers q' satisfying the condition $qq' \equiv \pm 1 \pmod{p}$.

Theorem 3. *The set $G_1^-(L_{p,q})$, $|p| \neq 2$ decomposes into a countable number of equivalence classes, whereas the sets $G_1^-(\mathbb{R}P^3)$, $G_1^-(L_{p,q} \# \mathbb{R}P^3)$, $G_1^-(M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1)))$ consist of a finite number of classes.*

Proof. By virtue of Theorem 2, flows of class $G_1^-(M^3)$ admit three types of manifolds 1) $L_{p,q}$; 2) $L_{p,q} \# \mathbb{R}P^3$; 3) $M(\mathbb{S}^2, (l_1, b_1), (l_2, b_2), (2, 1))$. Let us prove the proof separately for each of these cases.

1) According to Statement 1, two lens spaces $L_{p,q}$, $L_{p',q'}$ are homeomorphic if and only if $\bar{p} = \bar{p}'$ and either $\bar{q} = \bar{q}'$ or $\bar{q} = \tilde{q}'$. Whence it follows that $L_{p,q} \cong L_{\bar{p},\bar{q}}$ and, $L_{p',q'} \cong L_{\bar{p},\tilde{q}}$ if and only if at least one of the following conditions for $k \in \mathbb{Z}$ is satisfied:

$$p' = \bar{p}, q' = \bar{q} + k\bar{p}; \quad (20)$$

$$p' = -\bar{p}, q' = \bar{q} + k\bar{p}; \quad (21)$$

$$p' = \bar{p}, q' = -\bar{q} + k\bar{p}; \quad (22)$$

$$p' = -\bar{p}, q' = -\bar{q} + k\bar{p}; \quad (23)$$

$$p' = \bar{p}, q' = \tilde{q} + k\bar{p}; \quad (24)$$

$$p' = -\bar{p}, q' = \tilde{q} + k\bar{p}; \quad (25)$$

$$p' = \bar{p}, q' = -\tilde{q} + k\bar{p}; \quad (26)$$

$$p' = -\bar{p}, q' = -\tilde{q} + k\bar{p}; \quad (27)$$

By virtue of the 2 theorem, the lens $L_{\bar{p},\bar{q}}$, $\bar{p} \neq 2$ is a ambient manifold for flows with invariants

$$(\pm 1, n, p' + 2q', q'); \quad (28)$$

$$(p' + 2q', q', \pm 1, n), \quad (29)$$

where $n \in \mathbb{Z}$. Substituting the condition (20) into (28), we obtain sets of the form

$$(\pm 1, b_1, \bar{p} + 2(\bar{q} + k\bar{p}), \bar{q} + k\bar{p}).$$

From the definition of consistency, it follows that two sets of

$$(\pm 1, n_1, \bar{p} + 2(\bar{q} + k_1\bar{p}), \bar{q} + k_1\bar{p}), (\pm 1, n_2, \bar{p} + 2(\bar{q} + k_2\bar{p}), \bar{q} + k_2\bar{p}), \bar{q} + k_2\bar{p})$$

are consistent if and only if $k_1 = k_2, n_1 = n_2$. Thus, each representation of the lens $L_{p,q}$ in the form (20) gives rise to the family $(\pm 1, n, \bar{p} + 2(\bar{q} + k + k\bar{p}), \bar{q} + k\bar{p}), n, k \in \mathbb{Z}$ of pairwise non-consistent sets corresponding, by virtue of Theorem 2, to pairwise non-equivalent flows. If $|p| > 2$, then similar families are obtained from each of the representations (21), (22), (23), (23). It is directly verified that the sets of all four families are not pairwise equivalent. Finally, if $\bar{q} \neq \tilde{q}$ (equivalent to $q^2 \not\equiv \pm 1 \pmod{p}$), we obtain four more families of pairwise non-equivalent sets corresponding to the representations (24), (25), (??), (26), (26), (??). Adding to the list of sets, sets of type (29), we obtain a list of eight more pairwise non-equivalent sets, directly from which follows the result of the theorem in cases 1a), 1b).

In cases 1c); 1d), by directly substituting the pairs $\bar{p} = 0, \bar{q} = 1; \bar{p} = 1, \bar{q} = 0$ into the sets 1b), respectively, we obtain the announced lists of pairwise non-equivalent pairs.

In the case $|p| = 2$, the lens $L_{p,q}$ is an ambient manifold for the flows with invariants

$$(0, c, \pm 1, n); \quad (30)$$

$$(\pm 1, n, 0, d), \quad (31)$$

where $n \in \mathbb{Z}, c \in \{-2, -1, 1, 2\}, d \in \{-1, 1\}$. From the definition of consistency, it follows that the two sets

$$(0, c_1, \pm 1, n_1), (0, c_2, \pm 1, n_2)$$

are consistent if and only if $c_1 = \pm c_2, n_1 \equiv \pm n_2 \pmod{1}$. A similar statement is true for sets of the form (31), resulting in the announced list 1e).

2) By virtue of Theorem 2 the manifold $L_{p,q} \# \mathbb{R}P^3, |p| \neq 1$ is an ambient manifold for flows with invariants

$$(0, c, p', q'); \quad (32)$$

$$(p', q', 0, d), \quad (33)$$

where $n \in \mathbb{Z}, c \in \{-2, -1, 1, 2\}, d \in \{-1, 1\}$. It follows from the definition of consistency that for $|p| > 2$ the two sets of

$$(0, c_1, \bar{p}, \bar{q} + k_1\bar{p}), (0, c_2, \bar{p}, \bar{q} + k_2\bar{p})$$

are consistent if and only if $c_1 = c_2$, $k_1 \equiv k_2 \pmod{1}$. Thus, each representation of the lens $L_{p,q}$ in the form (20) gives rise to a family $(0, c, \bar{p}, \bar{q})$ of pairwise non-consistent sets corresponding, by virtue of Theorem 2, to pairwise non-equivalent flows. Similar families are obtained from each of the representations (21), (22), (22), (23), (24), (??), (26), (27), (27) if $\bar{q} \neq \tilde{q}$. Adding to the list of sets, the sets of type (33), we obtain the list of sets announced in (2a), 2b) of this theorem.

In cases 2c); 2d), by directly substituting into sets 2b) the pairs $\bar{p} = 0, \bar{q} = 1$; $\bar{p} = 2, \bar{q} = 1$, respectively, we obtain the announced lists of pairwise non-equivalent pairs.

3) By virtue of the 2 theorem, the manifold $M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1))$ is an ambient manifold for flows with invariants

$$(\alpha_1, \beta_1, \alpha_2, \beta_2). \quad (34)$$

By virtue of Statement 4 $M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1)) \cong M(\mathbb{S}^2, (\alpha'_1, \beta'_1), (\alpha'_2, \beta'_2), (2, 1))$ if and only if at least one of the following conditions is met:

$$\alpha'_1 = \alpha_1, \beta'_1 = \beta_1 + k_1\alpha_1, \alpha'_2 = \alpha_2, \beta'_2 = \beta_2 + k_2\alpha_2; \quad (35)$$

$$\alpha'_1 = \alpha_2, \beta'_1 = \beta_2 + k_1\alpha_2, \alpha'_2 = \alpha_1, \beta'_2 = \beta_1 + k_2\alpha_1, \quad (36)$$

where $k_1, k_2 \in \mathbb{Z}$. By virtue of Theorem refth:top. $M(\mathbb{S}^2, (\alpha_1, \beta_1), (\alpha_2, \beta_2), (2, 1))$ is an ambient manifold for flows with invariants

$$(\alpha'_1, \beta'_1, \alpha'_2, \beta'_2). \quad (37)$$

Substituting (35) into (37), we obtain sets of the form

$$(\alpha_1, \beta_1 + k_1\alpha_1, \alpha_2, \beta_2 + k_2\alpha_2).$$

It follows from the definition of consistency that all such sets are equivalent to the set $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. A similar situation is obtained with the relation (36). From where we obtain the sets announced in 3a), 3b). $\square$

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