ON ANOMALOUS DISSIPATION INDUCED BY TRANSPORT NOISE

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ABSTRACT. In this paper, we show that suitable transport noises produce anomalous dissipation of energy of solutions to the 2D Navier-Stokes equations and diffusion equations in all dimensions. The key ingredients are Meyers' type estimates for SPDEs with transport noise which are combined with recent scaling limits for such SPDEs. The former allow us to obtain, for the first time for such type of scaling limits, convergence in a space of positive smoothness uniformly in time. Compared to known results, one of the main novelties is that anomalous dissipation might take place even in presence of a transport noise of arbitrarily small intensity. Further, we discuss physical interpretations.

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1. Introduction and statement of the main results

The goal of this manuscript is to investigate the effect of transport noise on the anomalous dissipation of the 2D Navier-Stokes equations (NSEs in the following),

$$(1.1) \begin{cases} \partial_t u^{\nu} + (u^{\nu} \cdot \nabla) u^{\nu} = -\nabla p^{\nu} + \nu \Delta u^{\nu} \\ + \sqrt{2\mu} \sum_{k \in \mathbb{Z}_0^2} \left[-\nabla \widetilde{p}_n^{\nu} + \theta_k^{\nu} \left(\sigma_k \cdot \nabla \right) u^{\nu} \right] \circ \dot{W}_t^k & \text{on } \mathbb{T}^2, \\ \nabla \cdot u^{\nu} = 0 & \text{on } \mathbb{T}^2, \\ u^{\nu}(0, \cdot) = u_0 & \text{on } \mathbb{T}^2; \end{cases}$$

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and of diffusive scalars in all dimensions $d \ge 1$,

$$(1.2) \qquad \begin{cases} \partial_t \varrho^{\gamma} = \gamma \Delta \varrho^{\gamma} + \sqrt{c_d \mu} \sum_{k \in \mathbb{Z}_0^d} \sum_{1 \leq \alpha \leq d-1} \theta_k^{\gamma} (\sigma_{k,\alpha} \cdot \nabla) \varrho^{\gamma} \circ \dot{W}_t^{k,\alpha} & \text{on } \mathbb{T}^d, \\ \varrho^{\gamma}(0,\cdot) = \varrho_0 & \text{on } \mathbb{T}^d. \end{cases}$$

Here, \mathbb{T}^d denotes the d-dimensional torus, $\nu > 0$ the kinematic viscosity of the fluid, $\gamma > 0$ the diffusivity of the passive scalar, $(W^k)_{k \in \mathbb{Z}_0^2}$ and $(W^{k,\alpha})_{k \in \mathbb{Z}_0^d, \alpha \in \{1,\dots,d-1\}}$ families of complex Brownian motions, \circ the Stratonovich product, $\theta^{\nu} = (\theta_k^{\nu})_{k \in \mathbb{Z}_0^d} \in \ell^2$ and $\theta^{\nu} = (\theta_k^{\gamma})_{k \in \mathbb{Z}_0^d} \in \ell^2$ with $\mathbb{Z}_0^d = \mathbb{Z}^d \setminus \{0\}$, σ_k and $\sigma_{k,\alpha}$ divergence-free vector fields described in Subsection 2.2, and $\mu > 0$ the noises intensity (cf., (1.5) below).

Anomalous dissipation of energy in fluid flows has been proven experimentally to a large degree [73] and stands at the basis of turbulence theory [40, 50, 51, 52]. For this reason, it is sometimes referred to as the zeroth law of turbulence. Anomalous dissipation states that at high Reynolds numbers (i.e., $\nu \downarrow 0$) the averages of the energy dissipation rate $\langle \nu | \nabla u^{\nu} |^2 \rangle$ is uniformly bounded from below. More precisely,

(1.3)
$$\liminf_{\nu \downarrow 0} \langle \nu | \nabla u^{\nu} |^2 \rangle > 0.$$

In the above, u^{ν} is (typically) the solution of the 3D NSEs with kinematic viscosity $\nu > 0$ and the (unspecified) operator $\langle \cdot \rangle$ typically represents an ensemble average, e.g., space-time average or an expected value of it in case of random environments.

The physical mechanism behind the anomalous dissipation of energy is the transfer, as $\nu \downarrow 0$, of the energy from large to small spatial scales by the nonlinear convective term. This transference produces high gradients $|\nabla u^{\nu}|^2$ which cannot be compensated by the small multiplicative factor ν and therefore yielding a nontrivial limit energy dissipation rate as $\nu \downarrow 0$, cf., (1.3). In contrast to the clarity of the physical picture, anomalous dissipation is difficult to prove rigorously. In the mathematical community, anomalous dissipation for NSEs or diffusive passive scalars (e.g., the temperature) advected by turbulent flows has attracted a lot of interest in recent years. For passive scalars, anomalous dissipation by turbulent flows is at the basis of the corresponding theory of scalar turbulence [27, 71, 72]. It can be defined by replacing in (1.3) the viscosity ν and the velocity u^{ν} by the diffusivity $\gamma > 0$ and the intensity of the passive scalar ρ^{γ} (solving an advectiondiffusion such as (1.2)), respectively. In the deterministic setting, many results have been established in the context of anomalous dissipation. It is not possible to give here a complete overview of the deterministic results on anomalous dissipation and the reader is referred to, e.g., [12, 16, 20, 23, 24, 28] and the references therein.

Here, we are not aiming at capturing the sophisticated mechanics behind anomalous dissipation. Instead, in this manuscript, we prove that transport noises, localised at high frequencies and varying with the viscosity/diffusivity, lead to anomalous dissipation. In particular, anomalous dissipation is created by the 'turbulent flows' $\sum_{k\in\mathbb{Z}_0^2}\theta_k^\nu\sigma_k\,\dot{W}^k$ rather than the convective nonlinearity as expected in three-dimensional turbulence. Surprisingly enough, this is also true for noises of 'small intensities' and for 'nonlinear passive scalar' such as the 2D NSEs (here, we mean the turbulent flows differ from the advected quantities). This justifies that anomalous dissipation also occurs in two dimensions, although no anomalous dissipation is expected in two-dimensional turbulence. For comments on the three-dimensional

case, see Remark 1.2 below. Further details about a possible physical interpretation of our results are given in Subsection 1.2 below.

Before discussing works on anomalous dissipation and related topics in the context of stochastic fluid dynamics, let us first discuss the physical relevance of transport noise in stochastic fluid dynamics. Nowadays there are several derivations of NSEs with transport noise available in the literature, see e.g., [25, 37, 38, 47, 60, 61, and it is by now a well-established model in stochastic fluid dynamics [17, 18, 19, 22, 32, 36, 44, 45]. To some extent, at the basis of such derivations is the idea of separation of scales. A heuristic derivation using the latter principle is given in Subsection 1.2 below. This allows us to stress the physical relevance of the ν -dependence of transport noise in (1.1) and provide an interpretation of our results for 2D NSEs. Starting from the works [34, 41], the effect of transport noise on mixing and enhanced dissipation of NSEs or advection-diffusion equations is by now well-understood, see e.g., [31, 56, 30] (see also [1, 21, 29, 33, 35, 39, 54, 57] for related works). The same is not true for anomalous dissipation of energy. To the best of our knowledge, the only results on the anomalous dissipation for SPDEs with transport noise are given in [46, 68]. A comparison is postponed to Subsection 1.3.4 where a more detailed discussion on our contribution is possible. Finally, from a technical point of view, our proofs rely on a refinement of the scaling limit arguments [34, 41] in the parabolic setting which is interesting on its own. A detailed discussion is given in Subsection 1.3.2 below.

- 1.1. **Anomalous dissipation results.** Our main results show that, with a suitable transport noises, anomalous dissipation holds with:
 - small noise Theorems 1.1(1) and 1.3(1).
 - prescribed rate Theorems 1.1(2) and 1.3(2).

As highlighted above, compared to the existing literature, the main contribution of the current work is to reveal that anomalous dissipation can take place even in presence of a small transport noise; while previous works typically require noise with high intensity (as above, details are given in Subsection 1.3.4).

In the statement of our main results, we employ the following notation

$$(1.4) S_{\ell^2}^0 \stackrel{\text{def}}{=} \left\{ \theta = (\theta_k)_{k \in \mathbb{Z}_0^d} \in \ell^2 : \|\theta\|_{\ell^2} = 1 \text{ and } \#\{k : \theta_k \neq 0\} < \infty \right\}$$

for the set of normalized ℓ^2 -vectors with finitely non-zero components. Here, for simplicity, we did not display the dependence on the dimension $d \ge 1$.

Theorem 1.1 (Anomalous dissipation by transport noise – 2D NSEs). Let $N \ge 1$ and $\delta > 0$ be fixed. Then the following hold.

(1) (Small noise). For all $\mu > 0$, there exists a family $(\theta^{\nu})_{\nu \in (0,1)} \subseteq \mathcal{S}^{0}_{\ell^{2}}$ such that, for all mean-zero and divergence-free $u_{0} \in H^{\delta}(\mathbb{T}^{2}; \mathbb{R}^{2})$ satisfying $N^{-1} \leq \|u_{0}\|_{L^{2}}$ and $\|u_{0}\|_{H^{\delta}} \leq N$, we have

$$\inf_{\nu \in (0,1)} \mathbf{E} \int_0^1 \int_{\mathbb{T}^2} \nu |\nabla u^{\nu}|^2 \, \mathrm{d}x \, \mathrm{d}t > 0.$$

(2) (Prescribed rate). For all $\eta \in (0,1)$ and $\mu > -4\ln(1-\eta)$, there exists a family $(\theta^{\nu})_{\nu \in (0,1)} \subseteq S^0_{\ell^2}$ such that, for all mean-zero and divergence-free $u_0 \in$

 $H^{\delta}(\mathbb{T}^2;\mathbb{R}^2)$ satisfying $N^{-1} \leqslant \|u_0\|_{L^2}$ and $\|u_0\|_{H^{\delta}} \leqslant N$, we have

$$\inf_{\nu \in (0,1)} \mathbf{E} \int_0^1 \int_{\mathbb{T}^2} \nu |\nabla u^{\nu}|^2 \, \mathrm{d}x \, \mathrm{d}t \geqslant \frac{\eta}{2} \|u_0\|_{L^2}^2.$$

In the statements in (1)-(2), u^{ν} denotes the unique global smooth solution to (1.1).

As commented in (1.9) below, as $\theta \in \mathcal{S}_{\ell^2}^0$, the 'energy' of the transport noise $(\sqrt{2\mu} \theta_k \sigma_k)_{k \in \mathbb{Z}_0^2}$ is related to the intensity parameter μ :

(1.5)
$$\sqrt{2\mu} \sup_{\nu \in (0,1)} \|(\theta_k^{\nu} \sigma_k)_{k \in \mathbb{Z}_0^2}\|_{L^{\infty}(\mathbb{T}^2; \ell^2)} \leqslant \sqrt{2\mu}.$$

Since Theorem 1.1(1) holds for all $\mu > 0$, it indeed corcerns 'small' noises. Instead, Theorem 1.1(2) ensures that one can prescribe the rate of dissipation at the expense of considering a bounded but large transport noise, i.e., $\mu \gg 1$.

The case $u_0 = 0$ is excluded as no anomalous dissipation can take place. The existence of a global smooth solution of (1.1) with $\theta^{\nu} \in \mathcal{S}^0_{\ell^2}$ is proved in [9, Theorems 2.7 and 2.12]. An inspection of the proof of Theorem 1.1(1)-(2) shows that the time t = 1 can be replaced by any time t > 0; however, the corresponding choice of $(\theta^{\nu})_{\nu \in (0,1)}$ depends on such time t. In addition, we can take θ^{ν} to be constants for values of $\nu \in [\nu_{j+1}, \nu_j)$ where $(\nu_j)_{j \ge 1} \subseteq (0,1]$ is a sequence satisfying $\nu_1 = 1$, $\nu_{j+1} \le \nu_j$ and $\lim_{j \to \infty} \nu_j = 0$. Finally, the proof of Theorem 1.1 shows that θ^{ν} 's are concentrated at high frequencies, i.e.,

$$(1.6) \qquad \operatorname{supp} \theta^{\nu} \subseteq \{k \in \mathbb{Z}_0^2 : N^{\nu} \leqslant |k| \leqslant 2N^{\nu}\} \quad \text{ and } \quad \liminf_{\nu \downarrow 0} N^{\nu} = \infty.$$

Before going further, let us comment on the three-dimensional case.

Remark 1.2 (3D NSEs). The dimension restriction in Theorem 1.1 is related to the criticality of L^2 for the 2D NSEs. A version of Theorem 1.1 also holds for 3D NSEs provided, at a fixed viscosity $\nu > 0$, well-posedness and a-priori bounds for the 3D version of (1.1) hold in the sub-critical space $L^r(\mathbb{T}^3; \mathbb{R}^3)$ with r > 3. We refrain from formulating the latter assumption, as it implies the regularization by noise for 3D NSEs, which is one of the major open problems in stochastic fluid dynamics.

For passive scalars a version of Theorem 1.1 holds in all dimensions.

Theorem 1.3 (Anomalous dissipation by transport noise – passive scalars). Let $d \in \mathbb{N}_{\geq 1}$, $N \geq 1$ and $\delta > 0$ be fixed. Then the following hold.

(1) (Small noise). For all $\mu > 0$, there exists a family $(\theta^{\gamma})_{\gamma \in (0,1)} \subseteq \mathcal{S}^0_{\ell^2}$ such that, for all mean-zero initial datum $\varrho_0 \in H^{\delta}(\mathbb{T}^d)$ satisfying $N^{-1} \leqslant \|\varrho_0\|_{L^2}$ and $\|\varrho_0\|_{H^{\delta}} \leqslant N$, we have

$$\inf_{\gamma \in (0,1)} \mathbf{E} \int_0^1 \int_{\mathbb{T}^d} \gamma |\nabla \varrho^\gamma|^2 \, \mathrm{d}x \, \mathrm{d}t > 0.$$

(2) (Prescribed rate). For all $\eta \in (0,1)$ and $\mu > -\ln(1-\eta)$, there exists a family $(\theta^{\gamma})_{\gamma \in (0,1)} \subseteq \mathcal{S}^0_{\ell^2}$ such that, for all mean-zero initial datum $\varrho_0 \in H^{\delta}(\mathbb{T}^d)$ satisfying $N^{-1} \leq \|\varrho_0\|_{L^2}$ and $\|\varrho_0\|_{H^{\delta}} \leq N$, we have

$$\inf_{\gamma \in (0,1)} \mathbf{E} \int_0^1 \int_{\mathbb{T}^d} \gamma |\nabla \varrho^{\gamma}|^2 \, \mathrm{d}x \, \mathrm{d}t \geqslant \frac{\eta}{2} \|\varrho_0\|_{L^2}^2.$$

In the statements in (1)-(2), ϱ^{γ} denotes the unique global smooth solution to (1.2).

The comments below Theorem 1.1 extend to the above result, while the existence of a global smooth solution of (1.2) with $\theta^{\gamma} \in \mathcal{S}_{\ell^2}^0$ follows from, e.g., [6, Theorem 4.2]. In particular, arguing as in (1.5), Theorem 1.3(1) yields anomalous dissipation for passive scalars even in presence of a *small* transport noise.

In the next subsection, we provide a heuristic motivation for the ν -dependence of the transport noise in the context of NSEs with transport noise (1.1) and a possible physical interpretation of our results.

1.2. Heuristics for ν -dependent transport noise. Here, to motivate transport noise, we follow the heuristic motivations given in [34, Section 1.2] based on two scale arguments. Rigorous justifications of the transport noise in fluid dynamics models are given in e.g., [25, 38, 47, 60, 61].

As in [34, Section 1.2], let us assume that the velocity field u^{ν} of a turbulent fluid decomposes into large and small scales, i.e., $u^{\nu} = u_L^{\nu} + u_S^{\nu}$ where u_S^{ν} and u_L^{ν} are the 'small' and 'large' scales, respectively; and

(1.7)
$$\begin{cases} \partial_t u_L^{\nu} + ([u_S^{\nu} + u_L^{\nu}] \cdot \nabla) u_L^{\nu} = -\nabla p_L^{\nu} + \nu \Delta u_L^{\nu}, \\ \partial_t u_S^{\nu} + ([u_S^{\nu} + u_L^{\nu}] \cdot \nabla) u_S^{\nu} = -\nabla p_S^{\nu} + \nu \Delta u_S^{\nu}, \\ \nabla \cdot u_L^{\nu} = \nabla \cdot u_S^{\nu} = 0, \end{cases}$$

on \mathbb{T}^d , with $d \in \{2,3\}$. In particular, u^{ν} solves the (deterministic) NSEs. Heuristically, one could think that, in a turbulent regime, u_S^{ν} varies in time very rapidly compared to u_L^{ν} . Therefore, u_S^{ν} can be approximated, in time, by a white noise:

$$(1.8) u_S^{\nu}(t,x) \approx \sum_{k \in \mathbb{Z}_0^d} \sum_{1 \leqslant \alpha \leqslant d-1} \tilde{\theta}_k^{\nu} e^{2\pi \mathrm{i} k \cdot x} a_{k,\alpha} \dot{W}_t^{k,\alpha},$$

where $(a_{k,\alpha})_{\alpha\in\{0,\dots,d-1\}}\subseteq\mathbb{R}^d$ is an orthonormal basis of $k^\perp=\{k'\in\mathbb{R}^d:k'\cdot k=0\}$ (this fact ensures the incompressibility of the small scales). It is clear that using (1.8) in the first of (1.7) one obtains the d-dimensional version of (1.1), where \tilde{p}_n^{ν} is the corresponding 'noisy' component of the pressure p_L^{ν} . The choice of the Stratonovich formulation in (1.1) is due to Wong-Zakai type results which, roughly, ensures that (1.1) can be obtained as a limit of regular approximations of the white noises $\dot{W}_t^{k,\alpha}$. At this point, there are no physical motivations for the independence on $\nu>0$ of the coefficients $\tilde{\theta}_k^{\nu}$ in (1.8). In contrast, one expects that small scales are more active at the $Kolmogorov\ length\ scale \sim \nu^{-\ell_d}$ where $\ell_2=\frac{1}{2}$ and $\ell_3=\frac{3}{4}$, c.f., [74, pp. 350]. As u_S^{ν} in (1.8) is a model for small scales, then one expects that $k\mapsto \theta_k^{\nu}$ 'accumulates' at (high) frequencies $\sim \nu^{-\ell_d}$.

In light of the above facts and the support condition in our results (1.6), we conjecture that the anomalous dissipation in Theorem 1.1, which is physically not expected, is due to the incorrect scaling for supp θ^{ν} in Theorem 1.1; and therefore corresponds to an overestimation of the 'excitement' of small scales. More precisely, anticipating that $\theta^{\nu} = (\theta_k^{\nu})_{k \in \mathbb{Z}_0^2}$ is radially symmetric, we conjecture that $\lim_{\nu \downarrow 0} \nu^{1/2-\varepsilon} M^{\nu} = \infty$ for all $\varepsilon > 0$ where M^{ν} satisfies supp $\theta^{\nu} \subseteq \{|k| \leq M^{\nu}\}$ and $(\theta^{\nu})_{\nu \in (0,1)}$ is such that anomalous dissipation for (1.1) holds. Let us stress that we do not expect the latter for advection-diffusion equations as anomalous dissipation for passive scalars can also happen for d = 2. For 2D NSEs, the conjectured behaviour should be produced by the convective nonlinearity that might contrast anomalous dissipation. In the case of advection-diffusion equations, in

agreement with the fact that anomalous dissipation can be obtained by a (deterministic) Hölder continuous flow [12, 23], we instead expect that $M^{\gamma} \sim \gamma^{-\delta}$ for $\delta > 0$ is enough for obtaining anomalous dissipation for solutions to (1.2) (as above, M^{γ} satisfies supp $\theta^{\gamma} \subseteq \{|k| \leq M^{\gamma}\}$ for $\gamma > 0$). The proof of the above conjectures goes beyond the scope of this manuscript.

Next, we explore a consequence of the energy conservation for the small scales in combination with (1.8). It is reasonable to postulate that, although the small scales are more excited as $\nu \downarrow 0$, the corresponding 'energy' stays uniformly bounded in ν . On the one hand, due to the white-in-time ansatz (1.8), the energy cannot be defined directly. On the other hand, the kinetic energy $\mathcal{K}_S^{\nu,\phi}$ of the small scales u_S^{ν} at an observable $\phi \in L^2_{loc}([0,\infty))$ can be defined as

$$\mathcal{K}_{S}^{\nu,\phi}(t,x) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}_{0}^{d}} \widetilde{\theta}_{k}^{\nu} e^{2\pi i k \cdot x} \sum_{1 \leq \alpha \leq d-1} a_{k,\alpha} \int_{0}^{t} \phi(s) dW_{s}^{k,\alpha}$$

for $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^d$, and satisfies,

(1.9)
$$\mathbf{E} \| \mathcal{K}_{S}^{\nu,\phi}(t,\cdot) \|_{L^{2}(\mathbb{T}^{d})}^{2} \approx_{d} \| \widetilde{\theta}^{\nu} \|_{\ell^{2}}^{2} \| \phi \|_{L^{2}(0,t)}^{2} \text{ for } t > 0.$$

The requirement of bounded energy of the small scales, as $\nu \downarrow 0$, therefore implies

$$\sup_{\nu \in (0,1)} \|\widetilde{\theta}^{\nu}\|_{\ell^2} < \infty.$$

The above condition is indeed satisfied in Theorem 1.1 with d=2 and $\tilde{\theta}_k^{\nu}=\sqrt{2\mu}\,\theta_k^{\nu}$. In contrast to NSEs, physical motivations for the γ -dependence of the transport noise in advection-diffusion equations (1.2) are less clear. In any case, we will use the proof of Theorem 1.3 as a guideline for the one of Theorem 1.1.

To conclude, let us stress that some criticisms can be made about the decomposition (1.7) and the corresponding ansatz (1.8). Indeed, as commented in [34, Section 1.2], the above scale separation has never been established so strictly in real fluids. However, simplified models such as (1.7)-(1.8) can be used to understand basic features of certain phenomena. The reader is referred to [58] for discussions.

- 1.3. Strategy, novelty, physical interpretation and comparison. We begin by discussing the strategy used in the proofs of our main results.
- 1.3.1. Strategy. Here, we describe the strategy used to prove Theorem 1.3. The one of Theorem 1.1 is slightly more involved due to the presence of the convective nonlinearity. The main step in the proof of Theorem 1.3 can be roughly summarized as follows (cf., Corollary 4.5):

MAIN STEP. For each fixed $\mu > 0$ and $\gamma > 0$, there exists $\theta^{\gamma} \in \mathcal{S}_{\ell^2}^0$ such that the unique global solution ϱ^{γ} to (1.2) satisfies

(1.10)
$$\mathbf{E} \| \varrho^{\gamma}(1) \|_{L^{2}}^{2} \approx \| \varrho_{\text{det}}^{\gamma}(1) \|_{L^{2}}^{2}$$

where $\varrho_{\text{det}}^{\gamma}$ solves

(1.11)
$$\begin{cases} \partial_t \varrho_{\text{det}}^{\gamma} = (\gamma + \mu) \Delta \varrho_{\text{det}}^{\gamma} & \text{on } \mathbb{T}^d, \\ \varrho_{\text{det}}^{\gamma}(0, \cdot) = \varrho_0 & \text{on } \mathbb{T}^d. \end{cases}$$

Here, \approx means that the difference between the two quantities in (1.10) can be made as small as needed by choosing θ^{γ} appropriately.

The advantage of (1.11) compared to (1.2) is that the operator $\mu\Delta$ provides an additional dissipation/diffusion (sometimes referred to as *eddy viscosity*). It is important to note that the same is *not* true for the transport noise in (1.2). Indeed, due to the incompressibility of the noise coefficients $\nabla \cdot \sigma_{k,\alpha} = 0$ required below, the following energy balance holds:

(1.12)
$$\frac{1}{2} \|\varrho^{\gamma}(t)\|_{L^{2}}^{2} + \int_{0}^{t} \int_{\mathbb{T}^{d}} \gamma |\nabla \varrho^{\gamma}|^{2} dx ds = \frac{1}{2} \|\varrho_{0}\|_{L^{2}}^{2} \text{ a.s. for all } t > 0,$$

where ϱ^{γ} solves (1.2). Therefore, the transport noise in (1.2) with intensity $\mu > 0$ does not produce any additional dissipation.

If (1.10) holds, then the proof of Theorem 1.3 readily follows from the additional viscosity/diffusivity in (1.11):

$$2 \mathbf{E} \int_{0}^{1} \int_{\mathbb{T}^{d}} \gamma |\nabla \varrho^{\gamma}|^{2} dx dt \stackrel{(1.12)}{=} \|\varrho_{0}\|_{L^{2}(\mathbb{T}^{d})}^{2} - \mathbf{E} \|\varrho^{\gamma}(1)\|_{L^{2}(\mathbb{T}^{d})}^{2}$$

$$\stackrel{(1.10)}{\approx} \|\varrho_{0}\|_{L^{2}(\mathbb{T}^{d})}^{2} - \|\varrho_{\det}^{\gamma}(1)\|_{L^{2}(\mathbb{T}^{d})}^{2}$$

$$\stackrel{(i)}{\geqslant} (1 - e^{-\mu}) \|\varrho_{0}\|_{L^{2}(\mathbb{T}^{d})}^{2}$$

where in (i) we used that $\|\varrho_{\det}^{\gamma}(t)\|_{L^{2}(\mathbb{T}^{d})}^{2} \leq e^{-(\mu+\gamma)t}\|\varrho_{0}\|_{L^{2}(\mathbb{T}^{d})}^{2} \leq e^{-\mu t}\|\varrho_{0}\|_{L^{2}(\mathbb{T}^{d})}^{2}$ due to the increased viscosity/diffusivity in (1.11). Now, Theorem 1.3 follows by tuning the parameter $\mu > 0$ and taking the infimum over $\gamma \in (0, 1)$.

1.3.2. Novelty – Improved scaling limits via Meyers' estimates. The key ingredient in the proof of (1.10) is an improvement of a well-established scaling limit argument due to [29, 34]. Our improvement of such scaling limit argument reads roughly as follows (cf., Proposition 4.1):

For all $\mu, \gamma > 0$ and sequences $(\theta^n)_{n \ge 1} \subseteq \mathcal{S}^0_{\ell^2}$ (see (1.4)) satisfying

$$\lim_{n \to \infty} \|\theta^n\|_{\ell^{\infty}} = 0$$

it holds that

(1.14)
$$\lim_{n \to \infty} \varrho^n = \varrho_{\text{det}}^{\gamma} \text{ in probability in } C([0,1]; L^2(\mathbb{T}^d)),$$

where ϱ^n is the unique global solution to (1.2) with $\theta^{\gamma} = \theta^n$.

A sequence in $\mathcal{S}_{\ell^2}^0$ satisfying (1.13) is given in [56, eq. (1.9)] (see also (4.4)).

The main novelty and improvement compared to the above-mentioned works of (1.14) is that the limit is established in a space of zero smoothness uniformly in time. More precisely, in previous works, limits as in (1.14) were only established with $L^2(\mathbb{T}^d)$ replaced by $H^{-\varepsilon}(\mathbb{T}^d)$ for some $\varepsilon > 0$ (cf., [29, Proposition 3.7] and [34, Theorem 1.4]). Let us stress that such improvement is of central importance to establish (1.10) and thus for our approach to anomalous dissipation. Indeed, (1.14) and energy estimates readily imply (1.10) for a sufficiently large $n \ge 1$; while this is not true if one uses $H^{-\varepsilon}(\mathbb{T}^d)$ instead of $L^2(\mathbb{T}^d)$ in (1.14). Interestingly, we can also prove that (1.14) holds with $L^2(\mathbb{T}^d)$ replaced by $H^r(\mathbb{T}^d)$ for some $r(\gamma, \mu) > 0$. The latter fact is needed to deal with 2D NSEs, see the proof of Corollary 5.4.

The key tool behind (1.14) are the *stochastic Meyers' estimates* proven in Section 3. In particular, they imply the existence of $r_0(\gamma, \mu) > 0$ for which

(1.15)
$$\sup_{n\geqslant 1} \mathbf{E} \sup_{t\in[0,1]} \|\varrho^n(t)\|_{H^{r_0}(\mathbb{T}^d)}^2 < \infty$$

where ϱ^n is as in (1.14), i.e., the unique global solution to (1.2) with $\theta^{\gamma} = \theta^n$. After (1.15) is proved, then the claim (1.14) follows from a compactness argument.

The main difficulty behind the proof of (1.15) is the lack of uniformity of regularity of the noise coefficients $(\theta_k^n \sigma_{k,\alpha})_{k,\alpha}$ in (1.2) along the sequence $(\theta^n)_{n\geqslant 1}$ satisfying (1.13). More precisely, one has, for all r>0,

$$(1.16) \quad \sup_{n\geqslant 1} \|(\theta_k^n \sigma_{k,\alpha})_{k,\alpha}\|_{L^\infty(\mathbb{T}^d;\ell^2)} < \infty \quad \text{while} \quad \sup_{n\geqslant 1} \|(\theta_k^n \sigma_{k,\alpha})_{k,\alpha}\|_{H^r(\mathbb{T}^d;\ell^2)} = \infty.$$

Let us stress that the supremum over $n \ge 1$ in the above is essential it can hold that $\#\{k:\theta_k^n\ne 0\}<\infty$ and therefore $(\theta_k^n\sigma_{k,\alpha})_{k,\alpha}\in C^\infty(\mathbb{T}^d;\ell^2)$ for each fixed $n\ge 1$. In particular, (1.15) cannot be derived from well-known results on L^p -theory of SPDEs, as it always requires some degree of regularity of the coefficients and provides a high improvement of the regularity of solutions the SPDE under consideration, see e.g., [8, 53]. In contrast, Meyers' estimates provide a small improvement on the regularity of solutions of (S)PDEs with bounded, measurable and parabolic coefficients; which is exactly the setting one faces up with when dealing with the scaling limit (1.13)-(1.14). Indeed, the boundedness is given by (1.16), while the parabolicity is uniform due to the Stratonovich formulation of the transport noise, see the proof of Lemma 4.2 for details. To conclude, let us stress that, due to the lack of uniform smoothness as described in (1.16), (1.15) cannot hold with r_0 large. Counterexamples in the deterministic counterpart can be found in [13, 62].

1.3.3. Physical interpretation – Connection with homogenization. Arguing as in [1, Subsection 2.2], the scaling limit result of (1.13)-(1.14) can be interpreted as an homogenization result for the SPDE (1.2). Indeed, as discussed in Subsection 1.2, the transport noise term in (1.2) (or, more precisely, for 2D NSEs (1.1) for which a similar discussion applies) can be interpreted as the contribution of the 'small scales' of the fluid flow. Therefore, reasoning as in [1, Subsection 2.2], the limit (1.13) can be interpreted as 'zooming out' from small scales and the PDE (1.11) can be thought of as the 'effective equation' for the SPDE (1.2). In our context, the 'microscopic' parameter is $\varepsilon = \|\theta^n\|_{\ell^\infty} \downarrow 0$, cf., (1.13).

The homogenization viewpoint is also interesting from a mathematical perspective. Indeed, the main tool to achieve (1.14) are the (stochastic) Meyers' estimates which also play a fundamental role in the context of homogenization of PDEs (see e.g., [10, 11, 43, 70]) since, as it is well-known in homogenization, the only quantities remaining uniform along the process of zooming out from small scales (i.e., $\varepsilon \downarrow 0$) are measurability, boundedness and parabolicity.

1.3.4. Further comparison with the literature. As commented in Subsection 1.3.2, the main novelty consists of an improvement of the scaling limit (1.14). Let us note that such an improvement is also interesting on its own. Indeed, the latter can be used also to strengthen other results on the topic. For instance, arguing as in Section 5, one can check that the main results of [29] can be extended also to the case of $L^2(\mathbb{T}^d)$ being critical for the corresponding SPDE (here, we use criticality in the sense of [7, Section 3] and are implicitly assuming that [29, (H4)] holds with L^2 replaced by H^r with r > 0). For instance, this covers the case of the Keller-Segel model system in four dimensions, cf., [29, Subsection 2.1]. Moreover, the stochastic Meyers' estimates on domains (see Theorem 3.3) might be used in combination with the results in [30] to obtain a version of Theorem 1.3 on domains.

We are now in a position to give a comparison between our results and the one of [46, Section 3] and [68]. Our results are in spirit similar to the one in [68] which, roughly speaking, ensure that anomalous dissipation can happen if the noise is chosen according to the diffusivity $\gamma > 0$ (cf., the comments below [68, Theorem 1.1]). On the one hand, in our results, due to the application of the scaling limit in (1.13)-(1.14), we cannot provide an explicit dependence on $\sup \theta^{\gamma}$ on γ while the one in [68] are rather explicit. On the other hand, in our results, we have a sharp control of how the intensity of the noise affects anomalous dissipation and our methods are robust enough for applications to nonlinear SPDEs, e.g., 2D NSEs. From a technical point of view, the approaches are completely different and thus a comparison does not seem possible.

In [46], the authors showed total dissipations for passive scalars, and for 2D and (even) 3D weak solutions to the NSEs. Here, total dissipation means that all the initial energy $\|\varrho_0\|_{L^2} > 0$ is absorbed by the energy dissipation rates, i.e., $\lim_{t\uparrow 1} \|\varrho^{\gamma}(t)\|_{L^2} = 0$ a.s. for all $\gamma > 0$. Our results share some similarities with the one of [46, Section 3], where the authors considered transport noise. For instance, to the best of our knowledge, they were the first to use the constructions in [29, 41] in the context of anomalous dissipation. However, there are several differences between our results and the one in [46, Section 3], and in particular in the mechanism for creating anomalous dissipation. Firstly, instead of assuming a viscosity/diffusivity dependent θ in either (1.1) or (1.2), they considered a piecewise constant in time noise coefficient $\theta(t) = (\theta_k(t))_{k \in \mathbb{Z}_0^d}$ such that $\|\theta(t)\|_{\ell^2} \to \infty$ as $t \uparrow 1$, see [46, Subsection 2.1]. In particular, following the argument in Subsection 1.2, extending the identity (1.9) to the case of piecewise constant in time θ , one sees that their noise has 'infinite energy' as $t \uparrow 1$. This is not the case in our construction where, due to the requirement $\theta^{\nu} \in \mathcal{S}_{\ell^2}^0$ for all $\nu > 0$, we obtain finite energy of the noise as $\nu \downarrow 0$ and uniformly for $t \in [0,1]$. However, our method cannot deal with 3D NSEs in contrast to [46] (see also comments below Theorem 1.1). Secondly, from a technical point of view, the proofs are completely different. Indeed, in [46, Section 3 the authors rely on a careful study of the mixing properties of the transport noise and subsequently upgraded them to an anomalous dissipation result. This approach in particular requires $\sup_{t\in[0,1)}\|\theta(t)\|_{\ell^2}=\infty$. In our manuscript, instead, we employ the above-discussed refinement of the scaling limit arguments of [41, 29] via Meyers' estimates. To conclude, it would be interesting to see if the arguments used in [46, Section 4] to derive anomalous dissipation by advection via randomly forced NSEs can benefit from the use of Meyers' estimates. However, this goes beyond the scope of the current manuscript.

2. Preliminaries

2.1. **Notation.** Here, we collect the basic notation used in the manuscript. For given parameters p_1, \ldots, p_n , we write $R(p_1, \ldots, p_n)$ if the quantity R depends only on p_1, \ldots, p_n . For two quantities x and y, we write $x \leq y$, if there exists a constant C such that $x \leq Cy$. If such a C depends on above-mentioned parameters p_1, \ldots, p_n we either mention it explicitly or indicate this by writing $C(p_1, \ldots, p_n)$ and correspondingly $x \leq_{p_1, \ldots, p_n} y$ whenever $x \leq C(p_1, \ldots, p_n)y$. We write $x \approx_{p_1, \ldots, p_n} y$, whenever $x \leq_{p_1, \ldots, p_n} y$ and $y \leq_{p_1, \ldots, p_n} x$.

Below, $(\Omega, \mathscr{A}, (\mathscr{F}_t)_{t \geq 0}, \mathbf{P})$ denotes a filtered probability space carrying a sequence of independent standard Brownian motions which changes depending on

the SPDE under consideration. We write **E** for the expectation on $(\Omega, \mathscr{A}, \mathbf{P})$ and \mathscr{P} for the progressive σ -field. A process $\phi:[0,\infty)\times\Omega\to X$ is progressively measurable if $\phi|_{[0,t]\times\Omega}$ is $\mathscr{B}([0,t])\otimes\mathscr{F}_t$ measurable for all $t\geqslant 0$, where \mathscr{B} is the Borel σ -algebra on [0,t] and X a Banach space. Moreover, a stopping time τ is a measurable map $\tau:\Omega\to[0,\infty]$ such that $\{\tau\leqslant t\}\in\mathscr{F}_t$ for all $t\geqslant 0$. Finally, a stochastic process $\phi:[0,\tau)\times\Omega\to X$ is progressively measurable if $\mathbf{1}_{[0,\tau)\times\Omega}\phi$ is progressively measurable where $[0,\tau)\times\Omega\stackrel{\mathrm{def}}{=}\{(t,\omega)\in[0,\infty)\times\Omega:0\leqslant t<\tau(\omega)\}$ and $\mathbf{1}_{[0,\tau)\times\Omega}$ stands for the extension by zero outside $[0,\tau)\times\Omega$.

Next, we collect the notation for function spaces. We write $L^p(S,\mu;X)$ for the Bochner space of strongly measurable, p-integrable X-valued functions for a measure space (S,μ) and a Banach space X as defined in [48, Section 1.2b]. If $X=\mathbb{R}$, we write $L^p(S,\mu)$ and if it is clear which measure we refer to we also leave out μ . Finally, $\mathbf{1}_A$ dentoes the indicator function of $A\subseteq S$. Bessel-potential spaces are indicated as usual by $H^{s,q}(\mathbb{T}^d)$ where $s\in\mathbb{R}$ and $q\in(1,\infty)$. We also use the standard shorth-hand notation $H^{s,2}(\mathbb{T}^d)$ for $H^s(\mathbb{T}^d)$. We sometimes also employ Besov spaces $B^s_{q,p}(\mathbb{T}^d)$ which can be defined as the real interpolation space $(H^{-k,q}(\mathbb{T}^d),H^{k,q}(\mathbb{T}^d))_{(s+k)/2k,p}$ for $s\in\mathbb{R}$, $\mathbb{N}\ni k>|s|$ and $1< q,p<\infty$. The reader is referred to [15, 48] for details on interpolation and to [69, Section 3.5.4] for details on function spaces over \mathbb{T}^d . Finally, we let $\mathcal{A}(\mathbb{T}^d;\mathbb{R}^k)\stackrel{\mathrm{def}}{=} (\mathcal{A}(\mathbb{T}^d))^k$ and $\mathcal{A}(\cdot)\stackrel{\mathrm{def}}{=} \mathcal{A}(\mathbb{T}^d;\cdot)$ for $\mathcal{A}\in\{L^q,H^{s,q},B^s_{q,p}\}$ and $k\in\mathbb{N}$.

The Helmholtz projection $\mathbb P$ and its complement projection $\mathbb Q$. For an $\mathbb R^d$ -valued distribution $f=(f^i)_{i=1}^d\in\mathcal D'(\mathbb T^d;\mathbb R^d)$ on $\mathbb T^d$, let $\widehat{f^i}(k)=\langle e_k,f^i\rangle$ be k-th Fourier coefficients, where $k\in\mathbb Z^d,\ i\in\{1,\ldots,d\}$ and $e_k(x)=e^{2\pi\mathrm{i} k\cdot x}$. The Helmholtz projection $\mathbb P f$ for $f\in\mathcal D'(\mathbb T^d;\mathbb R^d)$ is given by

$$(\widehat{\mathbb{P}f})^{i}(k) \stackrel{\text{def}}{=} \widehat{f^{i}}(k) - \sum_{1 \leq i \leq d} \frac{k_{i}k_{j}}{|k|^{2}} \widehat{f^{j}}(k), \qquad (\widehat{\mathbb{P}f})^{i}(0) \stackrel{\text{def}}{=} \widehat{f^{i}}(0).$$

Formally, $\mathbb{P}f$ can be written as $f - \nabla \Delta^{-1}(\nabla \cdot f)$. We set

$$\mathbb{Q} \stackrel{\mathrm{def}}{=} \mathrm{Id} - \mathbb{P}.$$

From standard Fourier analysis, it follows that \mathbb{Q} and \mathbb{P} restrict to bounded linear operators on $H^{s,q}$ and $B^s_{q,p}$ for $s \in \mathbb{R}$ and $q,p \in (1,\infty)$. Finally, we can introduce function spaces of divergence-free vector fields: For $\mathcal{A} \in \{L^q, H^{s,q}, B^s_{q,p}\}$, we set

$$||f||_{\mathbb{A}(\mathbb{T}^d)} \stackrel{\text{def}}{=} \mathbb{P}(\mathcal{A}(\mathbb{T}^d; \mathbb{R}^d)),$$

endowed with the natural norm. ℓ^2 -values function spaces are defined analogously.

2.2. Structure of the noise. Here, we describe the quantities θ^{ν} , θ^{γ} , $\sigma_{k,\alpha}$, $W^{k,\alpha}$ and W^k appearing in the stochastic perturbation of (1.1) and (1.2), while $c_d \stackrel{\text{def}}{=} d/(d-1)$. In this subsection, we follow [29, 34]. In the following, we only describe the stochastic perturbation in (1.2), for the case of NSEs (1.1) it is enough to take d=2 in the following construction and omit the index α . Moreover, since $\gamma>0$ is fixed here, we simply write θ instead of θ^{γ} .

To begin, set $\mathbb{Z}_0^d \stackrel{\text{def}}{=} \mathbb{Z}^d \setminus \{0\}$. Throghout the manuscript $\theta = (\theta_k)_{k \in \mathbb{Z}_0^d} \in \ell^2(\mathbb{Z}_0^d)$ is normalized and radially symmetric, i.e.

$$(2.1) \qquad \|\theta\|_{\ell^2(\mathbb{Z}_0^d)} = 1 \quad \text{ and } \quad \theta_j = \theta_k \ \text{ for all } j,k \in \mathbb{Z}_0^d \ \text{ such that } |j| = |k|.$$

Next, we define the family of vector fields $(\sigma_{k,\alpha})_{k,\alpha}$. Here and in the following, we use the shorthand subscript 'k, α ' to indicate $k \in \mathbb{Z}_0^d$, $\alpha \in \{0, \ldots, d-1\}$. Let \mathbb{Z}_+^d and \mathbb{Z}_-^d be a partition of \mathbb{Z}_0^d such that $-\mathbb{Z}_+^d = \mathbb{Z}_-^d$. For any $k \in \mathbb{Z}_+^d$, let $\{a_{k,\alpha}\}_{\alpha \in \{1,\ldots,d-1\}}$ be a complete orthonormal basis of the hyperplane $k^{\perp} = \{k' \in \mathbb{R}^d : k \cdot k' = 0\}$, and set $a_{k,\alpha} \stackrel{\text{def}}{=} a_{-k,\alpha}$ for $k \in \mathbb{Z}_-^d$. Finally, we let

$$\sigma_{k,\alpha} \stackrel{\text{def}}{=} a_{k,\alpha} e^{2\pi i k \cdot x}$$
 for all $x \in \mathbb{T}^d$, $k \in \mathbb{Z}_0^d$, $\alpha \in \{1, \dots, d-1\}$.

By construction, we have that $\sigma_{k,\alpha}$ is smooth and divergence-free for all k,α .

Finally, we introduce the family of complex Brownian motions $(W^{k,\alpha})_{k,\alpha}$. Let $(B^{k,\alpha})_{k,\alpha}$ be a family of independent standard (real) Brownian motions on the above-mentioned filtered probability space $(\Omega, \mathscr{A}, (\mathscr{F}_t)_{t\geqslant 0}, \mathbf{P})$. Then, we set

(2.2)
$$W^{k,\alpha} \stackrel{\text{def}}{=} \begin{cases} B^{k,\alpha} + iB^{-k,\alpha}, & k \in \mathbb{Z}_+^d, \\ B^{-k,\alpha} - iB^{k,\alpha}, & k \in \mathbb{Z}_-^d. \end{cases}$$

In particular $\overline{W^{k,\alpha}} = W^{-k,\alpha}$ for all k, α .

As in [34, Section 2.3] or [29, Remark 1.1], by (2.1) and the definition of the vector fields $\sigma_{k,\alpha}$, at least formally, one has

(2.3)
$$\sqrt{c_d \mu} \sum_{k,\alpha} \theta_k(\sigma_{k,\alpha} \cdot \nabla) \varrho \circ \dot{W}_t^{k,\alpha} = \mu \Delta \varrho + \sqrt{c_d \mu} \sum_{k,\alpha} \theta_k(\sigma_{k,\alpha} \cdot \nabla) \varrho \, \dot{W}_t^{k,\alpha},$$

where we recall that $c_d = d/(d-1)$. To see the above, one uses that $\nabla \cdot \sigma_{k,\alpha} = 0$, (2.1) and the elementary identity (cf. [34, eq. (2.3)] or [29, eq. (3.2)])

(2.4)
$$\sum_{k,\alpha} \theta_k^2 \, \sigma_{k,\alpha} \otimes \overline{\sigma_{k,\alpha}} = \sum_{k,\alpha} \theta_k^2 \, a_{k,\alpha} \otimes a_{k,\alpha} = \frac{1}{c_d} \mathrm{Id}_{d \times d} \quad \text{on } \mathbb{T}^d.$$

Let us remark that the stochastic integration on the RHS(2.3) is understood in the Itô-sense. In the paper we will always understand the Stratonovich noise on the LHS(2.3) as the RHS(2.3), namely an Itô noise plus a diffusion term. However, note that the diffusion term $\mu\Delta\varrho$ does not provide any additional diffusion, as in energy estimates it balances the Itô correction coming from the Itô-noise, cf., (1.12).

A slightly more complicated situation arises in the reformulation of the Stratonovich noise in the 2D NSEs (1.1). Indeed, applying the Helmholtz projection on (1.1) to eliminate the auxiliary unknown pressure (see Subsection 2.1), the stochastic perturbation of (1.1) reads as

$$\sqrt{2\mu} \sum_{k \in \mathbb{Z}_0^2} \theta_k \, \mathbb{P}[(\sigma_k \cdot \nabla) u] \circ \dot{W}_t^k.$$

Reasoning as in [34, Section 2], by (2.4), formally it holds that

(2.5)
$$\sqrt{2\mu} \sum_{k \in \mathbb{Z}_0^2} \theta_k \, \mathbb{P}[(\sigma_k \cdot \nabla)u] \circ \dot{W}_t^k = \mu \Delta u + Q_{\theta}(u) + \sqrt{2\mu} \sum_{k \in \mathbb{Z}_0^2} \theta_k \, \mathbb{P}[(\sigma_k \cdot \nabla)u] \, \dot{W}_t^k$$

where

(2.6)
$$Q_{\theta}(u) \stackrel{\text{def}}{=} -2\mu \sum_{k \in \mathbb{Z}_0^2} \theta_k^2 \mathbb{P} [(\sigma_k \cdot \nabla) \mathbb{Q} [(\sigma_k \cdot \nabla) u]].$$

Further comments on the transformation (2.5) can be found in [9, Section 1].

2.3. Solution concept and well-posedness. Here, we define solutions to 2D NSEs (1.1) and to the scalar equation (1.2). We begin by recalling that the sequence of complex Brownian motions $(W^{k,\alpha})_{k,\alpha}$ induces an ℓ^2 -cylindrical Brownian motion \mathcal{W}_{ℓ^2} via the formula

$$\mathcal{W}_{\ell^2}(f) = \sum_{k,\alpha} \int_{\mathbb{R}_+} f_{k,\alpha}(t) \, dW_t^{k,\alpha} \quad \text{for} \quad f = (f_{k,\alpha})_{k,\alpha} \in L^2(\mathbb{R}_+; \ell^2).$$

Note that $W_{\ell^2}(f)$ is real-valued if $f_{k,\alpha} = f_{-k,\alpha}$ as $\overline{W^{k,\alpha}} = W^{-k,\alpha}$ for all k,α . Using this and the symmetry of $\sigma_{k,\alpha}$ under the reflection $k \mapsto -k$, one can check that the stochastic perturbation in (1.1) and (1.2) can be rewritten only using real-valued coefficients and real-valued Brownian motions $(B^{k,\alpha})_{k,\alpha}$ in (2.2). For instance,

(2.7)
$$\sum_{k,\alpha} \theta_k^{\gamma}(\sigma_{k,\alpha} \cdot \nabla) \varrho \, \dot{W}^{k,\alpha} = \sum_{k \in \mathbb{Z}_+^d, \alpha \in \{0, \dots, d-1\}} 2 \, \theta_k^{\gamma} \, (\Re \sigma_{k,\alpha} \cdot \nabla) \varrho \, \dot{B}^{k,\alpha} + \sum_{k \in \mathbb{Z}_-^d, \alpha \in \{0, \dots, d-1\}} 2 \, \theta_k^{\gamma} \, (\Im \sigma_{k,\alpha} \cdot \nabla) \varrho \, \dot{B}^{k,\alpha}$$

for any real-valued function ϱ . Hence, solutions to (1.1) and (1.2) are naturally real-valued. However, the complex formulation of the noise is more convenient for computations and has been widely employed in related works (e.g., [29, 31, 34, 56]).

We begin with defining solutions to the passive scalar equation (1.2). Below we use the reformulation of the Stratonovich noise in (1.2).

Definition 2.1 (Solutions – passive scalars). Fix $\gamma > 0$, $\theta^{\gamma} \in \ell^2$ and $\varrho_0 \in L^2(\mathbb{T}^d)$. Let $\tau^{\gamma} : \Omega \to [0, \infty]$ and $\varrho^{\gamma} : [0, \tau^{\gamma}) \times \Omega \to \mathbb{H}^1(\mathbb{T}^d)$ be a stopping time and a progressive measurable process, respectively.

• We say that $(\varrho^{\gamma}, \tau^{\gamma})$ is a local solution to (1.2) if the following are satisfied: $-\varrho^{\gamma} \in L^{2}_{loc}([0, \tau^{\gamma}); H^{1}(\mathbb{T}^{d})) \cap C([0, \tau^{\gamma}); L^{2}(\mathbb{T}^{d}))$ a.s.; -a.s. for all $t \in [0, \tau^{\gamma})$ it holds that

$$\varrho^{\gamma}(t) - \varrho_{0} = (\gamma + \mu) \int_{0}^{t} \Delta \varrho^{\gamma}(s) ds + \sqrt{c_{d}\mu} \int_{0}^{t} \mathbf{1}_{[0,\tau^{\gamma})} \Big((\theta_{k}^{\gamma} \sigma_{k,\alpha} \cdot \nabla) \varrho^{\gamma}(s) \Big)_{k,\alpha} d\mathcal{W}_{\ell^{2}}.$$

- A local solution $(\varrho^{\gamma}, \tau^{\gamma})$ to (1.2) is a said to be a unique local solution to (1.2) if for any local solution $(\chi^{\gamma}, \lambda^{\gamma})$ we have $\lambda^{\gamma} \leq \tau^{\gamma}$ a.s. and $\chi^{\gamma} = \varrho^{\gamma}$ a.e. on $[0, \lambda^{\gamma}) \times \Omega$.
- A unique local solution $(\varrho^{\gamma}, \tau^{\gamma})$ to (1.2) is said to be a unique global solution to (1.2) if $\tau^{\gamma} = \infty$ a.s.

Note that the deterministic and stochastic integral in Definition 2.1 are well-defined as a $H^{-1}(\mathbb{T}^d)$ -valued Bocher and $L^2(\mathbb{T}^d)$ -valued Itô integrals, respectively. The solutions of Definition 2.1 are weak in the PDE sense, but strong in the probabilistic sense. In case of global solutions, we only write ϱ^{γ} instead of $(\varrho^{\gamma}, \tau^{\gamma})$ if no confusion seems likely.

Existence and uniqueness of global solutions to (1.2) in the sense of Definition 2.1 is standard, see e.g., [55, Chapter 4]. In case θ_k^{γ} decays sufficiently fast as $|k| \to \infty$ (e.g., $\#\{k: \theta_k^{\gamma} \neq 0\} < \infty$), then the unique global solution ϱ^{γ} instantaneously improves its regularity in time and space, cf. [6, Theorems 2.7 and 4.2].

In the case of NSEs, the definition is analogous. As usual, we interpret the convective term $(u^{\nu}\cdot\nabla)u^{\nu}$ in its conservative form $\nabla\cdot(u^{\nu}\otimes u^{\nu})$ due to the divergencefree condition. Recall that Q_{θ} has been defined in (2.6).

Definition 2.2 (Solutions – 2D Naver-Stokes equations). Fix $\nu > 0$, $\theta^{\nu} \in \ell^2$ and $u_0 \in \mathbb{L}^2(\mathbb{T}^d)$. Let $\tau^{\nu}: \Omega \to [0, \infty]$ and $u^{\nu}: [0, \infty) \times \Omega \to \mathbb{H}^1(\mathbb{T}^d)$ be a stopping time and a progressive measurable process, respectively.

- We say that (u^{ν}, τ^{ν}) is a local solution to (1.1) if the following are satisfied: $- u^{\nu} \in L^{2}_{loc}([0, \infty); H^{1}(\mathbb{T}^{2}; \mathbb{R}^{2})) \cap C([0, \infty); L^{2}(\mathbb{T}^{2}; \mathbb{R}^{2})) \ a.s.;$ $- u^{\nu} \otimes u^{\nu} \in L^{2}_{loc}([0, \infty); L^{2}(\mathbb{T}^{2}; \mathbb{R}^{2})) \ a.s.;$ $- a.s. \ for \ all \ t \in \mathbb{R}_{+} \ it \ holds \ that$

$$u^{\nu}(t) - u_0 = \int_0^t \left[(\nu + \mu) \Delta u^{\nu}(s) + Q_{\theta^{\nu}}(u^{\nu}(s)) - \mathbb{P}[\nabla \cdot (u^{\nu}(s) \otimes u^{\nu}(s))] \right] ds$$
$$+ \sqrt{2\mu} \int_0^t \mathbf{1}_{[0,\tau^{\nu})} \left(\mathbb{P}[(\theta_k^{\nu} \sigma_k \cdot \nabla) u^{\nu}(s)] \right)_k d\mathcal{W}_{\ell^2}.$$

- A local solution (u^{ν}, τ^{ν}) to (1.1) is said to be a unique local solution to (1.1) if for any local solution (v^{ν}, λ^{ν}) we have $\lambda^{\nu} \leq \tau^{\nu}$ a.s. and $v^{\nu} = u^{\nu}$ a.e. on $[0,\lambda^{\nu})\times\Omega$.
- A unique local solution (u^{ν}, τ^{ν}) to (1.1) is said to be a unique global solution to (1.1) if $\tau^{\nu} = \infty$ a.s.

As above, the integrals in Definition 2.2 are well-defined and, in case of global solutions, we only write u^{ν} instead of (u^{ν}, τ^{ν}) if no confusion seems likely.

Existence and uniqueness of global solutions to (1.1) is standard, see e.g., [9, Appendix A] or [7, Theorem 3.4]. For additional details, the reader is referred to [36]. Instantaneous regularity results in case θ_k^{ν} decays sufficiently fast as $|k| \to \infty$ can be found in [9, Theorems 2.7 and 2.12].

3. Stochastic Meyers' estimates

In this section, we discuss maximal $L_t^p(L_x^q)$ -regularity estimates for parabolic SPDEs under only boundedness, measurability and parabolicity assumptions with p and q close to 2. In the deterministic case, such estimates have been first proven by Meyers' [59] via a perturbation argument. In the stochastic setting, such estimates have been proven in [14] where, due to the abstract setting, no extrapolation in space is given, i.e., q=2. Here, we give a direct proof of the stochastic Meyers' estimates following his original idea, which interestingly, also provides an improvement in the integrability in space. For simplicity, we mainly consider SPDEs on \mathbb{T}^d . Extensions to domains and more general operators are discussed in Subsection 3.3.

Below, $(W^n)_{n\geq 1}$ denotes a family of independent standard Brownian motions on a filtered probability space $(\Omega, \mathscr{A}, (\mathscr{F}_t)_{t \geq 0}, \mathbf{P})$ and $\mathbf{E} \stackrel{\text{def}}{=} \int_{\Omega} \cdot d\mathbf{P}$. Note that such Brownian motions might differ from the one introduced in Subsection 2.2.

3.1. Parabolic stochastic Meyers' estimates. In this subsection, we consider the following parabolic SPDEs

(3.1)
$$\begin{cases} \partial_t v = \kappa \Delta v + f + \sum_{n \ge 1} \left[(\xi_n \cdot \nabla) v + g_n \right] \dot{W}_t^n & \text{ on } \mathbb{T}^d, \\ v(0, \cdot) = 0 & \text{ on } \mathbb{T}^d, \end{cases}$$

where v is the unknown process and f, ξ_n and g_n are specified below. The case of systems of SPDEs is discussed in Subsection 3.3. In the following, we say that a progressively measurable process $v : \mathbb{R}_+ \times \Omega \to H^1(\mathbb{T}^d)$ is a *strong solution* to (3.1) if a.s. $v \in L^2_{loc}([0,\infty); H^1(\mathbb{T}^d))$, and for all $t \geq 0$,

$$v(t) = \int_0^t (\kappa \Delta v(s) + f(s)) \, \mathrm{d}s + \sum_{n \ge 1} \int_0^t \left[(\xi_n \cdot \nabla) v(s) + g_n(s) \right] \mathrm{d}W_s^n.$$

Here the equality is understood in $H^{-1}(\mathbb{T}^d)$. Similar definitions are employed for the SPDEs considered in this section.

The following complements [8, Theorem 1.2] in case of L^{∞} -transport noise.

Theorem 3.1 (Parabolic stochastic Meyers' estimates). Let $\kappa > 0$. Assume that $\xi_n = (\xi_j^n)_{j=1}^d : \mathbb{R}_+ \times \Omega \times \mathbb{T}^d \to \mathbb{R}^d$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}^d)$ -measurable for all $n \geq 1$, for which there exist $M_0 > 0$ and $\kappa_0 \in [0, \kappa)$ such that, a.e. on $\mathbb{R}_+ \times \Omega \times \mathbb{T}^d$,

(3.3)
$$\frac{1}{2} \sum_{n \ge 1} (\xi_n \cdot \eta)^2 \le \kappa_0 |\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^d.$$
 (parabolicity)

Then there exists $p_0(d, \kappa, \kappa_0, M_0) > 2$ such that, for all $T \in (0, \infty)$, $p \in [2, p_0]$, $q \in [2, p]$ and progressively measurable processes f and g satisfying

$$f \in L^p((0,T) \times \Omega; H^{-1,q}(\mathbb{T}^d))$$
 and $g = (g_n)_{n \geqslant 1} \in L^p((0,T) \times \Omega; L^q(\mathbb{T}^d; \ell^2)),$

there exists a unique strong solution $v \in L^p((0,T) \times \Omega; H^{1,q}(\mathbb{T}^d))$ to (3.1) and

(3.4)
$$||v||_{L^{p}((0,T)\times\Omega;H^{1,q}(\mathbb{T}^{d}))} \lesssim_{d,\kappa,\kappa_{0},M_{0},T} ||f||_{L^{p}((0,T)\times\Omega;H^{-1,q}(\mathbb{T}^{d}))} + ||g||_{L^{p}((0,T)\times\Omega;L^{q}(\mathbb{T}^{d};\ell^{2}))}.$$

Some remarks are in order. Firstly, it is well-known that the condition (3.3) with $\kappa_0 < \kappa$ is optimal in the parabolic regime. Secondly, by [4, Proposition 3.8], [63, Theorem 1.2] and the fact that the operator $v \mapsto -\Delta v$ on $H^{-1,q}(\mathbb{T}^d)$ with domain $H^{1,q}(\mathbb{T}^d)$ has a bounded H^{∞} -calculus of angle 0 (due to the periodic version of [49, Theorem 10.2.25]), it follows that the LHS(3.28) can be replaced by

(3.5)
$$||v||_{L^p(\Omega;C([0,T];B_{a,p}^{1-2/p}(\mathbb{T}^d)))}$$
, and

(3.6)
$$||v||_{L^p(\Omega; H^{\theta, p}(0, T; H^{1-2\theta, q}(\mathbb{T}^d)))}$$
 for $\theta \in [0, \frac{1}{2})$ provided $p > 2$.

The reader is referred to [49, Chapter 10] and [4, Subsection 2.2] for details on the H^{∞} -calculus and Banach-valued fractional Sobolev spaces, respectively. The appearance of Besov spaces in (3.5) is optimal in light of the trace method, see e.g. [15, Section 3.12] and [2, Theorem 1.2]. Finally, by [4, Proposition 3.10], the estimates (3.28)-(3.6) also hold with non-trivial initial data $v(\cdot,0) \in L^p_{\mathscr{F}_0}(\Omega; B^{1-2/p}_{q,p}(\mathbb{T}^d))$ provided on the RHS(3.28) one also add $\|v(\cdot,0)\|_{L^p(\Omega; B^{1-2/p}_q(\mathbb{T}^d))}$.

Proof. By a perturbation argument [8, Theorem 3.2], it is enough to prove the claim with $T = \infty$ and with Δv replaced $\Delta v - v$. The advantage is that the latter operator is invertible. This allows us to work on the half-line $\mathbb{R}_+ = (0, \infty)$ instead of intervals of finite length. In particular, this gives the independence of $p_0 > 2$ on

T of the maximal $L_t^p(L_x^q)$ -regularity estimates for (3.1). Hence, below, we consider

(3.7)
$$\begin{cases} \partial_t v = \kappa(\Delta v - v) + f + \sum_{n \ge 1} \left(\left[(\xi_n \cdot \nabla) v \right] + g_n \right) \dot{W}_t^n & \text{ on } \mathbb{T}^d, \\ v(0, \cdot) = 0 & \text{ on } \mathbb{T}^d. \end{cases}$$

The main idea is to regard (3.7) as a perturbation of the case $\xi_n \equiv 0$, and show the smallness of the transport noise part as (q,p) approaches p=q=2. To this end, we first analyze the constant in the maximal $L_t^p(L_x^q)$ -regularity estimate when $\xi_n \equiv 0$, i.e., for the stochastic heat equation with additive noise.

Step 1: (Analysis of constants – case $\xi_n \equiv 0$ and $f \equiv 0$) For each p > 2 there exists $K_p > 0$ satisfying $\lim_{p \downarrow 2} \sup_{2 \leqslant r \leqslant p} K_r < 1/\sqrt{2\kappa_0}$ such that, for all $q \in [2,p]$ and progressively measurable process $g \in L^p(\mathbb{R}_+ \times \Omega; L^q(\ell^2))$,

(3.8)
$$||v||_{L^p(\mathbb{R}_+ \times \Omega; H^{1,q})} \leq K_p ||g||_{L^p(\mathbb{R}_+ \times \Omega; L^q(\ell^2))},$$

where $v \in L^p(\mathbb{R}_+ \times \Omega; H^{1,q})$ is the unique strong solution to (3.7) with $\xi_n \equiv 0$ and $f \equiv 0$. The existence of a strong solution v to (3.7) with $\xi_n \equiv 0$ for which (3.22) holds follows [64, Theorem 7.1] and the above mentioned boundedness of the H^{∞} -calculus and invertibility of the operator $v \mapsto -\Delta v + v$ on $H^{-1,q}$ with domain $H^{1,q}$. For $p \geq 2$, let K_p be the optimal constant constant for which (3.22) holds for all $q \in [2, p]$. Since $\kappa_0 < \kappa$, to prove the claim of Step 1 it remains to prove that

(3.9)
$$\lim_{p \downarrow 2} \sup_{2 \leqslant r \leqslant p} K_r \leqslant 1/\sqrt{2\kappa}$$

To begin, note that $K_2 \leq 1/\sqrt{2\kappa}$. The latter readily follows by an application of the Itô's formula to $v \mapsto \|v\|_{L^2}^2$ (see e.g. [55, Theorem 4.2.5]) and taking expected values. Now, by complex interpolation (see, e.g., [48, Theorem 2.2.6]), for all p > 2,

$$K_{p_{\theta}} \leqslant K_2^{1-\theta} K_p^{\theta}$$
 where $\frac{1}{p_{\theta}} = \frac{1-\theta}{2} + \frac{\theta}{p}$.

Hence, (3.9) follows from the above.

Step 2: Conclusion. To prove the claim of Theorem 3.1, we employ the method of continuity [5, Proposition 3.13]. Hence, for $\lambda \in [0, 1]$, consider the SPDE

(3.10)
$$\begin{cases} \partial_t v_{\lambda} = \kappa(\Delta v_{\lambda} - v_{\lambda}) + f + \sum_{n \geq 1} \left(\lambda \left[(\xi_n \cdot \nabla) v_{\lambda} \right] + g_n \right) \dot{W}_t^n & \text{on } \mathbb{T}^d, \\ v_{\lambda}(0, \cdot) = 0 & \text{on } \mathbb{T}^d. \end{cases}$$

Note that, for $\lambda = 1$, the above reduced to (3.7). Now, by [5, Proposition 3.13] it is enough to prove the existence of $p_0 > 2$ and $C_0 > 0$ such that, for all $p \in [2, p_0]$, $q \in [2, p]$, $\lambda \in [0, 1]$ and progressively measurable processes $f \in L^p(\mathbb{R}_+ \times \Omega; H^{-1,q})$ and $g = (g_n)_{n \ge 1} \in L^p(\mathbb{R}_+ \times \Omega; L^q(\ell^2))$,

$$(3.11) ||v_{\lambda}||_{L^{p}(\mathbb{R}_{+} \times \Omega; H^{1,q})} \leq C_{0}(||f||_{L^{p}(\mathbb{R}_{+} \times \Omega; H^{-1,q})} + ||g||_{L^{p}(\mathbb{R}_{+} \times \Omega; L^{q}(\ell^{2}))})$$

where $v_{\lambda} \in L^{p}(\mathbb{R}_{+} \times \Omega; H^{1,q})$ is a strong solution to (3.10). The key point is the independence of C_{0} , p_{0} on $\lambda \in [0,1]$.

We begin with a reduction to the case $f \equiv 0$. Thus, let us assume that (3.11) holds with $f \equiv 0$. Since the operator $v \mapsto -\Delta v + v$ has a bounded H^{∞} -calculus of angle 0, it also has deterministic maximal $L_t^p(L_x^q)$ -regularity by [66, Theorem 4.4.5].

Arguing as in [65, Theorem 3.9], we can find a progressively measurable process $w \in L^p(\Omega; W^{1,p}(\mathbb{R}_+; H^{-1,q}) \cap L^p(\mathbb{R}_+; H^{1,q}))$ which satisfies, a.s. for all $t \ge 0$,

$$\partial_t w(t,\cdot) = \kappa(\Delta w(t,\cdot) - w(t,\cdot)) + f(t,\cdot), \qquad w(0,\cdot) = 0$$

on \mathbb{T}^d . Since $\tilde{v}_{\lambda} \stackrel{\text{def}}{=} v_{\lambda} - w$ solves (3.11) with $f \equiv 0$ and g_n replaced by $g_n + (\xi_n \cdot \nabla)w$, it is enough to prove (3.11) with $f \equiv 0$.

To prove (3.11) with $f \equiv 0$, let us note that (3.3) implies

$$\|((\xi_n \cdot \nabla)v)_{n \geqslant 1}\|_{L^q(\ell^2)} \leqslant \sqrt{2\kappa_0} \|\nabla v\|_{L^q} \text{ for all } v \in H^{1,q}.$$

Hence, by Step 1, if $f \equiv 0$, then for all $\lambda \in [0,1]$, $p \in [2,\infty]$ and $q \in [2,p]$,

$$||v_{\lambda}||_{L^{p}(\mathbb{R}_{+}\times\Omega;H^{1,q})} \leq K_{p}\left(||g||_{L^{p}(\mathbb{R}_{+}\times\Omega;L^{q}(\ell^{2}))} + ||((\xi_{n}\cdot\nabla)v)_{n\geqslant 1}||_{L^{p}(\mathbb{R}_{+}\times\Omega;L^{q}(\ell^{2}))}\right)$$
$$\leq K_{p}||g||_{L^{p}(\mathbb{R}_{+}\times\Omega;L^{q}(\ell^{2}))} + \sqrt{2\kappa_{0}}K_{p}||v_{\lambda}||_{L^{p}(\mathbb{R}_{+}\times\Omega;H^{1,q})}.$$

Again, by Step 1, there exists $p_0(\kappa_0, \kappa) > 2$ such that $\sup_{2 \le p \le p_0} K_p < 1/\sqrt{2\kappa_0}$. Thus, the previous estimate yields, for all $2 \le p \le p_0$ and $q \in [2, p]$,

$$||v_{\lambda}||_{L^{p}(\mathbb{R}_{+}\times\Omega;H^{1,q})} \leq (1-\sqrt{2\kappa_{0}}K_{p})^{-1}K_{p}||g||_{L^{p}(\mathbb{R}_{+}\times\Omega;L^{q}(\ell^{2}))}$$

Hence, (3.11) with $f \equiv 0$ holds with a constant independent of λ for $2 \leq p \leq p_0$. \square

3.2. Stochastic Meyers' estimates for the turbulent Stokes system. In this subsection, prove the Meyers' estimates for the turbulent Stokes system on \mathbb{T}^d , i.e.,

(3.12)
$$\begin{cases} \partial_t v = \kappa \Delta v + Q_{\xi} v + f + \sum_{n \geqslant 1} \left(\mathbb{P} \left[(\xi_n \cdot \nabla) v \right] + g_n \right) \dot{W}_t^n & \text{on } \mathbb{T}^d, \\ v(0, \cdot) = 0 & \text{on } \mathbb{T}^d, \end{cases}$$

here v is the unknown process, f, ξ_n and g_n are specified below. Finally, Q_{ξ} : $\mathbb{H}^1(\mathbb{T}^d) \to (\mathbb{H}^1(\mathbb{T}^d))^*$ is given by

$$\langle w, Q_{\xi} v \rangle \stackrel{\text{def}}{=} -\frac{1}{2} \sum_{n \geq 1} \int_{\mathbb{T}^d} \mathbb{Q}[(\xi_n \cdot \nabla) v] \cdot \mathbb{Q}[(\xi_n \cdot \nabla) w] \, \mathrm{d}x \text{ for } w \in \mathbb{H}^1(\mathbb{T}^d).$$

Although, it holds that $(\mathbb{H}^1)^* = \mathbb{H}^{-1}$ here we will not employ such identification as it is important to keep track of constants. Note that Q_{ξ} corresponds to the Itô-correction for the Stratonovich formulation of the transport noise for the turbulent Stokes system, see Subsection 2.2 and [9, Section 1].

Strong solutions to (3.12) can be defined similarly to the one of (3.1). The following complements [9, Theorem 3.2] in case of L^{∞} -transport noise.

Theorem 3.2 (Meyers' estimates – Turbulent Stokes system). Let $\kappa > 0$. Assume that $\xi_n = (\xi_j^n)_{j=1}^d : \mathbb{R}_+ \times \Omega \times \mathbb{T}^d \to \mathbb{R}^d$ is $\mathscr{P} \otimes \mathscr{B}(\mathbb{T}^d)$ -measurable for all $n \geq 1$ for which there exist $M_0 > 0$ and $\kappa_0 \in [0, \kappa)$ such that, a.e. on $\mathbb{R}_+ \times \Omega \times \mathbb{T}^d$,

(3.14)
$$\frac{1}{2} \sum_{n \ge 1} (\xi_n \cdot \eta)^2 \le \kappa_0 |\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^d.$$
 (parabolicity)

Then there exists $p_0(d, \kappa, \kappa_0, M_0) > 2$ such that, for all $T \in (0, \infty)$, $p \in [2, p_0]$, $q \in [2, p]$ and progressively measurable processes f and g satisfying

$$f \in L^p((0,T) \times \Omega; \mathbb{H}^{-1,q}(\mathbb{T}^d))$$
 and $g = (g_n)_{n \ge 1} \in L^p((0,T) \times \Omega; \mathbb{L}^q(\mathbb{T}^d;\ell^2)),$

there exists a unique strong solution $v \in L^p((0,T) \times \Omega; \mathbb{H}^{1,q}(\mathbb{T}^d))$ to (3.12) and

$$(3.15) ||v||_{L^{p}((0,T)\times\Omega;H^{1,q}(\mathbb{T}^{d};\mathbb{R}^{d}))} \lesssim_{d,\kappa,\kappa_{0},M_{0},T} ||f||_{L^{p}((0,T)\times\Omega;H^{-1,q}(\mathbb{T}^{d};\mathbb{R}^{d}))} + ||g||_{L^{p}((0,T)\times\Omega;L^{2}(\mathbb{T}^{d};\ell^{2}(\mathbb{N};\mathbb{R}^{d})))}.$$

The comments below Theorem 3.1 extend to the above result. In particular, the LHS(3.15) can be replaced by

$$\begin{split} & \|v\|_{L^p(\Omega; C([0,T]; \mathbb{B}^{1-2/p}_{q,p}(\mathbb{T}^d)))}, \ \text{ and } \\ & \|v\|_{L^p(\Omega; H^{\theta,p}(0,T; \mathbb{H}^{1-2\theta,q}(\mathbb{T}^d)))} \ \text{ for } \theta \in [0,\frac{1}{2}) \ \text{provided } p > 2. \end{split}$$

Moreover, due to [4, Proposition 3.10], the above estimates also hold if $v(\cdot,0) \in L^p_{\mathscr{F}_0}(\Omega; \mathbb{B}^{1-2/p}_{q,p}(\mathbb{T}^d))$ provided on the RHS(3.15) one also add $\|v(\cdot,0)\|_{L^p(\Omega;\mathbb{B}^{1-2/p}_{q,p}(\mathbb{T}^d))}$.

Proof. The proof is an extension of the one of Theorem 3.1. Since the deterministic component of the turbulent Stokes system (3.12) is more complicated than the one of (3.1), we perform the perturbation argument used in Theorem 3.1 twice. In particular, in the first step, we ensure that the deterministic part has maximal L^p -regularity which will be used to reduce to the case $f \equiv 0$ of (3.12).

Step 1: (Maximal L^p -regularity – deterministic problem) There exist $p_1 > 2$ and constants $(K_p)_{p \in [2,p_1]}$ such that, for all $p \in [2,p_1]$, $q \in [2,p]$ and $f \in L^p(\mathbb{R}_+;\mathbb{H}^{-1,q})$ the following Cauchy problem

$$(3.16) \partial_t w(t,\cdot) = \kappa(\Delta w(t,\cdot) - w(t,\cdot)) + Q_{\xi} w(t,\cdot) + f(t,\cdot), w(0,\cdot) = 0,$$

on \mathbb{T}^d , has a unique strong solution $w \in L^p(\mathbb{R}_+; \mathbb{H}^{1,q})$ satisfying

$$||w||_{L^p(\mathbb{R}_+:H^{1,q})} \leq K_p ||f||_{L^p(\mathbb{R}_+:H^{-1,q})}.$$

The proof of Step 1 follows as the proof of Theorem 3.1 regarding (3.16) as a perturbation of the case $\xi_n \equiv 0$. For clarity, we divide the proof into two substeps.

Substep 1a: (Analysis of constants – case $\xi_n \equiv 0$) For each p > 2 there exists $C_p > 0$ satisfying $\lim_{p \downarrow 2} \sup_{2 \leqslant r \leqslant p} C_r < 1/\kappa_0$ such that, for all $q \in [2,p]$ and $f \in L^p(\mathbb{R}_+; \mathbb{H}^{-1,q})$,

(3.17)
$$||w||_{L^p(\mathbb{R}_+;H^{1,q})} \leqslant C_p ||f||_{L^p(\mathbb{R}_+;H^{-1,q})}$$

where $w \in L^p(\mathbb{R}_+; \mathbb{H}^{1,q})$ is the unique strong solution to (3.16) with $\xi_n \equiv 0$. As in Step 1 of Theorem 3.1, the existence of such w follows from the boundedness H^{∞} -calculus of the operator $w \mapsto -\Delta w + w$ on $(\mathbb{H}^{1,q'})^*$ with domain $\mathbb{H}^{1,q}$. By interpolation and $\kappa_0 < \kappa$, it remains to show the validity of (3.17) with p = q = 2 and $C_2 = 1/\kappa$. To prove the latter, note that, by computing $\frac{d}{dt} \|w\|_{L^2}^2$ we obtain

$$\frac{1}{2} \|w(t)\|_{L^{2}}^{2} + \kappa \int_{0}^{t} \|w(s)\|_{H^{1}}^{2} ds = \int_{0}^{t} \langle w(s), f(s) \rangle ds$$

$$\leq \frac{1}{2\kappa} \int_{0}^{t} \|f(s)\|_{H^{-1}}^{2} ds + \frac{\kappa}{2} \int_{0}^{t} \|w(s)\|_{H^{1}}^{2} ds$$

for all $t \in \mathbb{R}_+$. The above immediately yields (3.17) with p = q = 2 and $C_2 = 1/\kappa$. Substep 1b: Proof of the claim of Step 1. We prove the claim of Step 1 by (the deterministic version of) the method of continuity [5, Proposition 3.13]. Thus, for all $\lambda \in [0, 1]$ consider, on \mathbb{T}^d ,

$$\partial_t w_\lambda = \kappa(\Delta w_\lambda - w_\lambda) + \lambda Q_\xi w_\lambda + f, \qquad w_\lambda(0, \cdot) = 0.$$

As in the proof of Theorem 3.1, it remains to prove an estimate for w_{λ} in $L^{p}(\mathbb{R}_{+}; H^{1,q})$ with constant independent of $\lambda \in [0,1]$. Firstly, by (3.13), it follows that

Secondly, we show that we can choose D_q in (3.18) such that

(3.19)
$$\lim_{q_1 \downarrow 2} \sup_{2 \leq q < q_1} D_{q_1} = \kappa_0.$$

By interpolation, it is enough to prove that (3.18) holds with $C_2 = \sqrt{2\kappa_0}$ whenever q = 2. Note that, for all $w \in \mathbb{H}^1$,

$$\|Q_{\xi}w\|_{(\mathbb{H}^{1})^{*}} = \frac{1}{2} \sup_{\phi \in \mathbb{H}^{1} : \|\phi\|_{H^{1}} \leqslant 1} \sum_{n \geqslant 1} \int_{\mathbb{T}^{d}} \mathbb{Q}[(\xi_{n} \cdot \nabla)v] \cdot \mathbb{Q}[(\xi_{n} \cdot \nabla)w] dx$$

$$\leqslant \frac{1}{2} \sup_{\phi \in \mathbb{H}^{1} : \|\phi\|_{H^{1}} \leqslant 1} \|(\mathbb{Q}[(\xi_{n} \cdot \nabla)w])_{n \geqslant 1}\|_{L^{2}(\ell^{2})} \|(\mathbb{Q}[(\xi_{n} \cdot \nabla)\phi])_{n \geqslant 1}\|_{L^{2}(\ell^{2})}$$

$$\leqslant \frac{1}{2} \sup_{\phi \in \mathbb{H}^{1} : \|\phi\|_{H^{1}} \leqslant 1} \|((\xi_{n} \cdot \nabla)w)_{n \geqslant 1}\|_{L^{2}(\ell^{2})} \|((\xi_{n} \cdot \nabla)\phi)_{n \geqslant 1}\|_{L^{2}(\ell^{2})}$$

$$\stackrel{(3.14)}{\leqslant} \kappa_{0} \|\nabla w\|_{L^{2}},$$

where (i) we used that $\|\mathbb{Q}\|_{\mathcal{L}(L^2)} = 1$ as \mathbb{Q} is an ortogonal projection on L^2 .

Now, similarly to Step 2 of Theorem 3.1, one can prove an a-priori estimate for $\|w_{\lambda}\|_{L^{p}(\mathbb{R}_{+};H^{1,q})}$ with constant independent of λ provided $2 \leq q \leq p$ is sufficiently small due to (3.19), $\kappa_{0} < \kappa$ and substep 1a.

Step 2: Conclusion. The proof of Theorem 3.2 follows by applying again the method of continuity. Hence, for $\lambda \in [0, 1]$, we consider

(3.20)
$$\begin{cases} \partial_t v_{\lambda} = \kappa(\Delta v_{\lambda} - v_{\lambda}) + Q_{\xi} v_{\lambda} + f + \sum_{n \geqslant 1} \left(\lambda \mathbb{P} \left[(\xi_n \cdot \nabla) v_{\lambda} \right] + g_n \right) \dot{W}_t^n, \\ v_{\lambda}(0, \cdot) = 0, \end{cases}$$

on \mathbb{T}^d , where $f \in L^p(\mathbb{R}_+ \times \Omega; (\mathbb{H}^{1,q'})^*)$ and $g \in L^p(\mathbb{R}_+ \times \Omega; L^q(\ell^2))$ are given progressively measurable processes. By the method of continuity [5, Proposition 3.13], it is enough to prove the existence of $p_0 > 2$ and $C_0 > 0$ such that, for all $p \in [2, p_0], q \in [2, p], \lambda \in [0, 1]$ and progressively measurable processes f and $g = (g_n)_{n \geqslant 1}$ as above,

where $v_{\lambda} \in L^p(\mathbb{R}_+ \times \Omega; H^{1,q})$ is a strong solution to (3.20). Without loss of generality, we may assume $p_0 \leq p_1$ where p_1 is as in Step 1 of the current proof. As in Step 2 of Theorem 3.2, due to Step 1, it is enough to prove (3.21) with $f \equiv 0$. We now again repeat the argument of Theorem 3.1 by analyzing first the constant in the energy inequality and afterwards, we argue by perturbation.

Substep 2a: (Analysis of constants – case $\lambda = 0$ and $f \equiv 0$) For each $p \in [2, p_1]$ there exists $K_p > 0$ satisfying $\lim_{p \downarrow 2} \sup_{2 \leq r \leq p} K_p = 1$ and

(3.22)
$$(\kappa - \kappa_0) \| (|v|^2 + |\nabla v|^2)^{1/2} \|_{L^p(\mathbb{R}_+ \times \Omega; L^q)}^2$$

$$+ \| (\mathbb{P}[(\xi_n \cdot \nabla)v])_{n \geqslant 1} \|_{L^p(\mathbb{R}_+ \times \Omega; H^{1,q})}^2 \leqslant K_p \| g \|_{L^p(\mathbb{R}_+ \times \Omega; L^q(\ell^2))}^2$$

where $v \in L^p(\mathbb{R}_+ \times \Omega; H^{1,q})$ is the strong solution to (3.7) with $\lambda = 0$ and $f \equiv 0$. The existence of such v follows from Step 1 and the argument in [65, Theorem 3.9] for the g-term. The intuition behind the RHS(3.22) is that such a quantity in the case p = q = 2 appears naturally in the energy balance with $K_2 = 1$. To see this, first note that, by combining the Itô formula [55, Theorem 4.2.5] and standard integration by parts argument, one has

$$(3.23) \ \kappa \|v\|_{L^2(\mathbb{R}_+ \times \Omega; H^1)} - \|(\mathbb{Q}[(\xi_n \cdot \nabla)v])_{n \geqslant 1}\|_{L^2(\mathbb{R}_+ \times \Omega; L^2(\ell^2))} \leqslant \|g\|_{L^2(\mathbb{R}_+ \times \Omega; L^2(\ell^2))}.$$

Since $\mathbb{Q} + \mathbb{P} = \mathrm{Id}_{L^2}$ and \mathbb{Q}, \mathbb{P} are orthogonal projections on L^2 ,

$$\begin{split} \| (\mathbb{P}[(\xi_n \cdot \nabla)v])_{n \geqslant 1} \|_{H^1}^2 &= \| ((\xi_n \cdot \nabla)v)_{n \geqslant 1} \|_{H^1}^2 - \| (\mathbb{Q}[(\xi_n \cdot \nabla)v])_{n \geqslant 1} \|_{H^1}^2 \\ & \stackrel{(3.14)}{\leqslant} \kappa_0 \| \nabla v \|_{L^2}^2 - \| (\mathbb{Q}[(\xi_n \cdot \nabla)v])_{n \geqslant 1} \|_{H^1}^2. \end{split}$$

Combing the above and (3.23), we get that (3.22) holds with q = p = 2 and $K_2 = 1$. We conclude by arguing by interpolation. To this end, consider the operator

$$\mathcal{S}_{q,p}: L^p_{\mathscr{P}}(\mathbb{R}_+ \times \Omega; \mathbb{L}^q(\mathbb{T}^d; \ell^2)) \to L^p(\mathbb{R}_+ \times \Omega; L^q(\mathbb{T}^d; \mathbb{R}^d \times \mathbb{R}^{d \times d} \times \ell^2)))$$
$$g \mapsto (\sqrt{\kappa - \kappa_0} v, \sqrt{\kappa - \kappa_0} \nabla v, (\mathbb{P}[(\xi_n \cdot \nabla v)])_{n \geqslant 1}),$$

where we norm the product space $\mathbb{R}^d \times \mathbb{R}^{d \times d} \times \ell^2$ with the norm $\|(\cdot,\cdot,\cdot)\|_{\mathbb{R}^d \times \mathbb{R}^{d \times d} \times \ell^2}^2 = \|\cdot\|_{\mathbb{R}^d}^2 + \|\cdot\|_{\mathbb{R}^{d \times d}}^2 + \|\cdot\|_{\ell^2}^2$. The latter choice comes from the fact that $\|\mathcal{S}_{2,2}\|_{\mathscr{L}} \leq 1$ as we have proved that (3.22) holds with $K_2 = 1$. The well-definiteness of $\mathcal{S}_{q,p}$ follows from the above-noticed existence and uniqueness of strong solutions $v \in L^p(\mathbb{R}_+ \times \Omega; H^{1,q})$ of (3.12) with data $g \in L^p_{\mathscr{P}}(\mathbb{R}_+ \times \Omega; \mathbb{L}^q(\mathbb{T}^d; \ell^2))$ due to Step 1. Thus the claim of Substep 2a follows by complex interpolation [48, Theorem 2.2.6] and the elementary inequality $a^{\alpha} + b^{\alpha} \leq (a+b)^{\alpha} \leq 2^{\alpha-1}(a^{\alpha} + b^{\alpha})$ for $\alpha > 1$.

Substep 2b: Proof of Theorem 3.2. In light of Substep 2a, the estimate (3.21) with constant independent of $\lambda \in [0,1]$ and $f \equiv 0$, now readily follows from $\kappa > \kappa_0$ and the boundedness of $(\xi_n)_{n \geq 1}$, see (3.13).

3.3. Extensions and comments. In this subsection, we would like to comment on possible extensions of the previous stochastic Meyers' estimates. The aim is to discuss the symmetric assumption used in [14, Theorem 1.2] (see Problem 1.5 there) in the context of divergence form SPDEs with transport noise by using our methods inspired by the original proof of Meyers [59]. Interestingly, the symmetry of the coefficients also comes in our approach. Finally, let us stress that the comments below also extend to the turbulent Stokes system (3.12) the situation is similar but some complications arise due to the presence of the Helmholtz projection.

Consider, on an open bounded set $\mathcal{O} \subseteq \mathbb{R}^d$ and for all $\alpha \in \{1, \ldots, n\}$ with $n \in \mathbb{N}$,

(3.24)
$$\begin{cases} \partial_t v^{\alpha} = \partial_i (a_{i,j}^{\alpha,\beta} \partial_j v^{\beta}) + f^{\alpha} + \sum_{n \geqslant 1} [\xi_{n,j}^{\alpha,\beta} \partial_j v^{\beta} + g_n^{\alpha}] \dot{W}_t^n, \\ v|_{\partial \mathcal{O}} = 0, \qquad v(0,\cdot) = 0, \end{cases}$$

where $v = (v^{\alpha})_{\alpha=1}^{n}$ is the unknown process and $a_{i,j}^{\alpha,\beta}, \xi_{i,n}^{\alpha} : \mathbb{R}_{+} \times \Omega \times \mathcal{O} \to \mathbb{R}$ are progressively measurable and satisfy, for some $M_{0}, \kappa_{0} > 0$ and a.e. on $\mathbb{R}_{+} \times \Omega \times \mathcal{O}$,

$$(3.25) |a_{i,j}^{\alpha,\beta}| + ||(\xi_{n,j}^{\alpha})_{n\geqslant 1}||_{\ell^2} \leqslant M_0, (boundedness)$$

$$(3.26) \quad \left(a_{i,j}^{\alpha,\beta} - \frac{1}{2} \sum_{n \geq 1} \xi_{n,i}^{\alpha} \xi_{n,j}^{\beta}\right) \eta_i^{\alpha} \eta_j^{\beta} \geqslant \kappa_0 |\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^{n \times d}. \quad \text{(parabolicity)}$$

Now we extend Theorem 3.1 to the case of the system (3.24), to show the natural appearance of the symmetry assumption on $a_{i,j}^{\alpha,\beta}$ to deal with the transport noise. Note that, strong solutions to (3.12) can be defined similarly to the one of (3.1) with $H^1(\mathbb{T}^d)$ replaced by $H^1_0(\mathbb{T}^d)$.

Theorem 3.3 (Parabolic stochastic Meyers' estimates – Systems). Suppose that $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain. Let $a_{i,j}^{\alpha,\beta}$ and b_j^{α} be progressively measurable processes such that (3.25)-(3.26) holds. Assume that, for all $\alpha, \beta \in \{1, \ldots, n\}$ and $i, j \in \{1, \ldots, d\}$,

(3.27)
$$a_{i,j}^{\alpha,\beta} = a_{j,i}^{\beta,\alpha} \quad a.e. \quad on \quad \mathbb{R}_+ \times \Omega \times \mathcal{O}.$$

Then there exists $p_0(d, \kappa_0, M_0) > 2$ such that, for all $T \in (0, \infty)$, $p \in [2, p_0]$, $q \in [2, p]$ and progressively measurable processes f and g satisfying

$$f \in L^p((0,T) \times \Omega; H^{-1,q}(\mathcal{O}; \mathbb{R}^n))$$
 and $g = (g_n)_{n \ge 1} \in L^p((0,T) \times \Omega; L^q(\mathbb{T}^d; \ell^2)),$

there exists a unique strong solution $v \in L^p((0,T) \times \Omega; H_0^{1,q}(\mathcal{O};\mathbb{R}^n))$ to (3.24) and

(3.28)
$$||v||_{L^{p}((0,T)\times\Omega;H^{1,q}(\mathcal{O};\mathbb{R}^{n}))} \lesssim_{d,\kappa_{0},M_{0},T} ||f||_{L^{p}((0,T)\times\Omega;H^{-1,q}(\mathcal{O};\mathbb{R}^{n}))} + ||g||_{L^{p}((0,T)\times\Omega;L^{q}(\mathcal{O};\ell^{2}))}.$$

As the proof is an easy extension of the one of Theorems 3.2 and 3.1, we only provide a sketch to highlight the role of the assumption (3.27).

Proof of Theorem 3.1 – Sketch. In absence of transport noise (i.e., $\xi_{n,j}^{\alpha} \equiv 0$), then the result coincides with the usual Meyers' estimate [59] and they are also valid without the assumption (3.27) (for other approaches see [26] and [10, Appendix B]). Let $p_1 > 2$ be the corresponding integrability improvement. Now, arguing as in the proof of Theorem 3.2, this fact shows that is enough to prove the claim with $f \equiv 0$. Now, we collect two main observations. Firstly, by (3.25) and $\kappa_0 > 0$, the parabolicity is 'self-improving' in the noise part, i.e., there exists $\varepsilon_0(M_0, \kappa_0) > 0$ such that (3.25)-(3.26) implies

$$\left(a_{i,j}^{\alpha,\beta} - \frac{1+\varepsilon_0}{2} \sum_{n \geq 1} \xi_{n,i}^{\alpha} \xi_{n,j}^{\beta}\right) \eta_i^{\alpha} \eta_j^{\beta} \geqslant \frac{\kappa_0}{2} |\eta|^2 \text{ for all } \eta \in \mathbb{R}^{n \times d}.$$

Secondly, the assumption (3.27) allows us to find a matrix value progressively measurable map $r=(r_{i,j}^{\alpha,\beta})_{i,j\in\{1,...,d\}}^{\alpha,\beta\in\{1,...,n\}}$ which satisfies the symmetry condition (3.27) and $a_{i,j}^{\alpha,\beta}=r_{i,k}^{\alpha,\gamma}r_{k,j}^{\gamma,\beta}$. Thus, the self-improved parabolicity can be rewritten as

$$\sum_{\gamma,k} (r_{k,i}^{\alpha,\gamma} \eta_i^{\alpha})^2 - \frac{1+\varepsilon_0}{2} \sum_{n\geqslant 1} (\xi_{n,i}^{\alpha} \eta_i^{\alpha})^2 \geqslant \frac{\kappa_0}{2} |\xi|^2.$$

Now, arguing as in Step 2 of Theorem 3.1, to prove the estimate (3.28), it is enough to study the operator

(3.29)
$$L^{p}_{\mathscr{P}}(\mathbb{R}_{+} \times \Omega; L^{q}(\mathbb{T}^{d}; \ell^{2})) \to L^{p}(\mathbb{R}_{+} \times \Omega; L^{q}(\mathcal{O}; \mathbb{R}^{n} \times \mathbb{R}^{n \times d}))$$
$$g \mapsto \left(v, (r_{k,i}^{\alpha, \gamma} \partial_{i} v^{\alpha})_{k \in \{1, \dots, d\}}^{\gamma \in \{1, \dots, n\}}\right)$$

with $\|(\cdot)\|_{\mathbb{R}^n \times \mathbb{R}^{n \times d}} \stackrel{\text{def}}{=} (|\cdot|_{\mathbb{R}^n}^2 + |\cdot|_{\mathbb{R}^{n \times d}})^{1/2}$ and where v is the unique solution in $L^p(\mathbb{R}_+ \times \Omega; H^{1,q}(\mathcal{O}; \mathbb{R}^n))$ of

$$\begin{cases} \partial_t v^{\alpha} = \partial_i (a_{i,j}^{\alpha,\beta} \partial_j v^{\beta}) - v^{\alpha} + \sum_{n \geqslant 1} g_n^{\alpha} \dot{W}_t^n, \\ v|_{\partial \mathcal{O}} = 0, \qquad v(0) = 0. \end{cases}$$

The conclusion follows by combining an interpolation between the cases p=2 and $p_1>2$ (see the beginning of the proof) and applying the method of continuity. \square

Thus, as the proof of Theorem 3.3 shows, it does not seem possible to avoid the symmetry assumption (3.27), which is used to build the operator in (3.29) which one interpolates. Thus, even from the above PDEs perspective, the assumption (3.27) seems necessary in case of transport noise.

4. Anomalous dissipation for passive scalars

This section is devoted to the proof of Theorem 1.3 which will also be a guideline for the one of Theorem 1.1. The strategy used below is outlined in Subsection 1.3.

4.1. Scaling limit at a fixed diffusivity. In this subsection, we prove the following scaling limit result for passive scalars (1.2) with fixed diffusivity $\gamma > 0$.

Proposition 4.1 (Scaling limit – passive equations). Fix $\gamma \in (0,1)$ and $\mu > 0$. Let $(\theta^n)_{n \ge 1} \subseteq \ell^2$ be a sequence of normalized radially symmetric coefficients (i.e., satisfying (2.1)) such that

$$\lim_{n\to\infty} \|\theta^n\|_{\ell^\infty} = 0.$$

Then there exists $\delta_0 = \delta_0(\gamma, \mu) > 0$ for which the following assertion holds. For all $\delta \in (0, \delta_0]$ and sequence $(\varrho_{0,n})_{n \geq 1} \subseteq H^{\delta}$ such that $\varrho_{0,n} \to \varrho_0$ in H^{δ} , we have

(4.1)
$$\lim_{n \to \infty} \mathbf{P} \left(\sup_{t \in [0,1]} \| \varrho_n^{\gamma}(t) - \varrho_{\det}^{\gamma}(t) \|_{L^2} \geqslant \varepsilon \right) = 0 \quad \text{for all } \varepsilon > 0,$$

where ϱ_n^{γ} and $\varrho_{\text{det}}^{\gamma}$ denote the unique global solution to (5.1) with $\theta^{\gamma} = \theta^n$ and initial data $\varrho_{0,n}$ and the unique global solution to

(4.2)
$$\begin{cases} \partial_t \varrho_{\text{det}}^{\gamma} = (\gamma + \mu) \Delta \varrho_{\text{det}}^{\gamma} & on \, \mathbb{T}^d, \\ \varrho_{\text{det}}^{\gamma}(0, \cdot) = \varrho_0 & on \, \mathbb{T}^d; \end{cases}$$

respectively.

Solutions to (4.2) are understood as in Definition 2.1 with trivial noise. In the proof of Proposition 4.1 below, we also show that, for all $r < \delta < \delta_0$ and $n \ge 1$,

$$\varrho_n^{\gamma} \in C([0,\infty); H^r) \text{ a.s.} \quad \text{ and } \quad \varrho_{\det}^{\gamma} \in C([0,\infty); H^r).$$

In particular, the norm in the claim (4.1) is well-defined.

As outlined in Subsections 1.3.1 and 1.3.2, the main novelty and key point in the above result is the presence of the L^2 -norm in (4.1), and the key behind such improvement is the use of the stochastic Meyers' estimates of Section 3. Actually, below we will even prove that

(4.3)
$$\lim_{n \to \infty} \mathbf{P} \Big(\sup_{t \in [0,1]} \| \varrho_n^{\gamma}(t) - \varrho_{\det}^{\gamma}(t) \|_{H^r} \geqslant \varepsilon \Big) = 0$$

for all $\varepsilon > 0$ and $r < \delta < \delta_0$, where δ_0 is as in Proposition 4.1. However, in contrast to the 2D NSEs analysed in Section 5 below, the above will not be needed.

Before going further, let us mention that a prototype example of a sequence satisfying the assumption of Proposition 4.1 is given by

$$(4.4) \theta^n = \frac{\Theta^n}{\|\Theta^n\|_{\ell^2}} \text{where} \Theta^n \stackrel{\text{def}}{=} \mathbf{1}_{\{n \leqslant |k| \leqslant 2n\}} \frac{1}{|k|^r} \text{for some } r > 0.$$

For details, see eq. (1.9) in [56] and the comments below it. In particular, the above sequence satisfies $\#\{k: \theta_k^n \neq 0\} < \infty$ for all $n \geq 1$.

Now, we turn to the proof of Proposition 4.1. The key ingredient is the following estimates that are uniform in the class normalized radially symmetric θ^{γ} .

Lemma 4.2 (Uniform in θ -estimates). Fix $\gamma \in (0,1)$ and $\mu > 0$. Then there exist $p_0(\gamma,\mu) > 2$ and $C_0(\gamma,\mu) > 0$ such that, for all $p \in [2,p_0]$ and all normalized radially simmetric $\theta^{\gamma} \in \ell^2$ (i.e., satisfying (2.1)) and $\varrho_0 \in B_{2,p}^{1-2/p}$, the unique global solution ϱ to (1.2) satisfies

$$\mathbf{E} \sup_{t \in [0,1]} \|\varrho^{\gamma}(t)\|_{B_{2,p}^{1-2/p}}^p + \mathbf{E} \int_0^1 \|\varrho^{\gamma}(t)\|_{H^1}^p \, \mathrm{d}t \leqslant C_0 \|\varrho_0\|_{B_{2,p}^{1-2/p}}^p.$$

Proof. The claimed estimate follows from the stochastic Meyers' inequality of Theorem 3.1 and the comments below it, see in particular (3.5) and the comments on non-trivial initial data for (3.1). For clarity, let us check the assumptions (3.2)-(3.3) with constants that are *uniform* in the class of normalized radially symmetric $\theta^{\gamma} \in \ell^2$. To this end, we employ the real reformulation (2.7) of the complex transport noise in (1.2). To this end, for $\alpha \in \{0, \ldots, d-1\}$ and $x \in \mathbb{T}^d$, we let

$$\xi_{k,\alpha}(x) \stackrel{\text{def}}{=} 2\sqrt{c_d\mu} \,\theta_k^{\gamma} \begin{cases} \Re[\sigma_{k,\alpha}(x)] = \cos(2\pi k \cdot x) \, a_{k,\alpha}, & k \in \mathbb{Z}_+^d, \\ \Im[\sigma_{k,\alpha}(x)] = \sin(2\pi k \cdot x) \, a_{k,\alpha}, & k \in \mathbb{Z}_-^d. \end{cases}$$

From the normalized condition $\|\theta^{\gamma}\|_{\ell^2} = 1$, it immediately follows that $\|(\xi_{k,\alpha})_{k,\alpha}\|_{\ell^2} \leq 2\sqrt{c_d\mu}$ which yields (3.2) with constant uniform in θ^{γ} . Finally, to check (3.3), note that (2.4) and $a_{-k,\alpha} = a_{k,\alpha}$ imply, for all $\eta \in \mathbb{R}^d$,

$$\frac{1}{2} \sum_{k,\alpha} (\xi_{k,\alpha} \cdot \eta)^2 = c_d \mu \sum_{k,\alpha} (\theta_k^{\gamma})^2 (a_{k,\alpha} \cdot \eta)^2 = \mu |\eta|^2.$$

Thus, the assumptions (3.2)-(3.3) with $(\xi_n)_{n\geqslant 1}$ replaced by an enumeration of $(\xi_{k,\alpha})_{k,\alpha}$ are uniform w.r.t. normalized radially simmetric $\theta^{\gamma} \in \ell^2$, and therefore the claim of Lemma 4.2 follows from Theorem 3.1 with $\kappa = \gamma + \mu$ and $\kappa_0 = \mu$. Here we also used that the leading differential operator in the Itô formulation of (1.2) is $(\gamma + \mu)\Delta$, see Definition 2.1.

To proceed further, note that, by the Itô formula (e.g., [55, Chapter 4]) and $\nabla \cdot \sigma_{k,\alpha} = 0$, the following energy equality holds for the global solution ϱ^{γ} of (1.2):

(4.5)
$$\frac{1}{2} \|\varrho^{\gamma}(t)\|_{L^{2}}^{2} + \int_{0}^{t} \int_{\mathbb{T}^{d}} \gamma |\nabla \varrho^{\gamma}|^{2} dx ds = \frac{1}{2} \|\varrho_{0}\|_{L^{2}}^{2} \quad \text{a.s. for all } t > 0.$$

The above will be used frequently below.

The following is the last ingredient in the proof of Proposition 4.1.

Lemma 4.3 (Time-regularity estimate). Let $\gamma \in (0,1)$, $\mu > 0$ and $a \in (1,\infty)$ be fixed. Fix $\varrho_0 \in L^2$, and assume that $\theta^{\gamma} \in \ell^2$ is normalized and radially symmetric.

Let ϱ^{γ} be the strong solution to (1.2), and set

$$\mathcal{M}(t) \stackrel{\text{def}}{=} \sqrt{c_d \mu} \sum_{k,\alpha} \theta_{k,\alpha}^{\gamma} \int_0^t (\sigma_{k,\alpha} \cdot \nabla) \varrho^{\gamma} \, dW_t^{k,\alpha} \qquad \forall t \geqslant 0.$$

Then there exists $s_0, r_1, K_0 > 0$ independent of (ϱ_0, θ) such that

$$\mathbf{E}[\|\mathcal{M}\|_{C^{s_0}(0,1;H^{-r_1})}^{2a}] \leqslant K_0 \|\theta^{\gamma}\|_{\ell^{\infty}}^{2a} \|\varrho_0\|_{L^2}^{2a},$$

(4.7)
$$\mathbf{E}[\|\varrho^{\gamma}\|_{C^{s_0}(0,1;H^{-r_1})}^{2a}] \leqslant K_0 \|\varrho_0\|_{L^2}^{2a}.$$

The key point in the above result is the presence of $\|\theta\|_{\ell^{\infty}}$ on the RHS(4.6).

Lemma 4.3 is well-known to experts, and it is a consequence of the structure of the noise described in Subsection 2.2 and of the energy estimate (4.5), see e.g., [29, Proposition 3.6] or [1, Lemma 6.3]. For brevity, we omit the proof.

Proof of Proposition 4.1. The argument used here is by now well-established, the reader is referred to e.g., [34, Theorem 1.4], [29, Proposition 3.7] or [1, Theorem 6.1]. We partially repeat it to highlight the fundamental role of the estimate in Lemma 4.2 with p > 2. Moreover, we prove the stronger result in (4.3).

Step 1: For all $s_0, r_1 > 0$ and $0 \le r < r_0$, set

$$\mathcal{Y} \stackrel{\text{def}}{=} C([0,1]; H^{r_0}) \cap C^{s_0}(0,1; H^{-r_1}) \quad and \quad \mathcal{X} \stackrel{\text{def}}{=} C([0,1]; H^r).$$

The embedding $\mathcal{Y} \hookrightarrow \mathcal{X}$ is compact. Moreover, for any $K \geqslant 1$, the following set is closed

$$\mathcal{X}_K \stackrel{\text{def}}{=} \Big\{ f \in \mathcal{X} \ : \ \sup_{t \in [0,1]} \| f(t) \|_{L^2}^2 + \int_0^1 \| \nabla f(t) \|_{L^2}^2 \, \mathrm{d}t \leqslant K \Big\}.$$

The final assertion follows from the first one and Fatou's lemma. It remains to prove that $\mathcal{Y} \hookrightarrow_{\mathbf{c}} \mathcal{X}$. By interpolation, for all $\tilde{r} \in (r, r_0)$ there exists $\tilde{s} > 0$ such that

$$C([0,1];H^{s_0}) \cap C^{s_0}(0,1;H^{-r_1}) \hookrightarrow C^{\tilde{s}}(0,1;H^{\tilde{r}}) \hookrightarrow_{\operatorname{comp}} C([0,1];H^{s}),$$

where we used the Ascoli-Arzelà theorem in the last embedding.

Step 2: Conclusion. We begin by collecting some useful facts. Let $p_0 > 2$ be as in Lemma 4.2. Fix $\delta_0 \in (0, 1 - 2/p_0)$, $\delta \in (0, \delta_0]$ and $r, r_0 \in [0, \delta)$ such that $r < r_0$. Finally, let us select $p \in (2, p_0]$ such that $\delta > 1 - 2/p > r_0$. In particular, we have the following embeddings at our disposal:

$$(4.8) H^{\delta} \hookrightarrow B_{2,p}^{1-2/p} \hookrightarrow H^{r_0}.$$

Now, since $\varrho_{0,n} \to \varrho_0$ in H^{δ} by assumption, we have $\sup_{n\geqslant 1} \|\varrho_{0,n}\|_{H^{\delta}} < \infty$. Hence, by Lemma 4.2–4.3 and the embedding (4.8),

(4.9)
$$\sup_{n\geq 1} \mathbf{E} \Big[\|\varrho_n^{\gamma}\|_{C([0,1];H^{r_0})}^2 + \|\varrho_n^{\gamma}\|_{C^{r_0}(0,1;H^{-s_1})}^2 \Big] < \infty.$$

Moreover, by the energy equality (4.5), we obtained the quenched estimate:

$$\sup_{t \in [0,1]} \|\varrho_n^{\gamma}(t)\|_{L^2}^2 + \int_0^1 \int_{\mathbb{T}^d} |\nabla \varrho_n^{\gamma}|^2 \, \mathrm{d}x \, \mathrm{d}t \leqslant N_{\gamma} \ \text{a.s.},$$

for some deterministic constant $N_{\gamma} \ge 1$ independent of n.

To prove (4.1), or even (4.3), it suffices to prove that for each sub-sequence $(n_k)_{k\geqslant 1}$ we can find a further sub-sequence $(n_{k_j})_{j\geqslant 1}$ for which

$$(4.11) \qquad \lim_{n_{k_{j}}\to\infty}\mathbf{P}\Big(\sup_{t\in[0,1]}\|\varrho_{n_{k_{j}}}^{\gamma}(t)-\varrho_{\mathrm{det}}^{\gamma}(t)\|_{H^{r}}\geqslant\varepsilon\Big)=0 \ \text{ for all } \varepsilon>0.$$

For notational convenience, below, we do not relabel sub-sequences. To begin, let $\mathcal{X}_{N_{\gamma}}$ be as in Step 1 with N_{γ} is as in (4.10). Moreover, denote by \mathcal{L}_n the law of ϱ_n^{γ} on $\mathcal{X}_{N_{\gamma}}$. By combining Prokhorov's theorem, Step 1, (4.9) and (4.10), it follows that the sequence $(\mathcal{L}_n)_{n\geq 1} \subseteq \mathcal{P}(\mathcal{X}_{N_{\gamma}})$ is tight (here \mathcal{P} the space of probability measures on the polish space $\mathcal{X}_{N_{\gamma}}$). In particular, there exists $\mu \in \mathcal{P}(\mathcal{X}_{N_{\gamma}})$ such that \mathcal{L}_n converges weakly to μ . It remains to prove that

$$\mu = \delta_{\varrho_{\text{det}}^{\gamma}}.$$

Indeed, by the Portmanteau theorem and the fact that the limit is deterministic, the previous implies (4.11).

To prove (4.12), arguing as either [1, Step 2, Theorem 6.1] or [29, Proposition 3.7], it follows that

$$\mu(f \in \mathcal{X}_{N_{\gamma}} : f \text{ is a solution to (4.2) with life-time } \geqslant 1) = 1.$$

In the above, we mean that the couple (f, 1) is a local solution to (4.2), cf., Definition 2.1 and the comments below Proposition 4.1. Now, the uniqueness of solutions to (4.2) in the class $\mathcal{X}_{N_{\gamma}}$ yields (4.12).

We conclude by discussing an interesting consequence of Proposition 4.1.

Remark 4.4 (Lack of L^2 gradients convergence). In the setting of Proposition 4.1, it holds that, in probability in C([0,1]),

$$\lim_{n \to \infty} \int_0^{\cdot} \int_{\mathbb{T}^d} \gamma |\nabla \varrho_n^{\gamma}|^2 dx ds = \int_0^{\cdot} \int_{\mathbb{T}^d} (\gamma + \mu) |\nabla \varrho_{\text{det}}^{\gamma}|^2 dx ds.$$

The above follows by arguing as in Subsection 1.3.1 and using Proposition 4.1:

$$2\int_{0}^{\cdot} \int_{\mathbb{T}^{d}} \gamma \left| \nabla \varrho_{n}^{\gamma} \right|^{2} dx ds = \|\varrho_{0}\|_{L^{2}(\mathbb{T}^{d})}^{2} - \|\varrho_{n}^{\gamma}(\cdot)\|_{L^{2}(\mathbb{T}^{d})}^{2}$$

$$\stackrel{n \to \infty}{\longrightarrow} \|\varrho_{0}\|_{L^{2}(\mathbb{T}^{d})}^{2} - \|\varrho_{\det}^{\gamma}(\cdot)\|_{L^{2}(\mathbb{T}^{d})}^{2} = 2\int_{0}^{\cdot} \int_{\mathbb{T}^{d}} (\gamma + \mu) \left| \nabla \varrho_{\det}^{\gamma} \right|^{2} dx ds,$$

where the last equality follows from the energy balance for (4.2).

In particular, if $\varrho_0 \neq 0$ (and therefore $\varrho_0 \notin \mathbb{R}$ as $\int_{\mathbb{T}^d} \varrho_0(x) dx = 0$), then the process ϱ_n^{γ} does not convergence in probability in $L^2(0,1;H^1(\mathbb{T}^d))$ to $\varrho_{\text{det}}^{\gamma}$.

4.2. **Proof of Theorem 1.3.** The key ingredient in the proof of Theorem 1.3 is the following consequence of Proposition 4.1 and the choice (4.4).

Corollary 4.5. Let $\gamma, \varepsilon, \delta \in (0,1)$ and $N, \mu > 0$ be fixed. Then there exists a normalized radially symmetric $\theta^{\gamma} \in \ell^2$ (i.e., satisfying (2.1)) for which the following assertion holds. For all $\varrho_0 \in H^{\delta}$ satisfying $\|\varrho_0\|_{H^{\delta}} \leq N$, the unique global solution ϱ^{γ} to (1.2) satisfies

(4.13)
$$\mathbf{P}\left(\sup_{t\in[0,1]}\|\varrho^{\gamma}-\varrho_{\det}^{\gamma}\|_{L^{2}}\leqslant\varepsilon\right)>1-\varepsilon,$$

where $\varrho_{\text{det}}^{\gamma}$ is the unique global solution to (4.2).

As the proof below shows, by using (4.3) instead of (4.1) in Proposition 4.1, we can replace the L^2 -norm in (4.19) by H^r for some r > 0 depending only on (δ, μ, γ) .

Proof. Let $\delta_0 > 0$ be as in Proposition 4.1. Without loss of generality, we can assume that $\delta < \delta_0$. Next, we set

$$\mathcal{B}_{N,\delta} \stackrel{\text{def}}{=} \{ \varrho_0 \in H^{\delta} : \|\varrho_0\|_{H^{\delta}} \leqslant N \}.$$

Fix $\varepsilon > 0$. Let $(\theta_n)_{n \ge 1}$ be as in (4.4), and due to the comments below it, we have $\lim_{n \to \infty} \|\theta^n\|_{\ell^{\infty}} = 0$. To conclude, it is enough to show that

(4.14)
$$\lim_{n \to \infty} \sup_{\varrho_0 \in \mathcal{B}_{N,\delta}} \mathbf{P} \Big(\sup_{t \in [0,1]} \|\varrho^n(\varrho_0) - \varrho_{\det}(\varrho_0)\|_{L^2} \geqslant \varepsilon \Big) = 0.$$

where $\varrho^n(\varrho_0)$ and $\varrho_{\text{det}}(\varrho_0)$ denote the solution to (1.2) with $\theta^{\gamma} = \theta^n$ and (4.2) both with initial data ϱ_0 , respectively. We prove (4.14) by contradiction. Indeed, assume that there exists a sub-sequence $(\varrho_{0,n_j})_{j\geqslant 1} \subseteq \mathcal{B}_{N,\delta}$ such that

(4.15)
$$\lim_{j \to \infty} \mathbf{P} \Big(\sup_{t \in [0,1]} \| \varrho_j - \varrho_{\det,j} \|_{L^2} \geqslant \varepsilon \Big) > 0.$$

where $\varrho_j = \varrho(\varrho_{0,n_j})$ and $\varrho_{\det,j} = \varrho(\varrho_{0,n_j})$. Note that, for all $\tilde{\delta} \in (0,\delta)$, there exists a (not-relabeled) sub-sequence (ϱ_{0,n_j}) such that $\varrho_{0,n_j} \to \varrho_0$ in $H^{\tilde{\delta}}$. Now the contradiction with (4.15) follows by applying Proposition 4.1 with $\delta = \tilde{\delta}$. Thus, (4.14) is proved.

Proof of Theorem 1.3. We split the proof into two cases.

(1): We begin by collecting some useful facts. Let (γ, N, δ) be as in the statement of Theorem 1.1. In particular,

$$(4.16) N^{-1} \leq \|\varrho_0\|_{L^2} \quad \text{and} \quad \|\varrho_0\|_{H^{\delta}} \leq N.$$

Moreover, by the mean-zero assumption on ϱ_0 and the Poincaré inequality, it follows that the unique global solution $\varrho_{\text{det}}^{\gamma}$ of (4.2) satisfies

(4.17)
$$\|\varrho_{\det}^{\gamma}(t)\|_{L^{2}}^{2} \leqslant e^{-\mu t} \|\varrho_{0}\|_{L^{2}}^{2} for all t > 0.$$

Applying Corollary 4.5 with (μ, N, δ) as above and $\gamma > 0$ fixed, we obtain a family of $(\theta^{\gamma})_{\gamma \in (0,1)} \in \ell^2$ for which the unique global solution ϱ^{γ} to (1.2) satisfies

(4.18)
$$\mathbf{P}\left(\sup_{t\in[0,1]}\|\varrho^{\gamma}-\varrho_{\det}^{\gamma}\|_{L^{2}}\leqslant\frac{\varepsilon}{2N}\right)>1-\varepsilon,$$

where $\varrho_{\rm det}^{\gamma}$ is the unique global solution of (4.2) and ε is chosen so that

$$(4.19) 0 < \varepsilon < \left(\frac{1 - e^{-\mu}}{2N}\right) \wedge \frac{1}{2}.$$

Now we turn to the proof of (1). Let τ be the stopping time defined as

$$(4.20) \tau \stackrel{\text{def}}{=} \inf \left\{ t \in [0,1] : \|\varrho^{\gamma}(t) - \varrho_{\text{det}}^{\gamma}(t)\|_{L^{2}} \geqslant \frac{\varepsilon}{2N} \right\} \text{where} \inf \varnothing \stackrel{\text{def}}{=} 1.$$

By (4.18), it follows that

$$(4.21) \mathbf{P}(\tau \ge 1) > 1 - \varepsilon.$$

Note that, by (4.5), (4.16) and (4.17) show that $\|\varrho^{\gamma}(t)\|_{L^2} \vee \|\varrho_{\text{det}}^{\gamma}(t)\|_{L^2} \leqslant N$ for all $t \in [0,1]$. In particular, by (4.20),

$$\mathbf{E} \big[\mathbf{1}_{\{\tau \geqslant 1\}} \big| \|\varrho^{\gamma}(1)\|_{L^{2}}^{2} - \|\varrho_{\det}^{\gamma}(1)\|_{L^{2}}^{2} \big] \leqslant 2N \, \mathbf{E} \big[\mathbf{1}_{\{\tau \geqslant 1\}} \|\varrho^{\gamma}(1) - \varrho_{\det}^{\gamma}(1)\|_{L^{2}} \big] \leqslant \varepsilon.$$

Combining the above with (4.21), we obtain

$$(4.22) \quad \mathbf{E} \big[\mathbf{1}_{\{\tau \geqslant 1\}} \| \varrho^{\gamma}(1) \|_{L^{2}}^{2} \big] \leqslant \varepsilon + (1 - \varepsilon) \| \varrho_{\det}^{\gamma}(1) \|_{L^{2}}^{2} \stackrel{(4.17)}{\leqslant} \varepsilon + (1 - \varepsilon) e^{-\mu} \| \varrho_{0} \|_{L^{2}}^{2}.$$

Hence, for all $\gamma \in (0,1)$,

$$(4.23) 2 \mathbf{E} \int_{0}^{1} \int_{\mathbb{T}^{2}} \gamma |\nabla \varrho^{\gamma}|^{2} dx dt \stackrel{(4.5)}{=} \|\varrho_{0}\|_{L^{2}}^{2} - \mathbf{E} \|\varrho^{\gamma}(1)\|_{L^{2}}^{2}$$

$$\stackrel{(i)}{\geqslant} (1 - \varepsilon) \|\varrho_{0}\|_{L^{2}}^{2} - \mathbf{E} [\mathbf{1}_{\{\tau \geqslant 1\}} \|\varrho^{\gamma}(1)\|_{L^{2}}^{2}]$$

$$\stackrel{(4.22)}{\geqslant} \|\varrho_{0}\|_{L^{2}}^{2} (1 - \varepsilon) (1 - e^{-\mu}) - \varepsilon.$$

where in (i) we used (4.21) and $\|\varrho^{\gamma}(1)\|_{L^2} \leq \|\varrho_0\|_{L^2}$ a.s. due to (4.5). Now, the claim of Theorem 1.1(1) follows from (4.16), (4.19) and the arbitrariness of $\gamma \in (0,1)$.

(2): The proof is analogous to the one of (1), provided one chooses the parameter ε also depending on the rate of dissipation η . Firstly, since $\mu > -\ln(1-\eta)$, it follows that $\eta_1 \stackrel{\text{def}}{=} 1 - e^{-\mu} > \eta$. Secondly, let $\eta_0 \stackrel{\text{def}}{=} \frac{\eta + \eta_1}{2} \in (\eta, \eta_1)$ and fix

$$0 < \varepsilon < \left(1 - \frac{\eta_0}{\eta_1}\right) \wedge \left(\frac{\eta_0 - \eta}{N}\right).$$

Then, it is clear that $\|\varrho_0\|_{L^2}^2(1-\varepsilon)(1-e^{-\mu})-\varepsilon \ge \eta \|\varrho_0\|_{L^2}^2$ as $\|\varrho_0\|_{L^2} \ge N^{-1}$, and therefore the assertion (2) of Theorem 1.3 follows from the lower bound (4.23). \square

5. Anomalous dissipation for 2D Navier-Stokes equations

This section is devoted to the proof of Theorem 1.1 and its extension to a subcritical surface quasi-geostrophic equation. The proof of Theorem 1.1 is similar to the one of Theorem 1.3 given in Section 4. However, there are additional difficulties related to the criticality of L^2 of the 2D Navier-Stokes nonlinearity. In particular, to deal with the nonlinearity it will be of central importance to perform the scaling limit in a space of positive smoothness uniformly in time.

This section is organized as follows. In Subsection 5.1 we first show the global well-posedness of the 2D NSEs with cut-off, where the cut-off is used to tame the nonlinearity. Secondly, in Subsection 5.2, we prove Theorem 1.1 by removing the cut-off and arguing as in the case of passive scalars. Finally, in Subsection ?? we comment on the extension to the surface quasi-geostrophic equation.

5.1. Scaling limit with cut-off at a fixed Reynolds number. We begin by introducing a 2D NSEs with a cut-off. For $\phi \in C^{\infty}([0,\infty))$ such that supp $\phi \subseteq [0,2]$ and $\phi = 1$ on [0,1], and parameters $r \in (0,1)$ and R > 0, consider

$$(5.1) \qquad \begin{cases} \partial_t v^{\nu} + \phi_{R,r}(v^{\nu}) \, \mathbb{P}[(v^{\nu} \cdot \nabla)v^{\nu}] = \nu \Delta v^{\nu} \\ + \sqrt{2\mu} \sum_{k \in \mathbb{Z}_0^2} \theta_k \, \mathbb{P}[(\sigma_k \cdot \nabla)v^{\nu}] \circ \dot{W}_t^k & \text{ on } \mathbb{T}^2, \\ v^{\nu}(0,\cdot) = u_0 & \text{ on } \mathbb{T}^2, \end{cases}$$

where

$$\phi_{R,r}(v^{\nu}) \stackrel{\text{def}}{=} \phi(R^{-1}||v^{\nu}||_{H^r}).$$

The choice of the norm in the cut-off function will be clear below when dealing with the criticality of 2D NSEs, cf., (5.5) below. Finally, with a slight abuse of notation, we do not display the dependence of v^{ν} on the parameters (R, r, γ) as we will argue for a fixed value of such parameters.

In contrast to the previous section, to combine Meyers' estimates of Theorem 3.2 with the criticality of the Navier-Stokes nonlinearity in L^2 , we need to consider

solutions to (5.1) that are more regular than the one in Definition 2.2 depending on a parameter p > 2 ruling the time integrability of solutions.

Definition 5.1 (p-solutions – 2D NSEs with cut-off). Let $p \in [2, \infty)$. Fix $\nu > 0$, $\theta^{\nu} \in \ell^2$ and $u_0 \in \mathbb{B}^{1-2/p}_{2,p}(\mathbb{T}^d)$. Let $\tau^{\nu} : \Omega \to [0, \infty]$ and $v^{\nu} : [0, \infty) \times \Omega \to \mathbb{H}^1(\mathbb{T}^d)$ be a stopping time and a progressive measurable process, respectively.

- We say that (v^{ν}, τ^{ν}) is a local p-solution to (5.1) if the following hold: $\begin{array}{l} -v^{\nu} \in L^{p}_{\mathrm{loc}}([0, \infty); H^{1}(\mathbb{T}^{2}; \mathbb{R}^{2})) \cap C([0, \infty); B^{1-2/p}_{2,p}(\mathbb{T}^{2}; \mathbb{R}^{2})) \ a.s.; \\ -v^{\nu} \otimes v^{\nu} \in L^{p}_{\mathrm{loc}}([0, \infty); L^{2}(\mathbb{T}^{2}; \mathbb{R}^{2})) \ a.s.; \\ -a.s. \ for \ all \ t \in \mathbb{R}_{+} \ \ it \ holds \ that \end{array}$
 - $v^{\nu}(t) u_0 = (\nu + \mu) \int_0^t \left(\Delta v^{\nu}(s) Q_{\theta^{\nu}}(v^{\nu}(s)) \right) ds$ $\int_0^t \phi_{R,r}(v^{\nu}(s)) \mathbb{P}[\nabla \cdot (v^{\nu}(s) \otimes u^{\nu}(s))] ds$ $+ \sqrt{2\mu} \int_0^t \mathbf{1}_{[0,\tau^{\nu})} \left(\mathbb{P}[(\theta_k^{\nu} \sigma_k \cdot \nabla) v^{\nu}(s)] \right)_k d\mathcal{W}_{\ell^2}.$
- A local p-solution (v^{ν}, τ^{ν}) to (5.1) is said to be a unique local p-solution to (5.1) if for any local solution (w^{ν}, λ^{ν}) we have $\lambda^{\nu} \leq \tau^{\nu}$ a.s. and $w^{\nu} = v^{\nu}$ a.e. on $[0, \lambda^{\nu}) \times \Omega$.
- A unique local p-solution (v^{ν}, τ^{ν}) to (5.1) is said to be a unique global p-solution to (5.1) if $\tau^{\nu} = \infty$ a.s.

For the optimality of the regularity of the initial data, the reader is referred to the comments below Theorem 3.1. Similar to Definition 2.2, if p=2, then we simply write 'solution' instead of 'p-solution'. The aim of this subsection is to prove the following result.

Proposition 5.2 (Scaling limit – 2D NSEs with cut-off). Fix $\nu \in (0,1)$ and $\mu > 0$. Let $(\theta_n)_{n\geqslant 1} \subseteq \ell^2$ be a sequence of normalized radially symmetric coefficients (i.e., satisfying (2.1)) such that

$$\lim_{n\to\infty} \|\theta^n\|_{\ell^\infty} = 0.$$

Then there exists $p_0(\nu,\mu) > 2$ for which the following assertion holds. If for $p \in (2,p_0]$, $r \in (0,1-2/p)$ and R > 0 we have

- (1) $u_0 \in \mathbb{B}^{1-2/p}_{2,p}$,
- (2) there exists a unique local solution $v_{\text{det}}^{\nu} \in C([0,1]; H^r) \cap L^2(0,1; H^1)$ on [0,1]

$$\begin{cases} \partial_t v_{\rm det}^{\nu} + \phi_{R,r}(v_{\rm det}) \, \mathbb{P}[(v_{\rm det}^{\nu} \cdot \nabla) v_{\rm det}^{\nu}] = \left(\nu + \frac{\mu}{4}\right) \Delta v_{\rm det}^{\nu} & on \, \mathbb{T}^2, \\ v_{\rm det}^{\nu}(0,\cdot) = u_0 & on \, \mathbb{T}^2; \end{cases}$$

then for all $r_0 \in (0, 1-2/p)$ and all sequences $(u_{0,n}) \subseteq \mathbb{B}_{2,p}^{1-2/p}$ such that $u_{0,n} \to u_0$ in $\mathbb{B}_{2,p}^{1-2/p}$ it holds that

(5.3)
$$\lim_{n \to \infty} \mathbf{P} \left(\sup_{t \in [0,1]} \|v_n^{\nu}(t) - v_{\det}^{\nu}(t)\|_{H^{r_0}} \geqslant \varepsilon \right) = 0 \quad \text{for all } \varepsilon > 0,$$

where v_n^{ν} is the unique global p-solution to (5.1) with $\theta = \theta^n$ and initial data $u_{0,n}$.

As for Proposition 4.1, the key point is that the Sobolev space in (5.3) has positive smoothness. Moreover, repeating the argument in Remark 4.4, there is no convergence of gradients of v^n to v_{det} in $L^2(0,1;L^2(\mathbb{T}^d))$. Here, to deal with the nonlinearity of the 2D NSEs, it will be of fundamental importance that we can choose $r_0 > r$. Finally, in contrast to the linear case, the uniqueness of (weak) solutions to (5.2) is not immediate and therefore we include it as an assumption.

The existence of a unique global p-solution to (5.1) is proven in Lemma 5.3 below. Unique local solutions to (5.2) on [0,1] are as in Definition 5.1 with trivial noise, p=2 and life-time $\geqslant 1$. Note that the condition $v_{\text{det}}^{\nu} \in C([0,1]; H^r)$ is not a-priori given in the requirements for being local solutions, this is why it has been displayed in (2). Similarly, $v_{\text{det}}^{\nu} \in L^2(0,1; H^1)$ is needed as life-time $\geqslant 1$ does not imply that $t \mapsto \|v(t)\|_{L^2}^2$ is integrable near t=1.

Next, we turn to the proof of Proposition 5.2. Following the argument of Subsection 4.1, the key ingredients are the following estimates which are uniform in the class of normalized radially symmetric θ^{ν} .

Lemma 5.3 (Uniform in θ -estimate – 2D NSEs with cut-off). Let $\nu, r \in (0, 1)$ and $\mu, R > 0$ be fixed. Then there exist $p_0(\nu, \mu) > 2$ and $C_0(\nu, r, \mu, R) > 0$ for which the following assertion holds. For all $p \in [2, p_0]$, $u_0 \in \mathbb{B}_{2,p}^{1-2/p}$, $r \in (0, 1-2/p)$ and all normalized radially symmetric $\theta^{\nu} \in \ell^2$ (i.e., satisfying (2.1)), there exists a unique global p-solution v^{ν} to (5.1) and

(5.4)
$$\mathbf{E} \sup_{t \in [0,1]} \|v^{\nu}(t)\|_{B_{2,p}^{1-2/p}}^p + \mathbf{E} \int_0^1 \|v^{\nu}(t)\|_{H^1}^p \, \mathrm{d}t \leqslant C_0 (1 + \|u_0\|_{B_{2,p}^{1-2/p}}^p).$$

As in the proof of Lemma 4.2, the key point is the independence of (p_0, C_0) on θ . As in the latter result, we key tools are the stochastic Meyers' estimates of Theorem 3.2. To handle the nonlinearity, we use the sub-criticality of H^r for all r > 0 in the case of 2D NSEs, see the analogous discussion for reaction-diffusion equations in [1, Subsection 2.3]. The sub-criticality of H^r is exploited via the following inequality (cf., [1, Lemmas 4.3 and 4.5]): For all $r < \frac{1}{2}$,

(5.5)
$$\|\mathbb{P}[\nabla \cdot (v \otimes v)]\|_{H^{-1}}^2 \lesssim \|v\|_{L^4}^2 \lesssim \|v\|_{H^{1/2}}^2$$
$$\lesssim \|v\|_{H^r}^{1+\kappa_r} \|v\|_{H^1}^{1-\kappa_r}$$

where $\kappa_r = r/(1-r) > 0$. The above follows from the Sobolev embedding $H^{1/2}(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$ as well as standard interpolation inequality. The sub-criticality of H^r is encoded in the power $1 - \kappa_r < 1$ for the H^1 -norm on the RHS(5.5). This fact will allow us to absorb the corresponding term via Young's inequality.

Proof of Proposition 5.3. As we argue at a fixed viscosity $\nu > 0$, we remove it from the notation and we simply write v instead of v^{ν} and similar. Let us begin by collecting some useful facts. Arguing as in the proof of Lemma 4.2, the assumptions of the stochastic Meyers' inequalities of Theorem 3.2 holds with constant independent of the choice of normalized radially symmetric $\theta \in \ell^2$, and we let $p_0 = p_0(\mu, \nu) > 2$ be the corresponding integrability parameter for which the estimate of Theorem 3.2 holds for all $p \in [2, p_0)$. Below $p \in (2, p_0]$ is fixed.

The proof is now split into three steps. We first prove the existence of a unique local p-solution (u, τ) with a corresponding blow-up criterion. Secondly, we prove

that such a localized version of (5.4) for such a local solution. In the third step, we prove that $\tau = \infty$ a.s. (i.e., (u, τ) is global) and that (5.4) holds.

Step 1: There exists a unique local p-solution (u, τ) to (5.1) and the following blow-up criterion hold

$$(5.6) \ \ \mathbf{P}\Big(\tau < T, \sup_{t \in [0,\tau)} \|v(t)\|_{B^{1-2/p}_{2,p}}^p + \int_0^t \|v(s)\|_{H^1}^p \, \mathrm{d}s < \infty\Big) = 0 \ \ \textit{for all} \ T \in (0,\infty).$$

To prove the existence of a unique local p-solution to (5.1), we employ [4, Theorem 4.8]. To this end, we rewrite (5.1) as a stochastic evolution equation on $X_0 \stackrel{\text{def}}{=} \mathbb{H}^{-1}$, i.e.,

(5.7)
$$dv + Av dt = F(v) dt + Bv dW_{\ell^2}, \qquad v(0) = u_0$$

where, for $u \in X_1 \stackrel{\text{def}}{=} \mathbb{H}^1$,

$$Au = -(\nu + \mu)\Delta u + Q_{\theta}u, \qquad F(u) = \phi_{R,r}(u)\mathbb{P}[\nabla \cdot (u \otimes u)],$$
$$Bu = ((\theta_k \sigma_{k,\alpha} \cdot \nabla)u)_{k,\alpha}.$$

Comparing Definition 5.1 with [4, Definition 4.4], one readily see that unique p-solution to (5.1) are equivalent to maximal L_0^p -solution to (5.7) (see also [5, Remark 5.6]). Set $X_\beta \stackrel{\text{def}}{=} H^{-1+2\beta}$ for $\beta \in (0,1)$. By [4, Theorem 4.8], it remains to show the existence of $1/2 < \beta \leqslant \varphi \leqslant 1$ such that $\varphi + \beta < 2 - 1/p$ such that, for all $n \geqslant 1$ and for all $u, v \in X_1$ such that $||u||_{B_2^{1-2/p}}, ||v||_{B_2^{1-2/p}} \leqslant n$,

$$(5.8) \ \|F(u) - F(v)\|_{X_0} \lesssim_n (\|u\|_{X_{3/4}} + \|v\|_{X_{3/4}}) \|u - v\|_{X_{3/4}} + (\|u\|_{X_\varphi} + \|v\|_{X_\varphi}) \|u - v\|_{X_\beta}.$$

To prove (5.8), let us write $F(u) - F(v) = I_{u,v} + J_{u,v}$ where

$$I_{u,v} = \phi_{R,r}(u) \mathbb{P}[\nabla \cdot (u \otimes u) - \nabla \cdot (v \otimes v)],$$

$$J_{u,v} = \mathbb{P}[\nabla \cdot (v \otimes v)] (\phi_{R,r}(u) - \phi_{R,r}(v)).$$

The first term on the RHS(5.8) clearly follows by estimating $I_{u,v}$ and using $X_{3/4} = H^{1/2}(\mathbb{T}^2) \hookrightarrow L^4(\mathbb{T}^2)$. By arguing as in (5.5), one can check that, for all $r < \frac{1}{2}$,

$$\left\|\mathbb{P}[\nabla\cdot(v\otimes v)]\right\|_{H^{-1}}^2\lesssim \|v\|_{H^r}^{1+\widetilde{\kappa}_r}\|v\|_{H^{1-r/2}}^{1+\widetilde{\kappa}_r}$$

where $\widetilde{\kappa}_r \stackrel{\text{def}}{=} r/(2-3r)$. Since $B_{2,p}^{1-2/p} \hookrightarrow H^r$ as 1-2/p > r by assumption, the above inequality shows that the second term on the RHS(5.8) readily follows from estimating $J_{u,v}$ with $\varphi = 1 - \frac{r}{4}$ and $\beta = 1 - \frac{2}{p} + \varepsilon$ where $\varepsilon > 0$ is arbitrary.

Finally, (5.6) follows from [5, Theorem 4.10(3)] applied to (5.7).

Step 2: Proof of $\tau = \infty$. We begin with a localization argument. Fix $T < \infty$. For all $m \ge 1$, let τ_m be the stopping time given by

$$\tau_m \stackrel{\text{def}}{=} \inf \left\{ t \in [0, \tau \wedge T) : \|u(t)\|_{B_{2,p}^{1-2/p}}^p + \int_0^t \|u(s)\|_{H^1}^p \, \mathrm{d}s \geqslant m \right\}$$

where inf $\varnothing \stackrel{\text{def}}{=} \tau \wedge T$. The Meyers' estimates of Theorem 3.2 and a standard localization argument (see e.g., [4, Proposition 3.12(b)]) ensure that existence of $C_1(\nu,\mu) > 0$ such that, for all $m \ge 1$, $p \in [2,p_0]$ and $u_0 \in \mathbb{B}_{2,p}^{1-2/p}$,

$$\mathbf{E} \sup_{t \in [0, \tau_m]} \|v(t)\|_{B_{2,p}^{1-2/p}}^p + \mathbf{E} \int_0^{\tau_m} \|v(t)\|_{H^1}^p \, \mathrm{d}t \leqslant C_1 \|u_0\|_{B_{2,p}^{1-2/p}}^p$$

+
$$C_1 \mathbf{E} \int_0^{\tau_m} \phi_{R,r}(\cdot, v^{\nu}) \| \mathbb{P}[\nabla \cdot (v \otimes v)] \|_{H^{-1}}^p dt$$
.

Here p_0 is as at the beginning of the proof.

Now, the sub-critical estimate (5.5) yields, for all $v \in H^1(\mathbb{T}^2; \mathbb{R}^2)$,

$$\begin{aligned} \phi_{R,r}(v) \left\| \mathbb{P}[\nabla \cdot (v \otimes v)] \right\|_{H^{-1}} &\leq K_r \phi_{R,r}(v) \|v\|_{H^r}^{1+\kappa_r} \|v\|_{H^1}^{1-\kappa_r} \\ &\leq K_r (2R)^{1+\kappa_r} \|v\|_{H^1}^{1-\kappa_r} \\ &\leq \frac{1}{2C_1} \|v\|_{H^1} + K_{r,R,\nu,\mu} \end{aligned}$$

where we used the Young inequality and that $\phi_{R,r}(\cdot,v) = 0$ if $||v||_{H^r} \ge 2R$. Hence, combining the previous estimates, we obtain, for all $m \ge 1$,

$$\mathbf{E} \sup_{t \in [0, \tau_m]} \|v(t)\|_{B_{2,p}^{1-2/p}}^p + \mathbf{E} \int_0^{\tau_m} \|v(t)\|_{H^1}^p \, \mathrm{d}t \leqslant 2C_1 \|u_0\|_{B_{2,p}^{1-2/p}}^p + K_{r,R,\nu,\mu}.$$

Let $C_0 \stackrel{\text{def}}{=} (2C_1) \vee K_{r,R,\nu,\mu}$. Note that C_0 is independent of $m \ge 1$ as are C_1 and $K_{r,R,\nu,\mu}$. Thus, since $\lim_{m\to\infty} \tau_m = \tau \wedge T$ a.s., the above implies

(5.9)
$$\mathbf{E} \sup_{t \in [0, \tau \wedge T)} \|v(t)\|_{B_{2,p}^{1-2/p}}^p + \mathbf{E} \int_0^{\tau \wedge T} \|v(s)\|_{H^1}^p \, \mathrm{d}s \le C_0 (1 + \|u_0\|_{B_{2,p}^{1-2/p}}^p).$$

Hence, $\|v(t)\|_{B_{2,p}^{1-2/p}}^p + \mathbf{E} \int_0^{\tau \wedge T} \|v(s)\|_{H^1}^p \, \mathrm{d}s < \infty$ a.s. on $\{\tau < T\}$, and therefore $\tau \ge T$ a.s. by (5.6). The arbitrariness of T implies $\tau = \infty$ a.s., as desired.

Step 3: Proof of (5.4). The estimate (5.4) follows from (5.9) and the fact that $\tau = \infty$ a.s. by Step 1.

Due to Lemma 5.3, the proof of Proposition 5.2 is very similar to the one of Proposition 4.1. Hence, we only give a sketch of it.

Proof of Proposition 5.2 – Sketch. The proof of Proposition 5.2 is completely analogous to the one of Proposition 4.1 as the content of Lemma 4.3 also holds with $(\varrho^{\gamma}, \gamma)$ replaced by (v^{ν}, ν) . Let us point out that the factor $\frac{\mu}{4}$ in the limiting equation (5.2) comes from the behaviour as $n \to \infty$ of the Itô-correction Q_{θ} with $\theta = \theta^n$ (cf., Definition 2.2), see [56, Theorem 3.1].

5.2. **Proof of Theorem 1.1.** Parallel to the proof of Theorem 1.3, the key ingredient in Theorem 1.1 is the following consequence of the above scaling limit.

Corollary 5.4. Let $\delta, \nu, \varepsilon \in (0,1)$ and $N, \mu > 0$ be fixed. Then there exists $\theta^{\nu} \in \ell^2$ such that

$$\|\theta^{\nu}\|_{\ell^2} = 1$$
 and $\#\{k : \theta_{\nu}^{\nu} \neq 0\} < \infty$,

and that for all $||u_0||_{\mathbb{H}^{\delta}} \leq N$ the unique global solution u^{ν} to (1.1) satisfies

$$\mathbf{P}\Big(\sup_{t\in[0,1]}\|u^{\nu}-u_{\mathrm{det}}^{\nu}\|_{L^{2}}\leqslant\varepsilon\Big)>1-\varepsilon$$

where u_{det} is the unique global solution to

(5.10)
$$\begin{cases} \partial_t u_{\text{det}}^{\nu} + \mathbb{P}[(u_{\text{det}}^{\nu} \cdot \nabla) u_{\text{det}}^{\nu}] = \left(\nu + \frac{\mu}{4}\right) \Delta u_{\text{det}}^{\nu} & on \ \mathbb{T}^2, \\ u_{\text{det}}^{\nu}(0, \cdot) = u_0 & on \ \mathbb{T}^2. \end{cases}$$

Proof. Let $p_0(\nu,\mu) > 2$ be as in Proposition 5.2. Fix $p \in (2,p_0)$ such that $1-2/p < \delta$ and r,r_0 such that $r < r_0 < 1-2/p$. Note that the choice of p yields $u_0 \in B_{2,p}^{1-2/p}$, cf. (4.8). Next, by Proposition A.1, there exists $K_0 = K_0(N,r_0) > 0$ such that the unique global solution u_{det} to (5.10) satisfies

(5.11)
$$\sup_{t \in [0,1]} \|u_{\det}^{\nu}(t)\|_{H^{r_0}} \leqslant K_0.$$

Next, to apply Proposition 5.2 with $R=K_0+1$ and (r,r_0) as above, we check that (5.2) has a unique solution on [0,1] and it is given by $u_{\rm det}^{\nu}$. To begin, as $u_{\rm det}^{\nu}$ is a unique global solution to (5.10) satisfying the bound (5.11), then it is also a local solution to (5.2) on [0,1] with $u_{\rm det}^{\nu} \in C([0,1];H^r) \cap L^2(0,1;H^1)$. It remains to discuss the uniqueness. To this end, we mimic a stopping-time argument. Let $w_{\rm det}$ be another global solution to (5.2) with $w_{\rm det}^{\nu} \in C([0,1];H^r)$. Let

$$e_{w_{\text{det}}^{\nu}} \stackrel{\text{def}}{=} \inf\{t \in [0,1] : \|w_{\text{det}}^{\nu}(t,\cdot)\|_{H^r} \geqslant R\} \quad \text{where} \quad \inf \varnothing \stackrel{\text{def}}{=} 1.$$

If $e_{w_{\mathrm{det}}^{\nu}}=1$, then $u_{\mathrm{det}}^{\nu}=w_{\mathrm{det}}^{\nu}$ by uniqueness of solutions to (5.10). If $e_{w_{\mathrm{det}}^{\nu}}<1$, then still using the uniqueness of solutions to (5.10) we get $u_{\mathrm{det}}^{\nu}=w_{\mathrm{det}}^{\nu}$ on $[0,e_{w_{\mathrm{det}}^{\nu}}]$. As $\sup_{t\in[0,e_{w_{\mathrm{det}}^{\nu}}]}\|u_{\mathrm{det}}^{\nu}(t)\|_{H^{r}}\leqslant\sup_{t\in[0,1]}\|u_{\mathrm{det}}^{\nu}(t)\|_{H^{r_{0}}}\leqslant K_{0}$ and $R>K_{0}$ by construction, it follows that $\|w_{\mathrm{det}}(e_{w_{\mathrm{det}}^{\nu}},\cdot)\|_{e_{w_{\mathrm{det}}^{\nu}}}< R$ which is a contradiction with the definition of $e_{w_{\mathrm{det}}^{\nu}}$ and $w_{\mathrm{det}}\in C([0,1];H^{r})$. This proves $e_{w_{\mathrm{det}}^{\nu}}=1$ and hence uniqueness of (5.2) follows from the one of solutions to (5.10).

Now, by Proposition 5.2 applied with $R = K_0 + 1$, (p, r, r_0) as above and θ^n as in (4.4), there exists $\theta^{\nu} \in \ell^2$ with the required properties such that

$$\mathbf{P}\Big(\sup_{t\in[0,1]}\|v^{\nu}-u_{\mathrm{det}}^{\nu}\|_{H^r}\leqslant 1\Big)>1-\varepsilon$$

where v^{ν} is the unique global *p*-solution to (5.1) with the above choice of (R, r). Set

$$\Omega_0 \stackrel{\text{def}}{=} \left\{ \sup_{t \in [0,1]} \|v^{\nu} - u_{\text{det}}^{\nu}\|_{H^r} \leqslant 1 \right\}.$$

Now, to conclude, it remains to prove that

(5.12)
$$u^{\nu}(t) = v^{\nu}(t) \text{ for all } t \in [0, 1] \text{ a.s. on } \Omega_0.$$

To see (5.12), define the following stopping time

$$\tau_0 \stackrel{\text{def}}{=} \inf \Big\{ t \in [0,1] : \sup_{t \in [0,1]} \|v^{\nu}(t)\|_{H^r} \geqslant K_0 + 1 \Big\}.$$

and $\inf \emptyset \stackrel{\text{def}}{=} 1$. Note that, by (5.11), we have

(5.13)
$$\tau_0 = 1 \quad \text{on } \Omega_0.$$

Moreover, $\phi_{R,r}(t,v^{\nu}) = 1$ for all $t \in [0,\tau_0]$. Hence, (v^{ν},τ_0) is a local solution to the 2D NSEs with transport noise (1.1). By uniqueness of u^{ν} we obtain

(5.14)
$$u^{\nu}(t) = v^{\nu}(t)$$
 a.s. for all $t \in [0, \tau_0]$,

cf., Definition 2.2. Hence, the claim
$$(5.12)$$
 follows from (5.13) and (5.14) .

Proof of Theorem 1.1. Due to Corollary 5.4, the proof of Theorem 1.1 now follows almost verbatim from the one of Theorem 1.3 given at the end of Subsection 4.2. The factor '4' in the condition $\mu > 4 \ln(1-\eta)$ of (2) comes from the one in (5.10). \Box

Appendix A. H^{δ} -estimates for two-dimensional fluids

The aim of this appendix is to prove the following

Proposition A.1 (H^{δ} -estimates – 2D fluids). Let $\kappa > 0$, $\delta \in (0,1]$ and let u be unique global solution to the 2D Navier-Stokes equations on \mathbb{T}^2 , i.e.,

(A.1)
$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \kappa \Delta u, \quad \nabla \cdot u = 0, \quad u(0) = u_0 \in \mathbb{H}^{\delta}(\mathbb{T}^2).$$

Then there exists a non-decreasing mapping $N:[0,\infty)\to [1,\infty)$ such that

$$(\mathrm{A}.2) \qquad \sup_{t \in [0,\infty)} \|u(t,\cdot)\|_{H^\delta} + \left\| u - \int_{\mathbb{T}^2} u_0 \, \mathrm{d}x \, \right\|_{L^2(\mathbb{R}_+;H^{1+\delta})} \leqslant N(\|u_0\|_{L^2}) \|u_0\|_{H^\delta}.$$

Here, $\mathbb{H}^{\delta}(\mathbb{T}^2)$ denotes the space of divergence-free vector field in $H^{\delta} = H^{\delta}(\mathbb{T}^2; \mathbb{R}^2)$, cf., Subsection 2.1. Moreover, solutions to (A.1) can be defined similarly to Definition 2.2 by writing the nonlinearities in the conservative form.

The above result might be known to experts, however, to the best of the author's knowledge, no direct reference is available. Therefore, we include the proof for the sake of completeness. Below, we deliberately do not use the vorticity formulation of 2D NSEs so that our arguments can also be extended to other situations, e.g., 2D NSEs on domains with no-slip boundary conditions. The case $\delta > 1$ of Proposition A.1 also holds and it follows from the case $\delta < 1$ and the sub-criticality of H^{δ} with $\delta > 0$ for (A.1). Details are left to the interested reader.

Proof. Inspired by [42], we prove the claimed estimate (A.2) by interpolation. For notational convenience, we assume $\kappa=1$. Note that, since $\int_{\mathbb{T}^2} u(t,x) \, \mathrm{d}x = \int_{\mathbb{T}^2} u_0(x) \, \mathrm{d}x$ for all $t \geq 0$ as $\nabla \cdot u = 0$ in $\mathcal{D}'(\mathbb{T}^2)$, without loss of generality, we may assume that $\int_{\mathbb{T}^2} u_0(x) \, \mathrm{d}x = 0$. For $\delta \in [0,1]$, let H_{mz}^{δ} be the subset of H^{δ} with mean zero. Recall that $H_{\mathrm{mz}}^{\delta} = (L_{\mathrm{mz}}^2, H_{\mathrm{mz}}^1)_{\delta,2}$ where $(\cdot, \cdot)_{\delta,2}$ is the real interpolation functor (see e.g., [15] for details). Hence

(A.3)
$$||u_0||_{H^{\delta}} \approx ||(2^{-j\delta}K_j(u_0))_{j\in\mathbb{Z}}||_{\ell^2}$$

where K_j is the K-functional computed at the dyadic time 2^j , i.e.,

(A.4)
$$K_j(u_0) = \inf_{u_0 = v_0 + w_0} (\|v_0\|_{L^2} + 2^j \|w_0\|_{H^1})$$

where the infimum is taken over all decompositions $u_0 = v_0 + w_0$ satisfying $v_0 \in L^2$ and $w_0 \in H^1$ with mean zero.

Next, we fix $u_0 \in H^{\delta}_{\mathrm{mz}}$. By definition of $K_j(u_0)$, for each $j \in \mathbb{Z}$, there exist $v_0^{(j)} \in L^2_{\mathrm{mz}}$ and $w_0^{(j)} \in H^1_{\mathrm{mz}}$ such that $u_0 = v_0^{(j)} + w_0^{(j)}$ and

$$K_j(u_0) \approx ||v_0^{(j)}||_{L^2} + 2^j ||w_0^{(j)}||_{H^1},$$

with implicit constant independent of $j \ge 1$. The minimality property implies $\|v_0^{(j)}\|_{L^2} \le \|u_0\|_{L^2}$, and therefore $\|w_0^{(j)}\|_{L^2} \le 2\|u_0\|_{L^2}$. Summarizing, we have,

(A.5)
$$||u_0||_{L^2} \ge \frac{1}{3} (||v_0^{(j)}||_{L^2} + ||w_0^{(j)}||_{L^2}),$$

(A.6)
$$||u_0||_{H^{\delta}} \approx ||(2^{-j\delta}[||v_0^{(j)}||_{L^2} + 2^j ||w_0^{(j)}||_{H^1}])|_{i\in\mathbb{Z}}||_{\ell^2}.$$

Following [42], we first prove estimates H^1 -estimates for

(A.7)
$$\partial_t v^{(j)} + \mathbb{P}[(v^{(j)} \cdot \nabla)v^{(j)}] = \Delta v^{(j)}, \qquad v^{(j)}(0) = v_0^{(j)};$$

and secondly an L^2 -estimates for

(A.8)
$$\partial_t w^{(j)} + \mathbb{P}(([v^{(j)} + w^{(j)}] \cdot \nabla)[v^{(j)} + w^{(j)}]) = +\Delta w^{(j)}, \quad w^{(j)}(0) = w_0^{(j)}$$

where \mathbb{P} is the Helmholtz projection, see Subsection 2.1.

The idea behind the decomposition (A.7)-(A.8) is that $u = v^{(j)} + w^{(j)}$ for all $j \in \mathbb{Z}$, by uniqueness of global solutions to (A.1).

Step 1: There exists a non-decreasing mapping $N:[0,\infty)\to [1,\infty)$ such that, for all $j\in\mathbb{Z}$,

(A.9)
$$\sup_{t \in [0,\infty)} \|w^{(j)}(t,\cdot)\|_{H^1} + \|w^{(j)}\|_{L^2(\mathbb{R}_+;H^2)} \le N(\|u_0\|_{L^2}) \|w_0^{(j)}\|_{H^1},$$

(A.10)
$$\sup_{t \in [0,\infty)} \|v^{(j)}(t,\cdot)\|_{L^2} + \|v^{(j)}\|_{L^2(0,T;H^1)} \le N(\|u_0\|_{L^2})\|v_0^{(j)}\|_{L^2},$$

where $(v_0^{(j)}, w_0^{(j)})$ are as above. We begin by proving (A.9). The existence of a unique global solution $w^{(j)} \in H^1_{loc}([0,\infty); \mathbb{L}^2(\mathbb{T}^2)) \cap L^2_{loc}([0,\infty); \mathbb{H}^2(\mathbb{T}^2))$ of (A.7) is standard. For instance, the latter fact can be proven by combining the local existence of [67, Theorem 1.2] and localized in time version of the estimate (A.10) proven below (see Step 3 in the proof of Lemma 5.3 for a similar situation).

Now, since $\int_{\mathbb{T}^d} w_0^{(j)} dx = 0$, the following energy estimate holds:

(A.11)
$$\sup_{t \in [0,\infty)} \|w^{(j)}(t,\cdot)\|_{L^2}^2 + \|w^{(j)}\|_{L^2(\mathbb{R}_+;H^1)}^2 \lesssim \|w_0^{(j)}\|_{L^2}^2.$$

where the implicit constant is independent of $j \in \mathbb{Z}$. Now, the Sobolev embeddings and standard interpolation inequalities imply

$$\|(w^{(j)} \cdot \nabla)w^{(j)}\|_{L^2} \lesssim \|w^{(j)}\|_{L^4} \|\nabla w^{(j)}\|_{L^4} \lesssim \|w^{(j)}\|_{L^4} \|w^{(j)}\|_{H^1}^{1/2} \|w^{(j)}\|_{H^2}^{1/2}.$$

Thus, with the aid of the above inequality, the Young inequality and integrating by parts we have, for some $C_0 > 0$ independent of $(j, w_0^{(j)})$,

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\nabla w^{(j)}\|_{L^{2}}^{2} + \|\Delta w^{(j)}\|_{L^{2}}^{2} = \left| \int_{\mathbb{T}^{d}} \mathbb{P}[(w^{(j)} \cdot \nabla)w^{(j)}] \cdot \Delta w^{(j)} \, \mathrm{d}x \right| \\
\leq C_{0} \|w^{(j)}\|_{L^{4}}^{2} \|w^{(j)}\|_{H^{1}} \|w^{(j)}\|_{H^{2}} + \frac{1}{4} \|\Delta w^{(j)}\|_{L^{2}}^{2} \\
\leq C_{1} \|w^{(j)}\|_{L^{4}}^{4} \|w^{(j)}\|_{H^{1}}^{2} + \frac{1}{2} \|w^{(j)}\|_{H^{2}}^{2}$$

a.e. on \mathbb{R}_+ . Due to (A.11), (A.9) now follows by applying the Gronwall lemma to the above estimate as well as the fact that (A.11) and interpolation imply

(A.12)
$$||w^{(j)}||_{L^4(\mathbb{R}_+;L^4)} \lesssim ||w_0^{(j)}||_{L^2} \lesssim ||u_0||_{L^2}^2.$$

The proof of (A.10) follows again the arguments leading to (A.11) and (A.12), see [3, Theorem 3.3] for a similar situation.

Step 2: Conclusion. Let us begin by recalling that $u = v^{(j)} + w^{(j)}$ for all $j \in \mathbb{Z}$ by the well-known uniqueness of global solutions $u \in H^1_{loc}([0,\infty); \mathbb{H}^{-1}(\mathbb{T}^2)) \cap L^2_{loc}([0,\infty); \mathbb{H}^1(\mathbb{T}^2))$ to (A.1). By (A.3), for all $t \in [0,\infty)$,

$$\begin{split} \|u(t,\cdot)\|_{H^{\delta}} &\leqslant \left\| \left(2^{-j\delta} [\|v^{(j)}(t,\cdot)\|_{L^{2}} + 2^{j} \|w^{(j)}(t,\cdot)\|_{H^{1}}] \right)_{j\geqslant 1} \right\|_{\ell^{2}} \\ &\leqslant N(\|u_{0}\|_{L^{2}}) \left\| \left(2^{-j\delta} [\|v^{(j)}_{0}\|_{L^{2}} + 2^{j} \|w^{(j)}_{0}\|_{H^{1}}] \right)_{j\geqslant 1} \right\|_{\ell^{2}} \\ &\approx N(\|u_{0}\|_{L^{2}}) \|u_{0}\|_{H^{\delta}}, \end{split}$$

where in the last step we used (A.6). To estimate the $L^2(\mathbb{R}_+; H^{1+\delta})$ -norm of u, one can argue analogously by noticing that $(L^2(\mathbb{R}_+; H^1), L^2(\mathbb{R}_+; H^2))_{\delta,2} = L^2(\mathbb{R}_+; H^{1+\delta})$ by [48, Theorem 2.2.6]. Hence, (A.2) follows similarly.

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