ONE-SIDE LIOUVILLE THEOREMS UNDER AN EXPONENTIAL GROWTH CONDITION FOR KOLMOGOROV OPERATORS

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ABSTRACT. It is known that for a possibly degenerate hypoelliptic Ornstein-Uhlenbeck operator

$$L = \frac{1}{2} \operatorname{tr}(QD^2) + \langle Ax, D \rangle = \frac{1}{2} \operatorname{div}(QD) + \langle Ax, D \rangle, \ x \in \mathbb{R}^N,$$

all (globally) bounded solutions of Lu = 0 on \mathbb{R}^N are constant if and only if all the eigenvalues of A have non-positive real parts (i.e., $s(A) \leq 0$). We show that if Q is positive definite and $s(A) \leq 0$, then any nonnegative solution v of Lv = 0 on \mathbb{R}^N which has at most an exponential growth is indeed constant. Thus under a non-degeneracy condition we relax the boundedness assumption on the harmonic functions and maintain the sharp condition on the eigenvalues of A. We also prove a related one-side Liouville theorem in the case of hypoelliptic Ornstein-Uhlenbeck operators.

1. INTRODUCTION

Let Q be a symmetric non-negative definite $N \times N$ matrix and let A be a real $N \times N$ matrix. The possibly degenerate Ornstein-Uhlenbeck operator (briefly OU operator) associated with (Q, A) is defined as

$$L = \frac{1}{2} \operatorname{tr}(QD^2) + \langle Ax, D \rangle = \frac{1}{2} \operatorname{div}(QD) + \langle Ax, D \rangle, \quad x \in \mathbb{R}^N.$$
(1)

We will assume the so-called Kalman controllability condition (see, for instance, Chapter 1 in [21]):

$$\operatorname{rank}[\sqrt{Q}, A\sqrt{Q}, \dots, A^{N-1}\sqrt{Q}] = N.$$
(2)

This is equivalent to the hypoellipticity of $L - \partial_t$ (see [12] and [13] for more details). Clearly, in the non-degenerate case when Q is positive definite we have that condition (2) holds.

We study one-side Liouville type theorems for L, i.e., we want to know when non-negative smooth solutions $u: \mathbb{R}^N \to \mathbb{R}$ to

$$Lu(x) = 0, \ x \in \mathbb{R}^N,$$

are constant. Such functions are called positive (non-negative) harmonic functions for L (see [14]). Positive harmonic functions are important in the

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study of the Martin boundary for L (see [3] for the non-degenerate twodimensional case and Chapter 7 in [14] for more information on the Martin boundary for non-degenerate diffusions).

Before stating our main result we discuss a known Liouville theorem concerning bounded harmonic functions (briefly BHFs) for L. To this purpose we introduce the spectral bound of A:

$$s(A) = \max\{Re(\lambda) : \lambda \in \sigma(A)\}.$$
(3)

where $\sigma(A)$ is the spectrum of A (i.e., set of the eigenvalues of A). By Theorem 3.8 of [16] it follows that all BHFs for the hypoelliptic OU operator L are constant if and only if $s(A) \leq 0$ (cf. Theorem 2.1 and see the comments in Section 2 for more details). Liouville theorems involving BHFs for diffusions are considered, for instance, in [14], [1], [16] and [18]. Liouville theorems involving BHFs for purely non-local OU operators are proved in [20] and [19]; see also Section 6. Recall that Liouville theorems for BHFs have probabilistic interpretations, in terms of absorption functions (see Chapter 9 in [14] and Section 6 in [16]) and in terms of successful couplings (see [20], [19] and the references therein).

Theorem 3.8 in [16] suggests the natural question if under the assumption $s(A) \leq 0$ we have more generally a one-side Liouville type theorem for L. In general, this is an open problem (see also Section 6). However one-side Liouville theorems for L have been proved in some cases under specific assumptions on Q and A (such assumptions imply that $s(A) \leq 0$). We refer to [8], [9], [10], [11] and the references therein; the papers [8], [9] and [10] contain also one-side Liouville theorems for other classes of Kolmogorov operators. We mention the main result in [11] where it is proved a one-side Liouville theorem assuming that Q is positive definite and that the norm of the exponential matrix e^{tA} is uniformly bounded when $t \in \mathbb{R}$ (cf. Theorem 2.4 and the related discussion in Section 2).

The main result of this work states that if Q is positive definite and $s(A) \leq 0$, then any positive harmonic function v for L which has at most an exponential growth is indeed constant. Thus under a non-degeneracy condition we relax the boundedness assumption on the harmonic functions of [16] and maintain the sharp condition on the eigenvalues of A (see Theorem 4.1 for the precise statement). We also prove a related one-side Liouville theorem under a sublinear growth condition which is valid for hypoelliptic OU operators (see Theorem 4.2).

The plan of the paper is the following one.

We recall and discuss known results in Section 2. In Section 3 we prove the convexity of positive harmonic functions for L under an exponential growth condition. This is a consequence of a result given in [17]. Such convexity property will be the starting point for the proof of our main result. We state and discuss Theorem 4.1 in Section 4 where we also present Example 1 to illustrate an idea of the proof in a significant case. The complete proof of the main result is given in Section 5. We finish the paper by presenting some open problems.

1.1. Notations. We denote by $|\cdot|$ the usual euclidean norm in any \mathbb{R}^k , $k \geq 1$. Moreover, $x \cdot y$ or $\langle x, y \rangle$ indicates the usual inner product in \mathbb{R}^k , $x, y \in \mathbb{R}^k$. The canonical basis in \mathbb{R}^k is denoted by $(e_i)_{i=1,...,k}$.

Let $k \geq 1$. Given a regular function $u : \mathbb{R}^k$, we denote by $D^2u(x)$ the $k \times k$ Hessian matrix of u at $x \in \mathbb{R}^k$, i.e., $D^2u(x) = (\partial_{x_ix_j}^2u(x))_{i,j=1,\ldots,k}$, where $\partial_{x_ix_j}^2u$ are the usual second order partial derivatives of u. Similarly we define the gradient $Du(x) \in \mathbb{R}^k$.

Given a real $k \times k$ matrix A, ||A|| denotes its operator norm and tr(A) its trace.

Given a symmetric non-negative definite $k \times k$ matrix Q we denote by N(0,Q) the symmetric Gaussian measure with mean 0 and covariance matrix Q (see, for instance, Section 1.7 in [2] or Section 2.2 in [5]). If in addition Q is positive definite than N(0,Q) has the density $\frac{1}{\sqrt{(2\pi)^k \det(Q)}} e^{-\frac{1}{2}\langle Q^{-1}x,x\rangle}$, $x \in \mathbb{R}^k$, with respect to the k-dimensional Lebesgue measure.

2. Some known results

First we introduce the Banach space $B_b(\mathbb{R}^N)$ of all Borel and bounded functions from \mathbb{R}^N into \mathbb{R} endowed with the supremum norm. We define the following semigroup of operators (P_t) acting on $B_b(\mathbb{R}^N)$:

$$(P_t f)(x) = P_t f(x) = \int_{\mathbb{R}^N} f(e^{tA}x + y)N(0, Q_t)dy, \ x \in \mathbb{R}^N, \ t > 0,$$
(4)

 $P_0 f = f, f \in B_b(\mathbb{R}^N)$; here $N(0, Q_t)$ is the Gaussian measure with mean 0 and covariance matrix

$$Q_t = \int_0^t e^{sA} Q e^{sA^*} ds$$

(see, for instance, Chapter 6 in [4] for more details). We are using exponential matrices e^{sA} and e^{sA^*} where A^* denotes the transpose of the matrix A; (P_t) is called the Ornstein-Uhlenbeck semigroup (brefly the OU semigroup).

Recall that the Kalman condition (2) is equivalent to the fact that Q_t is positive definite for t > 0 (cf. Section 1.3 in [21]). It is also equivalent to the strong Feller property of (P_t) and, moreover, to the fact that $P_t(B_b(\mathbb{R}^N)) \subset C_b^{\infty}(\mathbb{R}^N), t > 0$ (see, for instance, Chapter 6 in [4]).

Now we mention a special case of Theorem 3.8 in [16] (this covers also some classes of non-local OU operators).

Theorem 2.1 ([16]). Let us consider the hypoelliptic OU operator L (i.e., we assume (2)). Let $w \in C^2(\mathbb{R}^N)$ be a bounded solution to Lw(x) = 0 on \mathbb{R}^N (¹). Then w is constant if and only if $s(A) \leq 0$.

Remark 2.2. Note that a smooth bounded real function w is a solution to Lw(x) = 0 on \mathbb{R}^N if and only if it is a bounded harmonic functions for the OU semigroup, i.e.,

$$P_t w = w \text{ on } \mathbb{R}^N, t \ge 0.$$

This fact can be easily proved by using the Itô formula (we point out that the Itô formula will be also used in the proof of Proposition 3.2).

To prove \Leftarrow in the previous theorem one uses the next result, the proof of which uses control theoretic techniques.

¹L is hypoelliptic, so that every distributional solution to Lw = 0 is of class C^{∞} .

Theorem 2.3 ([15]). Let us consider the matrix $Q_t^{-1/2}e^{tA}$, t > 0 (this is well-defined by (2)). We have

$$\|Q_t^{-1/2}e^{tA}\| \to 0 \quad as \ t \to \infty \quad \Longleftrightarrow \quad s(A) \le 0.$$
(5)

Now we state a one-side Liouville type theorem proved in Theorem 1.1 of [11] which will be used in the sequel.

Theorem 2.4 ([11]). Let Q be a $N \times N$ positive definite matrix. Suppose that $\sup_{t \in \mathbb{R}} \|e^{tA}\| < \infty$ (²). Let v be a smooth non-negative solution to Lv = 0 on \mathbb{R}^N . Then v is constant.

Clearly, we can replace in the previous theorem the condition that v is non-negative by requiring that v is bounded from above or from below.

Remark 2.5. (i) Theorem 1.1 in [11] is proved when Q = I. On the other hand, if Q is positive definite then by using the change of variable $u(x) = l(Q^{-1/2}x)$, $l(y) = u(Q^{1/2}y)$, $y \in \mathbb{R}^N$, we can pass from an OU operator associated with (Q, A) to an OU operator associated with $(I, Q^{-1/2}AQ^{1/2})$. In particular, we have Lu(x) = 0, $x \in \mathbb{R}^N$, if and only if

$$\frac{1}{2} \triangle l(y) + \langle Q^{-1/2} A Q^{1/2} y, D l(y) \rangle = 0, \quad y \in \mathbb{R}^N.$$

Therefore, Theorem 1.1 in [11] holds more generally when Q is positive definite.

(ii) We do not know if the previous theorem holds replacing the assumption that Q is positive definite with the more general Kalman condition (2).

3. Positive harmonic function for the OU semigroup

Note that formula (4) is meaningful even if the Borel function f is only non-negative. Following [17] we say that a Borel function $u : \mathbb{R}^N \to \mathbb{R}_+$ is a positive harmonic function for the OU semigroup (P_t) if it satisfies

$$P_t u(x) = u(x), \quad x \in \mathbb{R}^N, \quad t \ge 0.$$
(6)

The next theorem is a special case of Theorem 5.1 in [17] which holds in infinite dimensions as well. Its proof uses Theorem 2.3 together with an idea of S. Kwapien (personal communication). We sketch its proof in Appendix.

Theorem 3.1. Assume the Kalman condition (2) and $s(A) \leq 0$. Consider a positive harmonic function u for the OU semigroup (P_t) . Then u is convex on \mathbb{R}^N .

In the next result we provide a sufficient condition under which positive harmonic functions for L are positive harmonic functions for the OU semigroup as well. For such result we do not need the Kalman condition (2).

Proposition 3.2. Let $u \in C^2(\mathbb{R}^N)$ be a non-negative solution to Lu(x) = 0on \mathbb{R}^N . Assume that u verifies the following exponential growth condition: there exists $c_0 > 0$ such that

$$|u(x)| \le c_0 e^{c_0 |x|}, \quad x \in \mathbb{R}^N.$$
 (7)

²This is equivalent to require that A is diagonalizable over \mathbb{C} with all the eigenvalues on the imaginary axis; in particular this implies that $s(A) \leq 0$.

Then we have $P_t u(x) = u(x), x \in \mathbb{R}^N, t \ge 0$ (³).

Proof. The proof uses stochastic calculus. Let us introduce the OU stochastic process starting at $x \in \mathbb{R}^N$ (see, for instance, page 232 in [7]). It is the solution to the following SDE

$$X_{t}^{x} = x + \int_{0}^{t} A X_{s}^{x} ds + \int_{0}^{t} \sqrt{Q} \, dW_{s}, \quad t \ge 0, \quad x \in \mathbb{R}^{N}.$$
(8)

Here $W = (W_t)$ is a standard N-dimensional Wiener process defined and adapted on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. The solution is given by

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A}\sqrt{Q}dW_s.$$

It is well-known that $\mathbb{E}[u(X_t^x)] = P_t u(x), t \ge 0, x \in \mathbb{R}^N$ (see, for instance, Section 5.1.2 in [5]).

By Itô's formula (see, for instance, Chapter 8 in [2] or Chapter 2 in [7]) we know that, \mathbb{P} -a.s.,

$$u(X_t^x) = u(x) + \int_0^t Lu(X_s^x) ds + M_t = u(x) + M_t, \ t \ge 0, \ x \in \mathbb{R}^N,$$

using the local martigale $M = (M_t)$, $M_t = \int_0^t Du(X_s^x) \sqrt{Q} dW_s$. Let us fix $x \in \mathbb{R}^N$. By using the stopping times $\tau_n^x = \inf\{t \ge 0 : X_t^x \in B_n\}$ (here B_n is the open ball of radius n and center 0) we find

$$\mathbb{E}[u(X_{t\wedge\tau_n^x}^x)] = u(x) + \mathbb{E}[M_{t\wedge\tau_n^x}].$$

By considering a C_b^2 -function u_n with bounded first and second derivatives on \mathbb{R}^N which coincides with u on B_{n+1} we obtain

$$M_{t\wedge\tau_n^x} = \int_0^{t\wedge\tau_n^x} Du_n(X_s^x) \sqrt{Q} dW_s, \quad t \ge 0, \ n \ge 1.$$

Since $\int_0^t Du_n(X_s^x)\sqrt{Q}dW_s$ is a martingale, by the Doob optional stopping theorem we know that

$$0 = \mathbb{E}\Big[\int_0^{t\wedge\tau_n^x} Du_n(X_s^x)\sqrt{Q}dW_s\Big] = \mathbb{E}[M_{t\wedge\tau_n^x}], \quad t \ge 0, \ n \ge 1.$$

We arrive at

$$\mathbb{E}[u(X_{t \wedge \tau_n^x}^x)] = u(x), \ t \ge 0, n \ge 1.$$
(9)

We fix t > 0. In order to pass to the limit in (9) we use that, for any $n \ge 1$, \mathbb{P} -a.s.,

$$|u(X_{t\wedge\tau_n^x}^x)| \le c_0 e^{c_0 |(X_{t\wedge\tau_n^x}^x)|} \le c_0 e^{c_0 \sup_{s\in[0,t]} |X_s^x|}.$$

It is known that there exists $\delta > 0$, possibly depending also on t, such that

$$\mathbb{E}\left[\exp\left(\delta\sup_{s\in[0,t]}\left|\int_{0}^{s}e^{(s-r)A}\sqrt{Q}\,dW_{r}\right|^{2}\right)\right]<\infty\tag{10}$$

(for instance, one can use Proposition 8.7 in [2] together with the estimate $\sup_{s \in [0,t]} \left| \int_0^s e^{sA} e^{-rA} \sqrt{Q} \, dW_r \right| \leq c_t \sup_{s \in [0,t]} \left| \int_0^s e^{-rA} \sqrt{Q} \, dW_r \right|$). It follows

³This result holds more generally assuming that there exists $\epsilon \in [0, 2)$ and $C_{\epsilon} > 0$ such that $|u(x)| \leq C_{\epsilon} e^{C_{\epsilon} |x|^{2-\epsilon}}$, $x \in \mathbb{R}^{N}$. Moreover, the hypothesis that u is non-negative is not necessary.

that $\mathbb{E}[e^{c_0 \sup_{s \in [0,t]} |X_s^x|}] < \infty$. Since \mathbb{P} -a.s. $\tau_n^x \to \infty$, we can pass to the limit in (9) as $n \to \infty$ by the dominated convergence theorem and get

$$\mathbb{E}[u(X_t^x)] = u(x), \quad t \ge 0.$$

The proof is complete since $\mathbb{E}[u(X_t^x)] = P_t u(x)$.

According to Theorem 3.1 and Proposition 3.2 we have

Corollary 3.3. Assume the Kalman condition (2) and $s(A) \leq 0$. Let u be a non-negative smooth solution to Lu(x) = 0 on \mathbb{R}^N which verifies the exponential growth condition (7).

Then u is a convex function on \mathbb{R}^N .

We obtain here a first one-side Liouville type theorem for possibly degenerate hypoelliptic OU operators. It holds under a sublinear growth condition and generalizes Theorem 2.1.

Theorem 3.4. Let us consider the OU operator L under the Kalman condition (2). Let $u \in C^2(\mathbb{R}^N)$ be a non-negative solution to Lu = 0 on \mathbb{R}^N . Assume the following growth condition: there exist $\delta \in [0,1)$ and $C_{\delta} > 0$ such that

$$|u(x)| \le C_{\delta} \left(1 + |x|^{\delta}\right), \ x \in \mathbb{R}^{N}.$$

$$(11)$$

If $s(A) \leq 0$ then we have that u is constant.

Proof. Since condition (11) implies the exponential growth condition (7) we can apply Corollary 3.3. The assertion follows since u is a convex function on \mathbb{R}^N .

4. A ONE-SIDE LIOUVILLE THEOREM UNDER AN EXPONENTIAL GROWTH CONDITION

The main result of the paper concerns non-degenerate OU operators L:

$$L = \frac{1}{2} \operatorname{tr}(QD^2) + \langle Ax, D \rangle, \qquad (12)$$

where Q is a positive definite $N \times N$ -matrix.

Theorem 4.1. Let us consider the OU operator L with Q positive definite. Let $u \in C^2(\mathbb{R}^N)$ be a non-negative solution to Lu(x) = 0 on \mathbb{R}^N . Suppose that $s(A) \leq 0$ holds. Suppose that u satisfies the exponential growth condition (7). Then u is constant.

In order to prove the result we need a first lemma which holds more generally for hypoelliptic OU operators.

Lemma 4.2. Let us consider the hypoelliptic OU operator L. Let $u \in C^2(\mathbb{R}^N)$ be a non-negative solution to Lu(x) = 0 on \mathbb{R}^N . Suppose that u verifies the exponential growth condition (7). Then if $s(A) \leq 0$ we have the following identity for any $x_0, x \in \mathbb{R}^N$

$$u(x) \ge u(x_0) - Du(x_0) \cdot x_0 + Du(x_0) \cdot e^{tA}x, \quad t \ge 0.$$
(13)

Proof. We know by Corollary 3.3 that for any $x_0 \in \mathbb{R}^N$ we have

$$u(y) \ge u(x_0) + Du(x_0) \cdot (y - x_0), \quad y \in \mathbb{R}^N.$$
 (14)

We apply the OU semigroup (P_t) to both sides of (14):

$$P_t u(x) \ge u(x_0) + Du(x_0) \cdot \int_{\mathbb{R}^N} [e^{tA}x - x_0 + z] N(0, Q_t) dz$$

= $u(x_0) - Du(x_0) \cdot x_0 + Du(x_0) \cdot e^{tA}x.$

Taking into account Proposition 3.2 we have that $P_t u = u, t \ge 0$, and the assertion follows.

To illustrate the proof of Theorem 4.1 we first examine an example when N = 3.

Example 1. We introduce

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad e^{tA}x = \begin{pmatrix} x_1 + tx_2 + \frac{t^2}{2}x_3 \\ x_2 + tx_3 \\ x_3 \end{pmatrix},$$

 $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Let Q can be any positive definite 3×3 matrix. We consider the OU operator associated to (Q, A).

We first prove that, for any $x_0 \in \mathbb{R}^3$, $\partial_{x_1} u(x_0) = \partial_{x_2} u(x_0) = 0$. Suppose by contradiction that

$$k = \partial_{x_1} u(x_0) \neq 0,$$

for some $x_0 \in \mathbb{R}^3$. Then we consider x = (0, 0, k) and by (13) we get

$$u(0,0,k) \ge u(x_0) - Du(x_0) \cdot x_0 + Du(x_0) \cdot e^{tA}(0,0,k)$$

= $u(x_0) - Du(x_0) \cdot x_0 + Du(x_0) \cdot \left(\frac{t^2}{2}k, tk, k\right)$
= $u(x_0) - Du(x_0) \cdot x_0 + \frac{t^2}{2}k^2 + t\partial_{x_2}u(x_0)k + \partial_{x_3}u(x_0)k, \quad t \ge 0$

Letting $t \to \infty$ we get a contradiction since $\frac{t^2}{2}k^2 + t\partial_{x_2}u(x_0)k$ tends to ∞ . It follows that $\partial_{x_1}u(x_0) = 0$. We have $u(x_1, x_2, x_3) = u(0, x_2, x_3)$ on \mathbb{R}^3 .

Suppose by contradiction that $l = \partial_{x_2} u(x_0) \neq 0$, for some $x_0 \in \mathbb{R}^3$. Then we consider x = (0, 0, l) and by (13) for any $t \geq 0$ we get

$$u(0,0,l) \ge u(x_0) - Du(x_0) \cdot x_0 + Du(x_0) \cdot (\frac{t^2}{2}l, tl, l)$$

= $u(x_0) - Du(x_0) \cdot x_0 + tl^2 + \partial_{x_3}u(x_0)l.$

Letting $t \to \infty$ we get a contradiction. It follows that $\partial_{x_2} u(x_0) = 0$, for any $x_0 \in \mathbb{R}^N$. We have obtained that $u(x_1, x_2, x_3) = u(0, 0, x_3) = v(x_3)$ on \mathbb{R}^3 .

Since $0 = Lu(x) = \frac{q_{33}}{2} \partial_{x_3 x_3}^2 u(0, 0, x_3) = \frac{q_{33}}{2} \frac{d^2}{dx_3^2} v(x_3), x_3 \in \mathbb{R}$, where $q_{33} = Qe_3 \cdot e_3$, we finally obtain that u = cost.

In the proof of Theorem 4.1 we will also use the following remarks.

Remark 4.3. Arguing as in (ii) of Remark 2.5 we note that it is enough to prove Theorem 4.1 when Q is replaced by

for some $\delta > 0$. Indeed by using the change of variable $u(x) = v(\delta^{1/2}x)$, $v(y) = u(\delta^{-1/2}y)$, we can pass from an OU operator associated with (Q, A) to an OU operator associated with $(\delta Q, A)$. We have

$$Lu(x) = 0, \ x \in \mathbb{R}^N \iff \frac{\delta}{2} tr(QD^2v(y)) + \langle Ay, Dv(y) \rangle = 0, \ y \in \mathbb{R}^N.$$

Remark 4.4. In the sequel we will always assume that in (12)

the matrix
$$A$$
 is in the real Jordan form, (15)

possibly replacing Q by PQP^* where P is a $N \times N$ real invertible matrix.

Note that PQP^* is still positive definite. Let us clarify the previous assertion. Let P be an invertible real matrix such that $PAP^{-1} = J$.

Using the change of variable u(x) = v(Px), $v(y) = u(P^{-1}y)$, we can pass from an OU operator associated with (Q, A) to an OU operator associated with (PQP^*, PAP^{-1}) . We have

$$Lu(x) = 0, \ x \in \mathbb{R}^N \iff \frac{1}{2}tr(PQP^*D^2v(y)) + \langle Jy, Dv(y) \rangle = 0, \ y \in \mathbb{R}^N.$$

We also remark that s(J) = s(A).

5. On the proof of Theorem 4.1

According to Remarks 4.3 and 4.4 we concentrate on proving the Liouville theorem for L in (12) assuming that A is in the real Jordan form. Moreover, when it will be needed we will replace Q by δQ with $\delta > 0$ small enough.

5.1. A technical lemma. Recall that (e_j) denotes the canonical basis in \mathbb{R}^N . Given a real $N \times N$ matrix C we write $C = B_0 \oplus B_1 \oplus ... \oplus B_n$ if

$$C = \begin{pmatrix} B_0 & 0 & \dots & 0 \\ 0 & B_1 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & B_n \end{pmatrix},$$

where B_i is a real $k_i \times k_i$ matrix, $i = 0, ..., n, k_i \ge 1$ and $k_0 + ... + k_n = N$. Let $x \in \mathbb{R}^N$. We write

$$x = (x_{B_0}, \dots, x_{B_n}) \tag{16}$$

where $x_{B_0} = (x_1^0, ..., x_{k_0}^0) = (x_1, ..., x_{k_0}),$

$$x_{B_i} = (x_1^i, \dots, x_{k_i}^i) = (x_{k_0 + \dots + k_{i-1} + 1}, \dots, x_{k_{i-1} + k_i}).$$
(17)

We say that $x_1^i, ..., x_{k_i}^i$ are the variables corresponding to B_i .

We also consider the related orthogonal projections $\pi_{B_i} : \mathbb{R}^N \to \mathbb{R}^{k_i}$:

$$\pi_{B_i} x = x_{B_i} \quad x \in \mathbb{R}^N, \, i = 0, ..., n.$$
 (18)

Clearly, we have

$$Cx = (B_0 x_{B_0}, ..., B_n x_{B_n}) = (B_0 \pi_{B_0} x, ..., B_n \pi_{B_n} x), \ x \in \mathbb{R}^N.$$
(19)

Now we write the $N \times N$ matrix A appearing in (12) with $s(A) \leq 0$ in the following real Jordan form:

$$A = S \oplus E_0 \oplus J(0, k_1) \oplus \dots \oplus J(0, k_p) \oplus J(0, d_1, g_1) \oplus \dots \oplus J(0, d_q, g_q) \oplus E_1,$$
(20)

 $p, q \geq 1$. Note that some of the previous blocks could be not present (for instance, it could be possible that $A = S \oplus E_1$). In the sequel we will examine the various possibile blocks in (20).

The block S is a $s \times s$ matrix, $s \ge 1$, and corresponds to the stable part of A (i.e., it corresponds to the eigenvalues of A with negative real part).

The block E_0 is the null $k_0 \times k_0$ matrix, $k_0 \ge 1$. The $2t \times 2t$ block E_1 , $t \ge 1$, corresponds to all possible simple complex eigenvalues $ih_1, ..., ih_t$ of A (with $h_1, ..., h_t \in \mathbb{R}$):

$$E_{1} = \begin{pmatrix} \begin{bmatrix} 0 & h_{1} \\ -h_{1} & 0 \end{bmatrix} & & 0 \\ & & \ddots & & \\ 0 & & & \begin{bmatrix} 0 & h_{t} \\ -h_{t} & 0 \end{bmatrix} \end{pmatrix}.$$

Moreover, $J(0, k_i)$ is the $k_i \times k_i$ Jordan block, $k_i \ge 2, i = 1, ..., p$,

$$J(0,k_i) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

such that $J(0, k_i)^{k_i}$ is the null $k_i \times k_i$ matrix. Finally, the $2g_j \times 2g_j$ Jordan block $J(0, d_j, g_j)$ is

$$J(0, d_j, g_j) = \begin{pmatrix} 0 & d_j & 1 & 0 & 0 & 0 & 0 & 0 \\ -d_j & 0 & 0 & 1 & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & 0 & d_j \\ 0 & \cdots & 0 & \ddots & \ddots & \ddots & -d_j & 0 \end{pmatrix},$$
(21)

where $d_j \in \mathbb{R}, g_j \ge 2, j = 1, ..., q$. If all blocks are present, according to (20) we have

$$s + k_0 + k_1 + \dots + k_p + 2[g_1 + \dots + g_q] + 2t = N.$$

Let us consider (20) and fix a real function $u(x_1, ..., x_N)$.

In the following definition we suppose that at least one Jordan block like $J(0, k_i)$ or $J(0, d_j, g_j)$ is present in (20).

We say that u is quasi-constant with respect to Jordan blocks like $J(0, k_i)$ or $J(0, d_j, g_j)$ if the following conditions hold:

(1) If the block $J(0, k_i)$ is present in formula (20) then u is constant in the variables

$$x_1^i, \dots, x_{k_i-1}^i,$$

where $x_{J(0,k_i)} = (x_1^i, ..., x_{k_i}^i)$ (cf. (17); i.e., if we consider only the variables $x_1^i, ..., x_{k_i}^i$, corresponding to the block $J(0, k_i)$, i = 1, ..., p, we say that u may only depend on $x_{k_i}^i$.

(2) If the block $J(0, d_j, g_j)$ is present in formula (20) then u is constant in the variables

$$x_1^j, \dots, x_{2g_j-2}^j,$$

where $x_{J(0,d_j,g_j)} = (x_1^j, ..., x_{2g_j}^j)$; i.e., if we consider only the variables $x_1^j, ..., x_{2g_j}^j$, corresponding to $J(0, d_j, g_j)$ we say that u may only depend on $x_{2g_j-1}^j$ and $x_{2g_j}^j$.

Lemma 5.1. We assume the hypotheses of Theorem 4.1 about L and u. Then u is constant in the following cases.

i) there is not the stable part S in (20). Moreover, blocks like $J(0, k_i)$ or $J(0, d_j, g_j)$ are not present.

ii) there is not the stable part S in (20). Moreover, u is quasi-constant with respect to Jordan blocks like $J(0, k_i)$ or $J(0, d_j, g_j)$.

iii) there is the stable part S in (20). Moreover, u is quasi-constant with respect to Jordan blocks like $J(0, k_i)$ or $J(0, d_j, g_j)$.

Proof. i) In this case L verifies the assumptions of Theorem 2.4 and the assertion follows (note that in such case we do not need to impose any growth condition on the non-negative function u).

To treat (ii) and (iii) we concentrate on the most difficult case when both blocks like $J(0, k_i)$ and $J(0, d_j, g_j)$ are present in (20) (otherwise, we can argue similarly).

ii) We start to study the term $Ax \cdot Du$ (cf. (12)).

Let $1 \leq i \leq p$. If u is constant in the variables $x_1^i, ..., x_{k_i-1}^i$ corresponding to the block $J(0, k_i)$ then, for any $x \in \mathbb{R}^N$ of the form

$$(0, ..., 0, x_{J(0,k_i)}, 0, ..., 0) = (0, ..., 0, x_1^i, ..., x_{k_i}^i, 0, ..., 0)$$

(cf. (16)) i.e., x has all the coordinates 0 possibly apart the coordinates $x_1^i, ..., x_{k_i}^i$, we have:

$$A(0, ..., 0, x_1^i, ..., x_{k_i}^i, 0, ..., 0) \cdot Du(x) = J(0, k_i)(x_1^i, ..., x_{k_i}^i) \cdot D_{(x_1^i, ..., x_{k_i}^i)}u(x)$$
$$= 0 \cdot \partial_{x_{k_i}^i} u(x) = 0, \qquad (22)$$

where $D_{(x_1^i,...,x_{k_i}^i)}u(x) \in \mathbb{R}^{k_i}$ denotes the gradient with respect to the $x_1^i,...,x_{k_i}^i$ variables of u at $x \in \mathbb{R}^N$.

Let $1 \leq j \leq q$. If u is constant in the variables $x_1^j, ..., x_{2g_i-2}^j$ corresponding to $J(0, d_j, g_j)$ then

$$A(0, ..., 0, x_1^j, ..., x_{2g_j}^j, 0, ..., 0) \cdot Du(x) = J(0, d_j, g_j)(x_1^j, ..., x_{2g_j}^j) \cdot D_g u(x)$$

$$= \left(d_j \, x_{2g_j}^j, -d_j \, x_{2g_j-1}^j \right) \cdot \left(\partial_{x_{2g_j-1}^j} u(x), \partial_{x_{2g_j}^j} u(x) \right)$$

$$= d_j \, x_{2g_j}^j \, \partial_{x_{2g_j-1}^j} u(x) - d_j \, x_{2g_j-1}^j \, \partial_{x_{2g_j}^j} u(x), \qquad (23)$$

where $D_g u(x) = D_{(x_1^j,...,x_{2g_j}^j)} u(x) \in \mathbb{R}^{2g_j}$ denotes the gradient with respect to the $x_1^j,...,x_{2g_i}^j$ variables of u at $x \in \mathbb{R}^N$.

Note that according to (20) and (19) we have, for any $x \in \mathbb{R}^N$,

$$Ax = \left(E_0 \pi_{E_0} x, J(0, k_1) \pi_{J(0, k_1)} x, \dots, J(0, k_p) \pi_{J(0, k_p)} x, J(0, d_1, g_1) \pi_{J(0, d_1, g_1)} x, \dots, J(0, d_q, g_q) \pi_{J(0, d_q, g_q)}, E_1 \pi_{E_1} x\right).$$

By the assumptions u depends only on m variables, $m \leq N$. Taking into account (22) and (23) we get, for any $x \in \mathbb{R}^N$,

$$Ax \cdot Du(x) = E_0 \pi_{E_0} x \cdot D_{E_0} u(x) + [d_1 x_{2g_1}^1 \partial_{x_{2g_1-1}^1} u(x) - d_1 x_{2g_1-1}^1 \partial_{x_{2g_1}^1} u(x)] + \dots + [d_q x_{2g_q}^q \partial_{x_{2g_q-1}^q} u(x) - d_q x_{2g_q-1}^q \partial_{x_{2g_q}^2} u(x)] + E_1 \pi_{E_1} x \cdot D_{E_1} u(x), \quad (24)$$

where D_{E_0} denotes the gradient with respect to the k_0 variables corresponding to E_0 and D_{E_1} denotes the gradient with respect to the 2t variables corresponding to E_1 .

We can set

$$u(x_1, ..., x_N) = v(x_1, ..., x_m) = v(x_C),$$
(25)

with $v : \mathbb{R}^m \to \mathbb{R}_+,$

$$x_C = (x_1, \dots, x_m) = (x_{E_0}, x_{2g_1-1}^1, x_{2g_1}^1, \dots, x_{2g_q-1}^q, x_{2g_q}^q, x_{E_1}).$$

By (24) we see that

$$Ax \cdot Du(x) = A_C x_C \cdot Dv(x_C)$$

for a suitable $m \times m$ -matrix A_C which is diagonalizable over the complex field with all the eigenvalues on the imaginary axis. We obtain

$$0 = \tilde{L}v(x_C) = \frac{1}{2} \operatorname{tr}(\tilde{Q}D^2v(x_C)) + A_C x_C \cdot Dv(x_C), \quad x_C \in \mathbb{R}^m,$$

for a suitable positive definite $m \times m$ matrix Q. By applying Theorem 2.4 we obtain that v is constant and so u is constant as well.

(iii) We start as in (ii). We denote the variables corresponding to the stable part S of A by

 $x_1, ..., x_s.$

By the assumptions u depends only on n variables, $n \leq N$. Moreover n = s + m where m is considered in (25). So we write

$$u(x_1, ..., x_N) = w(x_1, ..., x_s, x_{s+1}, ..., x_n) = w(x_S, x_C),$$
(26)

where $w : \mathbb{R}^n \to \mathbb{R}_+, x_S = (x_1, ..., x_s)$ and $x_C = (x_{s+1}, ..., x_n)$.

Arguing as before we obtain that there exists an $(n-s) \times (n-s)$ -matrix A_C such that, for any $x = (x_S, x_C) \in \mathbb{R}^n$,

$$0 = \tilde{L}w(x) = \frac{1}{2}\operatorname{tr}(\tilde{Q}D^2w(x)) + Sx_S \cdot D_Sw(x) + A_Cx_C \cdot D_Cw(x),$$

for a suitable positive definite $n \times n$ matrix \tilde{Q} . Moreover, D_S denotes the gradient with respect to the first s variables and D_C the gradient with respect to the $x_{s+1}, ..., x_n$ variables.

In the rest of the proof it is convenient to replace \tilde{Q} by

 δQ , for some $\delta > 0$ small enough to be chosen later.

(cf. Remark 4.3). We also recall that the matrix A_C is diagonalizable over the complex field with all the eigenvalues on the imaginary axis.

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To finish the proof we prove that w does not depend on the x_S -variable. Indeed once this is proved we can apply Theorem 2.4 and obtain that w is constant.

To obtain such assertion it will be important the exponential growth condition (7).

By the previous notation, $x = (x_S, x_C) \in \mathbb{R}^n$, since in particular w verifies the exponential growth condition (7) we have: $\tilde{P}_t w = w, t \ge 0$, i.e.,

$$\int_{\mathbb{R}^n} w(e^{tS}x_S + y_S, e^{tA_C}x_C + y_C) N(0, \tilde{Q}_t) dy_S dy_C = w(x_S, x_C), \quad (27)$$

 $t \geq 0,$ where $N(0, \tilde{Q}_t)$ is the Gaussian measure with mean 0 and covariance matrix

$$\tilde{Q}_t = \delta \int_0^t e^{s\tilde{A}} \tilde{Q} e^{s\tilde{A}^*} ds.$$

Here we are considering the $n \times n$ -matrix $\tilde{A} = S \oplus A_C$ so that

$$e^{t\tilde{A}} = e^{tS} \oplus e^{tA_C}$$

By Corollary 3.3 w is a convex function on \mathbb{R}^n . Applying a well-known result on convex functions (cf. Section 6.3 in [6]) we obtain in particular that, for any $x \in \mathbb{R}^n$, |x| > 1,

$$\sup_{|y| \le |x|} |Dw(y)| \le \frac{c(n)}{2|x|} \oint_{B(0,|2x|)} w(y) dy.$$

It follows that possibly replacing c_0 in (7) by another constant c > 0 we have

$$|Dw(x)| \le c e^{c|x|}, \quad x \in \mathbb{R}^n.$$
(28)

Let us fix $h_S \in \mathbb{R}^{n-s}$ and $x = (x_S, x_C) \in \mathbb{R}^n$. Differentiating both sides of (27) along the direction h_S we find

$$\int_{\mathbb{R}^n} D_S w(e^{tS} x_S + y_S, e^{tA_C} x_C + y_C) \cdot e^{tS} h_S N(0, \tilde{Q}_t) dy_S dy_C = D_S w(x_S, x_C) \cdot h_S.$$
(29)

where as before D_S denotes the gradient with respect to the first s variables. Recall that since the matrix S is stable there exist C > 0 and $\omega > 0$ such that

$$|e^{tS}h_S| \le Ce^{-\omega t}|h_S|, \ t \ge 0.$$

$$(30)$$

By (28) we infer

$$|D_S w(e^{tS} x_S + y_S, e^{tA_C} x_C + y_C)| \le c e^{c|e^{tS} x_S|} e^{c|e^{tA_C} x_C|} e^{c|y_S|} e^{c|y_C|}.$$

Note that $|e^{tA_C}x_C| = |x_C|, t \ge 0$. It follows that there exists a positive function $\lambda(x)$ (independent of $t \ge 0$) such that

$$|D_S w(e^{tS} x_S + y_S, e^{tA_C} x_C + y_C)| \le \lambda(x) e^{2c |(y_S, y_C)|}, \quad t \ge 0.$$
(31)

Setting $y = (y_S, y_C)$ it is not difficult to prove that there exists $c_1 > 0$ (independent of t) such that

$$\int_{\mathbb{R}^n} e^{2c \, |y|} N(0, \tilde{Q}_t) dy \le c_1 e^{c_1 \, c^2 \, \delta \, t}.$$
(32)

To this purpose we first remark that

$$\|\tilde{Q}_t\| \le \delta \int_0^t \|e^{s\tilde{A}}\| \|Q\| \|e^{s\tilde{A}^*}\| ds \le C_0 \delta t, \ t \ge 0,$$
(33)

for some constant $C_0 > 0$ independent of t. Then recall that if R is a $n \times n$ symmetric and non-negative definite matrix, using also the Fubini theorem, we obtain

$$\int_{\mathbb{R}^n} e^{r|y|} N(0,R) dy = \int_{\mathbb{R}^n} e^{r|R^{1/2}y|} N(0,I) dy$$
$$\leq \int_{\mathbb{R}^n} e^{r||R^{1/2}|| (|y_1|+\ldots+|y_n|)} N(0,I) dy = \left(\frac{2}{\sqrt{2\pi}} \int_0^\infty e^{r||R^{1/2}|| |y|} e^{-y^2/2} dy\right)^n$$
$$\leq 2^n e^{\frac{n}{2}r^2||R||}, \quad r \ge 0.$$

Combining the last computation and (33) we obtain (32). Using (29), (31) and (32) and we infer

$$\begin{aligned} |D_S w(x_S, x_C) \cdot h_S| \\ &= \left| \int_{\mathbb{R}^n} D_S w(e^{tS} x_S + y_S, e^{tA_C} x_C + y_C) \cdot e^{tS} h_S N(0, \tilde{Q}_t) dy_S dy_C \right| \\ &\leq \lambda(x) \left| e^{tS} h_S \right| \int_{\mathbb{R}^n} e^{2c \left| y \right|} N(0, \tilde{Q}_t) dy \\ &\leq c_1 C \lambda(x) \left| e^{c_1 c^2 \,\delta t} \, e^{-\omega t} |h_S|, \ t \geq 0. \end{aligned}$$

Since c_1 and c are independent of t and ω , choosing $\delta > 0$ small enough and passing to the limit as $t \to \infty$, we get

$$D_S w(x_S, x_C) \cdot h_S = 0$$

It follows that w does not depend on the x_S -variable. We have $w(x_S, x_C) = g(x_C)$ for a regular function $g : \mathbb{R}^{n-s} \to \mathbb{R}_+$. Moreover,

$$\tilde{L}w(x_S, x_C) = \frac{\delta}{2} \operatorname{tr}(\tilde{Q}_0 D_C^2 g(x_C)) + A_C x_C \cdot D_C g(x_C) = 0, \ x_C \in \mathbb{R}^{n-s},$$

where \tilde{Q}_0 is a positive definite $(n-s) \times (n-s)$ -matrix.

Applying Theorem 2.4 to the OU operator $\frac{\delta}{2} \operatorname{tr}(\tilde{Q}_0 D_C^2) + A_C x_C \cdot D_C$ we obtain that g is constant and this finishes the proof.

5.2. **Proof of Theorem 4.1.** We concentrate on the most difficult case when both blocks like $J(0, k_i)$ and $J(0, d_j, g_j)$ are present in (20). By Lemma 5.1 it is enough to show that u is quasi-constant with respect to the Jordan blocks

$$J(0, k_1), ..., J(0, k_p), J(0, d_1, g_1), ..., J(0, d_q, g_q).$$

I Step. We fix i = 1, ..., p, and consider the block $J(0, k_i)$ (see (20)). Let $x_1^i, ..., x_{k_i}^i$ be the variables corresponding to $J(0, k_i)$ according to (17). Let $x_0 \in \mathbb{R}^N$. We prove that $\partial_{x_k^i} u(x_0) = 0$ when $k = 1, ..., k_i - 1$.

To this purpose we first consider k = 1. We argue by contradiction and suppose that

$$\partial_{x_1^i} u(x_0) \neq 0, \tag{34}$$

for some $x_0 \in \mathbb{R}^N$. In order to apply Lemma 4.2, we first choose x having 0 in all the coordinates apart the coordinates $x_1^i, ..., x_{k_i}^i$, i.e., we have x =

 $(0, ..., 0, x_1^i, ..., x_{k_i}^i, 0, ..., 0). \text{ We find, setting } M_{x_0} = u(x_0) - Du(x_0) \cdot x_0$ $u(0, ..., 0, x_1^i, ..., x_{k_i}^i, 0, ..., 0) \ge M_{x_0} + Du(x_0) \cdot e^{tA}(0, ..., 0, x_1^i, ..., x_{k_i}^i, 0, ..., 0),$ $= M_{x_0} + D_{(x_1^i, ..., x_{k_i}^i)} u(x_0) \cdot e^{tJ(0, k_i)}(x_1^i, ..., x_{k_i}^i), \quad t \ge 0,$

where $D_{(x_1^i,...,x_{k_i}^i)}u(x_0)$ denotes the gradient with respect to the variables $x_1^i,...,x_{k_i}^i$. Recall that

$$e^{t J(0,k_i)} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \dots & \frac{t^{k_i-1}}{(k_i-1)!} \\ 0 & 1 & t & \dots & \dots & \frac{t^{k_i-2}}{(k_i-2)!} \\ 0 & 0 & 1 & \dots & \dots & \frac{t^{k_i-3}}{(k_i-3)!} \\ & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

Choosing further $x_1^1 = 0, ..., x_{k_i-1}^i = 0, x_{k_i}^i = \partial_{x_1^i} u(x_0)$ we find

$$u(0, ..., 0, \partial_{x_1^i} u(x_0), 0, ..., 0) \ge M_{x_0} + \partial_{x_1^i} u(x_0) \left[0 + t0 + ... + \frac{t^{k_i - 1}}{(k_i - 1)!} x_{k_i}^i \right]$$

+ $p(t, x_0) = M_{x_0} + (\partial_{x_1^i} u(x_0))^2 \frac{t^{k_i - 1}}{(k_i - 1)!} + p(t, x_0), \quad t \ge 0,$

where $p(t, x_0)$ is a polynomial in the *t*-variable which has degree less than $k_i - 1$. Letting $t \to \infty$ we find a contradiction since $(\partial_{x_1^i} u(x_0))^2 \frac{t^{k_i-1}}{(k_i-1)!} + p(t, x_0)$ tends to ∞ . Hence (34) cannot hold and we have proved that u does not depend on the x_1^i -variable.

Similarly, we prove that u does not depend on the x_2^i -variable as well.

Proceeding in finite steps, once we have proved that u does not depend on the variables $x_1^1, ..., x_{k-1}^i, k = 1, ..., k_i - 1$, we can show that for any $x_0 \in \mathbb{R}^N$ we have $\partial_{x_k^i} u(x_0) = 0$. To this purpose we argue by contradiction and suppose that

$$\partial_{x_{L}^{i}} u(x_{0}) \neq 0, \tag{35}$$

for some $x_0 \in \mathbb{R}^N$. In order to apply Lemma 4.2, we choose x having 0 in all the coordinates apart the coordinates $x_1^i, ..., x_{k_i}^i$. Moreover, choosing further $x_1^1 = 0, ..., x_{k_i-1}^i = 0, x_{k_i}^i = \partial_{x_k^i} u(x_0)$ we find (using that u does not depend on the variables $x_1^1, ..., x_{k-1}^i$)

$$u(0, ..., 0, \partial_{x_k^i} u(x_0), 0, ..., 0)$$

$$\geq M_{x_0} + \partial_{x_k^i} u(x_0) \left[x_k^i + t x_{x_{k+1}}^i + ... + \frac{t^{k_i - k}}{(k_i - k)!} x_{k_i}^i \right]$$

$$+ q_k(t, x_0) = M_{x_0} + (\partial_{x_k^i} u(x_0))^2 \frac{t^{k_i - k}}{(k_i - k)!} + q(t, x_0), \quad t \ge 0,$$

where $q(t, x_0)$ is a polynomial in the *t*-variable which has degree less than $k_i - k$.

Letting $t \to \infty$ we find a contradiction since $(\partial_{x_k^i} u(x_0))^2 \frac{t^{k_i-k}}{(k_i-k)!} + q(t,x_0)$ tends to ∞ . Hence (35) cannot hold and we have proved the assertion.

II Step. We fix j = 1, ..., q and consider the block $J(0, d_j, g_j)$ (see (20)). Let $x_1^j, ..., x_{2g_j}^j$ be the variables corresponding to $J(0, d_j, g_j)$. Let $x_0 \in \mathbb{R}^N$. We prove that $\partial_{x_k^j} u(x_0) = 0$ when $k = 1, ..., 2g_j - 2$.

We first consider k = 1. We argue by contradiction and suppose that

$$\partial_{x_{2}^{j}} u(x_{0}) \neq 0, \tag{36}$$

for some $x_0 \in \mathbb{R}^N$. As in I Step in order to apply Lemma 4.2, we choose x having 0 in all the coordinates apart the coordinates $x_1^j, \ldots, x_{2g_j}^j$. Moreover, we choose further

$$x_1^j = 0, ..., x_{2g_j-1}^j = 0, x_{2g_j}^j = \partial_{x_1^j} u(x_0)$$

and set $M_{x_0} = u(x_0) - Du(x_0) \cdot x_0$. By considering $t = T_n$, with

$$d_j \cdot T_n = \frac{\pi}{2} + 2n\pi, \quad n \ge 0 \tag{37}$$

we find

$$u(0, ..., 0, \partial_{x_1^j} u(x_0), 0, ..., 0) \ge M_{x_0}$$

+ $\partial_{x_1^j} u(x_0) \left[x_1^j \cos(d_j T_n) + x_2^j \sin(d_j T_n) + x_3^j T_n \cos(d_j T_n) + x_4^j T_n \sin(d_j T_n) + ... + x_{2g_j-1}^j \frac{T_n^{g_j-1}}{(g_j-1)!} \cos(d_j T_n) + x_{2g_j}^j \frac{T_n^{g_j-1}}{(g_j-1)!} \sin(d_j T_n) \right]$
+ $p(T_n, x_0) = M_{x_0} + (\partial_{x_1^j} u(x_0))^2 \frac{T_n^{g_j-1}}{(g_j-1)!} + p(T_n, x_0), \quad n \ge 0,$

where $p(t, x_0)$ is a polynomial in the *t*-variable which has degree less than $g_j - 1$. Letting $n \to \infty$ we find a contradiction since $(\partial_{x_k^j} u(x_0))^2 \frac{T_n^{g_j-1}}{(g_j-1)!} + p(T_n, x_0)$ tends to ∞ . Hence (36) cannot hold and we have proved that u does not depend on the x_1^j -variable.

Similarly, one can prove that u does not depend on the x_2^j -variable as well. We only note that in this case we choose x having 0 in all the coordinates apart the coordinates $x_1^j, \ldots, x_{2g_j}^j$. Moreover, $x_1^j = 0, \ldots, x_{2g_j-1}^j = 0, x_{2g_j}^j = \partial_{x_2^j} u(x_0)$ and define T_n such that $d_j \cdot T_n = 2n\pi, n \ge 0$. We have

$$u(0, ..., 0, \partial_{x_1^j} u(x_0), 0, ..., 0) \ge M_{x_0}$$

+ $\partial_{x_2^j} u(x_0) \left[-x_1^j \sin(d_j T_n) + x_2^j \cos(d_j T_n) - x_3^j T_n \sin(d_j T_n) + x_4^j T_n \cos(d_j T_n) \right]$
+ $... - x_{2g_j-1}^j \frac{T_n^{g_j-1}}{(g_j-1)!} \sin(d_j T_n) + x_{2g_j}^j \frac{T_n^{g_j-1}}{(g_j-1)!} \cos(d_j T_n) \right]$
+ $q(T_n, x_0), \quad n \ge 0,$

where $q(t, x_0)$ is a polynomial in the *t*-variable which has degree less than $g_j - 1$.

Proceeding in finite steps, once we have proved that u does not depend on the variables $x_1^1, ..., x_{k-1}^j, k = 1, ..., 2g_j - 2$, we show that for any $x_0 \in \mathbb{R}^N$ we have $\partial_{x_k^j} u(x_0) = 0$. To this purpose we suppose that k is even (we can proceed similarly if k is odd). We argue by contradiction and suppose that

$$\partial_{x_k^j} u(x_0) \neq 0,$$
 (38)

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for some $x_0 \in \mathbb{R}^N$. We choose x having 0 in all the coordinates apart the coordinates $x_1^j, ..., x_{2g_j}^j$. Moreover, we set $x_1^1 = 0, ..., x_{2g_j-1}^j = 0, x_{2g_j}^j = \partial_{x_k^j} u(x_0)$. We find (using that u does not depend on the variables $x_1^1, ..., x_{k-1}^j$) with T_n as in (37):

(0

$$u(0,...,0, \mathcal{O}_{x_{k}^{j}} u(x_{0}), 0, ..., 0) \geq M_{x_{0}}$$

+ $\partial_{x_{k}^{j}} u(x_{0}) \left[x_{k}^{j} \cos(d_{j}T_{n}) + x_{k+1}^{j} \sin(d_{j}T_{n}) + x_{k+2}^{j}T_{n} \cos(d_{j}T_{n}) + x_{k+3}^{j}T_{n} \sin(d_{j}T_{n}) \right]$
+ $... + x_{2g_{j}-1}^{j} \frac{T_{n}^{g_{j}-\frac{k+1}{2}}}{(g_{j}-\frac{k+1}{2})!} \cos(d_{j}T_{n}) + x_{2g_{j}}^{j} \frac{T_{n}^{g_{j}-\frac{k+1}{2}}}{(g_{j}-\frac{k+1}{2})!} \sin(d_{j}T_{n}) \right]$
+ $h(T_{n}, x_{0}) = M_{x_{0}} + (\partial_{x_{k}^{j}} u(x_{0}))^{2} \frac{T_{n}^{g_{j}-\frac{k+1}{2}}}{(g_{j}-\frac{k+1}{2})!} + h(T_{n}, x_{0}), \quad n \geq 0,$

where $h(t, x_0)$ is a polynomial in the *t*-variable which has degree less than $g_j - \frac{k+1}{2}$. Letting $n \to \infty$ we find a contradiction since $(\partial_{x_k^j} u(x_0))^2 \frac{T_n^{g_j - \frac{k+1}{2}}}{(g_j - \frac{k+1}{2})!} + h(T_n, x_0)$ tends to ∞ . Hence (38) cannot hold and we have proved the assertion. The proof is complete.

6. Some open problems

We list some open problems related to Liouville type theorems for OU operators.

(1) In general it is not known if under the Kalman condition (2) and the condition $s(A) \leq 0$ all non-negative smooth solutions v to Lv = 0 on \mathbb{R}^N are constant. This problem is also open in the non-degenerate case when we assume in addition that Q is positive definite.

(2) The papers [20] and [19] treat non-degenerate purely non-local OU operators ${\cal L}$

$$Lf(x) = \int_{\mathbb{R}^N} \left(f(x+y) - f(x) - \mathbf{1}_{\{|y| \le 1\}} \langle y, Df(x) \rangle \right) \nu(dy) + Ax \cdot Df(x),$$

 $x \in \mathbb{R}^N, f : \mathbb{R}^N \to \mathbb{R}$ bounded and smooth, where ν is a Lèvy measure. The hypotheses of [19] on ν improve the ones in [20]. Theorem 1.1 in [19] shows that under suitable hypotheses on ν and assuming $\sup_{t\geq 0} ||e^{tA}|| < \infty$ all bounded smooth harmonic functions for L are constant. It is not known if such result holds more generally under the assumption that $s(A) \leq 0$ (for instance, a matrix like A in Example 1 is not covered in [20] and [19]).

(3) In [11] the result below has been proved using probabilistic methods based on the known characterization of recurrence for OU stochastic processes. It seems that a purely analytic proof of such result is not known.

Theorem 6.1 (Theorem 6.1 in [11]). Let us consider hypoelliptic OU operator L. Let $v : \mathbb{R}^N \to \mathbb{R}$ be a non-negative C^2 -function such that $Lv \leq 0$ on \mathbb{R}^N . Then v is constant if the following condition holds: The real Jordan representation of B is $\begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix}$ where B_0 is stable and B_1 is at most of dimension 2 and of the form $B_1 = \begin{bmatrix} 0 \end{bmatrix}$ or $B_1 = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$ for some $\alpha \in \mathbb{R}$ (in this case we need $N \ge 2$).

Appendix A. Proof of Theorem 3.1

The proof of Theorem 3.1 is based on the following lemma which is a special case of an infinite dimensional result proved in Section 5 in [17]. We include the proof for the sake of completeness.

Lemma A.1. Let (P_t) be the OU semigroup. Assume (2) and $s(A) \leq 0$. Then for any non-negative Borel function $f : \mathbb{R}^N \to \mathbb{R}$, there results:

$$P_t f(x+a) + P_t f(x-a) \ge 2C_t(a) P_t f(x), \quad x, a \in \mathbb{R}^N,$$
(39)

where $C_t(a) = \exp[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2], t > 0$ (note that both sides in (39) can be $+\infty$).

Proof. We fix $x, a \in \mathbb{R}^N$ and set $N(0, Q_t) = N_{0,Q_t}$. By a direct computation we have

$$P_t f(x+a) = \int_{\mathbb{R}^N} f(e^{tA}x+y) \frac{dN_{e^{tA}a,Q_t}}{dN_{0,Q_t}}(y) N_{0,Q_t}(dy)$$
$$= \int_{\mathbb{R}^N} f(e^{tA}x+y) \exp[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2 + \langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y \rangle] N_{0,Q_t}(dy).$$

Note that the previous identity also holds in infinite dimensions by the Cameron-Martin formula (see, for instance, Chapter 1 in [4]). It follows that

$$\frac{1}{2}(P_t f(x+a) + P_t f(x-a))$$

$$= e^{-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2} \int_{\mathbb{R}^N} f(e^{tA}x+y) \frac{1}{2} \left(e^{\langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y \rangle} + e^{-\langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y \rangle} \right) N_{0,Q_t}(dy);$$

$$\geq \exp[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2] \int_{\mathbb{R}^N} f(e^{tA}x+y) N_{0,Q_t}(dy)$$

$$= C_t(a) P_t f(x), \text{ where } C_t(a) = \exp[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2].$$

Proof of Theorem 3.1. Let u be a positive harmonic function for (P_t) . By the previous lemma, we have, for any $x, a \in \mathbb{R}^N$,

$$\frac{1}{2}(u(x+a)+u(x-a)) = \frac{1}{2}(P_t u(x+a)+P_t u(x-a))$$

$$\geq \exp[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2]P_t u(x) = \exp[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2]u(x).$$

Passing to the limit as $t \to \infty$, we infer by Theorem 2.3

$$\frac{1}{2}(u(x+a)+u(x-a)) \ge u(x), \ x, a \in \mathbb{R}^N.$$

By a well-known result due to W. Sierpiśki this condition together with the measurability of u imply the convexity of u.

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References

- M. Bertoldi, S. Fornaro, Gradient estimates in parabolic problems with unbounded coefficients, Studia Math. 165, 2004, 221–254.
- [2] P. Baldi, Stochastic calculus. An introduction through theory and exercises. Universitext. Springer, Cham, 2017.
- [3] M. Cranston, S. Orey, U. Rösler, The Martin boundary of two dimensional Ornstein-Uhlenbeck processes, Probability, statistics and analyis, London Math. Soc. Lect. Note Ser. 79, Cambridge Univ. Press, 1983, 63-78.
- [4] G. Da Prato and J. Zabczyk, Second order partial differential equations in Hilbert spaces. London Mathematical Society Note Series, 293, Cambridge University Press, Cambridge, 2002.
- [5] G. Da Prato, J. Zabczyk, Stochastic equations in infinite dimensions. Second edition. Encyclopedia of Mathematics and its Applications, 152. Cambridge University Press, Cambridge, 2014.
- [6] Evans L. C., Gariepy R. F., Measure Theory And Fine Properties of Functions, Routledge 1992.
- [7] N. Ikeda, S. Watanabe, S., Stochastic Differential Equations and Diffusion Processes. North Holland-Kodansha, II edition (1989).
- [8] A. Kogoj, E. Lanconelli, An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations, 1, 2004, no. 1, 51-80.
- [9] A. E. Kogoj and E. Lanconelli, One-side Liouville theorems for a class of hypoelliptic ultraparabolic equations, in Geometric analysis of PDE and several complex variables, vol. 368 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2005, pp. 305-312.
- [10] A. E. Kogoj and E. Lanconelli. Liouville theorems for a class of linear secondorder operators with non-negative characteristic form. *Bound. Value Probl.*, Art. ID 48232, 16 pages, 2007.
- [11] A. E. Kogoj, E. Lanconelli, E. Priola, Harnack inequality and Liouville-type theorems for Ornstein-Uhlenbeck and Kolmogorov operators. Math. Eng. 2 (2020), no. 4, 680-697.
- [12] L. P. Kupcov. The fundamental solutions of a certain class of elliptic-parabolic second order equations. *Differ. Uravn.*, 8:1649–1660, 1716, 1972.
- [13] E. Lanconelli and S. Polidoro. On a class of hypoelliptic evolution operators. *Rend. Sem. Mat. Univ. Politec. Torino*, 52(1):29–63, 1994. Partial differential equations, II (Turin, 1993).
- [14] R. Pinsky, Positive Harmonic Functions and Diffusion, Cambridge Univ. Press, 1995.
- [15] E. Priola, J. Zabczyk, Null controllability with vanishing energy, SIAM J. Control Optim., 42, 2003, 1013-1032.
- [16] E. Priola, J. Zabczyk, Liouville theorems for nonlocal operators, J. Funct. Anal., 216, 2004, 455-490.
- [17] E. Priola, J. Zabczyk, Harmonic functions for generalized Mehler semigroups. SPDEs and applications-VII, pages 243-256, Lect. Notes Pure Appl. Math., 245, Chapman Hall/CRC, 2006, available at https://iris.unito.it/retrieve/handle/2318/62663/755313/PriolaZabczyk_Mehlerquad.pdf
- [18] E. Priola, F. Y. Wang, Gradient estimates for diffusion semigroups with singular coefficients. J. Funct. Anal. 236, 244-264, 2006.
- [19] R. L. Schilling, P. Sztonyk, J. Wang, On the coupling property and the Liouville theorem for Ornstein–Uhlenbeck processes, J. Evol. Equ. 12, 2012, no. 1, 119-140.
- [20] F. Y. Wang, Coupling for Ornstein-Uhlenbeck processes with jumps, Bernoulli 17, 2011, no. 4, 1136-1158.

[21] J. Zabczyk, Mathematical control theory-an introduction. Second edition. Birkhäuser/Springer, Cham, 2020.

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