On a Completion of Cohomological Functors Generalising Tate Cohomology II

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Abstract

Viewing group cohomology as a so-called cohomological functor, G. Mislin has generalised Tate cohomology from finite groups to all discrete groups by defining a completion for cohomological functors in [24]. For any cohomological functor $T^\bullet : \mathcal{C} \to \mathcal{D}$ we have constructed its Mislin completion $\widehat{T}^\bullet : \mathcal{C} \to \mathcal{D}$ in [15] under mild assumptions on the abelian categories $\mathcal{C}$ and $\mathcal{D}$ which generalises Tate cohomology to all $T1$ topological groups. In this paper we investigate the properties of Mislin completions. As their main feature, Mislin completions of Ext-functors detect finite projective dimension of objects in the domain category. We establish a version of dimension shifting, an Eckmann-Shapiro result as well as cohomology products such as external products, cup products and Yoneda products.

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1 Introduction

Because Tate cohomology and its generalisations are the main motivation of this work, we first outline Tate cohomology for a finite group G based on [1, p. 78–79]. For any module M over the group ring $\mathbb{Z}[G]$ and any integer $n \in \mathbb{Z}$ there is a Tate cohomology group $\widehat{H}^n(G, M)$. Note that $\widehat{H}^n(G, M) = H^n(G, M)$ for $n \geq 1$ and $\widehat{H}^n(G, M) = H_{-n-1}(G, M)$

for $n \leq -2$, whence Tate cohomology unites both group cohomology and group homology. According to [5, p. 136], Tate cohomology of G satisfies dimension shifting and possesses cup products. The former means that for any M there are $\mathbb{Z}[G]$-modules M_*, M^* such that $\widehat{H}^{n-1}(G, M_*) \cong \widehat{H}^n(G, M)$ and $\widehat{H}^{n+1}(G, M^*) \cong \widehat{H}^n(G, M)$ for every $n \in \mathbb{Z}$. Thus, if one desires to establish a property for every degree $\widehat{H}^n(G, -)$ of $\widehat{H}^\bullet(G, -)$, then one only needs to establish it for a carefully chosen degree n. If N is a $\mathbb{Z}[G]$-module and $m \in \mathbb{Z}$ another integer, then a cup product is a homomorphism of the form

$$\smile : \widehat{H}^n(G, M) \otimes \widehat{H}^m(G, N) \to \widehat{H}^{m+n}(G, M \otimes N)$$

where the tensor product $M \otimes N$ is a $\mathbb{Z}[G]$-module endowed with the diagonal action of G. Importantly for us, Tate cohomology for G vanishes on projectives according to [1, p. 79], meaning that $\widehat{H}^n(G, P) = 0$ for any projective $\mathbb{Z}[G]$-module P and $n \in \mathbb{Z}$. Then complete cohomology generalises Tate cohomology for finite groups and was introduced for all discrete groups by G. Mislin, P. Vogel, D. J. Benson and J. F. Carlson using different constructions all yielding isomorphic cohomology groups [21, p. 196–197].

In [15] we have generalised these constructions to a previously unknown extent. In particular, since our generalisation takes Tate cohomology to condensed mathematics, one can construct Tate cohomology for any topological group in which every point is closed, meaning for any $T1$ topological group. More specifically, condensed mathematics is a novel theory developed by D. Clausen and P. Scholze in 2018 [31]. In [32], P. Scholze writes that he wants "to make the strong claim that in the foundations of mathematics, one should replace topological spaces with condensed sets". Independently of whether this claim turns out to be true, it provides a very promising unified approach for studying topological groups, rings and modules [31, p. 6]. This is follow-up paper to [15] in which we investigate the properties of this generalisation of Tate cohomology and more generally, of Mislin completions.

Hence, let us provide a brief summary of Mislin completions. If $\mathcal{C}$, $\mathcal{D}$ denote two abelian categories, then a family of additive functors $(T^n : \mathcal{C} \to \mathcal{D})_{n \in \mathbb{Z}}$ is a cohomological functor if there are so-called connecting homomorphisms $(\delta^n : T^n \to T^{n+1})_{n \in \mathbb{Z}}$ satisfying two (natural) axioms [21, p. 201–202]. In particular, if G is a discrete group, R a discrete ring and A a discrete R-module, then setting $H^n_R(G, -) = 0$ and $\mathrm{Ext}^n_R(A, -) = 0$ for $n < 0$ renders group cohomology and Ext-functors into cohomological functors [21, p. 201], [24, p. 295]. We refer the reader unfamiliar with abelian categories to the account in [23, p. 249–257], [34, Tag 00ZX] or [38, Section A.4]. G. Mislin generalised Tate cohomology to all discrete groups by defining complete cohomology $\widehat{H}^\bullet_R(G, -)$ as a specific completion of ordinary group cohomology $H^\bullet_R(G, -)$ in his paper [24]. Following the convention from [21, p. 197/202], we call such a completion of a cohomological functor a Mislin completion. Formally, a Mislin completion of $T^\bullet : \mathcal{C} \to \mathcal{D}$ is a cohomological functor $\widehat{T}^\bullet : \mathcal{C} \to \mathcal{D}$ together with a morphism $\Phi^\bullet : T^\bullet \to \widehat{T}^\bullet$ such that $\widehat{T}^n(P) = 0$ for any projective $P \in \mathrm{obj}(\mathcal{C})$ and $n \in \mathbb{Z}$ and such that any morphism $T^\bullet \to V^\bullet$ to a cohomological functor $V^\bullet : \mathcal{C} \to \mathcal{D}$ also vanishing on projectives factors uniquely through $\Phi^\bullet$ [24, Definition 2.1]. By its universal property, any Mislin completion is unique up to isomorphism [21, p. 202]. In [15], we construct the Mislin completion $\widehat{T}^\bullet$ if $\mathcal{C}$ has enough projectives and in $\mathcal{D}$ all countable direct limits exist and are exact.

In order to present a key feature of Mislin completions, let us make a few comments on Ext-functors and group cohomology. If **Ab** denotes the category of abelian groups, then we do not only consider the "ordinary" unenriched Ext-functors $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathbf{Ab}$ arising as derived functors of the usual Hom-functors in $\mathcal{C}$, but also so-called enriched Ext-functors $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathcal{D}$ where the abelian category $\mathcal{D}$ does not need to be **Ab**. For instance, we consider enriched Ext-functors in condensed mathematics in [15, Theorem 7.1]. We remind the reader that group cohomology can be constructed as a specific Ext-functor. More specifically, if $\mathcal{C}_{R,G}$ is the respective category of module objects over the group ring (object) of G over R, one can define $H^n_R(G,-) := \mathrm{Ext}^n_{\mathcal{C}_{R,G}}(R,-)$ where G acts trivially on R. As with Ext-functors, group cohomology can be enriched or unenriched. Adhering to the terminology in [15], the Mislin completions of Ext-functors are called completed Ext-functors and the Mislin completion of group cohomology is termed complete cohomology. We say that an object has finite projective dimension if it admits a projective resolution of finite length [5, p. 152]. The group (object) G has finite cohomological dimension over R if the module object R has finite projective dimension in $\mathcal{C}_{R,G}$ [5, p. 184–185]. In this level of generality, at least the following two results pertain to condensed mathematics and thus to all $T1$ topological groups [15, Section 7]. Namely, we first obtain

Lemma 1.1 *(= Lemma 3.1)*

1. *Assume that $T^\bullet : \mathcal{C} \to \mathcal{D}$ is a cohomological functor where $\mathcal{C}$ has enough projectives and in $\mathcal{D}$ all countable direct limits exist and are exact. If $M \in \mathrm{obj}(\mathcal{C})$ has finite projective dimension, then $\widehat{T}^n(M) = 0$ for every $n \in \mathbb{Z}$. In particular, if every object in $\mathcal{C}$ has finite projective dimension such as in a category of modules over a ring of finite global dimension, then $\widehat{T}^\bullet = 0$ for any cohomological functor $T^\bullet$.*

2. *If one takes* ***enriched*** *Ext-functors $\mathrm{Ext}^n_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathcal{D}$ with $A \in \mathrm{obj}(\mathcal{C})$ of finite projective dimension, then $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,-) = 0$ for every $n \in \mathbb{Z}$. In particular, complete cohomology $\widehat{H}^\bullet_R(G,M)$ vanishes if the group object G has finite cohomological dimension or $M \in \mathcal{C}_{R,G}$ has finite projective dimension. As this applies to any condensed ring R and any condensed group G, this holds for any $T1$ topological group by taking its condensate.*

Then the main feature of completed Ext-functors is the following theorem which generalises Lemma 4.2.4 from [21]:

Theorem 1.2 *(= Theorem 3.2) If $\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,-) : \mathcal{C} \to \mathbf{Ab}$ denote completed unenriched Ext-functors for $A \in \mathrm{obj}(\mathcal{C})$, then the following are equivalent.*

1. *The object A has finite projective dimension.*

2. *$\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,-) = \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(-,A) = 0$ for any $n \in \mathbb{Z}$.*

3. *$\widehat{\mathrm{Ext}}^0_{\mathcal{C}}(A,A) = 0$.*

In particular, the zeroeth complete cohomology group detects whether a group (object) has finite cohomological dimension. This applies to any condensed group and thus to any $T1$ topological group.

We provide a (partial) version of dimension shifting for Mislin completions.

Theorem 1.3 *(= Theorem 4.3) Let $T^\bullet : \mathcal{C} \to \mathcal{D}$ be a cohomological functor where $\mathcal{C}$ has enough projectives and in $\mathcal{D}$ all countable direct limits exist and are exact. Then for every $M \in \mathrm{obj}(\mathcal{C})$ there is $M^* \in \mathrm{obj}(\mathcal{C})$ such that $\widehat{T}^{n+1}(M^*) \cong \widehat{T}^n(M)$ for every $n \in \mathbb{Z}$. In addition, if there is a monomorphism $f : M \to N$ in $\mathcal{C}$ with $\widehat{T}^k(N) = 0$ for every $k \in \mathbb{Z}$, then $\widehat{T}^{n-1}(\mathrm{Coker}(f)) \cong \widehat{T}^n(M)$.*

The conditions of this theorem are satisfied for instance if $\mathcal{C}$ is the category of discrete R-modules for a discrete commutative ring R and G is a discrete group together with a finite index subgroup of finite cohomological dimension. Another instance is if $\mathcal{C}$ is the category of profinite S-modules for a commutative profinite ring S and if K is a profinite group with an open subgroup of finite cohomological dimension (Example 4.4). Let us explain the terminology involved in the latter example. An inverse limit is a specific type of limit [39, p. 12]. A profinite group is a topological group that can be equivalently defined as an inverse system of finite discrete groups, as a closed subgroup of a cartesian product of finite discrete groups endowed with the product topology [39, Corollary 1.2.4] or as a (not necessarily finite) Galois group endowed with the Krull topology [39, Theorem 3.3.2]. Profinite spaces, rings, modules etc. are defined analogously [29, p. 1]. The completed group ring $S[[K]]$ is a profinite ring that is a profinite version of a (discrete) group ring [29, p. 171]. We note that the category of profinite $S[[K]]$-modules has enough projectives [37, p. 353]. Lastly, we follow the convention in [9, p. 235] that the cohomology groups $H^\bullet_S(K, M)$ are $U(S)$-modules where $U(S)$ denotes the ring S without its topology. The reader is referred to [29] and [39] for more background on profinite groups and to [33] and [37] for sources specialising on cohomology of profinite groups.

Moving back to greater generality, we recall that any $M \in \mathrm{obj}(\mathcal{C}_{R,H})$ can be turned into an object in $\mathcal{C}_{R,G}$ by induction $\mathrm{Ind}_H^G(M)$ and coinduction $\mathrm{Coind}_H^G(M)$. Contrarily, any $M \in \mathrm{obj}(\mathcal{C}_{R,G})$ can be turned into an object in $\mathcal{C}_{R,H}$ by restriction $\mathrm{Res}_H^G(M)$. Under specific circumstances we can retrieve an Eckmann-Shapiro type result.

Lemma 1.4 *(= Lemma 4.1)*

1. *If both $\mathrm{Res}_H^G(-)$ and $\mathrm{Coind}_H^G(-)$ are exact and preserve projective objects, then $\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A), B) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A, \mathrm{Coind}_H^G(B))$ as unenriched completed Ext-functors for every $n \in \mathbb{Z}$, $A \in \mathrm{obj}(\mathcal{C}_{R,G})$ and $B \in \mathrm{obj}(\mathcal{C}_{R,H})$. In particular, in the case where $A = R$ we have $\widehat{H}^n_R(H, B) \cong \widehat{H}^n_R(G, \mathrm{Coind}_H^G(B))$.*

2. *If $\mathrm{Ind}_H^G(-)$ and $\mathrm{Res}_H^G(-)$ are exact and preserve projectives, then for every $n \in \mathbb{Z}$, $A \in \mathrm{obj}(\mathcal{C}_{R,H})$ and $B \in \mathrm{obj}(\mathcal{C}_{R,G})$ observe for unenriched completed Ext-functors that $\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(\mathrm{Ind}_H^G(A), B) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A, \mathrm{Res}_H^G(B))$.*

Similar to before, the conditions of this lemma are satisfied by the category of discrete R-modules for a discrete commutative ring R and by a discrete group G together with a finite index subgroup. Or by the category of profinite S-modules for a commutative profinite ring S and by a profinite group K with an open subgroup (Example 4.2).

Let us investigate cohomology products. We can establish external products for completed unenriched Ext-functors and cup products for complete unenriched group cohomology under certain conditions. Recall that external products of unenriched Ext-functors descend from tensor products in the domain category where external products restricted to group cohomology give rise to cup products.

Theorem 1.5 *(= Theorem 6.2)*

1. *Let* $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *be a bi-additive functor and assume that* $P \otimes_{\mathcal{C}} Q$ *is projective whenever* $P, Q \in \operatorname{obj}(\mathcal{C})$ *are projective. Let* $A_\bullet$, $C_\bullet$ *be projective resolutions of* $A, C \in \operatorname{obj}(\mathcal{C})$ *such that the tensor product of resolutions* $A_\bullet \otimes_{\mathcal{C}} C_\bullet$ *is a projective resolution of* $A \otimes_{\mathcal{C}} C$. *If* $B, E \in \operatorname{obj}(\mathcal{C})$ *possess projective resolutions of a specific form, then for every* $m, n \in \mathbb{Z}$ *external products*

$$\vee : \widehat{\operatorname{Ext}}_{\mathcal{C}}^m(A, B) \otimes \widehat{\operatorname{Ext}}_{\mathcal{C}}^n(C, E) \to \widehat{\operatorname{Ext}}_{\mathcal{C}}^{m+n}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E)$$

can be defined for completed unenriched Ext-functors.

2. *If* R *is a ring object in* $\mathcal{C}$ *and* G *a group object, then we take the tensor product to be of the form* $\otimes_R : \mathcal{C}_{R,G} \times \mathcal{C}_{R,G} \to \mathcal{C}_{R,G}$. *Assume that there are natural isomorphisms* $\Theta'_M : M \otimes_R R \to M$ *and* $\Theta''_M : R \otimes_R M \to M$ *and that* $R_\bullet \otimes_R R'_\bullet$ *is a projective resolution of* R *for any two projective resolutions* $R_\bullet$, $R'_\bullet$ *of* R. *Then for every* $m, n \in \mathbb{Z}$ *there are cup products*

$$\smile : \widehat{H}_R^m(G, M) \otimes \widehat{H}_R^n(G, N) \to \widehat{H}_R^{m+n}(G, M \otimes_R N)$$

for completed unenriched group cohomology that descend from the above external products.

The conditions of this theorem are fulfilled for example by the category of discrete $R[G]$-modules for a discrete group G and a principal ideal domain R. Moreover, the restriction of A, B, C, E to R-modules needs to be projective. Analogously, the conditions are satisfied by the category of profinite $S[[K]]$-modules for a profinite group K and profinite commutative ring S with a unique maximal open ideal where the restriction of A, B, C, E to profinite S-modules needs to be projective (Example 6.4). In the latter case, there is a profinite tensor product $\otimes_S$ defined for profinite $S[[K]]$-modules [29, p. 177/191].

Remember that Yoneda products of unenriched Ext-functors descend from compositions of morphisms in the domain category. By generalising [3, p. 110], we construct Yoneda products for completed unenriched Ext-functors.

Theorem 1.6 *(= Theorem 6.6) Let* $F, H, J \in \operatorname{obj}(\mathcal{C})$. *If* $\otimes$ *denotes the tensor product in* **Ab**, *then for every* $m, n \in \mathbb{Z}$ *Yoneda products*

$$\circ : \widehat{\operatorname{Ext}}_{\mathcal{C}}^n(H, J) \otimes \widehat{\operatorname{Ext}}_{\mathcal{C}}^m(F, H) \to \widehat{\operatorname{Ext}}_{\mathcal{C}}^{m+n}(F, J)$$

can be defined for completed unenriched Ext-functors.

All of the above cohomology products are natural (Lemmaa 7.1) and compatible with connecting homomorphisms (Lemma 7.6) where external and cup products satisfy a version of graded commutativity (Proposition 7.5). We develop requirements under which they are associative (Lemma 7.2). We also demonstrate under which conditions cup products turn complete cohomology and Yoneda products turn completed Ext-functors into a graded ring with identity (Lemma 7.3).

Moreover, complete cohomology generalises Tate-Farrell cohomology. More specifically, F. T. Farrell generalised Tate cohomology of finite groups to discrete groups having a finite index subgroup of finite cohomological dimension in his paper [13] by using so-called complete resolutions. A complete resolution of an object A in $\mathcal{C}$ is a particular acyclic chain complex $(\overline{A}_n)_{n\in\mathbb{Z}}$ of projective objects in $\mathcal{C}$ that agrees with a projective resolution of M in sufficiently high degree [8, Definition 1.1]. Following [13, p. 158], we define the Tate-Farrell Ext-functor $\overline{\operatorname{Ext}}_{\mathcal{C}}^{\bullet}(A,B)$ as the cohomology of the cochain complex $\operatorname{Hom}_{\mathcal{C}}(\overline{A}_\bullet, B)$ where we the Hom-functor is unenriched. Then we show

Lemma 1.7 *(= Lemma 3.5) The Tate-Farrell Ext-functors $\overline{\operatorname{Ext}}_{\mathcal{C}}^{\bullet}(A,-)$ are isomorphic to the completed unenriched Ext-functors $\widehat{\operatorname{Ext}}_{\mathcal{C}}^{\bullet}(A,-)$ as cohomological functors.*

Using F. T. Farrell's work, P. Symonds deduces in [36, p. 34] that Tate-Farrell cohomology exists for every profinite group having an open subgroup of finite cohomological dimension. In particular, complete cohomology for profinite groups generalises P. Symonds Tate-Farrell cohomology taking coefficients in profinite modules (Example 3.6).

Let us mention a few examples. In [35], P. Symonds computes the Tate-Farrell cohomology of the Morava stabiliser group S_{p-1} with coefficients in the moduli space E_{p-1} for odd primes p. However, there are groups for which their complete cohomology cannot be calculated by a complete resolution. For instance, a free abelian group of countable rank $\bigoplus_{n\in\mathbb{N}}\mathbb{Z}$, $GL_n(K)$ for K a subfield of the algebraic closure of $\mathbb{Q}$, the Thompson group F and the free product $\ast_{n\in\mathbb{N}}\mathbb{Z}^n$ do not admit complete resolutions [8, Example 5.3], [10, p. 119–120]. By [24, p. 297–298], the cohomology of the first three examples is isomorphic to their complete cohomology while all cohomology groups of the last group are distinct from its complete cohomology groups [20, p. 432]. The zeroeth complete cohomology group of this last example is calculated in [10, Corollary 2.4 and Theorem A]. In [11], F. Dembegioti calculates the zeroeth complete cohomology group of a class of discrete polycyclic groups, but also explains why one cannot determine a general formula calculating the zeroeth complete cohomology groups for all polycyclic groups. Lastly, if p is a prime number, then pro-p groups are a class of profinite groups for which there is a rich theory of their group cohomology. See for instance [33, Chapter I] or [37]. One example of a pro-p group that is also a pro-p ring are the p-adic integers $\mathbb{Z}_p$ which are studied in [39, Section 1.5]. These examples give rise to the following questions.

Question 1.8 *As in [29, Example 3.3.8(c)], let $\prod_{\mathbb{N}}\mathbb{Z}_p$ denote the free abelian pro-p group over $\mathbb{N}$. Is its cohomology $H^{\bullet}_{\mathbb{Z}_p}(\prod_{\mathbb{N}}\mathbb{Z}_p,-)$ isomorphic to its complete cohomology $\widehat{H}^{\bullet}_{\mathbb{Z}_p}(\prod_{\mathbb{N}}\mathbb{Z}_p,-)$ as a cohomological functor?*

Question 1.9 *In accordance with [28, p. 137–140], denote by $G = \coprod_{n\in\mathbb{N}\cup\{\infty\}} G_n$ the following free pro-p product over the one-point compactification of the integers $\mathbb{N}\cup\{\infty\}$. Set $G_n = \mathbb{Z}_p^n$ for $n\in\mathbb{N}$ and $G_\infty = \{1\}$. Can one compute the zeroeth complete cohomology $\widehat{H}^0_{\mathbb{Z}_p}(G,A)$ for an $\mathbb{Z}_p[[G]]$-module A?*

On the other hand, we obtain the following positive result generalising Proposition 3.9 of [19] from discrete groups to pro-p groups.

Lemma 1.10 *(= Lemma 3.9) Let G be a pro-p group. Then G is finite if and only if $H^0_{\mathbb{Z}_p}(G,-)$ is naturally isomorphic to $\widehat{H}^0_{\mathbb{Z}_p}(G,-)$ and $H^n_{\mathbb{Z}_p}(G,-)\not\cong\widehat{H}^n_{\mathbb{Z}_p}(G,-)$ for any $n\geq 0$.*

Lastly, we note that the terms of the canonical morphism of the Mislin completion of unenriched Ext-functors $\mathrm{Ext}^{\bullet}_{\mathrm{C}}(A,-) \to \widehat{\mathrm{Ext}}^{\bullet}_{\mathcal{C}}(A,-)$ fit into a long exact sequence relating three distinct cohomological functors (Lemma 3.11).

For the reader's convenience, we showcase in Section 2 the relevant constructions of Mislin completions that we have generalised in [15]. We establish several properties of completed Ext-functors including the above feature of their canonical morphism in Section 3. More specifically, we show that zeroeth completed unenriched Ext-functors detect whether an object in the domain category has finite projective dimension, that completed unenriched Ext-functors generalise Tate-Farrell Ext-functors and that complete unenriched group cohomology detects finiteness of pro-p groups. Thereafter, in Section 4, we prove a (partial) version of dimension shifting and an Eckmann-Shapiro type lemma. To pave the way for cohomology products, we provide an overview of external products for unenriched Ext-functors and cup products for unenriched group cohomology in Section 5. Then, in Section 6, we construct external and Yoneda products for completed unenriched Ext-functors and cup products for completed unenriched group cohomology. We conclude by proving various properties of these cohomology products in Section 7.

Notation and terminology

We adopt the convention that the natural numbers $\mathbb{N}$ include 1, but do not include 0. We write $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Moreover, we adopt B. Poonen's convention from [27] that a ring is an abelian group together with a totally associative product, meaning a binary associative relation admitting an identity element. In particular, ring homomorphisms are understood to map identity elements to identity elements. We use the symbol $\varinjlim$ only to denote direct limits. If $\mathcal{D}$ is a category, then $\varinjlim_{\mathcal{D}}$ denotes a direct limit in this category and if I is a directed set, then $\varinjlim_{i\in I}$ denotes a direct limit indexed over I. We write $\varinjlim_{k\in\mathbb{N}}(M_k, \mu_k)$ if $\mu_k : M_k \to M_{k+1}$ are the morphisms giving rise to the direct limit. Lastly, we use the same numbering to label diagrams and equations.

2 Outline of constructions

This section provides an overview of the constructions of Mislin completion we have generalised in [15]. In order to present these, let us axiomatically define cohomological functors. For two abelian categories $\mathcal{C}$, $\mathcal{D}$ a family of additive functors $(T^n : \mathcal{C} \to \mathcal{D})_{n\in\mathbb{Z}}$ is called a cohomological functor if it satisfies the following two axioms [21, p. 201–202].

Axiom 2.1 *For every $n \in \mathbb{Z}$ and short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, there is natural connecting homomorphism $\delta^n : T^n(C) \to T^{n+1}(A)$.*

Being natural means in this context that for every commuting diagram in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

with exact rows there is a commuting diagram

$$\begin{array}{ccc} T^n(A) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \big\downarrow{\scriptstyle T^n(g)} & & \big\downarrow{\scriptstyle T^{n+1}(f)} \\ T^n(A') & \xrightarrow{\delta^n} & T^{n+1}(C') \end{array}$$

in $\mathcal{D}$.

Axiom 2.2 *For every short exact sequence* $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ *in* $\mathcal{C}$ *there is a long exact sequence*

$$\dots \xrightarrow{T^{n-1}(\pi)} T^{n-1}(C) \xrightarrow{\delta^{n-1}} T^n(A) \xrightarrow{T^n(\iota)} T^n(B) \xrightarrow{T^n(\pi)} T^n(C) \xrightarrow{\delta^n} T^{n+1}(A) \xrightarrow{T^{n+1}(\iota)} \dots$$

Assume that $(T^\bullet, \delta^\bullet)$ and $(U^\bullet, \varepsilon^\bullet)$ are cohomological functors from $\mathcal{C}$ to $\mathcal{D}$. Then a family of natural transformations $(\nu^n : T^n \to U^n)_{n\in\mathbb{Z}}$ is a morphism of cohomological functors if it satisfies the axiom [21, p. 202]

Axiom 2.3 *For every* $n \in \mathbb{Z}$ *and short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathcal{C}$*, the following square commutes:*

$$\begin{array}{ccc} T^n(C) & \xrightarrow{\delta^n} & T^{n+1}(A) \\ \big\downarrow{\scriptstyle \nu^n} & & \big\downarrow{\scriptstyle \nu^{n+1}} \\ U^n(C) & \xrightarrow{\varepsilon^n} & U^{n+1}(A) \end{array}$$

Let us generalise G. Mislin's Definition 2.1 from [24].

Definition 2.4 (Mislin completion) *Let* $(T^\bullet, \delta^\bullet)$ *be a cohomological functor from* $\mathcal{C}$ *to* $\mathcal{D}$*. Then its Mislin completion is a cohomological functor* $(\widehat{T}^\bullet, \widehat{\delta}^\bullet)$ *from* $\mathcal{C}$ *to* $\mathcal{D}$ *together with a morphism* $\nu^\bullet : T^\bullet \to \widehat{T}^\bullet$ *satisfying the following two conditions:*
1. *For every* $n \in \mathbb{Z}$ *and every projective* $P \in \mathrm{obj}(\mathcal{C})$ *we have* $\widehat{T}^n(P) = 0$.
2. *If* $(U^\bullet, \varepsilon^\bullet)$ *is any cohomological functor vanishing on projectives, then each morphism* $T^\bullet \to U^\bullet$ *factors uniquely as* $T^\bullet \xrightarrow{\nu^\bullet} \widehat{T}^\bullet \to U^\bullet$.

Hence, if $(U^\bullet, \varepsilon^\bullet)$ is another Mislin completion of $(T^\bullet, \delta^\bullet)$, then there is an isomorphism $\mu^\bullet : \widehat{T}^\bullet \to U^\bullet$, meaning that $\mu^n(M) : \widehat{T}^n(M) \to U^n(M)$ is an isomorphism for any $n \in \mathbb{Z}$ and $M \in \mathrm{obj}(\mathcal{C})$ [21, p. 202]. This allows us to state G. Mislin's definition from [24, p. 297] in greater generality.

Definition 2.5 (Axiomatic, Mislin) *For any* $A \in \mathrm{obj}(\mathcal{C})$ *we extend the (enriched or unenriched) Ext-functors to cohomological functor by setting* $\mathrm{Ext}^n_{\mathcal{C}}(A, -) = 0$ *for* $n < 0$ *and define completed Ext-functors as the Mislin completion* $(\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, -), \widehat{\delta}^\bullet)$*. Analogously, if* G *is a group object in* $\mathcal{C}$ *and* R *a ring object, then we extend group cohomology to a cohomological functor* $(H^\bullet_R(G, -), \delta^\bullet)$ *by imposing* $H^n_R(G, -) = 0$ *for* $n < 0$ *and define complete cohomology as the Mislin completion* $(\widehat{H}^\bullet_R(G, -), \widehat{\delta}^\bullet)$.

To ensure that complete cohomology exists, we introduce left satellite functors to present a construction due to G. Mislin. As we assume that $\mathcal{C}$ has enough projective objects, there is for any $M \in \mathrm{obj}(\mathcal{C})$ a short exact sequence $0 \to K \to P \to M \to 0$ in $\mathcal{C}$ with

P projective. For a cohomological functor $(T^\bullet, \delta^\bullet)$, we define the zeroeth left satellite functor of T^n as $S^0T^n := T^n$, the first left satellite functor as

$$S^{-1}T^n(M) := \mathrm{Ker}\big(T^n(K) \to T^n(P)\big)$$

and the k^{th} left satellite functor as $S^{-k}T^n := S^{-1}(S^{-k+1}T^n)$ for $k \geq 2$ [6, p. 36]. It is shown in [6, Section III.1] that left satellite functors do not depend on the choice of short exact sequence. Since they have been defined as kernels, it follows from Axiom 2.2 that $\delta^n : T^n \to T^{n+1}$ induces a morphism $\underline{\delta}^n : T^n(M) \to S^{-1}T^{n+1}(M)$ and therefore $S^{-k}\underline{\delta}^{n+k} : S^{-k}T^{n+k}(M) \to S^{-k-1}T^{n+k+1}(M)$ for any $k \in \mathbb{N}$ [21, p. 207–208]. Here, our assumption hits in that all countable direct limits exist in the codomain category $\mathcal{D}$ of $T^\bullet$.

Definition 2.6 *A partially ordered set $(I, \leq)$ is a directed set if for every $i, j \in I$ there is $k \in I$ such that $i, j \leq k$ [29, p. 1]. According to [29, p. 14], a diagram $\{D_i\}_{i\in I}$ in $\mathcal{D}$ indexed over a directed set is called a direct system in $\mathcal{D}$. More formally, I can be turned into a category whose objects are its elements and there is a unique morphism $i \to j$ whenever $i \leq j \in I$. Then a direct system is a covariant functor $I \to \mathcal{D}, i \mapsto D_i$. A direct limit $\varinjlim_{i\in I} D_i$ in $\mathcal{D}$ is a colimit of a direct system $\{D_i\}_{i\in I}$. Direct limits in $\mathcal{D}$ are called exact if for every direct system of short exact sequence $\{0 \to A_i \to B_i \to C_i \to 0\}_{i\in I}$ also*

$$0 \to \varinjlim_{i\in I} A_i \to \varinjlim_{i\in I} B_i \to \varinjlim_{i\in I} C_i \to 0$$

is a short exact sequence [34, Tag 079A].

Hence, we can extend G. Mislin's construction from [24, p. 293].

Definition 2.7 (Via satellite functors, Mislin) *The Mislin completion of a cohomological functor $(T^\bullet, \delta^\bullet)$ can be defined as $\widehat{T}^n(M) := \varinjlim_{k\in\mathbb{N}_0}(S^{-k}T^{n+k}(M), S^{-k}\underline{\delta}^{n+k})$ for any $M \in \mathrm{obj}(\mathcal{C})$ and $n \in \mathbb{Z}$. Accordingly, $\widehat{H}^n_R(G, M) := \varinjlim_{k\in\mathbb{N}_0}(S^{-k}H^{n+k}_R(G, M), S^{-k}\underline{\delta}^{n+k})$ is a definition of complete cohomology.*

In order that the above forms a cohomological functor, one needs to impose that all direct limits in $\mathcal{D}$ are exact. The reader might be aware that one can define cohomology of discrete groups and cohomology of profinite groups taking discrete coefficients by using that the respective category $\mathcal{C}$ has enough injectives [5, p. 61], [33, p. 9]. One could assume instead that $\mathcal{C}$ has enough injectives and in the category $\mathcal{D}$ all countable inverse limits exist and are exact. One could then perform a construction via right satellite functors dual to the above. This completion of group cohomology would have a universal property as a Mislin completion, except that it would vanish on injective objects instead of projective ones. However, cohomology of discrete groups as well as of profinite groups taking discrete coefficients already vanishes on injectives [5, p. 61], [33, p. 9]. Thus, such a completion would not yield Tate cohomology whence we assume that $\mathcal{C}$ has enough projectives instead of enough injectives.

Notation 2.8 *For the rest of the paper, $\mathcal{C}$ always denotes an abelian category with enough projectives and $\mathcal{D}$ an abelian category in which all countable direct limits exist and are exact.*

Let us go over to what we term the resolution construction that occurs in [7, Lemma B.3] and can be retrieved from page 299 in G. Mislin's paper [24]. If $(M_n)_{n\in\mathbb{N}_0}$ is a projective resolution of $M \in \mathrm{obj}(\mathcal{C})$, let us define $\widetilde{M}_0 := M$ and $\widetilde{M}_k := \mathrm{Ker}(M_{k-1} \to \widetilde{M}_{k-1})$ for $k \in \mathbb{N}$. This is called the k^{th} syzygy of $M_\bullet$ in the Gorenstein context [26, p. 89]. The choice of our notation is meant to reflect that our syzygies do not necessarily arise from a specific choice of projective resolution as in [21] and [24]. Since for every $k \in \mathbb{N}_0$ we have the short exact sequence $0 \to \widetilde{M}_{k+1} \to M_k \to \widetilde{M}_k \to 0$, there is a connecting homomorphism $\delta^{n+k} : T^{n+k}(\widetilde{M}_k) \to T^{n+k+1}(\widetilde{M}_{k+1})$ for every $n \in \mathbb{Z}$. Then the following definition makes it more apparent why Mislin completions vanish on projective objects.

Definition 2.9 (Resolutions, Mislin) *The Mislin completion of a cohomological functor $(T^\bullet, \delta^\bullet)$ can be defined as $\widehat{T}^n(M) := \varinjlim_{k\in\mathbb{N}_0}(T^{n+k}(\widetilde{M}_k), \delta^{n+k})$ for any $n \in \mathbb{Z}$ and $M \in \mathrm{obj}(\mathcal{C})$. Accordingly, $\widehat{H}^n_R(G, M) := \varinjlim_{k\in\mathbb{N}_0}(H^{n+k}_R(G, \widetilde{M}_k), \delta^{n+k})$ is a definition of complete cohomology.*

The next two constructions only give rise to completed unenriched Ext-functors. Let $A_\bullet$, $B_\bullet$ are projective resolutions of $A, B \in \mathrm{obj}(\mathcal{C})$ and let $\widetilde{f}_{n+k} : \widetilde{A}_{n+k} \to \widetilde{B}_k$ be a morphism for $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$ such that $n + k \geq 0$. Then we can write the commuting diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{A}_{n+k+1} & \longrightarrow & A_{n+k} & \longrightarrow & \widetilde{A}_{n+k} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{f}_{k+1}} & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle \widetilde{f}_k} & & \\
0 & \longrightarrow & \widetilde{B}_{k+1} & \longrightarrow & B_{k+1} & \longrightarrow & \widetilde{B}_k & \longrightarrow & 0
\end{array}$$

whose terms arise as follows. Since the bottom row is exact and the term A_{n+k} projective, there is a lift f_k of $\widetilde{f}_k$ making the right-hand square commute. Because $\widetilde{B}_{k+1} \to B_{k+1}$ is a kernel, there is a morphism $\widetilde{f}_{k+1}$ making the left-hand side commute. If $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denotes the (unenriched) Hom-functor in $\mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ is an abelian group by virtue of $\mathcal{C}$ being an abelian category. We define $\mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ to be the subgroup of $\mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ consisting of all morphisms factoring through a projective object and write the quotient as $[\widetilde{A}_{n+k}, \widetilde{B}_k]_{\mathcal{C}} := \mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)/\mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k)$ [21, p. 203]. As in the case of modules over a ring covered by [21, p. 204], we prove that

$$\begin{aligned}
t_{\widetilde{A}_{n+k}, \widetilde{B}_k} &: [\widetilde{A}_{n+k}, \widetilde{B}_k]_{\mathcal{C}} \to [\widetilde{A}_{n+k+1}, \widetilde{B}_{k+1}]_{\mathcal{C}}, \\
&\widetilde{f}_k + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k) \mapsto \widetilde{f}_{k+1} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_{n+k+1}, \widetilde{B}_{k+1})
\end{aligned}$$

is a well defined homomorphism. Through this we generalise the following definition from D. J. Benson and J. F. Carlson's paper [3, p. 109].

Definition 2.10 (Naïve construction, Benson & Carlson) *We can define the n^{th} completed unenriched Ext-functor as $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B) := \varinjlim_{k\in\mathbb{N}_0, n+k\geq 0}([\widetilde{A}_{n+k}, \widetilde{B}_k]_{\mathcal{C}}, t_{\widetilde{A}_{n+k}, \widetilde{B}_k})$ for $n \in \mathbb{Z}$. In particular, if $R_\bullet$ is a projective $R[G]$-resolution of $R \in \mathrm{obj}(\mathcal{C}_{R,G})$, we can define complete unenriched cohomology as $\widehat{H}^n_R(G, B) := \varinjlim_{k\in\mathbb{N}_0, n+k\geq 0}([\widetilde{R}_{n+k}, \widetilde{B}_k]_{\mathcal{C}}, t_{\widetilde{R}_{n+k}, \widetilde{B}_k})$.*

Lastly, we present what we call the hypercohomology construction of complete cohomology. We define the chain complex $(A'_n)_{n\in\mathbb{Z}}$ by $A'_n = A_n$ for $n \geq 0$ and $A'_n = 0$ for $n < 0$ and similarly $(B'_n)_{n\in\mathbb{Z}}$ [21, p. 209]. Define the hypercohomology complex $(\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n, d^n)_{n\in\mathbb{Z}}$ by having n-cochains $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n = \prod_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A'_{k+n}, B'_k)$.

To ease notation in the following, we view abelian groups as $\mathbb{Z}$-modules. If we denote by $a_n : A'_n \to A'_{n-1}$ and $b_n : B'_n \to B'_{n-1}$ the differentials induced from the respective projective resolution, we define for $n \in \mathbb{Z}$ the differential

$$\begin{aligned} d^n : \mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n &\to \mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n+1} \\ (\varphi_{n+k})_{k\in\mathbb{Z}} &\mapsto (b_{k+1} \circ \varphi_{n+k+1} - (-1)^n \varphi_{n+k} \circ a_{n+k+1})_{k\in\mathbb{Z}} \end{aligned} \tag{2.1}$$

Let us define the bounded complex $\mathrm{Bdd}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n\in\mathbb{Z}}$ as the subcomplex of $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n\in\mathbb{Z}}$ given by $\mathrm{Bdd}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n = \bigoplus_{k\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A'_{k+n}, B'_k)$. Now we define the Vogel complex $\mathrm{Vog}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_{n\in\mathbb{Z}}$ as the quotient complex $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n / \mathrm{Bdd}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_n$ [21, p. 209]. By this, we generalise Definition 1.2 from F. Goichot's paper [16] where he attributes it to P. Vogel on page 39.

Definition 2.11 (Hypercohomology, Vogel) *For $n \in \mathbb{Z}$ we can define the n^{th} completed unenriched Ext-functor as $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B) := H^n((\mathrm{Vog}_{\mathcal{C}}(A'_\bullet, B'_\bullet)_k, d^k)_{k\in\mathbb{Z}})$. We can thus define complete unenriched cohomology as $\widehat{H}^n_R(G, M) := H^n((\mathrm{Vog}_R(R'_\bullet, B'_\bullet)_k, d^k)_{k\in\mathbb{Z}})$.*

Let us remark why it is unlikely that the previous two constructions could yield Mislin completions of more general enriched Ext-functors. For the naïve construction, one aims to find a morphism $\mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k}, \widetilde{B}_k) \to \mathrm{Hom}_{\mathcal{C}}(\widetilde{A}_{n+k+1}, \widetilde{B}_{k+1})$. If one tries to lift along the morphisms induced by $A_{n+k} \to \widetilde{A}_{n+k}$ and $B_{n+k} \to \widetilde{B}_{n+k}$, one requires that $\mathrm{Hom}_{\mathcal{C}}(A_{n+k}, -)$ preserves epimorphisms. For unenriched Hom-functors, this role is exactly played by projective objects [38, Lemma 2.2.3]. For the hypercohomology construction we require that the coproduct $\mathrm{Bdd}_{\mathcal{C}}(A'_\bullet, B'_\bullet)$ maps into the product $\mathrm{Hyp}_{\mathcal{C}}(A'_\bullet, B'_\bullet)$. Moreover, as can be seen in [15, Subsection 6.1], the cohomology groups of the Vogel complex $\mathrm{Vog}_{\mathcal{C}}(A'_\bullet, B'_\bullet)$ correspond exactly to so-called almost chain maps modulo almost chain homotopy. However, this implies that the objects $\mathrm{Hom}_{\mathcal{C}}(A_p, B_q) \in \mathrm{obj}(\mathcal{D})$ describe morphisms of the form $A_p \to B_q$ in $\mathcal{C}$.

3 Completed Ext-functors and canonical morphisms

In this section, we establish several properties of completed Ext-functors and one for their canonical morphism. First, we show that zeroeth completed unenriched Ext-functors detect whether an object in the domain category has finite projective dimension. After providing results on when the terms of a cohomological functors agree with the ones of its Mislin completion, we show that completed unenriched Ext-functors generalise Tate-Farrell Ext-functors. We then move on to demonstrate that complete unenriched group cohomology detects when a pro-p group is finite. Lastly, we prove that the terms of the canonical morphism from unerniched Ext-functors to their Mislin completion fit into a long exact sequence relating three distinct cohomological functors.

Recall from Section 1 that an object in a category is said to have finite projective dimension if it admit a projective resolution of finite length. Remember that group cohomology as well as complete cohomology can be defined as a specific (completed) Ext-functor. Then we re-establish Lemma 4.2.3 from [21] in greater generality where we refer the reader to [15, Section 7] for an implementation of complete cohomology into condensed mathematics.

Lemma 3.1 *1. Assume that $T^\bullet : \mathcal{C} \to \mathcal{D}$ is a cohomological functor where $\mathcal{C}$ has enough projectives and in $\mathcal{D}$ all countable direct limits exist and are exact. If*

$M \in \mathrm{obj}(\mathcal{C})$ has finite projective dimension, then $\widehat{T}^n(M) = 0$ for every $n \in \mathbb{Z}$. In particular, if every object in $\mathcal{C}$ has finite projective dimension such as in a category of modules over a ring of finite global dimension, then $\widehat{T}^\bullet = 0$ for any cohomological functor $T^\bullet$.

2. *If one takes* ***enriched*** *Ext-functors $\mathrm{Ext}^n_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathcal{D}$ with $A \in \mathrm{obj}(\mathcal{C})$ of finite projective dimension, then $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -) = 0$ for every $n \in \mathbb{Z}$. In particular, complete cohomology $\widehat{H}^\bullet_R(G, M)$ vanishes if the group object G has finite cohomological dimension or $M \in \mathcal{C}_{R,G}$ has finite projective dimension. As this applies to any condensed ring R and any condensed group G, this holds for any $T1$ topological group by taking its condensate.*

Proof If M has finite projective dimension, then there is $m \in \mathbb{N}_0$ such that $\widetilde{M_k} = 0$ for any $k \geq m$. In particular, $T^{n+k}(\widetilde{M_k}) = 0$ for any $k \geq m$ and $\widehat{T}^n(M) = 0$ according to the resolution construction. On the other hand, if A has finite projective dimension, then there $m' \in \mathbb{N}_0$ such that $\mathrm{Ext}^{n+k}_{\mathcal{C}}(A, -) = 0$ for any $n + k \geq m'$. We conclude as before that $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -) = 0$. □

This allows us to re-establish the following version of a theorem that appears in the literature as [2, Proposition IX.1.3], [17, Theorem 4.11] and [18, Theorem 3.10].

Theorem 3.2 *If $\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, -) : \mathcal{C} \to$* **Ab** *denote completed unenriched Ext-functors for $A \in \mathrm{obj}(\mathcal{C})$, then the following are equivalent.*

1. *The object A has finite projective dimension.*
2. *$\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -) = \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(-, A) = 0$ for any $n \in \mathbb{Z}$.*
3. *$\widehat{\mathrm{Ext}}^0_{\mathcal{C}}(A, A) = 0$.*

In particular, the zeroeth complete cohomology group detects whether a group has finite cohomological dimension. This applies to any condensed group and thus to any $T1$ topological group.

Given the torsion-theoretic framework in which A. Beligiannis and I. Reiten work, this theorem follows from their definitions in [2]. On the other hand, S. Guo and L. Liang in [17] as well as J. Hu et al. in [18] prove this theorem using the hypercohomology construction. For completeness, we generalise a proof by the naïve construction found in [21, p. 205].

Proof As the second statement implies the third, it suffices by Lemma 3.1 to prove that A has finite projective dimension if $\widehat{\mathrm{Ext}}^0_{\mathcal{C}}(A, A) = 0$. By the construction of direct limits of abelian groups from [25, p. 261], there is $k \in \mathbb{N}_0$ such that $\mathrm{id}_{\widetilde{A}_k} + \mathcal{P}_{\mathcal{C}}(\widetilde{A}_k, \widetilde{A}_k) = 0$ in $[\widetilde{A}_k, \widetilde{A}_k]_{\mathcal{C}}$. Because unenriched Hom-functors are used, we conclude that $\mathrm{id}_{\widetilde{A}_k}$ factors through a projective. In particular, $\widetilde{A}_k$ is projective and A has finite projective dimension. □

The following generalisation of Lemma 2.3 in [24] provides a criterion determining when a cohomology group T^n of a cohomological functor agrees the corresponding cohomology group $\widehat{T}^n$ of its Mislin completion

Proposition 3.3 *Let $T^\bullet : \mathcal{C} \to \mathcal{D}$ be a cohomological functor. Then for $n \in \mathbb{Z}$ the following are equivalent.*

1. *For any $k \geq n$ the functor T^k vanishes on projective objects.*
2. *For any $k \geq n$ the functor T^k is naturally isomorphic to $\widehat{T}^k$.*

Proof As the second assertion implies the first, assume that T^k vanishes on projective objects for any $k \geq n$. Let $T^\bullet \to U^\bullet$ be a morphism of cohomological functors where $U^\bullet$ vanishes on projective objects. According to the proof of Lemma 3.2 in [15], this morphism factors uniquely as $T^\bullet \to T^\bullet\langle n\rangle \to U^\bullet$. Because $T^\bullet\langle n\rangle$ vanishes on projectives by our assumptions, we infer that it is a Mislin completion. In particular, we observe for any $k \geq n$ that $T^k = T^k\langle n\rangle \cong \widetilde{T}^k$. □

Let us present a criterion for when a cohomological functor vanishing on projectives is a Mislin completion. The following is a generalisation of Lemma 2.4 in [24].

Proposition 3.4 *Let $T^\bullet, V^\bullet : \mathcal{C} \to \mathcal{D}$ be cohomological functors where $V^\bullet$ vanishes on projective objects. Assume that $\Phi^\bullet : T^\bullet \to V^\bullet$ is a morphism such that there is $n \in \mathbb{Z}$ with the property that Φ^k is an isomorphism for any $k \geq n$. Then $V^\bullet$ together with $\Phi^\bullet$ forms a Mislin completion of $T^\bullet$.*

Proof Note that T^k vanishes on projectives for any $k \geq n$. It follows from the proof of Proposition 3.3 that there is a unique factorisation $\Phi^\bullet : T^\bullet \to T^\bullet\langle n\rangle \xrightarrow{\Psi^\bullet} V^\bullet$ where $T^\bullet\langle n\rangle$ is the Mislin completion of $T^\bullet$. According to the proof of Lemma 3.2 in [15], $\Psi^k = \Phi^k$ is an isomorphism for any $k \geq n$. Therefore, $\Psi^\bullet$ is an isomorphism of cohomological functors by Lemma 3.1 in [15] and $V^\bullet$ is a Mislin completion. □

Moving forward, a complete resolution of $A \in \mathrm{obj}(\mathcal{C})$ is an acyclic chain complex $(\overline{A}_n)_{n\in\mathbb{Z}}$ of projectives such that there is $n \in \mathbb{N}_0$ for which $(\overline{A}_k)_{k\geq n}$ agrees with a projective resolution of A and such that for any projective $P \in \mathrm{obj}(\mathcal{C})$ the cochain complex $\mathrm{Hom}_{\mathcal{C}}(\overline{A}_\bullet, P)$ is acyclic where $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denotes the unenriched Hom-functor [8, Definition 1.1]. In accordance with [13, p. 158], we define $\overline{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, B) := H^\bullet(\mathrm{Hom}_{\mathcal{C}}(\overline{A}_\bullet, B))$. By [8, Lemma 2.4], any two complete resolutions are chain homotopic and thus, Tate-Farrell Ext-functors do not depend on the choice of a complete resolution. In [17, Proposition 4.15], S. Guo and L. Liang demonstrate that the Tate-Farrell Ext-functor $\overline{\mathrm{Ext}}^n_{\mathcal{C}}(A, -)$ is naturally isomorphic to $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, -)$ for every $n \in \mathbb{Z}$. Then we obtain the following version of Theorem 4.6 from [18]

Lemma 3.5 *The Tate-Farrell Ext-functors $\overline{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, -)$ are isomorphic to the completed unenriched Ext-functors $\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, -)$ as cohomological functors.*

Apart from the proof found in [18], one can demonstrate this by using the proof of Theorem 1.2 in [8] together with Proposition 3.4.

Example 3.6 *P. Symonds constructs in [36, p. 34] Tate-Farrell cohomology for a profinite group with an open subgroup of finite cohomological dimension taking coefficients in profinite modules. Thus, complete cohomology generalises his Tate-Farrell cohomology.*

As mentioned in Section 1, for a prime number p the pro-p groups are a class of profinite groups for which there is a rich theory of their group cohomology. One example of a pro-p group that is also a pro-p ring are the p-adic integers $\mathbb{Z}_p$ which are studied in [39, Section 1.5]. If G is a profinite group, R a profinite commutative ring and M a profinite $R[[G]]$-module, recall the convention from [9, p. 235] seen in Section 1 that $H^n_R(G, M)$ is a $U(R)$-module where $U(R)$ denotes the ring R without its topology. To see that complete cohomology can determine whether a pro-p group is finite, we need the following description of $H^0_R(G, M)$ that has been already determined for discrete $R[[G]]$-modules M in [29, Lemma 6.2.1] and that can be retrieved from [39, p. 165].

Proposition 3.7 *Let G be a profinite group and R a profinite commutative ring. As any profinite $R[[G]]$-module M comes with a continuous G-action by [29, Proposition 5.3.6], define its $U(R)$-submodule*

$$M^G := \{m \in M \mid \forall g \in G : g \cdot m = m\} .$$

Then $H^0_R(G, M) \cong \mathrm{Hom}_{R[[G]]}(R, M) \cong M^G$.

Proof Because group cohomology is defined as a derived functor of $\mathrm{Hom}_{R[[G]]}(-, M)$, we see that $H^0_R(G, M) = \mathrm{Hom}_{R[[G]]}(R, M)$ by [38, p. 50]. Let $f : R \to M$ be a $R[[G]]$-module homomorphism. Remember that we follow B. Poonen's convention that any ring has a unit. As the $R[[G]]$-module R is endowed with a trivial G-action, we see that $f(1) \in M^G$. Moreover, the homomorphism f is uniquely determined by the value $f(1)$. Thus there is an injective $U(R)$-module homomorphism

$$\mathrm{ev} : \mathrm{Hom}_{R[[G]]}(R, M) \to M^G, f \mapsto f(1) .$$

Hence, assume that $m \in M^G$. Because M can be seen as a R-module, this gives rise to an R-module homomorphism $f : R \to M, r \mapsto r \cdot m$ that is also a continuous map between $R[[G]]$-modules. Given that R and z are fixed by the G-action, the homomorphism f is G-equivariant and thus an $R[G]$-module homomorphism where $R[G]$ denotes the usual group ring. The fact that $R[G]$ is dense in $R[[G]]$ by [29, Lemma 5.3.5] and that f is continuous imply that it is an $R[[G]]$-module homomorphism. In particular, ev is an isomorphism. □

Let us restrict our attention to the profinite ring $R = \mathbb{Z}_p$. By [29, p. 171], the completed group ring can be given by $\mathbb{Z}_p[[G]] = \varprojlim_{U \trianglelefteq G \text{ open}} \mathbb{Z}_p[G/U]$. In the case of the $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p[G/U]$ with $U \trianglelefteq G$ open, we can provide a more explicit description of $H^0_{\mathbb{Z}_p}(G, \mathbb{Z}_p[G/U])$.

Proposition 3.8 *For any profinite group G and any open subgroup $U \trianglelefteq G$ we have*

$$\mathbb{Z}_p[G/U]^G = \left\{(b_y)_{y \in G/U} \in \prod_{y \in G/U} \mathbb{Z}_p \mid \exists b \in \mathbb{Z}_p \, \forall y \in G/U : b_y = b\right\} .$$

If $V \trianglelefteq U \trianglelefteq G$ is another open subgroup and $G/V \to G/U$ denotes the projection homomorphism, then the induced homomorphism is given by

$$\mathbb{Z}_p[G/V] \to \mathbb{Z}_p[G/U], (b)_{y \in G/V} \to (|U : V| \cdot b)_{y' \in G/U} .$$

Proof Let $(b_y)_{y\in G/U} \in \mathbb{Z}_p[G/U]^G$. Then $(b_{g\cdot y})_{y\in G/U} = g\cdot(b_y)_{y\in G/U} = (b_y)_{y\in G/U}$ for any $g \in G$. According to [12, Section 0.9] the p-adic integers $\mathbb{Z}_p$ can be embedded into the p-adic rationals $\mathbb{Q}_p$ which are a topological field. In particular, one can embed $\mathbb{Z}_p[G/U]$ into the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p^{|G:U|}$. This implies that $b_y = b_{g\cdot y}$ for any $y \in G/U$ and $g \in G$, proving the first assertion. For the second assertion, if $f : X \to Y$ is a map between two finite discrete spaces, then the corresponding induced homomorphism is given by

$$\mathbb{Z}_p[f] : \mathbb{Z}_p[X] \to \mathbb{Z}_p[Y],\ (c_x)_{x\in X} \mapsto \Big(\sum_{x\in f^{-1}(y)} c_x\Big)_{y\in Y}.$$

Note now that the kernel of the group homomorphism $G/V \to G/U$ is given by U/V. □

This allows us to generalise Proposition 3.9 of [19] from discrete groups to pro-p groups.

Lemma 3.9 *Let G be a pro-p group. Then G is finite if and only if $H^0_{\mathbb{Z}_p}(G,-)$ is naturally isomorphic to $\widehat{H}^0_{\mathbb{Z}_p}(G,-)$ and $H^n_{\mathbb{Z}_p}(G,-) \ncong \widehat{H}^n_{\mathbb{Z}_p}(G,-)$ for any $n \geq 0$.*

Proof Given that G. Mislin states in [24] that complete cohomology agrees with Tate cohomology for a finite group G, one assertion follows from [1, p. 78–79] as we have seen at the start of Section 1. Assume that $H^0_{\mathbb{Z}_p}(G,-)$ is naturally isomorphic to $\widehat{H}^0_{\mathbb{Z}_p}(G,-)$ and that $H^n_{\mathbb{Z}_p}(G,-) \ncong \widehat{H}^n_{\mathbb{Z}_p}(G,-)$ for any $n \geq 0$. By Proposition 3.3, this is equivalent to saying that there is a projective $\mathbb{Z}_p[[G]]$-module P such that $H^0_{\mathbb{Z}_p}(G,P) \neq 0$. According to [29, Lemma 5.2.5], P is a quotient of a free profinite module $(\mathbb{Z}_p[[G]])[[X]]$ on a profinite space X where the quotient homomorphism has a (continuous) section. In particular, $H^0_{\mathbb{Z}_p}(G,(\mathbb{Z}_p[[G]])[[X]]) \neq 0$. By [29, Proposition 5.2.2], $(\mathbb{Z}_p[[G]])[[X]] = \varprojlim_{i\in I}(\mathbb{Z}_p[[G]])[X_i]$ where $X = \varprojlim_{i\in I} X_i$ with every X_i a finite discrete space. Denote the corresponding $\mathbb{Z}_p[[G]]$-homomorphisms by $f_{i,j} : (\mathbb{Z}_p[[G]])[X_j] \to (\mathbb{Z}_p[[G]])[X_i]$. We conclude from [38, p. 50] that $H^0_{\mathbb{Z}_p}(G,-) = \mathrm{Hom}_{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p,-)$ where Hom-functors of this form preserve limits according to [22, p. 116]. In particular, $\varprojlim_{i\in I} H^0_{\mathbb{Z}_p}(G,(\mathbb{Z}_p[[G]])[X_i]) \neq 0$ which is an inverse limit of $U(\mathbb{Z}_p)$-modules by our convention. If we use [29, p. 2–3] and Proposition 3.7, we can describe it as the submodule of the product $\prod_{i\in I} H^0_{\mathbb{Z}_p}(G,(\mathbb{Z}_p[[G]])[X_i])$ given by

$$\Big\{(a_i)_{i\in I} \in \prod_{i\in I} H^0_{\mathbb{Z}_p}(G,(\mathbb{Z}_p[[G]])[X_i]) \mid \forall i \leq j : f_{i,j}(a_j) = a_i\Big\}.$$

Thus, there is $i \in I$ such that $0 \neq a_i \in H^0_{\mathbb{Z}_p}(G,(\mathbb{Z}_p[[G]])[X_i])$. As is noted in [29, p. 167], $(\mathbb{Z}_p[[G]])[X_i] = \prod_{x\in X_i}\mathbb{Z}_p[[G]]$ which implies that $H^0_{\mathbb{Z}_p}(G,\mathbb{Z}_p[[G]]) \neq 0$. Assume by contradiction that G is infinite. Because the completed group ring can be given as $\mathbb{Z}_p[[G]] = \varprojlim_{U\trianglelefteq G \text{ open}} \mathbb{Z}_p[G/U]$ by [29, p. 171], we may conclude the existence of a nonzero element $b_{U_0} \in H^0_{\mathbb{Z}_p}(G,\mathbb{Z}_p[G/U_0])$ as before. Moreover, because G is infinite, we can also conclude that there is a countable sequence of open subgroups $U_{n+1} \precneqq U_n \trianglelefteq G$ and of elements $b_{U_n} \in H^0_{\mathbb{Z}_p}(G,\mathbb{Z}_p[G/U_n])$ such that for any $m \leq n \in \mathbb{N}_0$ the projection homomorphism $\mathbb{Z}_p[G/U_n] \to \mathbb{Z}_p[G/U_m]$ maps b_{U_n} to b_{U_m}. It follows from Proposition 3.8 that there is a $b \in \mathbb{Z}_p$ such that $b_{U_0} = (b)_{x\in G/U_0} \in \prod_{x\in G/U_0}\mathbb{Z}_p$. Since G is assumed to be a pro-p group, it also follows that b is divisible by arbitrarily large powers of p. However, this is impossible by [39, p. 26–27] and thus G is finite. □

In order to prove that the terms $\Phi^n : \mathrm{Ext}^n_{\mathcal{C}}(A,-) \to \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,-)$ fit into a long exact sequence relating three distinct cohomological functors, we generalise a remark from [21, p. 210] to

Proposition 3.10 *For unenriched Ext-functors* $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-):\mathcal{C}\to\mathbf{Ab}$ *the quotient map* $\mathrm{Hyp}_{\mathcal{C}}(A_\bullet,B_\bullet)_\bullet\to\mathrm{Vog}_{\mathcal{C}}(A_\bullet,B_\bullet)_\bullet$ *of chain complexes from the hypercohomology construction induces the canonical morphism* $\Phi^\bullet:\mathrm{Ext}^\bullet_{\mathcal{C}}(A,B)\to\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,B)$ *of cohomological functors from the definition of a Mislin completion.*

Proof Denote by $\Phi^\bullet:\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-)\to\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,-)$ the canonical morphism of cohomological functors established through the satellite functor construction in [15, Lemma 3.2]. If $\omega_\bullet:\widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,-)\to\mathrm{Ext}^\bullet_{Res\mathcal{C}}(A,-)$ is the isomorphisms of cohomological functors to the resolution construction from [15, Theorem 4.5], then $\omega_\bullet\circ\Phi^\bullet:\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-)\to\mathrm{Ext}^\bullet_{Res,\mathcal{C}}(A,-)$ is also a canonical morphism to the Mislin completion. By Diagram 4.6 of the proof of Lemma 4.1 in [15], each term $\omega_n\circ\Phi^n:\mathrm{Ext}^n_{\mathcal{C}}(A,B)\to\mathrm{Ext}^n_{Res,\mathcal{C}}(A,B)$ is a homomorphism to the direct limit occurring in the resolution construction as in Definition 2.9. If both $\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-)$ and $\mathcal{E}xt^\bullet_{Res,\mathcal{C}}(A,-)$ are taken as in [15, Definition 6.7], we denote by $\Psi^\bullet:\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-)\to\mathcal{E}xt^\bullet_{Res,\mathcal{C}}(A,-)$ an analogous morphism to the direct limit. Taking the isomorphisms of cohomological functors $\zeta_\bullet$ and $\zeta^\bullet$ also from [15, Definition 6.7], we see that the diagram

$$\begin{tikzcd}
\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-) \arrow[r, "\zeta_\bullet"] \arrow[d, "\omega_\bullet\circ\Phi^\bullet"] & \mathcal{E}xt^\bullet_{\mathcal{C}}(A,-) \arrow[d, "\Psi^\bullet"] \\
\mathrm{Ext}^{Res,\bullet}_{\mathcal{C}}(A,-) \arrow[r, "\zeta^\bullet"] & \mathcal{E}xt^\bullet_{Res,\mathcal{C}}(A,-)
\end{tikzcd}$$

commutes. Since $\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-)$ is a different description of the Ext-functors $\mathrm{Ext}^\bullet_{\mathcal{C}}(A,-)$ according to [15, Notation 6.1], $\Psi^\bullet$ also represents a canonical morphism to the Mislin completion. If $\vartheta^0_\bullet$ denote homomorphisms from the proof of [15, Lemma 6.9] and $\vartheta^\bullet$ the isomorphism of a cohomological functors from [15, Theorem 6.13], then the diagram

$$\begin{tikzcd}
\mathcal{E}xt^\bullet_{\mathcal{C}}(A,-) \arrow[d, "\Psi^\bullet"] \arrow[dr, "\vartheta^0_\bullet"] & \\
\mathcal{E}xt^\bullet_{Res,\mathcal{C}}(A,-) \arrow[r, "\vartheta^\bullet"] & \widehat{\mathcal{E}xt}^\bullet_{\mathcal{C}}(A,-)
\end{tikzcd}$$

is commutative. As before, we infer that $\vartheta^0_\bullet$ is a canonical morphism to the Mislin completion $\widehat{\mathcal{E}xt}^\bullet_{\mathcal{C}}(A,-)$. Note that we can restrict the quotient map of chain complexes

$$(\mathrm{Hyp}_{\mathcal{C}}(A_\bullet,B_\bullet)_n,d^n)_{n\in\mathbb{Z}}\to(\mathrm{Vog}_{\mathcal{C}}(A_\bullet,B_\bullet)_n,\overline{d}^n)_{n\in\mathbb{Z}}$$

to a homomorphism $\mathrm{Ker}(d^n)\to\mathrm{Ker}(\overline{d}^n)$ for any $n\in\mathbb{Z}$. Again by [15, Notation 6.1], the latter is equivalent to $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet,B_\bullet)\to\widehat{\mathrm{Hom}}_{\mathrm{Ch}(\mathcal{C})}(A[n]_\bullet,B_\bullet)$. By definition, this further descends to the homomorphism $\vartheta^0_n:\mathcal{E}xt^n_{\mathcal{C}}(A,B)\to\widehat{\mathcal{E}xt}^n_{\mathcal{C}}(A,B)$ as desired. $\square$

From this proposition we deduce

Lemma 3.11 *The short exact sequence of chain complexes*

$$0\to\mathrm{Bdd}_{\mathcal{C}}(A_\bullet,B_\bullet)_\bullet\to\mathrm{Hyp}_{\mathcal{C}}(A_\bullet,B_\bullet)_\bullet\to\mathrm{Vog}_{\mathcal{C}}(A_\bullet,B_\bullet)_\bullet\to 0$$

from the hypercohomology construction induces the long exact sequence

$$\ldots\to H^n(\mathrm{Bdd}_{\mathcal{C}}(A_\bullet,B_\bullet)_\bullet)\to\mathrm{Ext}^n_{\mathcal{C}}(A,B)\xrightarrow{\Phi^n}\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,B)\to H^{n+1}(\mathrm{Bdd}_{\mathcal{C}}(A_\bullet,B_\bullet)_\bullet)\to\ldots$$

where $\Phi^\bullet : \mathrm{Ext}^\bullet_{\mathcal{C}}(A,B) \to \widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A,B)$ is the canonical morphism from the definition of a Mislin completion. In particular, the terms Φ^n fit into a long exact sequence relating three distinct cohomological functors.

Remark 3.12 *This lemma is similar to Proposition 4.6 in S. Guo and L. Liang's paper [17]. Both results contain the same long exact sequence where our contribution lies in determining that it involves the terms of the canonical morphism to the Mislin completion.*

4 Dimension shifting and an Eckmann-Shapiro Lemma

One of the properties for Tate cohomology of finite groups that we have seen at the start of Section 1 is dimension shifting. As we only recover a partial version, we need an Eckmann-Shapiro type result in order to provide examples where dimension shifting holds in full generality.

For this, we formally define induction, coinduction and restriction as can be found in [5, p. 62–63 and p. 67]. Denote by $\mathcal{C}_{R,G}$ the subcategory of $\mathcal{C}$ of module objects over the group ring (object) $R[G]$. Let $\mathcal{C}$ contain a tensor product $-\otimes- : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ that is right-adjoint to an internal Hom-functor of the form $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$. Let H be a subgroup object of G. If we regard the group ring object $R[G]$ as an $(R[G], R[H])$-bimodule object, then we define restriction and coinduction as

$$\begin{aligned} \mathrm{Res}^G_H(-) &:= - \otimes_{R[G]} R[G] :\ \mathcal{C}_{R,G} \to \mathcal{C}_{R,H} \text{ and} \\ \mathrm{Coind}^G_H(-) &:= \underline{\mathrm{Hom}}_{\mathcal{C}_{R,H}}(R[G], -) :\ \mathcal{C}_{R,H} \to \mathcal{C}_{R,G}\,. \end{aligned}$$

These form adjoint functors in the sense that

$$\mathrm{Hom}_{\mathcal{C}_{R,H}}(\mathrm{Res}^G_H(A), B) \cong \mathrm{Hom}_{\mathcal{C}_{R,G}}(A, \mathrm{Coind}^G_H(B))$$

for every $A \in \mathcal{C}_{R,G}$ and $B \in \mathcal{C}_{R,H}$ where $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$ denotes the unenriched Hom-functor. Instead, if we regard $R[G]$ as an $(R[H], R[G])$-bimodule, then we define induction and restriction as $\mathrm{Ind}^G_H(-) := - \otimes_{R[H]} R[G] : \mathcal{C}_{R,H} \to \mathcal{C}_{R,G}$ and $\mathrm{Res}^G_H(-) := \underline{\mathrm{Hom}}_{\mathcal{C}_{R,G}}(R[G], -) : \mathcal{C}_{R,G} \to \mathcal{C}_{R,H}$. This redefinition of restriction allows us to conclude that $(\mathrm{Ind}^G_H(-), \mathrm{Res}^G_H(-))$ is a pair of adjoint functors. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves projectives if for every projective $P \in \mathrm{obj}(\mathcal{C})$ also $F(P) \in \mathrm{obj}(\mathcal{D})$ is projective.

Lemma 4.1 (Eckmann-Shapiro)

1. *If both $\mathrm{Res}^G_H(-)$ and $\mathrm{Coind}^G_H(-)$ are exact and preserve projectives, then for every $n \in \mathbb{Z}$, $A \in \mathrm{obj}(\mathcal{C}_{R,G})$ and $B \in \mathrm{obj}(\mathcal{C}_{R,H})$ we have that $\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,H}}(\mathrm{Res}^G_H(A), B) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A, \mathrm{Coind}^G_H(B))$ as unenriched completed Ext-functors. In particular, $\widehat{H}^n_R(H, B) \cong \widehat{H}^n_R(G, \mathrm{Coind}^G_H(B))$ if $A = R$.*

2. *If $\mathrm{Ind}^G_H(-)$ and $\mathrm{Res}^G_H(-)$ are exact and preserve projectives, then for every $n \in \mathbb{Z}$, $A \in \mathrm{obj}(\mathcal{C}_{R,H})$ and $B \in \mathrm{obj}(\mathcal{C}_{R,G})$ observe for unenriched completed Ext-functors that $\widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(\mathrm{Ind}^G_H(A), B) \cong \widehat{\mathrm{Ext}}^n_{\mathcal{C}_{R,G}}(A, \mathrm{Res}^G_H(B))$.*

Proof We only prove the assertion for the pair of adjoint functors $(\mathrm{Res}^G_H(-), \mathrm{Coind}^G_H(-))$ as the other one is analogous. For this we use the resolution construction. If $f : A \to C$

is a morphism in $\mathcal{C}_{R,H}$, then it follows from the naturality of the adjunction and the Five Lemma [6, Proposition I.1.1] that

$$\begin{aligned} \operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}_{R,H}}(\operatorname{Res}_H^G(f), B)\big) &\cong \operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}_{R,G}}(f, \operatorname{Coind}_H^G(B)\big) \text{ and} \\ \operatorname{Coker}\big(\operatorname{Hom}_{\mathcal{C}_{R,H}}(\operatorname{Res}_H^G(f), B)\big) &\cong \operatorname{Coker}\big(\operatorname{Hom}_{\mathcal{C}_{R,G}}(f, \operatorname{Coind}_H^G(B)\big) . \end{aligned}$$

To ease notation, write $\operatorname{Ker}(\operatorname{Res}(f), B)$ for $\operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}_{R,H}}(\operatorname{Res}_H^G(f), B)\big)$, $\operatorname{Ker}(f, \operatorname{Coind}(B))$ for $\operatorname{Ker}\big(\operatorname{Hom}_{\mathcal{C}_{R,H}}(f, \operatorname{Coind}_H^G(B)\big)$ and analogously $\operatorname{Coker}(\operatorname{Res}(f), B)$, $\operatorname{Coker}(f, \operatorname{Coind}(B))$. If $(A_l, a_l)_{l \in \mathbb{N}_0}$ is a projective resolution of A, then let $m, k \in \mathbb{N}_0$ with $m = n+k$ and consider the short exact sequence $0 \to \widetilde{B}_{k+1} \to B_k \to \widetilde{B}_k \to 0$. This gives rise to the diagram on the next page. The homomorphisms from the front to the back are isomorphisms arising from the above adjunction. By naturality of this adjunction, the front and back squares as well as the vertical squares running from front to back commute. Using the universal property of kernels and cokernels together with the naturality of the adjunction, we can deduce that also the horizontal squares running from front to back commute. Since we assume that restriction and coinduction are exact and preserve projectives, all rows are exact. Thus, the front and back side give rise to the connecting homomorphisms of the respective Ext-functors. In particular, this diagram gives rise to the commuting square

$$\begin{tikzcd} \operatorname{Ext}^m_{\mathcal{C}_{R,H}}(\operatorname{Res}_H^G(A), \widetilde{B}_k) \arrow[r, "\cong"] \arrow[d, "\delta^m"] & \operatorname{Ext}^m_{\mathcal{C}_{R,G}}(A, \operatorname{Coind}_H^G(\widetilde{B}_k)) \arrow[d, "\delta^m"] \\ \operatorname{Ext}^{m+1}_{\mathcal{C}_{R,H}}(\operatorname{Res}_H^G(A), \widetilde{B}_{k+1}) \arrow[r, "\cong"] & \operatorname{Ext}^{m+1}_{\mathcal{C}_{R,H}}(A, \operatorname{Coind}_H^G(\widetilde{B}_{k+1})) \end{tikzcd}$$

These squares form a direct system in whose direct limit we obtain the desired isomorphism. $\square$

For the following example, recall that we have defined profinite groups and their associated completed group rings in Section 1.

Example 4.2 *The conditions of Lemma 4.1 are satisfied in the following two instances.*

1. *Let $\mathcal{C}$ be the category of discrete R-modules for a discrete commutative ring R and let G be a discrete group together with a finite index subgroup H. The group rings $R[G]$ and $R[H]$ are taken to be discrete.*

2. *Let $\mathcal{C}$ be the category of profinite S-modules for a profinite commutative ring S and let K be a profinite group together with an open subgroup L. The group rings $S[[K]]$ and $S[[L]]$ are taken to be profinite.*

Proof Let us first clarify some points regarding the profinite setting. Since the subgroup L is open in K, it is of finite index [39, Lemma 0.3.1]. Although there is no internal Hom-functor for profinite modules in general, coinduction and restriction can be nevertheless defined because L is open subgroup of the profinite group K. Namely, for any $S[[L]]$-module M and any $S[[K]]$-module N, endowing $\operatorname{Coind}_L^K(M) = \operatorname{Hom}_{S[[L]]}(S[[K]], M)$ and $\operatorname{Res}_L^K(N) = \operatorname{Hom}_{S[[K]]}(S[[K]], N)$ with the compact-open topology turns them into profinite modules [37, p. 369–371]. As there is a tensor product for profinite modules [29, p. 177/191], induction and restriction can be defined in this case. By [4, Lemma 7.8] restriction is left adjoint to coinduction and induction left adjoint to restriction in the profinite case.

$$
\begin{tikzcd}
 & & & & & & & & \mathrm{Ext}^m_{\mathcal{C}_{R,G}}(A, \mathrm{Coind}_H^G(\widetilde{B}_k)) \arrow[dd] & \\
 & & & & & & \mathrm{Ext}^m_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A), \widetilde{B}_k) \arrow[urr] \arrow[dd] & & & \\
 & & & \mathrm{Coker}(a_m, \mathrm{Coind}(\widetilde{B}_{k+1})) \arrow[rr] \arrow[dd] & & \mathrm{Coker}(a_m, \mathrm{Coind}(B_k)) \arrow[rrr] \arrow[dd] & & & \mathrm{Coker}(a_m, \mathrm{Coind}(\widetilde{B}_k)) \arrow[r] \arrow[dd] & 0 \\
 & \mathrm{Coker}(\mathrm{Res}(a_m), \widetilde{B}_{k+1}) \arrow[urr] \arrow[rrr] \arrow[dd] & & & \mathrm{Coker}(\mathrm{Res}(a_m), B_k) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathrm{Coker}(\mathrm{Res}(a_m), \widetilde{B}_k) \arrow[urr] \arrow[r] \arrow[dd] & 0 & & \\
 & & 0 \arrow[r] & \mathrm{Ker}(a_{m+1}, \mathrm{Coind}(\widetilde{B}_{k+1})) \arrow[rr] \arrow[dd] & & \mathrm{Ker}(a_{m+1}, \mathrm{Coind}(B_k)) \arrow[rrr] & & & \mathrm{Ker}(a_{m+1}, \mathrm{Coind}(\widetilde{B}_k)) & \\
0 \arrow[r] & \mathrm{Ker}(\mathrm{Res}(a_{m+1}), \widetilde{B}_{k+1}) \arrow[urr] \arrow[rrr] \arrow[dd] & & & \mathrm{Ker}(\mathrm{Res}(a_{m+1}), B_k) \arrow[ur] \arrow[rr] & & \mathrm{Ker}(\mathrm{Res}(a_{m+1}), \widetilde{B}_k) \arrow[urr] & & & \\
 & & & \mathrm{Ext}^{m+1}_{\mathcal{C}_{R,G}}(A, \mathrm{Coind}_H^G(\widetilde{B}_{k+1})) & & & & & & \\
 & \mathrm{Ext}^{m+1}_{\mathcal{C}_{R,H}}(\mathrm{Res}_H^G(A), \widetilde{B}_{k+1}) \arrow[urr] & & & & & & & &
\end{tikzcd}
$$

As both the discrete and profinite case can be treated analogously from this point, we write $R[G]$ for either the discrete or profinite group ring of G over R and denote by H the finite index (open) subgroup. It follows from the description $\mathrm{Res}_H^G(-) = \mathrm{Hom}_{R[G]}(R[G], -)$ that restriction is exact. By [5, Proposition I.3.1], [29, Proposition 5.4.2] and [29, Proposition 5.7.1] the $R[H]$-module $R[G]$ is projective. We thus infer by [38, Lemma 2.2.3] that coinduction is exact and by [29, Proposition 5.5.3] and [38, p. 68] that induction is exact. According to the proof of [4, Corollary 7.9], induction preserves projectives. Because the (open) subgroup H is of finite index in G, $\mathrm{Ind}_H^G(M) \cong \mathrm{Coind}_H^G(M)$ for every $R[H]$-module M by [5, Proposition III.5.9] and [37, p. 371]. Due to this isomorphism and [38, Lemma 2.2.3], coinduction preserves projectives. □

Theorem 4.3 (Dimension shifting) *If $T^\bullet : \mathcal{C} \to \mathcal{D}$ is a cohomological functor, then for any $M \in \mathrm{obj}(\mathcal{C})$ there is $M^* \in \mathrm{obj}(\mathcal{C})$ such that $\widehat{T}^{n+1}(M^*) \cong \widehat{T}^n(M)$ for every $n \in \mathbb{Z}$. In addition, if there is a monomorphism $f : M \to N$ in $\mathcal{C}$ with $\widehat{T}^k(N) = 0$ for every $k \in \mathbb{Z}$, then $\widehat{T}^{n-1}(\mathrm{Coker}(f)) \cong \widehat{T}^n(M)$.*

Proof We consider the short exact sequence $0 \to \widetilde{M_1} \to M_0 \to M \to 0$ and set $M^* := \widetilde{M_1}$. Then the first assertion follows from Axiom 2.2 and Definition 2.4. The second assertion is deduced analogously. □

Example 4.4 *The conditions of Theorem 4.3 are satisfied in the following two instances.*

1. *Let $\mathcal{C}$ be the category of discrete R-modules for a discrete commutative ring R and let G be a discrete group together with a finite index subgroup H of finite cohomological dimension. The group rings $R[G]$ and $R[H]$ are taken to be discrete.*

2. *Let $\mathcal{C}$ be the category of profinite S-modules for a profinite commutative ring S and let K be a profinite group together with an open subgroup L of finite cohomological dimension. The group rings $S[[K]]$ and $S[[L]]$ are taken to be profinite.*

Proof As the discrete and profinite case can be treated largely analogously, we denote by $R[G]$ either the discrete or profinite group ring and by H the finite index (open) subgroup of finite cohomological dimension. Then by [5, Proposition III.5.9] and [37, p. 371] we deduce that there is a (continuous) injective $R[H]$-module homomorphism

$$\mathrm{const} : M \to \mathrm{Hom}_{R[H]}(R[G], \mathrm{Res}_H^G(M)) = \mathrm{Coind}_H^G(\mathrm{Res}_H^G(M))$$

defined by

$$\mathrm{const}(m) : R[G] \to \mathrm{Res}_H^G(M),\ x \mapsto \begin{cases} x \cdot m & \text{if } x \in R[H] \\ 0 & \text{if } x \notin R[H] \end{cases}$$

for every $m \in M$. Specifically in the profinite case, since const is a continuous R-module homomorphism that is invariant under the G-action, it is invariant under the ordinary group ring of G over R. Because the latter is dense in the profinite group ring $R[G]$ [29, Lemma 5.3.5], const is a continuous $R[G]$-module homomorphisms. Returning to full generality, we conclude by Lemma 3.1 and Example 4.2 that

$$\widehat{H}_R^n\big(G, \mathrm{Coind}_H^G(\mathrm{Res}_H^G(M))\big) \cong \widehat{H}_R^n(H, \mathrm{Res}_H^G(M)) = 0$$

for every $n \in \mathbb{Z}$. Specifically for discrete groups of finite virtual cohomological dimension this can be proved in a different manner. As mentioned in Section 1, Tate-Farrell cohomology can be defined by taking a complete resolution in this case [13, p. 157–158]. By [38, Lemma 2.3.4], $\widehat{H}_R^\bullet(G,-)$ vanishes on injectives where the category of discrete $R[G]$-modules has enough injectives [5, p. 61]. However, this argumentation does not pass through to profinite groups because the category of profinite $R[G]$-modules does not have enough injectives in general [37, p. 353]. □

5 External products of unenriched Ext-functors

In order to construct external and thus cup products of completed unenriched Ext-functors in the next section, we provide an outline of these products for unenriched Ext-functors in this section. As we have not found a treatment of external products when working in an abelian category $\mathcal{C}$ with enough projectives, we generalise them to this setting based on K. S. Brown's account of cup products in group cohomology found in [5, Chapter V].

The starting point of external products and thus of cup product are tensor products. We assume that there is a bi-additive functor suggestively written as $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. If $P, Q \in \mathrm{obj}(\mathcal{C})$ are projective, then we impose that also $P \otimes_{\mathcal{C}} Q$ is projective. We refer to the bifunctor $\otimes_{\mathcal{C}}$ as a tensor product. If $A, C \in \mathrm{obj}(\mathcal{C})$ and $(A_n, a_n)_{n\in\mathbb{N}_0}$, $(C_n, c_n)_{n\in\mathbb{N}_0}$ are corresponding projective resolutions, then we define the tensor product $A_\bullet \otimes_{\mathcal{C}} C_\bullet$ of $A_\bullet$ and $C_\bullet$ as the following chain complex. For $B_1, \dots, B_n \in \mathrm{obj}(\mathcal{C})$ there is a coproduct $\bigoplus_{i=1}^n B_i$ whose canonical monomorphisms we denote by $i_{n,B_k} : B_k \to \bigoplus_{i=1}^n B_i$ and canonical epimorphisms by $p_{n,B_k} : \bigoplus_{i=1}^n B_i \to B_k$; see [23, p. 250–251] for more detail. According to [23, p. 163] we form for $n \in \mathbb{N}_0$ the n-chains as

$$(A_\bullet \otimes_{\mathcal{C}} C_\bullet)_n := \bigoplus_{i=0}^{n} A_i \otimes_{\mathcal{C}} C_{n-i}$$

and the corresponding boundary map as

$$\begin{aligned} D_{n+1} := \sum_{k=0}^{n+1} \big(& i_{n+1,A_{k-1}\otimes_{\mathcal{C}} C_{n+1-k}} \circ (a_k \otimes_{\mathcal{C}} \mathrm{id}_{C_{n+1-k}}) \circ p_{n+2,A_k\otimes_{\mathcal{C}} C_{n+1-k}} \\ & + (-1)^k i_{n+1,A_k\otimes_{\mathcal{C}} C_{n-k}} \circ (\mathrm{id}_{A_k} \otimes_{\mathcal{C}} c_{n+1-k}) \circ p_{n+2,A_k\otimes_{\mathcal{C}} C_{n+1-k}} \big) : (A_\bullet \otimes_{\mathcal{C}} C_\bullet)_{n+1} \to (A_\bullet \otimes_{\mathcal{C}} C_\bullet)_n \end{aligned} \tag{5.1}$$

where it is understood that $a_0 = c_0 = 0$. Due to our assumptions, all terms of $A_\bullet \otimes_{\mathcal{C}} C_\bullet$ are projective. If $A'_\bullet$ and $C'_\bullet$ are different projective resolutions, then $A_\bullet \otimes_{\mathcal{C}} C_\bullet$ is chain homotopic to $A'_\bullet \otimes_{\mathcal{C}} C'_\bullet$ [23, p. 164]. Write $a : A_0 \to A$, $c : C_0 \to C$ for the augmentation maps. As can been inferred from [23, p. 164/229], one needs to impose conditions on $A_\bullet \otimes_{\mathcal{C}} C_\bullet$ to ensure that it is an acyclic complex and thus a projective resolution of $A \otimes_{\mathcal{C}} C$ with augmentation map $a \otimes_{\mathcal{C}} c : A_0 \otimes_{\mathcal{C}} C_0 \to A \otimes_{\mathcal{C}} C$.

External products arise from defining tensor products of chain maps. For this let $(B_n, b_n)_{\mathbb{N}_0}$, $(E_n, e_n)_{n\in\mathbb{N}_0}$ be chain complexes that vanish in negative degrees. As at the end of Section 2 when discussing the hypercohomology construction, we extend the projective resolutions $A_\bullet$, $C_\bullet$ to chain complexes indexed over $\mathbb{Z}$ by setting them to zero in negative degrees. For $m, n \in \mathbb{N}_0$ let $u_\bullet \in \mathrm{Hyp}_{\mathcal{C}}(A_\bullet, B_\bullet)_m$ and $v_\bullet \in \mathrm{Hyp}_{\mathcal{C}}(C_\bullet, E_\bullet)_n$ be cochains in the respective hypercohomology complex. Similar to the start of Subsection 6.1 in [15], we consider them as componentwise morphisms of the form $\{u_{m+k} : A_{m+k} \to B_k\}_{k\in\mathbb{Z}}$ and

$\{v_{n+l} : C_{n+l} \to E_l\}_{l\in\mathbb{Z}}$ which do not need to be chain maps. According to [5, p. 10], defining

$$(u_\bullet \otimes_{\mathcal{C}} v_\bullet)_k := \sum_{l=0}^{k} (-1)^{(m+l)n} i_{k,B_l\otimes_{\mathcal{C}}E_{k-l}} \circ (u_{m+l}\otimes_{\mathcal{C}} v_{n+k-l}) \circ p_{m+n+k,A_{m+l}\otimes_{\mathcal{C}}C_{n+k-l}} : \qquad (5.2)$$
$$(A_\bullet \otimes_{\mathcal{C}} C_\bullet)_{m+n+k} \to (B_\bullet \otimes_{\mathcal{C}} E_\bullet)_k \,.$$

for any $k \in \mathbb{N}_0$ yields a cochain $u_\bullet \otimes_{\mathcal{C}} v_\bullet \in \mathrm{Hyp}_{\mathcal{C}}(A_\bullet \otimes_{\mathcal{C}} C_\bullet, B_\bullet \otimes_{\mathcal{C}} E_\bullet)_{m+n}$. Since we have assumed that $\otimes_{\mathcal{C}}$ is bi-additive, this yields a homomorphism

$$\mathrm{Hyp}_{\mathcal{C}}(A_\bullet, B_\bullet)_m \otimes \mathrm{Hyp}_{\mathcal{C}}(C_\bullet, E_\bullet)_n \to \mathrm{Hyp}_{\mathcal{C}}(A_\bullet \otimes_{\mathcal{C}} C_\bullet, B_\bullet \otimes_{\mathcal{C}} E_\bullet)_{m+n},\ u_\bullet \otimes v_\bullet \mapsto u_\bullet \otimes_{\mathcal{C}} v_\bullet$$

where $\otimes$ denotes the tensor product of abelian groups. If we take the differential

$$d^{m+n} : \mathrm{Hyp}_{\mathcal{C}}(A_\bullet \otimes_{\mathcal{C}} C_\bullet, B_\bullet \otimes_{\mathcal{C}} E_\bullet)_{m+n} \to \mathrm{Hyp}_{\mathcal{C}}(A_\bullet \otimes_{\mathcal{C}} C_\bullet, B_\bullet \otimes_{\mathcal{C}} E_\bullet)_{m+n+1}$$

as in Equation 2.1, then it is also noted in the above source that

$$d^{m+n}(u_\bullet \otimes_{\mathcal{C}} v_\bullet) = (d^m u_\bullet) \otimes_{\mathcal{C}} v_\bullet + (-1)^m u_\bullet \otimes_{\mathcal{C}} (d^n v_\bullet)\,. \qquad (5.3)$$

Let $B, E \in \mathrm{obj}(\mathcal{C})$. Similar to the start of Subsection 6.1 in [15], we may consider the chain complex $i(B)_\bullet$ whose only nonzero terms is B in degree 0 and similarly $i(E)_\bullet$. If we define unenriched Ext-functors as derived functors of unenriched Hom-functors, then $\mathrm{Ext}^n_{\mathcal{C}}(A, B) = H^n(\mathrm{Hyp}_{\mathcal{C}}(A_\bullet, i(B)_\bullet))$ by definition of the corresponding differentials. We assume now explicitly that $A_\bullet \otimes_{\mathcal{C}} C_\bullet$ is a projective resolution of $A \otimes_{\mathcal{C}} C$. By this and Equation 5.3, the tensor product of cochains as in Equation 5.2 descends to a well defined homomorphism

$$\vee : \mathrm{Ext}^m_{\mathcal{C}}(A, B) \otimes \mathrm{Ext}^n_{\mathcal{C}}(C, E) \to \mathrm{Ext}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E) \qquad (5.4)$$
$$\big(u + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_m, B))\big) \otimes \big(v + \mathrm{Im}(\mathrm{Hom}(c_n, E))\big) \mapsto (-1)^{mn}(u \otimes_{\mathcal{C}} v) \circ p_{m+n,A_m\otimes_{\mathcal{C}}C_n}$$
$$+ \mathrm{Im}(\mathrm{Hom}(D_{m+n}, B \otimes_{\mathcal{C}} E))$$

that is termed an external product [5, p. 109–110].

Let us present some properties of external products of unenriched Ext-functors that we shall generalise to completed unenriched Ext-functors. To demonstrate that external products are natural as is mentioned in [5, p. 110], let $r : X \to A$, $s : Y \to C$, $f : B \to M$, $g : E \to N$ be morphisms in $\mathcal{C}$. If we take corresponding projective resolutions, assume that $X_\bullet \otimes_{\mathcal{C}} Y_\bullet$ is a projective resolution of $X \otimes_{\mathcal{C}} Y$ and consider lifts to chain maps $r_\bullet : X_\bullet \to A_\bullet$, $s_\bullet : Y_\bullet \to C_\bullet$ as in the Comparison Theorem [38, Theorem 2.2.6]. Due to Equation 5.3, the chain map $r_\bullet \otimes_{\mathcal{C}} s_\bullet : X_\bullet \otimes_{\mathcal{C}} Y_\bullet \to A_\bullet \otimes_{\mathcal{C}} C_\bullet$ only depends on r and s up to chain homotopy. Because the tensor product $\otimes_{\mathcal{C}}$ respects compositions of morphisms, the square

$$\begin{tikzcd}
\mathrm{Ext}^m_{\mathcal{C}}(A, B) \otimes \mathrm{Ext}^n_{\mathcal{C}}(C, E) \arrow[r, "\vee"] \arrow[d, "\mathrm{Ext}^m_{\mathcal{C}}(r,f)\otimes \mathrm{Ext}^n_{\mathcal{C}}(s,g)"] & \mathrm{Ext}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E) \arrow[d, "\mathrm{Ext}^{m+n}_{\mathcal{C}}(r\otimes_{\mathcal{C}} s, f\otimes_{\mathcal{C}} g)"] \\
\mathrm{Ext}^m_{\mathcal{C}}(X, M) \otimes \mathrm{Ext}^n_{\mathcal{C}}(Y, N) \arrow[r, "\vee"] & \mathrm{Ext}^{m+n}_{\mathcal{C}}(X \otimes_{\mathcal{C}} Y, M \otimes_{\mathcal{C}} N)
\end{tikzcd} \qquad (5.5)$$

commutes and external products are indeed natural.

As is proved in [5, p. 110–111], external products of Ext-functors respect connecting homomorphisms, which is important in the construction of external products of completed Ext-functors. More specifically, if $0 \to B'' \to B' \to B \to 0$ is a short exact sequence and E is such that $0 \to B'' \otimes_{\mathcal{C}} E \to B' \otimes_{\mathcal{C}} E \to B \otimes_{\mathcal{C}} E \to 0$ remains a short exact sequence, then the diagram

$$\begin{array}{ccc} \operatorname{Ext}^m_{\mathcal{C}}(A,B) \otimes \operatorname{Ext}^n_{\mathcal{C}}(C,E) & \xrightarrow{\vee} & \operatorname{Ext}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E) \\ \big\downarrow{\scriptstyle \delta^m \otimes \mathrm{id}} & & \big\downarrow{\scriptstyle \delta^{m+n}} \\ \operatorname{Ext}^{m+1}_{\mathcal{C}}(A,B'') \otimes \operatorname{Ext}^n_{\mathcal{C}}(C,E) & \xrightarrow{\vee} & \operatorname{Ext}^{m+n+1}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B'' \otimes_{\mathcal{C}} E) \end{array} \tag{5.6}$$

commutes. On the other hand, if $0 \to E'' \to E' \to E \to 0$ is a short exact sequence and B is such that $0 \to B \otimes_{\mathcal{C}} E'' \to B \otimes_{\mathcal{C}} E' \to B \otimes_{\mathcal{C}} E \to 0$ remains a short exact sequence, then the diagram

$$\begin{array}{ccc} \operatorname{Ext}^m_{\mathcal{C}}(A,B) \otimes \operatorname{Ext}^n_{\mathcal{C}}(C,E) & \xrightarrow{\vee} & \operatorname{Ext}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E) \\ \big\downarrow{\scriptstyle \mathrm{id} \otimes \delta^n} & & \big\downarrow{\scriptstyle (-1)^m \delta^{m+n}} \\ \operatorname{Ext}^{m}_{\mathcal{C}}(A,B) \otimes \operatorname{Ext}^{n+1}_{\mathcal{C}}(C,E'') & \xrightarrow{\vee} & \operatorname{Ext}^{m+n+1}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E'') \end{array} \tag{5.7}$$

commutes.

To conclude that external products are associative as mentioned in [5, p. 111], we impose that $\otimes_{\mathcal{C}}$ is associative in the following sense. For any objects X, Y, Z in $\mathcal{C}$ there is an isomorphism

$$\mathrm{assoc} : X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z) \to (X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z$$

and for any morphisms $f : X \to X'$, $g : Y \to Y'$, $h : Z \to Z'$ the diagram

$$\begin{array}{ccc} X \otimes_{\mathcal{C}} (Y \otimes_{\mathcal{C}} Z) & \xrightarrow{\mathrm{assoc}} & (X \otimes_{\mathcal{C}} Y) \otimes_{\mathcal{C}} Z \\ \big\downarrow{\scriptstyle f \otimes_{\mathcal{C}} (g \otimes_{\mathcal{C}} h)} & & \big\downarrow{\scriptstyle (f \otimes_{\mathcal{C}} g) \otimes_{\mathcal{C}} h} \\ X' \otimes_{\mathcal{C}} (Y' \otimes_{\mathcal{C}} Z') & \xrightarrow{\mathrm{assoc}} & (X' \otimes_{\mathcal{C}} Y') \otimes_{\mathcal{C}} Z' \end{array} \tag{5.8}$$

commutes. Moreover, we assume that tensor products distribute over finite products in the sense that there are isomorphisms

$$\begin{gathered} \mathrm{distr}_1 : (X \oplus Y) \otimes_{\mathcal{C}} Z \to (X \otimes_{\mathcal{C}} Z) \oplus (Y \otimes_{\mathcal{C}} Z) \text{ and} \\ \mathrm{distr}_2 : X \otimes_{\mathcal{C}} (Y \oplus Z) \to (X \otimes_{\mathcal{C}} Y) \oplus (X \otimes_{\mathcal{C}} Z) \end{gathered}$$

rendering the diagrams

$$\begin{array}{ccc} (X \oplus Y) \otimes_{\mathcal{C}} Z & \xrightarrow{\mathrm{distr}_1} & (X \otimes_{\mathcal{C}} Z) \oplus (Y \otimes_{\mathcal{C}} Z) \\ \big\downarrow{\scriptstyle (f \oplus g) \otimes_{\mathcal{C}} h} & & \big\downarrow{\scriptstyle (f \otimes_{\mathcal{C}} h) \oplus (g \otimes_{\mathcal{C}} h)} \\ (X \oplus Y) \otimes_{\mathcal{C}} Z & \xrightarrow{\mathrm{distr}_1} & (X \otimes_{\mathcal{C}} Z) \oplus (Y \otimes_{\mathcal{C}} Z) \end{array} \tag{5.9}$$

and

$$\begin{array}{ccc} X \otimes_{\mathcal{C}} (Y \oplus Z) & \xrightarrow{\mathrm{distr}_2} & (X \otimes_{\mathcal{C}} Y) \oplus (X \otimes_{\mathcal{C}} Z) \\ \big\downarrow{\scriptstyle f \otimes_{\mathcal{C}} (g \oplus h)} & & \big\downarrow{\scriptstyle (f \otimes_{\mathcal{C}} g) \oplus (f \otimes_{\mathcal{C}} h)} \\ X \otimes_{\mathcal{C}} (Y \oplus Z) & \xrightarrow{\mathrm{distr}_2} & (X \otimes_{\mathcal{C}} Y) \oplus (X \otimes_{\mathcal{C}} Z) \end{array} \tag{5.10}$$

commutative. It follows from this and Equation 5.4 that external products are associative, meaning that

$$\forall x \in \operatorname{Ext}^m_{\mathcal{C}}(A, B), y \in \operatorname{Ext}^n_{\mathcal{C}}(C, E), z \in \operatorname{Ext}^o_{\mathcal{C}}(F, G) : \; x \vee (y \vee z) = (x \vee y) \vee z \,. \tag{5.11}$$

Instead, if we assume that the tensor product $\otimes_{\mathcal{C}}$ is symmetric, then the external product $\vee$ is graded commutative which is demonstrated in [5, p. 111–112]. More specifically, let $\mathrm{swap} : X \otimes_{\mathcal{C}} Y \to Y \otimes_{\mathcal{C}} X$ be an isomorphism that is natural in both variables X and Y. For any two chain complexes $(X_n)_{n\in\mathbb{Z}}$, $(Y_n)_{n\in\mathbb{Z}}$ which vanish in negative degrees there is a chain isomorphism given by

$$\begin{aligned} \mathrm{swap}_k := \sum_{l=0}^{k} (-1)^{l(k-l)} i_{k+1, Y_{k-l}\otimes_{\mathcal{C}} X_l} \circ \mathrm{swap} \circ p_{k+1, X_l \otimes_{\mathcal{C}} Y_{k-l}} : \\ (X_\bullet \otimes_{\mathcal{C}} Y_\bullet)_k \to (Y_\bullet \otimes_{\mathcal{C}} X_\bullet)_k \end{aligned} \tag{5.12}$$

for any $k \in \mathbb{Z}$. In case $X_\bullet$, $Y_\bullet$ are projective resolutions of X and Y such that $X_\bullet \otimes_{\mathcal{C}} Y_\bullet$ is a projective resolution of $X \otimes_{\mathcal{C}} Y$, then $\mathrm{swap}_\bullet : X_\bullet \otimes_{\mathcal{C}} Y_\bullet \to Y_\bullet \otimes_{\mathcal{C}} X_\bullet$ is a chain map lifting $\mathrm{swap} : X \otimes_{\mathcal{C}} Y \to Y \otimes_{\mathcal{C}} X$ as in the Comparison Theorem [38, Theorem 2.2.6]. Moreover the diagram

$$\begin{tikzcd} \operatorname{Hyp}_{\mathcal{C}}(A_\bullet, B_\bullet)_m \otimes \operatorname{Hyp}_{\mathcal{C}}(C_\bullet, E_\bullet)_n \arrow[r] \arrow[d, "(-1)^{mn}\mathrm{swap}"] & \operatorname{Hyp}_{\mathcal{C}}(A_\bullet \otimes_{\mathcal{C}} C_\bullet, B_\bullet \otimes_{\mathcal{C}} E_\bullet)_{m+n} \arrow[d, "\operatorname{Hyp}_{\mathcal{C}}(\mathrm{swap}_\bullet, \mathrm{swap}_\bullet)_{m+n}"] \\ \operatorname{Hyp}_{\mathcal{C}}(C_\bullet, E_\bullet)_n \otimes \operatorname{Hyp}_{\mathcal{C}}(A_\bullet, B_\bullet)_m \arrow[r] & \operatorname{Hyp}_{\mathcal{C}}(C_\bullet \otimes_{\mathcal{C}} A_\bullet, E_\bullet \otimes_{\mathcal{C}} B_\bullet)_{m+n} \end{tikzcd} \tag{5.13}$$

commutes where the horizontal homomorphisms are taken as in Equation 5.2. Therefore

$$\forall x \in \operatorname{Ext}^m_{\mathcal{C}}(A, B), y \in \operatorname{Ext}^n_{\mathcal{C}}(C, E) : \operatorname{Ext}^{m+n}_{\mathcal{C}}(\mathrm{swap}, \mathrm{swap}) \circ (x \vee y) = (-1)^{mn} y \vee x \tag{5.14}$$

as desired.

In the case of group cohomology, external products give rise to cup products. Although our account is based on [5, p. 107–110], it differs from the source as our tensor product takes a slightly different form. For this let R be a ring object in $\mathcal{C}$ and G be a group object. In the following, we only work in the category $\mathrm{C}_{R,G}$ and take the tensor product to be of the form $\otimes_R : \mathcal{C}_{R,G} \times \mathcal{C}_{R,G} \to \mathcal{C}_{R,G}$. Given that $- \otimes_R -$ is a bifunctor, we assume that there are natural isomorphisms $\Theta'_M : M \otimes_R R \to M$ and $\Theta''_M : R \otimes_R M \to M$. If $R_\bullet$, $R'_\bullet$ are two projective resolutions of R, then we assume that $R_\bullet \otimes_R R'_\bullet$ is also a projective resolution of R. Consequently, the external product from Equation 5.4 becomes the so-called cup product

$$\smile : H^m_R(G, B) \otimes H^n_R(G, E) \to H^{m+n}_R(G, B \otimes_R E) \,. \tag{5.15}$$

Diagram 5.6 and 5.7 as well as Equation 5.11 pertain to hold for cup products, meaning that they are associative and respect connecting homomorphisms. Because the chain isomorphism $\mathrm{swap}_\bullet$ defined in 5.12 goes from one projective resolution of R to another, Equation 5.14 states for cup products that

$$\forall x \in H^m_R(G, B), y \in H^n_R(G, E) : H^{m+n}_R(G, \mathrm{swap}) \circ (x \smile y) = (-1)^{mn} y \smile x$$

where $H^{m+n}_R(G, \mathrm{swap})$ arises from $\operatorname{Ext}^{m+n}_{\mathcal{C}_{R,G}}(\mathrm{swap}, \mathrm{swap})$.

Contrary to external products, there is under certain conditions a multiplicative unit for cup products which is shown in [5, p. 111]. In particular, this turns group cohomology into a graded ring. We assume that there is a projective resolution $(R_\bullet, \partial_\bullet)$ of R with augmentation map $\varepsilon : R_0 \to R$ such that for any $k \in \mathbb{N}_0$ and any morphism $f : R_k \to M$ the diagram

$$\begin{tikzcd}
(R_\bullet \otimes_R R_\bullet)_k \arrow[r, "(\mathrm{id}\otimes_R\varepsilon)\circ p_{k+1,R_k\otimes_R R_0}"] \arrow[d, "(f\otimes_R\varepsilon)\circ p_{k+1,R_k\otimes_R R_0}"] & R_k \otimes_R R \arrow[r, "\Theta'_{R_k}"] & R_k \arrow[d, "f"] \\
M \otimes_R R \arrow[rr, "\Theta'_M"] & & M
\end{tikzcd} \tag{5.16}$$

commutes. Note that the top morphism forms part of a chain map $R_\bullet \otimes_R R_\bullet \to R_\bullet$ lifting $\Theta'_R : R \otimes_R R \to R$ and thus induces an isomorphism on the level of group cohomology. If we write $(\overline{R}_\bullet, \overline{\partial}_\bullet) := (R_\bullet, \partial_\bullet)$, then we assume further that the diagram

$$\begin{tikzcd}
(R_\bullet \otimes_R R_\bullet)_k \arrow[r, "(\varepsilon\otimes_R\mathrm{id})\circ p_{k+1,R_0\otimes_R R_k}"] \arrow[d, "(\varepsilon\otimes_R f)\circ p_{k+1,R_0\otimes_R R_k}"] & R \otimes_R \overline{R}_k \arrow[r, "\Theta''_{\overline{R}_k}"] & \overline{R}_k \arrow[d, "f"] \\
R \otimes_R M \arrow[rr, "\Theta''_M"] & & M
\end{tikzcd} \tag{5.17}$$

commutes. As before, the top morphism forms part of a chain map $R_\bullet \otimes_R R_\bullet \to \overline{R}_\bullet$ lifting $\Theta''_R : R \otimes_R R \to R$ and also induces an isomorphism in group cohomology. If we thus write $1 \in H^0_R(G, R)$ for the element arising from the augmentation map $\varepsilon : R_0 \to R$, then

$$\forall x \in H^n_R(G, M) : H^n_R(G, \Theta'_M)(x \smile 1) = x = H^n_R(G, \Theta''_M)(1 \smile x)\,.$$

If we define cup products with at least one element of negative degree to vanish, then the cup product turns $\bigoplus_{n\in\mathbb{Z}} H^n_R(G, R)$ into graded ring with identity 1.

6 Existence of cohomology products

In this section we establish external products and cup products under specific conditions and provide examples which meet these conditions. Moreover, we construct Yoneda products in full generality. For clarity, we introduce the following notation. Reserving the letter 'D' for boundary maps, we denote the arguments of external products usually by A, B, C and E. We write G for a group (object) and R for a ring (object). As we have used the letter 'I' for index sets previously, we denote the arguments for Yoneda products by F, H and J in order to distinguish them from the arguments of external products. As most constructions of these cohomology products involve taking direct limits, we require the following result that we have not found in the literature in this manner.

Proposition 6.1 *Let $(M_i, m_i)_{i\in\mathbb{N}}$, $(N_i, n_i)_{i\in\mathbb{N}}$ be direct systems of abelian groups and denote by $\otimes$ the tensor product of abelian groups. Then*

$$\varinjlim_{i\in\mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i) \cong (\varinjlim_{j\in\mathbb{N}} M_j) \otimes (\varinjlim_{k\in\mathbb{N}} N_k)\,.$$

Proof We demonstrate that $\varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i)$ satisfies the universal property of the tensor product $(\varinjlim_{j\in\mathbb{N}} M_j) \otimes (\varinjlim_{k\in\mathbb{N}} N_k)$ that can be found in [34, Tag 00CV] for instance. Denote by $M_i \times N_i$ the cartesian product and observe that the squares

$$\begin{tikzcd} M_i \times N_i \arrow[r, "q_i"] \arrow[d, "m_i \times n_i"] & M_i \otimes N_i \arrow[d, "m_i \otimes n_i"] \\ M_{i+1} \times N_{i+1} \arrow[r, "q_{i+1}"] & M_{i+1} \otimes N_{i+1} \end{tikzcd} \tag{6.1}$$

commute. Although $M_i \times N_i$ and $M_{i+1} \times N_{i+1}$ are abelian groups and $m_i \otimes n_i$ a homomorphism, we regard the former as sets and the latter as a function. In particular, Diagram 6.1 gives rise to a direct system of functions in whose direct limit we obtain

$$q := \varinjlim_{i \in \mathbb{N}} q_i : \varinjlim_{\mathbf{Set}, i \in \mathbb{N}} (M_i \times N_i, m_i \times n_i) \to \varinjlim_{\mathbf{Set}, i \in \mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i) \tag{6.2}$$

where the latter is taken to be a direct limit of sets. By [34, Tag 002W], the former direct limit can be given by

$$\varinjlim_{\mathbf{Set}, i \in \mathbb{N}} (M_i \times N_i, m_i \times n_i) \cong (\varinjlim_{\mathbf{Set}, j \in \mathbb{N}} M_j) \times (\varinjlim_{\mathbf{Set}, k \in \mathbb{N}} N_k)\,. \tag{6.3}$$

In particular, if $m_{i,\infty} : M_i \to (\varinjlim_{\mathbf{Set}, j \in \mathbb{N}} M_j)$ and $n_{i,\infty} : N_i \to (\varinjlim_{\mathbf{Set}, k \in \mathbb{N}} N_k)$ denote functions to the respective direct limit, then the function

$$m_{i,\infty} \times n_{i,\infty} : M_i \times N_i \to (\varinjlim_{\mathbf{Set}, j \in \mathbb{N}} M_j) \times (\varinjlim_{\mathbf{Set}, k \in \mathbb{N}} N_k) \tag{6.4}$$

represents the function to the direct limit $\varinjlim_{\mathbf{Set}, i \in \mathbb{N}} (M_i \times N_i, m_i \times n_i)$. According to [34, Tag 04AX], if $\sqcup$ denotes the disjoint union of sets, then the latter direct limit in Equation 6.2 can be constructed as

$$\varinjlim_{\mathbf{Set}, i \in \mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i) = \bigsqcup_{n \in \mathbb{N}} M_n \otimes N_n / \sim \tag{6.5}$$

where $x_i \in M_i \otimes N_i \sim y_j \in M_j \otimes N_j$ if there is $i, j \leq k$ such that x_i and y_j are mapped to the same element in $M_k \otimes N_k$. It follows from the proof of [29, Proposition 1.2.1] and [34, Tag 09WR] that it can be endowed with the structure of an abelian group such that it forms the colimit $\varinjlim_{i \in \mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i)$ of abelian groups and not just of sets. Same holds true for $\varinjlim_{i \in \mathbb{N}} M_i$ and $\varinjlim_{i \in \mathbb{N}} N_i$ from Equation 6.3. Hence, the function q from Equation 6.2 can be written as

$$q : (\varinjlim_{j \in \mathbb{N}} M_j) \times (\varinjlim_{k \in \mathbb{N}} N_k) \to \varinjlim_{i \in \mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i)\,.$$

To prove that q satisfies the above mentioned universal property, consider a bilinear function $a : (\varinjlim_{j \in \mathbb{N}} M_j) \times (\varinjlim_{k \in \mathbb{N}} N_k) \to A$ for an abelian group A. Composing with the homomorphisms $m_{i,\infty}$ and $n_{i,\infty}$ occurring in Equation 6.4 yields a bilinear function

$$M_i \times N_i \xrightarrow{m_{i,\infty} \times n_{i,\infty}} (\varinjlim_{j \in \mathbb{N}} M_j) \times (\varinjlim_{k \in \mathbb{N}} N_k) \xrightarrow{a} A\,.$$

By the universal property of the tensor product, there exists a unique homomorphism $b_i : M_i \otimes N_i \to A$ such that the square

$$\begin{tikzcd} M_i \times N_i \arrow[r, "q_i"] \arrow[d, "m_{i,\infty} \times n_{i,\infty}"] & M_i \otimes N_i \arrow[d, "b_i"] \\ (\varinjlim_{j \in \mathbb{N}} M_j) \times (\varinjlim_{k \in \mathbb{N}} N_k) \arrow[r, "a"] & A \end{tikzcd} \tag{6.6}$$

commutes. By this and Diagram 6.1 we infer that the triangle

$$\begin{tikzcd} M_i \otimes N_i \arrow[d, "m_i \otimes n_i"] \arrow[dr, "b_i"] & \\ M_{i+1} \otimes N_{i+1} \arrow[r, "b_{i+1}"] & A \end{tikzcd} \tag{6.7}$$

is commutative whence there is a homomorphism

$$b := \varinjlim_{i\in\mathbb{N}} b_i : \varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i) \to A$$

By Diagram 6.1, 6.6 and 6.7 we obtain the factorisation

$$a : \varinjlim_{\mathbf{Set}, i\in\mathbb{N}} (M_i \times N_i, m_i \times n_i) \xrightarrow{q} \varinjlim_{\mathbf{Set}, i\in\mathbb{N}} (M_i \otimes N_i, m_i \otimes n_i) \xrightarrow{b} A\,.$$

Because b is unique in this factorisation due to Equation 6.5 and the uniqueness of the homomorphisms b_i from Diagram 6.6 and 6.7, $\varinjlim_{i\in\mathbb{N}}(M_i \otimes N_i, m_i \otimes n_i)$ satisfies the universal property of the tensor product $(\varinjlim_{j\in\mathbb{N}} M_j) \otimes (\varinjlim_{k\in\mathbb{N}} N_k)$. $\square$

Theorem 6.2 *1. Let $\otimes_{\mathcal{C}} : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ be a bi-additive functor and assume that $P\otimes_{\mathcal{C}}Q$ is projective whenever $P, Q \in \mathrm{obj}(\mathcal{C})$ are projective. Let $A_\bullet$, $C_\bullet$ be projective resolutions of $A, C \in \mathrm{obj}(\mathcal{C})$ such that the tensor product of resolutions $A_\bullet \otimes_{\mathcal{C}} C_\bullet$ is a projective resolution of $A \otimes_{\mathcal{C}} C$. Let $B_\bullet$, $E_\bullet$ be projective resolutions of B, E such that for their syzygies $\widetilde{F}_k \in \{\widetilde{B}_k,\ \widetilde{E}_k\}$ the functors $- \otimes_{\mathcal{C}} \widetilde{F}_k, \widetilde{F}_k \otimes_{\mathcal{C}} - : \mathcal{C} \to \mathcal{C}$ are exact and preserve projectives. Then for every $m, n \in \mathbb{Z}$ external products*

$$\vee : \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A, B) \otimes \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C, E) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E)$$

can be defined for completed unenriched Ext-functors equivalently through the resolution and the hypercohomoology construction.

2. If R is a ring object in $\mathcal{C}$ and G a group object, then we take the tensor product to be of the form $\otimes_R : \mathcal{C}_{R,G} \times \mathcal{C}_{R,G} \to \mathcal{C}_{R,G}$. Assume that there are natural isomorphisms $\Theta'_M : M \otimes_R R \to M$ and $\Theta''_M : R \otimes_R M \to M$ and that $R_\bullet \otimes_R R'_\bullet$ is a projective resolution of R for any two projective resolutions $R_\bullet$, $R'_\bullet$ of R. Then for every $m, n \in \mathbb{Z}$ there are cup products

$$\smile: \widehat{H}^m_R(G, M) \otimes \widehat{H}^n_R(G, N) \to \widehat{H}^{m+n}_R(G, M \otimes_R N)$$

for completed unenriched group cohomology that descend from the above external products.

Proof As can be observed from the previous section, the existence of cup products follows from the existence of external products. Hence, we only prove the latter where we first use the resolution construction. Since we need to consider direct limit systems for this, let $k, l \in \mathbb{N}_0$ and write $K := m + k$ and $L := n + l$. Then by Diagram 5.6, Diagram 5.7 and [6, Proposition III.4.1] we see that the cube

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{Ext}^{K}_{\mathcal{C}}(A,\widetilde{B}_k)\otimes\mathrm{Ext}^{L}_{\mathcal{C}}(C,\widetilde{E}_l) \arrow[rr, "\vee"] \arrow[dl, "\delta^K\otimes\mathrm{id}"'] \arrow[dd, "\mathrm{id}\otimes\delta^L" near start] & & \mathrm{Ext}^{K+L}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{E}_l) \arrow[dl, "\delta^{K+L}"] \arrow[dd, "(-1)^K\delta^{K+L}"] \\
\mathrm{Ext}^{K+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1})\otimes\mathrm{Ext}^{L}_{\mathcal{C}}(C,\widetilde{E}_l) \arrow[rr, crossing over, "\vee" near end] \arrow[dd, "\mathrm{id}\otimes\delta^L"] & & \mathrm{Ext}^{K+L+1}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{E}_l) & \\
 & \mathrm{Ext}^{K}_{\mathcal{C}}(A,\widetilde{B}_k)\otimes\mathrm{Ext}^{L+1}_{\mathcal{C}}(C,\widetilde{E}_{l+1}) \arrow[rr, "\vee" near end] \arrow[dl, "\delta^K\otimes\mathrm{id}"'] & & \mathrm{Ext}^{K+L+1}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{E}_{l+1}) \arrow[dl, "\delta^{K+L+1}"] \\
\mathrm{Ext}^{K+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1})\otimes\mathrm{Ext}^{L+1}_{\mathcal{C}}(C,\widetilde{E}_{l+1}) \arrow[rr, "\vee"] & & \mathrm{Ext}^{K+L+2}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{E}_{l+1}) \arrow[from=uu, crossing over, "(-1)^{K+1}\delta^{K+L+1}" near start] &
\end{tikzcd}
\tag{6.8}
$$

commutes. As in [15, Lemma 4.6], one can construct for any sequence $o \in \{0,1\}^{\mathbb{N}}$ a direct system

$$\mathrm{Ext}^{m+P(o)_k}_{\mathcal{C}}(A,\widetilde{B}_{P(o)_k}) \otimes \mathrm{Ext}^{n+D(o)_k}_{\mathcal{C}}(C,\widetilde{E}_{D(o)_k}))_{k\in\mathbb{N}_0}$$

whose homomorphisms from one term to the next are given by

$$\begin{cases} \delta^{m+P(o)_k} \otimes \mathrm{id} & \text{if } P(o)_{k+1} = P(o)_k + 1 \\ \mathrm{id} \otimes \delta^{n+D(o)_k} & \text{if } P(o)_{k+1} = P(o)_k \end{cases}.$$

According to the above cube, this gives rise to a homomorphism

$$\varinjlim_{k\in\mathbb{N}_0} \vee : \varinjlim_{k\in\mathbb{N}_0} \big(\mathrm{Ext}_{\mathcal{C}}^{m+P(o)_k}(A, \widetilde{B}_{P(o)_k}) \otimes \mathrm{Ext}_{\mathcal{C}}^{n+D(o)_k}(C, \widetilde{E}_{D(o)_k})\big)$$
$$\to \varinjlim_{k\in\mathbb{N}_0} \mathrm{Ext}_{\mathcal{C}}^{m+n+k}(A \otimes_{\mathcal{C}} C, \widetilde{B}_{P(o)_k} \otimes_{\mathcal{C}} \widetilde{E}_{D(o)_k})$$

in the direct limit. Although the homomorphisms giving rise to the right hand direct limit involve changing signs, these have no effect as one can group the terms of the direct system such that these signs cancel each other. In particular, the right hand direct limit equals $\widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E)$ by the resolution construction. If the sequence o takes the value 0 only finitely many times, then there would be $d \in \mathbb{N}_0$ such that the above homomorphism would be of the form

$$\varinjlim_{k\in\mathbb{N}_0} \vee : \mathrm{Ext}_{\mathcal{C}}^{m+d}(A, \widetilde{B}_d) \otimes \widehat{\mathrm{Ext}}_{\mathcal{C}}^{n}(C, E) \to \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E)\,.$$

Since we do not wish to consider this to be an external product, we conclude that both values 0 and 1 occur infinitely often in o. By the proof of [15, Lemma 4.10] and Diagram 6.8, it does not matter which sequence o we choose as they all result in the same direct limits. Thus, if we choose o to alternate between 0 and 1, then the homomorphism in the direct limit takes the form

$$\varinjlim_{k\in\mathbb{N}_0} \vee : \varinjlim_{k\in\mathbb{N}_0} \big(\mathrm{Ext}_{\mathcal{C}}^{m+k}(A, \widetilde{B}_k) \otimes \mathrm{Ext}_{\mathcal{C}}^{n+k}(C, \widetilde{E}_k)\big) \to \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E)\,.$$

By Proposition 6.1, this results in the desirerd external product

$$\vee := \varinjlim_{k\in\mathbb{N}_0} \vee : \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m}(A, B) \otimes \widehat{\mathrm{Ext}}_{\mathcal{C}}^{n}(C, E) \to \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E)\,.$$

In order to translate this to the hypercohomology construction, consider two almost chain maps $\varphi_{\bullet+m} : A[m]_\bullet \to B_\bullet$ and $\psi_{\bullet+n} : C[n]_\bullet \to E_\bullet$. For $k \in \mathbb{N}_0$ write $\pi_k : B_k \to \widetilde{B}_k$ for the morphism to the k^{th} syzygy and define $\varphi'_{m+k} := \pi_k \circ \varphi_{m+k} : A_{m+k} \to \widetilde{B}_k$ where we define $\psi'_{n+k} : C_{n+k} \to \widetilde{E}_k$ analogously. According to [15, Lemma 6.6] and the proof of [15, Lemma 6.9], there is $\kappa \in \mathbb{N}_0$ such that $(\varphi_{m+k})_{k\geq 2\kappa}$ is a chain map that gives rise to the element $\varphi'_{m+2\kappa} + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_{m+2\kappa}, \widetilde{B}_{2\kappa})$ in $\mathrm{Ext}_{\mathcal{C}}^{m+2\kappa}(A, \widetilde{B}_{2\kappa})$. We see by the proofs of [15, Lemma 6.6] and of [15, Lemma 6.9] that

$$\delta^{m+K}\big(\varphi'_{m+K} + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_{m+K}, \widetilde{B}_K)\big) = \varphi'_{m+K+1} + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(a_{m+K+1}, \widetilde{B}_{K+1})$$

for any $K \geq 2\kappa$. Analogously, there is $\lambda \in \mathbb{N}_0$ such that $(\psi_{n+l})_{l\geq 2\lambda}$ is a chain map that gives rise to the element $\psi'_{n+2\lambda} + \mathrm{Im}(\mathrm{Hom}_{\mathcal{C}}(c_{n+2\lambda}, \widetilde{E}_{2\lambda})$ in $\mathrm{Ext}_{\mathcal{C}}^{n+2\lambda}(C, \widetilde{E}_{2\lambda})$. Given $K \geq 2\kappa$, $L \geq 2\lambda$, the external product of $(\varphi_{m+k})_{k\geq K}$ and $(\psi_{n+l})_{l\geq L}$ arises from

$$\varphi'_{m+K} \otimes_{\mathcal{C}} \psi'_{n+L} : A_{m+K} \otimes_{\mathcal{C}} C_{n+L} \to \widetilde{B}_K \otimes_{\mathcal{C}} \widetilde{E}_L$$

according to Equation 5.4. Observe that the diagrams

$$\begin{tikzcd} A_{m+K} \otimes_{\mathcal{C}} C_{n+L} \arrow[r, "\varphi_{m+K}\otimes_{\mathcal{C}}\psi'_{n+L}"] \arrow[dr, "\varphi'_{m+K}\otimes_{\mathcal{C}}\psi'_{n+L}"'] & B_K \otimes_{\mathcal{C}} \widetilde{E}_L \arrow[d, "\pi_K\otimes_{\mathcal{C}}\mathrm{id}"] \\ & \widetilde{B}_K \otimes_{\mathcal{C}} \widetilde{E}_L \end{tikzcd} \tag{6.9}$$

and

$$\begin{tikzcd}
A_{m+K} \otimes_{\mathcal{C}} C_{n+L} \arrow[r, "\varphi'_{m+K}\otimes_{\mathcal{C}}\psi_{n+L}"] \arrow[dr, "\varphi'_{m+K}\otimes_{\mathcal{C}}\psi'_{n+L}"] & \widetilde{B}_K \otimes_{\mathcal{C}} E_L \arrow[d, "\mathrm{id}\otimes_{\mathcal{C}}\pi_L"] \\
& \widetilde{B}_K \otimes_{\mathcal{C}} \widetilde{E}_L
\end{tikzcd} \tag{6.10}$$

commute. In order that the external product of $\varphi_{\bullet+m}$ and $\psi_{\bullet+n}$ is again an almost chain map modulo chain homotopy, a choice of a projective resolution of $B \otimes_{\mathcal{C}} E$ is required. Define the projective resolution $B_\bullet \otimes^o_{\mathcal{C}} E_\bullet$ by setting for any $k \in \mathbb{N}_0$ the k^{th} term to be

$$(B_\bullet \otimes^o_{\mathcal{C}} E_\bullet)_k := \begin{cases} B_{P(o)_k} \otimes \widetilde{E}_{D(o)_k} & \text{if } P(o)_{k+1} = P(o)_k + 1 \\ \widetilde{B}_{P(o)_k} \otimes E_{D(o)_k} & \text{if } P(o)_{k+1} = P(o)_k \end{cases}$$

and extend it to be zero in negative degrees. We define the componentwise morphism

$$\varphi_{\bullet+m} \otimes^o_{\mathcal{C}} \psi_{\bullet+n} : (A_\bullet \otimes_{\mathcal{C}} C_\bullet)[m+n]_\bullet \to B_\bullet \otimes^o_{\mathcal{C}} E_\bullet$$

by setting the k^{th} degree correspondingly to

$$\varphi_{m+P(o)_k} \otimes_{\mathcal{C}} \psi'_{n+D(o)_k} \circ p_{k+1, A_{m+P(o)_k} \otimes_{\mathcal{C}} C_{n+D(o)_k}} : (A_\bullet \otimes_{\mathcal{C}} C_\bullet)_{m+n+k} \to B_{P(o)_k} \otimes_{\mathcal{C}} \widetilde{E}_{D(o)_k} \text{ or}$$
$$\varphi'_{m+P(o)_k} \otimes_{\mathcal{C}} \psi_{n+D(o)_k} \circ p_{k+1, A_{m+P(o)_k} \otimes_{\mathcal{C}} C_{n+D(o)_k}} : (A_\bullet \otimes_{\mathcal{C}} C_\bullet)_{m+n+k} \to \widetilde{B}_{P(o)_k} \otimes_{\mathcal{C}} E_{D(o)_k}$$

whenever $m + P(o)_k, n + D(o)_k \geq 0$ and to be zero otherwise. Due to the construction of external products through the resolution construction, Diagram 6.9 and Diagram 6.10, $\varphi_{\bullet+m} \otimes^o_{\mathcal{C}} \psi_{\bullet+n}$ is a chain map in degrees k for which $P(o)_k \geq 2\kappa$, $D(o)_k \geq 2\lambda$ and thus an almost chain map. Likewise we conclude that

$$\begin{aligned} \widehat{\operatorname{Ext}}^m_{\mathcal{C}}(A,B) \otimes \widehat{\operatorname{Ext}}^n_{\mathcal{C}}(C,E) &\to \widehat{\operatorname{Ext}}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E) \\ (\varphi_{\bullet+m} + \widehat{\operatorname{Null}}_{\operatorname{Ch}(\mathcal{C})}(A[m]_\bullet, B_\bullet)) \otimes (\psi_{\bullet+n} + \widehat{\operatorname{Null}}_{\operatorname{Ch}(\mathcal{C})}(C[n]_\bullet, E_\bullet)) & \\ \mapsto (\varphi_{\bullet+m} \otimes^o_{\mathcal{C}} \psi_{\bullet+n} &+ \widehat{\operatorname{Null}}_{\operatorname{Ch}(\mathcal{C})}((A_\bullet \otimes_{\mathcal{C}} C_\bullet)[m+n]_\bullet, B_\bullet \otimes^o_{\mathcal{C}} E_\bullet) \end{aligned}$$

is the external product given by the hypercohomology construction. □

Remark 6.3 *In [3, p. 110], D. J. Benson and J. F. Carlson establish external products for completed unenriched Ext-functors of any category of modules over a ring through the naïve construction and thus cup products for complete unenriched cohomology of any discrete group. We do not know whether our constructions are equivalent to theirs, but we doubt that this is the case in general. They mention there that external products for completed Ext-functors cannot be defined through tensor products of almost chain maps as in Equation 5.2 because this does not yield an almost chain map in general.*

Example 6.4 *All conditions of Theorem 6.2 are satisfied in the following two instances.*

1. *Let $\mathcal{C}$ be the category of discrete $R[G]$-modules for a discrete group G and a principal ideal domain R. The restriction of A, B, C, E to R-modules needs to be projective.*

2. *Let $\mathcal{C}$ be the category of profinite $S[[H]]$-modules for a profinite group H and profinite commutative ring S with a unique maximal open ideal. Then the restriction of A, B, C, E to profinite S-modules needs to be projective.*

Remark 6.5 *The p-adic integers $\mathbb{Z}_p$ are an example of a profinite commutative ring with the unique maximal ideal $p\mathbb{Z}_p$ [39, p. 27]. The restriction of any p-torsionfree profinite $\mathbb{Z}_p[[G]]$-module to a $\mathbb{Z}_p$-module is projective [37, Corollary 2.1.2].*

Proof There is a tensor product $\otimes_R$ for discrete $R[G]$-modules and one $\widehat{\otimes}_S$ for profinite $S[[H]]$-modules [29, Section 5.5 and 5.8]. By [29, Proposition 5.5.3], tensoring with the coefficient ring R (resp. S) yields isomorphisms as required for tensor products. Because the coefficient ring is projective as a module over itself, all requirements for cup products are met as soon as all conditions on external products from Theorem 6.2 are met.

To prove the latter, denote by F any of A, B, C or E. By [5, p. 10–11] and [39, Lemma 9.8.2], there is a resolution $F_\bullet$ of F such that F_k is a free $R[G]$-module (resp. $S[[H]]$-module) for every $k \in \mathbb{N}_0$. Then F_k is free as an R-module (resp. S-module) where the result for discrete modules follows by construction and for profinite ones by [29, Corollary 5.7.2]. In particular, F_k is a projective R-module where one invokes [29, Proposition 5.4.2] for the profinite case. Because E is projective as an R-module (resp. S-module), the short exact sequence $0 \to \widetilde{F}_1 \to F_0 \to F \to 0$ is split and hence $\widetilde{F}_1$ projective as an R-module (resp. S-module). Inductively, we conclude for every $k \in \mathbb{N}_0$ that $\widetilde{F}_k$ is projective as an R-module (resp. S-module). According to [5, p. 29] and [29, Proposition 5.5.3], $\widetilde{F}_k$ is flat and according to [30, Theorem 9.8] and [39, Proposition 7.5.1], it is also free as an R-module. It then follows from the proof of [37, Proposition 3.3.2] that tensoring $\widetilde{F}_k$ with a projective $R[G]$-module (resp. $S[[H]]$-module) gives rise to another projective module.

This does not only prove B and E have projective resolutions of the desired form, but also that $P \otimes_R Q$ (resp. $P\widehat{\otimes}_S Q$) is projective whenever P and Q are projective. According to [39, Lemma 10.4.4], $A_\bullet \otimes_R C_\bullet$ is a projective resolution of $A \otimes_R C$ (resp. $A_\bullet \widehat{\otimes}_S C_\bullet$ of $A\widehat{\otimes}_S C$). $\square$

Contrary to external and cup products, we construct in the following Yoneda products in full generality. To this end, recall that for every $n \in \mathbb{Z}$ the completed unenriched Ext-functor $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Ab}$ forms a bifunctor that is additive in both variables by [15, Proposition 5.5] and [15, Proposition 6.3]. We generalise hereby the constructions of Yoneda products found in [3, p. 110] and prove that they are equivalent.

Theorem 6.6 *Let $F, H, J \in \mathrm{obj}(\mathcal{C})$. If $\otimes$ denotes the tensor product in $\mathbf{Ab}$, then for every $m, n \in \mathbb{Z}$ Yoneda products*

$$\circ : \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(H, J) \otimes \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(F, H) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(F, J)$$

can be defined for completed unenriched Ext-functors equivalently by the hypercohomology construction as composition of almost chain maps or by the naïve construction as a direct limit of the composition functors of the functors $[-,-]_{\mathcal{C}}$ from [15, Proposition 5.1].

Proof The definition of Yoneda products through the hypercohomology construction is analogous to their definition for unenriched Ext-functors, which is covered in [14, p. 166] for instance. Namely, if the almost chain map $f[n]_{\bullet+m} : F[m+n]_\bullet \to H[n]_\bullet$ is a representative an element in $\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(F, H)$ and $g_{\bullet+n} : H[n]_\bullet \to J_\bullet$ a representative of an element in $\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(H, J)$, then their composition $g_{\bullet+n} \circ f[n]_{\bullet+m} : F[m+n]_\bullet \to J_\bullet$ is again an almost chain map. If $f'_{\bullet+m}$ is chain homotopic to $f_{\bullet+m}$ and $g'_{\bullet+n}$ chain homotopic to $g_{\bullet+n}$, then

$g_{\bullet+n} \circ f[n]_{\bullet+m}$ is chain homotopic to $g_{\bullet+n} \circ f'[n]_{\bullet+m}$ which is in turn chain homotopic to $g'_{\bullet+n} \circ f'[n]_{\bullet+m}$. Hence,

$$g_{\bullet+n} \circ f[n]_{\bullet+m} + \widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C}}(F[m+n]_\bullet, J_\bullet)$$

is a well defined element in $\widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(F, J)$. Since this operation is bi-additive, it constitutes a Yoneda product

$$\widehat{\mathrm{Ext}}^{n}_{\mathcal{C}}(H, J) \otimes \widehat{\mathrm{Ext}}^{m}_{\mathcal{C}}(F, H) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(F, J)\,.$$

Moving over to the naïve construction, one can see in [15, Proposition 5.1] that for any $k \in \mathbb{N}_0$ with $n+m+k, n+k \geq 0$ the composition functor of morphisms descends to a bifunctor

$$\circ : [\widetilde{H}_{n+k}, \widetilde{J}_k]_{\mathcal{C}} \times [\widetilde{F}_{n+m+k}, \widetilde{H}_{n+k}]_{\mathcal{C}} \to [\widetilde{F}_{n+m+k}, \widetilde{J}_k]_{\mathcal{C}}\,.$$

In particular, if we take an element $f'_{n+m+k} + \mathcal{P}_{\mathcal{C}}(\widetilde{F}_{n+m+k}, \widetilde{H}_{n+k})$ in $[\widetilde{F}_{n+m+k}, \widetilde{H}_{n+k}]_{\mathcal{C}}$ and an element $g'_{n+k} + \mathcal{P}_{\mathcal{C}}(\widetilde{H}_{n+k}, \widetilde{J}_k)$ in $[\widetilde{H}_{n+k}, \widetilde{J}_k]_{\mathcal{C}}$, then they give rise to a well defined element $g'_{n+k} \circ f'_{n+m+k} + \mathcal{P}_{\mathcal{C}}(\widetilde{F}_{n+m+k}, \widetilde{J}_k)$ in $[\widetilde{F}_{n+m+k}, \widetilde{J}_k]_{\mathcal{C}}$. By [15, Proposition 5.2] and [15, Definition 5.3] we can pass to the direct limit to obtain

$$\varinjlim_{k\in\mathbb{N}_0} \big([\widetilde{H}_{n+k}, \widetilde{J}_k]_{\mathcal{C}} \times [\widetilde{F}_{n+m+k}, \widetilde{H}_{n+k}]_{\mathcal{C}}\big) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(F, J)\,.$$

Because this operation is bi-additive, this yields by Proposition 6.1 the second Yoneda product. Applying the isomorphism $\rho^{n+m} : \widehat{\mathcal{E}xt}^{n+m}_{\mathcal{C}}(F, J) \to BC^{n+m}_{\mathcal{C}}(F, J)$ from [15, Definition 6.16], we deduce from the proof of [15, Lemma 6.17] that

$$\rho^{n+m}\big(g_{\bullet+n}\circ f[n]_{\bullet+m}+\widehat{\mathrm{Null}}_{\mathrm{Ch}(\mathcal{C})}(F[n+m]_\bullet, J_\bullet)\big) = \big(\widetilde{g}_{n+2k}\circ \widetilde{f}_{n+m+2k}+\mathcal{P}_{\mathcal{C}}(\widetilde{F}_{n+m+2k}, \widetilde{J}_{2k})\big)_{k\geq K}$$

where $K \in \mathbb{N}_0$ is chosen such that both $(f_{n+m+k})_{k\geq 2K}$ and $(g_{n+k})_{k\geq 2K}$ are chain maps. This demonstrate that the two Yoneda products agree. □

Remark 6.7 *Using our constructions, we do not know whether cup products and Yoneda products agree in complete unenriched group cohomology. In contrast, D. J. Benson and J. F. Carlson prove in [3, p. 110] that Yoneda products agree with the cup products that they construct for complete cohomology of discrete groups.*

7 Properties of cohomology products

This final section is dedicated to the properties of external, cup and Yoneda products. First, we develop requirements for when these cohomology products are associative. Then we provide conditions under which cup products turn complete cohomology and Yoneda products turn completed Ext-functors into a graded ring with identity. We show that the canonical morphism from Ext-functors to their Mislin completion preserves cohomology products. Lastly, we prove that external and cup products satisfy a version of graded commutativity and that cohomology products are compatible with connecting homomorphisms. As we have only constructed these products for completed unenriched Ext-functors in the previous section, all Ext-functors are understood to be unenriched from this point on.

Lemma 7.1 *1. External products of completed Ext-functors and thus cup products of complete group cohomology are natural in both variables. More specifically, let $r : X \to A$, $s : Y \to C$, $f : B \to M$ and $g : E \to N$ be morphisms such that their objects satisfy all conditions in Theorem 6.2. Then the square*

$$\begin{array}{ccc}
\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B) \otimes \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C,E) & \xrightarrow{\ \vee\ } & \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, B \otimes_{\mathcal{C}} E) \\
\Big\downarrow {\scriptstyle \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(r,f) \otimes \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(s,g)} & & \Big\downarrow {\scriptstyle \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(r \otimes_{\mathcal{C}} s, f \otimes_{\mathcal{C}} g)} \\
\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(X,M) \otimes \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(Y,N) & \xrightarrow{\ \vee\ } & \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(X \otimes_{\mathcal{C}} Y, M \otimes_{\mathcal{C}} N)
\end{array}$$

commutes. In the case of cup products, we take the first variable of each completed Ext-functor to be the coefficient ring R and assume the existence of the required natural isomorphisms Θ' and Θ''.

2. *Yoneda products of completed Ext-functors*

$$\circ : \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(H,J) \otimes \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(F,H) \to \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(F,J)$$

are natural in the variable F and J.

Proof (1) To establish naturality via the resolution construction, let $k, l \in \mathbb{N}_0$, write $K := m + k$, $L := n + l$ and consider the lifts $f_\bullet : B_\bullet \to M_\bullet$, $g_\bullet : C_\bullet \to N_\bullet$ as in [15, Proposition 4.2]. By the naturality of connecting homomorphisms, Diagram 5.5 and Diagram 5.6, the cube on the next page is commutative. If we use Diagram 5.7 instead of Diagram 5.6, we find an analogous commuting cube with $\delta^K \otimes \mathrm{id}$ replaced $\mathrm{id} \otimes \delta^L$ and δ^{K+L} by $(-1)^K \delta^{K+L}$. This together with the construction of external products in proof of Theorem 6.2 implies the assertion.

(2) This assertion follows from [15, Definition 6.2] together with the definition of Yoneda products via the hypercohomology construction. ☐

Lemma 7.2 *1. Assume that the bifunctor $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is associative and distributes over finite products in the sense of satisfying Diagram 5.8, Diagram 5.9 and Diagram 5.10. If there are $A, B, C, E, F, H \in \mathrm{obj}(\mathcal{C})$ such that $A_\bullet \otimes_{\mathcal{C}} C_\bullet \otimes_{\mathcal{C}} H_\bullet$ is a projective resolution of $A \otimes_{\mathcal{C}} C \otimes_{\mathcal{C}} H$ and the corresponding pairs satisfy the conditions of Theorem 6.2, then*

$$\forall x \in \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B), y \in \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C,E), z \in \widehat{\mathrm{Ext}}^p_{\mathcal{C}}(F,H) : (x \vee y) \vee z = x \vee (y \vee z)\,.$$

2. *If the first variable of each completed Ext-functor is set to the coefficient ring R and the required natural isomorphisms Θ' and Θ'' exist, then also cup products for complete cohomology are associative.*

3. *Yoneda products of completed Ext-functors are associative.*

$$
\begin{tikzcd}
\mathrm{Ext}^K_{\mathcal{C}}(A,\widetilde{B}_k)\otimes\mathrm{Ext}^L_{\mathcal{C}}(C,\widetilde{E}_l) \arrow[rr, "\vee"] \arrow[dr, "\delta^K\otimes\mathrm{id}"] \arrow[dd, "\mathrm{Ext}^K_{\mathcal{C}}(r,\widetilde{f}_k)\otimes\mathrm{Ext}^L_{\mathcal{C}}(s,\widetilde{g}_l)"'] & & \mathrm{Ext}^{K+L}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{E}_l) \arrow[dr, "\delta^{K+L}"] \arrow[dd, "\widetilde{\mathrm{Ext}}^{K+L}_{\mathcal{C}}(r\otimes_{\mathcal{C}}s,\widetilde{f}_k\otimes_{\mathcal{C}}\widetilde{g}_l)"' near start] & \\
& \mathrm{Ext}^{K+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1})\otimes\mathrm{Ext}^L_{\mathcal{C}}(C,\widetilde{E}_l) \arrow[rr, "\vee" near start] \arrow[dd, "\mathrm{Ext}^{K+1}_{\mathcal{C}}(r,\widetilde{f}_{k+1})\otimes\mathrm{Ext}^L_{\mathcal{C}}(s,\widetilde{g}_l)" near start] & & \mathrm{Ext}^{K+L+1}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{E}_l) \arrow[dd, "\widetilde{\mathrm{Ext}}^{K+L+1}_{\mathcal{C}}(r\otimes_{\mathcal{C}}s,\widetilde{f}_{k+1}\otimes_{\mathcal{C}}\widetilde{g}_l)"] \\
\mathrm{Ext}^K_{\mathcal{C}}(X,\widetilde{M}_k)\otimes\mathrm{Ext}^L_{\mathcal{C}}(Y,\widetilde{N}_l) \arrow[rr, "\vee" near end] \arrow[dr, "\delta^K\otimes\mathrm{id}"] & & \mathrm{Ext}^{K+L}_{\mathcal{C}}(X\otimes_{\mathcal{C}}Y,\widetilde{M}_k\otimes_{\mathcal{C}}\widetilde{N}_l) \arrow[dr, "\delta^{K+L}"] & \\
& \mathrm{Ext}^{K+1}_{\mathcal{C}}(X,\widetilde{M}_{k+1})\otimes\mathrm{Ext}^L_{\mathcal{C}}(Y,\widetilde{N}_l) \arrow[rr, "\vee"] & & \mathrm{Ext}^{K+L+1}_{\mathcal{C}}(X\otimes_{\mathcal{C}}Y,\widetilde{M}_{k+1}\otimes_{\mathcal{C}}\widetilde{N}_l)
\end{tikzcd}
$$

Proof For Yoneda products this follows for instance from their definition through the hypercohomology construction. In order to establish associativity of external products, we first need to introduce some notation. In the proof of Theorem 6.2 we labelled direct systems by sequences $o \in \{0,1\}^{\mathbb{N}}$ taking the values 0 and 1 infinitely many times that lead to the definition of external products via the resolution construction. Now we consider sequences $o \in \{0,1,2\}^{\mathbb{N}}$ that take each value infinitely often. For $i \in \{0,1,2\}$ and $k \in \mathbb{N}$ we define $F_i(o)_k$ to be the number of times that the sequence o has taken the value i up to and including the k^{th} term. We set $F_i(o)_0 := 0$ and define $F(o) := (F_1(o)_k, F_2(o)_k, F_3(o)_k)_{k\in\mathbb{N}_0}$.

Then the element $(x \vee y) \vee z$ arises from a homomorphism going from the direct limit of

$$\big(\mathrm{Ext}_{\mathcal{C}}^{m+F_1(r)_k}(A, \widetilde{B}_{F_1(r)_k}) \otimes \mathrm{Ext}_{\mathcal{C}}^{n+F_2(r)_k}(C, \widetilde{E}_{F_2(r)_k}) \otimes \mathrm{Ext}_{\mathcal{C}}^{p+F_3(r)_k}(F, \widetilde{H}_{F_3(r)_k})\big)_{k\in\mathbb{N}_0} \tag{7.1}$$

to the direct limit of

$$\big(\mathrm{Ext}_{\mathcal{C}}^{m+n+p+k}(A \otimes_{\mathcal{C}} C \otimes_{\mathcal{C}} F, \widetilde{B}_{F_1(r)_k} \otimes_{\mathcal{C}} \widetilde{E}_{F_2(r)_k} \otimes_{\mathcal{C}} \widetilde{H}_{F_2(r)_k})\big)_{k\in\mathbb{N}_0} \tag{7.2}$$

where $r \in \{0,1,2\}^{\mathbb{N}}$ takes each value infinitely often. This uses that $A_\bullet \otimes_{\mathcal{C}} C_\bullet \otimes_{\mathcal{C}} F_\bullet$ is a projective resolution of $A \otimes_{\mathcal{C}} C \otimes_{\mathcal{C}} H$ and that external products of unerniched Ext-functors are associative which requires the other conditions stated in the lemma as can be seen in Section 5. The element $x \vee (y \vee z)$ arises from an analogous homomorphism between two direct limits arising from a different sequence $s \in \{0,1,2\}^{\mathbb{N}}$ taking each value infinitely often. In order to relate the corresponding direct systems, we tensor the terms in Diagram 6.8 with $\mathrm{Ext}_{\mathcal{C}}^{p+F_3(r)_k}(F, \widetilde{H}_{F_3(r)_k})$ and its corresponding identity map. If $\mathrm{Ext}_{\mathcal{C}}^{p+F_3(r)_k}(F, \widetilde{H}_{F_3(r)_k})$ appears as the right most term in the resulting tensor products, then no sign changes are required. If it occurs as the middle or left most term, then we need to multiply the corresponding connecting homomorphisms δ^{n+k+p} and $\delta^{n+k+p+1}$ on the left hand side with a factor of either $(-1)^{m+F_1(r)_k}$ or of $(-1)^{m+F_1(p)_k+n+F_2(r)_k}$ according to Diagram 5.6 and Diagram 5.7. This brings us into the situation of the proof of [15, Lemma 4.10]. More specifically, the direct limits in Equation 7.1 and Equation 7.2 arising from r and s agree whenever there are infinitely many indices $k \in \mathbb{N}_0$ for which the terms of the sequences $(F(r)_k)_{k\in\mathbb{N}_0}$ and $(F(s)_k)_{k\in\mathbb{N}_0}$ coincide.

However, they coincide only in finitely many terms in general. Thus, it suffices to construct a sequence $u \in \{0,1,2\}^{\mathbb{N}}$ inductively such that $F(u)$ has infinitely many terms in common with both $F(r)$ and $F(s)$. Note that the latter sequences coincide for $k = 0$. Assume that $\kappa \in \mathbb{N}_0$ is the last index for which $F(r)_\kappa = F(s)_\kappa$ and set $u_k := r_k$ for $1 \leq k \leq \kappa$. Inductively assume that there is $K \in \mathbb{N}_0$ such that $F(u)_K = F(r)_K$. As $F(u)_K \neq F(s)_K$, there are $\{h, i, j\} = \{0, 1, 2\}$ such that $F_h(u) > F_h(s)$ and $F_i(u) < F_i(s)$. We apply the following procedure for $k \geq K$. If $s_k = j$, we set $u_k := j$. In case $s_k = h$, set $u_k := i$ and in case $s_k = i$, set $u_k := h$. This yields an index $K' \geq K$ for which $F_h(u)_{K'} = F_h(s)_{K'}$ or $F_i(u)_{K'} = F_i(s)_{K'}$. If this is the case for the value h and we have that $F_j(u)_{K'} \neq F_j(s)_{K'}$, then we repeat the above procedure where we swap the roles of the values h and j. This provides us an index $K'' \geq K'$ such that $F(u)_{K''} = F(s)_{K''}$. The same procedures yield an index $K''' \geq K''$ such that $F(u)_{K'''} = F(r)_{K'''}$ from which we can inductively construct the desired desired sequence $u \in \{0, 1, 2\}$. □

Lemma 7.3 *For any $A \in \mathrm{obj}(\mathcal{C})$ denote by 1_A the element in $\widehat{\mathrm{Ext}}_{\mathcal{C}}^0(A, A)$ represented by the (almost) chain map $\mathrm{id}_\bullet : A_\bullet \to A_\bullet$.*

1. *Assume that Diagram 5.16 and Diagram 5.17 commute and that $R, M \in \mathrm{obj}(\mathcal{C}_{R,G})$ possess projective resolutions $R_\bullet$, $M_\bullet$ such that they satisfy all conditions ensuring the existence of cup products in complete cohomology. Then*

$$\forall n \in \mathbb{Z}, x \in \widehat{H}^n_R(G, M) : \widehat{H}^n_R(G, \Theta'_M)(x \smile 1_R) = x = \widehat{H}^n_R(G, \Theta''_M)(1_R \smile x)\,.$$

In particular, if cup products satisfy the conditions of Lemma 7.2, then they turn $\bigoplus_{n\in\mathbb{Z}} \widehat{H}^n_R(G, R)$ into a graded ring with identity 1_R.

2. *One has that*

$$\forall B \in \mathrm{obj}(\mathcal{C}), n \in \mathbb{Z}, y \in \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, B), z \in \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(B, A) : y \circ 1_A = y \text{ and } 1_A \circ z = z\,.$$

Hence, the Yoneda product turns $\bigoplus_{n\in\mathbb{Z}} \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A, A)$ *into a graded ring with identity* 1_A.

Proof As the assertion for Yoneda products follows from their construction via the hypercohomology construction, we prove the assertion for cup products by the resolution construction. For any $x \in \widehat{H}^n_R(G, M)$ there is $k \in \mathbb{N}_0$ and a morphism $f : R_{n+k} \to \widetilde{M_k}$ such that the element in $H^{n+k}_R(G, \widetilde{M_k})$ represented by f is mapped to x in the direct limit. By [15, Proposition 6.4] the augmentation map $\varepsilon : R_0 \to R$ gives rise to the element in $H^0_R(G, R) = \mathrm{Ext}^0_{\mathcal{C}_{R,G}}(R, R)$ represented by $\mathrm{id}_\bullet : R_\bullet \to R_\bullet$ which is mapped to $1_R \in \widehat{H}^0_R(G, R)$ in the direct limit. Then passing through an appropriate direct system and using Diagram 5.16 as well as Diagram 5.17 implies the desired identities. More specifically, the right hand morphism in Diagram 5.16 gives rise to x in the direct limit and the left hand morphism to $x \smile 1_R$. Taking a projective resolution of $M \otimes_R R$ whose first terms are of the form $M_l \otimes_R R$, we see that the bottom morphism gives rise to the isomorphism $\widehat{H}^n_R(G, \Theta'_M)$. The top morphism gives rise to isomorphisms in group cohomology and thus to an isomorphism in complete cohomoloogy after taking the direct limit. This proves that $\widehat{H}^n_R(G, \Theta'_M)(x \smile 1_R) = x$. An analogous argument using Θ''_M instead of Θ'_M and Diagram 5.17 instead of Diagram 5.16 demonstrates that $\widehat{H}^n_R(G, \Theta''_M)(1_R \smile x) = x$.$\square$

Proposition 7.4 *Let* $\Phi^\bullet : \mathrm{Ext}^\bullet_{\mathcal{C}}(A, -) \to \widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(A, -)$ *denote the canonical morphism of the Mislin completion.*

1. *Then* $\Phi^\bullet$ *preserves external and cup products whenever they exist. More specifically, if* A, B, C *and* E *are objects in* $\mathcal{C}$ *satisfying the conditions of Theorem 6.2, then*

$$\forall x \in \mathrm{Ext}^m_{\mathcal{C}}(A, B), y \in \mathrm{Ext}^n_{\mathcal{C}}(C, E) : \Phi^m(x) \vee \Phi^n(y) = \Phi^{m+n}(x \vee y)\,.$$

The corresponding statement for cup products is obtained by setting $A, C = R$ *and by assuming the existence of the required natural isomorphisms* Θ' *and* Θ''. *In particular, if the conditions of Lemma 7.3 are satisfied and the ring structure derived from the cup product is taken, then*

$$\bigoplus_{n\in\mathbb{Z}} \Phi^n(R) : \bigoplus_{n\in\mathbb{Z}} H^n_R(G, R) \to \bigoplus_{n\in\mathbb{Z}} \widehat{H}^\bullet_R(G, R)$$

is a ring homomorphism.

2. *Moreover,* $\Phi^\bullet$ *preserves Yoneda products, meaning that*

$$\forall \xi \in \mathrm{Ext}^m_{\mathcal{C}}(H, J), \upsilon \in \mathrm{Ext}^n_{\mathcal{C}}(F, H) : \Phi^n(\upsilon) \circ \Phi^m(\xi) = \Phi^{m+n}(\upsilon \circ \xi)\,.$$

In particular, if one takes the ring structure derived from the Yoneda product, then

$$\bigoplus_{n\in\mathbb{Z}} \Phi^n(F) : \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^n_{\mathcal{C}}(F, F) \to \bigoplus_{n\in\mathbb{Z}} \widehat{\mathrm{Ext}}^\bullet_{\mathcal{C}}(F, F)$$

is a ring homomorphism.

Proof (1) In the proof of Proposition 3.10 we have seen that the canonical morphism $\Phi^n : \mathrm{Ext}^n_{\mathcal{C}}(A,-) \to \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(A,-)$ can be taken as the canonical morphism to the direct limit occurring in the resolution construction. This together with the construction of external products via the resolution construction implies that $\Phi^\bullet$ preserves external and cup products. By the proof of Lemma 7.3, it induces the desired ring homomorphism because the identity element of $\bigoplus_{n\in\mathbb{Z}} H^n_R(G,R)$ arising from the augmentation map $\varepsilon : R_0 \to R$ is sent to the identity 1_R of $\bigoplus_{n\in\mathbb{Z}} \widehat{H}^\bullet_R(G,R)$.

(2) The definition of the Yoneda product via the hypercohomology construction from the proof of Theorem 6.6 is analogous to the one of Yoneda products of unenriched Ext-functors. Namely, instead of considering almost chain maps modulo chain homotopy we take chain maps modulo chain homotopy and compose them. Thus, $\Phi^\bullet$ preserves Yoneda products by Proposition 3.10. If the element 1_F from Lemma 7.3 is also taken to be a chain map modulo chain homotopy living in $\mathrm{Ext}^0_{\mathcal{C}}(F,F)$, then it represents the identity of $\bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^n_{\mathcal{C}}(F,F)$ with the ring structure derived from the Yoneda product. Because $\bigoplus_{n\in\mathbb{Z}} \Phi^n(F)$ maps the identity element to the identity element by Proposition 3.10, it is a ring homomorphism. $\square$

The following proposition states that external and cup products satisfy a form of graded commutativity.

Proposition 7.5 *Let A, B, C and E satisfy the conditions of Theorem 6.2 and assume that Diagram 5.13 commutes. If one takes the homomorphism* swap *as in the latter diagram, then the homomorphisms in the following commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}^{m+k}_{\mathcal{C}}(A,\widetilde{B}_k)\otimes\mathrm{Ext}^{n+k}_{\mathcal{C}}(C,\widetilde{E}_k) \arrow[r, "(-1)^{(m+k)(n+k)}\mathrm{swap}"] \arrow[d, "\delta^{m+k}\otimes\mathrm{id}"] & \mathrm{Ext}^{n+k}_{\mathcal{C}}(C,\widetilde{E}_k)\otimes\mathrm{Ext}^{m+k}_{\mathcal{C}}(A,\widetilde{B}_k) \arrow[d, "(-1)^{n+k}\mathrm{id}\otimes\delta^{m+k}"] \\
\mathrm{Ext}^{m+k+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1})\otimes\mathrm{Ext}^{n+k}_{\mathcal{C}}(C,\widetilde{E}_k) \arrow[r, "(-1)^{(m+k+1)(n+k)}\mathrm{swap}"] \arrow[d, "(-1)^{m+k+1}\mathrm{id}\otimes\delta^{n+k}"] & \mathrm{Ext}^{n+k}_{\mathcal{C}}(C,\widetilde{E}_k)\otimes\mathrm{Ext}^{m+k+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1}) \arrow[d, "\delta^{n+k}\otimes\mathrm{id}"] \\
\mathrm{Ext}^{m+k+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1})\otimes\mathrm{Ext}^{n+k+1}_{\mathcal{C}}(C,\widetilde{E}_{k+1}) \arrow[r, "(-1)^{(m+k+1)(n+k+1)}\mathrm{swap}"] & \mathrm{Ext}^{n+k+1}_{\mathcal{C}}(C,\widetilde{E}_{k+1})\otimes\mathrm{Ext}^{m+k+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1})
\end{tikzcd} \tag{7.3}$$

give rise to the homomorphism

$$\widehat{\mathrm{swap}} : \widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B)\otimes\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C,E) \to \widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C,E)\otimes\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B)$$

in the direct limit. Moreover, the diagram

$$\begin{tikzcd}
\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B)\otimes\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C,E) \arrow[r, "\vee"] \arrow[d, "\widehat{\mathrm{swap}}"] & \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(A\otimes_{\mathcal{C}} C, B\otimes_{\mathcal{C}} E) \arrow[d, "\widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(\mathrm{swap},\mathrm{swap})"] \\
\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C,E)\otimes\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B) \arrow[r, "\vee"] & \widehat{\mathrm{Ext}}^{m+n}_{\mathcal{C}}(B\otimes_{\mathcal{C}} E, A\otimes_{\mathcal{C}} C)
\end{tikzcd} \tag{7.4}$$

commutes. The corresponding statements for cup products are obtained by imposing that $A, C = R$ and by assuming the existence of the required natural isomorphisms Θ' and Θ''.

Proof If we apply Proposition 6.1, the homomorphisms on the left hand side of Diagram 7.3 do not seem to yield $\widehat{\mathrm{Ext}}^m_{\mathcal{C}}(A,B)\otimes\widehat{\mathrm{Ext}}^n_{\mathcal{C}}(C,E)$ in the direct limit according to the resolution construction. However, if we concatenate four copies of Diagram 7.3, the sign issues vanish yielding the direct limit of the desired form. For the same reason the homomorphisms on the right hand side result in the correct direct limit from which we obtain the homomorphism $\widehat{\text{swap}}$. Next, we form with Diagram 5.13 the following direct system of commuting squares. We connect its left hand homomorphisms via Diagram 7.3 to each other. We connect its top and bottom homomorphisms via Diagram 5.6 and Diagram 5.7 where the factor of $(-1)^m$ is assigned to the term $\mathrm{id}\otimes\delta^n$ instead of the term δ^{m+n} in the latter diagram. According to Equation 5.14, the right hand morphisms are of the form $\mathrm{Ext}^{m+n+2k}(\mathrm{swap},\mathrm{swap})$ and connected by connecting homomorphism. Then Diagram 7.4 results as the direct limit of these squares. $\square$

Lemma 7.6 *1. Let $0\to B\to B'\to B''\to 0$ and $0\to E\to E'\to E''\to 0$ be short exact sequences in $\mathcal{C}$. If for $\mathcal{B}\in\{B,B',B''\}$ and $\mathcal{E}\in\{E,E',E''\}$ the objects A, $\mathcal{B}$, C, E as well as A, B, C, $\mathcal{E}$ satisfy the conditions of Theorem 6.2, then the diagrams*

$$\begin{CD}
\widehat{\mathrm{Ext}}_{\mathcal{C}}^{m}(A,B)\otimes\widehat{\mathrm{Ext}}_{\mathcal{C}}^{n}(C,E) @>{\vee}>> \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(A\otimes_{\mathcal{C}}C,B\otimes_{\mathcal{C}}E)\\
@V{\widehat{\delta}^m\otimes\mathrm{id}}VV @VV{\widehat{\delta}^{m+n}}V\\
\widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+1}(A,B'')\otimes\widehat{\mathrm{Ext}}_{\mathcal{C}}^{n}(C,E) @>{\vee}>> \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n+1}(A\otimes_{\mathcal{C}}C,B''\otimes_{\mathcal{C}}E)
\end{CD}\tag{7.5}$$

and

$$\begin{CD}
\widehat{\mathrm{Ext}}_{\mathcal{C}}^{m}(A,B)\otimes\widehat{\mathrm{Ext}}_{\mathcal{C}}^{n}(C,E) @>{\vee}>> \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(A\otimes_{\mathcal{C}}C,B\otimes_{\mathcal{C}}E)\\
@V{\mathrm{id}\otimes\widehat{\delta}^n}VV @VV{(-1)^m\widehat{\delta}^{m+n}}V\\
\widehat{\mathrm{Ext}}_{\mathcal{C}}^{m}(A,B)\otimes\widehat{\mathrm{Ext}}_{\mathcal{C}}^{n+1}(C,E'') @>{\vee}>> \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n+1}(A\otimes_{\mathcal{C}}C,B\otimes_{\mathcal{C}}E'')
\end{CD}\tag{7.6}$$

commute. The corresponding diagrams for cup products are obtained by setting $A,C=R$ and by assuming the existence of the required natural isomorphisms Θ' and Θ''.

2. For any $F,H,J\in\mathrm{obj}(\mathcal{C})$ and any short exact sequence $0\to J\to J'\to J''\to 0$ the diagram

$$\begin{CD}
\widehat{\mathrm{Ext}}_{\mathcal{C}}^{n}(H,J)\otimes\widehat{\mathrm{Ext}}_{\mathcal{C}}^{m}(F,H) @>{\circ}>> \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n}(F,J)\\
@V{\widehat{\delta}^n\circ\mathrm{id}}VV @VV{\widehat{\delta}^{m+n}}V\\
\widehat{\mathrm{Ext}}_{\mathcal{C}}^{n+1}(H,J'')\otimes\widehat{\mathrm{Ext}}_{\mathcal{C}}^{m}(F,H) @>{\circ}>> \widehat{\mathrm{Ext}}_{\mathcal{C}}^{m+n+1}(F,J'')
\end{CD}$$

commutes.

Proof The statement about Yoneda products follows their definition via the hypercohomology construction and the construction of the connecting homomorphism found in [15, Definition 6.5]. Let us argue that Diagram 7.5 commutes. It follows from Diagram 6.8

and the proof of Theorem 6.2 that the diagram

$$
\begin{tikzcd}[column sep=small]
& \mathrm{Ext}^K_{\mathcal{C}}(A,\widetilde{B}_k)\otimes\mathrm{Ext}^L_{\mathcal{C}}(C,\widetilde{E}_l) \arrow[rr, "\vee"] \arrow[dl, "(-1)^k\delta^K\otimes\mathrm{id}"'] \arrow[dd, "\mathrm{id}\otimes\delta^L"] & & \mathrm{Ext}^{K+L}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{E}_l) \arrow[dl, "(-1)^k\delta^{K+L}"'] \arrow[dd, "(-1)^K\delta^{K+L}"] \\
\mathrm{Ext}^{K+1}_{\mathcal{C}}(A,\widetilde{B}''_{k+1})\otimes\mathrm{Ext}^L_{\mathcal{C}}(C,\widetilde{E}_l) \arrow[rr, "\vee", crossing over] \arrow[dd, "\mathrm{id}\otimes\delta^L"] & & \mathrm{Ext}^{K+L+1}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}''_{k+1}\otimes_{\mathcal{C}}\widetilde{E}_l) & \\
& \mathrm{Ext}^K_{\mathcal{C}}(A,\widetilde{B}_k)\otimes\mathrm{Ext}^{L+1}_{\mathcal{C}}(C,\widetilde{E}_{l+1}) \arrow[rr, "\vee"] \arrow[dl, "(-1)^k\delta^K\otimes\mathrm{id}"'] \arrow[dd, "\delta^K\otimes\mathrm{id}"] & & \mathrm{Ext}^{K+L+1}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_k\otimes_{\mathcal{C}}\widetilde{E}_{l+1}) \arrow[dl, "(-1)^k\delta^{K+L+1}"'] \arrow[dd, "\delta^{K+L+1}"] \\
\mathrm{Ext}^{K+1}_{\mathcal{C}}(A,\widetilde{B}''_{k+1})\otimes\mathrm{Ext}^{L+1}_{\mathcal{C}}(C,\widetilde{E}_{l+1}) \arrow[rr, "\vee", crossing over] \arrow[dd, "\delta^{K+1}\otimes\mathrm{id}"] & & \mathrm{Ext}^{K+L+2}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}''_{k+1}\otimes_{\mathcal{C}}\widetilde{E}_{l+1}) \arrow[from=2-3, "(-1)^{K+1}\delta^{K+L+1}", crossing over] & \\
& \mathrm{Ext}^{K+1}_{\mathcal{C}}(A,\widetilde{B}_{k+1})\otimes\mathrm{Ext}^{L+1}_{\mathcal{C}}(C,\widetilde{E}_{l+1}) \arrow[rr, "\vee"] \arrow[dl, "(-1)^{k+1}\delta^{K+1}\otimes\mathrm{id}"'] & & \mathrm{Ext}^{K+L+2}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}_{k+1}\otimes_{\mathcal{C}}\widetilde{E}_{l+1}) \arrow[dl, "(-1)^{k+1}\delta^{K+L+2}"'] \\
\mathrm{Ext}^{K+2}_{\mathcal{C}}(A,\widetilde{B}''_{k+2})\otimes\mathrm{Ext}^{L+1}_{\mathcal{C}}(C,\widetilde{E}_{l+1}) \arrow[rr, "\vee"] & & \mathrm{Ext}^{K+L+3}_{\mathcal{C}}(A\otimes_{\mathcal{C}}C,\widetilde{B}''_{k+2}\otimes_{\mathcal{C}}\widetilde{E}_{l+1}) \arrow[from=4-3, "\delta^{K+L+2}", crossing over] &
\end{tikzcd}
\tag{7.7}
$$

is commutative. The left hand side of Diagram 7.7 gives rise to the left hand homomorphisms of Diagram 7.5, the front side to the bottom homomorphism and the back side to the top homomorphism. After concatenating four copies of Diagram 7.7, the resulting direct system from the right hand side gives rise to the right hand homomorphism in

Diagram 7.5. Again by Diagram 6.8 and the proof of Theorem 6.2, it follows inductively for $l \in \mathbb{N}_0$ that the diagram

$$\begin{tikzcd}
& \operatorname{Ext}^K_{\mathcal{C}}(A, \widetilde{B}_k) \otimes \operatorname{Ext}^L_{\mathcal{C}}(C, \widetilde{E}_l) \arrow[rr, "\vee"] \arrow[dl, "(-1)^{m+l}\mathrm{id}\otimes\delta^L"'] \arrow[dd, "\delta^K\otimes\mathrm{id}"] & & \operatorname{Ext}^{K+L}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{E}_l) \arrow[dl, "(-1)^l\delta^{K+L}"] \arrow[dd, "\delta^{K+L}"] \\
\operatorname{Ext}^K_{\mathcal{C}}(A, \widetilde{B}_k) \otimes \operatorname{Ext}^{L+1}_{\mathcal{C}}(C, \widetilde{E}''_{l+1}) \arrow[rr, "\vee"] \arrow[dd, "\delta^K\otimes\mathrm{id}"] & & \operatorname{Ext}^{K+L+1}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, \widetilde{B}_k \otimes_{\mathcal{C}} \widetilde{E}''_{l+1}) \arrow[dd, "\delta^{K+L+1}"] & \\
& \operatorname{Ext}^{K+1}_{\mathcal{C}}(A, \widetilde{B}_{k+1}) \otimes \operatorname{Ext}^L_{\mathcal{C}}(C, \widetilde{E}_l) \arrow[rr, "\vee"] \arrow[dl, "(-1)^{m+l}\mathrm{id}\otimes\delta^L"'] \arrow[dd, "\mathrm{id}\otimes\delta^L"] & & \operatorname{Ext}^{K+L+1}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, \widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{E}_l) \arrow[dl, "(-1)^{l+1}\delta^{K+L+1}"] \arrow[dd, "(-1)^{m+1}\delta^{K+L+1}"] \\
\operatorname{Ext}^{K+1}_{\mathcal{C}}(A, \widetilde{B}_{k+1}) \otimes \operatorname{Ext}^{L+1}_{\mathcal{C}}(C, \widetilde{E}''_{l+1}) \arrow[rr, "\vee"] \arrow[dd, "\mathrm{id}\otimes\delta^{L+1}"] & & \operatorname{Ext}^{K+L+2}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, \widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{E}''_{l+1}) \arrow[dd, "(-1)^{m+1}\delta^{K+L+2}"] & \\
& \operatorname{Ext}^{K+1}_{\mathcal{C}}(A, \widetilde{B}_{k+1}) \otimes \operatorname{Ext}^{L+1}_{\mathcal{C}}(C, \widetilde{E}_{l+1}) \arrow[rr, "\vee"] \arrow[dl, "(-1)^{m+l+1}\mathrm{id}\otimes\delta^{L+1}"'] & & \operatorname{Ext}^{K+L+2}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, \widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{E}_{l+1}) \arrow[dl, "(-1)^{l+2}\delta^{K+L+2}"] \\
\operatorname{Ext}^{K+1}_{\mathcal{C}}(A, \widetilde{B}_{k+1}) \otimes \operatorname{Ext}^{L+2}_{\mathcal{C}}(C, \widetilde{E}''_{l+2}) \arrow[rr, "\vee"] & & \operatorname{Ext}^{K+L+3}_{\mathcal{C}}(A \otimes_{\mathcal{C}} C, \widetilde{B}_{k+1} \otimes_{\mathcal{C}} \widetilde{E}''_{l+2}) &
\end{tikzcd}$$

is commutative. As before, it give rise to a direct system of commuting squares in whose direct limit we obtain Diagram 7.6. $\square$

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References

[1] A. Adem and R. J. Milgram. *Cohomology of Finite Groups*, volume 309 of *Grundlehren der mathematischen Wissenschaften / A Series of Comprehensive Studies in Mathematics*. Springer-Verlag Berlin Heidelberg New York, 2nd edition, 2004.

[2] A. Beligiannis and I. Reiten. *Homological and Homotopical Aspects of Torsion Theories*, volume 188 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2007.

[3] D. J. Benson and J. F. Carlson. Products in Negative Cohomology. *Journal of Pure and Applied Algebra*, 82(2):107–129, 1992.

[4] M. Boggi and G. Corob Cook. Continuous Cohomology and Homology of Profinite Groups. *Documenta Mathematica*, 21:1269–1312, 2016.

[5] K. S. Brown. *Cohomology of Groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag New York, Inc., 1982.

[6] H. Cartan and S. Eilenberg. *Homological Algebra*, volume 19 of *Princeton Mathematical Series*. Princeton University Press, 1956.

[7] O. Celikbas, L. W. Christensen, L. Liang, and G. Piepmeyer. Complete Homology over Associative Rings. *Israel Journal of Mathematics*, 221(1):1–24, 2017.

[8] J. Cornick and P. H. Kropholler. On Complete Resolutions. *Topology and its Applications*, 78(3):235—250, 1997.

[9] G. Corob Cook. On Profinite Groups of Type FP_∞. *Advances in Mathematics*, 294:216–255, 2016.

[10] F. Dembegioti. On the Zeroeth Complete Cohomology. *Journal of Pure and Applied Algebra*, 203(1–3):119–132, 2005.

[11] F. Dembegioti. On the Zeroeth Complete Cohomology of Certain Polycyclic Groups. *Communications in Algebra*, 36(5):1927–1941, 2008.

[12] J. D. Dixon, M. P. F. du Sautoy, A. Mann, and D. Segal. *Analytic Pro-p Groups*, volume 61 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2nd edition, 1999.

[13] F. T. Farrell. An Extension of Tate Cohomology to a Class of Infinite Groups. *Journal of Pure and Applied Algebra*, 10(2):153–161, 1977.

[14] S. I. Gelfand and Yu. I. Manin. *Methods of Homological Algebra*. Springer Monographs in Mathematics. Springer-Verlag Berlin Heidelberg, 2nd edition, 2003.

[15] M. Gheorghiu. On a Completion of Cohomological Functors Generalising Tate Cohomology I, May 2024. To Appear.

[16] F. Goichot. Homologie de Tate-Vogel Équivariante. *Journal of Pure and Applied Algebra*, 82(1):39–64, 1992.

[17] S. Guo and L. Liang. Stable Functors and Cohomology Theory in Abelian Categories. *arXiv e-prints*, page arXiv:2006.13562v3, October 2023.

[18] J. Hu, D. Zhang, T. Zhao, and P. Zhou. Complete Cohomology for Extriangulated Categories. *Algebra Colloquium*, 28(4):701–720, 2021.

[19] J. H. Jo. Projective Complete Cohomological Dimension of a Group. *International Mathematics Research Notices*, 2004(13):621–636, 2004.

[20] J. H. Jo. Complete Homology and Related Dimensions of Groups. *Journal of Group Theory*, 12(3):431–448, 2009.

[21] P. H. Kropholler. Hierarchical Decompositions, Generalized Tate Cohomology, and Groups of Type $(\mathrm{FP})_\infty$. In *Combinatorial and Geometric Group Theory, Edinburgh 1993*, volume 204 of *London Mathematical Society Lecture Note Series*, pages 190–216. Cambridge University Press, January 1995.

[22] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer New York, NY, 2nd edition, 1978.

[23] S. Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag Berlin Heidelberg, 1995.

[24] G. Mislin. Tate Cohomology for Arbitrary Groups via Satellites. *Topology and its Applications*, 56:293–300, 1994.

[25] M. S. Osborne. *Basic Homological Algebra*, volume 196 of *Graduate Texts in Mathematics*. Springer-Verlag New York, Inc., 2000.

[26] M. Paganin. On Some Generalizations of Tate Cohomology: an Overview. *Pro Mathematica*, 29(58):63–115, 2016.

[27] B. Poonen. Why All Rings should have a 1. `https://math.mit.edu/~poonen/papers/ring.pdf`, June 2018. This website has been consulted on 25 October 2023.

[28] L. Ribes. *Profinite Graphs and Groups*, volume 66 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics.* Springer Nature, 2017.

[29] L. Ribes and P. Zalesskii. *Profinite Groups*, volume 40 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics.* Springer-Verlag Berlin Heidelberg, 2nd edition, 2010.

[30] J. J. Rotman. *Advanced Modern Algebra.* Prentice Hall, 2002.

[31] P. Scholze. Lectures on Condensed Mathematics. `https://www.math.uni-bonn.de/people/scholze/Condensed.pdf`, May 2019. This website has been consulted on 21 February 2023.

[32] P. Scholze. Liquid Tensor Experiment. `https://xenaproject.wordpress.com/2020/12/05/liqui` December 2020. This website has been consulted on 25 September 2023.

[33] J.-P. Serre. *Cohomologie Galoisienne*, volume 5 of *Lecture Notes in Mathematics.* Springer-Verlag Berlin Heidelberg, 5th edition, 1994.

[34] The Stacks project authors. The Stacks Project. `https://stacks.math.columbia.edu`, 2024.

[35] P. Symonds. The Tate-Farrell Cohomology of the Morava Stabilizer Group s_{p-1} with Coefficients in e_{p-1}. In *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory*, volume 346 of *Contemporary Mathematics*, pages 485–402. American Mathematical Society, 2004.

[36] P. Symonds. Permutation Complexes for Profinite Groups. *Commentarii Mathematici Helvetici*, 82(1):1–37, 2007.

[37] P. Symonds and T. Weigel. Cohomology of p-adic Analytic Groups. In *New Horizons in Pro-p Groups*, volume 184 of *Progress in Mathematics*, pages 347–408. Birkhäuser, Boston, 2000.

[38] C. A. Weibel. *An Introduction to Homological Algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, 1994.

[39] J. S. Wilson. *Profinite Groups*, volume 19 of *London Mathematical Society Monographs New Series.* Oxford University Press Inc., New York, 1998.