k-CONVEX HYPERSURFACES WITH PRESCRIBED WEINGARTEN CURVATURE IN WARPED PRODUCT MANIFOLDS

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Abstract. In this paper, we consider Weingarten curvature equations for k-convex hypersurfaces with $n < 2k$ in a warped product manifold $\overline{M} = I \times_\lambda M$. Based on the conjecture proposed by Ren-Wang in [26], which is valid for $k \geq n-2$, we derive curvature estimates for equation $\sigma_k(\kappa) = \psi(V, \nu(V))$ through a straightforward proof. Furthermore, we also obtain an existence result for the star-shaped compact hypersurface Σ satisfying the above equation by the degree theory under some sufficient conditions.

1. Introduction

Let (M, g') be a compact Riemannian manifold and I be an open interval in $\mathbb{R}$. The warped product manifold $\overline{M} = I \times_\lambda M$ is endowed with the metric

$$\overline{g}^2 = dr^2 + \lambda^2(r) g', \tag{1.1}$$

where $\lambda : I \to \mathbb{R}^+$ is a positive C^2 differential function. Let Σ be a compact star-shaped hypersurface in $\overline{M}$, thus Σ can be parametrized as a radial graph over $\overline{M}$. Specifically speaking, there exists a differentiable function $r : M \to I$ such that the graph of Σ can be represented by

$$\Sigma = \{X(u) = (r(u), u) \mid u \in M\}.$$

In this paper, we consider the following prescribed Weingarten curvature equation in warped product manifold $\overline{M}$

$$\sigma_k(\kappa(V)) = \psi(V, \nu(V)), \quad \forall\ V \in \Sigma, \tag{1.2}$$

where $V = \lambda \frac{\partial}{\partial r}$ is the position vector field of hypersurface Σ in $\overline{M}$, σ_k is the k-th elementary symmetric function, $\nu(V)$ is the outward unit normal vector field along the

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hypersurface Σ and $\kappa(V) = (\kappa_1, \cdots, \kappa_n)$ are the principle curvatures of hypersurface Σ at V.

Curvature estimates for equation (1.2) in $\mathbb{R}^{n+1}$ has been studied extensively. When $k = 1$ and $k = n$, the equation is quasi-linear equation and Gauss curvature equation respectively, then the corresponding curvature estimates follow from the classical theory of quasi-linear PDEs and Monge-Ampère type equations in [3]. When ψ is independent of ν, curvature estimates were proved by Caffarelli-Nirenberg-Spruck [4] for a general class of fully nonlinear operators F, including $F = \sigma_k$ and $F = \frac{\sigma_k}{\sigma_l}$. When ψ depends only on ν, curvature estimates were proved by Guan-Guan [10]. Curvature estimates were also proved for equation of prescribing curvature measures problem in [11, 12], where $\psi(X, \nu) = \langle X, \nu\rangle \tilde{\psi}(X)$. Ivochkina [15, 16] considered the Dirichlet problem of equation (1.2) and obtained curvatute estimates under some extra conditions on the dependence of ψ on ν.

In recent years, there are many progresses on establishing curvature estimates for equation (1.2) in case $2 \leq k \leq n-1$. When $k = 2$, curvature estimates for admissible solutions of equation (1.2) were obtained by Guan-Ren-Wang [13]. They also established curvature estimates of convex solutions for general k, see a simpler proof in Chu [7]. Subsequently, Spruck-Xiao [27] extended 2-convex case to space forms and gave a simple proof for the Euclidean case. In [24, 25], Ren-Wang proved curvature estimates for $k = n-1$ and $n-2$, respectively. They also proved curvature estimates for equation (1.2) with $n < 2k$ in [26] based on a concavity conjecture.

Moreover, some results have been obtained by Li-Oliker [21] on unit sphere, Barbosa-de Lira-Oliker [2] on space forms, Jin-Li [17] on hyperbolic space, Andrade-Barbosa-de Lira [1] on warped product manifolds, Li-Sheng [19] for Riemannain manifold equipped with a global normal Gaussian coordinate system. In particular, Chen-Li-Wang [6] generalized the results in [13, 24] to $(n-1)$-convex hypersurfaces in warped product manifolds.

Inspired by the above works, it is natural to consider extending Ren-Wang's results in [24, 25, 26] from Euclidean space to warped product manifolds. Here we introduce the following conjecture:

Conjecture 1.1. *Let $\kappa = (\kappa_1, \cdots, \kappa_n) \in \Gamma_k$ with $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$ and $n < 2k$. Assume that there exist constants N_0, N_1 such that $N_0 \leq \sigma_k(\kappa) \leq N_1$. If there exist* *constants K and B such that $\kappa_1 \geq B$, then*

$$\kappa_1 \left(K(\sum_j \sigma_k^{jj}(\kappa)\xi_j)^2 - \sigma_k^{pp,qq}(\kappa)\xi_p\xi_q \right) - \sigma_k^{11}(\kappa)\xi_1^2 + \sum_{j\neq 1} a_j \xi_j^2 \geq 0,$$

for any $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. Here $a_j = \sigma_k^{jj}(\kappa) + (\kappa_1 + \kappa_j)\sigma_k^{11,jj}(\kappa)$.

The main theorem is as follows.

Theorem 1.1. *Let r_1, r_2 be constants with $r_1 < r_2$, M be a compact Riemannian manifold, $\overline{M}$ be the warped product manifold with the metric (1.1) and Γ be an open neighborhood of unit normal bundle of M in $\overline{M} \times \mathbb{S}^n$. Assume that λ is a positive C^2 differential function with $\lambda' > 0$ and Conjecture 1.1 holds. Suppose ψ satisfies*

$$\psi(V, \nu) > C_n^k \zeta^k(r) \qquad \forall\ r \leq r_1, \tag{1.3}$$

$$\psi(V, \nu) < C_n^k \zeta^k(r) \qquad \forall\ r \geq r_2 \tag{1.4}$$

and

$$\frac{\partial}{\partial r}(\lambda^k \psi(V, \nu)) \leq 0 \qquad \forall\ r_1 < r < r_2, \tag{1.5}$$

where $V = \lambda \frac{\partial}{\partial r}$ and $\zeta(r) = \lambda'(r)/\lambda(r)$. Then there exists a $C^{4,\alpha}$, k-convex, star-shaped and closed hypersurface Σ in the annulus domain $\{(r, u) \in \overline{M} \mid r_1 \leq r \leq r_2\}$ that satisfies equation (1.2) for any $\alpha \in (0, 1)$.

Remark 1.2. *The key to prove Theorem 1.1 is to obtain curvature estimates (Theorem 3.4) for this Hessian type equation in warped product manifold. Compared to the proof in Euclidean space by Ren-Wang [26], we give a straightforward proof. Note that Conjecture 1.1 is weaker than the one proposed by Ren-Wang in [24, 25, 26].*

It is worth noting that Conjecture 1.1 holds for $k \geq n - 2$, which was proved in Ren-Wang [24, 25, 26]. Thus we can directly get the following results.

Corollary 1.1. *Let $k \geq n - 2$. M, $\overline{M}$, Γ, λ and ψ are proposed in Theorem 1.1, then there exists a $C^{4,\alpha}$, k-convex, star-shaped and closed hypersurface Σ in $\{(r, u) \in \overline{M} \mid r_1 \leq r \leq r_2\}$ that satisfies equation (1.2) for $\alpha \in (0, 1)$.*

The organization of the paper is as follows. In Sect. 2 we start with some preliminaries. C^0, C^1 and C^2 estimates are given in Sect. 3. In Sect. 4 we prove theorem 1.1.

After we completed our paper, we found that Wang independently proved the corresponding curvature estimates for $k = n-1, n-2$ in Theorem 4.1 of [29]. It also provides a new perspective to prove the global curvature estimates.

2. Preliminaries

2.1. **Star-shaped hypersurfaces in the warped product manifold.** Let M be a compact Riemannian manifold with the metric g' and I be an open interval in $\mathbb{R}$. Assuming $\lambda : I \to \mathbb{R}^+$ is a positive differential function and $\lambda' > 0$, the manifold $\overline{M} = I \times_\lambda M$ is called the warped product if it is endowed with the metric

$$\overline{g}^2 = dr^2 + \lambda^2(r) g'.$$

The metric in $\overline{M}$ is denoted by $\langle \cdot, \cdot \rangle$. The corresponding Riemannian connection in $\overline{M}$ will be denoted by $\overline{\nabla}$. The usual connection in M will be denoted by ∇'. The curvature tensors in M and $\overline{M}$ will be denoted by R and $\overline{R}$, respectively.

Let $\{e_1, \cdots, e_{n-1}\}$ be an orthonormal frame field in M and let $\{\theta_1, \cdots, \theta_{n-1}\}$ be the associated dual frame. The connection forms θ_{ij} and curvature forms Θ_{ij} in M satisfy the structural equations

$$d\theta_i = \sum_j \theta_{ij} \wedge \theta_j, \quad \theta_{ij} = -\theta_{ji}, \tag{2.1}$$

$$d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj} = \Theta_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \theta_l. \tag{2.2}$$

An orthonormal frame in $\overline{M}$ may be defined by $\overline{e}_i = \frac{1}{\lambda} e_i, 1 \leq i \leq n-1$, and $\overline{e}_0 = \frac{\partial}{\partial r}$. The associated dual frame is that $\overline{\theta}_i = \lambda \theta_i$ for $1 \leq i \leq n-1$ and $\overline{\theta}_0 = dr$. Then we have the following lemma (See [14]).

Lemma 2.1. *Given a differentiable function $r : M \to I$, its graph is defined by the hypersurface*

$$\Sigma = \{(r(u), u) : u \in M\}.$$

Then the tangential vector takes the form

$$X_i = \lambda \overline{e}_i + r_i \overline{e}_0,$$

where r_i are the components of the differential $dr = r_i\theta^i$. The induced metric on Σ has

$$g_{ij} = \lambda^2(r)\delta_{ij} + r_i r_j,$$

and its inverse is given by

$$g^{ij} = \frac{1}{\lambda^2}(\delta_{ij} - \frac{r^i r^j}{v^2}).$$

We also have the outward unit normal vector of Σ

$$\nu = -\frac{1}{v}\bigg(\lambda\overline{e}_0 - r^i\overline{e}_i\bigg),$$

where $v = \sqrt{\lambda^2 + |\nabla' r|^2}$ with $\nabla' r = r^i e_i$. Let h_{ij} be the second fundamental form of Σ in term of the tangential vector fields $\{X_1, ..., X_n\}$. Then,

$$h_{ij} = -\langle \overline{\nabla}_{X_j} X_i, \nu\rangle = \frac{1}{v}\bigg(- \lambda r_{ij} + 2\lambda' r_i r_j + \lambda^2\lambda'\delta_{ij}\bigg)$$

and

$$h^i_j = \frac{1}{\lambda^2 v}(\delta_{ik} - \frac{r^i r^k}{v^2})\bigg(- \lambda r_{kj} + 2\lambda' r_k r_j + \lambda^2\lambda'\delta_{kj}\bigg),$$

where r_{ij} are the components of the Hessian $\nabla'^2 r = \nabla' dr$ of r in M.

The Codazzi equation is a commutation formula for the first order derivative of h_{ij} given by

$$h_{ijk} - h_{ikj} = \overline{R}_{0ijk} \tag{2.3}$$

and the Ricci identity is a commutation formula for the second order derivative of h_{ij} given by

Lemma 2.2. *Let $\overline{X}$ be a point of Σ and $\{E_0 = \nu, E_1, \cdots, E_n\}$ be an adapted frame field such that each E_i is a principal direction and $\omega_i^k = 0$ at $\overline{X}$. Let (h_{ij}) be the second quadratic form of Σ. Then, at the point $\overline{X}$, we have*

$$h_{ii11} - h_{11ii} = h_{11}h_{ii}^2 - h_{11}^2 h_{ii} + 2(h_{ii} - h_{11})\overline{R}_{i1i1} + h_{11}\overline{R}_{i0i0} - h_{ii}\overline{R}_{1010} + \overline{R}_{i1i0;1} - \overline{R}_{1i10;i}. \tag{2.4}$$

Proof. See [6, Lemma 2.2]. □

Consider the function

$$\tau = \langle V, \nu\rangle, \qquad \Lambda(r) = \int_0^r \lambda(s)ds$$

with the position vector field

$$V = \lambda(r)\frac{\partial}{\partial_r}.$$

Then we need the following lemma for τ and Λ.

Lemma 2.3.

$$\nabla_{E_i}\Lambda = \lambda\langle \overline{e}_0, E_i\rangle E_i, \tag{2.5}$$

$$\nabla_{E_i}\tau = \sum_j \nabla_{E_j}\Lambda h_{ij}, \tag{2.6}$$

$$\nabla^2_{E_i,E_j}\Lambda = \lambda' g_{ij} - \tau h_{ij} \tag{2.7}$$

and

$$\nabla^2_{E_i,E_j}\tau = -\tau\sum_k h_{ik}h_{kj} + \lambda' h_{ij} + \sum_k \left(h_{ijk} - \overline{R}_{0ijk}\right)\nabla_{E_k}\Lambda. \tag{2.8}$$

Proof. See Lemma 2.2, Lemma 2.6 and Lemma 2.3 in [9], [17] or [6] for the proof. □

2.2. **k-th elementary symmetric functions.** Let $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{R}^n$, then we recall the definition of elementary symmetric function for $1 \le k \le n$

$$\sigma_k(\kappa) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \kappa_{i_1}\kappa_{i_2}\cdots\kappa_{i_k}.$$

Definition 2.1. *A C^2 regular hypersurface $M \subset \mathbb{R}^{n+1}$ is called k-convex if its principal curvature vector $\kappa(X) \in \overline{\Gamma}_k$ for all $X \in M$. For a domain $\Omega \subset \mathbb{R}^n$, a function $u \in C^2(\Omega)$ is called admissible if its graph is k-convex. Here Γ_k is the Gårding's cone*

$$\Gamma_k = \{\kappa \in \mathbb{R}^n : \sigma_m(\kappa) > 0, \quad m = 1, \cdots, k\}.$$

Denote $\sigma_{k-1}(\kappa|i) = \frac{\partial\sigma_k}{\partial\kappa_i}$ and $\sigma_{k-2}(\kappa|ij) = \frac{\partial^2\sigma_k}{\partial\kappa_i\partial\kappa_j}$, then we list some properties of σ_k which will be used later.

Lemma 2.4. *If $\kappa \in \Gamma_k$ and $\kappa_1 \ge \cdots \ge \kappa_k \ge \cdots \ge \kappa_n$, then we have*

(a) For any $1 \le l < k$, we have

$$\sigma_l(\kappa) \ge \kappa_1\kappa_2\cdots\kappa_l,$$

(b)

$$\sigma_k(\kappa) \le C_n^k\kappa_1\cdots\kappa_k,$$

(c)

$$\sigma_{k-1}(\kappa|k) \ge C(n,k)\sigma_{k-1}(\kappa),$$

(d)

$$-\kappa_i < \frac{(n-k)\kappa_1}{k},$$

if $\kappa_i \le 0$, $1 \le i \le n$,

(e)

$$\sum_i \sigma_{k-1}(\kappa|i)\kappa_i^2 \ge \frac{k}{n}\sigma_1(\kappa)\sigma_k(\kappa).$$

Proof. See Proposition 1.2.7, 1.2.9, Corollary 1.2.11 in [5], Lemma 2.2 in [22] and Lemma 8, 9 in [26] for the proof. □

Lemma 2.5. *Assume that* $\kappa = (\kappa_1, \cdots, \kappa_n) \in \Gamma_k$. *Then for any given indices* $1 \le i, j \le n$, *if* $\kappa_i \ge \kappa_j$, *we have*

$$|\sigma_{k-1}(\kappa|ij)| \le \sqrt{\frac{k(n-k)}{n-1}}\sigma_{k-1}(\kappa|j).$$

Proof. See Lemma 6 in [26] and the proof was given in [23]. □

Lemma 2.6. *Let* $\kappa = (\kappa_1, \cdots, \kappa_n) \in \Gamma_k$ *with* $\kappa_1 \ge \kappa_2 \ge \cdots \ge \kappa_n$ *and* $n < 2k$. *Assume that* $\sigma_k(\kappa) \ge N_0 > 0$. *Then for any* $1 \le i, j \le n$ *with* $i \ne j$, *if* $\kappa_i \ge \kappa_1 - \frac{\sqrt{\kappa_1}}{n}$, *we have*

$$\frac{2\kappa_i(1-e^{\kappa_j-\kappa_i})}{\kappa_i-\kappa_j}\sigma_k^{jj}(\kappa) \ge \sigma_k^{jj}(\kappa) + (\kappa_i+\kappa_j)\sigma_k^{ii,jj}(\kappa),$$

when κ_1 *is sufficiently large.*

Proof. See Lemma 13 in [26]. □

Lemma 2.7. *For any* $\epsilon \in (0,1)$, *there exists a positive constant* $\delta < 4\epsilon$ *such that the function* $f(x) = x - (1-\epsilon)(1-e^{-x})(x+\delta) > 0$ *for any* $x \in (0, +\infty)$.

Proof. See Lemma 2.9 in [28]. □

3. The priori estimates

In order to prove Theorem 1.1, we use the degree theory for nonlinear elliptic equation developed in [20] and the proof here is similar to [21, 17, 1, 19]. First, we consider the family of equations for $0 \le t \le 1$, $n < 2k$

$$\sigma_k(h^i_j) = \widetilde{\psi}, \tag{3.1}$$

where $\widetilde{\psi} = t\psi(r,u,\nu) + (1-t)\varphi(r)C_n^k\zeta^k(r)$, $\zeta(r) = \lambda'(r)/\lambda(r)$ and φ is a positive function which satisfies the following conditions:

(a) $\varphi(r) > 0$,

(b) $\varphi(r) \geq 1$ for $r \leq r_1$,

(c) $\varphi(r) \leq 1$ for $r \geq r_2$,

(d) $\varphi'(r) < 0$.

3.1. C^0 **Estimates.** Now, we can prove the following proposition which asserts that the solution of equation (3.1) have uniform C^0 bound.

Proposition 3.1. *Under the assumptions* (1.3) *and* (1.4) *mentioned in Theorem 1.1, if the k-convex hypersurface $\overline{\Sigma} = \{(r(u), u) \mid u \in M\} \subset \overline{M}$ satisfies the equation* (3.1) *for a given $t \in (0, 1]$, then*

$$r_1 < r(u) < r_2, \quad \forall\ u \in M.$$

Proof. Assume $r(u)$ attains its maximum at $u_0 \in M$ and $r(u_0) \geq r_2$, then recalling

$$h^i_j = \frac{1}{\lambda^2 v}(\delta_{ik} - \frac{r^i r^k}{v^2})\Big(- \lambda r_{kj} + 2\lambda' r_k r_j + \lambda^2\lambda'\delta_{kj}\Big),$$

which implies together with the fact the matrix r_{ij} is non-positive definite at u_0

$$h^i_j(u_0) = \frac{1}{\lambda^3}\Big(- \lambda r_{ij} + \lambda^2\lambda'\delta_{ij}\Big) \geq \frac{\lambda'}{\lambda}\delta_{ij}.$$

Thus, we have at u_0

$$\sigma_k(h^i_j) \geq C^k_n \zeta^k(r).$$

So, we arrive at u_0

$$t\psi(r, u, \nu) + (1 - t)\varphi(r) C^k_n \zeta^k(r) \geq C^k_n \zeta^k(r).$$

Thus, we obtain at u_0

$$\psi(r, u, \nu) \geq C^k_n \zeta^k(r),$$

which is in contradiction with (1.4). Thus, we have $r(u) < r_2$ for $u \in M$. Similarly, we can obtain $r(u) > r_1$ for $u \in M$. □

Now, we prove the following uniqueness result.

Proposition 3.2. *For $t = 0$, there exists an unique k-convex solution of the equation* (3.1), *namely $\Sigma_0 = \{(r(u), u) \in \overline{M} \mid r(u) = r_0\}$, where r_0 satisfies $\varphi(r_0) = 1$.*

Proof. Let Σ_0 be a solution of (3.1) for $t = 0$, then

$$\sigma_k(h^i_j) - \varphi(r)C^k_n\zeta^k(r) = 0.$$

Assume $r(u)$ attains its maximum r_{max} at $u_0 \in M$, then we have at u_0

$$h^i_j = \frac{1}{\lambda^3}\Big(- \lambda r_{ij} + \lambda^2\lambda'\delta_{ij}\Big),$$

which implies together with the fact the matrix r_{ij} is non-positive definite at u_0

$$\sigma_k(h^i_j) \geq C^k_n\zeta^k(r).$$

Thus, we have by the equation (3.1)

$$\varphi(r_{max}) \geq 1.$$

Similarly,

$$\varphi(r_{min}) \leq 1.$$

Thus, since φ is a decreasing function, we obtain

$$\varphi(r_{min}) = \varphi(r_{max}) = 1.$$

We conclude

$$r(u) = r_0$$

for any $(r(u), u) \in \overline{M}$, where r_0 is the unique solution of $\varphi(r_0) = 1$. □

3.2. C^1 **Estimates.** In this section, we establish gradient estimates for equation (3.1).

Theorem 3.3. *Under the assumption* (1.5)*, if the closed star-shaped k-convex hypersurface $\Sigma = \{(r(u), u) \in \overline{M} \mid u \in M\}$ satisfying the curvature equation* (3.1) *and λ has positive upper and lower bound. Then there exists a constant C depending only on $n, k, \|\lambda\|_{C^1}, \inf r, \sup r, \inf \widetilde{\psi}$, $\|\widetilde{\psi}\|_{C^1}$ and the curvature $\overline{R}$ such that*

$$|\nabla r| \leq C.$$

Proof. As the treatment in [6], it is sufficient to obtain a positive lower bound of τ. If V is parallel to the normal direction ν of at u_0, we can obtain the lower bound of τ. Thus, our result holds. So we assume V is not parallel to the normal direction ν at u_0 and derive a contradiction. More details can refer to Lemma 3.1 in [6]. □

3.3. C^2 **Estimates.** Under the assumptions (1.3)-(1.5), from Theorem 3.1 and 3.3 we know that there exists a positive constant C_0 depending on $\inf_\Sigma r$ and $\|r\|_{C^1}$ such that

$$\frac{1}{C_0} \leq inf_\Sigma \tau \leq \tau \leq \sup_\Sigma \tau \leq C_0.$$

Theorem 3.4. *Let Σ be a closed star-shaped k-convex hypersurface satisfying equation* (3.1) *and the assumptions of Theorem 1.1 with $n < 2k$. Then there exists a constant C depending only on $n, k, \|\lambda\|_{C^1}, \|r\|_{C^1}, \inf \lambda', \inf r, \sup r, \inf \widetilde{\psi}$, $\|\widetilde{\psi}\|_{C^1}$ and the curvature $\overline{R}$ such that for $1 \leq i \leq n$*

$$|\kappa_i(u)| \leq C, \quad \forall\ u \in M.$$

Proof. Taking the allxillary function

$$Q = \log \kappa_1 - A\tau + B\Lambda,$$

where $A, B > 1$ are constants to be determined later. Suppose Q attains its maximum at V_0. We can choose a local orthonormal frame $\{E_1, E_2, \cdots, E_n\}$ near V_0 such that (h_{ij}) is diagonalized. Without loss of generality, we may assume κ_1 has multiplicity m, then

$$h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_1 = \cdots = \kappa_m > \kappa_{m+1} \geq \cdots \geq \kappa_n \quad \text{at } V_0.$$

As the perturbation argument in [7], we need to perturb h_{ij} by a diagonal matrix T which satisfies

$$T_{ij} = \delta\delta_{ij}(1 - \delta_{1i}), \quad T_{ij,p} = T_{11,ii} = 0 \quad \text{at } V_0,$$

$\delta < 1$ is a sufficiently small constant to be determined later. Thus we define $\widetilde{h}_{ij} = h_{ij} - T_{ij}$ and denote its eigenvalues by $\widetilde{\kappa}_1 \geq \widetilde{\kappa}_2 \geq \cdots \geq \widetilde{\kappa}_n$. It then follows that $\kappa_1 \geq \widetilde{\kappa}_1$ near V_0 and

$$\widetilde{\kappa}_i = \begin{cases} \kappa_1, & \text{if } i = 1, \\ \kappa_i - \delta, & \text{if } i > 1, \end{cases} \quad \text{at } V_0.$$

Thus $\widetilde{\kappa}_1 > \widetilde{\kappa}_2$ at V_0, then $\widetilde{\kappa}_1$ is smooth at V_0. We consider the new function

$$\widetilde{Q} = \log \widetilde{\kappa}_1 - A\tau + B\Lambda.$$

It still attains its maximum at V_0. Since $\widetilde{\kappa}_1 = \kappa_1$ at V_0, then at V_0 we have

$$0 = \widetilde{Q}_i = \frac{\widetilde{\kappa}_{1,i}}{\widetilde{\kappa}_1} - A\tau_i + B\Lambda_i = \frac{\widetilde{\kappa}_{1,i}}{\widetilde{\kappa}_1} - A\sum_j h_{ij}\Lambda_j + B\Lambda_i, \tag{3.2}$$

and

$$
\begin{aligned}
0 \geq \sigma_k^{ii}\widetilde{Q}_{ii} &= \sigma_k^{ii}(\log\widetilde{\kappa}_1)_{ii} - A\sigma_k^{ii}\tau_{ii} + B\sigma_k^{ii}\Lambda_{ii} \\
&= \sigma_k^{ii}(\log\widetilde{\kappa}_1)_{ii} - A\sigma_k^{ii}\{-\tau h_{ii}^2 + \lambda' h_{ii} + \sum_l (h_{iil} - \overline{R}_{0iil})\Lambda_l\} \\
&\quad + B\sigma_k^{ii}(\lambda' g_{ii} - \tau h_{ii}).
\end{aligned} \tag{3.3}
$$

Note that

$$
\begin{aligned}
h_{11ii} &= h_{ii11} + h_{11}^2 h_{ii} - h_{ii}^2 h_{11} + \overline{R}_{0ii1;1} + \overline{R}_{01i1;i} + h_{i1}\overline{R}_{0i01} + h_{1i}\overline{R}_{01i0} \\
&\quad -2h_{11}\overline{R}_{1ii1} + h_{11}\overline{R}_{0ii0} - 2h_{ii}\overline{R}_{i1i1} + h_{ii}\overline{R}_{0101}.
\end{aligned} \tag{3.4}
$$

We divide our proof in three steps. For convenience, we will use a unified notation C to denote a constant depending on $n, k, \|\lambda\|_{C^1}, \|r\|_{C^1}, \inf\lambda', \inf r, \sup r, \inf\widetilde{\psi}$, $\|\widetilde{\psi}\|_{C^2}$ and the curvature $\overline{R}$.

Step 1: We show that

$$
\begin{aligned}
0 &\geq -\frac{1}{\kappa_1}\sum_{p\neq q}\sigma_k^{pp,qq}h_{pp1}h_{qq1} + 2\sum_{p>1}\frac{\sigma_k^{pp}h_{1pp}^2}{\kappa_1(\kappa_1 - \widetilde{\kappa}_p)} - \frac{\sigma_k^{11}h_{111}^2}{\kappa_1^2} - \frac{C}{\kappa_1}\sum_{p>m}\sigma_k^{11,pp} \\
&\quad + (A\tau - 1)\sigma_k^{ii}h_{ii}^2 + (B\lambda' - CA - \frac{C}{\delta\kappa_1})\sum_i \sigma_k^{ii} - C(h_{11} + A + B).
\end{aligned} \tag{3.5}
$$

The following calculations are all at V_0. By Lemma 3.1 in [7], we know that

$$
\widetilde{\kappa}_{1,i} = h_{11i}, \quad \widetilde{\kappa}_{1,ii} = h_{11ii} + 2\sum_{p>1}\frac{h_{1pi}^2}{\kappa_1 - \widetilde{\kappa}_p}.
$$

Differentiating (3.1) twice, we obtain

$$
\sigma_k^{ii}h_{iij} = d_V\widetilde{\psi}(\nabla_j V) + d_\nu\widetilde{\psi}(\nabla_j\nu) = \lambda' d_V\widetilde{\psi}(E_j) + h_{jl}d_\nu\widetilde{\psi}(E_l) \tag{3.6}
$$

and

(3.7)

$$
\begin{aligned}
&\sigma_k^{ii}h_{ii11} + \sigma_k^{ij,pq}h_{ij1}h_{pq1} \\
=& d_V\widetilde{\psi}(\nabla_{11}V) + d_V^2\widetilde{\psi}(\nabla_1 V, \nabla_1 V) + 2d_V d_\nu\widetilde{\psi}(\nabla_1 V, \nabla_1\nu) + d_\nu^2\widetilde{\psi}(\nabla_1\nu, \nabla_1\nu) + d_\nu\widetilde{\psi}(\nabla_{11}\nu) \\
\geq& -C - Ch_{11}^2 + \sum_l h_{l11}d_\nu\widetilde{\psi}(E_l).
\end{aligned}
$$

Without loss of generality, we assume that $\kappa_1 \geq 1$, then by (3.4) and (3.7)

$$\begin{aligned}
\sigma_k^{ii}(\log \widetilde{\kappa}_1)_{ii} &= \frac{\sigma_k^{ii}\widetilde{\kappa}_{1,ii}}{\widetilde{\kappa}_1} - \frac{\sigma_k^{ii}\widetilde{\kappa}_{1,i}^2}{\widetilde{\kappa}_1^2} \\
&= \frac{\sigma_k^{ii}h_{11ii}}{\kappa_1} + 2\sum_{p>1}\frac{\sigma_k^{ii}h_{1pi}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} - \frac{\sigma_k^{ii}h_{11i}^2}{\kappa_1^2} \\
&\geq -\frac{\sigma_k^{ij,pq}h_{ij1}h_{pq1}}{\kappa_1} + 2\sum_{p>1}\frac{\sigma_k^{ii}h_{1pi}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} + \frac{1}{\kappa_1}\sum_l h_{l11}d_\nu\widetilde{\psi}(E_l) \\
&\quad -\frac{\sigma_k^{ii}h_{11i}^2}{\kappa_1^2} - \sigma_k^{ii}h_{ii}^2 - C\sum_i \sigma_k^{ii} - Ch_{11} - C. \qquad (3.8)
\end{aligned}$$

Combining (3.2), (3.6) and Codazzi equation, we get

$$\begin{aligned}
\frac{1}{\kappa_1}\sum_l h_{l11}d_\nu\widetilde{\psi}(E_l) &= \frac{1}{\kappa_1}\sum_l(h_{11l}+\overline{R}_{01l1})d_\nu\widetilde{\psi}(E_l) \\
&= \sum_l(A\sum_j h_{lj}\Lambda_j - B\Lambda_l)d_\nu\widetilde{\psi}(E_l) + \sum_l\frac{1}{\kappa_1}\overline{R}_{01l1}d_\nu\widetilde{\psi}(E_l) \\
&= A\sum_j\sigma_k^{ii}h_{iij}\Lambda_j - A\lambda'\sum_j d_V\widetilde{\psi}(E_j)\Lambda_j - B\sum_l\Lambda_l d_\nu\widetilde{\psi}(E_l) \\
&\quad +\sum_l\frac{1}{\kappa_1}\overline{R}_{01l1}d_\nu\widetilde{\psi}(E_l). \qquad (3.9)
\end{aligned}$$

Putting (3.8)-(3.9) into (3.3), we obtain

$$\begin{aligned}
0 &\geq -\frac{\sigma_k^{ij,pq}h_{ij1}h_{pq1}}{\kappa_1} + 2\sum_{p>1}\frac{\sigma_k^{ii}h_{1pi}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} - \frac{\sigma_k^{ii}h_{11i}^2}{\kappa_1^2} \\
&\quad +(A\tau-1)\sigma_k^{ii}h_{ii}^2 - (A\lambda'+B\tau)k\widetilde{\psi} \\
&\quad +(B\lambda'-CA)\sum_i\sigma_k^{ii} - Ch_{11} - C(A+B). \qquad (3.10)
\end{aligned}$$

Since

$$\begin{aligned}
(3.11)\quad & -\frac{\sigma_k^{ij,pq}h_{ij1}h_{pq1}}{\kappa_1} + 2\sum_{p>1}\frac{\sigma_k^{ii}h_{1pi}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} \\
\geq\ & -\frac{1}{\kappa_1}\sum_{p\neq q}\sigma_k^{pp,qq}h_{pp1}h_{qq1} + \frac{1}{\kappa_1}\sum_{p\neq q}\sigma_k^{pp,qq}h_{pq1}^2 + 2\sum_{p>1}\frac{\sigma_k^{pp}h_{1pp}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} + 2\sum_{p>1}\frac{\sigma_k^{11}h_{1p1}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} \\
\geq\ & -\frac{1}{\kappa_1}\sum_{p\neq q}\sigma_k^{pp,qq}h_{pp1}h_{qq1} + \frac{2}{\kappa_1}\sum_{p>m}\sigma_k^{11,pp}h_{1p1}^2 + 2\sum_{p>1}\frac{\sigma_k^{pp}h_{1pp}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} + 2\sum_{p>1}\frac{\sigma_k^{11}h_{1p1}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)}.
\end{aligned}$$

When $\kappa_p \geq 0$, then by choosing $\kappa_1 \geq 2\delta$, we have

$$\frac{\frac{1}{2}\kappa_1 + \widetilde{\kappa}_p}{\kappa_1 - \widetilde{\kappa}_p} = \frac{\frac{1}{2}\kappa_1 + \kappa_p - \delta}{\kappa_1 - \kappa_p + \delta} \geq 0.$$

When $\kappa_p < 0$, then by choosing $\kappa_1 \geq \frac{2k\delta}{3k-2n}$, we have

$$\frac{\frac{1}{2}\kappa_1 + \widetilde{\kappa}_p}{\kappa_1 - \widetilde{\kappa}_p} = \frac{\frac{1}{2}\kappa_1 + \kappa_p - \delta}{\kappa_1 - \kappa_p + \delta} = -1 + \frac{3}{2(1 - \frac{\kappa_p}{\kappa_1} + \frac{\delta}{\kappa_1})} \geq -1 + \frac{3}{2(1 - \frac{\kappa_p}{\kappa_1} + \frac{3}{2} - \frac{n}{k})} \geq 0.$$

Hence by Cauchy-Schwarz inequality, Codazzi equation and choosing $\kappa_1 \geq \max\{2\delta, \frac{2k\delta}{3k-2n}\}$, we derive

$$\begin{aligned}
(3.12)\qquad & \frac{2}{\kappa_1}\sum_{p>m}\sigma_k^{11,pp}h_{1p1}^2 + 2\sum_{p>1}\frac{\sigma_k^{11}h_{1p1}^2}{\kappa_1(\kappa_1 - \widetilde{\kappa}_p)} - \sum_{p>1}\frac{\sigma_k^{pp}h_{11p}^2}{\kappa_1^2} \\
= \;& \frac{2}{\kappa_1}\sum_{p>m}\sigma_k^{11,pp}(h_{11p} + \overline{R}_{01p1})^2 + 2\sum_{p>1}\frac{\sigma_k^{11}(h_{11p} + \overline{R}_{01p1})^2}{\kappa_1(\kappa_1 - \widetilde{\kappa}_p)} - \sum_{p>1}\frac{\sigma_k^{pp}h_{11p}^2}{\kappa_1^2} \\
\geq \;& \frac{3}{2}\sum_{p>m}\frac{(\sigma_k^{pp} - \sigma_k^{11})h_{11p}^2}{\kappa_1(\kappa_1 - \kappa_p)} + \frac{3}{2}\sum_{p>1}\frac{\sigma_k^{11}h_{11p}^2}{\kappa_1(\kappa_1 - \widetilde{\kappa}_p)} - \sum_{p>1}\frac{\sigma_k^{pp}h_{11p}^2}{\kappa_1^2} \\
& -\frac{C}{\kappa_1}\sum_{p>m}\sigma_k^{11,pp} - \frac{C}{\delta\kappa_1}\sum_i \sigma_k^{ii} \\
\geq \;& \sum_{p>m}\sigma_k^{11}\frac{h_{11p}^2}{\kappa_1^2}\frac{\frac{1}{2}\kappa_1 + \widetilde{\kappa}_p}{\kappa_1 - \widetilde{\kappa}_p} - \frac{C}{\kappa_1}\sum_{p>m}\sigma_k^{11,pp} - \frac{C}{\delta\kappa_1}\sum_i \sigma_k^{ii} \\
\geq \;& -\frac{C}{\kappa_1}\sum_{p>m}\sigma_k^{11,pp} - \frac{C}{\delta\kappa_1}\sum_i \sigma_k^{ii}.
\end{aligned}$$

Putting (3.11)-(3.12) into (3.10), we obtain (3.5).

Step 2: Next we show that

$$\sum_{p>m}\sigma_k^{11,pp} \leq C\sum_i \sigma_k^{ii}.$$

We shall discuss into two cases.

Case 1. If $\sigma_{k-1} \geq \sigma_{k-2}$. Accorrding to Lemma 2.5, since $\kappa_1 \geq \kappa_p$ for $p > m$, then we have

$$\sum_{p>m}\sigma_k^{11,pp} \leq \sum_{p>m}|\sigma_{k-1}(\kappa|1p)| \leq \sqrt{\frac{k(n-k)}{n-1}}\sum_p \sigma_{k-1}(\kappa|p) \leq \sqrt{\frac{k(n-k)}{n-1}}\sum_i \sigma_k^{ii}.$$

Case 2. If $\sigma_{k-1} \leq \sigma_{k-2}$, by Lemma 2.4 we know that

$$\kappa_1 \cdots \kappa_{k-1} \leq \sigma_{k-1} \leq \sigma_{k-2} \leq C_n^{k-2}\kappa_1 \cdots \kappa_{k-2},$$

which implies that $\kappa_{k-1} \le C$. Then we divide into two sub-cases to discuss for $p > m$. Without loss of generality, we assume that $\kappa_1 \ge 1$.

Subcase 2.1: If $2\kappa_p \le \kappa_1$, then for $p > m$

$$\sigma_k^{11,pp} = \frac{\sigma_k^{pp} - \sigma_k^{11}}{\kappa_1 - \kappa_p} \le \frac{\sigma_k^{pp} - \sigma_k^{11}}{\frac{\kappa_1}{2}} \le 2\sigma_k^{pp} \le C \sum_i \sigma_k^{ii}.$$

Subcase 2.2: For sufficiently large κ_1, if $2\kappa_p > \kappa_1$, by $\kappa_{k-1} \le C$, we have $m < p \le k-1$, then by Lemma 2.4

$$\sigma_k^{11,pp} = \sigma_{k-2}(\kappa|1p) \le C\frac{\kappa_2 \cdots \kappa_k}{\kappa_p} \le C\kappa_1 \cdots \kappa_{k-1} \le C\sigma_{k-1} \le C\sigma_{k-1}(\kappa|k) \le C\sum_i \sigma_k^{ii}.$$

Step 3: By concavity of $\sigma_k^{\frac{1}{k}}$, we get

$$-\frac{\epsilon}{\kappa_1}\sum_{p\neq q} \sigma_k^{pp,qq} h_{pp1}h_{qq1} \ge -\epsilon\frac{k-1}{k}\frac{(\widetilde{\psi}_1)^2}{\widetilde{\psi}\kappa_1} \ge -C\epsilon\kappa_1. \tag{3.13}$$

According to Conjecture 1.1, we have

$$\begin{aligned}
& -\frac{1-\epsilon}{\kappa_1}\sum_{p\neq q}\sigma_k^{pp,qq}h_{pp1}h_{qq1} + 2\sum_{p>1}\frac{\sigma_k^{pp}h_{1pp}^2}{\kappa_1(\kappa_1 - \widetilde{\kappa}_p)} - \frac{\sigma_k^{11}h_{111}^2}{\kappa_1^2} \\
\ge\ & -\frac{K(1-\epsilon)}{\kappa_1}(\sigma_k^{jj}h_{jj1})^2 - \frac{1-\epsilon}{\kappa_1^2}\sum_{j>1}a_jh_{jj1}^2 - \frac{\epsilon}{\kappa_1^2}\sigma_k^{11}h_{111}^2 + 2\sum_{p>1}\frac{\sigma_k^{pp}h_{1pp}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} \\
\ge\ & -CK(1+\kappa_1) - \epsilon\frac{\sigma_k^{11}h_{111}^2}{\kappa_1^2} + 2\sum_{1<p\le m}\frac{\sigma_k^{11}h_{1pp}^2}{\delta\kappa_1} + 2\sum_{p>m}\frac{\sigma_k^{pp}h_{1pp}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} \\
& -\frac{1-\epsilon}{\kappa_1^2}\sum_{1<p\le m}a_ph_{pp1}^2 - \frac{1-\epsilon}{\kappa_1^2}\sum_{p>m}a_ph_{pp1}^2.
\end{aligned} \tag{3.14}$$

Lemma 2.6 implies that $a_p = \sigma_k^{11} + 2\kappa_1\sigma_k^{11,pp} \leq 2\kappa_1\sigma_k^{pp}$ for $1 < p \leq m$. Hence by Cauchy-Schwarz inequality and choosing $\delta \leq \frac{3}{4}$, we derive

$$\begin{aligned}
& 2\sum_{1<p\leq m} \frac{\sigma_k^{11}h_{1pp}^2}{\delta\kappa_1} - \frac{1-\epsilon}{\kappa_1^2}\sum_{1<p\leq m} a_p h_{pp1}^2 \\
\geq\ & \frac{3}{2}\sum_{1<p\leq m} \frac{\sigma_k^{11}h_{pp1}^2}{\delta\kappa_1} - \frac{C\sum_i \sigma_k^{ii}}{\delta\kappa_1} - \frac{1-\epsilon}{\kappa_1^2}\sum_{1<p\leq m} a_p h_{pp1}^2 \\
\geq\ & \sum_{1<p\leq m} \sigma_k^{11}\frac{h_{pp1}^2}{\kappa_1}\left(\frac{3}{2\delta} - 2(1-\epsilon)\right) - \frac{C\sum_i \sigma_k^{ii}}{\delta\kappa_1} \\
\geq\ & -\frac{C\sum_i \sigma_k^{ii}}{\delta\kappa_1}.
\end{aligned} \tag{3.15}$$

For $p > m$, according to Lemma 2.6, when κ_1 is sufficiently large, we get

$$\frac{2\kappa_1(1-e^{\kappa_p-\kappa_1})}{\kappa_1-\kappa_p}\sigma_k^{pp} \geq \sigma_k^{pp} + (\kappa_1+\kappa_p)\sigma_k^{11,pp} = a_p.$$

Then choosing $\theta = 1-\sqrt{1-\epsilon}$, we have

$$\begin{aligned}
& 2\sum_{p>m} \frac{\sigma_k^{pp}h_{1pp}^2}{\kappa_1(\kappa_1-\widetilde{\kappa}_p)} - \frac{1-\epsilon}{\kappa_1^2}\sum_{p>m} a_p h_{pp1}^2 \\
\geq\ & \sum_{p>m} \frac{\sigma_k^{pp}h_{pp1}^2}{\kappa_1}\left(\frac{2(1-\theta)}{\kappa_1-\kappa_p+\delta} - \frac{2(1-\epsilon)(1-e^{\kappa_p-\kappa_1})}{\kappa_1-\kappa_p}\right) - \frac{C_\theta}{\delta\kappa_1}\sum_i \sigma_k^{ii} \\
=\ & \sum_{p>m} \frac{2(1-\theta)\sigma_k^{pp}h_{pp1}^2}{\kappa_1(\kappa_1-\kappa_p)(\kappa_1-\kappa_p+\delta)}(\kappa_1-\kappa_p-(1-\theta)(1-e^{\kappa_p-\kappa_1})(\kappa_1-\kappa_p+\delta)) \\
& -\frac{C_\theta}{\delta\kappa_1}\sum_i \sigma_k^{ii} \\
\geq\ & -\frac{C_\theta}{\delta\kappa_1}\sum_i \sigma_k^{ii},
\end{aligned} \tag{3.16}$$

the last inequality comes from Lemma 2.7 by choosing $\delta < 4\theta$. Here C_θ is a constant depending only on θ.

Using (3.2) and Cauchy-Schwarz inequality, we derive

$$\begin{aligned}\epsilon\frac{\sigma_k^{11}h_{111}^2}{\kappa_1^2} &= \epsilon\sigma_k^{11}(A\sum_j h_{1j}\Lambda_j - B\Lambda_1)^2\\ &\leq C\epsilon A^2\sum_i \sigma_k^{ii}h_{ii}^2 + C\epsilon B^2\sigma_k^{11}\\ &\leq \frac{A}{2C_0}\sum_i \sigma_k^{ii}h_{ii}^2,\end{aligned} \tag{3.17}$$

by choosing $h_{11} \geq \frac{CB}{A}$ and $\epsilon \leq \frac{1}{CC_0A}$. Then combining Step 1-Step 2, (3.13)-(3.17) and the fact that $\sigma_k^{ii}h_{ii}^2 \geq C\kappa_1$, we obtain

$$\begin{aligned}0 &\geq \left(\frac{A}{2C_0} - 1\right)\sigma_k^{ii}h_{ii}^2 + \left(B\lambda' - CA - \frac{C_\theta + C}{\delta\kappa_1}\right)\sum_i \sigma_k^{ii}\\ &\quad -C(K+1)\kappa_1 - C(A+B+K)\\ &\geq \kappa_1 - C(A+B+K),\end{aligned}$$

by choosing $\kappa_1 \geq \frac{C_\theta + C}{\delta}$, $B \geq \frac{CA+1}{\inf\limits_{r_1 \leq r \leq r_2}\lambda'}$, $A \geq 2C_0(\frac{1}{C} + K + 2)$. Then we can derive $\kappa_1 \leq C(A+B+K)$, the proof is completed. □

4. The proof of Theorem 1.1

In this section, we use the degree theory for nonlinear elliptic equation developed in [20] to prove Theorem 1.1. The proof here is similar to [1, 17, 19]. So, only sketch will be given below.

After establishing the priori estimates in Theorem 3.1, Theorem 3.3 and Theorem 3.4, we know that the equation (3.1) is uniformly elliptic. From [8], [18], and Schauder estimates, we have

$$|r|_{C^{4,\alpha}(M)} \leq C \tag{4.1}$$

for any k-convex solution M to the equation (3.1), where the position vector of Σ is $X = (r(u), u)$ for $u \in M$. We define

$$C_0^{4,\alpha}(M) = \{r \in C^{4,\alpha}(M) : \Sigma \text{ is } k\text{-convex}\}.$$

Let us consider

$$F(.;t) : C_0^{4,\alpha}(M) \to C^{2,\alpha}(M),$$

which is defined by

$$F(r,u;t)=\sigma_k(h^i_j)-t\psi(r,u,\nu)-(1-t)\varphi(r)C^k_n\zeta^k(r).$$

Let

$$\mathcal{O}_R=\{r\in C^{4,\alpha}_0(M):|r|_{C^{4,\alpha}(M)}<R\},$$

which clearly is an open set of $C^{4,\alpha}_0(M)$. Moreover, if R is sufficiently large, $F(r,u;t)=0$ has no solution on $\partial\mathcal{O}_R$ by the priori estimate established in (4.1). Therefore the degree $\deg(F(.;t),\mathcal{O}_R,0)$ is well-defined for $0\le t\le 1$. Using the homotopic invariance of the degree, we have

$$\deg(F(.;1),\mathcal{O}_R,0)=\deg(F(.;0),\mathcal{O}_R,0).$$

Theorem 3.2 shows that r_0 which satisfies $\varphi(r_0)=1$ is the unique solution to the above equation for $t=0$. Direct calculation shows that

$$F(sr_0,u;0)=(1-\varphi(sr_0))C^k_n\zeta^k(sr_0).$$

Then

$$\delta_{r_0}F(r_0,u;0)=\frac{d}{ds}|_{s=1}F(sr_0,u;0)=-r_0\varphi'(r_0)C^k_n\zeta^k(r_0),$$

where $\delta F(r_0,u;0)$ is the linearized operator of F at r_0. Clearly, $\delta_wF(r_0,u;0)$ takes the form

$$\delta_wF(r_0,u;0)=-a^{ij}w_{ij}+b^iw_i-\varphi'(r_0)C^k_n\zeta^k(r_0)w,$$

where (a^{ij}) is a positive definite matrix. Since $-\varphi'(r_0)C^k_n\zeta^k(r_0)>0$, thus $\delta F(r_0,u;0)$ is an invertible operator. Therefore,

$$\deg(F(.;1),\mathcal{O}_R;0)=\deg(F(.;0),\mathcal{O}_R,0)=\pm 1.$$

So, we obtain a solution at $t=1$. This completes the proof of Theorem 1.1.

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