

Several Special Solutions of Open WDVV Equations

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Abstract

The Witten-Dijkgraaf-Verlinde-Verlinde(WDVV) equations appeared in the study of two-dimensional topological field theories in the early 1990s[15][5]. An extension of the WDVV equations, called the open WDVV equations, was introduced by A.Horev and J.P.Solomon[12]. In this paper, we give some particular solutions to the open WDVV equations.

1 Introduction

The WDVV equations are the following systems of equations¹:

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F}{\partial t^\delta \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^3 F}{\partial t^\nu \partial t^\gamma \partial t^\alpha}, \quad (1)$$

that hold for $1 \leq \alpha, \beta, \gamma, \delta \leq N$. Here $F = F(t^1, \dots, t^N)$ is an analytic function defined on some open sets $M \subset \mathbb{C}^N$. $\eta = (\eta_{\alpha\beta})$ is an $N \times N$ dimensional constant non-degenerate matrix and $(\eta^{\alpha\beta}) := \eta^{-1}$. We use the convention of summing over repeated Greek indices. The corresponding Frobenius manifolds on M can be defined due to the work of Dubrovin [7].

Particular examples of solutions to WDVV equations are interesting. B.Dubrovin constructs closed Frobenius structures on the space of orbits of Coxeter groups, where all of solutions to WDVV equations are polynomials [6]. B.Dubrovin gives the closed potential of extended affine group $A_1^{(1)}$ in [7]. For a particular choice of basis of the root systems, B.Dubrovin and Y.Zhang construct Frobenius structures on the orbit spaces of the extended affine Weyl groups [9]. In fact, the geometric structure revealed in [9] does not depend on the choice of basis of root systems in case A_l , B_l , C_l and D_l , according to the work of B.Dubrovin, Y.Zhang [9](case A_l) and B.Dubrovin, I.Strachan, Y.Zhang, D.Zuo [8](case B_l , C_l , D_l). In these cases, the solutions are triangular polynomials. In [13], M.Kontsevich and Yu.I.Manin show that solutions of quantum cohomology type are triangular series.

Recently, another system of equations appears in the research of Gromov-Witten theory [12]. Let $F = F(t^1, \dots, t^N)$ be a solution of the WDVV equations. The open WDVV equations associated to F are the following system of equations, whose solution $F^o = F^o(t^1, \dots, t^N, s)$ depends on another variable s :

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^2 F^o}{\partial t^\nu \partial t^\gamma} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta} \frac{\partial^2 F^o}{\partial s \partial t^\gamma} = \frac{\partial^3 F}{\partial t^\gamma \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^2 F^o}{\partial t^\nu \partial t^\alpha} + \frac{\partial^2 F^o}{\partial t^\gamma \partial t^\beta} \frac{\partial^2 F^o}{\partial s \partial t^\alpha}, \quad (2)$$

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\mu} \eta^{\mu\nu} \frac{\partial^2 F^o}{\partial t^\nu \partial s} + \frac{\partial^2 F^o}{\partial t^\alpha \partial t^\beta} \frac{\partial^2 F^o}{\partial s^2} = \frac{\partial^2 F^o}{\partial s \partial t^\beta} \frac{\partial^2 F^o}{\partial s \partial t^\alpha}, \quad (3)$$

that hold for $1 \leq \alpha, \beta, \gamma, \delta \leq N$. We consider solutions satisfying the following quasihomogenous condition [3]

$$E^\gamma \frac{\partial F^o}{\partial t^\gamma} + \frac{1-d}{2} s \frac{\partial F^o}{\partial s} = \frac{3-d}{2} F^o + D_\gamma t^\gamma + \tilde{D} s + E, \quad D_\gamma, \tilde{D}, E \in \mathbb{C} \quad (4)$$

and additional conditions

$$\frac{\partial^2 F^o}{\partial t^1 \partial t^\alpha} = 0, \quad \frac{\partial^2 F^o}{\partial t^1 \partial s} = 1. \quad (5)$$

¹These systems of equations are called WDVV associativity equations in Dubrovin's original literature [7]. Besides the associativity equations, the WDVV equations in [7] also contain homogeneous equations. For simplicity, we call the associativity part the WDVV equations in this paper.

A generalization of Frobenius manifolds, called F-manifolds, was introduced by C.Hertling and Y.Manin [11]. Flat F-manifolds were studied by E.Getzler[10] and Manin[14]. According to A.Adam [1] and P.Rossi, open WDVV equations can be interpreted as the rank-1 extension of the Dubrovin-Frobenius manifold to flat F-manifold.

It is natural to consider the solutions of open WDVV equations associated to the Dubrovin-Frobenius manifolds of other finite irreducible Coxeter groups, extended affine Weyl groups and quantum cohomology.

A.Adam proves the existence of polynomial solutions to open WDVV equations associated to the extension of A_N singularity [1]. A.Basalaev and A.Buryak give the explicit solution in [3]. In fact, more generally, for finite irreducible Coxeter groups A_N , B_N , D_N and $I_2(k)$, A.Basalaev and A.Buryak also give the following explicit solutions to the associated open WDVV equations: [3] [2]

$$A_N: F_{A_N}^o = F_{A_N}^o(t^1, \dots, t^N, s), \text{ with}$$

$$\frac{\partial^{m+k} F_{A_N}^o}{\partial t^{\alpha_1} \dots \partial t^{\alpha_m} (\partial s)^k} \Big|_{t^1 = \dots = t^N = s=0} = \begin{cases} (m+k-2)!, & \text{if } \sum_{i=1}^m (N+2-\alpha_i) + k = N+2, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_N: F_{B_N}^o(t^1, \dots, t^N, s) := F_{A_{2N-1}}^o(t^1, 0, t^2, 0, \dots, t^{N-1}, 0, t^N, s).$$

$$D_N: F_{D_N}^o(t^1, \dots, t^N, s) := \frac{s^{2N-1}}{2^{N-2}(2N-1)(2N-2)} + \frac{(t^N)^2}{2s} + \sum_{k=1}^{N-1} \frac{v_k^D s^{2k-1}}{2^{k-1}(2k-1)}, \text{ with}$$

$$v_b^D = \sum_{\alpha_1, \dots, \alpha_{N-1} \geq 0, \sum_{k=1}^{N-1} (N-k)\alpha_k = N-b} \frac{(|\alpha| + 2b - 3)!}{(2b-2)!} \prod_{k=1}^{N-1} \frac{(t^k)^{\alpha_k}}{\alpha_k!}.$$

$I_2(k-1)$:

$$F_{I_2(k-1)}^o(t^1, t^2, s) = \begin{cases} st^1 + (t^2)^q + \sum_{m=0}^{q-1} \frac{2^{2m+1}}{(2m+2)!} \frac{(2q-1)^m (q-m+1)_{m-2} (q+1)_{m-3}}{(q-1)^m q^m} s^{2q-2m} (t^2)^m, & k = 2q, \\ st^1 + \sum_{m=0}^q b_m s^{2q-2m+1} (t^2)^m, & k = 2q+1. \end{cases}$$

with $(a)_n = a(a+1) \dots (a+n-1)$. b_n satisfies the following recursion relations

$$\begin{aligned} b_q^2 &= \frac{2(2q+1)(2q-1)}{q}, \\ b_{s-q} &= -\frac{\sum_{m=s-q+1}^{q-1} m(1+2m+2q-2s)(m(-1+4q)+s-2q(1+s))b_m b_{s-m}}{2q^2(1+4q-2s)(-1+2q-s)b_q}, \\ q \leq s &\leq 2q-1. \end{aligned}$$

For the extended affine Weyl group case $A_1^{(1)}$, A.Basalaev and A.Buryak obtain the following open solutions [4]

$$F_{A_1^{(1)}}^o(t^1, t^2, s) = t^1 s \pm 2\alpha^{-1} e^{\frac{t^2}{2}} \sinh(\alpha(s+\beta)), \quad \alpha, \beta \in \mathbb{C}.$$

In this paper, we obtain two new solutions to open WDVV equations, which are associated to the Coxeter group H_3 and the extended affine Weyl group $A_2^{(1)}$.

2 Open solutions related to Coxeter groups H_3 and extended affine Weyl groups $A_2^{(1)}$

In the following section, we give two solutions to open WDVV equations associated to the Dubrovin-Frobenius manifolds of the Coxeter group H_3 and the extended affine Weyl group $A_2^{(1)}$. They are new as far as we know.

2.1 H_3 case

The Dubrovin-Frobenius potential for Coxeter group H_3 [7] is

$$F_{H_3}^c(t^1, t^2, t^3) = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}t^1(t^2)^2 + \frac{1}{6}(t^2)^3(t^3)^2 + \frac{1}{20}(t^2)^2(t^3)^5 + \frac{1}{3960}(t^3)^{11}. \quad (6)$$

In this case, the constant $d = \frac{4}{5}$ and we consider solutions F^o satisfying the quasihomogeneous equation

$$t^1 \frac{\partial F^o}{\partial t^1} + \frac{3}{5}t^2 \frac{\partial F^o}{\partial t^2} + \frac{1}{5}t^3 \frac{\partial F^o}{\partial t^3} + \frac{1}{10}s \frac{\partial F^o}{\partial s} = \frac{11}{10}F^o. \quad (7)$$

Proposition 2.1 *A solution of the open WDVV equations for H_3 is*

$$F_{H_3}^o(t^1, t^2, t^3, s) = st^1 + \frac{(t^2)^2}{2s} + t^2 \left(\frac{(t^3)^2 s}{2} - \frac{t^3 s^3}{4} + \frac{s^5}{40} \right) - \frac{(t^3)^2 s^7}{32} + \frac{(t^3)^3 s^5}{8} - \frac{5(t^3)^4 s^3}{24} + \frac{(t^3)^5 s}{10} + \frac{t^3 s^9}{288} - \frac{s^{11}}{7040}. \quad (8)$$

Note that it has a simple pole at $s = 0$, which is similar to the case D_N .

2.2 $A_2^{(1)}$ case

The Dubrovin-Frobenius potential for extended affine Weyl group $A_2^{(1)}$ [7] is

$$F_{A_2^{(1)}}^c = \frac{1}{2}(t^1)^2 t^3 + \frac{1}{2}t^1(t^2)^2 - \frac{1}{24}(t^2)^4 + t^2 e^{t^3}. \quad (9)$$

In this case, the constant $d = 1$ and we consider solutions F^o satisfying the quasihomogeneous equation

$$t^1 \frac{\partial F^o}{\partial t^1} + \frac{1}{2}t^2 \frac{\partial F^o}{\partial t^2} + \frac{3}{2} \frac{\partial F^o}{\partial t^3} = F^o. \quad (10)$$

Using the methods of characteristics, we obtain the following solution to $A_2^{(1)}$:

Proposition 2.2 *A solution of the open WDVV equations for $A_2^{(1)}$ is*

$$F_{A_2^{(1)}}^o(t^1, t^2, t^3, s) = st^1 + (t^2)^2 \sum_{p,q \geq 0} a_{p,q} s^p (t^2 - 3\log(t^3))^q, \quad (11)$$

$$a_{p,q} = \begin{cases} \frac{(-1)^{\frac{p-1}{2}} 2^{2p-6} \sum_{j=1}^{\frac{p+1}{2}} j^q (\sum_{i=1}^j i^{-2j+p+3} b_{i,j})}{p!q!}, & p \text{ is odd}, \\ \frac{(-1)^{p/2} \left(\sum_{j=2}^{\frac{p}{2}+1} (2j-1)^q (\sum_{i=1}^j (-1)^{i+1} (4i-2)^{-2j+p+4} d_{i,j}) + 2^{p-1} \right)}{2^{q-1} p!q!}, & p \text{ is even}. \end{cases} \quad (12)$$

The coefficients $b_{i,j}, d_{i,j}$ satisfy the following recursion relations:

$$b_{1,1} = -8, \quad d_{1,1} = 1. \quad (13)$$

For $p = 2k$, the relation is

$$\begin{aligned} \tilde{A}_{k,q} + \sum_{j=0}^q \sum_{r=2}^{k+1} \sum_{m=1}^r \tilde{B}_{k,q,m,r,j} d_{m,r} + \sum_{i=0}^k \sum_{j=0}^q \sum_{t=1}^{-i+k+1} \sum_{s=1}^t \tilde{C}_{k,q,s,t,i,j} b_{s,t} \\ + \sum_{i=0}^k \sum_{j=0}^q \sum_{t=1}^i \sum_{s=1}^t \tilde{D}_{k,q,s,t,i,j} b_{s,t} + \sum_{i=0}^k \sum_{j=0}^q \sum_{t=1}^{-i+k+1} \sum_{s=1}^t \sum_{r=2}^{i+1} \sum_{m=1}^r \tilde{E}_{k,q,s,t,m,r,i,j} b_{s,t} d_{m,r} \\ + \sum_{i=0}^k \sum_{j=0}^q \sum_{t=1}^i \sum_{s=1}^t \sum_{r=2}^{-i+k+2} \sum_{m=1}^r \tilde{F}_{k,q,s,t,m,r,i,j} b_{s,t} d_{m,r} = 0. \end{aligned} \quad (14)$$

For $p = 2k + 1$, the relation is

$$\begin{aligned} & \sum_{j=0}^q \sum_{r=1}^{k+1} \sum_{m=1}^r \tilde{G}_{k,q,m,r} b_{m,r} + \sum_{i=0}^k \sum_{j=0}^q \sum_{r=2}^{i+1} \sum_{m=1}^r \tilde{H}_{k,q,m,r} d_{m,r} \\ & + \sum_{i=0}^k \sum_{j=0}^q \sum_{t=2}^{-i+k+2} \sum_{s=1}^t \tilde{I}_{k,q,s,t} d_{s,t} + \sum_{i=0}^k \sum_{j=0}^q \sum_{r=1}^{i+1} \sum_{t=1}^{-i+k+1} \sum_{m=1}^r \sum_{s=1}^t \tilde{J}_{k,q,m,r,s,t} b_{m,r} b_{s,t} \\ & + \sum_{i=0}^k \sum_{j=0}^q \sum_{r=2}^{i+1} \sum_{t=2}^{-i+k+2} \sum_{m=1}^r \sum_{s=1}^t \tilde{K}_{k,q,m,r,s,t} d_{m,r} d_{s,t} = 0, \end{aligned} \quad (15)$$

with coefficients given by

$$\tilde{A}_{k,q} = \sum_{j=0}^q \frac{(-1)^k 2^{-j+2k-2}}{j!(2k)!(q-j)!}, \quad (16)$$

$$\tilde{B}_{k,q,m,r,j} = \frac{(36r^2 - 72r + 35)(2r-1)^j (-1)^{k+m} 2^{-j+2k-2r+3} (2m-1)^{2k-2r+4}}{j!(2k)!(q-j)!}, \quad (17)$$

$$\tilde{C}_{k,q,s,t,i,j} = \frac{(-1)^{k+1} (1-2t) (-2i+2k+1) 2^{-2i-j+4k-5} t^{q-j} s^{-2(i-k+t-2)}}{(2i)!(j)! (-2i+2k+1)!(q-j)!}, \quad (18)$$

$$\tilde{D}_{k,q,s,t,i,j} = \frac{(-1)^{k+1} (2t-1) (-i+k+1) t^{j+1} s^{2i-2t+2} 2^{2i+j+2k-q-5}}{(2i-1)! j! (-2i+2k+2)!(q-j)!}, \quad (19)$$

$$\begin{aligned} \tilde{E}_{k,q,s,t,m,r,i,j} = & \frac{(-1)^{k+m} (2r-2t-1) (-2i+2k+1) (2r-1)^{j+1} (2m-1)^{2i-2r+4} 2^{-2i-j+4k-2r} s^{-2(i-k+t-2)} t^{q-j}}{(2i)! j! (-2i+2k+1)!(q-j)!}, \end{aligned} \quad (20)$$

$$\begin{aligned} \tilde{F}_{k,q,s,t,m,r,i,j} = & \frac{(-1)^{k+m} (-2r+2t+1) (-i+k+1) (2r-1)^{q-j} (2m-1)^{-2i+2k-2r+6} 2^{2i+j+2k-q-2r} s^{2i-2t+2} t^{j+1}}{(2i-1)! j! (-2i+2k+2)!(q-j)!}, \end{aligned} \quad (21)$$

$$\tilde{G}_{k,q,m,r} = \frac{(-1)^{k+1} 16^{k-1} (9r^2 - 9r + 2) r^j m^{2k-2r+4}}{j!(2k+1)!(q-j)!}, \quad (22)$$

$$\tilde{H}_{k,q,m,r} = \frac{(-1)^{k+m+1} 2^{2k-q-2r+8} (r-1) (-i+k+1) (2r-1)^{j+1} (2m-1)^{2i-2r+4}}{(2i)! j! (-2i+2k+2)!(q-j)!}, \quad (23)$$

$$\tilde{I}_{k,q,s,t} = \frac{(-1)^{k+s+1} 2^{2k-q-2t+8} (1-t) (-i+k+1) (2t-1)^{q-j} (2s-1)^{-2i+2k-2t+6}}{(2i)! j! (-2i+2k+2)!(q-j)!}, \quad (24)$$

$$\tilde{J}_{k,q,m,r,s,t} = \frac{(-1)^{k+1} 2^{4k-7} (r-t) r^{j+1} m^{2i-2r+4} t^{q-j} s^{-2(i-k+t-2)}}{(2i+1)! j! (2k-2i)!(q-j)!}, \quad (25)$$

$$\begin{aligned} \tilde{K}_{k,q,m,r,s,t} = & \frac{(-1)^{k+m+s} 2^{2k-q-2r-2t+13} (r-t) (-i+k+1) (2r-1)^{j+1} (2m-1)^{2i-2r+4} (2t-1)^{q-j} (2s-1)^{-2i+2k-2t+6}}{(2i)! (j)! (-2i+2k+2)!(q-j)!}. \end{aligned} \quad (26)$$

Example 2.1 Some values of $b_{p,q}$ are

$$\begin{aligned}
b_{1,1} &= -8, \quad b_{1,2} = -2, \quad b_{1,3} = -\frac{5}{2}, \quad b_{1,4} = -\frac{21}{4}, \quad b_{1,5} = -\frac{231}{16}, \quad b_{1,6} = -\frac{3003}{64}, \quad b_{1,7} = -\frac{21879}{128}, \\
b_{1,8} &= -\frac{692835}{1024}, \quad b_{1,9} = -\frac{11685817}{4096}, \\
b_{2,2} &= 2, \quad b_{2,3} = 16, \quad b_{2,4} = 168, \quad b_{2,5} = 2112, \quad b_{2,6} = 30030, \quad b_{2,7} = 466752, \\
b_{2,8} &= 7759752, \quad b_{2,9} = 135980416, \\
b_{3,3} &= -\frac{27}{2}, \quad b_{3,4} = -\frac{2187}{4}, \quad b_{3,5} = -\frac{649539}{32}, \quad b_{3,6} = -\frac{98513415}{128}, \quad b_{3,7} = -\frac{3875799213}{128}, \\
b_{3,8} &= -\frac{1265272270353}{1024}, \quad b_{3,9} = -\frac{106704628133103}{2048}, \\
b_{4,4} &= 384, \quad b_{4,5} = 45056, \quad b_{4,6} = 4100096, \quad b_{4,7} = 347602944, \quad b_{4,8} = 28894494720, \quad b_{4,9} = 2399280300032, \\
b_{5,5} &= -\frac{859375}{32}, \quad b_{5,6} = -\frac{888671875}{128}, \quad b_{5,7} = -\frac{161865234375}{128}, \\
b_{5,8} &= -\frac{207000732421875}{1024}, \quad b_{5,9} = -\frac{62346649169921875}{2048}, \\
b_{6,6} &= 3582306, \quad b_{6,7} = 1734623424, \quad b_{6,8} = 556163635320, \quad b_{6,9} = 150090026385024.
\end{aligned}$$

Some values of $d_{p,q}$ are

$$\begin{aligned}
d_{1,1} &= 1, \quad d_{1,2} = -\frac{1}{64}, \quad d_{1,3} = -\frac{21}{512}, \quad d_{1,4} = -\frac{2145}{8192}, \quad d_{1,5} = -\frac{323323}{131072}, \quad d_{1,6} = -\frac{30421755}{1048576}, \\
d_{1,7} &= -\frac{3305942145}{8388608}, \quad d_{1,8} = -\frac{1590158171745}{268435456}, \quad d_{1,9} = -\frac{824543781407775}{8589934592}, \\
d_{2,2} &= -\frac{1}{64}, \quad d_{2,3} = -\frac{567}{1024}, \quad d_{2,4} = -\frac{312741}{8192}, \quad d_{2,5} = -\frac{235702467}{65536}, \quad d_{2,6} = -\frac{427708145475}{1048576}, \\
d_{2,7} &= -\frac{1756913199480945}{33554432}, \quad d_{2,8} = -\frac{1971842247550780605}{268435456}, \quad d_{2,9} = -\frac{2366260407369698510385}{2147483648}, \\
d_{3,3} &= -\frac{525}{1024}, \quad d_{3,4} = -\frac{1340625}{8192}, \quad d_{3,5} = -\frac{3608515625}{65536}, \quad d_{3,6} = -\frac{42441064453125}{2097152}, \\
d_{3,7} &= -\frac{269038260498046875}{33554432}, \quad d_{3,8} = -\frac{905851823096923828125}{268435456}, \quad d_{3,9} = -\frac{3202573724839324951171875}{2147483648}, \\
d_{4,4} &= -\frac{1030029}{8192}, \quad d_{4,5} = -\frac{38038627627}{262144}, \quad d_{4,6} = -\frac{292292272742925}{2097152}, \quad d_{4,7} = -\frac{2178975938039761245}{16777216}, \\
d_{4,8} &= -\frac{32681271562328539043985}{268435456}, \quad d_{4,9} = -\frac{498218295256742735716364505}{4294967296}, \\
d_{5,5} &= -\frac{24546728349}{262144}, \quad d_{5,6} = -\frac{561238628492295}{2097152}, \quad d_{5,7} = -\frac{9431269774195483755}{16777216}, \\
d_{5,8} &= -\frac{285795767967445744227765}{268435456}, \quad d_{5,9} = -\frac{8310236142024805237932527775}{4294967296}, \\
d_{6,6} &= -\frac{310532064755055}{2097152}, \quad d_{6,7} = -\frac{28582541707068569715}{33554432}, \quad d_{6,8} = -\frac{895748274557821906298385}{268435456}, \\
d_{6,9} &= -\frac{24086181828592215443355930675}{2147483648}.
\end{aligned}$$

Some values of $a_{p,q}$ are

$$\begin{aligned}
a_{0,0} &= 1, \quad a_{0,1} = \frac{1}{2}, \quad a_{0,2} = \frac{1}{8}, \quad a_{0,3} = \frac{1}{48}, \quad a_{0,4} = \frac{1}{384}, \quad a_{0,5} = \frac{1}{3840}, \quad a_{0,6} = \frac{1}{46080}, \\
a_{0,7} &= \frac{1}{645120}, \quad a_{0,8} = \frac{1}{10321920}, \quad a_{0,9} = \frac{1}{185794560}, \\
a_{1,0} &= -\frac{1}{2}, \quad a_{1,1} = -\frac{1}{2}, \quad a_{1,2} = -\frac{1}{4}, \quad a_{1,3} = -\frac{1}{12}, \quad a_{1,4} = -\frac{1}{48}, \quad a_{1,5} = -\frac{1}{240}, \quad a_{1,6} = -\frac{1}{1440}, \\
a_{1,7} &= -\frac{1}{10080}, \quad a_{1,8} = -\frac{1}{80640}, \quad a_{1,9} = -\frac{1}{725760}, \\
a_{2,0} &= -\frac{5}{2}, \quad a_{2,1} = -\frac{7}{4}, \quad a_{2,2} = -\frac{13}{16}, \quad a_{2,3} = -\frac{31}{96}, \quad a_{2,4} = -\frac{85}{768}, \quad a_{2,5} = -\frac{247}{7680}, \quad a_{2,6} = -\frac{733}{92160}, \\
a_{2,7} &= -\frac{313}{184320}, \quad a_{2,8} = -\frac{1313}{4128768}, \quad a_{2,9} = -\frac{19687}{371589120}, \\
a_{3,0} &= \frac{1}{3}, \quad a_{3,1} = -\frac{2}{3}, \quad a_{3,2} = -\frac{4}{3}, \quad a_{3,3} = -\frac{10}{9}, \quad a_{3,4} = -\frac{11}{18}, \quad a_{3,5} = -\frac{23}{90}, \quad a_{3,6} = -\frac{47}{540}, \\
a_{3,7} &= -\frac{19}{756}, \quad a_{3,8} = -\frac{191}{30240}, \quad a_{3,9} = -\frac{383}{272160}, \\
a_{4,0} &= -\frac{7}{24}, \quad a_{4,1} = -\frac{179}{48}, \quad a_{4,2} = -\frac{1199}{192}, \quad a_{4,3} = -\frac{6779}{1152}, \quad a_{4,4} = -\frac{36119}{9216}, \quad a_{4,5} = -\frac{187139}{92160}, \\
a_{4,6} &= -\frac{955199}{1105920}, \quad a_{4,7} = -\frac{4834379}{15482880}, \quad a_{4,8} = -\frac{24346919}{247726080}, \quad a_{4,9} = -\frac{122259539}{4459069440}, \\
a_{5,0} &= -\frac{76}{15}, \quad a_{5,1} = -\frac{256}{15}, \quad a_{5,2} = -\frac{428}{15}, \quad a_{5,3} = -\frac{1388}{45}, \quad a_{5,4} = -\frac{1097}{45}, \quad a_{5,5} = -\frac{3407}{225}, \\
a_{5,6} &= -\frac{10457}{1350}, \quad a_{5,7} = -\frac{31847}{9450}, \quad a_{5,8} = -\frac{96497}{75600}, \quad a_{5,9} = -\frac{291407}{680400}, \\
a_{6,0} &= -\frac{2401}{144}, \quad a_{6,1} = -\frac{95467}{1440}, \quad a_{6,2} = -\frac{738613}{5760}, \quad a_{6,3} = -\frac{5558491}{34560}, \quad a_{6,4} = -\frac{8191361}{55296}, \quad a_{6,5} = -\frac{297250507}{2764800}, \\
a_{6,6} &= -\frac{2134462933}{33177600}, \quad a_{6,7} = -\frac{15212619451}{464486400}, \quad a_{6,8} = -\frac{21570744737}{1486356480}, \quad a_{6,9} = -\frac{761828479147}{133772083200}.
\end{aligned}$$

Note that the solution is a Taylor series of variables s and $t^2 - 3\log(t^3)$.

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